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CLASSIFICATION PROBLEMS IN DIFFERENTIAL TOPOLOGY—III

APPLICATIONS TO SPECIAL CASES

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THE MAIN object of the paper is to make explicit the results of the preceding papers of the series; [I], classifying handlebodies $M \in \mathscr{H}(m, k, s)$, and [II], classifying diffeomorphisms of such handlebodies; in the two cases $m = 2s + 1$, $m = 2s$. This lays the foundation for the diffeomorphism classification of almost closed, $(s - 1)$-connected, $(2s + 1)$-manifolds which will appear subsequently.

We first perform some necessary computation of homotopy groups and homomorphisms. These are then applied to the cases mentioned. Finally, if $N \in \mathscr{H}(2s + 1, k, s)$, and the manifold M is obtained from ∂N by deleting the interior of an imbedded $2s$-disc, then $M \in \mathscr{H}(2s, 2k, s)$. We relate the invariants of N to those of M and close by giving (for $s \geqslant 4$) a necessary and sufficient condition for a diffeomorphism of M to be extendable to one of N.

We continue our numbering from papers [I] and [II].

§7. CALCULATIONS OF HOMOTOPY GROUPS

We first list a number of known results. The stable homotopy of the orthogonal group is due to Bott [17]; the remaining results to Kervaire [18], except in low dimensions.

PROPOSITION (4). (i) *For $r \geqslant s + 1$, the homotopy groups $\pi_{s-1}(SO_r)$ are stable under suspension, and depend (up to isomorphism) only on the residue of s modulo 8. They are as follows:*

$s(mod\ 8)$:	0	1	2	3	4	5	6	7
$\pi_{s-1}(SO)$:	Z	Z_2	Z_2	0	Z	0	0	0.

(ii) *Composition with the nonzero element of the stable 1-stem $\pi_s(S^{s-1}) \cong Z_2$ yields epimorphisms $\pi_{s-1}(SO) \longrightarrow \pi_s(SO)$ if $s \equiv 0, 1$ (mod 8).*
(iii) *For $s > 8$, the group $\pi_{s-1}(SO_s)$ still depends only on the residue of s modulo 8. We have the table:*

$s(mod\ 8)$:	0	1	2	3	4	5	6	7
$\pi_{s-1}(SO_s)$:	$Z + Z$	$Z_2 + Z_2$	$Z + Z_2$	Z_2	$Z + Z$	Z_2	Z	Z_2.

If s is even, the homomorphisms defined by the projection and suspension

$$\pi : \pi_{s-1}(SO_s) \longrightarrow \pi_{s-1}(S^{s-1}) \cong Z$$

$$S : \pi_{s-1}(SO_s) \longrightarrow \pi_{s-1}(SO_{s+1}) \cong \pi_{s-1}(SO)$$

induce a monomorphism $(\pi, S) : \pi_{s-1}(SO_s) \longrightarrow Z + \pi_{s-1}(SO)$. *The image has index 2; in fact* $\pi(x)$ *is always even. If s is odd, we choose a homomorphism*

$$\phi : \pi_{s-1}(SO_s) \longrightarrow Z_2$$

which is an isomorphism if $s \equiv 3, 5, 7 \ (mod\ 8)$, *and is such that* (ϕ, S) *induce an isomorphism if* $s \equiv 1 \ (mod\ 8)$.

$(\phi, S) : \pi_{s-1}(SO_s) \rightarrow Z_2 + \pi_{s-1}(SO)$

(iv) *If* $s = 3, 7$ *then* $\pi_{s-1}(SO_s) = 0$.

If $s = 5, 6$ *the results of* (iii) *are valid.*

If $s = 4, 8$ *the result of* (iii) *is modified only since* $\pi(x) + S(x)$ (*instead of* $\pi(x)$) *is even.*

Next we must calculate the map of Lemma (2)

$$F : \pi_{s-1}(SO_{m-s}) \times \pi_r(S^s) \longrightarrow \pi_{r-1}(SO_{m-r})$$

in certain cases. Putting $r = s+1$, $m = 2s + a + 1$, this becomes

$$F : \pi_{s-1}(SO_{s+a+1}) \times \pi_{s+1}(S^s) \longrightarrow \pi_s(SO_{s+a}).$$

We now fix ξ as the nonzero element of $\pi_{s+1}(S^s)$; for $s \geqslant 3$, this is a suspension element. By Lemma (5, 1), we have a homomorphism

$$F_a' : \pi_{s-1}(SO_{s+a+1}) \longrightarrow \pi_s(SO_{s+a}),$$

defined if $s \geqslant 3$, $s + a \geqslant 4$, $s + 2a \geqslant 4$. Now it follows from the definition of F that if it is suspended till all groups are stable, we get just the induced bundle, defined by composition $\pi_{s-1}(SO) \times \pi_r(S^s) \longrightarrow \pi_{r-1}(SO)$. In our case, stable compositions are described in Proposition (4, ii), and this determines the map F_a' if $a \geqslant 2$ (and $s \geqslant 3$).

We are also interested in the cases $a = 1$ ($s \geqslant 3$), $a = 0$ ($s \geqslant 4$). We shall need further unstable homotopy groups than those listed above; they may also be found in [18], and we shall use them without further reference. We shall also use Lemma (5, 3) in the form: $F_{a+1}'(Sz) = SF_a'(z)$, when the right-hand side is defined (in fact, we have already referred above to "suspending" F_a').

LEMMA (18). *Let* $s \geqslant 4$. *Then the homomorphisms*

$$F_1' : \pi_{s-1}(SO_{s+2}) \longrightarrow \pi_s(SO_{s+1}), \qquad F_0' : \pi_{s-1}(SO_{s+1}) \longrightarrow \pi_s(SO_s)$$

are zero unless $s \equiv 0$ *or* 1 (*mod* 8). *When* $s \equiv 1$ (*mod* 8), F_1' *is an isomorphism onto the torsion subgroup. When* $s \equiv 0$ (*mod* 8), $S \circ F_1'$ *is an isomorphism, so we can choose* ϕ *with* $\phi \circ F_1' = 0$.

Proof. The domains of F_1' and F_0' are stable groups, and so vanish unless $s \equiv 0, 1, 2$ or 4

(mod 8). Now we have the commutative diagram

$$\begin{array}{ccccc} \pi_{s-1}(SO_{s+1}) & \xrightarrow{S} & \pi_{s-1}(SO_{s+2}) & \xrightarrow{S} & \pi_{s-1}(SO_{s+3}) \\ \downarrow {\scriptstyle F'_0} & & \downarrow {\scriptstyle F'_1} & & \downarrow {\scriptstyle F'_2} \\ \pi_s(SO_s) & \xrightarrow{S} & \pi_s(SO_{s+1}) & \xrightarrow{S} & \pi_s(SO_{s+2}) \end{array}$$

where the maps in the upper row are isomorphisms. If $s \equiv 0$ or 1 (mod 8), F'_2 and the maps in the lower row are onto, so F'_0 and F'_1 are nonzero, and the descriptions given above follow.

Next let $s \equiv 2$ (mod 8). Use the diagram which extends the above one term to the left: replacing groups by isomorphic copies, it is

$$\begin{array}{ccccc} \mathbf{Z}+\mathbf{Z}_2 & \xrightarrow{S} & \mathbf{Z}_2 & \xrightarrow{S} & \mathbf{Z}_2 \\ \downarrow {\scriptstyle F'_{-1}} & & \downarrow {\scriptstyle F'_0} & & \downarrow {\scriptstyle F'_1} \\ \mathbf{Z}_8 & \xrightarrow{S} & \mathbf{Z}_4 & \xrightarrow{S} & \mathbf{Z}_2. \end{array}$$

The maps in the upper row give isomorphisms of the torsion subgroups; those in the lower row are onto. It follows that F'_0 and F'_1 are zero.

Finally, when $s \equiv 4$ (mod 8) and $s > 4$, we use the diagram

$$\begin{array}{ccccc} \pi_{s-1}(SO_{s-2}) & \xrightarrow{S^3} & \pi_{s-1}(SO_{s+1}) & \xrightarrow{S} & \pi_{s-1}(SO_{s+2}) \\ \downarrow {\scriptstyle F'_{-3}} & & \downarrow {\scriptstyle F'_0} & & \downarrow {\scriptstyle F'_1} \\ \pi_s(SO_{s-3}) & \xrightarrow{S^3} & \pi_s(SO_s) & \xrightarrow{S} & \pi_s(SO_{s+1}). \end{array}$$

Here, the maps in the upper row are isomorphisms, and $\pi_s(SO_{s-3})$ vanishes; the desired result again follows.

Observe that the argument breaks down when $s = 4$: F'_{-3} is not defined in this case. We conjecture that, nevertheless, F'_0 and F'_1 are zero. This is the only gap in our arguments when $s = 4$; if it can be filled (either with or against our conjecture), the results of this paper will all extend at once (with appropriate modification) to the case $s = 4$.

§8. THE FIRST STABLE RANGE

We use the term "first stable range" in a somewhat vague sense, to denote that all relevant homotopy groups are stable. This distinguishes it from the wider "second stable range", in which Haefliger's imbedding theorems [3], and our own results, are valid.

Consider the classification problem for handlebodies $N \in \mathscr{H}(n, k, s)$ for $n \geq 2s+1$. If $s \geq 2$, then also $2n \geq 4s+2 \geq 3s+4$ and Theorem (2) is applicable. With the notation of that theorem, $H = H_s(N)$ is a free abelian group of rank k, and a complete set of invariants is given by

$$\alpha : H \longrightarrow \pi_{s-1}(SO_{n-s})$$
$$\lambda : H \times H \longrightarrow \pi_s(S^{n-s}).$$

But since $n \geqslant 2s+1$, $\pi_s(S^{n-s}) = 0$ and $\pi_{s-1}(SO_{n-s})$ is stable. Thus $\lambda = 0$, and by Theorem (1), α is a homomorphism.

Now by Proposition (4, i), we know what $\pi_{s-1}(SO)$ is: in particular, it is cyclic. By a standard result on free abelian groups, we can choose a basis $e_1, \ldots, e_k$ for H with $\alpha(e_i) = 0$ for $i > 1$, and the subgroup $Im\,\alpha$ of $\pi_{s-1}(SO)$ determines the whole system.

LEMMA (19). *Let $s \geqslant 2$, $n \geqslant 2s+1$. For $s \equiv 3, 5, 6, 7$ (mod 8), all elements of $\mathscr{H}(n, k, s)$ are diffeomorphic. For $s \equiv 1, 2$ (mod 8), there are two types, one with trivial and one with nontrivial tangent bundle. For $s \equiv 0$ (mod 4), there is one type for each nonnegative integer.*

We can give explicit descriptions of all types. Since λ vanishes, by the Corollary to Theorem (2), any direct sum splitting of H determines one of N, so we can write N as a sum of $N_i \in \mathscr{H}(n, 1, s)$, where e_i is a basis of $H_s(N_i)$. Then N_i is diffeomorphic to the D^{n-s}-bundle over S^s defined by $\alpha(e_i) \in \pi_{s-1}(SO_{n-s})$; and we may suppose for $i > 1$ that $\alpha(e_i) = 0$ and so $N_i \cong S^s \times D^{n-s}$.

Next, we classify the diffeomorphisms of these handlebodies. By Lemma (10), their homotopy classes are in (1–1) correspondence with automorphisms of H preserving α. Thus if $\alpha = 0$, we obtain all automorphisms of the free abelian group H. If $\alpha \neq 0$, and $s \equiv 1, 2$ (mod 8), we have just those automorphisms which leave invariant the subgroup Ker α of index 2; these form a subgroup of index $2^k - 1$ in the group of all automorphisms. If $\alpha \neq 0$ and $s \equiv 0$ (mod 4), the automorphisms leave Ker α (a direct summand of rank $k-1$) invariant, and induce the identity on Coim α.

Now consider diffeomorphisms homotopic to the identity. Then (by Lemma (11)) we have obstructions

$$\beta_i \in \pi_s(SO_{n-s}), \qquad \mu_{ij} \in \pi_{s+1}(S^{n-s}).$$

For the first stable range, make the restriction $n \geqslant 2s+2$ (the case $n = 2s+1$ is considered later), and $s \geqslant 3$, so that the results of [II] do in fact apply. Then μ vanishes, and the β_i lie in the stable group $\pi_s(SO)$. Let $e_1, \ldots, e_k$ be a basis of H such that $\alpha(e_i) = 0$ for $i > 1$. Then the formula (Lemma 15) for the indeterminacy of β_i reduces (since λ is zero) to

$$\beta_i' = \beta_i + F(S\alpha_1, \xi_{1i}).$$

By Lemma (18), in the stable range, for $\gamma \in \pi_{s-1}(SO_{n-s-1})$, $\xi \in \pi_{s+1}(S^s)$, $F(\gamma, \xi)$ vanishes except when $s \equiv 0, 1$ (mod 8), $\xi \neq 0$, and γ is an odd multiple of the generator. It will be convenient to have a special name for this situation.

We say that a handlebody $N \in \mathscr{H}(n, k, s)$, with $n \geqslant 2s$, is in the *exceptional case* if either $s \equiv 1$ (mod 8), $S\alpha$ is nonzero, or $s \equiv 0$ (mod 8), and Im$(S\alpha)$ contains an odd multiple of the generator of $\pi_{s-1}(SO)$. (For $n = 2s$, $2s+1$, there might also be a case when F is nonzero with $s = 4$.)

Applying Theorem (4), we now obtain

LEMMA (20). *Let $n \geqslant 2s+2$, $s \geqslant 3$, $N \in \mathscr{H}(n, k, s)$. Then homotopic diffeomorphisms of (N rel D^n) are quasi-diffeotopic if $s \equiv 2, 4, 5, 6$ (mod 8), or in the exceptional case. Otherwise, the obstruction is a homomorphism $\beta : H \longrightarrow \pi_s(SO)$.*

§9. HANDLEBODIES IN $\mathscr{H}(2s, k, s)$

The next handlebodies to classify are those in $\mathscr{H}(2s, k, s)$. We shall again apply Theorem (2)—here we need $s \geqslant 3$ for its validity—and since now neither α nor λ vanishes, the full structure of Theorem (1) is brought into play. In this case, λ is a symmetric or skew-symmetric bilinear map of $H \times H$ to $\pi_s(S^s)$, which we identify with the integers. The most interesting case is when λ is unimodular—i.e. the determinant $\det \lambda(e_i, e_j) = \pm 1$. This case was discussed at some length in [11], from the same standpoint as here, so we need not give a full description. We recall one or two facts about the classification. Instead of α, it is simpler to consider related invariants, for by Proposition (4, iii), if s is even,

$$S \oplus \pi : \pi_{s-1}(SO_s) \longrightarrow \pi_{s-1}(SO) \oplus \pi_{s-1}(S^{s-1})$$

is a monomorphism, so instead of α we consider $\chi = S\alpha$ and $\pi\alpha$. The advantage of this is (Theorem (1)) that $S\pi\alpha(x) = \lambda(x, x)$, so $\pi\alpha$ can be forgotten, and the addition formula for α shows that χ is a homomorphism. The relation between S and π shows that if $s \neq 4, 8$ then $\lambda(x, x)$ is always even, whereas if s is 4 or 8, $\lambda(x, x)$ has the same parity as $\chi(x)$ (which, in this case, also lies in an infinite cyclic group). When s is odd, of course, $\pi\alpha = 0$, and we replace α by $\chi = S\alpha$, again a homomorphism, and (unless s is 3 or 7) $\phi\alpha$, which is not, but takes values in Z_2 and satisfies

$$\phi\alpha(x+y) = \phi\alpha(x) + \phi\alpha(y) + \lambda(x, y) \qquad (\text{mod } 2).$$

In the case when λ is unimodular and s odd, a complete classification was given in Lemma (5) of [11]. We take the opportunity here of pointing out that a similar classification can be given when s is even, if we assume that the unimodular quadratic form λ has zero signature. Since λ is unimodular, it induces an isomorphism of H on $Hom(H, Z)$; we let $\hat{\chi}$ be the element of H which corresponds by this isomorphism to χ in the case when χ is a homomorphism to Z. When χ is a homomorphism to Z_2, we can "lift" it to a homomorphism to Z (H being free), and hence define $\hat{\chi}$, which is determined modulo $2H$.

THEOREM (5). *For a handlebody in $\mathscr{H}(2s, r, s)$, where s is even, $s \geqslant 4$, λ unimodular with zero signature, then r is even, say $r = 2k$, and we can choose a basis $\{e_i, f_i : 1 \leqslant i \leqslant k\}$ with*

$$\lambda(e_i, e_j) = \lambda(f_i, f_j) = 0 \qquad \lambda(e_i, f_j) = \delta_{ij}$$

(except that if s is 4 or 8 we may have $\lambda(e_1, e_1) = 1$) and

if $s \equiv 6$ (mod 8), $\chi = 0$;

if $s \equiv 2$ (mod 8), $\hat{\chi} = 0$, f_1 or $e_1 + f_1$ (mod $2H$);

if $s \equiv 0$ (mod 4), $s \neq 4, 8$, $k = 1$, $\hat{\chi} = ae_1 + bf_1$ $(0 \leqslant |a| \leqslant b)$;

$$k \geqslant 2, \quad \hat{\chi} = 0 \quad \text{or} \quad = d(e_1 + Nf_1) \qquad (d > 0);$$

If $s = 4$ *or* 8, $\lambda(e_1, e_1) = 0$, $\hat{\chi}$ *as above with* a, b *(resp.* d*) even,*

$$\lambda(e_1, e_1) = 1, k = 1, \quad \hat{\chi} = ae_1 + bf_1 \qquad (0 \leqslant b, a + b \text{ ; } a \text{ even, } b \text{ odd}) \text{ ;}$$

$$k \leqslant 2, \quad \hat{\chi} = d(2e_1 + Nf_1) \qquad (d > 0 \text{ ; } d, N \text{ odd}).$$

The cases given are all inequivalent.

Proof. If $s \neq 4$ or 8, λ is an even quadratic form; it is then well-known that bases $\{e_i, f_i\}$ exist on which λ takes values as above (see e.g. [19], corollary to the basic lemma). Now we use the main results (Theorems (4) and (6)) of [19], which state that if $k \geqslant 2$, the orthogonal group of λ is transitive on vectors of given divisor, norm, and type. As λ is even, we need not worry about type, and 0, $d(e_1 + Nf_1)$ $(d > 0)$ gives a complete set of representatives of orbits of the orthogonal group. A similar result holds when $s = 4$, 8, on recalling that $\hat{\chi}$ is then a characteristic vector. When $k = 1$, the orthogonal group only contains 4 elements, and the classification is trivial.

For the case $s \equiv 2 \pmod 8$, reducing $ae_1 + bf_1$ mod 2, we see $\hat{\chi}$ may be taken as 0, e_1, f_1 or $e_1 + f_1$. Since e_1, f_1 are interchangeable, we can avoid taking e_1. The cases $\hat{\chi} = f_1$, $e_1 + f_1$ are inequivalent since, in this case, $\lambda(\hat{\chi}, \hat{\chi})$ (modulo 4) is an invariant.

This completes the proof; similar classifications can be given whenever λ is unimodular and strongly indefinite, so that the results of [19] apply.

We mention at this point another extension of our results which has become possible since [11] was written. We refer the reader to the remark on p. 183. It has now been proved by Adams [16] that, in fact, J is a monomorphism in the stable range when $n \equiv 1, 2 \pmod 8$. It follows that in all cases considered in [11], the combinatorial classification coincides with the differential—at least, for almost closed manifolds. Moreover, in each case if $n \not\equiv 0 \pmod 4$, the classification coincides with that according to homotopy type of the corresponding closed manifold.

We now remark that if $N \in \mathscr{H}(2s + 1, k, s)$, then ∂N is a closed, $(s - 1)$-connected, $2s$-manifold, with $H_s(\partial N)$ of rank $2k$. Hence, if $s \geqslant 3$, by a result of Smale, ∂N admits a handle decomposition with one O-handle, $2k$ s-handles, and one $2s$-handle, and (see below) this may also be seen directly for any s. Thus if M is obtained from ∂N by deleting the interior of an imbedded disc D^{2s}, we have $M \in \mathscr{H}(2s, 2k, s)$. We next enquire how the invariants of N determine those of M. In fact, as remarked after Lemma (19), if $\{e_i' : 1 \leqslant i \leqslant k\}$ is a base of $H_s(N)$, N is a sum of k elements of $H(2s + 1, 1, s)$—i.e. D^{s+1}-bundles over S^s, which are classified by elements $\alpha(e_i')$ of $\pi_{s-1}(SO_{s+1})$. Its boundary is the connected sum of their boundaries, and correspondingly M is the sum of k elements of $H(2s, 2, s)$, derived from these bundles as M was from N. But the invariants of an S^s-bundle over S^s were computed in [11].

Write α_i' for $\alpha(e_i') \in \pi_{s-1}(SO_{s+1})$; pick an element $\alpha_i \in \pi_{s-1}(SO_s)$ which suspends to α_i' (suspension is certainly onto in this dimension). This corresponds to choosing a cross-section of the bundle; denote its homology class by e_i and that of a fibre by f_i. Then in the

S^s-bundle over S^s, we have

$$\alpha(e_i) = \alpha_i \qquad \alpha(f_i) = 0$$

$$\lambda(e_i, e_i) = S\pi\alpha_i \qquad \lambda(e_i, f_i) = 1 \qquad \lambda(f_i, f_i) = 0.$$

Observe that unless $s = 4, 8$, the reduction α_i can be chosen with $S\pi\alpha_i = 0$, and if $s = 4, 8$ we may suppose $S\pi\alpha_i = 0$ or 1, so that it vanishes if $\alpha_i' = 0$. If s is even, this condition completely determines the reduction; if s is odd, we impose instead the condition $\phi\alpha_i = 0$. We thus obtain

LEMMA (21). *Let $N \in \mathscr{H}(2s+1, k, s)$, $s \geq 3$, and choose a base $\{e_i'\}$ for $H_s(N)$ such that $\alpha(e_i') = 0$ for $i > 1$. Then for the corresponding $M \in \mathscr{H}(2s, 2k, s)$, $H_s(M)$ has a base $\{e_i, f_i\}$ such that if $i : M \subset N$,*

(1) *$i_*(e_i) = e_i'$, $i_*(f_i) = 0$.*

(2) *$\lambda(e_i, e_j) = \lambda(f_i, f_j) = 0$, $\lambda(e_i, f_j) = \delta_{ij}$, except that $\lambda(e_1, e_1) = 1$ if $s = 4$ or 8 and $\alpha(e_1')$ is an odd multiple of the generator.*

(3) *$\chi(e_i) = \chi(f_i) = 0$ except for $\chi(e_1) = \alpha(e_1')$.*

(4) *$\phi(e_i) = \phi(f_i) = 0$.*

Notice that this implies that if s is even, the signature of M vanishes, and when s is odd, its Arf invariant does. A closer inspection, using Lemma (5) of [11] and Theorem (5) above, yields

COROLLARY *$M \in \mathscr{H}(2s, r, s)$ appears in the list above if and only if*

(1) *λ is unimodular.*

(2) *For s even, the signature and $\lambda(\hat{\chi}, \hat{\chi})$ vanish.*

(3) *For s odd, Φ and $\phi(\hat{\chi})$ vanish.*

This suggests an interpretation of the Grothendieck groups of [11] as cobordism groups of some kind. We shall work out the details of this in a later section.

§10. DIFFEOMORPHISMS OF HANDLEBODIES: FIRST UNSTABLE CASES

First let $N \in \mathscr{H}(n, k, s)$ with $n = 2s + 1$ and $s \geq 3$, and so $2n \geq 3s + 5$. We have already discussed the homotopy classification of diffeomorphisms of N: we next give the quasi-diffeotopy classification of those homotopic to 1. We must consider obstructions.

$$\beta_i \in \pi_s(SO_{s+1}) \qquad \mu_{ij} \in \pi_{s+1}(S^{s+1}) \cong Z.$$

Since λ is zero, Lemma 14 shows that the μ_{ij} are well-defined. By Lemma (15), the indeterminacy of the β_i is given by

$$\beta_i' = \beta_i + \Sigma_j F(S\alpha_j, \xi_{ji}).$$

and if we choose the base of $H_s(N)$ as usual, this reduces to $\beta_i' = \beta_i + F(S\alpha_1, \xi_{1i})$, so that the indeterminacy of each β_i is just $F(S\alpha_1, \eta)$ where η is the nonzero element of $\pi_{s+1}(S^s)$. Now this was computed in Lemma (18); it is nonzero precisely in the exceptional case, when its image is characterised by $\phi = 0$ (when $s \equiv 0 \pmod 8$) and by $\pi = 0$ (when $s \equiv 1 \pmod 8$). As in the preceding paragraph, we shall modify our invariants slightly. It is convenient to use the formulae of Theorem (1), with μ and β replacing λ and α, to extend μ_{ij} and β_i to maps

$$\mu : H \times H \longrightarrow Z$$

$$\beta : \quad H \quad \longrightarrow \pi_s(SO_{s+1})$$

(cf. proof of Theorem (2)). Then, as before, if s is odd we replace β by $S\beta$ and $\pi\beta$ (and $\pi\beta$ is determined by μ): if s is even, replace β by $S\beta$ and $\phi\beta$. Then we have

LEMMA (22). *Let $N \in \mathscr{H}(2s+1, k, s)$, $s \geqslant 3$, $s \neq 4$. The obstruction to quasi-diffeotopy of homotopic diffeomorphisms of (N rel D^{2s+1}) is given by an $(s+1)$-symmetric bilinear map $\mu : H \times H \longrightarrow Z$, a homomorphism $S\beta : H \longrightarrow \pi_s(SO)$ and, if s is even ($\neq 6$), a map $\phi\beta : H \longrightarrow Z_2$ satisfying $\phi\beta(x+y) = \phi\beta(x) + \phi\beta(y) + \mu(x, y) \pmod 2$.*

In the exceptional case, $S\beta$ must be omitted.

Here, of course, $S\beta$ and μ satisfy the usual relation: if s is odd, $\mu(x, x)$ is even, except that if s is 3 or 7, identify $\pi_s(SO)$ with Z; then $S\beta(x) + \mu(x, x)$ is even. The case $s = 4$ is distinguished because we were unable in Lemma (18) to compute $F_1' : \pi_3(SO_6) \longrightarrow \pi_4(SO_5)$ (when $s = 3$, F_1' is clearly zero, as its domain is a zero group). If F_1' is zero, the above result holds: if not, $\phi\beta$ must be omitted when α is nonzero modulo 2.

We now consider diffeomorphisms of a handlebody $M \in \mathscr{H}(2s, k, s)$. This case is the deepest which we have to consider in detail; it also brings us to the central point in our whole investigation. It is not surprising, then, that it is rather more complicated than the other cases; to render the problem feasible, we shall suppose λ nonsingular. We also omit discussion of the group of homotopy classes of diffeomorphisms as too complicated; even the simplest cases give an orthogonal or a symplectic group over the integers.

We assume $s \geqslant 4$ in order to be able to use Theorem (4) though, as before, the case $s = 4$ is somewhat embarrassing. First consider μ: we have obstructions $\mu_{ij} \in \pi_{s+1}(S^s)$, and we identify this group of order 2 with Z_2. Observe that signs can now be omitted. By Lemma (14), the change in the μ_{ij}, corresponding to a choice of elements $\xi_{ri} \in \pi_{s+1}(S^s)$, which we also regard as integers modulo 2, is given by

$$\mu'_{ij} = \mu_{ij} + \Sigma_r \lambda_{jr} \xi_{ri} + \Sigma_r \lambda_{ir} \xi_{rj}$$

since composition now reduces to multiplication. We regard the λ_{ij}, μ_{ij} and ξ_{ij} as defining matrices $\mathbf{L}$, $\mathbf{M}$, $\mathbf{N}$; elements μ_{ii} may be chosen arbitrarily: the most reasonable choice being $S\beta_i$. The above equation now reads

$$\mathbf{M}' = \mathbf{M} + \mathbf{LN} + (\mathbf{LN})^t,$$

where t denotes the transpose. Now if we choose $\mathbf{P}$ to equal $\mathbf{M}$ above the diagonal and zero on and below it, since $\mathbf{L}$ is nonsingular, we can solve $\mathbf{LN} = \mathbf{P}$, and we obtain correspondingly $\mathbf{M}' = \mathbf{M} + \mathbf{P} + \mathbf{P}^t$ whence the off-diagonal elements of $\mathbf{M}'$ vanish. Thus the μ_{ij} do not represent genuine obstructions, but to keep them zero, we may only allow changes by ξ_{ij} which form a matrix $\mathbf{N}$ such that $\mathbf{LN}$ is symmetric.

We must now consider the obstructions β_i: by Lemma (15), their indeterminacy is given by

$$\beta_i' = \beta_i + \Sigma_j F(S\alpha_j, \xi_{ji}) + \Sigma_j \partial(\lambda_{ij} \circ \xi_{ji}).$$

Consider the second term first. The $\Sigma_j \lambda_{ij} \xi_{ji}$ are just the diagonal terms of LN, and N can be chosen with LN symmetric and these arbitrary. If F vanishes, we can disregard the first term, and we then see that the indeterminacy of β is precisely the image of $\partial : \pi_{s+1}(S^s) \longrightarrow \pi_s(SO_s)$. Since this coincides with the kernel of the suspension, in this case $S\beta$ is the desired obstruction.

If F does not vanish, then (if $s \neq 4$) by Lemma (18), we are in the exceptional case. Recall that in the previous paragraph we introduced an element $\hat\chi$ of H (which is certainly well-determined modulo 2) which, by duality, gives rise to χ. Then if θ generates $\pi_{s-1}(SO_{s+1})$,

$$\Sigma_j F(S\alpha_j, \xi_{ji}) = \Sigma_j F(\chi(e_j), \xi_{ji})$$
$$= \Sigma_j F(\lambda(\hat\chi, e_j)\theta, \xi_{ji}) = F(\theta, n_i),$$

where $n_i = \Sigma_j \lambda(\hat\chi, e_j)\xi_{ji}$. Since we are in the exceptional case, $\hat\chi$ is nonzero modulo 2; since we are only concerned with its value modulo 2, we can take it as e_1. We now have

$$n_i = \Sigma_j \lambda_{1j}\xi_{ji} = (\mathrm{LN})_{1i}$$

giving the first row of the matrix LN. Now LN is only restricted to be symmetric; the only relation between the first row and the principal diagonal is that their first elements coincide. Thus if $i \neq 1$, the indeterminacy subgroup of β_i is generated by $\partial\eta$ and $F_0'(\theta)$; if $i = 1$, it is generated by $\partial\eta + F_0'(\theta)$. Now (by Lemma (18)), $\partial\eta$ and $F_0'(\theta)$ are distinct elements of order 2 in $\pi_s(SO_s)$. So when $i \neq 1$, we see that since β_i can be varied by $\partial\eta$, we need only consider $S\beta_i$ (as above); since β_i can also be varied by $F_0'(\theta)$, $S\beta_i$ can be varied by $SF_0'(\theta) = F_1'(S\theta)$, and we have a situation as in Lemma (22); in fact the only remaining invariant is $\phi S\beta_i$, when $s \equiv 0 \pmod 8$. For $i = 1$, the indeterminacy subgroup is only half as big, so there is an extra invariant in $\mathbf{Z}_2$. To make this definite, we choose a subgroup Ω of $\pi_s(SO_s)$ which if $s \equiv 0 \pmod 8$ is mapped isomorphically by ϕS to $\mathbf{Z}_2$, and if $s \equiv 1 \pmod 8$ is zero (in fact if $s \equiv 0 \pmod 8$, $\pi_s(SO_s) \cong \mathbf{Z}_2 + \mathbf{Z}_2 + \mathbf{Z}_2$, so this is certainly possible). Define $\omega \in \mathbf{Z}_2$ to be zero if the indeterminacy can be so chosen that $\beta_1 \in \Omega$, and nonzero otherwise. Then ω and (if $s \equiv 0 \pmod 8$) $\phi S\beta_1$ are independent, and completely describe β_1 up to its indeterminacy.

LEMMA (23). *Let $M \in \mathscr{H}(2s, k, s)$ (where $s \geqslant 5$) have nonsingular λ. Then the precise obstruction to quasi-diffeotopy rel D^{2s} of two homotopic diffeomorphisms is in general a homomorphism $S\beta : H_s(M) \longrightarrow S\pi_s(SO_s)$, but in the exceptional case an invariant $\omega \in \mathbf{Z}_2$ and (when $s \equiv 0 \bmod 8$) a homomorphism $\phi S\beta : H_s(M) \longrightarrow \mathbf{Z}_2$.*

A similar discussion can be given when $s = 4$ for each of the possible alternative assumptions about F_0'; again, if F_0' is zero, the case $s = 4$ is included in the above discussion: if not (so $S\alpha$ is nonzero mod 2) we have a single invariant $\omega \in \mathbf{Z}_2$.

If we have a diffeomorphism h not homotopic to the identity, it may still be possible to compute some of the invariants. If e_1 is a basis element of $H_s(M)$, with $h_*(e_1) = e_1$, we can construct a cylinder $S^s \times I$ just as in §5, and compute β_1. In particular if $s \equiv 0 \pmod 4$ and $\chi \neq 0$, since h preserves α, it will preserve the associated primitive vector e_1 of $\hat{\chi}$. Thus ω can still be defined in the exceptional case when $s \equiv 0 \pmod 8$. In fact this is also possible when $s \equiv 1 \pmod 8$ though since $\hat{\chi}$ is only then determined modulo 2, a more complicated construction is sometimes necessary. As the reader has no doubt already surmised, the invariant ω assumes particular importance in the sequel.

§11. OBSTRUCTIONS TO EXTENDING DIFFEOMORPHISMS

We now combine the ideas of the last three sections. Let $N \in \mathscr{H}(2s+1, k, s)$, $s \geqslant 5$. We have defined a manifold $M \in \mathscr{H}(2s, 2k, s)$ by deleting the interior of a disc D^{2s} imbedded in ∂N. Each diffeomorphism of N leaving D^{2s} invariant induces a diffeomorphism of M. We are now ready to ask, conversely, which diffeomorphisms of M extend to diffeomorphisms of N. The concept of quasi-diffeotopy is just sufficiently precise to permit us to completely resolve this question.

We need some notational conventions. Let $i: M \longrightarrow N$ be the inclusion. We shall use $H, \lambda, \alpha, \mu, \beta$ to refer to the customary invariants of M, and attach primes to refer to N. We choose bases $\{e_i'\}$ of H' and $\{e_i, f_i\}$ of H as in Lemma (21). Recall that $i_*(e_i) = e_i'$, $i_*(f_i) = 0$, and so the f_i form a basis for the kernel, K, of $i_*: H \longrightarrow H'$.

Let h' be a diffeomorphism of N which leaves D^{2s} fixed, h the induced diffeomorphism of M. Then h_*' is an automorphism of H', say $h_*'(e_i') = \Sigma_j a_{ij} e_j'$. Now $h'i = ih$, so also $h_*' i_* = i_* h_*$. Thus h_* preserves K. So we can write:

$$h_*(e_i) = \Sigma_j (a_{ij} e_j + b_{ij} f_j), \qquad h_*(f_i) = \Sigma_j c_{ij} f_j$$

for appropriate b_{ij}, c_{ij}. Now h_* preserves λ, so

$$\begin{aligned} \delta_{ij} = \lambda(e_i, f_j) &= \lambda(h_*(e_i), h_*(f_j)) \\ &= \lambda(\Sigma_k(a_{ik} e_k + b_{ik} f_k), \Sigma_l c_{jl} f_l) \\ &= \Sigma_{kl} a_{ik} c_{jl} \delta_{kl} = \Sigma_k a_{ik} c_{jk}. \end{aligned}$$

Thus the matrix of c_{ij} is the transposed inverse of the matrix of a_{ij}, so is determined by it. However, the a_{ij} do not determine the b_{ij}, as we shall see below.

Now suppose h' homotopic to the identity. Then the above reduce to

$$h_*(e_i) = e_i + \Sigma_j b_{ij} f_j, \qquad h_*(f_i) = f_i.$$

We wish to determine $b_{ij} = \lambda(e_j, h_*(e_i)) - \lambda(e_j, e_i)$.

LEMMA (24). *We have* $b_{ij} = \mu_{ij}'$.

Proof. We calculate μ_{ij}' by attempting to define a diffeomorphism Q of $N \times I$ agreeing with 1 on $N \times 0$ and with h' on $N \times 1$: μ_{ij}' is then defined by an intersection number. Altering h' by a diffeotopy, we may take it as the identity on $D^{2s+1} \cup (D_j^s \times O)$, so Q can be taken as the identity on the product of this with I. Since $\lambda' = 0$, $D_i^s \times O$ and $D_j^s \times O$ may be completed to disjoint embedded spheres S_i^s, S_j^s in N, and μ_{ij}' is the intersection number of

$Q(S_j^s \times I) = S_j^s \times I$ with $Q(S_i^s \times I)$—or, projecting on N by π_1—of S_j^s with $\pi_1 Q(S_i^s \times I)$. Since this intersection number is well-defined, it is unaltered by taking S_i^s in M representing e_i (of course, $\pi_1 Q(S_i^s \times I)$ will not lie in M). Let y be the homology class defined by $\pi_1 Q(S_i^s \times I)$ in $H_{s+1}(N, \partial N)$; then $\partial y = h_*(e_i) - e_i$. Using $\cap$ to denote intersection numbers, we now have

$$\begin{aligned} \mu'_{ij} &= e'_j \cap y = i_*(e_j) \cap y = e_j \cap \partial y \\ &= e_j \cap (h_*(e_i) - e_i) = b_{ij}. \end{aligned}$$

We observe that μ'_{ij} was indeed well-defined, with zero indeterminacy, and that if the formulae of Theorem (1) are used to extend μ' to a map of $H \times H$ to Z, the above argument is also valid for $i = j$. We can deduce from the Lemma that

$$b_{ji} = \mu'_{ji} = (-1)^{s+1}\mu_{ij} = (-1)^{s+1} b_{ij},$$

but this already follows from

$$\begin{aligned} \lambda(e_i, e_j) &= \lambda(h_*(e_i), h_*(e_j)) \\ &= \lambda(e_i + \Sigma_k b_{ik} f_k, e_j + \Sigma_l b_{jl} f_l) \\ &= \lambda(e_i, e_j) + b_{ji} + (-1)^s b_{ij}. \end{aligned}$$

Hence, in particular, for s even $b_{ii} = 0$. We now observe that if (as by Lemma (21) we may) we suppose $\phi(e_i) = \phi(f_i) = 0$, then for s odd and not 3 or 7, b_{ii} is even. For h_* preserves ϕ so, reckoning modulo 2,

$$\begin{aligned} 0 = \phi(h_*(e_i)) &= \phi(e_i + \Sigma_j b_{ij} f_j) \\ &= \phi(e_i) + b_{ii} + \phi(\Sigma_j b_{ij} f_j) = b_{ii}. \end{aligned}$$

The discussion of β is easier than that of μ; in fact we have

LEMMA (25). *For $x \in H$, $\beta'(i_*(x)) = S\beta(x)$.*

Proof. Observe that, just as we defined α in general in §1, we can also define β (as in §5) for more general manifolds than handlebodies. Also, β is natural for inclusions of manifolds of the same dimension (this was already used in Lemma (17)). But we have $M \times I \subset N$, and normal bundles in $M \times I$ are just the suspensions of those in M. The result follows.

We observe that in the non-exceptional case, β' and $S\beta$ are both primary obstructions (Lemmas (22) and (23)). In the exceptional case, apart from ω, we have nothing if $s \equiv 1$ (mod 8), and $\phi\beta'$ and $\phi S\beta$ if $s \equiv 0$ (mod 8). Thus β (with its indeterminacy) does conveniently determine β' up to *its* indeterminacy. (If there is an exceptional case with $s = 4$, $\beta(e_i)$ and $\beta'(e'_i)$ are both totally indeterminate.)

We are now ready to consider the problem of when a given diffeomorphism h of M is the restriction of a diffeomorphism of N. Firstly, it is clear that h_* must leave K invariant. Suppose this satisfied; then if we write $h_*(e_i) = \Sigma_j (a_{ij} e_j + b_{ij} f_j)$, this defines an automorphism T of H' by $T(e'_i) = \Sigma_j a_{ij} e'_j$. Since h_* is induced by the diffeomorphism h, it preserves α, hence T preserves α' (as in Lemma (25), $\alpha'(e'_i) = S\alpha(e_i)$). Now λ' is zero, and we can appeal to Lemma (10) to see that T does correspond to a diffeomorphism h' of N. Modifying this by a diffeotopy if necessary, we may (by the Disc Theorem) suppose that it leaves M invariant. Now multiply h by the inverse of the restriction of h' to M: we see that we can suppose $a_{ij} = \delta_{ij}$. It now follows from calculations above that $b_{ji} = (-1)^{s+1} b_{ij}$, and that $h_*(f_i) = f_i$.

We now repeat the argument. Since the image of $\pi : \pi_s(SO_{s+1}) \longrightarrow \pi_s(S^s) \cong Z$ is zero for s even, the whole group for $s = 3$ or 7, and of index 2 otherwise, it always contains b_{ii}, so we can choose $\beta_i \in \pi_s(SO_{s+1})$ with $\mu_{ii} = \pi\beta_i = -b_{ii}$. By Lemma (13), there is a diffeomorphism h'' of N, homotopic to the identity, with obstructions $\mu_{ij} = -b_{ij}$ $(i < j)$ and β_i as above. We may suppose that it leaves M invariant; multiplying h by its restriction to M, we reduce b_{ij} to zero. Thus we may suppose h homotopic to the identity.

We may again repeat the argument, using Lemmas (13) and (25), to reduce $\beta(e_i)$ (or rather its suspension) to zero since, by a remark above, the indeterminacy of $S\beta(e_i)$ coincides with that of $\beta'(e'_i)$. We are left with $\beta(f_i)$, which are genuine obstructions—by Lemma (25) they must vanish for h to be extendable.

Observe that the classifications relative to discs of diffeomorphisms of M and N are adequate for this problem. For the kernel of $\tilde{\pi}_0(\mathrm{Diff}\,(N \text{ rel } D^{2s+1})) \longrightarrow \tilde{\pi}_0(\mathrm{Diff}\, N)$ is the image of Z_2 defined by extending to N the diffeotopy of D^{2s+1} defined by continuous rotation through 2π about a great D^{2s-1}. We get the same element by taking D^{2s+1} as a half ball, meeting M in a disc D^{2s}, normal to D^{2s-1}. Thus the diagram

$$\begin{array}{ccccc}
Z_2 & \longrightarrow & \tilde{\pi}_0(\mathrm{Diff}\,(N \text{ rel } D^{2s+1})) & \longrightarrow & \tilde{\pi}_0(\mathrm{Diff}\, N) \\
\Vert & & \downarrow i_* & & \downarrow i_* \\
Z_2 & \longrightarrow & \tilde{\pi}_0(\mathrm{Diff}\,(M \text{ rel } D^{2s})) & \longrightarrow & \tilde{\pi}_0(\mathrm{Diff}\, M)
\end{array}$$

is commutative and exact, so there is a (1–1) correspondence between cosets of $i_*\tilde{\pi}_0(\mathrm{Diff}\, N)$ in $\tilde{\pi}_0(\mathrm{Diff}\, M)$ and the cosets in the relative groups.

THEOREM (6). *Let $N \in \mathscr{H}(2s+1, k, s)$ with $s \geqslant 4$, and $M = \partial N - \mathring{D}^{2s}$. Then a diffeomorphism h of M extends to one of N if and only if*
(i) *h leaves invariant K, the kernel of $i_* : H_s(M) \longrightarrow H_s(N)$*
(ii) *certain obstructions, defined by $\beta(x)$ for $x \in K$, vanish.*
More precisely, if $h_* = 1$, we have in general a homomorphism $S\beta : K \longrightarrow S\pi_s(SO_s)$, and *in the exceptional case an invariant $\omega \in Z_2$ and, if $s \equiv 0 \pmod 8$, a homomorphism $\phi S\beta : K \longrightarrow Z_2$.*

It is perhaps worth indicating that our arguments could have been addressed more directly to this problem, so we outline an

Alternative Proof. The necessity of condition (i) is evident; suppose it satisfied. We can extend h to a prismatic neighbourhood of M by a product map. Recall that we write $N = N' \cup (\partial N' \times I)$, where $N' = D^{2s+1} \cup_{i=1}^{k} (D_i^s \times D_i^{s+1})$. We next seek to extend h to $O \times D_i^{s+1}$. Now $h(O \times \partial D_i^{s+1})$ is already given; by condition (i), its homology class vanishes, so the sphere is nullhomotopic in N. Hence by Haefliger's theorem [3] we can extend h to an imbedding of $O \times D_i^{s+1}$. Next, we must make these discs disjoint: the proof that this is possible is essentially the same (using Haefliger's theorem) as the proof that the μ_{ij} have total indeterminacy. In order to extend to $D_i^s \times D_i^{s+1}$, we need only check that the normal bundles behave correctly; the obstructions here are just the $\beta(f_i)$, and the problem reduces to the algebraic one of calculating the indeterminacies of these. If these obstructions also vanish, we can calculate the homology and show that the part of N on which h is defined is a deformation retract; using Smale's theorems [9], we can show that what is left is a disc D^{2s+1} attached along a D^{2s} in its boundary; it is now easy to extend the diffeomorphism to this.

Theorem (6) is our key result, and motivated the development of the entire theory up to this point. We shall use it later to give a complete diffeomorphism classification of almost closed, $(s-1)$-connected $(2s+1)$-manifolds.

The proof above leads to one further observation: that we can often define β even on elements of H not left fixed by h_*. For instead of comparing any chosen framing of the normal bundle with that induced by h_*, observe that if $x \in K$, a representative s-sphere in M bounds discs in N, so we obtain a class of framings of its normal bundle. If also $h_*(x) \in K$, we can compare the two classes and obtain an invariant $\beta(x)$: it is easily seen that this has the same indeterminacy as previously computed.

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(References 1–15 can be found in the previous papers in this series: I. Classification of Handlebodies, *Topology* 2 (1963), 253–261; and II. Diffeomorphisms of Handlebodies, *Topology* 2 (1963), 263–272. However, they are reproduced here for the benefit of our readers.)

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