DIFFEOMORPHISMS OF 4-MANIFOLDS

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Recent advances in differential topology (due notably to S. Smale) have mainly referred to manifolds of high dimension—that is, of dimension not less than 5. From a different approach, much is also known about manifolds of dimensions up to 3. The 4-dimensional case, while certainly no less interesting than the others, has appeared to need deeper results. Our aim in this and a subsequent paper is to show that methods now available give considerable—though by no means complete—information on the structure of 4-manifolds.

Our main idea is a simple construction for diffeomorphisms which was suggested by a classical construction for surfaces. This, with some number theory, gives results like the following. For a 4-manifold M, we write $Q(M)$ for the quadratic form defined by intersection numbers on $H_2(M)$.

THEOREM 2. *Let N be a simply-connected, closed oriented 4-manifold. Suppose that either*

(i) *$Q(N)$ is indefinite, or*

(ii) *The rank of $H_2(N)$ is at most* 8.

Then, if M is the connected sum $N \# (S^2 \times S^2)$, every automorph of $Q(M)$ is induced by a diffeomorphism of M.

Using this, and a few imbeddings of S^2 in simple manifolds such as $S^2 \times S^2$, we obtain results on representation of 2-dimensional homology classes by imbedded 2-spheres, such as

THEOREM 3. *Let M be as in Case* (i) *above. Then every primitive, ordinary element of $H_2(M)$ is represented by an imbedded 2-sphere.*

The contrast of this with the non-imbedding theorem of Kervaire and Milnor [**1**] is interesting; their result refers specifically to characteristic elements of $H_2(M)$; ours to ordinary elements.

We use the number-theoretic terminology of [**6**]. A quadratic form Q will be associated with a symmetric bilinear map L of a free abelian group H to the integers $\mathbf{Z}$, $L: H \times H \to \mathbf{Z}$. Let $\{e_i\}$ be a basis for H, and $L(e_i, e_j) = a_{ij}$. Then if a_{ii} is even for each i, so is any

$$Q\left(\sum_1^n x_i e_i\right) = L\left(\sum_1^n x_i e_i, \sum_1^n x_i e_i\right) = \sum_1^n a_{ii} x_i^2 + 2 \sum_{i<j} a_{ij} x_i x_j,$$

and we call Q *even*; otherwise, Q is *odd*. Also, Q is *nonsingular* if

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$\det(a_{ij}) = \pm 1$; all the forms in this paper will be nonsingular, by duality. An element $x = \Sigma x_i e_i$ of H is *primitive* if the coefficients x_i are coprime, so that $x = dy$ $(y \in H)$ implies $d = \pm 1$. If $L(x, y) = Q(y)$ (mod 2) for all $y \in H$, x is *characteristic*, otherwise it is *ordinary*; it can be seen that characteristic elements always exist, and if Q is nonsingular, they are unique (modulo $2H$). If, for example, M^4 is closed and simply-connected, $H = H^2(M)$, and L is induced by the cup product, Q is non-singular by Poincaré duality, and $x \in H$ is characteristic if and only if its mod 2 reduction is $w_2(M)$—thus Q is even if and only if $w_2(M) = 0$.

Over the rational numbers, any nonsingular Q can be reduced to the form

$$\sum_{1}^{p} x_i^2 - \sum_{p+1}^{n} x_i^2,$$

and the *signature* $2p-n$ is an invariant of Q. We give Q names as follows: *negative definite*, if $p = 0$; *positive definite*, if $p = n$; *indefinite* otherwise, and in particular, *almost definite* if $p = 1$ or $n-1$, *strongly indefinite* if $2 \leqslant p \leqslant n-2$.

Unless otherwise stated, all manifolds in this paper shall be compact oriented differential 4-manifolds M with boundary ∂M such that $H_1(\partial M) = 0$. This condition has the effect that Lefschetz duality for M has the same form as Poincaré duality for a closed manifold: it means that either ∂M is empty (M is closed) or each component of ∂M is a homology 3-sphere. We shall always use "imbedding" to mean imbedding as a smooth submanifold.

1. *2-sphere bundles over S^2.*

We first consider some elementary manifolds, which will later be used somewhat in the character of "building bricks". Bundles over S^2 with group SO_3 are classified by $\pi_1(SO_3)$, which is a group of order 2 (Steenrod [4; pp. 99, 115]). We write S for the product $S^2 \times S^2$, and T for the 2-sphere bundle associated to the nonzero element of the group; both are, of course, simply-connected closed 4-manifolds.

Next consider reductions of the group to SO_2, classified by $\pi_1(SO_2)$, which we identify with the group of integers. This maps onto $\pi_1(SO_3)$, so both bundles can be reduced. Write T_k for the S^2-bundle associated with the reduction given by the integer k. Write x for the class in $H_2(T_k)$ of the sphere imbedded as the cross-section, corresponding to the "south pole", y for the class of a sphere imbedded as a fibre. Intersection numbers are easy to calculate: clearly, $y.y = 0$ (two fibres do not meet); $x.y = 1$ (we choose the orientation of y to ensure this); and $x.x$ is the obstruction to existence of a cross-section of the associated circle bundle, so equals the Euler class, k.

For homology bases of S, T we shall reduce the bundles as T_0, T_1. The relation of these bases to those given by a reduction as T_{2k}, T_{2k+1} is that for the latter manifolds, the chosen base is $x+ky$, y (in terms of the former). This is clear from the definitions.

Further elementary manifolds are P, the complex projective plane, and Q, the same but with the opposite orientation.

LEMMA 1. *The connected sum* $P \# Q \cong T$.

Proof. The proof is given by Steenrod [**4**; pp. 135–136]. We briefly sketch it. In fact the sub-bundle of T_1 with the equator as fibre is a Hopf bundle (since the Euler class is 1), so forms a 3-sphere, and decomposes T. The complements are D^2-bundles over S^2, with the same Euler class, so are diffeomorphic to a neighbourhood of $P_1(\mathbf{C})$ in $P_2(\mathbf{C})$, with complement a disc. The result follows by taking care of orientations.

We usually denote the homology class of the complex projective line in P and Q by u and v respectively. It is clear that in the above diffeomorphism u and v become the classes x and $x-y$ representing the canonical cross-sections at south and north poles. One checks that this is compatible with the given intersection numbers ($u.u = 1$, $v.v = -1$).

We remark on the following diffeomorphisms (which, in accordance with our terminology, preserve orientation): for P and Q, complex conjugation; for S the interchange of factors in $S^2 \times S^2$, and the simultaneous orientation reversal in each. We shall use these with

LEMMA 2. *Let h_i be a diffeomorphism of M_i ($i = 1, 2$). There exists a diffeomorphism h of $M_1 \# M_2$ whose induced homology map is the direct sum of those induced by the h_i.*

Proof. Let $j_i : D^4 \to M_i$ be an imbedding. By the Disc Theorem, $h_i \circ j_i$ is isotopic to j_i. By the Isotopy Extension Theorem [**3**], there is a diffeomorphism k_i of M_i, isotopic to the identity, such that $k_i \circ h_i \circ j_i = j_i$. Hence, replacing h_i by $k_i \circ h_i$, we may suppose that h_i keeps $j_i(D^4)$ fixed.

If we now define the connected sum by imbedding discs in the interiors of the $j_i(D^4)$, we observe that h_1 and h_2 fit together to give a diffeomorphism h of the sum, which clearly has the desired properties.

2. *The basic construction.*

We first explain the general idea. In the theory of surfaces, the classification implies that forming connected sums of the real projective plane with torus or Klein bottle leads to the same result. This is explained as follows. Formation of connected sum of a surface with torus or Klein bottle may be described as "adding a handle"—remove two discs, and join their boundaries by a tube; the two cases are distinguished by the orientations used for the two discs. But for the projective plane, this is no distinction: a disc may be carried round an orientation reversing path and back to its former position.

We now attempt to describe this rigorously, at the same time doubling the dimensions involved.

Let N be any 4-manifold, which it is convenient to assume homologically 1-connected, *i.e.* that $H_1(N)=0$. Hence the homology invariants reduce to the intersection form on the free abelian group $H_2(N)$, which must be nonsingular. Let $f\colon S^1\times D^3\to N$ be an imbedding; and derive M by a spherical modification, that is to say, delete the interior of the image of f, and glue in its place $D^2\times S^2$. We shall define diffeomorphisms of M by translating the attaching circle $f(S^1\times O)$ round "paths in N".

Suppose given an isotopy of S^1 in N, with initial and final position given by $f(S^1\times O)$. By the isotopy extension theorem, this is induced by an isotopy of N on itself. The final map h of the isotopy may not preserve $S^1\times D^3$ pointwise, but by the tubular neighbourhood theorem we may suppose that it induces a bundle map of $S^1\times D^3$ (over S^1) on itself. This defines a diffeomorphism of M onto a manifold M' (obtained from N as was M, but with a different attaching map) induced by h on the common part of M and N, and by the identity on the attached $D^2\times S^2$.

We next calculate the effect of our constructions on homology. We have the exact sequence

$$0\to H_3\big(N,\,N-f(S^1\times D^3)\big)\to H_2\Big((N-f(S^1\times D^3)\big)\Big)\to H_2(N)\to 0$$

with first group isomorphic to $H_3(S^1\times D^3,\ S^1\times S^2)\cong \mathbf{Z}$. Denote a generator —or rather its image in $H_2\big(N-f(S^1\times D^3)\big)$—by y; this is represented by $f(1\times S^2)$. We also have

$$0\to H_2\big(N-f(S^1\times D^3)\big)\to H_2(M)\to H_2(D^2\times S^2,\ S^1\times S^2)\to 0$$

with last group infinite cyclic. We choose an inverse image x of the generator in $H_2(M)$ as follows: let C be a surface in $N-f(S^1\times D^3)$ spanning $S^1\times 1$ (not necessarily nonsingular) and take x as the class of the 2-cycle formed from C and $D^2\times 1$. Observe that $y.y=0$, $x.y=1$ (if orientations are suitably chosen); $x.x$ may be anything. Then the group generated by x, y is in fact an orthogonal direct summand of $H_2(M)$ (this follows since the matrix of intersection numbers on it has determinant -1), and the exact sequences above then induce an isomorphism which we use to identify $H_2(N)$ with the orthogonal complement of x and y in $H_2(M)$.

For M' we define basic classes x', y' similarly, using the same surface C. Write ξ for homology classes in $H_2(N)$; ω for the class of the surface traced out by $S^1\times O$ under the isotopy, and E for the isomorphism of $H_2(M)$ on $H_2(M')$ induced by the diffeomorphism.

LEMMA 3. $E(\xi)=\xi-(\xi.\omega)y',\ E(y)=y',\ E(x)=x'+\omega.$

Proof. Certainly $E(y)=y'$ since a 2-sphere representing y is fixed by a diffeomorphism. Now the isotopy "drags" the boundary of C round a surface in N [which, w.l.o.g., we suppose disjoint from $f(S^1\times O)$] which represents ω, by definition. Hence the image of the cycle defining x gives one defining x' together with a surface which represents ω, and $E(x)=x'+\omega$.

Finally, the diffeomorphism of N is isotopic to the identity, so induces the identity map of homology. It follows from the first exact sequence above that for any ξ, $E(\xi)-\xi$ is a multiple of y', say $E(\xi)=\xi+ay'$. But

$$0=x.\xi.\therefore=E(x).E(\xi)=(x'+\omega)(\xi+ay')=a+(\omega.\xi)$$

and so $a=-\xi.\omega$, which concludes the proof.

In order to use this to prove results, we need the existence of isotopies of S^1 in N representing nontrivial classes ω; this is the crucial point of the whole process. In fact, we prove

LEMMA 4. *Suppose ω spherical. Then there is an isotopy of S^1 in N, with initial and final map $f(S^1\times O)$, representing ω.*

Proof. We can certainly find a map of a torus representing ω; for map a torus by projection to $f(S^1\times O)$, and a sphere representing ω; join by an arc, and hence define a map of the connected sum—another torus—into N.

Now in these dimensions (*i.e.* for 1-manifolds in 4-manifolds) every homotopy may be replaced by an isotopy. This is a standard result, due in principle to Whitney; we sketch the proof. In fact we have a map of $S^1\times I$ (defining the above torus) into N, and the projection on I, hence a product map to $N\times I$. Move this slightly into "general position" (formally this would be expressed using Thom's transversality theorem). Then it becomes an imbedding, but still represents a homotopy—hence an isotopy—of S^1 in N.

3. *Simply-connected manifolds.*

Now assume N simply-connected. Then any two circles are homotopic, so—by the argument of Lemma 4—isotopic, and so any one spans an imbedded 2-disc, and lies in the interior of an imbedded 4-disc in N. The modification may now be defined by removing such a 4-disc, and glueing in an alternative manifold, obtained from D^4 by the corresponding modification. But these alternative manifolds are precisely the 2-sphere bundles S and T over S^2, with a disc removed. Hence in general, if $f(S^1\times O)$ is homotopic to zero, $M\cong N\# S$ or $N\# T$. Again, for N simply-connected, any 2-dimensional homology class is spherical (by the Hurewicz theorem). We now apply the results of the preceding section, using the 2-disc spanning S^1 as C. Note that this induces a reduction of the group of the normal bundle of S^1 from SO_3 to SO_2 (the directions normal to the disc); accordingly we distinguish the bundles T_k.

THEOREM 1. *Let N be a simply-connected 4-manifold, $\omega \in H_2(N)$ with $\omega^2 = r$. There is a diffeomorphism of $N \# T_{k+r}$ on $N \# T_k$ inducing*

$$\xi \to \xi - (\xi . \omega) y', \quad x \to x' + \omega, \quad y \to y'$$

in 2-dimensional homology.

COROLLARY 1. *Let N have odd quadratic form. Then $N \# S \cong N \# T$.*

We can choose any ω with ω^2 odd.

COROLLARY 2. *If $\omega^2 = 2s$, $N \# T_k$ admits a diffeomorphism inducing $E_\omega{}^1$: $\xi \to \xi - (\xi . \omega) y$, $x \to x + \omega - sy$, $y \to y$ on $H_2(N \# T_k)$.*

We recognise this as the $E_\omega{}^1$ of [**6**]. This seems a particularly pleasant and natural way of deriving the fact that these transformations, due to Siegel, do indeed give automorphs of the quadratic form (this is, of course, trivial to verify but hard to predict).

We continue to suppose N simply-connected, and try to see which automorphs of $Q(N \# S)$ can be represented. We use the notation of [**6**]. Then, by reversing the order in $S^2 \times S^2$, we see that $E_\omega{}^2$, as well as $E_\omega{}^1$, can be realised. Assume that $Q(N)$ is either indefinite, or definite and of rank not exceeding 8. Then by [**6**; (6.12)], the group of automorphs generated by $E_\omega{}^1$ and $E_\omega{}^2$ contains all those of determinant and spinor norm 1.

We can improve on even this result. For by Lemma 2, diffeomorphisms of S induce ones of $N \# S$ (taking the identity on N), and by the remark preceding that lemma, we obtain all automorphs of $Q(S)$ (called U in [**6**]) from diffeomorphisms of S. Hence the determinant and spinor norm can be ± 1 independently: for $Q(N)$ even, it follows that we obtain the whole orthogonal group. To prove this for $Q(N)$ odd, we need only find some diffeomorphism inducing an automorph of spinor norm ± 2. But by Theorem 1, Corollary 1, $N \# S \cong N \# T$; by Lemma 2, any diffeomorphism of T induces one of the sum; by Lemma 1, $T \cong P \# Q$, and we can use complex conjugation in P; which has spinor norm 2. Thus we have

THEOREM 2. *Let N be a simply-connected 4-manifold with boundary ∂N such that $H_1(\partial N) = 0$, and such that $Q(N)$ is indefinite, or has rank at most 8. Then any automorph of $Q(N \# S)$ can be represented by a diffeomorphism of $N \# S$.*

Examples of such manifolds N are connected sums of copies of P, Q and S (*cf.* Milnor [**2**]). The theorem deals with most of these.

COROLLARY. *Let M be a connected sum of copies of P, Q, S and T. Exclude the case when $Q(M)$ is almost definite, of rank greater than* 10. *Then any automorph of $Q(M)$ can be represented by a diffeomorphism.*

Proof. First suppose M has no summand S or T. Then it is a sum of copies of P and Q; moreover (Lemma 1) we do not have both. Suppose w.l.o.g. M a sum of copies of P. Then M admits as diffeomorphisms permutations of the summands and (Lemma 2) diffeomorphisms of them—say complex conjugation. But by (1.2) of [**6**], this shows that we have all the automorphs already.

Now if one of the summands of M is S, the result follows at once from the theorem. If not, M is a sum of copies of P and Q, since by Lemma 1, $T \cong P \# Q$; and we suppose that each occurs. If there are more than two summands, M is a sum of T with a manifold with odd quadratic form, hence (Theorem 1, Corollary 1) admits S as summand, and the result again follows.

The only remaining case is $M \cong P \# Q \cong T$; here we appeal to (1.4) of [**6**]—stating that only four automorphs exist—and Lemma 2, using the identity and complex conjugation in each of the two summands.

4. *Application to imbedding spheres.*

We continue to suppose $M \cong N \# S$, N a simply connected 4-manifold, and now observe that the existence of imbedded spheres in S, together with the diffeomorphisms of M, now leads to a wide variety of imbedded spheres.

THEOREM 3. *Let N be a simply-connected 4-manifold, with boundary ∂N such that $H_1(\partial N) = 0$, and $Q(N)$ indefinite. Let $\xi \in H_2(N \# S)$ be primitive and ordinary. Then ξ is represented by an imbedded S^2, with simply-connected complement.*

Proof. Since $Q(N \# S)$ is strongly indefinite, the main results (Theorems 4 and 6) of [**5**] show that its orthogonal group is transitive on primitive ordinary vectors of given square. We shall show that for each integer r, some primitive ordinary $x_r \in H_2(N \# S)$ with $x_r{}^2 = r$ is representable by $f: S^2 \to N \# S$; then the result follows. For if $\xi^2 = r$, A is an automorph of $Q(N \# S)$ with $x_r A = \xi$, and (by Theorem 2), h a diffeomorphism of $N \# S$ with induced homology map A, then $h \circ f$ is the required imbedding.

Now $N \# S = N \# T_0 \cong N \# T_{2k}$ (by §1), and in T_{2k} we have the canonical cross-section representing x_{2k}, with $x_{2k}^2 = 2k$, and with simply-connected complement. The same sphere in $N \# T_{2k}$ may now be chosen. If ξ^2 is odd, $Q(N)$ is odd, and so $N \# S \cong N \# T \cong N \# T_{2k+1}$ and the canonical cross-section now represents x_{2k+1}, with $x_{2k+1}^2 = 2k+1$.

If $Q(N)$ is not indefinite, but definite and of rank at most 8, we can use Theorem 2, but our appeal to [5] fails. We can still obtain results, and cite the following:

Let ξ be a primitive ordinary class in $H_2(P \# P \# Q)$; if $-8 < \xi^2 < 16$, then ξ is represented by an imbedded S^2 with simply-connected complement.

For the relevant discussion of transitivity (not in detail) see [6].

We observe that if a non-primitive class ξ is represented by a sphere S^2 in a homologically 1-connected manifold M, then $H_2(M-S^2) \simeq \mathbf{Z}_r$, where r is the divisor of ξ (this is very easy); thus the requirement of simply-connected complement is natural for primitive vectors. For a few others, imbeddings can be deduced from:

LEMMA 5. *Let $f: S^2 \to M$ represent $\xi \in H_2(M)$. If $\xi^2 = 0$, then any $n\xi$, and if $\xi^2 = \pm 1$, then 2ξ can also be represented by an imbedded sphere.*

Proof. By a result which we have already used, the self-intersection of an imbedded 2-manifold gives the Euler class of the normal bundle. Then for $\xi^2 = 0$, we have a trivial normal bundle, so f can be extended to imbed $S^2 \times I$. We thus choose n disjoint imbeddings, and join the spheres by tubes to obtain a single sphere, representing $n\xi$.

For $\xi^2 = \pm 1$, we can find a cross-section of the normal disc bundle of S^2 meeting the zero cross-section in only one point, transversely. Thus we have two S^2's representing ξ, which meet transversely at a single point. We remove a small neighbourhood of the point—this leaves as boundary two circles linked in S^3, which we span by an annulus $S^1 \times I$ (with desired orientation), thus again obtaining an imbedded sphere.

Theorem 3 refers only to ordinary $\xi \in H_2(M)$; in other words, we insist that the dual cohomology class modulo 2 of ξ shall not be the second Stiefel class w_2. We briefly consider the opposite—"characteristic"—case. The argument leading to Theorem 3 remains valid in principle, but the basic representing spheres are now much harder to construct.

LEMMA 6. *Let $Q(N)$ be even (i.e. $w_2(N) = 0$), and let $M = N \# kP \# lQ$ where $k \geqslant 1$, $l \geqslant 1$, and $(k, l) \neq (1, 1)$. Exclude the case when $Q(M)$ is almost definite. Then every primitive characteristic element of $H_2(M)$, of square $k-l$, is represented by an imbedded S^2, with simply-connected complement.*

Proof. Since $k \geqslant 1$, $l \geqslant 1$, M has a summand T, and since either k or l is larger, we can deduce from Theorem 1 that it admits a summand S, and hence from Theorem 2 that any automorph can be realised by a diffeomorphism. Referring again to Theorem 4 of [5], we see that the orthogonal group is transitive on primitive characteristic vectors of given square, so it will be sufficient to represent one such.

Now a generator, u, of $H_2(P)$ is represented by a complex projective line, which is certainly a 2-sphere, and its complement is a disc. Correspondingly for $v \in H_2(Q)$. We take $k+l$ such 2-spheres, one in each of the later summands of M, and observe that they are disjoint. Hence we can connect them by tubes to obtain a single 2-sphere, still with simply-connected complement. The homology class is characteristic, of norm $k-l$.

As before, we can obtain results not included by this, but they are much weaker (except for the exclusion of the case $k = l = 1$, which can probably be avoided). Notice that for a given manifold M, we have only succeeded in representing characteristic $\xi \in H_2(M)$ with a single value of ξ^2. In a sequel to this paper, we shall prove that for some manifolds, several values of ξ^2 can appear.

5. *The non-simply-connected case.*

Our arguments have not always needed simple connectivity, and we now briefly discuss what survives without it. In §2, the hypothesis was not needed, except as a convenience in calculations. As in §3, we observe that if $f: S^1 \times D^3 \to N$ is null-homotopic, the manifold M obtained by a spherical modification is of the form $N \# T_k$. The proof of Theorem 1 now shows:

(5.1) *Theorem* 1 *holds for N homologically* 1*-connected, provided we assume ω spherical.*

(5.2) *Assume for some spherical $\omega \in H_2(N)$ that ω^2 is odd. Then $N \# S \cong N \# T$.*

The proof is as before; we have no need to compute the induced homology map, so no hypotheses on N are necessary.

Theorem 2 appears to break down completely—we do not see how (even in favourable circumstances) to characterise the representable automorphs, without further information on spherical classes in $H_2(N)$. The applications to imbedding spheres also break down, but we can show

(5.3) *Suppose N homologically* 1*-connected, that ξ, $\omega \in H_2(N)$ are spherical, and $\omega . \xi = 1$, ω^2 is even. Then ξ can be represented by an imbedded sphere in $N \# S$.*

First suppose ξ^2 even. Then $E^2_{-\omega}\,\xi = \xi + x$, and

$$E^1_{-\xi}(\xi + x) = x + (\tfrac{1}{2}\xi^2)\,y.$$

But by (5.1), $E^2_{-\omega}$ and $E^1_{-\xi}$ are represented by diffeomorphisms, and as before, $x + (\frac{1}{2}\xi^2)\,y$ is representable by an S^2 in $S^2 \times S^2$.

If ξ^2 is odd, we must interpret $E^1_{-\xi}$ as corresponding to a diffeomorphism of $N \# S$ on $N \# T$; otherwise the argument is just the same.

We make several remarks about this result.

(i) The condition $\omega . \xi = 1$ implies that ξ is primitive, and that ω^2 is even implies that ξ is ordinary—indeed, they are almost equivalent conditions.

(ii) The result is not included in any of our results on simply-connected manifolds.

(iii) Nevertheless, it is unsatisfactory; under suitable conditions on the signature we would prefer to replace $\xi, \omega \in H_2(N)$ in the hypothesis by ξ, $\omega \in H_2(N \# S)$. We could obtain extended results by transforming ξ by a sequence of E-maps, but again do not see how to obtain a useful result without further information on spherical classes in $H_2(N)$.

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