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K-Theory and Analytic Isomorphisms

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Abstract. If s is a central nonzerodivisor of a ring A, let B denote the s-adic completion of A. By a theorem of Karoubi, there is a long exact sequence relating the K-theory of A, B, $A[s^{-1}]$, and $B[s^{-1}]$. This sequence was first exploited by Vorst in his thesis. We give two applications of the Karoubi sequence: (1) an example of a 2-dimensional normal domain with $NK_0 \neq 0$, answering a question of Murthy, and (2) a complete computation of K_2 of an (affine) seminormal curve over an algebraically closed field.

This paper is an exploitation of a technique originally developed by Karoubi for L-theory in [K], and first applied to K-theory by Vorst in [V]. The basic idea is that if s is a nonzerodivisor of A and if B is the s-adic completion of A, there is a "Mayer-Vietoris sequence"

$$\dots K_*(A) \to K_*(A[s^{-1}]) \oplus K_*(B) \to K_*(B[s^{-1}] \dots .$$

The present author first became interested in this sequence as a tool for analyzing K_2 of the node $A = k(x, y]/(y^2 = x^2 + x^3)$. In [V], Vorst was able to show that $NK_2(A) \neq 0$ for the node; in §4 of this paper we show that $K_2(A) = K_2(k) \oplus K_3(k) \oplus k^+$.

The appearance of the K_3 summand is not surprising, since the Karoubi-Villamayor group is $KV_2(A) = K_2(k) \oplus K_3(k)$. In §4 this computation is generalized to show that for any reduced 1-dimensional seminormal ring A, finitely generated over a field k, and with only rational singularities, we have $K_2(A) = KV_2(A) \oplus \mathrm{nil}\, K_2(A)$, where $\mathrm{nil}\, K_2(A)$ is a finite dimensional vector space over k determined by the singularities of A.

Our other application (in §2) of Karoubi's technique is to a bad noetherian ring discovered by Nagata. This ring is a 2-dimensional local normal domain, and we show that $NK_0 \neq 0$. This answers an old question of Murthy. The interpretation is that there are projective $A[t]$-modules which are not even

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stably extended from A. We also show that $NK_{-1}(A)\neq 0$, so there are projective $A[s,s^{-1},t]$-modules which are not even stably a direct sum of modules extended from $A[s,t]$, $A[s^{-1},t]$, and $A[s,s^{-1}]$.

In §3 we descend from Nagata's bad ring to find geometric examples of (normal) UFD's with NK_0 and K_{-1} nonzero. We then prove a generalization due to Swan, and use it to show that for any field

$$NK_0(k[x,y,s,t]/(x^3-y^2-st))\neq 0,$$
$$K_{-1}(k[x,y,s,t]/(x^2+x^3-y^2-st))\neq 0.$$

Both rings are 3-dimensional UFD's with the only singularity at the origin.

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§1. The Localization MV-Sequences

In this section we develop the machinery necessary to do the calculations in the later sections. Following [B], we will let $\mathbf{H}(A)$ denote the category of (right) A-modules having finite resolutions by finitely generated projective A-modules. If S is a multiplicative set of central nonzerodivisors of A we write $\mathbf{H}_S(A)$ for the subcategory of S-torsion modules. A superscript of n indicates that we restrict to modules of homological dimension at most n.

The central notion of this paper is that of analytic isomorphism. A ring homomorphism $i: A\to B$ is an *analytic isomorphism along* S if S is a multiplicatively closed set of central nonzerodivisors of A for which $i(S)$ consists of central nonzerodivisors of B, and if $A/sA\cong B/sB$ for every $s\in S$.

Theorem 1.1 (Karoubi [K]). *Let $i: A\to B$ be a ring homomorphism, $S\subset A$ a multiplicatively closed set of central nonzerodivisors for which $i(S)$ consists of central nonzerodivisors of B. Then $B\otimes_A$: $M\mapsto B\otimes_A M$ is an exact functor from $\mathbf{H}_S^1(A)$ to $\mathbf{H}_S^1(B)$.*

If i is an analytic isomorphism along S, then $B\otimes_A$ is a natural equivalence $\mathbf{H}_S^n(A)\cong\mathbf{H}_S^n(B)$, $n=1,2,\ldots$, and $\mathbf{H}_S(A)\cong\mathbf{H}_S(B)$.

Proof. What is proven on p. 401 of [K] is that $\mathrm{Tor}_1^A(B,M)=0$ for $M\in\mathbf{H}_S^1(A)$, that $\mathbf{H}_S^1(A)\to\mathbf{H}_S^1(B)$ is well-defined, and that when i is an analytic isomorphism, $\mathbf{H}_S^1(A)\cong\mathbf{H}_S^1(B)$. It follows that if $0\to M_2\to M_1\to M_0\to 0$ is an exact sequence in $\mathbf{H}_S^1(A)$ then $0\to B\otimes M_2\to B\otimes M_1\to B\otimes M_0\to 0$ is exact, i.e., that $B\otimes_A: \mathbf{H}_S^1(A)\to\mathbf{H}_S^1(B)$ is exact. Finally, we assume that i is an analytic isomorphism. The categories of S-torsion modules (over A and B) are equivalent, and (by [B], p. 435), the $\mathbf{H}_S$ are the full subcategories of objects having finite resolutions by objects in the equivalent subcategories $\mathbf{H}_S^1$. The objects of $\mathbf{H}_S^n$ are those with resolutions of length $n-1$, whence the rest of the theorem.

We note that the "resolution theorem" [B, Q] shows that $K_*(\mathbf{H}_S^1)=K_*(\mathbf{H}_S^n)=K_*(\mathbf{H}_S)$ for all $*\geqq 0$. It also holds for $*<0$ if we adopt the following notation (see pp. 658, 664 of [B]):

Definition. Let $\mathbf{C}$ be a functor from a category $\mathscr{A}$ of rings to exact categories. We assume that $\mathscr{A}$ contains the inclusions $C \subset C[t] \subset C[t, t^{-1}]$ and the involution $t \to t^{-1}$ of $C[t, t^{-1}]$ whenever C belongs to $\mathscr{A}$. Then for $n > 0$ we define $K_{-n}\mathbf{C}$ by setting

$$K_{-n}\mathbf{C}(C) = L^n K_0 \mathbf{C}(C),$$

the cokernel of $K_{1-n}\mathbf{C}(C[t]) \oplus K_{1-n}\mathbf{C}(C[t^{-1}]) \to K_{1-n}\mathbf{C}(C[t, t^{-1}])$. Two important cases will be $K_{-n}\mathbf{H}(C) = K_{-n}(C)$ and $K_{-n}\mathbf{H}_S(C)$.

For $j \in \mathbf{Z}$, we will call C "$K_j\mathbf{C}$-regular" if $K_j\mathbf{C}(C) = K_j\mathbf{C}(C[t_1, \ldots, t_p])$ for all p. This is equivalent to requiring the functors $N^i K_j \mathbf{C}, i \geqq 1$, to vanish on C. We will consider $N^i K_j \mathbf{C}(C)$ to be the intersection of the kernels of the maps

$$\text{“}t_q = 0\text{”}: K_j\mathbf{C}(C[t_1, \ldots, t_p]) \to K_j\mathbf{C}(C[t_1, \ldots, \hat{t}_q, \ldots, t_p]).$$

Whenever we refer to "the" map $NK_j\mathbf{C} \to K_j\mathbf{C}$, we will mean the map induced by "$t-1$". These functors are defined in a slightly different (but equivalent) way on p. 658 of [B].

Corollary 1.2. *Under the assumptions of Theorem* 1.1, *there is a map of exact sequences*

$$\begin{array}{ccccccc}
\ldots K_{*+1}(S^{-1}A) & \xrightarrow{\partial} & K_*\mathbf{H}_S^1(A) & \longrightarrow & K_*(A) & \longrightarrow & K_*(S^{-1}A) \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\ldots K_{*+1}(S^{-1}B) & \xrightarrow{\partial} & K_*\mathbf{H}_S^1(B) & \longrightarrow & K_*(B) & \longrightarrow & K_*(S^{-1}B) \ldots
\end{array}$$

This sequence is valid for all integers $*$.

Proof. By [Q] and [G] there are fibrations $Q\mathbf{H}_S^1(A) \to Q\mathbf{L} \to Q\mathbf{H}(S^{-1}A)$, $Q\mathbf{H}_S^1(B) \to Q\mathbf{H}(B) \to Q\mathbf{H}(S^{-1}B)$, where $\mathbf{L}$ is the subcategory of modules in $\mathbf{H}(A)$ which are Tor-independent from B. Note that $Q\mathbf{P} \subset Q\mathbf{L} \subset Q\mathbf{H}$ are homotopy equivalences by the resolution theorem. The ad hoc category $\mathbf{L}$ is such that $Q(B \otimes_A)$ is a map of fibrations. The result follows for $* \geqq 0$. Carter [C] has shown that the localization sequences extend to $* < 0$, and the naturality of his splicing construction is sufficient to show that $B \otimes_A$ is a map of exact sequences for $* < 0$ as well.

Theorem 1.3 (Vorst [V]). *Let $A \to B$ be an analytic isomorphism along S. Then:*

(i) *There is a long exact sequence*

$$\ldots K_*(A) \to K_*(S^{-1}A) \oplus K_*(B) \to K_*(S^{-1}B) \ldots,$$

defined for all $* \in \mathbf{Z}$. For fixed S, this sequence is natural in $A \to B$.

(ii) *Assume $S^{-1}A$ and $S^{-1}B$ are K_1-regular. Then there is a long exact sequence in Karoubi-Villamayor K-theory*

$$\ldots KV_*(A) \to KV_*(S^{-1}A) \oplus KV_*(B) \to KV_*(S^{-1}B) \ldots.$$

This sequence is also natural, and there is a natural transformation from the sequence of (i) *to this sequence.*

(iii) *Assume* $S^{-1}A$ and $S^{-1}B$ *are regular. Then there is a Mayer-Vietoris sequence*

$$\ldots K_*(A) \to KV_*(A) \oplus K_*(B) \to KV_*(B) \ldots .$$

Proof. We have long exact localization sequences for A and B, and a map between these sequences, as follows:

$$\begin{array}{ccccccc}
\ldots K_*\mathbf{H}_S(A) & \longrightarrow & K_*(A) & \longrightarrow & K_*(S^{-1}A) & \longrightarrow & K_{*-1}\mathbf{H}_S(A)\ldots \\
\| & & \downarrow & & \downarrow & & \| \\
\ldots K_*\mathbf{H}_S(B) & \longrightarrow & K_*(B) & \longrightarrow & K_*(S^{-1}B) & \longrightarrow & K_{*-1}\mathbf{H}_S(B)\ldots .
\end{array}$$

If $S^{-1}A$ and $S^{-1}B$ are K_1-regular, then in the diagram we can replace Quillen's K-theory K_* by Karoubi-Villamayor K-theory KV_* and retain exactness by [W3]. (The definition of the $KV_*\mathbf{H}_S$ is given in [W3].) The two long exact sequences claimed in (i) and (ii) follow easily from the diagram, as does the natural transformation. Finally (iii) follows from the fact that $K_*(C) = KV_*(C)$ for any regular ring (see [G]), and from the diagram

$$\begin{array}{ccccccc}
\ldots K_{*+1}(S^{-1}B) & \longrightarrow & K_*(A) & \longrightarrow & K_*(S^{-1}A) \oplus K_*(B) & \longrightarrow & K_*(S^{-1}B)\ldots \\
\| & & \downarrow & & \downarrow & & \| \\
KV_{*+1}(S^{-1}B) & \longrightarrow & KV_*(A) & \longrightarrow & KV_*(S^{-1}A) \oplus KV_*(B) & \longrightarrow & KV_*(S^{-1}B)\ldots .
\end{array}$$

Remarks. Karoubi's Theorem 1.1 was first used in [K] to produce a corresponding sequence for the functors ${}_\varepsilon L_*$. The sequence for K_*, $* \geqq 0$, was first used by Vorst in [V] in the case $B = \varprojlim A/sA$ to study the K_2-regularity of 1-dimensional noetherian rings.

Corollary 1.4. *Let k be a commutative subring of A containing S, and let R be a flat k-algebra. Then if* $i\colon A \to B$ *is an analytic isomorphism along S, so is* $i \otimes_k R\colon A \otimes_k R \to B \otimes_k R$. *In particular, for every* $i \geqq 0$ *there is an exact sequence*

$$\ldots N^i K_*(A) \to N^i K_*(S^{-1}A) \oplus N^i K_*(B) \to N^i K_*(S^{-1}B) \ldots .$$

When $S^{-1}A, S^{-1}B$ *are regular rings, the ring A is* K_**-regular if and only if B is.*

Proof. The point of flatness is that the property of S being a set of nonzero-divisors is preserved by $\otimes R$ (it suffices to have the $\mathrm{Tor}_1^k(R, A/sA)$ vanish for all $s \in S$). The first claim follows from $(A \otimes R)/(s) = A/(s) \otimes R = B/(s) \otimes R = B \otimes R/(s)$.

To prove existence and exactness of the sequence of $N^i K_*$'s, we use induction on i (the case $i=0$ being Karoubi's theorem as $N^0 K_* = K_*$). By naturality of the sequence of $N^i K_*$'s, the sequence for the pair (A, B) is a natural summand of the sequence for the pair $(A[t], B[t])$. The complementary summand is the sequence of $N^{i+1} K_*$'s for the pair (A, B). This establishes the long exact sequence for each i; the last claim of the corollary follows from the fact (cf. [Q]) that regular rings are K_*-regular for all $*$.

Proposition 1.5. *For any ring R, let* $R((x))$ *denote* $R[[x]][x^{-1}]$. *Then* $K_*(R((x))) = K_*(R[[x]]) \oplus NK_*(R) \oplus K_{*-1}(R)$ *and*

$$N^i K_*(R((x))) = N^i K_*(R[[x]]) \oplus N^{i+1} K_*(R) \oplus N^i K_{*-1}(R).$$

Proof. The sequence of the analytic isomorphism $R[x] \to R[[x]]$ is

$$\dots N^i K_*(R[x]) \to N^i K_*(R[x, x^{-1}]) \oplus N^i K_*(R[[x]]) \to N^i K_*(R((x))) \dots.$$

By the Fundamental Theorem in [GQ], $N^i K_*(R[x, x^{-1}]) = N^i K_*(R[x]) \oplus N^{i+1} K_*(R) \oplus N^i K_{*-1}(R)$, whence the result.

At this point, it becomes clear that a notation is needed to avoid writing $(\varprojlim A/(x^n A))[x^{-1}]$ and the like. We propose to write $A_x[x^{-1}]$. More generally, we will write $A_{\hat{I}}$ for the I-adic completion $\varprojlim A/I^n$, and if I is generated by the (central) element x, we will write $A_{\hat{x}}$ for $A_{\hat{I}}$. If A is noetherian, $A_{\hat{I}}$ may be obtained by a sequence of analytic isomorphisms of the type $B \to B_{\hat{x}}$. In fact, if $I = J + xA$, then $A_{\hat{I}} = (A_{\hat{J}})_{\hat{x}}$, as may be seen for example from the description $A_{\hat{x}} = A[[X]]/(X - x)$ in §23 of [Mat].

We would like to say that $S^{-1} A_{\hat{I}}$ is regular when $S^{-1} A$ is. Unfortunately, this is not so: we will give counterexamples in §2. When A is "excellent", though, it is true enough to let us prove the following result.

Proposition 1.6. *Let A be an excellent ring and I an ideal such that* A_p *is regular for every prime not containing I (i.e.,* $Sing(A) \subseteq V(I)$*). Assume I does not consist of zerodivisors. Then*

(i) *For all* $* \in \mathbf{Z}$, *A is* K_**-regular iff* $A_{\hat{I}} = \varprojlim A/I^n$ *is* K_**-regular.*
(ii) *For all* $* \leqq -2$, $K_*(A) = K_*(A_{\hat{I}})$
(iii) $K_{-1}(A)$ *maps onto* $K_{-1}(A_{\hat{I}})$.

Remarks. Excellent rings include rings which are finitely generated over a field or over $\mathbf{Z}$. If A is an excellent ring, $f \in A$, and J is an ideal of A, then $A_{\hat{J}}[f^{-1}]$ is regular whenever $A[f^{-1}]$ is. This is the crucial property of excellent rings used in the proposition, and follows from [EGA] IV (7.8.3) (v). I am indebted to S.C. Geller and J. Lipman for this reference. In §2 we will show that if A is "Nagata's bad ring", the ring $A_{\hat{x}}[x^{-1}]$ is very far from regular, even though $A[x^{-1}]$ is Dedekind; of course A is not an excellent ring.

Result (i) is almost certainly known by Vorst, and is stated when A has isolated singularities in his thesis [V] (and in the attached *Stellingen*).

Proof. We first show that I can be generated by nonzerodivisors. Let J be the ideal generated by all nonzero divisors of I, and $\{p_i\}$ the maximal associated primes of zero. If $x \in I - J$, then (by reindexing) $x \in p_1 \cap \dots \cap p_r$, $x \notin p_{r+1} \cup \dots \cup p_n$. Choose $y \in J \cap p_{r+1} \cap \dots \cap p_n - (p_1 \cup \dots \cup p_r)$; such an element always exists since no p_i $(i = 1, \dots, r)$ contains any of the ideals in $\{J, p_{r+1}, \dots, p_n\}$, and so cannot contain their intersection. Since $x + y$ lies in no p_i, it is a nonzero divisor contained in I. Thus $x + y \in J$, whence the contradiction $x \in J$, showing that $I = J$.

We now prove the theorem by induction on the number of nonzerodivisor generators of $I = (x, \dots, y, z)A$. Since $A_{\hat{I}} = (A_{\hat{J}})_{\hat{z}}$, the map $A_{\hat{J}} \to A_{\hat{I}}$ is an analytic isomorphism along $\{z^n\}$. By the above remarks, and the hypothesis that $A[z^{-1}]$ is regular, both $A_{\hat{I}}[z^{-1}]$ and $A_{\hat{J}}[z^{-1}]$ are regular. Result (i) now follows from Corollary 1.4. Theorem 1.3 implies (ii) and (iii) since for regular rings $K_*=0$ whenever $*<0$. Done.

As another application, we sketch the derivation of some known formulas in the K-theory of group rings. Let G be a finite group of exponent dividing n and take $S=\{n^i\}$. The map from $\mathbf{Z}[G]$ to $\Pi\{\mathbf{Z}_p[G]: p|n\}$ is an analytic isomorphism along S, where by $\mathbf{Z}_p$ we mean the p-adic completion of $\mathbf{Z}$. Define the idele groups of $K_*(\mathbf{Q}[G])$ relative to $\mathbf{Z}$, n as in [R] and [C]:

$$JK_*(\mathbf{Q}[G])=\{(\alpha_p)\in\Pi K_*(\mathbf{Q}_p[G]): \alpha_p\in\operatorname{im} K_*(\mathbf{Z}_p[G]) \text{ a.e.}\}.$$

Proposition 1.7. (see Reiner [R], Carter [C]).

(i) *$N^i K_j\mathbf{Z}[G])\cong\Pi\{N^i K_j(\mathbf{Z}_p[G]): p|n\}$ for $i\geqq 1$ or $j\leqq -2$.*

(ii) *If $\tilde{K}_0(\mathbf{Z}[G])$ denotes the kernel of $K_0(\mathbf{Z}[G])\to K_0(\mathbf{Q}[G])$, then*

$$\tilde{K}_0(\mathbf{Z}[G])\cong JK_1(\mathbf{Q}[G])/\operatorname{im} K_1(\mathbf{Q}[G])\operatorname{im}\Pi K_1(\mathbf{Z}_p[G]).$$

(iii) *$\tilde{K}_0(\mathbf{Z}[G])$ maps onto $\tilde{K}_0\left(\mathbf{Z}\left[\frac{1}{n}\right][G]\right)$, hence onto $\operatorname{Pic}(R_n)$. Here R_n is the integral closure of $\mathbf{Z}\left[\frac{1}{n}\right]$ in the center of $\mathbf{Q}[G]$.*

(iv) *The following is a fin. gen. free resolution of $K_{-1}(\mathbf{Z}[G])$:*

$$0\to\mathbf{Z}\to K_0(\mathbf{Q}[G])\oplus\prod_{p|n}K_0(\mathbf{F}_p[G])\to\prod_{p|n}K_0(\mathbf{Q}_p[G])\to K_{-1}(\mathbf{Z}[G])\to 0.$$

Proof. It is well known [S] that $\mathbf{Z}\left[\frac{1}{n}\right][G]$ and $\mathbf{Q}_p[G]$ are regular, that the image of $K_0(\mathbf{Z}[G])$ in $\prod K_0(\mathbf{Z}_p[G])$ is $\mathbf{Z}$, that $K_0\left(\mathbf{Z}\left[\frac{1}{n}\right][G]\right)=K_0(\mathbf{Q}[G])\oplus\tilde{K}_0\left(\mathbf{Z}\left[\frac{1}{n}\right][G]\right)$, and that $\tilde{K}_0\left(\mathbf{Z}\left[\frac{1}{n}\right][G]\right)$ is a torsion group mapping onto $\operatorname{Pic}(R_n)$. Moreover, the groups $K_0(\mathbf{Z}_p[G])=K_0(\mathbf{F}_p[G])$ and $K_0(\mathbf{Q}[G])$ are torsion-free (cf. [B], p. 449). The long exact sequence of Theorem 1.3 for N^iK_j immediately yields (i) and the exact sequences

$$K_1\left(\mathbf{Z}\left[\frac{1}{n}\right][G]\right)\oplus\prod_{p|n}K_1(\mathbf{Z}_p[G])\to\prod_{p|n}K_1(\mathbf{Q}_p[G])\to\tilde{K}_0(\mathbf{Z}[G])$$

$$\to\tilde{K}_0\left(\mathbf{Z}\left[\frac{1}{n}\right][G]\right)\to 0,\ 0\to\mathbf{Z}\to K_0(\mathbf{Q}[G])\oplus\prod_{p|n}K_0(\mathbf{F}_p[G])\to\prod_{p|n}K_0(\mathbf{Q}_p[G])$$

$$\to K_{-1}(\mathbf{Z}[G])\to\prod_{p|n}K_{-1}(\mathbf{Z}_p[G])\to 0.$$

From the first sequence we deduce (iii). Since $\tilde{K}_0\left(\mathbf{Z}\left[\frac{1}{n}\right][G]\right)=0$ for large n, we obtain (ii) by passing to the limit over n. If we can show that

$K_{-1}(\mathbf{Z}_p[G])=0$ for all $p|n$, the second sequence will imply (iv). For this we quote the argument of [C]: a maximal $\mathbf{Z}_p$-order Γ containing $\mathbf{Z}_p[G]$ is p-adically complete, so that $K_0(\Gamma)\to K_0(\Gamma/n\Gamma)$ is onto. From the Mayer-Vietoris of the conductor square (c.f. [B], p. 677), $K_{-1}(\mathbf{Z}_p[G])$ is a summand of $K_{-1}(\Gamma)$ $\oplus K_{-1}(\mathbf{Z}[\mathbf{G}]/(n))$, which is zero.

Remark. Carter has completely computed $K_{-1}(\mathbf{Z}[G])$ in [C2]: it is $\mathbf{Z}^r\oplus(\mathbf{Z}/2)^s$, where r and s are integers determined by the representations of G. He has also shown in [C] that $N^i K_j(\mathbf{Z}[G])=0$ if $j\leqq -2$.

Remark. The argument of (iii) can be applied to [B] (1.10), p. 538 to show that $K_0\mathbf{Z}[G]$ maps onto Pic(R_0), where R_0 is the closure of $\mathbf{Z}$ in $\mathbf{Q}[G]$. Note that Pic(R_n) is a quotient of Pic(R_0).

§2. Nagata's Bad Ring

By the phrase "*Nagata's bad ring*" we mean Nagata's Example 6 in the appendix "Examples of bad noetherian rings" of [N]. Here is a description of the ring, which will be denoted A. Choose a field K of characteristic $p\neq 0$ with an infinite set $\{b_1, c_1, b_2, c_2, \ldots\}$ of p-independent elements. Then $R=K^p[[x,y]][K]$ is a regular local 2-dimensional ring whose completion (at the maximal ideal) is $\hat{R}=K[[x,y]]$. Set $d=\Sigma(b_i x^i+c_i y^i)$, $e=d^p\in R$, and define $A=R[d]=R[z]/(z^p-e)$. Then A, the bad ring, is a normal local 2-dimensional ring which is analytically ramified; its completion is $\hat{A}=\hat{R}[z]/(z-d)^p$.

In this section we will study this bad ring. The main property of interest will be that $NK_0(A)\neq 0$, i.e., there is a projective $A[t]$-module which is not even stably free. This answers problem IV of [B], also raised in [MP], and on p. 66 of [L]. Since A is normal we have $N\,\mathrm{Pic}(A)=0$; these projective $A[t]$-modules are indecomposable of rank 2.

We will also show that $K_{-1}(A[t])$ and $NK_{-1}(A)$ are nonzero, so that there are even rank 2 projective $A[t,s,s^{-1}]$-modules which are not even stably a direct sum of projective modules extended from $A[s,s^{-1}]$, $A[t,s]$, and $A[t,s^{-1}]$. However, $N^iK_j(A)=0$ if $j\leqq -2$, so that every projective $A[t_1,\ldots,t_m,s_1,s_1^{-1},\ldots,s_n,s_n^{-1}]$-module $(n\geqq 2)$ is stably a direct sum of projective modules extended from the subrings $A[t_1,\ldots,t_m,s_1,\ldots,s_n][s_i^{-1}]$ and from the variants of these rings obtained by replacing a subset of the s_j's by their inverses s_j^{-1}.

In §3 we will give explicit descriptions of these nonextended projective modules, and give some geometric examples of normal rings with $NK_0\neq 0$.

Theorem 2.1 (Bourbaki). *If A is a Krull domain, then* $\mathrm{Pic}(A)=\mathrm{Pic}(A[t])=\mathrm{Pic}(A[t,t^{-1}])$, *and* $C(A)=C(A[t])=C(A[t,t^{-1}])$.

Proof. Since Pic(A) is a subgroup of the class group $C(A)$, the first statement follows from the second. $C(A)\cong C(A[t])$ is wellknown (cf. [B], p. 147 and [Sam], p. 22). Now $t\in A[t]$ is a prime element, so by Nagata's Theorem (on p. 21 of [Sam]) we have $C(A[t])\cong C(A[t,t^{-1}])$.

Corollary 2.2. $\mathrm{Pic}(A)=\mathrm{Pic}(A[t])=\mathrm{Pic}(A[t,t^{-1}])=0$ *for Nagata's bad ring A.*

Proof. As A is local, all projective A-modules are free.

Remark. A is not a UFD. In fact, if $p\neq 2$ and we define $\beta_i, \gamma_i (i=1,2,\ldots)$ by $\beta_1 = \gamma_1 = 1$, $(\Sigma \beta_i x^i)^2 - (\Sigma \gamma_i y^i)^2 = w = d - x(b_1 + b_2 x - x) - y(c_1 + c_2 y + y)$, then we have

$$[\Sigma(\beta_i x^i + \gamma_i y^i)^p] \cdot [\Sigma(\beta_i x^i - \gamma_i y^i)^p] = w^p.$$

The bracketed terms are irreducible in A since there is no unit u of $K[[x,y]]$ with $u\Sigma(\beta_i x^i + \gamma_i y^i) \in A$.

To understand the K-theory of A, we will use the analytic isomorphism $A \to A_{\hat{y}}$. Being normal and 1-dimensional, the ring $A[y^{-1}]$ is regular. This transfers interest to the rings $A_{\hat{y}}$ and $A_{\hat{y}}[y^{-1}]$. Note that $A_{\hat{y}}[y^{-1}]$ is not regular, reflecting the lack of excellence in the ring A (cf. Proposition 1.6). In fact, if we set $B = A/yA$ we have:

Proposition 2.3. *We have $A_{\hat{y}} = B[[y]]$. This is a 2-dimensional noetherian local domain which does not have finite normalization; $A_{\hat{y}}[y^{-1}] = B((y))$ does not have finite normalization either.*

Proof. That $A_{\hat{y}} = \varprojlim A/y^n A$ is $B[[y]]$ is clear from the definition. The ring B is the ring of Nagata's (E3.2), on p. 206 of [N]: B is a domain whose normalization $\bar{B}$ is not a finite B-module. The integral extensions $B[[y]][\bar{B}]$ and $B((y))[\bar{B}]$ of $\bar{B}[[y]]$ and $\bar{B}((y))$ are not finite, yet have the same quotient field.

We will need the following useful result.

Proposition 2.4. *Suppose R is a ring, x a nonzerodivisor in R such that $R[x^{-1}]$ is regular. Suppose also that there is an analytic isomorphism $R \to C[[x]]$ along $\{x^n\}$ with $C = R/xR$. Then there are isomorphisms*

(a) $N^i K_*(R) \cong N^i K_*(C) \oplus N^{i+1} K_{*+1}(C), i \geqq 1,$ all $*$

(b) $K_*(R) \cong K_*(C) \oplus NK_{*+1}(C), * \leqq -2.$

Moreover, $N^i K_*(C) \to N^i K_*(R)$ is the transfer map for all $i \geqq 0$.

Proof. The sequence of Theorem 1.3 and Corollary 1.4 is

$$\ldots N^i K_*(C[[x]]) \to N^i K_*(C((x))) \to N^i K_{*-1}(R) \ldots,$$

where either $* < 0$ or $i \geqq 1$. By Proposition 1.5 this sequence splits, yielding parts (*a*), (*b*). Finally, the map $N^i K_*(C) \to N^i K_*(R)$ is obtained by chasing the diagram

$$\begin{array}{ccc} & N^i K_{*+1}(C((x))) & \\ & \downarrow\uparrow & \\ N^i K_*(C) \rightleftarrows & N^i K_* \mathbf{H}_x(C[[x]]) & = N^i K_* \mathbf{H}_x(R) \\ \downarrow & & \downarrow \\ 0 & & N^i K_*(R). \end{array}$$

It is clear that the map is induced from the restriction of scalars $\mathbf{H}(C) \to \mathbf{H}(R)$, and hence is the transfer map.

Remark 2.5. The sequence of Theorem 1.3 also yields a partial description of $K_{-1}(R)$: there is an exact sequence

$$K_0(R) \to K_0(R[x^{-1}]) \to NK_0(C) \oplus K_{-1}(C) \to K_{-1}(R) \to 0.$$

Unfortunately, it seems difficult to determine the left-hand maps in general.

Presaging the main theorem, we analyze the K-theory of $B = A/yA$. In [N] it is shown that $B/xB = K[d]/(d^p = 0) = K[\varepsilon]$ and that $B_{\hat{x}}$ is $K[[x]][\varepsilon] = K[\varepsilon][[x]]$.

Theorem 2.6. *$K_*(B) = N^i K_*(B) = 0$ for $* < 0$, but*

$$NK_0(B) = N^2 U(K[\varepsilon]) \neq 0.$$

More generally, for $i \geqq 1$ and all integers $*$ we have

$$N^i K_*(B) = N^i K_*(K[\varepsilon]) \oplus N^{i+1} K_{*+1}(K[\varepsilon]),$$

where $K[\varepsilon] = B/xB$, $\varepsilon^p = 0$, and the map $N^i K_*(K[\varepsilon]) \to N^i K_*(B)$ is the transfer map.

Proof. We invoke Proposition 2.4 and Remark 2.5, utilizing the analytic isomorphism $B \to K[\varepsilon][[x]]$ along $\{x^n\}$. The theorem now follows from the fact that (for $* < 0$ and $i \geqq 1$) $N^i K_1(K[\varepsilon]) = N^i U(K[\varepsilon])$, $N^i K_0(K[\varepsilon]) = N^i K_*(K[\varepsilon]) = K_*(K[\varepsilon]) = 0$, which follows for example from [B] XII (10.1), p. 685.

Remark. Since B is local, $K_0(B) = \mathbf{Z}$. The ring $B[x^{-1}]$ is a subfield of $K((x))$, and we also have an exact sequence

$$K_2(B[x^{-1}]) \to NK_2(K[\varepsilon]) \oplus K_1(K[\varepsilon]) \to SK_1(B) \to 0.$$

From this one can also show that $SK_1(B) \neq 0$.

Theorem 2.7. *Let A denote Nagata's bad ring, $B = A/yA$. Then A is a 2-dimensional local normal domain satisfying*

(i) $NK_0(A) = N^2 K_1(B) \oplus NK_0(B) \neq 0$,

(ii) $NK_{-1}(A) = N^2 K_0(B) = N^3 U(K[\varepsilon]) \neq 0$,

(iii) $N^i K_j(A) = 0$ *for* $j \leqq -2$,

(iv) $K_j(A) = 0$ *for* $j \leqq -2$,

(v) *There is an exact sequence*

$$0 \to \operatorname{Pic}(A[y^{-1}]) \to NK_0(B) \to K_{-1}(A) \to 0,$$

(vi) $N^i K_j(A) = N^i K_j(B) \oplus N^{i+1} K_{j+1}(B)$ *for all* $i \geqq 1$, *and the inclusion* $N^i K_j(B) \to N^i K_j(A)$ *is the transfer map.*

Proof. There is an analytic isomorphism $A \to B[[y]]$ along $\{y^n\}$ with $A[y^{-1}]$ regular. The conjunction of Proposition 2.4 and Theorem 2.6 suffice to prove all but part (v), which follows from Remark 2.5.

Remark. In an earlier version of this paper, I used the analytic isomorphism $A_{\mathfrak{y}} \to \hat{A} = K[[x, y]][\varepsilon]$ to produce isomorphisms

$$NK_0(A) = NK_2(\hat{A}[x^{-1}, y^{-1}])/\mathrm{im}(NK_2(\hat{A}[x^{-1}]) \oplus NK_2(\hat{A}[y^{-1}]))$$
$$NK_{-1}(A) = NU(\hat{A}[x^{-1}, y^{-1}])/NU(\hat{A}[x^{-1}])\,NU(\hat{A}[y^{-1}]).$$

I am extremely grateful to R.G. Swan for showing me how to simplify the proof. The diligent reader will be able to prove these identities, armed with (1.5), (2.6), and (2.7). I do not know if $K_{-1}(A) \neq 0$, but I suspect so.

We conclude this section with an elementary result for rings of small Krull dimension. Note that the first part of Theorem 2.6 follows from part (i). Proposition 2.8 was proven on p. 688 of [B] for the case in which the ring B has finite normalization.

Proposition 2.8. *Let B be a commutative noetherian ring.*
(o) *If* $\dim(B) = 0$, *then B is K_*-regular for $* \leqq 0$ and $K_*(B) = 0$ for $* < 0$.*
(i) *If* $\dim(B) = 1$, *then B is K_*-regular for $* \leqq -1$ and $K_*(B) = 0$ for $* < -1$.*
(ii) *If* $\dim(B) = 2$, *let $\bar{B}$ denote the normalization of B. For $* \leqq -2$, B is K_*-regular iff $\bar{B}$ is. For $* < -2$, $K_*(B) = K_*(\bar{B})$.*

Proof. As $N^i K_j(B) = N^i K_j(B/\mathrm{nil}(B))$ for $j \leqq 0$, we can replace B by $B/\mathrm{nil}(B)$ to assume that B is reduced. Case (o) is XII (10.1) on p. 685 of [B]. Now assume $\dim(B) > 0$. The normalization $\bar{B}$ of B is the direct limit of finite integral extensions C of B, so the groups $N^i K_j(\bar{B})$ are the direct limit of the $N^i K_j(C)$. For each C there is a conductor square

$$\begin{array}{ccc} B & \longrightarrow & C \\ \downarrow & & \downarrow \\ B/I & \longrightarrow & C/I \end{array}$$

which gives rise to a long exact sequence for $j \leqq 1$ by [B], XII (8.3), p. 677:

$$\ldots N^i K_j(B) \to N^i K_j(C) \oplus N^i K_j(B/I) \to N^i K_j(C/I) \ldots.$$

Since $\dim(B/I)$, $\dim(C/I)$ are smaller than $d = \dim(B)$, we can use induction on d to see that (for $d = 1$ or 2) $N^i K_j(B) = N^i K_j(C)$ if either $j < -d$ or $j = -d$ and $i > 0$. Passing to the direct limit, we obtain $N^i K_j(B) = N^i K_j(\bar{B})$ for the same range of i and j. This proves that (i) implies (ii). Finally, since $\bar{B}$ is regular when $\dim(B) = 1$, we have $N^i K_j(B) = 0$ if either $i > 0$ or $j < 0$, proving (i).

Questions 2.9. If A is a commutative noetherian ring of Krull dimension d, is A K_*-regular for $* \leqq -d$? Is $K_*(A) = 0$ for $* < -d$?

When $d = 0, 1$ this follows from Proposition 2.6. When $d = 2$ the normalization of A is again noetherian (see p. 120 of [N]), so the question reduces to the case in which A is a normal domain. It is possible that if A is normal we also have $K_{-d}(A) = 0$.

§3. Geometric Examples

In this section we give examples of normal rings (in fact UFD's), finitely generated over any field k, with $NK_0 \neq 0$, $K_{-1} \neq 0$. Much of this section is due to Swan; I am grateful for his permission to include his results here.

The arguments of §2 fail for rings of finite type over fields, calling for a somewhat different approach. This is because such rings are universally Japanese: if C is a local domain which is universally Japanese then $\hat{C}$ has no nilpotent elements (see [Mat], p. 236, or [EGA] IV.) Nagata's bad ring A is not universally Japanese since $A/yA = B$ does not have finite normalization.

The following argument of Swan shows that for every field k of characteristic $p \neq 0$ there are normal rings finitely generated over k with $NK_0, K_{-1} \neq 0$. Given k, choose $K \supseteq k$ with $[K:K^p] = \infty$ and construct Nagata's bad ring A. The ring A is the direct colimit of finitely generated normal domains over k, and NK_0 et. al. preserve direct colimits, so some of these domains must have $NK_0 \neq 0$, etc.

In order to get our hands on explicit examples by this descent approach, we must analyze $NK_0(B) = N^2 U(K[\varepsilon])$ and the transfer map $NK_0(B) \to NK_0(A)$. This descent culminates in Theorem 3.3. We then introduce a simpler approach, due to Swan, climaxing in the examples of Theorem 3.6. We conclude the section by giving a descent argument for K_{-1} and NK_{-1}. We give an indecomposable rank 2 projective $A_0[t, s, s^{-1}]$-module in Corollary 3.10 which represents a non-zero element of $NK_{-1}(A_0)$, where A_0 is the affine ring of a factorial hypersurface in $\mathbf{A}_k^5$.

We note that, since the generators of the projective modules are of degree at most 3 in t, the product of the A_0 over all $p \neq 0$ gives examples in characteristic 0 as well.

Preparatory to analyzing $NK_0(B)$, we give a graded description of $N^2 U(K[\varepsilon])$. We will think of $N^2 U(K[\varepsilon])$ as a subgroup of $U(K[\varepsilon][x^{-1}, t])$, which is a subgroup of $K_1(K((x))[\varepsilon, t])$ by Proposition 1.5. With this convention, the elements of $N^2 U(K[\varepsilon])$ are those units in $K[x^{-1}, \varepsilon, t]$ congruent to 1 modulo $(x^{-1} t\varepsilon)$. We can grade these units by the degree in x^{-1}: each element of $N^2 U(K[\varepsilon])$ is of the form $x^{-n}(x^n + t\varepsilon g)$ for a unique pair $(n, t\varepsilon g)$ in $\mathbf{N} \times (t\varepsilon K[x, \varepsilon, t] - xt\varepsilon K[x, \varepsilon, t])$. The set $N^2 U(K[\varepsilon])$ is in $1-1$ correspondence with those pairs for which $t\varepsilon g$ has x-degree less than n.

With this convention, the map $N^2 U(K[\varepsilon]) \to NK_0(B)$ is induced by the composition

$$K_1(K((x))[\varepsilon, t]) \xrightarrow{\partial} K_0 \mathbf{H}_x(K[[x]][\varepsilon, t]) = K_0 \mathbf{H}_x(B[t]) \to K_0(B[t]).$$

From the description of ∂ on p. 493 of [B], $\partial(x^{-n}) = -n[K[\varepsilon, t]]$ and $\partial(x^n + t\varepsilon g) = K[[x]][\varepsilon, t]/(x^n + t\varepsilon g) = B[t]/I$. It is clear that I is an invertible $B[t]$-ideal and that $[I]-1$ represents the image of $1 + x^{-n} t\varepsilon g$ in $NK_0(B)$.

In order to better describe I, we need to review the analytic isomorphism $B[t] \to K[[x]][\varepsilon, t]$. The sequence of elements

$$d_N = d - \sum_{i=1}^{N-1} b_i x^i = \sum_{i=N}^{\infty} b_i x_i$$

of B converges to ε in $B_{\hat{x}}=K[[x]][\varepsilon]$. When $N\geqq np$, the element d_N maps to ε in the ring $B/(x^{np})=K[x,\varepsilon]/(x^{np},\varepsilon^p)$. As $(x^n+t\varepsilon g)^p=x^{np}$ in $K[[x]][\varepsilon,t]$, we have

$$B[t]/I=B[t]/(x^{np},x^n+td_Nf)=K[x,\varepsilon,t]/(x^{np},\varepsilon^p,x^n+t\varepsilon g)$$

for some f in $B[t]$. (d_Nf is unique up to a multiple of x^{np}.) We can find an f in $K[x,d_N,t]\subset B[t]$ of x-degree less than n and d_N-degree less than $p-1$ in the following way. There are unique $g_i\in K[x,t]$ $(i=1,\ldots,p-1)$ of x-degree less than n for which $t\varepsilon g=\Sigma t\varepsilon^i g_i$ in $K[x,\varepsilon,t]$; we set

$$f=g_1+d_N g_2+\ldots+d_N^{p-2}g_{p-1}.$$

Adding $d_N^{p-1}g_p$ to f does not change d_Nf modulo x^{np}. Since $B[t]/(x^{np},d_N^{p-1})$ has as $K[t]$-basis $\{d_N^i x^j: 0\leqq i<p-1,\ 0\leqq j<np\}$, it makes sense to say that f has "x-degree less than $n \bmod (x^{np},d_N^{p-1})$": every $d_N^i x^j$ occurring in $f \bmod (x^{np},d_N^{p-1})$ has $j<n$. In summary:

Lemma 3.1. *There is a $1-1$ correspondence between elements of $NK_0(B)$ and pairs (n,f), where $n\geqq 0$ and f is an element of $B[t]/(x^{np},d_N^{p-1})$ of x-degree less than $n \bmod (x^{np},d_N^{p-1})$. Here $N\geqq np$ is arbitrary. The pair (n,f) corresponds to the ideal $I=(x^{np},x^n+td_Nf)$ of $B[t]$.*

Swan has pointed out that it is possible to take a much more naive approach in studying $NK_0(B)$. The fact that $NK_0(B)\neq 0$ is a reflection of the lack of seminormality of B: the nonextended projective $B[t]$-modules are of the form "free$\oplus$ideal". For convenience, we cite Theorem 1 of [BC]:

Theorem 3.2. *Let D be an integral domain. The following are equivalent:*
(a) *D is seminormal.*
(b) $N\,\mathrm{Pic}(D)=0$.
(c) *For every q in the quotient field of D, q^2 and $q^3\in D$ implies that $q\in D$.*

If D is not seminormal, and $q\notin D$ is such that $q^2,q^3\in D$, then the "Schanuel module" $I=(q^2,1+tq)$ is a rank 1 projective $B[t]$-module not extended from B.

It is easy to write down a q in the quotient field of B with $q^2,q^3\in B$, $q\notin B$. For example, $x^{-i}d_N^j$ is such an element if $3i\leqq Np$ and $p/2\leqq j<p$. The corresponding Schanuel modules are

$$x^iI=(x^{-i}d_N^{2j},x^i+td_N^j).$$

Either way, we suppose given a projective $B[t]$-ideal I. The transfer of the element $[I]-1$ of $K_0(B[t])$ is the element $[P]-2$ of $K_0(A[t])$, where P is the projective $A[t]$-module defined by exactness of the sequence

$$0\to P\to A[t]^2\to I\to 0.$$

Since $NK_0(B)$ is a summand of $NK_0(A)$ under transfer, whenever $0\neq[I]-1\in NK_0(B)$ the class $[P]-2$ is a nonzero element of $NK_0(A)$. We can now descend to a finitely generated example.

Theorem 3.3. *Let k be any field of characteristic $p\neq 0, N\geqq p$, and set $A_0=k[u,v,x,y,z]/(z^p=ux^{Np}+vy^{Np})$. Then A_0 is a 4-dimensional (normal) UFD with $NK_0(A_0)\neq 0$.*

Proof. A_0 is a domain, and y is a prime element with $A[y^{-1}]=k[u,x,y,y^{-1},z]$, a UFD. By Nagata's theorem (see p. 21 of [Sam]), A is a UFD. There is a natural map $A_0 \to A$ sending z to $d_N = d - \Sigma(b_i x^i + c_i y^i)$, u to $(b_N + \Sigma b_{N+i} x^i)^p$, and v to $(c_N + \Sigma c_{N+i} y^i)^p$. This map sends $B_0 = A_0/yA_0$ to B and sends the $B_0[t]$-ideal

$$I_\pm = (x^p, x \pm t z^{p-1})$$

to the invertible $B[t]$-ideal associated to the pair $(1, \pm d_N^{p-2})$ under the correspondence of Lemma 3.1. It is easy to see that $I_+ I_- = x^2 B_0[t]$, so the $I_\pm$ are projective $B_0[t]$-modules. Define the projective $A_0[t]$-module P to be the kernel of the obvious surjection $A_0[t]^2 \to I_+$; we will show that $[P]-2$ is a nonzero element of $NK_0(A_0)$. If we apply $A \otimes_{A_0}$ we obtain the exact sequence

$$0 \to \mathrm{Tor}_1^{A_0}(A, I_+) \to A \otimes P \to A[t]^2 \to A \otimes I_+ \to 0.$$

Since I_+ is a y-torsion A_0-module, so is $\mathrm{Tor}(A, I_+)$. As $\mathrm{Tor}(A, I_+)$ is a submodule of the projective $A[t]$-module $A \otimes P$ it must be zero. This shows that $[A \otimes P]-2$ is the nonzero element of $NK_0(A)$ corresponding to $[A \otimes I_+]-1 \in NK_0(B)$. Thus $[P]-2$ must be nonzero as it maps to a nonzero element.

Remark. It is relatively easy to show that $P \subseteq A_0[t]^2$ is generated by the 4 elements $(y,0)$, $(0,y)$, $(x+t z^{p-1}, -x^p)$, and

$$(t^2 u z^{p-2} x^{Np-2} - 1, x^{p-2}(x - t z^{p-1})).$$

Since the degree of t is at most 2 in the relevant data, we can take the product of the A's over all p to get an example in characteristic zero.

In order to motivate a simpler approach, due to Swan, we set $C_0 = A_0/(v,y) = k[u,x,z]/(z^p - u x^{Np})$. Since $A_0/(y) = B_0 = C_0[v]$, we have a commutative diagram:

$$\begin{array}{ccccc} A_0 & \xrightarrow{v=0} & C_0[y] & & \\ {\scriptstyle y=0}\downarrow & & \downarrow & \searrow & \\ & & & & C_0[[y]]. \\ B_0 = C_0[v] & \xrightarrow{v=0} & C_0 & \swarrow & \end{array}$$

Note that $NK_0(B_0) \to NK_0(C_0)$ is a split surjection, and that the $[I_\pm]-1 \in NK_0(B_0)$ map to nonzero elements. Since $A_0[y^{-1}]$ is regular, we could have also proven Theorem 3.3 by invoking the following result.

Theorem 3.4 (Swan). *Let ϕ: $A \to C[[y]]$ be a ring map, and let s be a nonzero-divisor in A for which $\phi(s)=y$. Assume that $A[s^{-1}]$ is regular and that (for some j) the induced map*

$$NK_j(A/sA) \to NK_j(C)$$

is nonzero. Then $NK_j(A) \neq 0$, and the transfer map $NK_j(A/sA) \to NK_j(A)$ is nonzero.

Similarly, if $j < -1$ (or if $j = -1$ and $K_0(A) \to K_0(A[s^{-1}])$ is onto), then $K_j(A) \neq 0$ whenever the induced map $K_j(A/sA) \to K_j(C)$ is nonzero.

Remark. In applications we will actually map A to $C[y]$.

Proof. We will only prove the NK_j part, as the proof of the K_j part is entirely identical. Since every third term in the relevant localization sequence is zero, we have $NK_j\mathbf{H}_s(A)\cong NK_j(A)$. By Theorem 1.1, ϕ induces a homomorphism from $NK_j\mathbf{H}_s(A)$ to $NK_j\mathbf{H}_y(C[[y]])$. By Proposition 1.5, the latter group is $NK_j(C)\oplus N^2K_{j+1}(C)$. The theorem now follows from commutativity of the diagram

$$\begin{array}{ccc} NK_j(A/sA) & \longrightarrow & NK_j(\mathbf{H}_s(A))\cong NK_j(A) \\ \downarrow & & \downarrow \\ NK_j(C) & \rightleftarrows & NK_j(\mathbf{H}_y(C[[y]]). \end{array}$$

Corollary 3.5. *Let k be a field of any characteristic, and choose integers $a,b\geqq 2$, $c\geqq 1$ with $a+b\geqq 5$. The ring $A=k[u,v,x,y,z]/(z^a=ux^b+vy^c)$ is a 4-dimensional (normal) UFD with $NK_0(A)\neq 0$.*

Proof. (cf. proof of Theorem 3.3). A is a UFD since $A[y^{-1}]$ is. $C=A/(v,y)=k[u,x,z]/(z^a=ux^b)$ is not seminormal by Theorem 3.2: z^{a-1}/x is not in C but its square and cube are. Applying Theorem 3.4 to the map "$v=0$": $A\to C[y]$ finishes the proof.

The following family of examples is due to Swan. Let $C=k[x_1,\dots,x_n]/(f)$ be a domain and define $A=k[x_1,\dots,x_n,s,t]/(f-st)$. Send $A\to C[y]$, by $\phi(x_i)=x_i$, $\phi(s)=y$, $\phi(t)=0$. The induced map from $A/s=C[t]$ to C is the split surjection "$t=0$". On the other hand, $A[s^{-1}]=k[x_1,\dots,x_n,s,s^{-1},t]$ is regular and $K_0(A)\to K_0(A[s^{-1}])$ is onto. By Swan's Theorem 3.4, $NK_j(A)\neq 0$ whenever $NK_j(C)\neq 0$, and for $j<0$ $K_j(A)\neq 0$ whenever $K_j(C)\neq 0$. As s is prime and $A[s^{-1}]$ is a UFD, Nagata's Theorem (in [Sam]) implies that A is a UFD. We record this as:

Theorem 3.6 (Swan). *Let k be a field, $C=k[x_1,\dots,x_n]/(f)$ a domain. Then $A=k[x_1,\dots,x_n,s,t]/(f-st)$ is a UFD for which*

(a) *$NK_j(A)\neq 0$ if $NK_j(C)\neq 0$, all integers j.*
(b) *$K_j(A)\neq 0$ if $K_j(C)\neq 0$, all integers $j<0$.*

In particular, $NK_0\neq 0$ for $A=k[x,y,s,t]/(x^3-y^2-st)$, and $K_{-1}\neq 0$ for $A=k[x,y,s,t]/(x^2+x^3-st)$.

Remark. If $C=k[x_1,\dots,x_n]/(f)$ is not a domain, A is still a normal domain, since the singular locus is contained in the codimension 2 set $s=t=0$. In particular, $A=k[x_0,\dots,x_n,s,t]/(x_0\dots x_n(1-\Sigma x_i)-st)$ is an $(n+2)$-dimensional normal domain with $K_{-n}(A)\neq 0$ since in this case $K_{-n}(C)=\mathbf{Z}$ by [DW].

We conclude this section with a descent argument for K_{-1} and NK_{-1}, similar in spirit to the analysis of NK_0. By Theorem 2.7, there are surjections $NK_0(B)\to K_{-1}(A)$, $N^2K_0(B)\to NK_{-1}(A)$, where A is Nagata's bad ring. We will think of $K_{-1}(A)$ as a subgroup of $K_0(A[s,s^{-1}])$ and $NK_{-1}(A)$ as a subgroup of $K_0(A[t,s,s^{-1}])$. The group $NK_0(B)$ might be thought of as a subgroup of $K_0(B[y^{-1}])\subseteq K_0(B((y)))$; we shall think of it as a subgroup of $K_1(B((y))[s,s^{-1}])$ by multiplication by s. With this convention, we have to study the composite

$$K_1(B((y))[s,s^{-1}]) \xrightarrow{\partial} K_0\mathbf{H}_y(B[[y]][s,s^{-1}])$$
$$= K_0\mathbf{H}_y(A[s,s^{-1}]) \to K_0(A[s,s^{-1}]).$$

Similarly, $N^2K_0(B)$ is to be a subgroup of $K_1(B((y))[t,s,s^{-1}])$, and we need to study the analogous composite to $K_0(A[t,s,s^{-1}])$. We first describe the embedding of $NK_0(B[t])$ in $K_1(B((y))[t,s,s^{-1}])$.

Proposition 3.7. *If P is a projective R-module given as the image of the idempotent matrix e in $M_n(R)$, then multiplication by s sends $[P]\in K_0(R)$ to $s\cdot[P]=[I+(s-1)e]$ in $K_1(R[s,s^{-1}])$. In particular, if $\Sigma a_i b_i=1$ for $a_i,b_i\in\mathrm{frac}(R)$, then $I=(b_1,\dots,b_n)$ is the image of the idempotent $e=(a_i b_j)$, so $s\cdot[I]$ is represented by the matrix $I+(s-1)(a_i b_j)$.*

Proof. By [B] XII (7.4), p. 663, we have $s\cdot[P]=[(P[s,s^{-1}],s)]=[(\mathrm{Im}(e)\oplus\mathrm{Ker}(e), s\oplus 1)]=[(R^n,\ se+(1-e))]=[(R^n,\ 1+(s-1)e)]$. The second sentence is easy to verify, given the first sentence.

As an application, we take $b_1=x^p$, $b_2=x+t\,y^{-1}d_N^{p-1}$, $a_1=t^2y^{-2}u\,d_N^{p-2}x^{Np-p-2}$, and $a_2=x^{-2}(x-t\,y^{-1}d_N^{p-1})$ in $B[t,y^{-1}]$, where $u=(d_N/x^N)^p$. This shows that $s\cdot[I_+]$ is represented by the matrix $y^{-3}M$, where

$$M=\begin{pmatrix} y^3+(s-1)t^2yu\,d_N^{p-2}x^{Np-2} & (s-1)t^2u\,d_N^{p-3}x^{Np-p-1}(y\,d_N+t\,u\,x^{Np-1}) \\ (s-1)y^2x^{p-2}(x\,y-t\,d_N^{p-1}) & s\,y^3-(s-1)t^2yu\,d_N^{p-2}x^{Np-2} \end{pmatrix}$$

is a matrix in $M_2(B[[y]][t,s,s^{-1}])$ and $I_+=(x^p,x+t\,y^{-1}z^{p-1})$ is an invertible $B[t,y^{-1}]$-ideal (cf. the proof of Theorem 3.3).

The map ∂ sends $s\cdot([I_+]-1)$ to $[\mathrm{coker}(M)]-3[B[t,s,s^{-1}]]$ in $K_0\mathbf{H}_y(B[[y]][t,s,s^{-1}])=K_0\mathbf{H}_y(A[t,s,s^{-1}])$. Thus the image of $s\cdot([I_+]-1)$ in $K_0(A[t,s,s^{-1}])$ is the transfer of $[\mathrm{coker}(M)]$.

Lemma 3.8. *Set $F=B[[y]][t,s,s^{-1}]^2$. Then the image of $M\in\mathrm{End}(F)$ contains y^6F, and the following sequence is exact:*

$$F/y^6F \xrightarrow{\bar{M}} F/y^6F \longrightarrow \mathrm{coker}(M) \longrightarrow 0.$$

Proof. The matrix $y^{-3}M$ is of the form $se+(1-e)$, where $\mathrm{Im}(e)=I_+$. The inverse of this matrix is $s^{-1}e+(1-e)=y^{-3}N$, where N is the matrix obtained from M by replacing s by s^{-1}. We thus have $y^6\cdot 1_F=y^6(y^{-3}M)(y^{-3}N)=MN$, showing that $y^6F=M(N(F))$. Exactness of the sequence is immediate.

Now $B[[y]][t,s,s^{-1}]/(y^6)\cong A[t,s,s^{-1}]/(y^6)$, so it suffices to lift M to A to get generators for the projective $A[t,s,s^{-1}]$-module P defined by exactness of the sequence

$$0\to P\to A[t,s,s^{-1}]^2\to \mathrm{coker}(M)\to 0.$$

We set $d_N=d-\Sigma(b_ix^i+c_iy^i)$, $u=b_N+\Sigma b_{N+i}x^i$ in A, and obtain:

Proposition 3.9. *The element $[P]-2$ is a nonzero element of the subgroup $NK_{-1}(A)$ of $K_0(A[t,s,s^{-1}])$, where P is the submodule of $A[t,s,s^{-1}]^2$ generated by $(y^6,0)$, $(0,y^6)$, and the columns of the matrix M above.*

Now there is submodule P_0 of $A_0[t,s,s^{-1}]^2$ defined by the corresponding generators, where A_0 is the ring of Theorem 3.3. When $Np \geqq 6$ there is an isomorphism $A_0/(y^6) \cong B_0[y]/(y^6)$. The above proof goes through and shows that P_0 is the kernel of the evident map $A_0[t,s,s^{-1}]^2 \to \operatorname{coker}(M)$.

Corollary 3.10. *P_0 is a projective $A_0[t,s,s^{-1}]$-module representing a nonzero element in $NK_{-1}(A_0) \subseteq K_{-1}(A_0[t])$ whenever $Np \geqq 12$.*

Proof. Since there is an evident surjection $P_0 \otimes A \to P$, it is enough to show P_0 is projective, and for this it suffices to show that coker(M) has homological dimension 1 over $A_0[t,s,s^{-1}]$. We copy the argument on p. 401 of [K]: there is an exact sequence

$$0 \to B_0[y,t,s,s^{-1}]^4 \xrightarrow{M \oplus y^6 N} B_0[y,t,s,s^{-1}]^4 \to \operatorname{coker}(M) \oplus \operatorname{coker}(y^6 N) \to 0.$$

By the identity

$$\begin{pmatrix} M & 0 \\ 0 & y^6 N \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y^6 & -N \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ M & 1 \end{pmatrix} \begin{pmatrix} 1 & -N \\ 0 & y^6 \end{pmatrix}$$

it follows that coker$(M \oplus y^6 N)$, and hence coker(M), has homological dimension 1 over A_0 (we have to use the remark on the bottom of p. 401 of [K] to resolve coker$(M \oplus y^6 N)$), which is an $A_0[t,s,s^{-1}]/(y^{12})$-module).

Remark. If we set $t=1$ in the definition of P_0, we get a projective $A_0[s,s^{-1}]$-module representing that element of $K_{-1}(A_0)$ which is the image of the class of $I = (x^p, x+y^{-1} d_N^{p-1})$ in $NK_0(B_0)$ under the map of Theorem 2.7 (v).

§4. K_2 of Seminormal Curves

In this section we compute $K_2(A)$, where A is the coordinate ring of a seminormal curve over a field whose singularities split completely. In this case we get $K_2(A) = KV_2(A) \oplus V$, where V is a vector space over the ground field:

$$V \cong \bigoplus_{p \in \operatorname{Max}(A)} (A/p)^{\binom{n}{2}}, \qquad n = n(p) = \dim_{A/p}(\bar{A}/p\bar{A}).$$

A special case in point is the node $A = k[x,y]/(y^2 = x^2 + x^3)$, where $KV_2(A) = K_2(k) \oplus K_3(k)$ and $V \cong k$.

The idea is to choose f in the conductor from $\bar{A}$ to A, and use the exact sequences of the analytic isomorphism $A \to A_{\hat{f}}$. The latter ring is a direct sum of complete local seminormal rings C_i, one for each prime p_i of A containing f. Set $K_i = \bar{A}/p_i$. Since A is seminormal, $K_i = \Pi\, k_{ij}$, a product of fields. By §3 of [D] we have $C_i = k_i \oplus t K_i[[t]]$. Therefore, there are analytic isomorphisms from $B_i = k_i \oplus t K_i[t]$ to C_i. Now $K_2(B_i)$ is "known" from the computations of [DK], and we merely have to assemble the pieces to compute $K_2(A)$. For example, it is clear from Corollary 1.4 that for all $n \geqq 1$

$$N^n K_*(A) \cong N^n K_*(A_{\hat{f}}) \cong \prod_i N^n K_*(B_i).$$

Definition 4.1. If R is any ring with unit, $\mathrm{nil}K_*(R)$ will denote the kernel of the map $K_*(R)\to KV_*(R)$.

For $*\leqq 0$ we have $\mathrm{nil}K_*(R)=0$ by definition. The group $\mathrm{nil}K_1(R)$ is the subgroup of $K_1(R)$ generated by unipotent matrices, and is the image of $NK_1(R)\to K_1(R)$. In general, the image of $NK_*(R)\to K_*(R)$ will be a subgroup of $\mathrm{nil}K_*(R)$. We have the following general formula: $\mathrm{nil}K_*(R[t])=\mathrm{nil}K_*(R)\oplus NK_*(R)$ ($*\geqq 1$ only).

Our first task is to compute $K_2(B_i)$ by computing $\mathrm{nil}K_2(B_i)$ and $KV_2(B_i)$, and then to use this to compute $K_2(C_i)$.

Proposition 4.2. *Let $k\to K$ be a map of regular rings, and set $B=k\oplus tK[t]=\{f(t)\in K[t]:f(0)\in k\}$. Then $KV_*(B)=K_*(k)$, and is a summand of $K_*(B)$.* The complementary summand is $\mathrm{nil}K_*(B)=K_*(B,tK[t])$.

Proof. There is a cartesian square

$$\begin{array}{ccc} B & \longrightarrow & K[t] \\ \downarrow & & \downarrow \\ k & \longrightarrow & K. \end{array}$$

This gives rise to a long exact sequence in Karoubi-Villamayor K-theory, since $K[t]\to K$ is a Gl-fibration (see [K-V], appendix 7), from which everything easily follows.

Proposition 4.3 (Vorst). $\mathrm{nil}K_1(B)=\Omega_{K/k}$. *In particular, B is K_1-regular iff $\Omega_{K/k}=0$.*

Proof. $K_1(B)=K_1(k)\oplus K_1(B,I)$, where $I=tK[t]$. Since I is the conductor (from $K[t]$) and $K_1(K[t],I)=0$ the group $\mathrm{nil}K_1(B)=K_1(B,I)$ is the kernel of excision. This can be easily computed, using the method of [GW], as a subquotient of $K_2(K\oplus I/I^2)=K_2(K)\oplus K_2(K\oplus I/I^2,I/I^2)$. It is the cokernel of $K_2(k\oplus I/I^2,I/I^2)\to K_2(K\oplus I/I^2,I/I^2)$, which is $I/I^2\otimes\Omega_{K+I/k+I}=\Omega_{K/k}$ by Lemma 2.1 of [GW]. Finally, B is K_1-regular iff $\mathrm{nil}\,K_1(B)=\mathrm{nil}\,K_1(B[x_1,\dots,x_n])=0$, which holds iff

$$\Omega_{K/k}\otimes_K K[x_1,\dots,x_n]=\Omega_{K[x_1,\dots]/k[x_1,\dots]}=0, \qquad \text{i.e., iff } \Omega_{K/k}=0.$$

Corollary 4.4. *Let $k\to K$ be a map of regular rings, and set $C=k\oplus tK[[t]]$, the t-adic completion of $B=k\oplus tK[t]$. Then*

$$KV_*(C)=K_*(k)\oplus K_*(K[[t]],tK[[t]]).$$

There is an exact sequence

$$0\to K_*(C)/\mathrm{nil}K_*(C)\to KV_*(C)\to \mathrm{nil}K_{*-1}(B)\to \mathrm{nil}K_{*-1}(C)\to 0.$$

In particular, the cokernel of $K_2(C)\to KV_2(C)$ is $\Omega_{K/k}$.

Proof. There is a cartesian square for C analogous to the one used for B (replace $K[t]$ by $K[[t]]$). The computation of $KV_*(C)$ follows easily from this. By Theorem 1.3, there is an exact sequence

$$K_*(k)\oplus K_*(C)\to KV_*(C)\to K_{*-1}(B)\to K_{*-1}(k)\oplus K_{*-1}(C)\to KV_{*-1}(C).$$

The second claim follows from this sequence and (4.2). The last claim follows from this, (4.3), and the fact that nil $K_1(C)=0$ (because $K_1(C)=U(C)\oplus SK_1(k)$, e.g. by [B, p. 450]).

Remark. If K is an inseparable field extension of k, this provides us with a 1-dimensional seminormal, complete local ring for which $K_2 \to KV_2$ is not onto.

Remark. The groups nil$K_2(B)$, nil$K_2(C)$ are the images of the maps $NK_2(B)\to K_2(B)$, $NK_2(C)\to K_2(C)$ in this case. This may be seen from the Gersten-Anderson spectral sequence $E^1_{pq}=K_qR[t_1,\ldots,t_p]\Rightarrow KV_{p+q}R$ (see [G] or [W2] for a description).

The crucial point is that, since $N^pK_q(B)=N^pK_q(C)$, we have $E^2_{pq}(B)=E^2_{pq}(C)$ for $p\geqq 2$. On the other hand, it is easy to compute that $E^2_{p1}(B)=0$ for $p\neq 0$, given Proposition 4.3. Thus $E^2_{21}=0$ for both B and C, which implies that $NK_2\to K_2\to KV_2$ is exact for both B and C. Using the ideas of [W2], we can also show that $E^2_{22}=0$, so that nilK_3 is the image of $NK_3\to K_3$ for both B and C. I suspect, but do not know, that the same is true of the higher nil K_* groups.

We can reduce to the case when K is a domain via the following result.

Proposition 4.5 (Dennis-Krusemeyer). *Suppose $K=\Pi K_j$ $(j=1,\ldots,n)$, and that $k\to K$ is a map of regular rings. Let B, C be as before, and set $B_j=k\oplus tK_j[t]$, $C_j=k\oplus tK_j[[t]]$. Then*

$$\mathrm{nil}K_2(B)=\Pi\,\mathrm{nil}K_2(B_j)\oplus\prod_{i<j}(K_i\otimes_k K_j)$$

$$\mathrm{nil}K_2(C)=\Pi\,\mathrm{nil}K_2(C_j)\oplus\prod_{i<j}(K_i\otimes_k K_j).$$

In particular, if all the K_j equal k, then $K_2(B)=K_2(k)\oplus k^{\binom{n}{2}}$, and $K_2(C)=KV_2(C)\oplus k^{\binom{n}{2}}$.

Proof. Set $I_j=tK_j[t]$, so that $B=k\oplus I_1\oplus\ldots\oplus I_n$ in the notation of [DK]. By (2.10) and (3.1) of [DK] we have

$$K_2(B)=K_2(k)\oplus\Pi K_2(B_j,I_j)\oplus\prod_{i<j}(I_i/I_i^2\otimes I_j/I_j^2),$$

and $K_2(C)$ is given by replacing B_j by C_j. The tensor product is taken over k^e, and it is easy to show that $I_i/I_i^2\otimes I_j/I_j^2\cong K_i\otimes_k K_j$. Since nil$K_2(B_j)=K_2(B_j,I_j)$, the computation of nil$K_2(B)$ follows. Now nil$K_2(C)$ is the kernel of $K_2(C,\oplus I_j)\to K_2(K[[t]],t)=\Pi K_2(K_j[[t]],t)$ by Corollary 4.4, and similarly for nil$K_2(C_j)$. The computation of nil$K_2(C)$ now follows by naturality.

Remark. I do not know what the groups nil$K_2(B_j)$ are. When K_j is a separable field extension of k of degree d and $N=d(d-1)/2$, then k^N is a direct summand of nil$K_2(B_j)$, $K_2(B_j)$, nil$K_2(C_j)$, and $K_2(C_j)$. This may be proven using the following idea of Vorst [V]: choose a galois extension L of k containing K_j, and note that $B_j\otimes L=L\otimes tL^d[t]$. Now nil $K_2(B_j\otimes L)$ is an N-dimensional vector space over L. The trace of the galois action corresponds to the composition nil $K_2(B_j\otimes L)\to$ nil$K_2(B_j)\to$ nil$K_2(B_j\otimes L)$, showing that the image of nil$K_2(B_j)$ in nil$K_2(B_j\otimes L)$ is the invariant k-vector space, which has dimension N.

We return to the computation of $K_2(A)$, where A is a reduced 1-dimensional seminormal ring, finitely generated over a field. Choose f in the conductor from the normalization $\bar{A}$ to A, and set $J=\underline{\mathrm{rad}}(f)=p_1\cap\ldots\cap p_m$, $k_i=A/p_i$. Since A is seminormal, J is also a radical ideal of $\bar{A}$. Thus $\bar{A}/p_i\bar{A}=\Pi K_{ij}$, where $\{K_{ij}\}$ are the residue fields of the primes of $\bar{A}$ over p_i.

By §3 of [D], $A_{\hat{f}}$ is the product over i of the complete local rings $C_i = k_i \oplus t\Pi K_{ij}[[t]]$. Set

$$B_{ij}=k_i\oplus tK_{ij}[t], \qquad C_{ij}=k_i\oplus tK_{ij}[[t]].$$

Since C_i is local, $K_1(C_i)$ consists only of units (see p. 450 of [B]), and so $K_1(A_{\hat{f}}) = KV_1(A_{\hat{f}})$. By Theorem 1.3, there is a long exact sequence ending in

$$\begin{aligned} K_3(A)\to KV_3(A)\oplus \Pi K_3(C_i)\to \Pi KV_3(C_i)\to K_2(A)\\ \to KV_2(A)\oplus \Pi K_2(C_i)\to \Pi KV_2(C_i)\to \mathrm{nil}K_1(A)\to 0. \end{aligned} \tag{4.6}$$

Remark. The group $\mathrm{nil}K_2(A)$ is the image of the map $NK_2(A)\to K_2(A)$, and similarly for $\mathrm{nil}K_3(A)$. This follows from the second remark after Corollary 4.4, since $N^pK_q(A)=N^pK_q(A_{\hat{f}})$ implies that for $p\geqq 2$ we have $E^2_{pq}(A)=\Pi E^2_{pq}(C_i)$ in the Gersten-Anderson spectral sequence.

Theorem 4.7. *Let A be a reduced 1-dimensional seminormal ring, finitely generated over a field. If $\bar{A}$, $\{k_i, K_{ij}\}$ are as above then there is an exact sequence*

$$0\to K_2(A)/\mathrm{nil}K_2(A)\to KV_2(A)\to \Pi\Omega_{K_{ij}/k_i}\to \mathrm{nil}\,K_1(A)\to 0.$$

(i) *$K_2(A)$ maps onto $KV_2(A)$ if either $\bar{A}=K[t]$ (K will be a product of fields) or all the K_{ij} are separable extensions of the k_i.*

(ii) *If $\mathrm{nil}K_2(B_{ij})=\mathrm{nil}K_2(C_{ij})$ for all (i,j), then $K_3(A)$ maps onto $KV_3(A)$ and*

$$\mathrm{nil}K_2(A)=\prod_{(i,j)}\mathrm{nil}K_2(C_{ij})\oplus\prod_{\substack{j<k\\ \text{all } i}}(K_{ij}\otimes_{k_i}K_{ik}).$$

Proof. The exact sequence is an immediate consequence of (4.4) and (4.6). Also by (4.4), (4.5) and (4.6), we derive an exact sequence

$$\begin{aligned} 0\to K_3(A)/\mathrm{nil}K_3(A)\to KV_3(A)\to \prod_{(i,j)}\ker(\mathrm{nil}K_2(B_{ij})\to \mathrm{nil}\,K_2(C_{ij}))\\ \to \mathrm{nil}K_2(A)\to \prod_{(i,j)}\mathrm{nil}K_2(C_{ij})\oplus\prod_{j<k}(K_{ij}\otimes K_{ik})\to 0. \end{aligned}$$

Statement (ii) follows from this, and (i) follows when the K_{ij} are separable over the k_i (compare with [V]).

Finally, we assume that $\bar{A}=K[t]$, and write $\Pi\Omega$ for the term $\Pi\Omega_{K_{ij}/k_i}$. Now $\Pi\Omega$ is a quotient of the summand $\Pi K_2(K_{ij}[[t]],(t))$ of $K_2(A_{\hat{f}})$. It follows that the map $KV_2(A)\to\Pi\Omega$ factors through the composition

$$K_2(K)=K_2(\bar{A})\to K_2(\Pi K_{ij}[[t]])/K_2(\Pi K_{ij})=\Pi K_2(K_{ij}[[t]],(t)).$$

This composition is zero since $K\to\Pi K_{ij}[[t]]$ factors through ΠK_{ij}. Done.

We will say that *the singularities* (of A) *split completely* if the maps $k_i \to K_{ij}$ are all isomorphisms. Equivalently, for every prime P of $\bar{A}$ the fields $A/A \cap P$ and $\bar{A}/P$ are isomorphic under the natural correspondence.

Corollary 4.8. *Let A be a reduced 1-dimensional seminormal ring, finitely generated over a field. Assume A's singularities split completely. Then*

$$K_2(A) = KV_2(A) \oplus \operatorname{nil} K_2(A),$$

where $\operatorname{nil} K_2(A)$ *is* $\oplus k_i^{N_i}$, $N_i = n_i(n_i - 1)/2$, *where* n_i *is the number of primes of* $\bar{A}$ *lying over the prime* p_i *of* A.

Proof. The theorem implies everything but the fact that the inclusion $\operatorname{nil} K_2(A) \to K_2(A)$ is split by the map $K_2(A) \to \prod K_2(C_i) \to \prod \operatorname{nil} K_2(C_i)$, which is supplied by Proposition 4.5.

Example 4.9. Let A be the coordinate ring of any seminormal curve defined over an algebraically closed field k. Then $K_2(A) = KV_2(A) \oplus$ a finite-dimensional vector space over k, whose dimension is $\sum n_i(n_i - 1)/2$. The group $KV_2(A)$ is determined up to extension by the Mayer-Vietoris sequence of the conductor square:

$$K_3(\bar{A}) \oplus \prod_i K_3(k) \to \prod_{(i,j)} K_3(k) \to KV_2(A)$$
$$\to K_2(\bar{A}) \oplus \prod_i K_2(k) \to \prod_{(i,j)} K_2(k).$$

As a special case we can recover the computations of [DR] for K_2 of n lines in the plane (no three through any point). As another special case, we see that K_2 of the node is $K_2(k) \oplus K_3(k) \oplus k$.

Example 4.10. More generally, let $k \subseteq k_1 = k(\alpha) = k(\beta)$ be any finite field extension, where α and β are non-conjugate elements. Set $A = \{f(t) \in k[t] : f(\alpha) = f(\beta)\}$, the ring obtained from $k[t]$ by glueing the primes $t = \alpha$, $t = \beta$ together. There is a cartesian square

$$\begin{array}{ccc} A & \longrightarrow & k[t] \\ \downarrow & & \downarrow \\ k_1 & \longrightarrow & k_1 \oplus k_1, \end{array}$$

from which we deduce that $KV_*(A) = K_*(k) \oplus K_{*+1}(k_1)$ and that $K_2(A) = K_2(k) \oplus K_3(k_1) \oplus k_1$.

We can describe the additive copy of k_1 as follows. Let $f_1, f_2 \in k[t]$ be the minimum polynomials of α, β, and set $f = f_1 f_2$. Then $A_{\hat{f}} = k_1[[t_1, t_2]]/(t_1 t_2 = 0)$. The copy of k_1 in $K_2(A)$ injects into $K_2(A_{\hat{f}})$ in such a way that $w \in k_1$ maps to the symbol $\langle w t_1, t_2 \rangle$. This follows from (4.6) and the computations of [DK]. R.K. Dennis has pointed out that (as a subring of $k_1[t]$) A is a "ring with norm", so that the image of $K_2(2, A)$ in $K_2(A)$ is just $K_2(k)$ by Bak's Theorem [DB] [Str]. Thus, generators in $St(A)$ for the elements corresponding to $K_3(k_1) \oplus k_1$ cannot come from relations on 2×2 matrices.

References

[B] Bass, H.: Algebraic K-theory. New York: Benjamin 1968

[B2] Bass, H.: Some problems in 'classical' algebraic K-theory. Lecture Notes in Math. 342, New York: Springer-Verlag 1973

[BC] Brewer, J.W., Costa, D.L.: Seminormality and projective modules over polynomial rings. J. Algebra **58**, 208–216 (1979)

[C] Carter, D.: Localization in lower K-theory. Comm. Algebra **8**, 603–622 (1980)

[C2] Carter, D.W.: Lower K-theory of finite groups. Preprint (1979)

[D] Davis, E.D.: On the geometric interpretation of seminormality. Proc. Amer. Math. Soc. **68**, 1–5 (1978)

[DB] Dennis, R.K.: Bak's theorem on K_2 of polynomial extensions. preprint (1974)

[DK] Dennis, R.K., Krusemeyer, M.: $K_2(A[x,y]/xy)$, a problem of Swan, and related computations. J. Pure Appl. Algebra **15**, 125–148 (1979)

[DR] Dayton, B.H., Roberts, L.G.: K_2 of n lines in the plane. J. Pure Appl. Algebra **15**, 1–9 (1979)

[DW] Dayton, B.H., Weibel, C.A.: K-theory of hyperplanes. Trans Amer. Math. Soc. **257**, 119–141 (1980)

[EGA] Grothendieck, A.: Eléments de géométrie algébrique IV. Publ. Math. I.H.E.S. Vol. 24, Paris, 1964

[G] Gersten, S.M.: Higher K-theory of rings. Lecture Notes in Math. 341, New York: Springer-Verlag 1973

[GQ] Grayson, D.: Higher Algebraic K-theory: II (after D. Quillen), Lecture Notes in Math. 551, New York: Springer-Verlag 1973

[GW] Geller, S.C., Weibel, C.A.: K_2 measures excision for K_1. Proc. Amer. Math. Soc. in press (1980)

[K] Karoubi, M.: Localisation de formes quadratiques I. Ann. Sci. École Norm. Sup., 4[e] série, t. **7**, 63–95 (1971)

[K-V] Karoubi, M., Villamayor, O.: K-theorie algebrique et K-theorie topologique I. Math. Scand. **28**, 265–307 (1971)

[L] Lam, T.Y.: Serre's Conjecture. Lecture Notes in Math. 635, New York: Springer-Verlag 1978

[Mat] Matsumura, H.: Commutative Algebra. New York: Benjamin 1970

[MP] Murthy, M.P., Pedrini, C.: K_0 and K_1 of polynomial rings. Lecture Notes in Math. 342, New York: Springer-Verlag 1973

[N] Nagata, M.: Local Rings. New York: Interscience 1962

[Q] Quillen, D.: Higher Algebraic K-theory I. Lecture Notes in Math. 341, New York: Springer-Verlag 1973

[R] Reiner, I.: Class groups and picard groups of group rings and orders. CBMS Regional Conf. series in Math. No. 26, AMS, Providence, 1976

[S] Swan, R.: K-theory of finite groups and orders. Lecture Notes in Math. 149, New York: Springer-Verlag 1970

[Sam] Samuel, P.: Lectures on unique factorization domains. Tata Institute of Fundamental Research, Lectures on Math. and Physics v. 30, Bombay, 1964

[Str] Strooker, J.R.: The fundamental group of the general linear group. J. Algebra **48**, 477–508 (1977)

[v.d.K] van der Kallen, W.: Sur le K_2 des nombres duax. C.R. Acad. Sci. Paris, Sér. A, t. **273**, 1204–1207 (1971)

[V] Vorst, A.: Localization of the K-theory of polynomial extensions. Math. Ann. **244**, 33–53 (1979)

[W1] Weibel, C.: The homotopy exact sequence in algebraic K-theory, Comm. Algebra **6**, 1635–1646 (1978)

[W2] Weibel, C.: Nilpotence in algebraic K-theory. J. Algebra **61**, 298–307 (1979)

[W3] Weibel, C.: KV-theory of categories. Preprint (1979)

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