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CLASSIFICATION PROBLEMS IN DIFFERENTIAL TOPOLOGY—IV

THICKENINGS

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THIS PAPER is concerned with defining and establishing some basic properties of a functor of CW-complexes, which we will call the *thickening* functor: roughly speaking, for K a CW-complex, $\mathscr{T}^m(K)$ denotes the set of diffeomorphism classes of m-manifolds homotopy equivalent to K. Our object is to set up some algebraic machinery for computing this functor in some special cases: in particular, we obtain comprehensive generalisations of the results of [I].

It should be noted that there is already in existence an extensive technique, originating from work of S. P. Novikov [13], for giving diffeomorphism classifications of smooth manifolds. The overlap of results is very small, since to apply this technique to a manifold with boundary M, one needs to start from the homotopy type of the CW pair $(M, \partial M)$; whereas we start simply from the homotopy type of M.

The arguments of this paper are applicable equally to differential and to piecewise linear manifolds. We make the convention that, throughout the paper, the terms "manifold" and "homeomorphism" are to be interpreted consistently in either the differential or the piecewise linear sense.

We apologise to readers anticipating the classification of $(n-1)$-connected $(2n+1)$-manifolds announced in [I]: this paper had its origin in the discovery of a gap in our original argument—and provides adequate techniques to fill the gap.

The notion of thickening is in part motivated by results of Mazur in [11]; in particular his "non-stable neighbourhood theorem". We will not state this theorem here: it is the special case of our embedding theorem which corresponds to simple homotopy equivalence. However, Mazur brought out the relation existing between thickenings (in the sense below), i.e. simple homotopy equivalences $\phi: K^k \to M^m$ of a complex to a manifold which satisfies $\pi_1(\partial M) \cong \pi_1(M)$, and a natural (but hard to define) concept involving a parallelism between a cell decomposition of K and a handle decomposition of M. Here again, we shall give no details, but the parallelism is quite clear from the proof of our imbedding theorem.

It is convenient to mention at this point one simple homotopy-theoretic consequence of this, viz. that the pair $(M, \partial M)$ is $(m-k-1)$-connected, (for there is a handle decomposition of M based on ∂M with handle dimensions $\geqq (m-k)$). This can easily be shown without handles: since $\pi_1\, \partial M = \pi_1 M$ by hypothesis, it is enough to consider the universal covers and show $H_i(\tilde{M}, \partial\tilde{M}) = 0$ for $i < m-k$. By duality, this is equivalent to $H^j\tilde{K} = H^j\tilde{M} = 0$ for $j > k$ (with infinite chains), which is trivially true.

§1. DEFINITION

Let K^k be a finite CW-complex with base point $*$ and dimension k, $\phi : K^k \to M^m$ a simple homotopy equivalence (preserving the base point) of K and a manifold M of dimension $m \geqq k+3$. We suppose also that the base point $*$ of M lies in the boundary ∂M; that the inclusion $i : \partial M \to M$ induces an isomorphism $i_* : \pi_1(\partial M) \to \pi_1(M)$ of fundamental groups, and that the tangent space to M at $*$ is oriented. Then we say that ϕ defines a *thickening* (or, more precisely, an *m-thickening*) of K.

In fact, a thickening is to be an equivalence class of such ϕ; we define $\phi_1 : K \to M_1$ and $\phi_2 : K \to M_2$ to be *equivalent* if there is a homeomorphism $h : M_1 \to M_2$, preserving $*$ and the given orientations of the tangent space there, such that $h\phi_1 \simeq \phi_2 : (K, *) \to (M_2, *)$.

We write $\mathscr{T}^m(K)$ for the set of equivalence classes. Then $\mathscr{T}^m(K)$—the set of thickenings of K—is to be our object of study. The extra conditions in the definition are inserted to make the theory of our functor a little easier (not harder!). Note that ϕ is *not* assumed to be an imbedding. As an example, Smale's solution of the Poincaré conjecture shows that for $m \geqq 6$, $\mathscr{T}^m(*)$ contains only one element. In fact, we will always suppose $m \geqq 6$: to obtain a good theory for $m < 6$, one needs (at present) a more subtle definition.

§2. BASIC OPERATIONS

We have two generally defined and two sometimes defined operations.

Product. We leave to the reader to verify that the ordinary Cartesian product induces an operation

$$\times : \mathscr{T}^m(K) \times \mathscr{T}^n(L) \to \mathscr{T}^{m+n}(K \times L)$$

for all finite CW-complexes K^k and L^l, and $m \geqq k+3$, $n \geqq l+3$. The product is associative; also commutative (up to the usual change by $(-1)^{mn}$ of orientation at the base point). We will not in fact use this operation, but rather products by $I = [0, 1]$, which are not included as a special case. Endow I with the usual orientation, and $* = 0$: then multiplication (on the right) by I defines an operation

$$S : \mathscr{T}^m(K) \to \mathscr{T}^{m+1}(K)$$

which we call suspension.

Sum. Let $\phi : K^k \to M^m$, $\psi : L^l \to N^m$ represent elements $\alpha \in \mathscr{T}^m(K)$, $\beta \in \mathscr{T}^m(L)$. Write $K \vee L$ for the one-point union of K and L; then we will define $\alpha + \beta \in \mathscr{T}^m(K \vee L)$.

Let D^{m-1} denote the standard $(m-1)$-disc, with a base point $*$ on its boundary, and $f_1 : (D^{m-1}, *) \to (\partial M^m, *)$ resp. $f_2 : (D^{m-1}, *) \to (\partial N^m, *)$ imbeddings such that the standard orientation of D^{m-1}, followed by the inward (resp. outward) normal induces the given orientation at $*$ of M (resp. N). By a standard result, this determines f_1 and f_2 up to isotopy.

Now form P^m from $M^m \cup N^m$ by identifying points corresponding to each other under $f_2 f_1^{-1}$. In the differential case, the definition of a differential structure on P involves some choices (see [12, Theorem 1.4]), but these will not affect the class of the result; in the piecewise linear case, P inherits at once a structure of that kind.

Now ϕ and ψ define a simple homotopy equivalence $K \vee L \to P$; the orientations of M and N at * agree to define one of P; the condition on fundamental groups is readily verified, so we have defined a thickening of $K \vee L$. This clearly depends only on α and β, so we have a map

$$+ : \mathcal{T}^m(K) \times \mathcal{T}^m(L) \to \mathcal{T}^m(K \vee L)$$

defined whenever K and L have dimensions $\leqq (m-3)$. Addition is also clearly commutative and associative.

Intersection. Let M^m be a manifold, K^k and L^l CW-complexes, $f : K \to M$ and $g : L \to M$ continuous maps. Consider the product $f \times g : K \times L \to M \times M$. In the special case when f and g are imbeddings, their intersection may be measured by the inverse image under $f \times g$ of the diagonal. Let us write N for a tubular neighbourhood (in the smooth case) or regular neighbourhood (in the piecewise linear case) of the diagonal in $M \times M$. Form T from $M \times M$ by shrinking the closure of the complement of N to a point—or equivalently from N by shrinking the frontier to a point. (We sometimes write M^τ for T.)

Now consider the homotopy class of the composite

$$K \times L \xrightarrow{f \times g} M \times M \to T.$$

Observe that T contains a class of m-spheres—for example, if N is well-chosen, the image of $M \times *$ in T is a sphere S^m. We wish to argue that the map $K \times L \to T$ is in the image, by inclusion, of a well-determined homotopy class of maps $K \times L \to S^m$, i.e. a class in the cohomotopy set $\pi^m(K \times L)$. This follows if the pair (T, S^m) is $(k + l + 1)$-connected. Now T is clearly simply-connected ($m \geqq 2$), and by the Thom isomorphism $H_r(M, *) \xrightarrow{\simeq} H_{m+r}(T, S^m)$, we see that it is sufficient to have M $(k + l - m + 1)$-connected. Finally if (as is always the case with us) $k, l \leqq m - 2$, then the projection $K \times L \to K \wedge L$ (i.e. the map which identifies the subspace $K \vee L$ to a point) induces a bijection of $\pi^m(K \wedge L)$ on $\pi^m(K \times L)$.

Since the construction is entirely in the framework of homotopy classes, we have associated to the homotopy classes of $f : K^k \to M^m$ and $g : L^l \to M^m$ an intersection invariant in $\pi^m(K \wedge L)$. The hypotheses necessary for the construction are $k, l \leqq m - 2$, and that M^m is $(k + l - m + 1)$-connected.

One could of course go on to define triple (and higher) intersections in an analogous manner. Also observe that since T and S^m are simply-connected, there is no need to preserve base points in the construction.

We have used above (by applying the Thom isomorphism) the fact that T is the Thom space of the tangent bundle of M. Following a suggestion of Atiyah, another construction can be performed at this point: use the classifying map of the tangent bundle τ, $M \to BO_m$ or $M \to BPL_m$, and the induced map of T to the universal Thom space. In this way, we obtain an element of the m^{th} co-bordism, (rather than co-homotopy) group of $K \wedge L$. Even better would be to combine the two constructions: if M is r-connected, to map it to the r-connected covering of BO_m or BPL_m, and use r-connected co-bordism. In this paper we shall stick to cohomotopy, on the grounds that co-bordism is too crude, and r-connected co-bordism too complicated.

Induced thickening. We shall give in §7 the proof of the following result.

EMBEDDING THEOREM. *Suppose M^m a manifold, K^k a finite CW-complex, $f: K \to M$ a $(2k - m + 1)$-connected map, and $k \leqq m - 3$. Then there is a compact submanifold N^m of M^m, with $\pi_1(\partial N) = \pi_1(N)$, and a simple homotopy equivalence $g : K \to N$, such that $g \simeq f: K \to M$. We may suppose that the homotopy keeps the base point fixed. If f is $(2k - m + 2)$-connected, the submanifold N will be unique up to concordance in M.*

COROLLARY (2.1). *If K^k and L^l are finite CW-complexes, $k, l \leqq m - 3$, and $f: K \to L$ is a $(2k - m + 2)$-connected map (preserving base points) then f induces a map $f^* : \mathscr{T}^m(K) \leftarrow \mathscr{T}^m(L)$.*

Proof of Corollary. Given a thickening of L, represented by $\phi : L \to M$, apply the theorem to $\phi f: K \to M$. There results a simple homotopy equivalence $g : K \to N$, with $\pi_1(\partial N) = \pi_1(N)$ and N unique up to homeomorphism (all respecting the base point). Then g defines the, required thickening of K (we use the orientation at $* \in \partial N \subset N$ induced from that of M.

Our construction is evidently functorial (i.e. $1^* = 1$, $(fg)^* = g^* f^*$) but we cannot present $\mathscr{T}^m$ as a functor without losing a number of properties: e.g. the above corollary may apply to two maps, but not to their composition.

§3. OTHER OPERATIONS

The above operations can be combined in many ways, of which we now indicate a few.

Trivial thickening. In the corollary to the embedding theorem take L to be a point, and use the thickening defined by the disc D^m (which, at least for $m \geqq 6$, is the only one). We deduce that: if K^k is $(2k - m + 1)$-connected, just one class in $\mathscr{T}^m(K)$ is represented by manifolds M^m which can be imbedded in $\mathbf{R}^m$.

Intersection. By taking the intersection of the homotopy classes induced by the inclusions of K and L in $K \vee L$, we obtain a map which we write

$$\lambda : \mathscr{T}^m(K \vee L) \to \pi^m(K \wedge L).$$

This is defined if K^k and L^l are both $(k + l - m + 1)$-connected.

Selfintersection. Similarly, we can take the intersection of an inclusion map with itself. Hence, if K^k is $(2k - m + 1)$-connected, a map $\pi : \mathscr{T}^m(K) \to \pi^m(K \wedge K)$.

Additive structure. If K^k is at least $(2k - m + 2)$-connected, any map $\nabla : K \to K \vee K$ induces a map ∇^* of thickenings and hence, by composition,

$$\mathscr{T}^m(K) \times \mathscr{T}^m(K) \xrightarrow{+} \mathscr{T}^m(K \vee K) \xrightarrow{\nabla^*} \mathscr{T}^m(K)$$

a composition operation on the set $\mathscr{T}^m(K)$. If, in particular, ∇ endows K with the structure of H'-space, the composition admits the trivial thickening as a 2-sided unit. If ∇ is also homotopy associative, then the addition is associative (using transitivity of induced thickenings and an evident commutative diagram); and, using the theorem [10] that there exists a map $e : K \to K$ which acts as an inverse map for ∇, we see that $\mathscr{T}^m(K)$ has the structure of a group, which is abelian if ∇ is homotopy commutative.

For example, for all $r \leqq m - 3$, $\mathscr{T}^m(S^r)$ is an abelian group. For $3 \leqq r \leqq m - 4$, and any integer n, $\mathscr{T}^m(S^{r-1} \cup_n e^r)$ is an abelian group.

Simple homotopy. Let $f : K^k \to L^t$ be a homotopy equivalence which need not be simple, g a homotopy inverse, $m \geqq k + 3, l + 3$. By our result on induced thickenings, g^* is a 2-sided inverse to f^* which is therefore bijective. This shows that although simple homotopy entered the definition of $\mathscr{T}^m(K)$, it will not figure in our calculations.

It is not difficult to picture the bijection geometrically. For if the thickening $\psi : L \to N^m$ induces $\phi : K \to M^m$, and we deform M a little to lie in the interior of N, then the region between (i.e. the closure of N–M) is an h-cobordism of ∂M to ∂N. But, by a remark of Milnor, h-cobordisms W (of dimension $\geqq 6$) with one end (say ∂N) given are classified up to diffeomorphism by the Whitehead torsion $\tau(W, \partial N)$. So the manifolds related by bijections f^* are obtained from one another by glueing h-cobordisms (which we know) onto the boundary.

It is of course precisely the absence to date of a proof of the h-cobordism theorem in low dimensions which necessitates our hypothesis $m \geqq 6$ throughout.

§4. FORMAL PROPERTIES OF THE OPERATIONS

(4.1) Suspension commutes with addition. (This is immediate).

(4.2) Induced maps commute with addition; i.e. if $f : K \to L$, $f' : K' \to L'$, $\alpha \in \mathscr{T}^m(L)$, $\alpha' \in \mathscr{T}^m(L')$ are such that $(f \vee f')^*(\alpha + \alpha')$ is defined, then $f^*\alpha$ and $f'^*\alpha'$ are defined, and

$$(f \vee f')^*(\alpha + \alpha') = f^*\alpha + f'^*\alpha' \in \mathscr{T}^m(K \vee K')$$

For f and f' are at least as highly connected as $f \vee f'$, and if N, N' represent α, α' and $M \subset N$, $M' \subset N'$ represent $f^*\alpha, f'^*\alpha'$, then in $N + N'$, M and M' intersect in the base disc D^{m-1} on the boundary, so form a sum $M + M'$, which clearly represents the thickening induced by $f \vee f'$.

(4.3) If K, L are H'-spaces such that $\mathscr{T}^m K$ and $\mathscr{T}^m L$ have group structures, and if $f : K \to L$ is primitive and induces a map f^*, then f^* is a homomorphism.

For $f \vee f$ will then also induce a map of thickenings; we have $(f \vee f)^*(\alpha + \beta) = f^*\alpha + f^*\beta$ by (2) for the exterior sum, and since f is primitive, $\nabla_K \circ (f \vee f) \simeq f \circ \nabla_L$ induce the same map of $\mathscr{T}^m$.

(4.4) If K is an H'-space, then writing $[K : M]$ for the set of (based) homotopy classes of maps of K to M, the H'-structure on K defines a group structure on $[K : M]$. Intersections defined a map which we now write as

$$\lambda' : [K : M] \times [L : M] \to \pi^m(K \wedge L)$$

if M is $(k + l - m + 1)$-connected; we now assert that this map is linear in the first variable.

Note that $\pi^m(K \wedge L)$ has *a priori* two group structures: from the H'-structure of K, and addition in the cohomotopy group: a well-known result asserts that these agree.

The proof of linearity is essentially trivial: it merely consists in performing the construction used to define the intersection for $K \vee K$ and L, and observing that the construction is natural throughout.

Similarly if L is an H'-space, λ' is linear in the second variable.

(4.5) If K, L are H'-spaces, then $K \vee L$ has a natural H'-structure. The map

$$\lambda : \mathcal{T}^m(K \vee L) \to \pi^m(K \wedge L)$$

is a homomorphism, if K, L are $(2 \max(k, l) - m + 2)$-connected (so that $\mathcal{T}^m(K \vee L)$ is a monoid).

Recall again that the three group structures on $\pi^m(K \wedge L)$ all agree. Then if $\alpha, \beta \in \mathcal{T}^m(K \vee L)$, and we form $\alpha + \beta$, the induced map of $(K \vee K) \wedge (L \vee L)$ to S^m is trivial on two of the four component copies of $K \wedge L$: on the other two we have representatives of $\lambda(\alpha)$ and $\lambda(\beta)$. The result follows.

(4.6) We can also generalize the results of Theorem 1 of [I] to our present context. Suppose $x \in [K^k : M^m]$ a class represented by $(2k - m + 2)$-connected maps $f : K \to M$; then we will write $\alpha(x)$ for the element of $\mathcal{T}^m K$ which, according to Theorem 1, is induced by f. Our present α and λ' generalise the α and λ of [I].

We have already proved λ' bilinear; the symmetry of λ' is obvious. So is the formula $\lambda'(x, x) = \pi\alpha(x)$ for $x \in [K : M]$ as above: the selfintersection of f can be calculated equally well using any neighbourhood N of the image. This follows from the homotopy commutative diagram

$$\begin{array}{ccccc} & \nearrow & N \times N & \longrightarrow & T_N \\ K \times K & & \downarrow & & \uparrow \\ & \searrow & M \times M & \longrightarrow & T_M \end{array}$$

The addition formula for α lies somewhat deeper. We must now suppose K^k $(2k - m + 1)$-connected and M^m $(2k - m + 2)$-connected, so that any map from K or from $K \vee K$ to M is $(2k - m + 2)$-connected. Given $x, y \in [K : M]$, we can form $\alpha(x \vee y) \in \mathcal{T}^m(K \vee K)$, and if K is an H'-space, $\alpha(x + y) = \nabla^*\alpha(x \vee y)$. Let i_1, i_2 be the standard inclusions of K in $K \vee K$: if K is $(2k - m + 2)$-connected, these induce maps i_1^*, i_2^* and we form the product

$$\mathcal{T}^m(K \vee K) \xrightarrow{(i_1{}^*, i_2{}^*, \lambda)} \mathcal{T}^m(K) \times \mathcal{T}^m(K) \times \pi^m(K \wedge K).$$

We shall prove in (6.3) that if $2m \geqq 3k + 3$ and K is $(2k - m + 2)$-connected, (i_1^*, i_2^*, λ) is a bijection. In this case, $\alpha(x \vee y)$ will be determined by

$$(i_1^*, i_2^*, \lambda)\alpha(x \vee y) = (\alpha(x), \alpha(y), \lambda'(x, y)),$$

using the (trivial) formula $\lambda'(x, y) = \lambda\alpha(x \vee y)$ when the right side is defined.

For the addition theorem, we suppose K admits a map $\nabla : K \to K \vee K$ giving it an H'-structure which is homotopy commutative and associative. Then $K \vee K$ also acquires an H'-structure, and i_r is an H'-map, so by (4.3) induces a homomorphism of groups of thickenngs. Also, by (4.5), λ is a homomorphism. Thus (i_1^*, i_2^*, λ) is an isomorphism of groups. Now as ∇^* is a group homomorphism, we have

$$\begin{aligned}\alpha(x + y) &= \nabla^*\alpha(x \vee y) = \nabla^*(\alpha(x), \alpha(y), \lambda'(x, y)) \\ &= \alpha(x) + \alpha(y) + \partial_s\lambda'(x, y)\end{aligned}$$

if we write $\partial' : \pi^m(K \wedge K) \to \mathscr{T}^m(K \vee K)$ for the injection of the third component, and $\partial_s = \nabla^*\partial'$.

§5. THE SUSPENSION SEQUENCE AND STABLE THICKENINGS

We first recall a well-known result. Let K^k and L^t be finite based CW-complexes, with L c-connected. Then the map induced by suspension

$$[K : L] \xrightarrow{S} [SK : SL]$$

is surjective if $k = 2c + 1$ and bijective if $k \leqq 2c$.

SUSPENSION THEOREM. *There exists a map* $\partial : \pi^{m+1}(SK \wedge K) \to \mathscr{T}^m(K)$, *defined if* $2m \geqq 3k + 3$ *and* K^k *is* $(2k - m + 1)$*-connected, such that the sequence*

$$\mathscr{T}^{m+2}(SK) \xrightarrow{S^{-1}\circ\pi} \pi^{m+1}(SK \wedge K) \xrightarrow{\partial} \mathscr{T}^m(K) \xrightarrow{S} \mathscr{T}^{m+1}(K) \xrightarrow{\pi} \pi^{m+1}(K \wedge K)$$

is exact. For exactness at $\mathscr{T}^{m+1}(K)$, *it is enough to have* $2m \geqq 3k + 2$, *and* K $(2k - m)$*-connected. If K admits a homotopy commutative and associative* H'*-structure, and is* $(2k - m + 2)$*-connected, then* $\partial \circ S = \partial_s$ *is the map defined above.*

The proof of this result will be given in §8.

Since (evidently), if $\dim K = k$, $\pi^m(K \wedge K)$ vanishes for $m > 2k$, the Suspension Theorem implies that the suspension map

$$S : \mathscr{T}^m(K) \to \mathscr{T}^{m+1}(K)$$

is surjective for $m \geqq 2k$ and monomorphic for $m \geqq 2k + 1$. In fact, S is bijective if $m \geqq 2k + 1$: the proof is essentially the same. [We have $N^{m+1} = M^m \times I$ and $M'^m \times I$. Then M'^m can be "pushed" off $M \times 1$ in $\partial(M \times I)$, since each is a thickening of K^k, and the manifold has dimension $m \geqq 2k + 1$. So we can suppose $M' \subset M$. But the inclusion is a simple homotopy equivalence; applying [16, Theorem 6.4] we deduce $M = M' + (\partial M' \times I)$, so both define the same thickening.]

Write $\mathscr{T}(K)$ for the constant value of $\mathscr{T}^m(K)$, $m \geqq 2k + 1$. We can now determine $\mathscr{T}(K)$ (in the differentiable case, this result is due to Mazur [11]). Take the tangent bundle of M. This has a classifying map:

$$K \to BO \qquad \text{or} \qquad K \to BPL$$

in the smooth or piecewise linear cases; preserving base points in either. Thus we have a natural transformation

$$\tau(K) : \mathscr{T}(K) \to [K : BQ]$$

where Q denotes O, resp. PL.

PROPOSITION 5.1. *For any K, $\tau(K)$ is a bijection.*

Proof. Let $\phi : K \to M_0^m$ be the trivial thickening where, e.g. $m = 2k + 1$, defining $* \in \mathscr{T}(K)$. Since M_0 is by definition parallelizable, $\tau(*)$ is the class of the constant map $K \to BQ$.

To prove τ surjective, take any map $K \to BQ$. Since, by definition, Q is the limit of the Q_n, the map can be factorised $K \to BQ_n$, and so induces a bundle over K with fibre $\mathbf{R}^n$. As ϕ is a homotopy equivalence, we obtain a corresponding bundle ξ over M_0. Note that the tangent bundle of the total space is the direct sum of a trivial bundle and of the bundle induced from ξ. Hence the thickening α induced by ϕ followed by the zero cross-section of ξ has $\tau(\alpha)$ in the required class.

To prove τ injective, we may suppose (after suspension) that the thickenings $\phi_1 : K \to M_1^m$ and $\phi_2 : K \to M_2^m$ have the same tangent bundle. According to the classification of immersions, due to Hirsch [8] in the differential case and to Haefliger and Poenaru [6] in the piecewise linear case (Hirsch's statement does not include the case we need, but the arguments in [6] show how to fill the gap), there is an immersion $\psi : M_1^m \to M_2^m$ with $\psi\phi_1 \simeq \phi_2$ (relative to *) and preserving orientation at *. Now as $m \geqq 2k + 1$, we may suppose ϕ_1 an imbedding, and also perform a small regular homotopy of ψ to make $\psi\phi_1$ an imbedding. As M_1 can be shrunk to a small neighbourhood of $\phi_1 K$ (i.e. the identity map is isotopic to an imbedding into a neighbourhood of $\phi_1 K$) and ψ is an immersion which imbeds $\phi_1 K$, hence also some neighbourhood, we may suppose ψ an imbedding. But then we can use the s-cobordism theorem as usual to show that M_1 and M_2 define the same thickening.

The Proposition shows that in the stable range, $\mathscr{T}$ is representable. In particular, there are Mayer–Vietoris sequences. We will extend these in §6 (with some complications) to the metastable range. Also, combined with the Suspension Theorem, it gives a method for calculating $\mathscr{T}^m(K)$ in the metastable range.

§6. THE MAYER–VIETORIS THEOREM

We observed in §5 that the stable functor $\mathscr{T}$ is representable, so if L is a subcomplex of K, the induced sequence

$$\mathscr{T}(L) \leftarrow \mathscr{T}(K) \leftarrow \mathscr{T}(K/L)$$

is exact. It cannot remain exact beyond the stable range, for there may be thickenings $\phi : K \to M^m$, inducing the trivial thickening of L, but such that some r-cycle of L has non-zero intersection number with an $(m-r)$-cycle of M. However it turns out that, if we consider our generalised intersection invariant, the above is (in a certain range) essentially the only obstruction to exactness of the sequence.

For reasons of symmetry, it will be more convenient to study a Mayer–Vietoris type of sequence; as our proof will proceed by induction on added cells, we first consider the situation where thickenings of $K \cup e^r$ and of $K \cup e^s$ induce the same thickening M^m of K. By the arguments of Theorem 1, we may write these thickenings as $M \cup h^r$, $M \cup h^s$, where $h^r(h^s)$ denotes an attached r-handle (s-handle). We wish to attach these two handles simultaneously, and so we must deform their attaching maps (into ∂M) to be disjoint. This is possible if and only if the induced imbeddings

$$\bar{f} : S^{r-1} \to \partial M \qquad \bar{g} : S^{s-1} \to \partial M$$

can be deformed so as to be disjoint. Here we use a result of Haefliger [4, p. 169] in the differential case; the corresponding result in the PL case has been obtained by C. Weber: any homotopy of $\bar{f} \times \bar{g}$ to a map into $\partial M \times \partial M$ minus the diagonal is induced by an isotopy of $\bar{f}$ and $\bar{g}$ to disjoint positions, provided that $2(r + s - m) < \min(r, s) - 1$; if, moreover, $2(r + s - m) < \min(r, s) - 2$ then the isotopy is also unique up to isotopy.

We next observe that finding a homotopy of $\bar{f} \times \bar{g}$ is equivalent to a problem on our intersection invariant (when this is defined). For this invariant provides maps

$$F : (K \cup e^r) \times (K \cup e^r) \to S^m \qquad G : (K \cup e^s) \times (K \cup e^s) \to S^m$$

whose restrictions to $K \times K$ are homotopic (since the two thickenings induce the same thickening of K). Modifying F (or G) by the appropriate homotopy, we may suppose $F|K \times K = G|K \times K$. Then F and G combined define a map of all of $(K \cup e^r \cup e^s) \times (K \cup e^r \cup e^s)$ except the cells $e^r \times e^s$ and $e^r \times e^s$: we have the induced map on the boundary $\partial(D^r \times D^s) \xrightarrow{a} S^m$. Comparing this with the intersection invariant of $\bar{f}$ and $\bar{g}$, which is a map $S^{r-1} \times S^{s-1} \xrightarrow{b} S^{m-1}$, we observe that a is obtained by extending over $D^r \times S^{s-1}$ into one hemisphere and $S^{r-1} \times D^s$ into the other, so is moreorless the suspension of b: thus a nullhomotopy of a induces one of b.

To justify the above, we need a hypothesis that the intersections are all well-defined, which gives

LEMMA 6.1. *With the notation above, suppose* K^k $(2k - m + 1)$*-connected; similarly for* $K \cup e^r$ *and* $K \cup e^s$*; let* $2m \geqq r + s + \max(r, s, k) + 2$. *Then any extension of* $F \cup G$ *over* $e^r \times e^s$ *is induced by a thickening of* $K \cup e^r \cup e^s$. *If* $2m \geqq r + s + \max(r, s, k) + 3$, *the resulting thickening is uniquely determined.*

It remains only to check that the intersection invariant of $\bar{f}$ and $\bar{g}$ is defined. Now if $r \geqq k$, $K \cup e^r$ has dimension r; since $r \leqq m - 3$, so $r - 1 \geqq (2r - m + 1)$, to suppose it $(2r - m + 1)$-connected is equivalent to supposing that K is. So our hypothesis implies K $(2\max(r, s, k) - m + 1)$-connected, hence certainly $M \simeq K$ is $(r + s - m)$-connected. But as M is a thickening of K^k, $(M, \partial M)$ is $(m - k - 1) \geqq (r + s - m + 1)$-connected. Hence ∂M is $(r + s - m)$-connected as required.

We observe also, for the case $r < k$, since K is $(2k - m + 1)$-connected, that the assumption that $K \cup e^r$ also is equivalent to the inequality $r > 2k - m + 1$ (which holds anyway if $r \geqq k$). Thus the hypotheses of the lemma may be rewritten as:

(H) *Let* $l = \max(r, s, k)$. *Then* K *is* $(2l - m + 1)$*-connected*, $l \leqq m - 3$, $2m \geqq r + s + l + 2$, *and* $m \geqq 2k - \min(r, s) + 2$.

We now integrate this into a theorem.

MAYER–VIETORIS THEOREM. *Suppose given finite connected CW-complexes with* $A^a \cap B^b = C^c$, $A^a \cup B^b = D^d$, *thickenings* $\alpha : A \to L^m$, $\beta : B \to M^m$ *and a homeomorphism of the induced thickenings of C, so we can suppose the intersection invariants* $F : A \times A \to S^m$, $G : B \times B \to S^m$ *agree on* $C \times C$. *Then if* $H : A \times B \to S^m$ *extends* $F|A \times C$ *and* $G|C \times B$, *there is a thickening* $\delta : D \to P^m$ *whose intersection invariant extends F, G and H; provided A is* $(2a - m + 1)$*-connected (similarly for B, C, D),* $d \leqq m - 3$, *and* $2m \geqq a + b + d + 2$. *If* $2m \geqq a + b + d + 3$, *the thickening* δ *is uniquely determined.*

Proof. We attach cells to C in order of increasing dimension:

$$C = A_0 \subset A_1 \subset \cdots \subset A_k = A \qquad A_i = A_{i-1} \cup e^{\alpha_i}; \quad \alpha_{i-1} \leqslant \alpha_i;$$

similarly for B. We now obtain a thickening of $A_i \cup B_j$, by induction on $i + j$, extending thickenings already obtained of $A_{i-1} \cup B_j$ and of $A_i \cup B_{j-1}$ (which, by the inductive hypothesis, agree on $A_{i-1} \cup B_{j-1}$), and with intersection invariant given by F on $A_i \times A_i$, by G on $B_j \times B_j$ and by H on $A_i \times B_j$. The induction step is performed using the lemma above; it remains, then, to verify the hypothesis of the lemma. The dimensional restrictions are evidently satisfied in virtue of $d \leqq m - 3$ and $2m \geqq a + b + d + 2$, so we need only check the connectivity conditions. We shall check in the above that A_i may be taken $(2 \max(c, \alpha_i) - m + 1)$-connected); the same arguments will then apply to $A_i \cup B_j$.

Suppose i is chosen so that $\alpha_{i-1} \leqq c \leqq \alpha_i$. Then we want to show A_j $(2c - m + 1)$-connected for $j < i$, and $(2\alpha_j - m + 1)$-connected for $j \geqq i$. But the latter condition is immediate: A is obtained from A_j by attaching cells of dimension $\geqq \alpha_j$, hence (A, A_j) is $(\alpha_j - 1)$-connected; A is $(2a - m + 1)$-connected, so A_j is connected at least up to the dimension $\min(\alpha_j - 2, 2a - m + 1) \geqq 2\alpha_j - m + 1$. As to the other case, note that if C and A_1 are both $(2c - m + 1)$-connected, then so is (A_1, C), so we must have $\alpha_1 \geqq 2c - m + 2$. If this does not hold for the given complexes, we shall replace them by (simple) homotopy equivalent complexes.

We are considering only connected complexes, so may certainly suppose that each has only one O-cell. Hence $\alpha_1 \geqq 1$, so if the above condition is violated, $1 \leqq \alpha_1 \leqq 2c - m + 1$ and all of A, B, C, D are simply-connected. Then we can suppose that there are no 1-cells, so all the subcomplexes also are simply-connected. But now any set of chain groups consistent with the homology structure of the pair (A, C) can be realised by a pair of cell-complexes, according to an argument of Milnor (see [15, Proposition 4.1, and Lemma 1.2]). Thus we can suppose α_1 equal to $1 +$ the connectivity of (A, C), and so $\alpha_1 \geqq \min(2a - m + 2, 2c - m + 3) \geqq 2c - m + 2$.

This proves the existence clause of the theorem: the same induction using the uniqueness part of the lemma establishes also the uniqueness clause.

COROLLARY 6.2. *Let* L^l *be a connected subcomplex of* K^k, *and suppose* $\max(k, l + 1) \leqq m - 3$, $2m \geqq k + l + \max(k, l + 1) + 3$, *L is* $(2l - m + 1)$*-connected, K* $(2k - m + 1)$*-connected, and K/L* $\{2 \max(k, l + 1) - m + 1\}$*-connected. Let* α *be a thickening of* K^k *inducing the trivial thickening of L, let* $F : K \wedge K \to S^m$ *be the intersection invariant of K, and* *suppose the induced nullhomotopy of* $L \wedge L$ *extends over* $K \wedge L$. *Then* α *is induced from a thickening of* K/L.

Apply the theorem with $C = L$, $A = K$, $B =$ the cone on L: the corollary is then immediate.

COROLLARY 6.3. *Let* K^k *and* L^l *be finite connected CW-complexes,* $m - 3 \geqq k \geqq l$, $2m \geqq 2k + l + 3$, K *and* L $(2k - m + 2)$*-connected. Let* $i_1 : K \to K \vee L$ *and* $i_2 : L \to K \vee L$ *be the inclusion maps. Then*

$$(i_1^*, i_2^*, \lambda) : \mathscr{T}^m(K \vee L) \to \mathscr{T}^m K \times \mathscr{T}^m L \times \pi^m(K \wedge L)$$

is bijective.

This is the case of the theorem when C is a point: note in this case that $(K \vee L) \wedge (K \vee L) = (K \wedge K) \vee (K \wedge L) \vee (L \wedge K) \vee (L \wedge L)$, so the "extension" of the theorem is simply a map defined on $K \wedge L$. The hypotheses of the theorem have been strengthened: for i_1^* to be well-defined we need i_1 $(2k - m + 2)$-connected and hence also L. If $k > l$, it would suffice for K to be $(2k - m + 1)$-connected.

Note that for $\alpha \in \mathscr{T}^m(K)$, $\beta \in \mathscr{T}^m(L)$, we have $\alpha + \beta \in \mathscr{T}^m(K \vee L)$, and evidently (if it is defined)

$$(i_1^*, i_2^*, \lambda)(\alpha + \beta) = (\alpha, \beta, 0).$$

Thus the above Corollary implies, except for the weakening of the connectivity assumption, the

PROPOSITION 6.4. *Let* K^k *and* L^l *be finite connected CW-complexes,* $m - 3 \geqq k \geqq l$, $2m \geqq 2k + l + 3$, K $(k + l - m + 1)$*-connected, and* L $(2k - m + 1)$*-connected. Then the sequence*

$$\mathscr{T}^m K \times \mathscr{T}^m L \xrightarrow{+} \mathscr{T}^m(K \vee L) \xrightarrow{\lambda} \pi^m(K \wedge L)$$

is exact.

Proof. We give a direct proof which justifies the weakening of the connectivity condition.

Since L is $(2k - m + 1)$-connected, so is the map $K \to K \vee L$, hence by the embedding theorem there is a thickening (not unique):

$$\phi_1 : K \to M_1 \subset M$$

induced by a thickening M of $K \vee L$. Similarly we obtain

$$\phi_2 : L \to M_2 \subset M.$$

Now assume that the λ-invariant of M vanishes. Then $\phi_1 \times \phi_2$ is homotopic to a map which avoids the diagonal. Now since M_1 is a thickening of K, so admits a handle decomposition with all handles of dimension $\leqq k$; similarly for M_2; and $2m \geqq 2k + l + 3$, the result of Haefliger already used [4, p. 169], modified as on [4, p. 173]—or the corresponding result in the PL case—shows that the inclusion of M_1 is isotopic to a position avoiding M_2.

Thus we may suppose M_1 and M_2 disjoint: connect them by a tube, and we now have $M_1 + M_2 \subset M$. A simple application of the s-cobordism theorem (or rather of [16, 6.4]) now shows $M \cong M_1 + M_2$, and completes the proof.

Under suitably restrictive assumptions, we can even replace the homotopy hypothesis in the criterion (6.4) for a connected sum splitting by a homology hypothesis.

PROPOSITION 6.5. *Suppose M^m an $(r-1)$-connected manifold with ∂M a homotopy sphere, and $m \leqq 3r-2$, $r \geqq 3$. Suppose given a homotopy equivalence $h: K_1 \vee K_2 \to M$ such that for $i \neq 0, m$, the map $\phi: H^i(K_1) \to H_{m-i}(K_1)$ induced by cap product D with the fundamental class of $(M, \partial M)$ is an isomorphism. Then M is a boundary-connected sum $M_1 + M_2$, with M_i a thickening of K_i.*

Proof. ϕ is induced by the natural inclusion and projection and the following sequence of isomorphisms

$$H^i(K_1 \vee K_2) \xleftarrow[\cong]{h^*} H^i(M) \xrightarrow[\cong]{D} H_{m-i}(M, \partial M) \xleftarrow{\cong} H_{m-i}(M) \xleftarrow[\cong]{h_*} H_{m-i}(K_1 \vee K_2).$$

We apply the imbedding theorem to the map $f: K_1 \to M$ induced by h. Since M (and hence K_1) is $(r-1)$-connected, so is f. Since K_1 is simply-connected, and its homology vanishes above dimension $(m-r)$, we may suppose $\dim K_1 = m - r \leqq m - 3$. Then the theorem applies, and we obtain a submanifold N of Int M, homotopy equivalent to K_1.

Join ∂N to ∂M by an arc in $M - N$; thicken the arc, and add to N, giving M_1; then $\partial M_1 \cap \partial M$ is a disc D^{m-1}. The fundamental class of $(M, \partial M)$ induces that of $(M_1, \partial M_1)$; our hypothesis gives isomorphisms $H^i(M_1) \to H_{m-i}(M_1)$ whose composites $H^i(M_1) \to H_{m-i}(M_1) \to H_{m-i}(M_1, \partial M_1)$ are the isomorphisms induced by cap products. Hence the map $H_{m-i}(M_1) \to H_{m-i}(M_1, \partial M_1)$ is an isomorphism for $i \neq 0, m$; since ∂M_1 is simply-connected it is a homotopy sphere.

Now if $m \leqq 5$, our assumptions imply M contractible, and the result is trivial. Otherwise, the generalised Poincaré conjecture [14] shows that ∂M_1 is obtained by attaching two discs D^{m-1}. Hence the relative boundary of M_1 is a disc D^{m-1}; this cuts M into M_1 and M_2 (say) and $M = M_1 + M_2$ by the definition of sum.

Finally we note that if we take M, and identify M_1 to a point, the result is homeomorphic to M_2. The composite $K_2 \to K_1 \vee K_2 \xrightarrow{h} M \to M_2$ is a map of simply-connected spaces which (by the Five Lemma) induces homology isomorphisms. By results of Whitehead, it is then a simple homotopy equivalence.

Remark. If we started with a closed manifold L^m, we can remove an m-disc to obtain M^m, and use the above to give useful sufficient conditions for separating L^m by a homotopy S^{m-1}. Only in the PL-case, however, are we then able to write L as a connected sum.

§7. PROOF OF THE IMBEDDING THEOREM

The essential step is contained in the following result, conjectured in part by the author and proved by J. Hudson [9]:

PROPOSITION 7.1. *If V^v and M^m are connected manifolds with boundary, $m \geqq v + 3$, $f: (V, \partial V) \to (M, \partial M)$ is a map, and $(V, \partial V)$ is $(2v - m)$-connected, $(M, \partial M)$ is $(2v - m + 1)$-connected, then f is homotopic to an imbedding $g: (V, \partial V) \to (M, \partial M)$. If also $(V, \partial V)$ is*

$(2v - m + 1)$-connected and $(M, \partial M)$ is $(2v - m + 2)$-connected, and g_1, g_2 are imbeddings homotopic to f, there is a homeomorphism h of $(M, \partial M)$, homotopic to the identity, with $h \circ g_1 = g_2$.

In accordance with our conventions, this result is to be interpreted in either the smooth or the piecewise linear sense. The uniqueness clause of Hudson's result is actually slightly more precise, in that he obtains a homeomorphism H of $(M, \partial M) \times I$ extending the identity on $(M, \partial M) \times 0$ and h on $(M, \partial M) \times 1$; i.e. g_1 is concordant to g_2. However, the above will suffice for our purposes, and in our case (V a disc) it can actually be shown that g_1 and g_2 are isotopic.

We first need a reformulation of Proposition (7.1); the result will then follow by induction.

LEMMA 7.2. *Let $j: N^m \subset \text{Int}\, M^m$ be c-connected; suppose that N has a handle decomposition with handles of dimension $\leqq d \leqq m - 3$, and that $s \leqq \min\{\frac{1}{2}(m + c - 1), m + c - d - 1, m - 3\}$. Then any $\alpha \in \pi_s(M, N)$ is representable by an imbedding $f: D^s \to M^m$ with $f^{-1}(N) = S^{s-1}$.*

Proof. Write $W = \partial N$, $V =$ the closure of $M - N$, so $V \cup N = M$, $V \cap N = W$. Suppose (V, W) is $(r - 1)$-connected and $s \leqq \min(m - 3, \frac{1}{2}(m + r - 2))$. Then, by (7.1), any $\beta \in \pi_s(V, W)$ can be represented by a disc imbedded as desired. Thus the lemma will follow if we prove

(a) (V, W) is c-connected, so we can take $r = c + 1$,

(b) the map $\pi_s(V, W) \to \pi_s(M, N)$ induced by inclusion is onto.

Now our hypothesis gives a handle decomposition of N; the dual decomposition is based on W and has handle dimensions $\geqq (m - d) \geqq 3$. So the inclusion $W \subset N$ is $(m - d - 1)$-connected, and induces an isomorphism $\pi_1(W) \xrightarrow{\simeq} \pi_1(N)$. Now if $c = 0$, (a) is trivial, if $c = 1$, van Kampen's theorem gives

$$\pi_1(M) = \pi_1(N) *_{\pi_1(W)} \pi_1(V) \cong \pi_1(V),$$

so as $\pi_1(N)$ maps onto $\pi_1(M)$, $\pi_1(W)$ must map onto $\pi_1(V)$. For $c \geqq 1$, the same argument shows that all four fundamental groups are isomorphic. Hence applying the Blakers–Massey theorem [2, Theorem 1] to the triad $(M; N, V)$—or rather to its universal cover—we find that (M, N) c-connected implies (V, W) c-connected.

Now since (V, W) is c-connected and (N, W) is $(m - d - 1)$-connected, the Blakers–Massey theorem as extended by Toda [19, 1.23] shows that the triad $(M; N, V)$ is $(m + c - d - 1)$-connected. As $s \leqq (m + c - d - 1)$, it follows that $\pi_s(V, W)$ maps onto $\pi_s(M, N)$ (here, use $\pi_1(W) \cong \pi_1(N)$ if $s = 1$).

This proves the lemma: we now prove the existence clause of the theorem.

We suppose K formed from $*$ by attaching cells in increasing order of dimension: we now prove, by induction on the number of cells in L, the

ASSERTION. *There exist a thickening $h: L \to N' \subset M$ and a homotopy* (rel $*$) *of $f | L$ to h in M.* The existence clause of the theorem is the special case $L = K$ of the assertion. The induction starts trivially with $L = *$. Let e^s be the next cell of K (so $\dim L \leqq s$). This defines a homotopy class $\alpha_1 \in \pi_s(K, L)$. We may suppose $f|L = h$; then $f_*\alpha_1 \in \pi_s(fK, hL)$ defines a class $\alpha \in \pi_s(M, N')$.

Now check that α satisfies the conditions of (7.2). We have $s = d \leqq m - 3$. Since L contains the $(s-1)$-skeleton of K, if f is c-connected (so $c \geqq 2k - m + 1 \geqq 2s - m + 1$), the pair (M, N') is c-connected if $s > c$ and $(s-1)$-connected if $s \leqq c$. The hypotheses of (7.2) are thus satisfied, and we can imbed D^s to represent α. If we thicken the disc, we have attached an s-handle h^s to N'; since the disc represents α, we can perform a homotopy of $f|(L \cup e^s)$ to make it a homotopy equivalence (evidently simple) into $N' \cup h^s$. This completes the induction, and the existence part of the conclusion follows.

As to the uniqueness, we must go through the same argument, and sharpen all the hypotheses to obtain homotopy isomorphisms, and uniqueness of discs up to concordance at each stage; we leave the details to the reader.

Remark. We have apparently obtained more than was claimed: the map $K \to N$ is not merely a simple homotopy equivalence, but we have a handle decomposition of N which resembles the cell decomposition of K. However, a result of Mazur [11, VIII] mentioned in the introduction shows that (assuming $\pi_1(\partial N) \cong \pi_1(N)$) this is in fact no sharper a condition, at least in dimensions $\geqq 6$. In dimensions <6, it might well be taken as the definition of a thickening. However, the resulting theory will be trivial in dimensions $\leqq 4$; in dimension 5 one would need the following (apparently weaker than the Poincaré conjecture)

CONJECTURE. *Let $M^5 \subset \operatorname{Int} N^5$ be a simple homotopy equivalence of compact 5-manifolds, each admitting a handle decomposition with handles of dimension $\leqq 2$. Then if W is the closure of $N - M$, $W \cong \partial M \times I$.*

If this holds, all the results of this paper could be developed in dimension 5, with an appropriately modified definition of thickening. Of course, if the full s-cobordism theorem held, no modifications would be necessary.

§8. PROOF OF THE SUSPENSION THEOREM

Exactness at $\mathscr{T}^{m+1}(K)$. For π to be defined, we must assume K to be $(2k - m)$-connected. It is then clear that $\pi \circ S = 0$, for given (ϕ, M) representing an element of $\mathscr{T}^m(K)$. then $\phi \times 0 : K \to M \times 0$ is homotopic in $M \times I$ to $\phi \times 1 : K \to M \times 1$, and the two images do not intersect. (As remarked earlier, there is no need to preserve a base point when computing intersections).

Conversely, let $\phi : K \to N^{m+1}$ represent $x \in \mathscr{T}^{m+1}(K)$ such that $\pi(x) = 0$. We will show that this hypothesis implies ϕ homotopic to a map $\psi : K \to \partial N$. Now $(N, \partial N)$ is $(m-k)$-connected; since ϕ is a homotopy equivalence, the homotopy exact sequence of the triple $K \to \partial N \subset N$ shows that $\pi_i(N, \partial N) \cong \pi_{i-1}(\psi)$, so ψ is $(m-k-1)$-connected. As $2m \geqq 3k + 2$, it follows that ψ is $(2k - m + 1)$-connected, so by the Imbedding Theorem we can find a submanifold M of ∂N (not unique) such that $\psi \simeq \phi'$, where $\phi' : K \to M$ is a simple homotopy equivalence, defining $y \in \mathscr{T}^m(K)$.

Let $L \cong \partial M \times I$ be a collar attached to M in ∂N. Introduce corners of N along ∂L. Then we may regard N as a cobordism of M to some manifold M' which, along the edge

L, is a product cobordism. Since the submanifold M constructed using Theorem 1 has a k-dimensional spine, whose codimension $m - k \geqq 3$ by hypothesis, the complement M' has the same fundamental group as ∂N, and hence as N. Now the inclusion $M \subset N$ is a simple homotopy equivalence; as $m \geqq 6$, the strong form [16, Theorem 6.4] of the s-cobordism theorem implies $N \cong M \times I$, whence $S(y) = x$.

It remains to produce the required homotopy of ϕ to a map ψ. We must first recall the definition of π; or rather the discussion surrounding it. Let V denote the tubular (or regular) neighbourhood of the diagonal ΔN in $N \times N$, C the closure of its complement. We write $T = N \times N/C$, and then $\phi : K \to N$ induces

$$K \times K \xrightarrow{\phi \times \phi} N \times N \to N \times N/C = T;$$

our hypothesis is equivalent to the assumption that the composite is nullhomotopic. We assert that if K is $(2k - m)$-connected, it follows from this that $\phi \times \phi$ is homotopic to a map into C. Now if $m \geqq 2k$, since the codimension $(m + 1)$ of ΔN exceeds the dimension of $K \times K$, this holds without hypothesis. Otherwise, all the spaces in question are simply-connected, and it will suffice to prove that the identification induces isomorphisms

$$\pi_i(N \times N, C) \to \pi_i(N \times N/C)$$

for $i \leqq 2k$. But since $(N \times N, C)$ is m-connected, and so the connectivity of C equals that of K, which $\geqq 2k - m$, the assertion follows from [2, Theorem 2].

Now let N' be the complement of a collar neighbourhood of ∂N in N, $i : N' \to N$ the inclusion. Since K, N, and N' all have the same homotopy type, the assertion above shows that $\phi \times i : K \times N' \to N \times N$ is homotopic to a map into C, and hence to a map into $\text{Int}(N \times N - \Delta N)$. But the projection of $\text{Int}(N \times N - \Delta N)$ onto the second factor $\text{Int}\, N$ is a fibration (it is well known to be locally trivial); the above homotopy projects to one in $\text{Int}\, N$, whose inverse lifts to a homotopy of the constructed map $K \times N' \to C$ to a map of the form $\psi' \times i$. Thus $\phi \simeq \psi'$, and the image of ψ' lies in a collar neighbourhood of ∂N; a further evident homotopy sends ψ' to a map $\psi : K \to \partial N$.

We observe that the thickening of K obtained by this construction depends on the homotopy class of ψ, and hence eventually on the chosen homotopy of $\phi \times \phi$: this suggests how to proceed to define ∂.

In fact, if we assume K to be $(2k - m + 1)$-connected, then the same sequence of arguments shows that if we choose a homotopy class of nullhomotopies of the induced map $K \times K \to T$, there is induced a particular homotopy class of maps $\psi : K \to \partial N$, homotopic in N to ϕ.

Exactness at $\mathscr{T}^m(K)$. We define ∂ as follows. Choose M_0^m to be the trivial thickening of K, $N_0 = M_0 \times I$, $\psi_0 = \phi \times O : K \to \partial N_0$. Then the induced map $K \times K \to T$ is constant. Now an $x \in \pi^{m+1}(SK \wedge K)$ defines (using the same sequence of isomorphisms as in the definition of intersections) a class of maps $S(K \times K) \to T$, and hence a nullhomotopy of the constant map $K \times K \to T$. By the remark above, starting with ψ_0, this induces a homotopy class of maps $\psi : K \to \partial N_0$. We define ∂x to be the thickening induced by ψ: we need $2m \geqq 3k + 3$ for this to be well-determined.

The same argument as in the exactness proof at $\mathscr{T}^{m+1}(K)$ shows that N_0 itself represents the suspension of this induced thickening: $S\partial x$ is trivial.

Conversely, if M^m represents a thickening whose suspension is trivial, there is a homeomorphism of $M \times I$ onto N_0. This carries $\phi : K \to M$ to a map $\psi_1 : K \to \partial N_0$, and our thickening is induced by ψ_1. Now there is a homotopy ψ_t in N of ψ_0 to ψ_1; form the product $\psi_t \times i : K \times N' \to N_0 \times N_0 \to T$. This gives a homotopy of the constant map to itself, and determines a homotopy class of maps $S(K \times K) \to T$, and hence a class $x \in \pi^{m+1}(SK \wedge K)$. By the definition of ∂, ∂x is the given thickening.

Exactness at $\pi^{m+1}(SK \wedge K)$. Let x represent an element of $\pi^{m+1}(SK \wedge K)$; we have seen that if K is $(2k - m + 1)$-connected, x gives rise to a unique homotopy class of maps $\psi : K \to \partial N_0$, and if also $2m \geqq 3k + 3$, that ψ induces a unique thickening $K \to M^m \subset \partial N_0$. Also, the s-cobordism theorem shows that (ignoring corners) $N_0 \cong M \times I$.

Hence the induced thickening ∂x (represented by ϕ, M) is trivial if and only if there is a homeomorphism of M on the trivial thickening M_0^m, i.e. (as $N_0 = M_0^m \times I$ by definition) a homeomorphism h of N_0 on itself throwing $M_0 \times O$ on M.

Suppose this satisfied. As M_0 is induced by the map $K \to *$ we have an imbedding $i : M_0^m \subset D^m$, inducing $N_0 = M_0 \times I \subset D^m \times I \subset S^{m+1}$. Now take two copies of D^{m+2}, and attach along the copies of N_0 imbedded in the boundary by k, using h as attaching map. We obtain an $(m + 2)$-manifold W, with the homotopy type of the suspension of N_0, i.e. of K: a homotopy equivalence is given by extending $\phi : K \to N_0$ by maps of two cones on K into the two discs: this still preserves the base point, etc. Also, the complement C of $k(N_0)$ in S^{m+1} is simply-connected, as by van Kampen's theorem

$$1 = \pi_1(S^{m+1}) = \pi_1(k(N_0)) *_{\pi_1(\partial C)} \pi_1(C) \cong \pi_1 C$$

since $\pi_1(\partial C) \to \pi_1(k\, N_0)$ is an isomorphism. Hence ∂W is simply-connected. It follows that W defines a thickening $y \in \mathscr{T}^{m+2}(SK)$.

We now show that $\pi(y)$ equals the suspension of x, $Sx \in \pi^{m+1}(SK \wedge SK)$. It will be convenient to represent W as the union of $N_0 \times I$ and of two copies of D^{m+2}, where k : $N_0 \times O \subset \partial D_0^{m+2}$ and $k \circ h$: $N_0 \times 1 \subset \partial D_1^{m+2}$ are used as attaching maps. Similarly, represent SK as the union of two cones C_0K, C_1K and an equatorial belt $K \times I$. Then we define a homotopy equivalence $\omega : SK \to W$ as ψ_0 on $K \times O \to N_0 \times O$, extended radially to $C_0K \to D_0^{m+2}$; as ψ_t on $K \times I \to N_0 \times I$, and $k \circ h \circ \psi_1 = k \circ \psi_0$ extended radially to $C_1K \to D_1^{m+2}$.

Now $\psi_0(K) \subset M_0 \times O \subset N_0$, and we have an imbedding $j : M_0^m \subset D^m$, inducing

$$k : N_0 = M_0 \times I \subset D^m \times I \subset S^{m+1}.$$

We can deform ψ_0 to ψ_0', whose image lies in $M_0 \times \varepsilon \subset D^m \times \varepsilon$; likewise deform ω on C_0K to ω', disjoint from it, by taking the linear extension of ψ_0', joined up to a vertex slightly above the centre of D_0^{m+2}. A similar trick works on C_1K, so we can assume that the images of ω and ω' meet only inside $N_0 \times I$. A slight further deformation takes ψ_0' and ψ_1' into the interior of N_0: eventually we may suppose $\psi_0' = \psi_1' = \phi$, and $\psi_t' = \phi$ is a constant homotopy.

Recall that x is defined by the homotopy

$$K \xrightarrow{\phi \times \psi_t} N_0 \times N_0 \to N_0 \times N_0/C \leftarrow S^{m+1}.$$

We have now shown that ∂y is represented by

$$(K \times I) \times (K \times I) \xrightarrow{(\phi \times 1) \times \psi} (N_0 \times I) \times (N_0 \times I) \to (N_0 \times I) \times (N_0 \times I)/C' \leftarrow S^{m+2}.$$

This is essentially the same map, except that we have introduced an extra factor I (resulting eventually in a suspension) which is mapped by the identity, thus giving the suspension map, as required.

It also follows from the previous discussion that $\partial \circ (S^{-1}\pi) = 0$, provided we know that any thickening of SK can be obtained by attaching two copies of D^{m+2} along N_0. We defer the proof of this till §9. It remains, then, only to prove $\partial \circ S = \partial_s$, when the latter is defined.

Let $x \in \pi^{m+1}(SK \wedge K)$. Then, as above, if M_0^m is the trivial thickening of K and $N_0 = M_0 \times I$, x induces a map $\psi : K \to \partial N_0$, homotopic in N_0 to the natural map ψ_0. If, instead of starting with ψ_0, we start with a constant map, x similarly induces $\chi : K \to \partial N_0$. It follows from our assumptions that

$$\psi_0 \vee \chi : K \vee K \to \partial N_0$$

induces an m-thickening of K. The theorem will follow if we check

(1) The homotopy class $[\psi] = [\psi_0] + [\chi]$.

(2) χ induces the trivial thickening of K.

(3) $S\lambda'([\psi_0], [\chi]) = x$.

For by (2) and (3), $\alpha(\psi_0 \vee \chi)$ is determined by $(i_1^*, i_2^*, \lambda)\alpha(\psi_0 \vee \chi) = (O, O, S^{-1}x)$, and thus is equal to $\partial' S^{-1}x$, and by (1), $\alpha(\psi) = \partial_s S^{-1}(x)$, whereas by definition $\alpha(\psi) = \partial(x)$.

Proof of (1). We interpreted x by replacing S^{m+1} first, by $(N \times N/C)$, then by the pair $(N \times N, C)$, and observed that projection on the second factor N induced (up to homotopy) a fibration, with fibre the pair $(N, \partial N)$. Thus we interpret x by an element ξ of $\pi_1(N, \partial N; K)$.

Now the boundary $\partial_* : \pi_1(N, \partial N; K) \to \pi_0(\partial N; K) = [K : \partial N]$ is a homomorphism of abelian groups, and our definitions above amount to $[\chi] = \partial_* \xi$, $[\psi] = [\psi_0] + \partial_* \xi$. The result follows.

Proof of (2). By definition, χ is nullhomotopic in N_0. Thus it extends to a map of the cone on K into N_0. We assert that we can relativise the embedding theorem to find a thickening of CK (hence a copy of D^{m+1}) in N_0, meeting ∂N_0 in a manifold representing $\alpha(\chi)$. But then this manifold is imbeddable in S^m, and so determines the trivial thickening.

The proof given in §7, in fact, applies to the present situation with only slight changes. First, (7.1) is true if ∂M is replaced by an $(m-1)$-dimensional submanifold of ∂M throughout (Hudson's proof covers this case). Next, alter (7.2) to allow N to meet ∂M: write W for the relative boundary (i.e. the closure of $\partial N - \partial M$), and let the assumed handle decomposition of N be based on $N \cap \partial M$. Then the statement and proof need no further alteration.

Finally, if M_2 is the thickening induced by χ, we start by taking a collar neighbourhood $M_2 \times I$, and attach handles, corresponding to cells of CK not in K, in order of increasing dimension. Since K is $(2k - m + 2)$-connected, and we can choose a cell decomposition with each cell of K having dimension $\geqq (2k - m + 3)$, we can always take $c = 2k - m + 2$ in applying the lemma. There are then no difficulties in the induction.

Proof of (3). The left hand side is the suspension of the map defined by

$$K \times K \xrightarrow{\psi_0 \times \chi} \partial N_0 \times \partial N_0 \to (\partial N_0)^\tau \leftarrow S^m$$

But by the definition of χ, χ extends to a map $\chi' : CK \to N_0$, and x is represented by

$$K \times CK \xrightarrow{\psi_0 \times \chi'} N_0 \times N_0 \to N_0{}^\tau \leftarrow S^{m+1}.$$

Here, this last diagram is somewhat inaccurate: in the original discussion, ψ_0 was replaced by a map into the interior of N_0. If such replacement is not made, then (as we just saw) $K \times K$ is mapped into S^m, and this map is extended over $K \times CK$ into one hemisphere (since the first copy of K maps to $M_0 \times 0$, the arrow joining a point of it to a point of the second copy always points upward). The replacement is a homotopy, and has the effect of pushing $K \times K$ "downwards" in S^{m+1} to a point. So, up to homotopy, we have the usual definition of suspension.

§9. THICKENINGS OF SUSPENSIONS

We will now prove a result which was needed at one point in the last section.

THEOREM (9.1). *Suppose K^k is $(2k - m)$-connected, $\alpha \in \mathcal{T}^{m+2}(SK)$. Then if $\phi : K \to N_0 \subset S^{m+1}$ is the trivial thickening of K, there is a homeomorphism h of N_0 such that α is obtained by glueing two discs D^{m+2} along N_0 by h.*

Proof. We prove the result by induction on the number of cells of K. Write $K = L \cup_f e^k$. Then applying the imbedding theorem to the inclusion $SL \subset SK$, we see that α induces a thickening β of SL, represented (say) by $SL \to Y$, and α is represented by a manifold formed from Y by attaching a $(k + 1)$-handle. Also, L has less cells than K, so the induction hypothesis applies to give a description of β. (Note that the lemma is trivial for K a point, so the induction starts without trouble).

Now N_0 also induces the trivial thickening T_0 of L: in fact we have $N_0 = T_0 \cup h^k$, by the inductive proof of the imbedding theorem. Let $g : S^{k-1} \times D^{m-k+1} \to \partial T_0$ be the attaching map of h^k, $\bar{g}$ its restriction to $S^{k-1} \times O$. We have seen that Y is of the form $D_a^{m+2} \cup_{h_0} D_b^{m+2}$, where h_0 is a homeomorphism of the imbedded copies of T_0; and we know that a manifold representing α is formed from Y by adding a handle h^{k+1}. Suppose we can show that the attaching map F of h^{k+1} can be chosen such that the attaching sphere S^k has one hemisphere D_-^k the core of h^k in $N_0 \subset \partial D_a^{m+2}$, and the other hemisphere D_+^k in ∂D_b^{m+2}. Then we can perform an isotopy of $F : S^k \times D^{m-k+1} \to \partial Y$, fixed on $S^k \times O$, to make Im F meet D_b^{m+2} in $D_+^k \times D^{m-k+1}$, and D_a^{m+1} in the handle h^k. If we attach h^{k+1} to D_b^{m+2} by $F|(D_+^k \times D^{m-k+1})$, then (ignoring corners in the smooth case) we are attaching two

$(m+2)$-discs along a common $(m+1)$-face, and the result is again an $(m+2)$-disc D_c^{m+2} and our thickening is expressed as $D_a^{m+2} \cup D_c^{m+2}$, where the discs meet in $N_0 \subset \partial D_a^{m+2}$, as asserted.

Thus it remains only to deform $F|S^k \times O$ till the intersection with ∂D_a^{m+2} is as required. Now any two k-discs are isotopic, so we can certainly perform an isotopy of F to make $F|D_-^k$ as required. Thus the essential point is to show that $F|D_+^k$ can be deformed (relative to the boundary) off ∂D_a^{m+2} into ∂D_b^{m+2}.

We first show that this can be achieved by a homotopy. This follows at once from the assertion that the image under $\pi_k \partial Y \to \pi_k(\partial Y, \partial Y \cap \partial D_b^{m+2})$ of the class of F is represented by $F|D_-^k$. Write Z for $\partial Y \cap \partial D_b^{m+2}$. Now observe that if the centre of D_a^{m+2} is removed, we can deformation retract what is left onto ∂D_a^{m+2}. Thus also $\partial Y \cup D_b^{m+2}$ is a deformation retract of Y minus a point. Consider the triad $(\partial Y \cup D_b^{m+2}; \partial Y, D_b^{m+2})$. Here, the pair $(\partial Y, Z)$ has the same homology (and also connectivity, both having the same fundamental group) as $(\partial Y \cap \partial D_a^{m+2}, \partial Y \cap \partial D_a^{m+2} \cap \partial D_b^{m+2})$, or as $(\partial D_a^{m+2}, T_0)$, hence its connectivity exceeds by 1 that of T_0, so $(\partial Y, Z)$ is $(2k - m + 1)$-connected. The pair (D_b^{m+2}, Z) has connectivity 1 greater than that of Z, and Z is homotopy equivalent to $\partial D_b^{m+2} - T_0$, so is m-dual to T_0, or to L, so Z is $(m - k - 1)$-connected. By the Blakers–Massey theorem 2, Theorem 1] (see also [19, 1.23]), the triad is $(k+1)$-connected. Hence

$$\pi_k(\partial Y, Z) \cong \pi_k(\partial Y \vee D_b^{m+2}, D_b^{m+2}) \cong \pi_k(Y, *).$$

But the homotopy class in $Y \simeq SL$ of F is that of the suspension of f; $F|D_-^k$ was chosen to represent this, and our assertion follows.

We now need to treat two cases separately

Case 1. $2m \geqq 3k + 2$

Let $G : (D_+^k, S^{k-1}) \to (\partial Y, Z)$ be the map just constructed. We assert that G is homotopic (rel S^{k-1}) to an imbedding. Indeed, the dimension condition $2(m+1) \geqq 3(k+1)$ is satisfied; by the theorem of Haefliger [3] or Irwin (see [17] or [18]) it is sufficient to check Z $2k - (m+1) + 1$-connected. But we saw above that Z was $(m - k - 1) \geqq (2k - m + 1)$-connected.

Collating G with $F|D_-^k$ we obtain an imbedding $F' : S^k \to \partial Y$ homotopic to F. We now show F' isotopic to F, which completes the proof in this case. It suffices to appeal to the isotopy theorems corresponding to the above, and observe that ∂Y is $2k - (m+1) + 2$-connected and $2(m+1) > 3(k+1)$. The connectivity of ∂Y follows, for example, from that of Z and the fact that the pair $(\partial Y, Z)$ (as we have just seen) has the same connectivity as Y, or as SL, hence at least $(2k - m + 1)$.

Case 2. *K is $(r-1)$-connected, where $2r \geqq k + 1$*

We observed above that Z is Spanier–Whitehead m-dual to L. A corresponding remark applies to $\partial Y \cap D_a^{m+2}$, which thus has a deformation retract a finite complex L^* homotopy equivalent to one of dimension $(m - r)$: indeed, we may take L^* of dimension $(m - r)$ by applying results of Smale (see [14], or [16, Theorem 5.5]).

We shall use the relative version of the theorem of Haefliger quoted in §6, to obtain an isotopy of $F|D_+^k$ to a position disjoint from L^* (which can then be pulled back further into Z). Our result about the class of F in $\pi_k(\partial Y, Z)$ shows that there is a homotopy which accomplishes this; to apply the result in question, we need only check the dimension conditions

$$2(m+1) \geqq k + (m-r) + \max(k, m-r) + 3,$$

which reduce to $2r \geqq k + 1$ and $r \geqq (2k - m + 1)$: our two hypotheses.

It remains only to observe that as K is $(2k - m)$-connected, we can take $r = 2k - m + 1$; thus if Case 2 does not apply, we have $k \geqq 2r = 2(2k - m + 1)$, and so Case 1 does.

Note: A simpler proof of Theorem 9 has now been found by J. P. E. Hodgson.

§10. CALCULATIONS

Our intention here is not so much to give specific calculations as to demonstrate the utility of the concepts introduced above. Note that in §6 we have already performed a calculation: of thickenings in the stable range. Our next example extends this to the metastable range in a very special case.

PROPOSITION (10.1). *Let K^k be a smooth compact manifold, $2m \geqq 3k + 3$, $m \geqq k + 3$. Then smooth thickenings are determined by*

$$\mathscr{T}^m(K) \cong [K : BO_{m-k}].$$

Proof. Given a map $K \to BO_{m-k}$, take the induced disc bundle E over K, and the inclusion of K as zero cross-section: this evidently defines a thickening. (If K has no boundary, it is necessary to deform the base-point to ∂E: since the fibre is a disc, hence contractible, this can be done essentially uniquely). Thus we have a natural transformation

$$\alpha : [K : BO_{m-k}] \to \mathscr{T}^m(K).$$

Conversely, given a thickening $\phi : K^k \to M^m$, by a theorem of Haefliger [3], ϕ is homotopic to a smooth imbedding in Int M. Moreover, if $2m \geqq 3k + 4$, any two such imbeddings are isotopic, so the normal bundles are the same, and define a homotopy class of classifying maps $K \to BO_{m-k}$. Thus we have a map $\beta : \mathscr{T}^m(K) \to [K : BO_{m-k}]$. Evidently $\beta\alpha = 1$, and the relation $\alpha\beta = 1$ follows by a (by now familiar) application of the s-cobordism theorem.

In the case $2m = 3k + 3$, we assert that two homotopic imbeddings are regularly homotopic, which suffices for our argument. The proof of this is essentially the same as for Lemma 1 of [1]; we shall omit it.

Remark. In the piecewise linear case everything goes through without change, except where we come to use the normal bundle. Thus we have a natural transformation

$$\alpha : [K : BPL_{m-k}] \to \mathscr{T}^m(K),$$

but cannot easily define an inverse. In fact, using the ideas of [7] we obtain easily a natural transformation

$$\beta : \mathscr{T}^m(K) \to [K : BPL_{m,k}]$$

and the composite $\beta \circ \alpha$ is induced by the natural inclusion $PL_{m-k} \subset PL_{m,k}$. It is conjectured in [7] that this inclusion induces isomorphisms of homotopy groups in the metastable range: were this so, we could conclude as above. Even failing this, we conjecture that β is always bijective in the metastable range. (The author now understands that this conjecture has been proved by Haefliger with $PL_{m,k}$ replaced by $\widetilde{PL}_{m,k} = \lim_{r \to \infty} PL_{m+r,k+r}$).

The next calculation is taken from a paper of Haefliger [5, Theorems 5.3 and 5.7]. The author had previously obtained part of this result, but Haefliger's proof is so clear that there seems no need to repeat it. For comparing our statement of the result with his, note that the proof of our embedding theorem shows that thickenings in $\mathscr{T}^{n+q}(S^n)$ coincide with handlebodies in $\mathscr{H}(n+q, 1, n)$.

PROPOSITION 10.2 (HAEFLIGER). *Let $q \geqq 3$. Then the group $\mathscr{T}^{n+q}(S^n)$ is isomorphic to the n^{th} homotopy group of the quadruple*

$$\begin{array}{ccc} * & \longrightarrow & G_q \\ \downarrow & & \downarrow \\ SO\ (\text{or } SPL) & \longrightarrow & G \end{array}$$

Here, G_q denotes the space of maps $S^{q-1} \to S^{q-1}$ of degree 1 (with the compact-open topology), $G_q \subset G_{q+1}$ by suspension and $G = \lim G_q$; similarly $SO = \lim SO_q$ (and $SO_q \subset G_q$ in an obvious way) and for SPL. As usual, $*$ denotes the base point. The homotopy group of the quadruple is taken in the sense of Eckmann–Hilton. Of course, if SO is used in the above it classifies smooth thickenings: use SPL for piecewise-linear thickenings.

Further work of Haefliger gives also a calculation of $\mathscr{T}^{n+q}$ of a bouquet of spheres S^n of the same dimension. Note that in the metastable range, $\pi_n(SO, SO_q) \cong \pi_n(G, G_q)$ and our two results coincide. Note also that in the same range, $\pi_n(G, G_q) = \pi_n(PL, PL_{n+q,n})$, so the conjecture above is verified in the case of spheres.

Our last result is less precise, but will be useful in computations using the suspension sequence. We need the

BARRATT–MAHOWALD THEOREM [1]. *The natural fibration $\Omega^{8s}BSO(n+8s) \to \Omega^{8s}BSO$ has a cross-section over the $(n+4s-7)$-skeleton.*

COROLLARY. *Let $\dim L \leqq n+4s-7$. Then the homomorphism $[S^{8s}L : BSO(n+8s)] \to [S^{8s}L : BSO]$ is a split epimorphism.*

PROPOSITION (10.3). *Let $\dim L = l \leqq n+4s-7$; suppose L r-connected, write $K^k = S^{8s}L$. Then*

$$\mathscr{T}^{2k+n-r+1}(K) \to \mathscr{T}(K)$$

is a split epimorphism (smooth case only).
As we already know that $\mathscr{T}^{2k}(K) \to \mathscr{T}(K)$ is epi (though not necessarily split) the result is only really useful if $r \geqq n+2$: in any case, only if K is connected up to a large dimension.

Proof. As K is $(8s+r)$-connected, of dimension $(8s+l)$, there is a trivial thickening M_0 in $\mathscr{T}^{8s+2l-r+1}(K)$. The corollary above shows that $[K : BSO(n+8s)] \to [K : BSO] =$

$\mathscr{T}(K)$ is a split epimorphism: for $\alpha \in \mathscr{T}(K)$, take the image in $[K : BSO(n + 8s)]$ and let N_α be the total space of the corresponding disc bundle, over M_0. Thus N_α defines an element of $\mathscr{T}^{2t+16s+n-r+1}(K)$; it remains only to show that the splitting map defined by $\alpha \to N_\alpha$ is a homomorphism.

Now in the same dimension where K has a trivial thickening, Theorem 1 also shows that the "diagonal" map $K \to K \vee K$ is such that the trivial thickening $M_0 + M_0$ of $K \vee K$ induces a thickening (not unique) of K. But as $M_0 + M_0$ is induced by imbedding in Euclidean space, so is this induced thickening, which thus must be M_0. Now if $\alpha, \beta \in \mathscr{T}(K)$ have images $\alpha', \beta' \in [K : BSO(n + 8s)]$, we form N_α, N_β and $N_\alpha + N_\beta$. As we have an imbedding $\delta : M_0 \subset M_0 + M_0$, and the induced bundle is classified by $\alpha' + \beta'$, we have an imbedding of bundle spaces $N_{\alpha+\beta} \subset N_\alpha + N_\beta$. It follows that our construction is additive, as required.

COROLLARY. *Take* $L = S^2 \cup_\alpha e^3$, $K^k = S^{8s}L$. *Then* $\mathscr{T}^m K \to \mathscr{T} K$ *is a split epimorphism if* $m \geqq 12s + 23$, i.e. $2m \geqq 3k + 37$.

We conjecture that this result can be substantially improved.

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