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AMPHICHEIRALS ACCORDING TO TAIT AND HASEMAN

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ABSTRACT

In this essay, we present our interpretation of the work of P. Tait and M. Haseman on achiral alternating knots. We have tried to elucidate the meaning of several concepts used by them such as: Amphicheirals, skew-amphicheirals, 1st or 2nd order, 1st or 2nd class, distorsions, etc. We also comment on some of their many pictures.

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Mathematics Subject Classification 2000: 57M25, 57M27

1. Chirality

In this essay, we present the work of P. Tait and M. Haseman on achiral alternating knots. We are aware of the danger of misunderstanding mathematical texts written in a style very different from the one we use today. It is therefore more appropriate to say that this is our interpretation of their work. Fortunately, Tait and Haseman's papers are now available on the Net. See [15]. This is why we have decided not to reproduce here all their knot pictures. We ask the reader to consult directly the original.

We work in the differentiable category. Knots and links are infinitely differentiable submanifolds of the 3-sphere S^3. In this setting, it is useful to be aware of J. Cerf's theorem [4] which says that a degree $+1$ diffeomorphism $\varphi : S^3 \to S^3$ is isotopic to the identity. We restrict our attention to knots, i.e. connected 1-dimensional submanifolds of S^3, although some results are also valid for links.

Definition 1.1. A knot K in S^3 is said to be *achiral* if there exists a diffeomorphism $h : S^3 \to S^3$ such that:

(1) $h(K) = K$.
(2) h reverses the orientation of S^3.

We call h a *mirror diffeomorphism* for K. A knot K is said to be *periodically* (or rigidly) achiral if there exists a mirror diffeomorphism h for K which satisfies the following condition:

(3) There exists an integer $n \geq 2$ such that $h^n = id$. Note that n is necessarily even.

Achirality has two variants depending on what the action of h on K is. A knot K is said to be *positively achiral* if there exists a mirror diffeomorphism h such that h preserves the orientation of K. It is *negatively achiral* if there exists an h which reverses the orientation of K.

Suppose that $h : S^3 \to S^3$ is periodic. Smith theory implies that the set Fix(h) of fixed points of h is either S^2 or S^0. If we assume that K is prime (as we do), then Fix(h) $= S^0$. In this case, K is +achiral if Fix(h) $\cap K = \emptyset$ and −achiral if Fix(h) $\subset K$.

2. Tait's Amphicheirals

During the month of January 1867, the Scottish physicist Peter Guthrie Tait proceeded in Edinburgh to experiments on smoke rings. His goal was to visualize Hermann von Helmholtz's theorems about vortices in a perfect fluid. William Thomson was present and a few weeks later Thomson stated his hypothesis that atoms are knotted vortices in the ether. Initially with the purpose of classifying atoms, Tait decided to classify knots. He soon recognized the importance of chirality questions.

We examine now how he expressed himself in a series of papers published between 1876 and 1885. Tait considers almost exclusively knot diagrams drawn on the sphere S^2, rather than on the plane R^2. His diagrams are alternating and prime (i.e. indecomposable with respect to connected sum of diagrams). In particular, they do not have "nugatory crossings". *From now on, we shall always assume that diagrams are alternating and prime.*

Tait created the word "*amphicheiral*", built on the greek roots amphi = on both sides and chir = hand. Presumably, Tait meant that an object is amphicheiral if it is both left-handed and right-handed. "Amphicheiral" was widely used until the 1980s when it was gradually replaced by "achiral". A good mnemotechnic trick is to remember that both words begin with the same letter "a". A knot which is not achiral is chiral. The word "chiral" was coined in 1893 by W. Thomson (then Lord Kelvin). Tait uses "amphicheiral" to refer to a diagram of an achiral knot. But he advances by successive approximations. Here is his first definition [18, Sec. 1].

Definition 2.1 (Tait). An amphicheiral knot is one which can be deformed into its own perversion.

Here knot means diagram. The perversion of a diagram is the new diagram obtained by exchanging overpasses with underpasses at each crossing. This name was already used by J.-B. Listing in [12]. It comes from the latin verb "pervertere" which means to turn over. Deformed means "there exists a diffeomorphism $\varphi : S^2 \to S^2$, preserving the orientation of S^2". A mirror diffeomorphism h is essentially obtained by extending φ radially in both hemispheres and then composing it with the reflection through the 2-sphere on which the diagram is drawn. (As pointed out by Ray Lickorish, if we wish to perform the extension in the differentiable category, we should refer to [16]. It results from its main theorem that an orientation preserving diffeomorphism of the 2-sphere extends to the 3-ball). In [18, Sec. 2] Tait sketches a proof of the following fact: the diffeomorphism φ is conjugate to a rotation of angle π, with axis a line which cuts the 2-sphere S^2 in the middle of two opposite arcs of the diagram. We shall call the corresponding mirror diffeomorphism a *Tait involution*. In order to better visualize the involution, Tait draws the diagram in the plane, sending one of the two fixed points to infinity. The diagram has two threads which go to infinity and the rotation center is easy to see. It is obvious that the knot is −achiral. In [17, Secs. 47 and 48], Tait gave procedures to construct such diagrams. He observed that the number c of crossings is necessarily even and he showed that these diagrams exist for every even $c \geq 4$.

We conclude this section by quoting a few lines from Tait's paper about Listing. See [19, Sec. 23]. These lines are quite interesting if we wish to grasp Tait's view of the question of chirality. Notice the use of the word "curious" twice. "There is one very curious point about knots which, so far as I know, has yet no analogue elsewhere. In general the perversion of a knot (i.e. its image in a plane mirror) is not congruent with the knot itself. Thus, as in fact Listing points out, it is impossible to change even the simple form [a left-handed trefoil] into its image [a right-handed trefoil]. But I have shown that there exists at least one form, for every even number of crossings which is congruent to its own perversion. The unique form with four crossings gave me the first hint of this curious fact."

3. Tait' s Distorsions

In order to go further, we need to recall an important notion introduced by Tait, which is at the basis of his work on alternating knots. Consider Fig. 1 representing a portion of a diagram.

The two dotted circles do not belong to the diagram. They bound the two discs A and B, which contain what we do not see of the diagram. It is easy to see that the diagram can be transformed by an isotopy in 3-space into the diagram represented in Fig. 2.

The disc B is unmoved, while the disc A is modified by a rotation of angle π and axis a line contained in the projection plane. We remark that this way to represent the transformation is essentially due to Mary Haseman [7]. Such a transformation was called a *distorsion* by Tait. John Conway in his famous article [5] calls it a *flype*. This word was given a different meaning by Tait. Today, flype is widely used in Conway's sense.

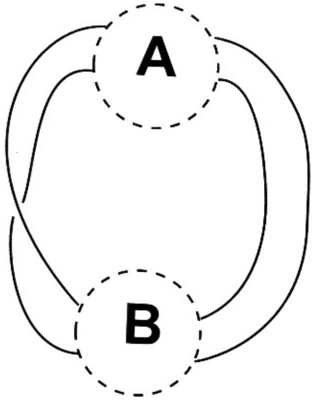

Fig. 1.

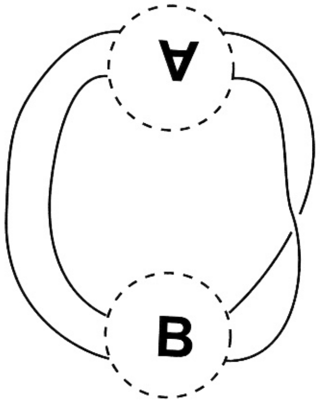

Fig. 2.

Tait's second definition for an amphicheiral is essentially the following.

Definition 3.1 (Tait). A diagram is amphicheiral if it can be distorted into its own perversion.

Clearly "distorted" means transformed by a finite sequence of distorsions (and diffeomorphisms of S^2).

Let us now recall the Flyping Theorem of William Menasco and Morwen Thistlethwaite [14], which makes lawful the main principle used by Tait in his attempt at knot classification.

Theorem 3.2 (Menasco–Thistlethwaite). *Two prime alternating diagrams represent the same isotopy class of knots if and only if one can be transformed into the other by a finite sequence of flypes (and diffeomorphisms of* S^2*).*

Actually, the theorem proved by Menasco and Thistlethwaite is stronger because "knot" can be replaced by "oriented link". As a consequence, if an alternating prime knot is achiral, then any minimal diagram representing it can be distorted into its own perversion. Hence, Tait's second definition coincides with the present-day one.

4. Tait's First Order Amphicheirals

We now try to make clear some of Tait's definitions.

Definition 4.1. (1) A diagram D is an *amphicheiral of first order and first class* if it can be transformed into its own perversion by a Tait involution.

(2) A diagram D is an *amphicheiral of first order and second class* if it is not of first class, but if it can be distorted into one of first class.

We shall shorten this by saying that D is of type $(1, 1)$ (respectively, $(1, 2)$). Tait and Haseman knew very well how to construct diagrams of second class. See for instance [18, Sec. 6]. It sufficies to have a diagram of first class and perform distorsions which are not symmetrical with respect to Tait involution. If the original diagram is complicated enough, the new one will not be invariant by any Tait involution. Similarly, by performing symmetrical distorsions, one can expect to find achiral knots which have several different diagrams which admit Tait involutions. Let us also observe that 1st order implies −achiral.

The question of the existence of a possible third class will be addressed below in Sec. 9.

5. Planar Graphs Associated to an Alternating Diagram

We introduce now a concept much used by Tait and Haseman. It goes back to Listing [12]. If D is a diagram in S^2, let us call *region* (determined by D) a connected component of $S^2 - D$. Following Listing, let us label δ and λ the sectors located in the neighborhood of a crossing point, as shown in Fig. 3.

The letter δ stands for "dexiotrop" which means turning towards the right. The letter λ stands for "laeotrop" = turning towards the left. Listing had his own opinion

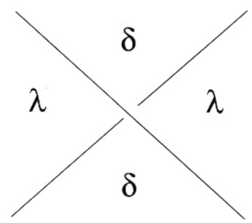

Fig. 3.

about the meaning of "turning towards right or left". It is opposite to Maxwell's, which is the one usually accepted today.

Tait observes that, because D is alternating, all sectors which belong to the same region are either all δ or all λ. Two adjacent regions have sectors with a different label. As a consequence, if we consider the checkerboard associated to a diagram, the colors (black or white) correspond to the labels (δ or λ). Hence, we shall call δ or λ the *color* of a region.

We write Δ for the planar graph (embedded in S^2!) associated to the regions of color δ and write Λ for the planar graph associated to the color λ. Recall that (for instance) the planar graph Λ is constructed in the following way. In each region R of color λ, we choose a point S_R which will be the vertex of the graph Λ corresponding to that region. Let P be a crossing point contiguous to two regions R and R' of color λ. We associate to P an edge A_P joining S_R to $S_{R'}$ and going through P via the two sectors of color λ near P. Hence, the number of vertices of the graph Λ is equal to the number of regions of color λ and the number of edges of Λ is equal to the number c of crossings of D. An example is given in Fig. 4.

For more details, see for instance [1, end of the last chapter]. Each of these two graphs is called "partition grouping" by Tait and "compartment symbol" by Haseman. Both knew quite well that it is important to consider these graphs as embedded in S^2. It is easy to see that they determine each other: they are "duals" in S^2. From any of them, we can reconstruct the initial diagram D.

6. Equivalences of Planar Graphs

Definition 6.1. Two planar graphs Γ and Γ' are $\pm$equivalent if there exists a diffeomorphism $\varphi : S^2 \to S^2$ of degree ± 1 such that $\varphi(\Gamma) = \Gamma'$.

The following proposition was probably known to Tait and Haseman. See, for instance [17, Sec. 47] where "with the same grouping" is written in italics.

Proposition 6.2. *Let D be a diagram in S^2. Then D is invariant by a Tait involution (i.e. D is of type* (1,1)*) if and only if Δ is $+$equivalent to Λ.*

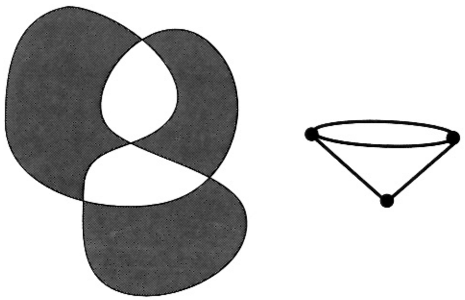

Fig. 4.

Here are the broad lines of the proof.

(1) In one direction, the implication is clear. The rotation of angle π induces (in fact is) a $+$equivalence between Δ and Λ.
(2) Conversely, let $\psi : S^2 \to S^2$ be a diffeomorphism of degree $+1$ such that $\psi(\Delta) = \Lambda$. As we can reconstruct the diagram D from Δ or Λ, we can modify ψ by an isotopy such that $\psi(D) = D$.
(3) As each region determined by D is a 2-cell, we can modify ψ by another isotopy to make ψ periodic of period say $n \geq 2$.
(4) From Smith theory, we deduce that this periodic ψ has two fixed points. These fixed points cannot be in the interior of a region, because ψ exchanges the colors. They cannot be at crossing points of D because D represents a knot and not a link. Hence, each fixed point is in the middle of an arc of D. It is noteworthy that most of this argument was known to Tait. See [18, Sec. 2].
(5) Choose one of the two arcs of D which contain a fixed point of ψ. As ψ preserves the orientation of S^2 and exchanges the two regions adjacent to the arc, ψ reverses the orientation of the arc. Hence, ψ^2 preserves the orientation of the arc. As ψ^2 is of finite order, ψ^2 is the identity on the arc. From Tutte's arguments in [21, Sec. 3], we deduce that ψ^2 is the identity. Hence, ψ is conjugate to a Tait involution.

Summary.

(1) "D is of type $(1,1)$" is equivalent to "Δ and Λ are $+$equivalent";
(2) "D is of type $(1,2)$" is equivalent to "we can modify D by distorsions in order that Δ and Λ be $+$equivalent".

Compare with the definitions proposed by Tait in [18, Sec. 13].

Let us now consider $-$equivalence. Up to the vocabulary used, Tait asks if it is possible that Δ and Λ be $-$equivalent. This is what he means in [18, Sec. 9] when he says "left and right meshes" are "similar but not congruent". He adds that this is a "curious question" which could mean that he was mostly interested in $-$achiral knots and diagrams invariants by a Tait involution. However, Tait immediately gives an answer to the question and indicates how this can happen. It suffices to have a diagram drawn in the sphere S^2 which is invariant by the antipodal map. As it is not easy to draw pictures on a sphere, he proposes a clever method to draw these on a usual sheet of paper. See [18, Secs. 10–12]. For the moment, let us just remark that if a diagram $D \subset S^2$ is invariant by the antipodal map, then the knot $K \subset S^3$ represented by D has a mirror diffeomorphism with two fixed points away from K and hence K is $+$achiral. Here are now Tait's definitions for the second order of amphicheirality. See [18, Sec. 13].

Definition 6.3. (1) A diagram D is an *amphicheiral of second order and first class* if Δ is $-$equivalent to Λ.

(2) A diagram D is an *amphicheiral of second order and second class* if it is not of first class, but if it can be distorted into one of first class.

We shall talk of amphicheirals of type $(2,1)$ (respectively, $(2,2)$). Tait seems to believe that the only way to obtain amphicheirals of type $(2,1)$ is via the antipodal map. He is wrong here, as we shall see in the next section.

7. Mary Haseman's Skew-Amphicheirals

Mary Gertrude Haseman now enters our story. Interesting information about her can be obtained from [15]. She received her PhD at Bryn Mawr College in 1918 with a thesis titled: "On Knots, with a Census of the Amphicheirals with Twelve Crossings". It was published in the same journal which welcomed Tait's papers: *The Transactions of the Royal Society of Edinburgh*. She is truly a disciple of Tait (she was not his student, because Tait died in 1901). She understood him very well. Indeed her explanations prove useful to understand Tait's writings, which are often obscure! She introduced the name "tangle" in order to define precisely the distorsions. She clearly draws the auxiliary circle which cuts the diagram in four points (today called a Conway circle). Her work is mainly devoted to the study of amphicheirals. As with Tait, she is more interested in diagrams in S^2 rather than in knots in S^3. Her diagrams are alternating. Her main discovery is what she calls *skew-amphicheiral* diagrams. Let us present a definition for it in present-day language.

Definition 7.1. A diagram is *skew-amphicheiral* if it is invariant by a rotatory reflection of even order $n \geq 4$.

More precisely, for Haseman, the rotatory reflection is the composition of a rotation of angle $2\pi/n$ whose axis is a line perpendicular to the plane which contains the diagram, followed by a reflection through a sphere Σ centered at the intersection point O between the plane and the line. The sphere Σ cuts the rotation axis in two points, one above the plane and one below. They are the two fixed points of the transformation. The plane cuts Σ in a circle γ centered at O. What we see in the plane containing the diagram is a rotation of angle $2\pi/n$ centered at O followed by a reflection through the circle γ. If $n = 2$, such a rotatory reflection is conjugate to the antipodal map and hence Haseman generalizes Tait.

A typical example is provided by Conway diagram 10^* which is Rolfsen 10_{123}. In Tait's plates, it is represented by the knot 10_{38} and also by picture E. Obviously, the diagram E is invariant by a rotatory reflection Φ of order 10. But Φ^5 is of order 2 and hence conjugate to the antipodal map. Maybe, this is the fact which prevented Tait from discovering the skew-amphicheirals. Proposition 7.2 is the counterpart of Proposition 6.2 for second order amphicheirals.

Proposition 7.2. *Let D be a diagram in S^2. Then D is invariant by a rotatory reflection of even order $n \geq 2$ if and only if Δ is $-$equivalent to Λ.*

Here are the broad lines of the proof.

(1) Suppose that D is invariant by a rotatory reflection Φ. By definition, Φ reverses the orientation of S^2. This implies that Φ exchanges the colors δ and λ. As a consequence, Φ induces a $-$equivalence between Δ and Λ. That implication was clearly known to Haseman.

(2) Let us prove the converse, and let $\psi : S^2 \to S^2$ be a diffeomorphism of degree -1 such that $\psi(\Delta) = \Lambda$. As in the proof of Proposition 6.2, we can modify ψ by an isotopy to make $\psi(D) = D$ with ψ of finite order. As ψ reverses the orientation of S^2, by Smith theory we have $\mathrm{Fix}(\psi) = S^1$ or $\emptyset$. Because D is assumed to be prime, we have $\mathrm{Fix}(\psi) = \emptyset$.

(3) Consider the diffeomorphism $\psi^2 : S^2 \to S^2$. It preserves the orientation of S^2 and hence $\mathrm{Fix}(\psi^2) = S^2$ or S^0. If $\mathrm{Fix}(\psi^2) = S^2$, then ψ is of order 2 and hence conjugate to the antipodal map.

(4) If $\mathrm{Fix}(\psi^2) = S^0$, then ψ^2 is conjugate to a rotation of order say $d \geq 2$. Because $\psi^2(\Delta) = \Delta$ and $\psi^2(\Lambda) = \Lambda$, the two fixed points of ψ^2 must be situated one inside a region of color δ and one of color λ (crossing points are excluded because D represents a knot).

(5) It follows that ψ is conjugate to a rotatory reflection of order $2d \geq 4$.

8. Haseman's Tabulations

At the end of [7], Haseman presents a list of 61 amphicheiral diagrams with 12 crossings. It is known today that there are 54 achiral alternating knot types with 12 crossings. See [9, 20]. All of them are represented in Haseman's list, seven of them twice. Among them, 16 types are both $+$ and $-$achiral, 37 types are $-$achiral and there is just 1 type which is only $+$achiral. In Haseman's list, this last type is represented by the diagrams 59 and 60. Of course, these pictures do not have strands which go to infinity.

Haseman's favorite knot seems to be the one represented by the 61st (and last) diagram (see Fig. 5 which represents Haseman's diagram 61 slightly modified). It is as if she had placed it here as a sweet ... The corresponding knot is also represented by the diagram 6 of [7] and by Figs. 2 and 3 of [8] as well as by 4, $4'$, $4''$ and $4'''$ of [7]. Without taking diffeomorphisms into account, there are 16 minimal diagrams for the knot type which differ by flypes. It is probably that abundance which pushed Haseman to write the sequel [8] to [7] which corrects certain statements of [7] about that knot.

The knot is both $+$ and $-$achiral, and its symmetry group is D_4. It has diagrams which are simultaneously of type $(1,1)$ and $(2,2)$; others are simultaneously of type $(1,2)$ and $(2,1)$, still others of type $(1,2)$ and $(2,2)$! They are the result of symmetrical or unsymmetrical distorsions. On the diagram 61 (represented in Fig. 5), one can clearly see the rotatory reflection which generates the subgroup $C_4 \subset D_4$. This is typically a case of skew-amphicheiralism because the knot cannot

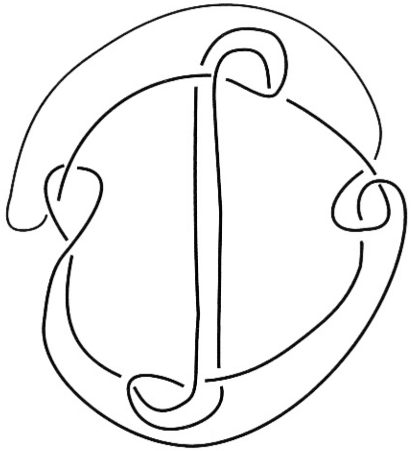

Fig. 5.

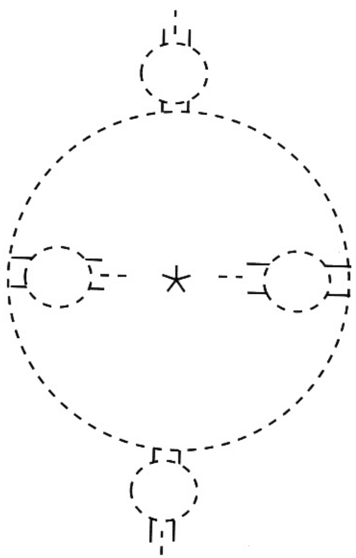

Fig. 6.

have a diagram which is invariant by the antipodal map. Haseman realized that quite well. Let us remark that the order of the rotatory reflection is equal to 4. Hence, one cannot argue as in the case of Conway knot 10^*, where n could be written as $n = 2d$ with d odd.

Among the diagrams presented by Haseman, let us also point out [8, p. 600, Fig. 6]. The diagram has 24 crossings and the corresponding knot is +achiral. Its symmetry group is isomorphic to C_4 (thank you Knotscape!). The rotatory

reflection of order 4 which generates the group is clearly visible in the picture given by Haseman.

The last two knot types we have just been talking about are AAA: Alternating, Achiral and Arborescent in the sense of F. Bonahon and L. Siebenmann [2] (i.e. Algebraic in Conway's sense). When such a knot is +achiral, it very often (but not always as we shall see in Sec. 9) has a diagram which is invariant by a rotatory reflection of order 4. We propose to call *Haseman symmetry* the rotatory reflection of order 4 whose principle is illustrated in Fig. 6.

This symmetry is the composition of a rotation of angle $\pi/2$ centered at the starred point, followed by a reflection through the large dotted circle.

This phenomenon was also observed by Alain Caudron (student of Larry Siebenmann) in his thesis, for knots which are achiral and arborescent. See [3].

9. Amphicheirals of Third Class

We now try to clear things up, without betraying too much Tait and Haseman.

Definition 9.1. (1) We define an *amphicheiral of first order* to be a minimal diagram representing an alternating knot which is −achiral.

(2) We define an *amphicheiral of second order* to be a minimal diagram representing an alternating knot which is +achiral.

(3) *First class* means that the symmetry is visible in the diagram.

(4) *Second class* means that the diagram is not of first class but that it is obtained by distorsions from one of first class.

By "visible" we mean that the diagram is invariant by a diffeomorphism $S^2 \to S^2$ of finite order, which essentially describes the mirror diffeomorphism. In the first order case, it is a Tait involution. In the second order case, it is a rotatory reflection of even order.

There is an important question which remains open. Tait knew about it and he admitted to have no answer. See [18, Sec. 8].

Question 1. Do there exist third class amphicheirals ?

By definition an *amphicheiral of third class* would be a minimal diagram representing an alternating achiral knot that could not be distorted into one on which the symmetry is visible. By the Flyping Theorem, all minimal diagrams are equivalent via distorsions. Hence Question 1 is equivalent to:

Question 2. Do there exist alternating achiral knots having no diagram on which the symmetry is visible?

By Propositions 6.2 and 7.2, this is also equivalent to asking:

Question 3. Do there exist alternating achiral knots having no minimal diagram such that Δ is equivalent to Λ?

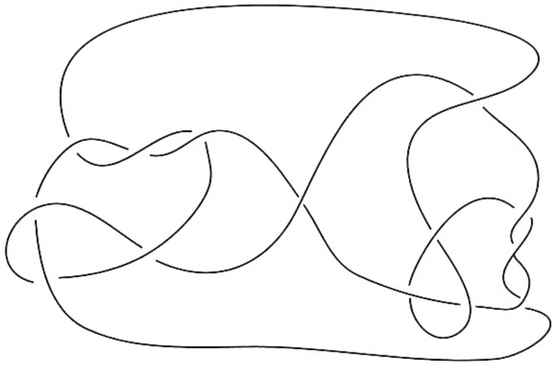

Fig. 7.

In this last form, the question was raised by Lou Kauffman as problem 845 in [10]. For +achiral knots, we know today that the answer is yes. It was given by O. Dasbach and S. Hougardy in [6]. See Fig. 7. The example provided by the authors is very interesting. In fact, one can show that it is the simplest (c minimal) of a quite large family of knots all having that property. Dasbach–Hougardy example is an AAA knot with 14 crossings. The planar Bonahon–Siebenmann trees which encode any of its minimal diagrams have a Haseman symmetry subtly broken. We really need distorsions to proceed from any minimal diagram to its perversion. However, the symmetry group of the knot is also C_4.

The question of the existence of 3rd class amphicheirals for alternating −achiral knots remains open. It is known today that a hyperbolic −achiral knot has always a mirror diffeomorphism of order two. See [11] or [13]. As all achiral alternating knots are hyperbolic, the question is this: Can we find a mirror involution which is a Tait involution?

10. Final Comments

1. For $c \leq 10$ there are 20 prime alternating achiral knots. Tait exhibits them in plate VII. Among them, 13 are both + and −achiral, while 7 are only −achiral. Hence, all of them are (at least) −achiral. For each of them, Tait produces a diagram which is invariant by what we call a Tait involution. He also observes that some of them are +achiral as well. See [18, Sec. 12].

2. Tait payed a little visit to achiral knots with 12 crossings. He considered the knot represented by figures A, B and C in Plate VII. In [18, Sec. 12], he says that the diagram is equivalent by distorsions to its perversion but that it is not "amphicheiral in the ordinary sense". He means that the knot represented by those three diagrams is +achiral (the diagrams A and B are obviously invariant by the antipodal map) but that it is not −achiral. He is right. This is the unique alternating knot with 12 crossings which has this property. Knotscape tells us that its symmetry group is C_2

(it is coded as 12a427). Mary Haseman represents it in Figs. 59 and 60 as we have already seen. She does not quote Tait for that matter.

3. Haseman has proposed (following a suggestion of Charlotte A. Scott, her thesis advisor) a clever variation on the Gauss–Tait word to encode a diagram, which she calls the "intrinsic symbol". Her notation has indeed the advantage of being independent of the symbols used to name the crossings. The intrinsic symbol is in fact equivalent to the cord diagram, without the necessity to draw a picture.

4. Here and there, there are little mistakes in Haseman. For instance, some planar graphs Δ and Λ are apparently not correctly drawn. In [7], there are a few wrong statements about the knot represented by the diagram 61, which are corrected in [8]. About the skew-amphicheirals (with $n \geq 4$) she saw that the ones she discovered are all also −achiral. She says that this might be always true, but adds however that counterexamples might exist for large c. In fact, the 24-crossing knot represented by [8, Fig. 6] is such a counterexample. She does not mention this property.

5. Finally, let us raise high our hat to the remarkable exploration work done by Tait and Haseman. Both have certainly drawn an enormous quantity of diagrams. This impressive and hidden work has resulted in the publication of the plates, which are incredibly exhaustive.

11. Historical Sources Available on the Net

Thanks to Josef Przytycki and Andrew Ranicki, many papers of great historical interest are now available on the Net. See [15].

Tait's papers which address chirality questions are: [17] see Secs. 13, 17, 47, 48 and [18] see Secs. 1–16 and also plate VII at the end of the paper.

Interesting from a historical and psychological viewpoint is Tait's paper about Listing's Topologie: [19].

Haseman's papers are two: [7, 8].

Let us not forget Listing's pioneering work: [12]. Knots are treated in pp. 859–866.

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MÖBIUS TRANSFORMATIONS OF POLYGONS AND PARTITIONS OF 3-SPACE

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ABSTRACT

The image of a polygonal knot K under a spherical inversion of $\mathbb{R}^3 \cup \infty$ is a simple closed curve made of arcs of circles, perhaps some line segments, having the same knot type as the mirror image of K. But suppose we reconnect the vertices of the inverted polygon with straight lines, making a new polygon $\widehat{K}$. This may be a different knot type. For example, a certain 7-segment figure-eight knot can be transformed to a figure-eight knot, a trefoil, or an unknot, by selecting different inverting spheres. *Which knot types can be obtained from a given original polygon K under this process?* We show that for large n, most n-segment knot types cannot be reached from one initial n-segment polygon, using a single inversion or even the whole Möbius group.

The number of knot types is bounded by the number of complementary domains of a certain system of round 2-spheres in $\mathbb{R}^3$. We show the number of domains is at most polynomial in the number of spheres, and the number of spheres is itself a polynomial function of the number of edges of the original polygon. In the analysis, we obtain an exact formula for the number of complementary domains of any collection of round 2-spheres in $\mathbb{R}^3$. On the other hand, the number of knot types that can be represented by n-segment polygons is exponential in n.

Our construction can be interpreted as a particular instance of building polygonal knots in non-Euclidean metrics. In particular, start with a list of n vertices in $\mathbb{R}^3$ and connect them with arcs of circles instead of line segments: Which knots can be obtained? Our polygonal inversion construction is equivalent to picking one fixed point $p \in \mathbb{R}^3$ and replacing each edge of K by an arc of the circle determined by p and the endpoints of the edge.

Keywords: Polygonal knot; spherical inversion; Möbius transformation; knot energy.

Mathematics Subject Classification 2000: 57M25

1. Introduction

Inversion of 3-space through a sphere is a well-known transformation of $\mathbb{R}^3 \cup \infty$. If $S_{\mathbf{p},r}$ is the round sphere of radius r centered at the point $\mathbf{p}$, the mapping

$$\rho(\mathbf{x}) = \mathbf{p} + \frac{r^2(\mathbf{x}-\mathbf{p})}{|\mathbf{x}-\mathbf{p}|^2}$$