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CLASSIFICATION PROBLEMS IN DIFFERENTIAL TOPOLOGY—VI

CLASSIFICATION OF $(s-1)$-CONNECTED $(2s+1)$-MANIFOLDS

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IN THIS paper we first complete the diffeomorphism classification of almost-closed $(s-1)$-connected $(2s+1)$-manifolds P. We compute the Grothendieck group $\mathscr{G}_s^{2s+1}$ of such P modulo as those obtained from boundaries of handlebodies. The diffeomorphism class of the boundary of P depends only on the class of P in $\mathscr{G}_s^{2s+1}$, which has order at most 8. We give applications of this result, and an interpretation of $\mathscr{G}_s^{2s+1}$ as a cobordism group. In a final section, we compare our information on exotic spheres with a construction due to Milnor.

We continue our numbering from the previous papers of the series [I], [II], [III].

The appearance of this paper has been delayed for three years by the discovery of a serious gap in the argument. This has not affected the results materially, but it has led us to write [IV], and we will also need a few calculations based on [IV]. To preserve continuity of style, these are deferred to another paper [VII] and the present paper retains almost its original form, except for the introduction of canonical thickenings, which seem to be a potentially important tool.

§12A. HOMOLOGY INVARIANTS

Before we tackle our main classification problem, as promised above, it is convenient to list the invariants which will occur later and to enumerate their properties. We shall speak of our manifolds as closed. If P is bounded by a homotopy sphere—i.e. is almost-closed in the sense of [15]—we need only add a cone to the boundary to obtain a space which we can treat as a closed manifold.

Let, then, P be a closed $(s-1)$-connected $(2s+1)$-manifold. The only nonzero homology groups are those in dimensions $0, s, s+1, 2s+1$. If we suppose P oriented (we assume $s \geqslant 2$ throughout, so P is certainly orientable) the first and last of these are free cyclic, with preferred generators. We write G for $H_s(P)$. By duality, $H_{s+1}(P)$ is a free abelian group, of the same torsion-free rank as G; in fact,

$$H_{s+1}(P) \cong H^s(P) \cong H_s(P) \pitchfork Z \cong G \pitchfork Z,$$

where we borrow the symbol $\pitchfork$ for a group of homomorphisms, as all the other relevant notation and terminology, from our paper [Q]. There is a further element of structure, the linking invariant (see [11] p. 253, [8], [14]). This may be described as follows. The exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{S} \to 0$ of coefficient groups induces an exact cohomology sequence

$$H^s(P; \mathbb{Q}) \to H^s(P; \mathbb{S}) \to H^{s+1}(P; \mathbb{Z}) \to H^{s+1}(P; \mathbb{Q}).$$

Since $\mathbb{S}$ is injective, $H^s(P; \mathbb{S}) \cong H_s(P) \pitchfork \mathbb{S}$. It is clear that the kernel of the central map above is the set of homomorphisms $H_s(P) \to \mathbb{S}$ which annihilate the torsion subgroup; the image is the torsion subgroup of $H^{s+1}(P)$. Writing τ for torsion subgroup, we thus have a natural isomorphism (without any condition on P)

$$\tau H_s(P) \pitchfork \mathbb{S} \cong \tau H^{s+1}(P)$$

and so a nonsingular bilinear pairing

$$\tau H_s(P) \otimes \tau H^{s+1}(P) \to \mathbb{S}.$$

But we can identify $H^{s+1}(P)$ with $H_s(P)$ using Poincaré duality. Thus, writing G^* for the torsion subgroup of G, we have a nonsingular bilinear form

$$b : G^* \otimes G^* \to \mathbb{S}.$$

We recall the geometrical definition of b by linking numbers. Let $x \in G^*$ have order r, represent x by a chain ξ, and let $\partial \zeta = r\xi$. Then if $y \in G^*$ is represented by the chain η, we have

$$b(x, y) = \frac{1}{r} \zeta \cap \eta \qquad (\text{mod } 1).$$

We also recall that b is $(-1)^{s-1}$-symmetric; indeed, if $\partial\psi = q\eta$, and ψ and ζ meet normally,

$$\begin{aligned} \partial(\zeta \cap \psi) &= \partial\zeta \cap \psi + (-1)^s \zeta \cap \partial\psi \qquad (\text{since } \dim \zeta = s) \\ &= \psi \cap r\xi + (-1)^s \zeta \cap q\eta \end{aligned}$$

and the Kronecker index of the 0-chain on the left is clearly zero; that on the right is $qr(b(y, x) + (-1)^s b(x, y))$, and so $b(y, x) = (-1)^{s+1} b(x, y)$ as required.

We shall study $b(x, x)$ rather more carefully, and will do this geometrically since although our hypotheses can be somewhat weakened (cf. Browder [3]), the homotopy theory is somewhat complex. Represent x by an imbedded sphere S^s (since P is $(s-1)$-connected, $\pi_s(P) \cong H_s(P)$). Let B be a tubular neighbourhood of S^s: this is a disc bundle defined by $\alpha(x) \in \pi_{s-1}(SO_{s+1})$. Let S_1^s be a cross-section of the associated sphere bundle ∂B and S_2^s a fibre, and let $\alpha_1 \in \pi_{s-1}(SO_s)$ be the characteristic class of the normal bundle of S_1^s in ∂B: we have $S\alpha_1 = \alpha(x)$. Write X for the manifold $P - \mathring{B}$; let y_i be the homotopy class of S_i in X. Then y_2 has infinite order, for if some chain in X is bounded by nS_2^s, on filling in with discs (fibres of B) we obtain a cycle in P, with intersection number n with x; since x is a torsion class, $n = 0$. Since P is formed from X by attaching an $(s+1)$-cell and a $(2s+1)$-cell, y_2 generates the kernel of $i_* : H_s(X) \to H_s(P)$. Now $i_*(y_1) = x$ has order r, so $i_*(ry_1) = 0$, and thus $ry_1 = \lambda y_2$ for some well-determined integer λ. Moreover, $b(x, x) = \lambda/r$ (mod 1).

We consider the effect of changing the cross-section S_1. Homotopy classes of cross-sections are in (1–1) correspondence with the integers, and $\sim S_1 + nS_2$ in ∂B for various n. We find accordingly (using Theorem 1)

$$\alpha_1' = \alpha_1 + \partial(n)$$
$$\lambda' = \lambda + nr$$

where $\partial : \pi_s(S^s) \to \pi_{s-1}(SO_s)$. Now if s is odd, and not 3 or 7, we demand $\phi(\alpha_1') = 0$. This determines n modulo 2, and hence λ/r modulo 2; we call the result $q(x)$. If s is even, then $\pi(\alpha_1)$ is even except, perhaps, if $s = 2$, 4 or 8. But if s is 4 or 8, the homomorphism $\alpha : G \to \pi_{s-1}(SO_{s+1})$ annihilates the torsion class x, so again $\pi(\alpha_1)$ is even. The condition $\alpha_1' = 0$ completely determines n and hence the rational number $\lambda/r(s+2)$; we denote it by $q(x)$.

LEMMA 26. *For $s \geqslant 4$, $s \neq 7$ we have*

$$q(x+y) - q(x) - q(y) = b(x, y) + b(y, x).$$

I.e. if s is odd, the left hand side is $2b(x, y)$ (mod 2); *if s is even, it is zero.*

Proof. For $x \in G^*$, we construct S, B, S_1, S_2 as above, and similarly find T, C, T_1, T_2 for y, with C disjoint from B. Join B to C by a tube. Let $rx = ry = 0$. We note that (as above) the choice of S_1, T_1 determines $q(x)$ etc. as rational numbers; we shall prove the formula in this sense.

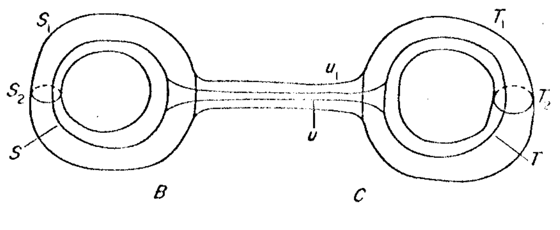
In $P - \mathring{B}$ we have the homologies

$$rS_1 \sim rq(x)S_2 \qquad rT_1 \sim rb(y, x)S_2$$

and in $P - \mathring{C}$ similarly

$$rS_1 \sim rb(x, y)T_2 \qquad rT_1 \sim rq(y)T_2.$$

But we can join S and T by a tube to obtain an imbedded sphere U representing $x + y$. A corresponding cross-section U_1 is obtained by joining S_1 and T_1, and both S_2 and T_2 occur as fibres to the normal bundle.



By the above, if B, C and the tube are deleted from P, (the tube makes no real difference),

$$rU_1 \sim r\{q(x) + b(y, x)\}S_2 + r\{b(x, y) + q(y)\}T_2$$

so deleting only the neighbourhood of U, when $S_2 \sim T_2 \sim U_2$,

$$rU_1 \sim r\{q(x) + q(y) + b(x, y) + b(y, x)\}U_2.$$

But we have $rU_1 \sim rq(x+y)U_2$, for the characteristic class of the normal bundle of U_1 in the boundary of a tubular neighbourhood of U is the sum of the corresponding elements for S and T so ϕ (if s is odd), π (if s is even) vanishes for it. The result now follows.

COROLLARY 1. *If $s \geqslant 5$ is odd, $s \neq 7$, then q defines a homogeneous quadratic form on G^*, with associated bilinear map $2b$.*

The lemma proves all except homogeneity; this follows by reversing the orientations of all the spheres involved.

It seems that q is in fact a homotopy type invariant; we will show this elsewhere.

COROLLARY 2. *If s is even, $s \geqslant 4$, then q is zero, so b is strictly skew-symmetric* (see [Q, p. 282]).

For q is then a homomorphism of a torsion group into a torsion-free group.

Some cases of this result were obtained in [14].

COROLLARY 3. *If $s = 2$, $b(x, x)$ is 0 or $\frac{1}{2}$ according as $\alpha(x)$ is zero or not.*

The argument above shows that if $\alpha(x) = 0$, then $b(x, x) = 0$. But, identifying $\pi_1(SO_2)$ with Z, it will also prove $q(x) = \lambda/r - \alpha_1/2$ a homomorphism, and so zero. We write $c(x)$ for $b(x, x)$ when s is even. Then c is a homomorphism; we shall describe the last result by saying, simply, $\alpha = 2c$.

LEMMA 27. *For $s \geqslant 2$, a closed $(s-1)$-connected P^{2s+1} has homology invariants: A group $G = H_s(P)$; a nonsingular, $(-1)^{s+1}$-symmetric bilinear $b: G^* \otimes G^* \to \mathbf{S}$, with $c = \frac{1}{2}\alpha$ if $s = 2$, $c = 0$ if $s > 2$ is even; a quadratic $q: G^* \to \mathbf{Q}$ (mod $2\mathbf{Z}$) with associated bilinear map $2b$ if s is odd, $s \neq 3, 7$.*

§12B. TANGENTIAL INVARIANTS

We have further invariants of P (as above): we next investigate the λ, α of [1]. We shall consider them both in dimension s and in $(s+1)$; in the latter case we denote them by μ, β. Intersections of s-cycles and $(s+1)$-cycles were discussed above. Now for dimensional reasons, λ is zero and α a homomorphism, $\alpha: G \to \pi_{s-1}(SO_{s+1})$. This is defined for any value of s.

By Theorem 1, provided $s \geqslant 4$, we have mappings

$$\beta: \pi_{s+1}(P) \to \pi_s(SO_s) \qquad \mu: \pi_{s+1}(P) \times \pi_{s+1}(P) \to \pi_{s+1}(S^s)$$

with the usual properties. As is well-known (and easy to prove),

$$\pi_{s+1}(P) \cong H_{s+1}(P) \oplus (H_s(P) \otimes Z_2),$$

for e.g. we can replace P by an equivalent CW-complex, with cells only in dimensions 0, s, $s+1$, $2s+1$. The projection on the first summand is the Hurewicz map; the injection of the second is found by composing elements of $H_s(P) \cong \pi_s(P)$ with the generator η of $\pi_{s+1}(S^s)$. But by results of §1 (recall that F_0' was defined in §7 as a special case of the F of Lemma 2), for $x \in \pi_s(P)$, $y \in \pi_{s+1}(P)$ we have

$$\beta(x \circ \eta) = F_0'\alpha(x),$$

$$\mu(x \circ \eta, y) = \lambda(x, y) \circ \eta.$$

These, with the addition formulae (Theorem 1) show that values of β and μ on the second summand above contribute no new invariants—these must then be sought on the first summand. This, however, is not uniquely defined, so β and μ have indeterminacies. Essentially, these are as in Lemmas 14 and 15, and the calculation as for Lemma 22, but as the situation is somewhat different, we give the details.

Let $e_1, \dots, e_r$ be a base for the free part of $G = H_s(P)$, $f_1, \dots, f_r$ the dual base of $\hat{G} = H_{s+1}(P)$, and $f_1', \dots, f_r'$ some corresponding elements of $\pi_{s+1}(P)$. We identify $\pi_s(P)$ with $H_s(P)$, and use x, y for typical torsion elements, so $x \in G^*$. By the formulae above,

$$\begin{aligned}\mu(f_i' + (\Sigma_k \xi_{ik} e_k + x) \circ \eta, f_j' + (\Sigma_l \xi_{jl} e_l + y) \circ \eta) &= \mu(f_i', f_j') + \Sigma_k \xi_{ik} \mu(e_k \circ \eta, f_j') \\ &\quad + \Sigma_l \xi_{jl} \mu(f_i', e_l \circ \eta) \\ &= \mu(f_i', f_j') + (\xi_{ij} + \xi_{ji})\eta,\end{aligned}$$

(signs may be ignored as all terms have order 2), and this may be made zero for $i \neq j$ by choosing the ξ_{ij} appropriately. So μ contributes no new invariant, and we are still free to modify the f_i' using a symmetric matrix ξ_{ij} and compositions $x \circ \eta$.

Define again the exceptional case as that in which F_0' is nonzero; we recall from [III] that this happens when $\alpha(G)$ is nonzero modulo 2 and $s = 0, 1 \pmod 8$, perhaps also when $s = 4$. Apart from this case,

$$\beta(f_i' + (\Sigma_j \xi_{ij} e_j + x) \circ \eta) = \beta(f_i') + \xi_{ii} \partial \eta,$$

so $\partial\eta$ generates the indeterminacy subgroup of $\beta(f_i')$ for each i, and $S\beta$ remains as a new invariant. In the exceptional case, the image of F_0' is cyclic of order 2, and

$$\beta(f_i' + (\Sigma_j \xi_{ij} e_j + x) \circ \eta) = \beta(f_i') + \xi_{ii} \partial \eta + F_0' \alpha(\Sigma_j \xi_{ij} e_j) + F_0' \alpha(x).$$

Thus if $F_0'\alpha(x)$ can be nonzero, we have an indeterminacy independent of the above and (cf. §10) find the desired invariant is $\phi S\beta$ ($s = 0 \pmod 8$) or 0 (otherwise). If, however, $F_0'\alpha$ vanishes on G^*, we may choose our basis with $F_0'\alpha(e_1) \neq 0$, $F_0'\alpha(e_i) = 0$ for $i > 1$, and the above reduces to

$$\beta(f_i) + \xi_{ii} \partial \eta + \xi_{i1} F_0'(0),$$

so that for $i > 1$ we have $\phi S\beta(f_i)$ or 0 as above, but for $i = 1$ an extra invariant ω as in Lemma 23.

Note that if $s = 0 \pmod 4$, the image of α is torsion-free, so certainly $\alpha(G^*) = 0$. Here we identify $\pi_{s-1}(SO_{s+1})$ with $\mathbf{Z}$, and so $\alpha : G \to \mathbf{Z}$ with $\hat{\alpha} \in \hat{G}$. We observe that in the case above this agrees with $f_1 \pmod{2\hat{G}}$.

LEMMA 28. *For $s \geqslant 4$, the manifold P^{2s+1} has tangential invariants: a homomorphism $\alpha : G \to \pi_{s-1}(SO_{s+1})$; in the non-exceptional case a homomorphism $S\beta : \hat{G} \to S\pi_s(SO_s)$; in the exceptional case (a) if $s = 0 \pmod 8$ a homomorphism $\phi S\beta : \hat{G} \to \mathbf{Z}_2$ and (b) if $\alpha(G^*) = 0$, an element $\omega \in \mathbf{Z}_2$.*

Notice that ω was defined as part of $\beta(\hat{\alpha})$, modulo indeterminacy, if $s = 0 \pmod 4$. The linearity of $S\beta$ follows from the addition formula for β.

We shall frequently consider connected sums of manifolds, and so formulate a complement to the last two lemmas. By the *system of invariants* of P we understand the group G, together with such of the mappings and elements (b, q, α, $S\beta$, $\phi S\beta$, ω) as are appropriate, satisfying the conditions of Lemmas 27 and 28: also write $\omega = 0$ in the non-exceptional case. The *direct sum* of two such systems, suffixed by 1 and 2, is given by the group $G = G_1 \oplus G_2$, and if

$$x = x_1 + x_2,\ y = y_1 + y_2,$$
$$b(x, y) = b_1(x_1, y_1) + b_2(x_2, y_2)$$
$$q(x) = q_1(x_1) + q_2(x_2) \qquad \text{(similarly } \alpha,\ S\beta,\ \phi S\beta)$$
$$\omega = \omega_1 + \omega_2 \qquad \text{if } \alpha(G^*) = 0.$$

COMPLEMENT. *The system of invariants of $P_1 \# P_2$ is the direct sum of the corresponding systems for P_1, P_2.*

For certainly $H_s(P_1 \# P_2) \cong H_s(P_1) \oplus H_s(P_2)$, and a sphere representing a given class in P_1 continues to do so in $P_1 \# P_2$, retains the same normal bundle and intersection with another sphere in P_1, and fails to intersect a sphere in P_2.

If two spheres, representing $\hat{\alpha}_1$ in P_1 and $\hat{\alpha}_2$ in P_2 are joined by a tube, we get a sphere representing $\hat{\alpha}$ in P, and the normal bundles just add: this proves the result for ω.

The result (and proof) is valid for all s if the invariants involving β are deleted.

The above results will not be adequate for our problem, on account of the presence of torsion in the homology groups of P. We next extract the simplest invariant: the stable tangent bundle of P, which can be regarded as an element of the known group $KO(P)$. This group lies in the exact sequence

$$0 \to H^{s+1}(P; \pi_s(SO)) \to KO(P) \to H^s(P; \pi_{s-1}(SO)) \to 0 \qquad (1)$$

This sequence arises by calculating $KO(P)$ by standard obstruction theory; it can also be obtained from the spectral sequence of the generalized cohomology theory KO. It splits except when $s \equiv 1 \pmod 8$ and P has torsion of order 2. If we exclude the exceptional case, our element of $KO(P)$ is determined by invariants

$$\hat{\alpha} \in H^s(P; \pi_{s-1}(SO)) \qquad \hat{\beta} \in H^{s+1}(P; \pi_s(SO)).$$

In the exceptional case, we cannot define a true homology invariant $\hat{\beta}$, and will ignore it. Note that only if $s \equiv 0$ or $1 \pmod 8$ are all three terms of (1) nonzero.

In Lemma 28 we considered the unstable group $S\pi_s(SO_s)$. It is easily shown (or we may use the calculations of Kervaire [7], cf. [III]) that

If $s = 1, 3, 7$, $S : S\pi_s(SO_s) \to \pi_s(SO)$ is injective, with image of index 2.

If $s \neq 1, 3, 7$ is odd, or if $s = 2, 4$ or 8, S as above is bijective.

If $s \neq 2, 4, 8$ is even, S is a split surjection with kernel of order 2.

We will require a cohomology class in $H^{s+1}(P; S\pi_s(SO_s))$ in the non-exceptional case. Most of this provided by $\hat{\beta}$; in the third case above we need an additional invariant (say $\hat{\phi}$) in

$H^{s+1}(P : Z_2)$. Such an invariant is constructed in [VII]. In the first case above, the injective map S need not induce an injective cohomology map: thus $\hat{\beta}$ is not adequate for our purposes. As the case $s = 3$ was already excluded in [III], we will exclude the cases $s = 1, 2, 3, 7$ for the remainder of this paper, except for odd comments.

§12C. THE CANONICAL THICKENING

In this subparagraph we suppose throughout that $s \equiv 1 \pmod 8$. Then Lemma [illegible] defined an invariant ω in the case $\alpha(G^*) = 0$, and we proved it additive. Our present objective is to extend the definition to all manifolds P, and again find an addition formula. This will necessitate a much closer investigation.

First consider the classifying map of the stable tangent bundle of P, $P \to BSO$. Since P is $(s-1)$-connected, this factorises through a map, determined uniquely up to homotopy, of P to the $(s-1)$-connected covering of BSO. According to Bott [2], the homotopy group of this last is cyclic of order 2 in dimensions s and $(s+1)$, and zero in dimension $(s+2)$; moreover, the first k-invariant is nonzero. It is an easy corollary that, up to homotopy, it admits a cell decomposition of the form $(S^s \cup_2 e^{s+1}) \cup$ cells of dimension $\geqslant s+3$. We write BY for the $(s+1)$-skeleton $S^s \cup_2 e^{s+1}$, and conclude that our classifying map can be regarded as a map $\tau : P \to BY$ (each determines the other up to homotopy).

[illegible] $(s+1)$-duality of P with [illegible] Spanier–Whitehead [illegible] dual of BY. We choose this so that it also admits a cell decomposition $S^s \cup_2 e^{s+1}$. Then since $[AY : P]$ is a stable group, S-duality gives an isomorphism

$$D : [P : BY] \to [AY : P].$$

Thus $D(\tau)$ determines a homotopy class of maps $AY \to P$, which we will call canonical. By the embedding theorem of [IV], this class determines a $(2s+1)$-thickening of AY. This thickening—or the corresponding element of $\mathscr{T}^{2s+1}(AY)$—we will call the *canonical thickening*.

We next need the computation [VII] of $\mathscr{T}^{2s+1}(AY)$. Note that given a $(2s+1)$-thickening $\phi : AY \to M$, we can form $M/\partial M$, which will consist of a $(2s+1)$-cell attached to an S-dual of AY. Up to homotopy, we can write

$$M/\partial M = BY \cup_f e^{2s+1}.$$

Then ϕ determines the homotopy class of f in $\pi_{2s}(BY)$. Also, $\pi_{2s+2}(S^2 BY)$ is stable, and [18, II] there is a split short exact sequence

$$0 \to Z_8 \to \pi_{2s}(BY) \to \pi_{2s+2}(S^2 BY) \to 0.$$

Let $\omega : \pi_{2s}(BY) \to Z_8$ be a splitting. Then f is determined by $\omega(f)$ and $S^2 f$. Moreover, the stable homotopy class of f is determined by the tangent bundle of M, which is classified (cf. above) by a map $\tau : AY \to BY$. Then it is shown in [VII] that

$$(\tau, \omega(f)) : \mathscr{T}^{2s+1}(AY) \to [AY : BY] \oplus Z_8 \approx Z_4 \oplus Z_8$$

is an isomorphism.

The invariant ω we seek is $\omega(f)$, calculated for the canonical thickening. Observe that the canonical thickening—hence also $\omega(f)$—is additive. This is immediate, as the boundary-connected sum $P = P_1 + P_2 \cong P_1 \vee P_2$, and τ_P is given by τ_{P_i} on the summands; similarly D preserves the summands, hence $D(\tau_P) = D(\tau_{P_1}) + D(\tau_{P_2})$. Our assertion now follows from Property (4.2) in [IV].

To justify our neglect of τ on the canonical thickening, we observe that it is in fact determined by the value of $\omega(f)$ on the canonical thickening. For since D is defined by Poincaré duality, the above τ—i.e. the composite $\tau_P \circ D(\tau_P)$—can be identified with the composite $AY \to M \to M/\partial M \leftarrow BY$. But by [18, II] this is determined by the Hopf invariant of f, and so (up to sign) is the same as $\omega(f)$ reduced mod 4. We also observe that the map $AY \to BY$ determined by duality is then determined by our quadratic form q. In fact, the canonical map $AY \to P$ induces a map of the subcomplex S^s, and hence a homology class $\hat{\chi} \in G$ (with $2\hat{\chi} = 0$). Then $q(\hat{\chi})$ is a multiple of $\frac{1}{2}$, calculated mod 2. Reducing mod 1 gives $b(\hat{\chi}, \hat{\chi})$, which is equivalent to the functional cup product induced by f, and hence to the Hopf invariant of f, reduced mod 2. Thus $q(\hat{\chi})$ (mod 2) is equivalent to the Hopf invariant of f, and hence to $\omega(f)$ (Mod 4). We choose the isomorphism of Z_8 on the value group of $\omega(f)$ so that

$$\omega(f) \equiv 2q(\hat{\chi}) \qquad (\text{mod } 4).$$

Observe that $\alpha(G^*) = 0 \Leftrightarrow \hat{\chi} = 0$. When this happens, $\omega(f)$ is a multiple of 4, defined modulo 8: this is clearly equivalent to the invariant ω constructed above for this case.

We sum up our discussion in

LEMMA 29. *When $s \equiv 1$ (mod 8) we can define an invariant of P, $\omega(f) \in Z_8$. This is additive for connected sums, and satisfies $\omega(f) \equiv 2q(\chi)$ (mod 4). If $\alpha(G^*) = 0$, the invariant ω of Lemma 28 is determined by $\omega(f) \equiv 4\omega$ (mod 8).*

§13. THE DIFFEOMORPHISM CLASSIFICATION

Again let P be a closed, $(s - 1)$-connected $(2s + 1)$-manifold, $s \geqslant 3$. By a theorem of Smale [12], P admits a handle decomposition with one 0-handle, k s-handles, k $(s + 1)$-handles and one $(2s + 1)$-handle, for some (large enough) value of k. Write N for the union of the 0-handle and s-handles. N is a thickening of a bouquet of s-spheres representing a set of k generators of $G \cong \pi_s(P)$. It follows by Proposition 1 that these generators determine N up to diffeotopy, and by Proposition 2 that the integer k is already sufficient. But the union, N', of the $(2s + 1)$-handle and $(s + 1)$-handles satisfies the same conditions; hence, in particular, it is diffeomorphic to N.

Moreover by Proposition 1 the imbeddings of the s-spheres are unique up to a diffeomorphism of P; taking a smooth regular neighbourhood [6] we see that the submanifold N is unique up to a diffeomorphism of P. Hence the imbedding $i_1 : N \to P$ is unique up to replacement by hi_1f, where f, h are diffeomorphisms of N and P respectively. Now P is obtained from the two copies N, N' of N by matching the boundaries. We have just seen that the decomposition of P into N and N' is unique up to a diffeomorphism h; likewise the attaching map $i_1^{-1}i_2$ (on ∂N) is unique up to possible replacement by $(hi_1f_1)^{-1}(hi_2f_2) = f_1^{-1}(i_1^{-1}i_2)f_2$; in other words, up to multiplication on the left and on the right by diffeomorphisms which extend to ones of N. We observe that this discussion has made no real use of $(s-1)$-connectivity of P.

We can argue similarly if P is almost-closed. Here, the $(2s+1)$-handle is missing. It is convenient to move the "hole" downwards. Define N as above, N' as its complement. Now delete from N' a tube $D^{2s} \times I$ joining the two boundary components. This does not change the diffeomorphism type of P, but it makes $\partial N'$ connected, from which it follows that N' is now, like N, a handlebody and so—by the uniqueness, as before—N' is diffeomorphic to N. These are attached to give P by a diffeomorphism of the boundaries, but with a disc deleted from each (we ought also to pay attention to corners arising, but it is easily seen that these make no essential difference to the argument). This diffeomorphism, just as before, is nearly unique.

LEMMA 30. *Let $s \geqslant 3$, P be a closed (resp. almost-closed) $(s-1)$-connected $(2s+1)$-manifold, such that $H_s(P)$ admits k generators. Then there is a unique $N \in \mathscr{H}(2s+1, k, s)$, and a diffeomorphism h of ∂N (resp. of M), unique up to left and right multiplication by the group S of diffeomorphisms which extend to N, such that P is obtained by glueing two copies of N together by h.*

Here, as in §11, M denotes ∂N with the interior of a disc D^{2s} removed.

It is clear that this lemma, taken with Theorem 6, solves in principle the classification problem, in the almost-closed case. We shall proceed to solve it in practice. Let us first recall Theorem 6.

A diffeomorphism h of M extends to one of N if and only if

(i) *h_* leaves invariant K, the kernel of $i_* : H_s(M) \to H_s(N)$;*

(ii) *certain obstructions, defined by $\beta(x)$ for $x \in K$, vanish.*

We shall take the two obstructions in turn. Let us temporarily proceed as if the Theorem, with (ii) deleted, were true for all $s \geqslant 2$. Then if $h_*(K) = h'_*(K)$, $h^{-1} \circ h'$ leaves K invariant and so extends to N; multiplying h on the right by this gives h', so h and h' are in the same right coset of S. Similarly we see that two right cosets, determined by the positions K_2 and K_2' of $h_*(K)$, are in the same double coset of S if and only if M has a diffeomorphism g with $g_*(K) = K$ and $g_*(K_2) = K_2'$. Thus double cosets are in (1-1) correspondence with isomorphism classes of triples (H, K, K_2). (Recall that $H = H_s(M)$.)

If we remark that by the Corollary to Lemma 2 of [Q] (which is easily extended to take into account the further elements of structure we have) any two kernels K in H are equivalent, we see that the above problem is precisely that to which the main part of [Q] was addressed. We borrow the notation of that paper, and refer particularly to Theorems 1 and 2 and the Proposition which extends them. Now by [15] or by [III] §9, the given structure on $H = H_s(M)$ consists of:

a $(-1)^s$-symmetric nonsingular bilinear $\lambda : H \times H \to Z$;

a homomorphism $\chi : H \to \pi_{s-1}(SO)$;

if $s \neq 3, 7$ is odd, a homogeneous quadratic $\phi : H \to Z_2$;

related by:

if $s \neq 2, 4, 8$ is even, then λ is even;
if $s = 2, 4, 8$ and $x \in H$, then $\chi(x)$ and $\lambda(x, x)$ have the same parity; the associated bilinear map of ϕ is λ (mod 2).

Now (K_1, K_2) is a pair of kernels in this structure. The corresponding invariants are the rank k and:

an abelian group G, admitting k generators;
a $(-1)^{s+1}$-symmetric nonsingular bilinear $b : G^* \times G^* \to \mathbf{S}$;
if $s \neq 3, 7$ is odd, a homogeneous quadratic $q : G \to \mathbf{Q}$ (mod 2);
a homomorphism $\chi : G \to \pi_{s-1}(SO)$;

related by

if $s \neq 2, 4, 8$ is even, b is strictly skew;
if $s = 2, 4, 8$, c and χ determine the same homomorphism $G^* \to \mathbf{Z}_2$ (so if $s = 4, 8$ $c = 0$, as $\chi : G \to \mathbf{Z}$);
the associated bilinear map of q is $2b$ (mod 2).

Since P has decompositions for all large enough k, we can drop k from this list.

LEMMA 31. *The invariants listed above coincide with those of Lemma 27, together with* $\alpha : G \to \pi_{s-1}(SO_{s+1})$.

Proof. We must first identify G. From our present point of view, P is constructed from M by attaching to it two copies of N. Up to homotopy, attaching a copy of N is equivalent to adding k $(s+1)$-cells, by generators of K_1, or K_2. Thus $H_s(P) \cong H_s(M)/K_1 + K_2 = H/K_1 + K_2 = G$, and our definitions agree.

Since each $g \in G$ is represented by a sphere in M, the normal bundle in P is the suspension of that in M, and so coincides with the χ above.

The above description of P by cells gives chain groups $C_{s+1}(P) \cong K_1 \oplus K_2$, $C_s(P) = H$, and the boundary operator is induced by the inclusion of K_1 and minus that of K_2. Other chain groups vanish, except $C_0(P)$. Thus

$$H_{s+1}(P) = Z_{s+1}(P) = \{(x, x) : x \in K_1 \cap K_2\} \cong K_1 \cap K_2.$$

To calculate intersections we need the relevant chains to be transverse, so deform M slightly into N_1. Then chains in $K_2 \subset C_{s+1}(P)$ do not meet those in $C_s(P)$, which lie in M; and if $x \in K_1$, the corresponding $(s+1)$-chain meets M in the s-chain given by $x \in H$, and so meets an s-chain $y \in C_s(P) = H$ with intersection number $\lambda(x, y)$. In particular, this determines the intersection numbers of homology classes.

The b of Lemma 27 was defined as follows. Let $x, y \in G$ have finite order, and represent them by chains ξ, η with $r\xi = \partial\zeta$, then

$$b(x, y) = (\zeta \cap \eta)/r \qquad \text{(mod 1)}.$$

In the above situation we have $\zeta = \zeta_1 \oplus \zeta_2$ with $\zeta_i \in K_i$, and $(\zeta \cap \eta)/r = (1/r)\lambda(\zeta_1, \eta)$. But with the notation of §2 of [Q]. $\xi = \zeta_1/r - \zeta_2/r$ with $\zeta_i/r \in K_i'$ and if likewise $\eta = \eta_1 + \eta_2$, we there defined

$$b(x, y) = \bar{\lambda}(\zeta_1/r, \eta_2) = \bar{\lambda}(\zeta_1/r, \eta_1 + \eta_2) = (1/r)\lambda(\zeta_1, \eta) \qquad (\text{mod } 1).$$

Hence the two definitions of b coincide.

Finally we must identify q when $s \neq 3, 7$ is odd. Note that the choice of $\xi \in H$ representing $x \in G$ already gives a representative sphere S^s in M, and so a reduction of the group of its normal bundle to SO_s. If $\phi(\xi) = 0$, the q of Lemma 27 is $(\zeta \cap \xi)/r = (1/r)\lambda(\zeta_1, \xi)$ (mod 2)—in general it is $\phi(\xi) + (1/r)\lambda(\zeta_1, \xi)$—and again this coincides with the alternative definition of q.

It is now time to take into account (ii) of Theorem 6. Let the diffeomorphisms h_1, h_2 of M, used to construct the manifolds P_1, P_2, agree in the invariants of Lemma 31. What we proved above amounts to this, that we can modify h_2 by multiplication on left and right by diffeomorphisms which extend to N, with the result that $h_1 \simeq h_2$. The obstructions to quasi-diffeotopy of h_1 to h_2—or equivalently, of $h_1^{-1} \circ h_2$ to 1—now consist, by Lemma 23, of a homomorphism $S\beta : H \to S\pi_s(SO_s)$ or, in the exceptional case, of $\omega \in \mathbf{Z}_2$ and, if $s \equiv 0$ (mod 8), of a homomorphism $\phi S\beta : H \to \mathbf{Z}_2$. Now modifying h_2 by multiplying on the right (resp. left) by a general diffeomorphism g, homotopic to 1 and extending to N, has the effect of altering $S\beta$ or $\phi S\beta$ by a general homomorphism zero on K_1 (resp. K_2).

We can replace $S\beta : H \to S\pi_s(SO_s) = F$, say, by a dual element of $\hat{H} \otimes F$, or—using the isomorphism of H with its dual—an element of $H \otimes F$. We are allowed to change this by any element of $K_2 \otimes F$ (resp. $K_1 \otimes F$): we obtain an element of the quotient group

$$H \otimes F/\text{Im}\,(K_1 + K_2) \otimes F = G \otimes F.$$

Thus the obstruction can be identified with an element of $H^{s+1}(P; S\pi_s(SO_s))$ or, in the exceptional case, $\omega \in \mathbf{Z}_2$ and, if $s \equiv 0$ (mod 8), an element of $H^{s+1}(P; \mathbf{Z}_2)$. Moreover, $S\pi_s(SO_s)$ was computed at the end of §12B. We can thus replace the above by:

(1) In the non-exceptional case, an element of $H^{s+1}(P; \pi_s(SO))$ or—if $s = 3$ or 7—of $H^{s+1}(P; \mathbf{Z})$, where $\mathbf{Z}$ is regarded as a subgroup of index 2 in $\pi_s(SO)$;

(2) if $s \neq 2, 4, 8$ is even, an element of $H^{s+1}(P; \mathbf{Z}_2)$, and

(3) in the exceptional case, an element of $\mathbf{Z}_2$.

Now exclude the cases $s = 3, 7$.

We must identify these obstructions to diffeotopy with the invariants of Lemmas 28 and 29. Observe that they are defined to compare framings of certain s-spheres in ∂N. In the handle decomposition of P, these spheres are the attaching spheres of the $(s + 1)$-handles. Now consider the obstructions in turn. The first determines the stable framing used to attach the $(s + 1)$-handles, and hence the stable tangent bundle of P; and conversely, hence is determined by $\hat{\beta}$. The second refers to the unstable group $\pi_s(SO_{s+1})$, and the homomorphism to $\mathbf{Z}_2$ (when defined) detects the image of

$$\mathbf{Z} \approx \pi_{s+1}(S^{s+1}) \xrightarrow{\partial} \pi_s(SO_{s+1})$$

(which has order 2 for s even and $\neq 2, 4, 8$). Now the invariant $\hat{\phi}$ defined in [VII] has the property of also detecting the effect on the attaching map (in $P \times I$) of changing the framing by this amount. Thus $\hat{\phi}$ determines the obstruction under (2).

The third and last obstruction arose by considering framings in ∂N on an s-sphere representing the homology class $\hat{\chi} \in H = H_s(\partial N)$. If $s \equiv 0 \pmod 4$, $\hat{\chi}$ is well-determined, and is trivial in N: thus a representative s-sphere bounds a disc in N. This, together with the core of an attached handle (or any disc spanning it in N') forms an $(s+1)$-sphere, embedded (we may suppose smoothly) in P. Comparison of the framings induced on S^s from the two discs determines the normal bundle in P of the $(s+1)$-sphere. It is now clear that the extra obstruction in this case is determined by the invariant ω defined in §12B. If $s \equiv 1 \pmod 8$, $\hat{\chi}$ is only determined modulo $2H$. Since $\chi(K_1) = 0$, and K_1 is a kernel, we may choose $\hat{\chi}_1 \in K_1$. Set $\hat{\chi}_2 = h_*(\hat{\chi}_1) \in K_2$: this also is dual to χ. But we can not in general choose $\hat{\chi}_2 = \hat{\chi}_1$. When we can, the above argument suffices. It is easily seen that this choice is possible precisely when $\alpha(G^*) = 0$. Now $\hat{\chi}_1 - \hat{\chi}_2 = 2z$ for some $z \in H$. We claim that the class of z in $G \approx \pi_s(P)$ is that of the restriction of the canonical map to $S^s \subset AY$. For the canonical map was defined from α by duality, and hence by the form b; now $2\{z\} = 0$, and if $x \in G^*$ has representative ξ,

$$b(\{z\}, x) = \tfrac{1}{2}\lambda(\hat{\chi}_1, \xi) = \tfrac{1}{2}\chi(\xi) \pmod 1.$$

Thus the canonical map can be defined as follows: map S^s by z; map two further spheres by $\hat{\chi}_1$ and $\hat{\chi}_2$, and join by a homotopy in ∂N corresponding to the equation $\hat{\chi}_1 - \hat{\chi}_2 = 2z$; add suitable $(s+1)$-discs spanning $\hat{\chi}_1$ and $\hat{\chi}_2$ in N and N'. Hence changing the framing on one of the spheres $\hat{\chi}_i$ will change the framing used to attach the $(s+1)$-handle in the canonical thickening. Hence in this case the final obstruction to diffeomorphism of manifolds P is detected by the canonical thickening, and hence by $\omega(f)$. This completes the proof of our main result.

THEOREM 7. *For $s \geq 4$, $s \neq 7$, diffeomorphism classes of almost-closed $(s-1)$-connected $(2s+1)$-manifolds P are in (1–1) correspondence with sets of invariants as defined in §12, viz,*

A finitely generated abelian group $G = H_s(P)$.

A nonsingular bilinear $b : G^* \times G^* \to S$ such that for s even, $b(x, x) = 0$.

If s is odd, a quadratic $q : G^* \to Q/2Z$ with associated bilinear map $2b$.

A homomorphism $\alpha : G \to \pi_{s-1}(SO)$.

In the non-exceptional case, a class $\hat{\beta} \in H^{s+1}(P; \pi_s(SO)) \approx G \otimes \pi_s(SO)$.

If $s \neq 4, 8$ is even, a class $\hat{\phi} \in H^{s+1}(P; Z_2) \approx G \otimes Z_2$.

In the exceptional case, if $4 \mid s$ an element $\omega \in Z_2$, and if $s \equiv 1 \pmod 8$ an element $\omega(f) \in Z_8$ such that $\omega(f) \equiv 2q(\chi) \pmod 4$.

If P_1, P_2 are two manifolds of the type considered, the invariants of $P_1 + P_2$ are the direct sum of those of P_1, P_2 where defined; in the non-exceptional case, ω and $\omega(f)$ must be interpreted as 0.

These details all follow from Lemmas 27, 28 and 29.

We conjecture that $\hat{\beta}$ can be defined, and the above proved, when $s = 3, 7$. When $s = 2$, $\hat{\beta}$ would be in a zero group anyway and the result, for *closed* manifolds, has been proved by Barden [1]. When $s = 1$, there is a fundamental group, and the problem is very much harder. The case $s = 0$ is trivial.

COROLLARY. *A splitting $G = G_1 \oplus G_2$ corresponds to a splitting $P = P_1 + P_2$ if and only if G_1^* is orthogonal to G_2^* for b.*

For in this case we have induced splittings of G^*, $\hat{G}$, and the invariant functions on G induce those on the G_i, except if only one G_i is exceptional, when we have some choice for $\hat{\beta}$, or if both are, when we have choices for $\omega(f)$ and forming the sum we recover the invariants of G (except if both G_i are exceptional, when the chosen $\omega_i(f)$ must not be unrelated).

We shall make much use of this in the sequel, and observe that we could also prove it directly, using Haefliger's theorems [4] to imbed an $(s+1)$-complex of the appropriate homotopy type for G_1, thickening it and joining to the boundary by a tube, and then using duality to show that its relative boundary in P is contractible, and hence a disc (cf. [IV, 6.5]).

§14. BOUNDARIES OF HANDLEBODIES

We should now like to proceed to give a complete classification of closed $(s-1)$-connected $(2s+1)$-manifolds; we are, alas, unable to do this. However, we apply techniques similar to those of [15] to gain a considerable amount of information, and reduce the remaining problem to bare essentials. Our method rests on having at hand a sufficiency of examples of closed $(s-1)$-connected $(2s+1)$-manifolds. In fact we have the manifolds $P = \partial L$, where $L \in \mathscr{H}(2s+2, k, s+1)$; it is clear that such P are indeed closed, $(s-1)$-connected, and $(2s+1)$-dimensional. Moreover L, hence also P, is s-parallelisable (N.B. Here s is an integer, and does not stand for stable). So α vanishes for P. The key result of this paragraph is the converse.

THEOREM 8. *Let P be an almost-closed, $(s-1)$-connected $(2s+1)$-manifold, with $\alpha = 0$. Then there is a handlebody $L \in \mathscr{H}(2s+2, k, s+1)$ with $P = \partial L - \mathring{D}^{2s+1}$.*

A result similar to this was obtained by Kervaire and Milnor [8] in the case where P is parallelisable (in our notation, $\hat{\beta} = \hat{\phi} = 0$) and independently by the author [14] without this restriction. It is easy to verify using these papers that the result holds for all $s \geqslant 0$, provided if $s = 2$ that we assume that ∂P is a 4-sphere. Here we confine ourselves to the case $s \geqslant 4$, where the result follows from the theory above. When $s = 3$, $\alpha \equiv 0$; we feel that this case of the result will be the key to extending Theorem 7.

Proof. By Theorem 2, L is determined by $H = H_{s+1}(L)$,

$$\alpha : H \to \pi_s(SO_{s+1}) \qquad \lambda : H \times H \to Z$$

satisfying the conditions of Theorem 1. Naturally, we do not now suppose λ unimodular. Up to dimension $2s$, the homotopy type of L is determined from that of ∂L by attaching k $(s+1)$-cells and a $(2s+2)$-cell. We used this in [15], where we showed in particular that the homology of ∂L could be calculated from chain groups

$$C_{s+1} = H, \qquad C_s = H \pitchfork Z = \hat{H}.$$

and boundary homomorphism (notation of [Q])

$$\partial = A\lambda : H \to \hat{H}.$$

(See also below.) Thus $G = H_s(P) \cong \hat{H}/A\lambda(H)$ and $H_{s+1}(P) \cong \mathrm{Ker}\,(A\lambda)$. As a check, note that $\hat{G}$ can be identified with the group of homomorphisms $\hat{H} \to \mathbf{Z}$ which annihilate $A\lambda(H)$, and so with the subgroup of H λ-orthogonal to H, i.e. the radical $\mathrm{Ker}\,(A\lambda) \cong H_{s+1}(\partial L)$.

We next calculate b for ∂L. Let $x, y \in G^*$, with $rx = 0$, have representatives $\xi, \eta \in \hat{H}$, with $r\xi = A\lambda(\zeta)$. Then $b(x, y) = (\zeta \cap \eta)/r \pmod 1$. The intersection of chains is given by the natural dual pairing of H and $\hat{H}$, thus b is entirely determined by λ. We assert that this is the invariant of λ studied in [Q] §7. For if $A\lambda$ is a monomorphism, and imbeds H as a subgroup of finite index in $\hat{H}$, we can identify $\hat{H}$ with the H' of [Q]. The natural pairing extends uniquely to a bilinear map $\hat{\lambda} : \hat{H} \times \hat{H} \to \mathbf{Q}$. But now $b(x, y) = (\zeta \cap \eta)/r = \hat{\lambda}(\zeta/r, \eta) = \hat{\lambda}(\xi, \eta)$ is precisely the induced product of [Q].

Next suppose s odd, $s \neq 3, 7$; we must determine q. This was defined geometrically, so we must give geometrical descriptions of our chains. Say L has a presentation

$$f : \bigcup_{i=1}^{k} (S_i^s \times D_i^{s+1}) \to S^{2s+1}.$$

Then ∂L is obtained from S^{2s+1} by deleting the $S_i^s \times D_i^{s+1}$ and replacing by $D_i^{s+1} \times S_i^s$; we write Y for the intermediate stage. We take $1 \times S_i^s$ as basis for our s-chains, and our $(s+1)$-chains come from those of $(\partial L, Y)$ represented by $D_i^{s+1} \times 1$: these are to be completed in some way in Y, with boundaries a linear combination of the $1 \times S_i^s$. It is now clear that ∂ is given by the linking numbers of f, and that these are dual bases for intersection numbers in ∂L, which justifies two remarks above.

To define q, we took a tubular neighbourhood B of a representative S^s. Let y_1, y_2 be the homology classes of a cross-section and fibre of the bundle ∂B in $\partial L - \mathring{B}$, and set $ry_1 = \lambda y_2$, $q(S^s) = \lambda/r \pmod 2$, provided the cross-section had trivial normal bundle in ∂B. Here we shall only calculate q on the generators $1 \times S_i^s$ (its other values can be deduced from b, since it is quadratic). This lies in $S_i^s \times S_i^s$ in ∂L, and it is natural to choose a cross-section which lies in $\dot{Y}$: this forms part of a framing, so satisfies the triviality condition. Now if the bounding $(s+1)$-chain ζ_i is appropriately chosen, near the boundary it will consist of a certain number of copies of the mapping cylinder of the projection of the chosen cross-section. Deleting $\zeta \cap B$, we have a chain in $\partial L - \mathring{B}$ with boundary $ry_1 - \lambda y_2$, where λ is the intersection number of ζ and $S_i^s \times 1$. Thus if $\partial\zeta = r(S_i^s \times 1)$, we have $q(S_i^s \times 1) = (\zeta \cap (S_i^s \times 1))/r, \pmod 2$. As for b, we recognise this as the algebraic construction of [Q] §7—and λ is even precisely when $s \neq 3, 7$.

Now α vanishes for ∂L, so the exceptional case does not arise, and the only remaining invariants are $\hat{\beta}$ and $\hat{\phi}$. As the normal bundle of ∂L in L is trivial, the stable normal bundle of ∂L is induced from that of L, so $\hat{\beta}$ is determined by α_L. Proceeding more algebraically, we replace the homomorphism $S\alpha_L : H \to \pi_s(SO)$ by a dual element $\hat{\alpha}_L \in \hat{H} \otimes \pi_s(SO)$, and deduce from the above remark that this has image $\hat{\beta} \in G \otimes \pi_s(SO)$. Now suppose $s \neq 4, 8$

is even. Write H' for the subgroup of elements $h \in H$ with $A\lambda(h) \in 2\hat{H}$. Clearly $2H \subset H'$. Moreover, if $y \in H'$,

$$\phi(x + y) = \phi(x) + \phi(y) \qquad \text{for all } x \in H.$$

Hence, in particular, ϕ induces a homomorphism $H' \to Z_2$ with $2H \to 0$. This defines a dual element $\hat{\phi}$ of

$$\mathrm{Hom}(H'/2H, Z_2) \approx \hat{H}/(2\hat{H} + A\lambda(H)) \approx G \otimes Z_2.$$

This agreees with the $\hat{\phi}$ defined in §12B. For that $\hat{\phi}$ was induced from a functorial invariant of $(2s + 2)$-thickenings of complexes with s- and $(s + 1)$-cells only, using the invariant of $P \times I$. But $P \times I \subset L$, so $\hat{\phi}$ for P is induced from $\hat{\phi}$ for L, and all we have done is to express this algebraically.

We now prove the theorem. Write $H_s(P)$ as the direct sum of its torsion subgroup and a free part, then by the Corollary to Theorem 7, P splits correspondingly, and it will be enough to take the summands separately.

If G is free, take $H = \hat{G}$, $\lambda = 0$ and $\alpha = S\beta$. By Theorem 2 this defines a handlebody L; by the discussion above, ∂L has the same invariants as P, so by Theorem 7, $P \cong \partial L - \mathring{D}^{2s+1}$.

If G is finite, by Theorem 6 of [Q] there is an H and a $\lambda : H \times H \to Z$ with nonzero determinant inducing the given (G, b), and if $s \neq 3, 7$ is odd, we may suppose that λ is even and induces q; in any case λ is $(-1)^{s+1}$-symmetric. Define $S\alpha : H \to \pi_s(SO)$ to be induced by $\hat{\beta} \in G \otimes \pi_s(SO) \approx (\hat{H}/A\lambda(H)) \otimes \pi_s(SO)$. If $s \neq 4, 8$ is even, extend $H'/2H$ to a basis of $H/2H$, and choose $\phi\alpha$ on the remaining generators (e.g. as zero): this choice, plus the values on H', then determines $\phi\alpha$ uniquely on H by the usual formulae.

By Theorem 2, this (H, α, λ) determines an L; by the calculation above, ∂L and P have the same invariants, and the result again follows.

§15. THE GROTHENDIECK GROUP

If P is an almost-closed $(2s + 1)$-manifold, then by definition ∂P is a homotopy $2s$-sphere, so by a theorem of Smale [12] if $s \geqslant 3$ it determines an element of the Thom group Γ_{2s}[13]. We should like to be able to say which element. It is elementary that $\partial(P_1 + P_2) = \partial P_1 \# \partial P_2$, so ∂ gives a homomorphism of the additive monoid of almost-closed manifolds to Γ_{2s}. Theorem 8 gives a large number of elements in the kernel of this homomorphism, for it states that their boundaries bound also discs. This suggests

DEFINITION. *The Grothendieck group $\mathscr{G}_s^{2s+1}$ is the abelian group with one generator $[P]$ for each diffeomorphism class of almost-closed, $(s - 1)$-connected $(2s + 1)$-manifolds, and relators $[P_1 + P_2] - [P_1] - [P_2]$ for any two such, and also $[P]$ when $\alpha(P) = 0$.*

Equivalently, $\mathscr{G}_s^{2s+1}$ is the universal group for all additive functions on the set of manifolds P with values in an abelian group, which vanish when $\alpha(P) = 0$. Since ∂ is such a function, it induces a homomorphism

$$\vartheta : \mathscr{G}_s^{2s+1} \to \Gamma_{2s}$$

which we refer to as the obstruction, for $\vartheta(P)$ is the obstruction to "closing" the almost-closed P, by filling in a disc spanning the boundary.

We have also met other additive functions: for example, ω is one when $s \equiv 0 \pmod 8$, Also if $s \equiv 0 \pmod 4$ we have an element $\hat{\alpha} \in \hat{G}$ which corresponds by duality to $\alpha : G \to \pi_{s-1}(SO_{s+1}) \cong \mathbf{Z}$. Now if $s \equiv 4 \pmod 8$, and perhaps $s \neq 4$, $S\beta(\hat{\alpha})$ depends only on P, is additive, and vanishes when α does. Similarly if $s \equiv 0 \pmod 8$ we have $\phi S\beta(\hat{\alpha})$. Finally let $s \equiv 1 \pmod 8$. Then the canonical thickening gives an additive function $\omega(f)$. We now prove

THEOREM 9. *For $s \geqslant 3$, the groups $\mathscr{G}_s^{2s+1}$ are as follows:*

For $s \equiv 2, 3, 5, 6, 7 \pmod 8$, zero.
For $s \equiv 0 \pmod 8$, $(\omega, \phi S\beta(\hat{\alpha}))$ defines an isomorphism onto $\mathbf{Z}_2 + \mathbf{Z}_2$.
For $s \equiv 1 \pmod 8$, $\omega(f)$ defines an isomorphism onto $\mathbf{Z}_8$.
For $s \equiv 4 \pmod 8$, $S\beta(\hat{\alpha})$ defines an isomorphism onto $\mathbf{Z}_2$, except perhaps if $s = 4$, when $S\beta(\hat{\alpha})$ or ω does.

Proof. If $s \equiv 3, 5, 6, 7 \pmod 8$ then $\pi_{s-1}(SO_{s+1})$ vanishes, α is necessarily zero, and the result is trivial (even for $s = 3$).

Now let $s \equiv 2 \pmod 8$, $s > 2$. Let X be a manifold with $H_s(X) \cong \mathbf{Z}_2 + \mathbf{Z}_2$ and α nonzero. Since $c = 0$, by [Q] Lemma 7, (G, b) is unique up to isomorphism; it is easily seen that $\hat{\gamma}$ (which determines α) can be chosen as the first generator, so by Theorem 7, X is essentially unique. Given any P, form $P + X$. The corresponding $\hat{\gamma}$ is nonzero, and since it has a component in X, has height 1. Write $H_s(P + X) = G^* + F$ with G^* finite and F free. By the proof of Theorem 3 of [Q] there is an orthogonal splitting $G^* = G_1 \oplus G_2$, with $G_2 \cong \mathbf{Z}_2 + \mathbf{Z}_2$ and $\hat{\gamma} \in G_2$. By Theorem 7, Corollary, the splitting $H_s(P + X) = (F + G_1) + G_2$ determines a splitting $P + X = P_1 + P_2$. But $\alpha(P_1) = 0$, and P_2 has the form of X above. Hence $[P] + [X] = 0 + [X]$ and so $[P] = 0$.

If $s \equiv 0 \pmod 4$, $\pi_{s-1}(SO_{s+1}) \cong \mathbf{Z}$. Write θ for a generator, and write X^β for a manifold with $H_s(X^\beta) = \mathbf{Z}$, with generator x, $\alpha(x) = \theta$, $\hat{x}$ the dual base of $H_{s+1}(X^\beta)$, and β the appropriate invariant (one or two of $S\beta$, $\phi S\beta$, ω) evaluated on $\hat{x}$. By Theorem 7, this is determined up to diffeomorphism by β. Given any P, form $P + X^\circ$. Then $H_s(P + X^\circ) \cong H_s(X^\circ) \oplus \operatorname{Ker} \alpha$. Correspondingly $P + X^\circ$ splits, say as $X^\beta + P'$, with $\alpha(P') = 0$. Thus $[P] = [X^\beta] - [X^\circ]$. In each case, β is determined uniquely by the additive invariants already known (one or two of $S\beta(\hat{\alpha})$, $\phi S\beta(\hat{\alpha})$, ω), which thus define a monomorphism of $\mathscr{G}_s^{2s+1}$. Since they are already independent on the X^β, they define an isomorphism.

Finally let $s \equiv 1 \pmod 8$. If, for a given P, $b(\hat{\gamma}, \hat{\gamma}) \neq 0$, then b is nonsingular on the subgroup $\{\hat{\gamma}\}$, so by Lemma 1 of [Q], G^* splits as $\{\hat{\gamma}\} \oplus \{\hat{\gamma}\}^\circ$. Since likewise we can choose a complement F to G^* in G on which α vanishes, and the splitting $G = (F + \{\hat{\gamma}\}^\circ) + \{\hat{\gamma}\}$ determines, say $P = P_1 + P_2$ where $\alpha(P_1) = 0$, we have simply $[P] = [P_2]$ where $H_s(P_2) \cong \mathbf{Z}_2$ and $\alpha(P_2) \neq 0$. Now the map induced by τ from P_2 to BY is a homotopy equivalence, since each has the form $S^s \cup_2 e^{s+1}$ and the map induces homology isomorphisms. Hence the canonical map $AY \to P_2$ is also a homotopy equivalence, and P_2 is diffeomorphic to the canonical thickening. But canonical thickenings form a group Z_8; P_2 corresponds to a generator, since for it τ induces a homotopy equivalence $AY \to BY$. So there are four such manifolds; let X° be a fixed one. Now if, for P, $b(\hat{\chi}, \hat{\gamma}) = 0$, we form $P + X^\circ$ and split as above, to obtain an equation $[P] = [X] - [X^\circ]$. Hence $\mathscr{G}_s^{2s+1}$ has at most 8 elements: the $[X]$ and the $[X] - [X^\circ]$. Since these are precisely distinguished by the 8 values of $\omega(f)$ the result follows.

§16. CLOSED AND ALMOST-CLOSED MANIFOLDS

From the last paragraph we deduce that for any almost-closed $(s-1)$-connected P^{2s+1} with $[P] = 0$, ∂P is diffeomorphic to S^{2s}, so we can fill in by D^{2s+1} to obtain a closed manifold. Lacking further information about the obstruction homomorphism $\theta : \mathscr{G}_s^{2s+1} \to \Gamma_{2s}$, we can say no more. When $[P] = 0$, filling in by various diffeomorphisms of S^{2s} on ∂P gives closed manifolds related to each other by forming connected sums with elements of Γ_{2s+1}: on the problem of classifying these we are again silent.

However we are now able to say a little about the corresponding problem a dimension lower (thus fulfilling a promise made in [15]). Let M be a closed, $(s-1)$-connected $2s$-manifold, $N = M - \mathring{D}^{2s}$. We recall the exact sequence of Theorem 3:

$$\Gamma_{2s+1} \to \tilde{\pi}_0(\text{Diff } M) \to \tilde{\pi}_0(\text{Diff } N) \xrightarrow{\gamma} \Gamma_{2s} \to \{M/N\} \to 0.$$

It follows that $\{M/N\}$ may be identified with the set of cosets of Γ_{2s} modulo the image of γ, and we shall do our best towards finding this image—though, to be sure, this will still not give an explicit classification for such M (that must be sought in properties of manifolds they bound). Our contribution amounts to the following. The map γ may be factorised as

$$\tilde{\pi}_0(\text{Diff } N) \xrightarrow{\kappa} \mathscr{G}_s^{2s+1} \xrightarrow{\theta} \Gamma_{2s}.$$

Now $\tilde{\pi}_0(\text{Diff } N)$is known explicitly by Theorem 4, and we present below an explicit (except for $\omega(f)$) calculation of κ. This will reduce the problem to that of describing θ. It also has the following consequence.

COROLLARY. *Let M be a closed, $(s-1)$-connected $2s$-manifold, m the order of the subgroup of elements of Γ_{2s} determining spheres T^{2s} with $M \# T \cong M$. Then for* $s \equiv 2, 3, 5, 6, 7 \pmod 8$, $m = 1$; *for* $s \equiv 4 \pmod 8$, $m \leqslant 2$; *for* $s \equiv 0 \pmod 8$, $m \leqslant 4$; *and for* $s \equiv 1 \pmod 8$, $m \leqslant 8$.

This holds for all s since if $s \leqslant 3$, $\Gamma_{2s} = 0$. The result extends that of our paper [16]; indeed, the definition of κ was essentially given there.

Let f be a diffeomorphism of S^{n-1} keeping the hemisphere D_-^{n-1} fixed. Form B_f from $S^{n-1} \times [0, 1]$ by identifying each $(x, 1)$ with $(fx, 0)$, and T_f from B_f by a spherical modification on $D_-^{n-1} \times [0, 1]/(x, 1) = (x, 0)$—i.e. on $D_-^{n-1} \times S^1$—replacing it by $S^{n-2} \times D^2$.

HILFSSATZ. *T_f can be obtained by attaching two discs D^n by the diffeomorphism f of the boundary.*

The proof is in [16]; the idea is to cut the entire figure in half (at $t = 0$ and $t = \frac{1}{2}$), observe that the two halves of T_f are discs, and that the attaching map is the identity except on $D_-^{n-1} \times 1$.

Now suppose N^n almost-closed, $\partial N \cong S^{n-1}$, and g a diffeomorphism of N. Then $\gamma(g)$ is defined by attaching two discs along $S^{n-1} \cong \partial N$ by the diffeomorphism $f = g \mid S^{n-1}$. Using the disc theorem, we may suppose that f leaves a hemisphere D_-^{n-1} fixed. Form B_g from $N \times [0, 1]$ by identifying each $(x, 1)$ with $(gx, 0)$: we observe that $\partial B_g = B_f$ contains the submanifold $D_-^{n-1} \times S^1$. Form T_g from B_g by attaching a handle $D^{n-1} \times D^2$ along $D_-^{n-1} \times S^1$; then $\partial T_g = T_f$ which, by the Hilfssatz, defines $\gamma(g)$.

We have thus constructed a manifold whose boundary represents $\gamma(g)$. It is clear that if N is k-connected, so is the manifold T_g. For our application, take for N an $(s-1)$-connected $2s$-manifold; then T_g is an almost-closed, $(s-1)$-connected $(2s+1)$-manifold such that ∂T_g represents $\gamma(g)$. We define $\kappa(g) = [T_g]$; then $\theta\kappa(g) = \gamma(g)$ as promised.

To describe κ explicitly, it is now necessary to calculate the invariants of T_g. We remark that the comparative ease with which such calculations are performed demonstrates the practicability of our system of invariants. For $N \in \mathscr{H}(2s, k, s)$ let $e_1, \ldots, e_k$ denote a set of generating k-spheres, and also the corresponding homology basis of $H = H_s(N)$. Write α' and λ for the invariants of N.

As chains in T_g take the e_i and $f_i = \{e_i \times I\}$. These are clearly sufficient for homotopy type. The chain groups C_s, C_{s+1} are both naturally isomorphic to H. The boundary is given by

$$\partial\{x \times I\} = g_*(x) - x,$$

thus $G \cong \operatorname{Coker}(g_* - 1)$, $\hat{G} \cong \operatorname{Ker}(g_* - 1)$—and indeed, since g_* preserves λ, $\operatorname{Ker}(g_* - 1)$ is the λ-annihilator of $\operatorname{Im}(g_* - 1)$. For since $\lambda(x, y) = \lambda(g_*(x), g_*(y))$, we have

$$\lambda(x, g_*(y) - y) = 0 \Leftrightarrow \lambda(g_*(x) - x, g_*(y)) = 0,$$

and the first vanishes for all y if and only if x is λ-orthogonal to $\operatorname{Im}(g_* - 1)$; the second (since λ is unimodular) only if $g_*(x) - x = 0$.

To make s-chains and $(s+1)$-chains transverse, so as to define intersections, we deform $e_i = e_i \times 0$ to $e_i \times \frac{1}{2}$. It is now clear that

$$\{x \times I\} \cap y = \lambda(x, y).$$

We find b as usual: let $\xi, \eta \in H$ represent $x, y \in G^*$, and $r\xi = g_*\zeta - \zeta$; then $b(x, y) = (1/r)\lambda(\zeta, \eta) \pmod 1$. As to q, we observe that an $(s+1)$-chain always approaches its boundary in a direction normal to N; thus if the boundary is ξ, with tubular neighbourhood B, we choose a cross-section of ∂B in one component of $\partial B - N$; its normal bundle is clearly then $\phi(\xi)$. Thus $q(x) = \phi(\xi) + (1/r)\lambda(\zeta, \xi) \pmod 2$.

As usual, $N \times I \subset T_g$, so normal bundles in N must be suspended for those in T_g, and indeed since $S\alpha' : H \to \pi_{s-1}(SO_{s+1})$ is g_*-invariant, it vanishes on $\operatorname{Im}(g_* - 1)$, and defines a homomorphism α of the quotient group G.

We must now show how to compute the β invariants which are, of course, not determined by the homotopy class of g. If $x \in \hat{G}$, the representative $\xi \in H$ has $g_*(\xi) = \xi$, and we have already in [11] defined an "obstruction" $\beta(\xi)$ for g^{-1}, modulo a certain indeterminacy. We assert that this gives the normal bundle of an $(s+1)$-sphere in the homology class x

in T_g. First, we must construct such a sphere. Let $S^s \subset N$ represent x. By appropriate diffeotopies, we can suppose that g leaves fixed a half-disc D^{2s}_- in N, meeting ∂N in D^{2s-1}_-; that S^s meets D^{2s}_- in a hemisphere D^s_-, and that $g(S^s) = S^s$. Let i be an imbedding of $S^{s-1} \times I$ in D^{2s}_- with $i(S^{s-1} \times 0) = \partial D^s_-$, $i(S^{s-1} \times 1) = \Sigma^{s-1} \subset D^{2s-1}$, and write $C = i(S^{s-1} \times I)$. Then $(D^s_+ \cup C) \times S^1 \cup \Sigma^{s-1} \times D^2$ is a suitable sphere in T_g; by suitable choice of i, we can make the various parts fit together smoothly. We first remove the disc $D^s_+ + [\frac{1}{4}, 1]$ from this sphere and frame the remaining disc. This gives a framing for $\partial(D^s_+ \times [\frac{1}{4}, 1]$ which, up to homotopy, is given (say) first at $D^s_+ \times 1$, and (since S^1 spans D^2 in T_g) induced by a product along $\partial D^s_+ \times [\frac{1}{4}, 1]$. By definition, the obstruction to extending the framing over $D^s_+ \times [\frac{1}{4}, 1]$ is just $\beta(x)$. We have not shown how to calculate $\hat{\beta}$ or $\hat{\phi}$ (we will not need this in general), nor $\omega(f)$ when $s \equiv 1 \pmod 8$, except if $\hat{\alpha}$ can be chosen in H with $g_*(\hat{\alpha}) = \hat{\alpha}$. This is easily seen to be equivalent to $\alpha(G^*) = 0$. Indeed, if this is not satisfied, [III] gave no precise definition of the corresponding obstruction for g, though if $g_1 \simeq g_2$, we know how to compute a difference element. So we have to refer back to the formal definition of §12C for each particular case. (One would take an S^s in N representing $\hat{\alpha}$, take the image of $S^s \times I$ in B_g, get rid of the 1-cycle as above, and tidy up the ends.)

To describe $\kappa(g)$, it is sufficient by Theorem 9 to give the defining invariants of $[T_g]$. If $s \equiv 0 \pmod 4$, $S\alpha' : H \to \mathbf{Z}$ determines by duality $\hat{\alpha} \in H$, fixed by g_*, and so in $\hat{G}$. Then the relevant invariants $S\beta(\hat{\alpha})$, $\phi S\beta(\hat{\alpha})$, ω are directly defined (Lemma 11). If $s \equiv 1 \pmod 8$, we choose an $\hat{\alpha} \in H$ dual to $S\alpha'$. Since g_* preserves α, and $\hat{\alpha}$ is unique mod $2H$, $g_*(\hat{\alpha}) - \hat{\alpha} = 2\zeta$ for some ζ. Then for any $\eta \in H$, using $\{\eta\}$ for the class of η in G, and supposing this of finite order,

$$b(\{\zeta\}, \{\eta\}) = \tfrac{1}{2}\lambda(\hat{\alpha}, \eta) = \tfrac{1}{2}S\alpha'(\eta) \qquad (\text{mod } 1),$$

so that $\{\zeta\} = \hat{\chi}$. Hence

$$q(\hat{\chi}) = \phi(\hat{\alpha}) + \tfrac{1}{2}\lambda(\hat{\alpha}, \zeta) \qquad (\text{mod } 2).$$

Let us summarise our findings. Let M^{2s} be closed and $(s-1)$-connected, $N = M - \mathring{D}^{2s}$. Then (Theorem 3) for $T \in \Gamma_{2s}$, $M \# T \cong M$ if and only if N has a diffeomorphism g with $T = \gamma(g) = \theta\kappa(g)$. Moreover, we have just computed $\kappa(g)$.

THEOREM 10. *Suppose M^{2s} closed and $(s-1)$-connected, T an exotic sphere representing $\tau \in \Gamma_{2s}$. Then $M \# T \cong M$ if and only if (i) $\tau = 0$ or (ii) $S\alpha : H_s(M) \to \pi_{s-1}(SO_{s+1})$ is nonzero modulo 2 and $\tau \in \theta(\mathscr{G}^{2s+1}_s)$.*

Proof. For $s \leqslant 3$, $\Gamma_{2s} = 0$ and the result is trivial.

If $S\alpha$ is zero modulo 2 for M, then for any diffeomorphism g of N, α is zero modulo 2 for T_g, so by Theorem 9, $\kappa(g) = [T_g] = 0$. It remains to be shown that if $S\alpha$ is nonzero modulo 2, then for any $X \in \mathscr{G}^{2s+1}_s$, N has a diffeomorphism g with $\kappa(g) = X$. The only cases that arise are $s \equiv 0 \pmod 4$ and $s \equiv 1 \pmod 8$.

First let $s \equiv 0 \pmod 4$. Since $S\alpha : H \to \mathbf{Z}$ is nonzero modulo 2, $\hat{\alpha}$ is not in $2H$. We choose a basis for H with $\hat{\alpha}$ an odd multiple of e_1. By Lemma 13, N has a diffeomorphism homotopic to 1 with arbitrary β_1. By the addition formulae, since $\pi_s(SO_s)$ has exponent 2, $\beta_1 = \beta(\hat{\alpha})$. As this is arbitrary, so is $\kappa(g)$.

Next let $s \equiv 1 \pmod 8$). We take a symplectic basis $\{e_i, f_i\}$ for H with $\hat{\alpha} = e_1 \pmod{2H}$ Consider the automorphism T of H which leaves all e_i, f_i fixed except $T(e_1) = e_1 + 2f_1$. This is the identity modulo $2H$, so preserves ϕ and by Theorem 4 corresponds to diffeomorphisms. Now $\zeta = f_1$ so $q(\hat{\chi}) = \phi(e_1) + \frac{1}{2} = \pm\frac{1}{2}$. Thus $\omega(f)$ is a generator of $\mathbf{Z}_8$. The corresponding element of $\theta(\mathscr{G}_s^{2s+1})$ generates the whole subgroup, so the result again follows.

§17. GROTHENDIECK GROUPS AND COBORDISM GROUPS

We shall now give an interpretation of the Grothendieck groups of §15, and of those of [15], as cobordism groups (in the latter case it was suggested by Milnor that such an interpretation should exist).

Consider the set of closed, $(s-1)$-connected, n-manifolds, where $s \geqslant 2$. These are orientable, and we suppose them all oriented. Write $M_1 \sim_s M_2$ if for some $(s-1)$-connected W, $\partial W = M_1 \cup (-M_2)$. As usual, this is an equivalence relation, and the equivalence classes form a group under the addition defined by the connected sum. We denote this group by Ω_s^n: the definition is due to Milnor [9].

Next consider the almost-closed, $(s-1)$-connected n-manifolds. Here $M_1 \sim_s M_2$ if there is an $(s-1)$-connected manifold W with corners separating ∂W into three parts with closures $M_1, -M_2$, and an h-cobordism between ∂M_1 and ∂M_2. This likewise is an equivalence relation and compatible with sums, and we obtain a group $\mathscr{A}_s^n$.

LEMMA 32. *There is an exact sequence for* $s \leqslant n$

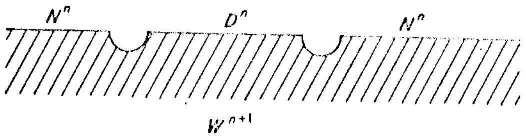
$$\ldots \mathscr{A}_s^{n+1} \xrightarrow{\rho} \Theta^n \xrightarrow{\sigma} \Omega_s^n \xrightarrow{\tau} \mathscr{A}_s^n \xrightarrow{\rho} \Theta^{n-1} \ldots$$

Here Θ^n is the Milnor group of homotopy spheres [8]; by results of Smale (see e.g. [12]) it is isomorphic to Γ_n except, perhaps, if $n = 3, 4$.

Proof. The map ρ is defined by taking the boundary, σ by observing that a homotopy n-sphere is $(s-1)$-connected if $s \leqslant n$, and τ by deleting the interior of an imbedded disc in a representative manifold. These constructions are evidently compatible with the equivalence relations and with sums, so define homomorphisms.

The kernel of σ is the set of homotopy n-spheres which bound $(s-1)$-connected manifolds; this is exactly the image of ρ. A manifold W^n is in the kernel of ρ if and only if ∂W is h-cobordant to S^{n-1}, i.e. if and only if W, with the h-cobordism glued on, has boundary S^{n-1}, and is obtained from a closed manifold by deleting a disc. But clearly this modified manifold $\sim_s W$.

Now $\sigma\tau = 0$, for if we take a homotopy sphere and delete a disc, we have just remarked that it $\sim_s$ a disc, and this defines the zero of $\mathscr{A}_s^n$. Finally let M^n represent an element of Ker τ. Then if $N^n = M^n - \mathring{D}^n$, an $(s-1)$-connected W^{n+1} exists, the three parts of whose boundary are N^n, a disc D^n, and an h-cobordism of their boundaries. Rounding the corners, we find a V^{n+1}, still $(s-1)$-connected, whose boundary is obtained by filling in ∂N with a homotopy disc, so has the form $N^n \# T^n$ for some homotopy sphere T^n. It follows that the class of N^n in Ω_s^n is minus that of T^n, and so in the image of σ.



We observe that an almost-closed $(s-1)$-connected N^{2s-1} is a homotopy disc. Hence the above sequence ends

$$\Theta^{2s+1} \to \Omega_s^{2s+1} \to \mathscr{A}_s^{2s+1} \to \Theta^{2s} \to \Omega_s^{2s} \to \mathscr{A}_s^{2s} \to \Theta^{2s-1} \to \Omega_s^{2s-1} \to 0.$$

This is another formulation of our obstruction theory for closing almost-closed manifolds, though by its generality it loses some of the power of Theorem 3. Our object is to identify $\mathscr{A}_s^{2s}$, $\mathscr{A}_s^{2s+1}$ with the Grothendieck groups.

LEMMA 33. *Let M^{n-1}, N^n be $(s-1)$-connected, $\partial N = M$, $n \geq 2s+1$, $s \geq 2$, and suppose $S\alpha : H_s(M) \to \pi_{s-1}(SO)$ onto. Also if $n = 2s+1$ and $\pi_{s-1}(SO) \neq 0$, suppose the rank of $H_s(M)$ at least 4. Then there is an $(s-1)$-connected N', with $\partial N' = M$ and $H_s(N', M) = 0$.*

Proof. We have to kill the group $H_s(N, M) = H_s(N)/i_* H_s(M)$; we shall do it by surgery. Choose any generator and represent by $x \in H_s(N)$. Let $y \in H_s(M)$ have $S\alpha(y) = \alpha(x)$, and replace x by $x - y$. Now if the sphere S^s in N represents this class, it has a trivial normal bundle; accordingly we can perform a spherical modification. If $n > 2s+1$, this kills the generator represented by x, and we proceed by induction.

If, however, $n = 2s+1$, new homology is in general introduced, and more care must be exercised. In the case $s = 2$ we gave a full proof (with fewer hypotheses) in [17]. A similar argument is valid in general. If, in fact, x represents an element of infinite order in $H_s(N, M)$, the modification automatically decreases the rank of this group by one. So we can suppose $H_s(N, M)$ finite. To simplify further we choose an $x \in H_s(N)$ with (i) $\alpha(x) = 0$, (ii) x determines a nonzero element of $H_s(N, M)$, (iii) x is primitive in $H_s(N)$. It is easily seen that when (iii) is satisfied, no new homology is introduced. Note that we now have, by duality, rank $H_s(N) = \frac{1}{2}$ rank $H_s(M)$. If this is at least 2, since $\pi_{s-1}(SO)$ is cyclic one easily verifies that an x must exist satisfying (i)–(iii), and we are done. If not, $\pi_{s-1}(SO) = 0$, and either the rank is 1, when we choose x as a generator, or zero, when we appeal to [8] or [14] for a proof of the result.

We recall from [15] that we computed (for $s \geq 3$) the Grothendieck group $\mathscr{G}_s$ of almost-closed, $(s-1)$-connected $2s$-manifolds: it was (unnaturally) isomorphic to $\mathbf{Z} \oplus \pi_{s-1}(SO_s)$. The class of $S^s \times S^s$ (with the interior of a disc removed) was interpreted as the generator of the first summand $\mathbf{Z}$; we shall factor this out and write $\mathscr{G}_s^{2s}$ for the quotient, isomorphic to the second summand.

THEOREM 11. *There are natural isomorphisms*

$$\kappa_1 : \mathscr{G}_s^{2s} \to \mathscr{A}_s^{2s} (s \geq 3), \qquad \kappa_2 : \mathscr{G}_s^{2s+1} \to \mathscr{A}_s^{2s+1} (s \geq 4).$$

Proof. Since addition in $\mathscr{A}_s^{2s}$ is defined by the usual sum of bounded manifolds, by the universal mapping property there is a homomorphism of $\mathscr{G}_s$ to $\mathscr{A}_s^{2s}$. But $S^s \times S^s$ bounds

$S^s \times D^{s+1}$, so determines $0 \in \Omega_s^{2s}$, and *a fortiori* $0 \in \mathscr{A}_s^{2s}$. Thus we have an induced homomorphism κ_1 of the quotient group $\mathscr{G}_s^{2s}$. Similarly since, by Theorem 8, an $(s-1)$-connected $(2s+1)$-manifold with zero α bounds an s-connected manifold, it determines $0 \in \mathscr{A}_s^{2s+1}$. Again the universal property gives us a homomorphism κ_2 of $\mathscr{G}_s^{2s+1}$. Since $\mathscr{A}_s^{2s}, \mathscr{A}_s^{2s+1}$ are defined by equivalence classes of almost-closed $(s-1)$-connected manifolds, κ_1 and κ_2 are onto. It remains to prove their kernels zero.

We can find an almost-closed $(s-1)$-connected M_1^{2s} with $S\alpha$ onto and the rank of $H_s(M)$ at least 2—e.g. by using the classification (Theorem 1). Now if M_2^{2s} gives an element of Ker κ_1, so does $M = M_1 + (-M_1) + M_2$. Thus M bounds an $(s-1)$-connected N (with appropriate corners, which may be removed). The hypotheses of Lemma 33 are now fulfilled; by that lemma, we may suppose $H_s(N, M) = 0$. So $H^{s+1}(N) = 0$, N has the homotopy type of a wedge of s-spheres, so by Smale [12] is a handlebody, $N \in \mathscr{H}(2s+1, k, s)$ for some k. But now, by the Corollary to Lemma 21, M determines $0 \in \mathscr{G}_s^{2s}$.

The second case is similarly reduced to showing that if an $(s-1)$-connected M^{2s+1} bounds N^{2s+2} with (N, M) s-connected, M must determine $0 \in \mathscr{G}_s^{2s+1}$. First let $s \equiv 0 \pmod 4$. Then (with an obvious notation) $\alpha_M : H_s(M) \to \mathbf{Z}$ is induced by $\alpha_N : H_s(N) \to \mathbf{Z}$, so for $\hat{\alpha}_M \in H_{s+1}(M)$, $\hat{\alpha}_N \in H_{s+2}(N, M)$ we have $\hat{\alpha}_M = \partial_* \hat{\alpha}_N$. Thus a representative S^{s+1} for $\hat{\alpha}_M$ is null-homologous in N. So its homotopy class in N has the form $y \circ \eta$, $y \in \pi_s(N)$, η the generator of $\pi_s(N)$, so we can write $y = i_*(x)$. Modifying the homotopy class of S^{s+1} in M by $x \circ \eta$, we still (Proposition 1) obtain an imbedding in M, now null-homotopic in N. By the relative version of Proposition 1 if $s \geqslant 5$ (Lemma 1 if $s = 4$), it therefore bounds an imbedded (immersed) disc in N, and so has trivial normal bundle in M. Hence whichever of $S\beta(\hat{\alpha})$, $\phi S\beta(\hat{\alpha})$, ω are defined all vanish for M, and so $[M] = 0$.

Finally let $s \equiv 1 \pmod 8$. As (N, M) is s-connected, N is homotopy equivalent to an $(s+1)$-dimensional complex, and so its stable tangent bundle also can be regarded as a map $N \to BY$, inducing the stable tangent bundle of M by $M \subset N \to BY$. Proceeding to S-duals we see that the canonical map $AY \to M$ is null-homotopic in N (it factors through the desuspension of N/M). We take the relative thickening induced by the nullhomotopy $(CAY, AY) \to (N, M)$: here C denotes cone (cf. [IV, p. 89 end]). As CAY is contractible, this is a disc D^{2s+2} with the canonical thickening for M lying in its boundary, S^{2s+1}. Hence the canonical thickening is trivial, and this completes the proof of the theorem

§18. MILNOR'S EXOTIC SPHERES

In his paper [10], Milnor gives a construction which associates to two elements $\alpha_1 \in \pi_m(SO_{n+1})$, $\alpha_2 \in \pi_n(SO_{m+1})$ (if $m = n$, a side-condition is necessary) a diffeomorphism of $S^m \times S^n$, and hence a differential structure on S^{m+n+1}, determining an element $T(\alpha_1, \alpha_2)$ of Γ_{m+n+1}. He has also proved (see notes of Princeton lectures on *Differentiable Structures*, 1961) that the map

$$T : \pi_m(SO_{n+1}) \times \pi_n(SO_{m+1}) \to \Gamma_{m+n+1}$$

is bilinear. Moreover he constructs a manifold W^{m+n+2} with this exotic sphere as boundary. In fact, W is defined by glueing handles of dimensions $m+1$, $n+1$ to a disc D^{m+n+2}, where the attaching maps have linking number 1, and the α_i are used to describe the normal bundles. It follows at once from [5] that in the stable range, this description characterises W. We now investigate the relation of his constructions to ours: it turns out that in cases of overlap our results are slightly more precise than the above.

First let $m = n = s - 1$. Then $W \in \mathscr{H}(2s, 2, s)$ and has intersection matrix

$$\begin{pmatrix} \pi(\alpha_1) & 1 \\ 1 & \pi(\alpha_2) \end{pmatrix}.$$

This is unimodular if either $\pi(\alpha_i)$ vanishes; in fact (unless s is 4 or 8 and a $\pi(\alpha_i)$ odd) by choosing a different homology basis we can suppose each $\pi(\alpha_i) = 0$. It is now trivial to calculate the invariants which determine class in the Grothendieck group $\mathscr{G}_s^{2s}$.

If s is even, the signature σ vanishes. We may identify $S\alpha_1$, $S\alpha_2$ with integers χ_1, χ_2; which are even if $s = 4$, 8, zero if $s = 6$ (mod 8), and only defined modulo 2 if $s = 2$ (mod 8). Then $\chi = \chi_2 e_1 + \chi_1 e_2$ and so $\chi^2 = 2\chi_1\chi_2$. Accordingly, in the first four cases of [15], the invariants are $(0, \chi_1\chi_2)$, $(0, \frac{1}{4}\chi_1\chi_2)$, $(0, \chi_1\chi_2 \pmod 2))$, (0). In each case σ vanishes, but by taking suitable χ_i we get all possible values subject to this.

If s is odd, in Case 7, $\mathscr{G}_s^{2s}$ vanishes anyway. In Case 6, Φ determines class in $\mathscr{G}_s^{2s}$; here $\Phi = 1$ if $\phi(\alpha_1)$ and $\phi(\alpha_2)$ are 1, and vanishes otherwise. Case 5 is more interesting; here the defining invariants of $\mathscr{G}_s^{2s}$ are $\phi_i = \phi(\alpha_i)$, $\chi_i = \chi(\alpha_i)$ $(i = 1, 2)$, then $\chi = \chi_2 e_1 + \chi_1 e_2$ and so

$$\Phi = \phi_1\phi_2 \pmod 2 \qquad \phi(\chi) = \phi_1\chi_2 + \phi_2\chi_1 + \chi_1\chi_2 \pmod 2,$$

and again all elements of $\mathscr{G}_s^{2s}$ appear.

Let us write $\pi_{s-1}^0(SO_s)$ for the kernel of $\pi : \pi_{s-1}(SO_s) \to Z$. The above formulae show that T can be refined to a symmetric bilinear map

$$U : \pi_{s-1}^0(SO_s) \times \pi_{s-1}^0(SO_s) \to \mathscr{G}_s^{2s},$$

with $T = \theta \circ U$. Moreover the cokernel of U is mapped monomorphically by the signature. We would like to speak of the kernel, so replace U by the associated homomorphism U' of the symmetric tensor product. This last is isomorphic to Z if $s = 0$ (mod 4), to 0 if $s = 6$ (mod 8), 3 or 7, to Z_2 if $s = 2, 3, 5, 7$ (mod 8), and to $Z_2 + Z_2 + Z_2$ if $s = 1$ (mod 8). In each case except the last, U' is a monomorphism. In the last case it has kernel Z_2. Thus if $\phi_1 = \chi_1 = \chi_2 = 1$, $\phi_2 = 0$, we have $U(\alpha_1, \alpha_2) = 0$ and so also $T(\alpha_1, \alpha_2) = 0$.

Next consider the case $m = s - 1$, $n = s$. Let $\alpha \in \pi_{s-1}(SO_{s+1})$, $\beta \in \pi_s(SO_s)$, and form the corresponding W. This is an almost-closed $(s-1)$-connected $(2s+1)$-manifold, so can be described by our invariants (if $s \geq 4$). In fact $G \cong Z$, so b and q do not appear, and α and β are determined by the elements above. The corresponding element of $\mathscr{G}_s^{2s+1}$ is zero unless α is nonzero modulo 2, when β determines appropriate invariants $S\beta$, $\phi S\beta$, and/or ω: we have $q(\hat{\chi}) = 0$ since there is no torsion.

Hence again we can refine T to a homomorphism

$$U : \pi_{s-1}(SO_{s+1}) \otimes \pi_s(SO_s) \to \mathscr{G}_s^{2s+1}$$

with $\theta \circ U = T$; U is onto except when $s \equiv 1$ (mod 8) in which case the cokernel is mapped isomorphically by $q(\xi)$. If $\pi_{s-1}(SO_{s+1})$ vanishes, so does $\mathscr{G}_s^{2s+1}$; if it does not, the domain of $U \cong Z_2 \otimes \pi_s(SO_s) \cong \pi_s(SO_s)$—except when $s \equiv 2$ (mod 8)—and the kernel of U reflects the indeterminacy of β. This suggests a more general result.

THEOREM 12. *Let $r > s$, $2s \geqslant r + 3$, $\xi \in \pi_r(S^s)$. Then Milnor's map*

$$T : \pi_{s-1}(SO_r) \times \pi_{r-1}(SO_s) \to \Gamma_{r+s-1}$$

has the property that

$$T(\alpha, \beta) = T(\alpha, \beta + F(\alpha, \xi) + \partial\xi).$$

Proof. We consider presentations of the corresponding W by taking D^{r+s}, adding an s-handle and then an r-handle. By the relative version of Proposition 1, we can alter the homotopy class of the central r-disc of the r-handle by any element ξ of $\pi_r(S^s) \subset \pi_r(W)$. By a result of Smale [12], adding the new (thickened) handle still gives a manifold diffeomorphic to W. But by Theorem 1, the framing β has changed to $\beta + F(\alpha, \xi) + \partial\xi$.

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