Proc. Natl. Acad. Sci. USA
Vol. 85, pp. 5362–5363, August 1988
Mathematics

Homotopy invariance of η-invariants

SHMUEL WEINBERGER

Department of Mathematics, University of Chicago, Chicago, IL 60637

Communicated by Louis Nirenberg, March 21, 1988

ABSTRACT **Intersection homology and results related to the higher signature problem are applied to show that certain combinations of η-invariants of the signature operator are homotopy invariant in various circumstances.**

Statement of Results. Atiyah *et al.* (1–3) introduced the η-invariant in the course of their study of the spectral asymmetry of elliptic operators on odd dimensional Riemannian manifolds. That is, for any self-adjoint elliptic operator A on an odd-dimensional manifold, define

$$\eta_A(s) = \sum (\text{sign } \lambda)\lambda^{-s}, \qquad \lambda \text{ ranges over nonzero eigenvalues of } A,$$

$$\eta(A) = \eta_A(0), \qquad \text{by analytic continuation.}$$

They made the fundamental discovery of its connection to the index problem for manifolds with boundary. Using this connection they obtained an invariant of smooth manifolds by twisting the signature operator by a flat bundle (given by a representation ρ of the fundamental group) and comparing to the untwisted operator, by observing that this difference is independent of the Riemannian metric. The invariant so obtained, denoted $\eta_\rho(M)$, is not in general a homotopy invariant (it distinguishes the linear lens spaces from one another), but if the flat bundles used in its definition have a structure group that is free abelian, Neumann has shown (4) that it is a homotopy invariant and, in fact, gave a homotopical method for computing it. Our main goal is the following.

THEOREM 1. *If π is the fundamental group of a complete hyperbolic (or flat) manifold, or is torsion-free poly-finite-or-cyclic, or lies in the Cappell–Waldhausen class of groups, then all representations $\rho:\pi_1(\mathrm{M}) = \pi \to \mathrm{U(n)}$, the Atiyah–Patodi–Singer invariant, $\eta_\rho(\mathrm{M})$, is an oriented homotopy invariant.*

The Cappell–Waldhausen class of groups is the smallest class of groups containing the trivial group and closed under amalgamated free products and HNN extensions. It includes all torsion-free fundamental groups of surfaces and of irreducible sufficiently large three manifolds. If Thurston's geometrization conjecture were correct (up to homotopy type), *Theorem 1* would be true for all torsion-free fundamental groups of three manifolds. It seems conceivable, but perhaps is reckless to conjecture, that *Theorem 1* is true of all torsion-free groups. (Amusingly, *Theorem 1* is false for all residually finite, or virtually torsion-free, groups that contain nontrivial torsion.)

An interesting question is how to interpret this invariant homotopically. My proof makes use of intersection homological ideas as well as the deep work of Kasparov (5), Yamasaki (6) [extending earlier work of Farrell and Hsiang (7)], and Cappell (8) on the higher signature problem. The method also leads to some less interesting (to me) results for other special situations.

THEOREM 2. *If the Novikov conjecture (see below) holds for a group π, then the difference of η_ρ for homotopy equivalent manifolds is a rational number and the question of homotopy invariance depends only on the homotopy class of the representation ρ. For all groups, η_ρ is the same for the total spaces of fiber homotopy equivalent bundles over the same base manifold if ρ factors through the fundamental group of the base.*

Neumann's theorem follows from either theorem; for *Theorem 2* note that the Novikov conjecture for the free abelian case is classical and that elementary linear algebra implies that $\mathrm{Hom}(\mathbf{Z}^n{:}\,U(k))$ is connected.

Ideas of Proofs. The Novikov conjecture asserts that if

$$f{:}M \to B\pi$$

is a map, then the generalized Pontrjagin number

$$f_*(L(M)\cap[M]) \in H_*(B\pi{:}\mathbf{Q})$$

is an oriented homotopy invariant (where $L(M)$ is the Hirzebruch L-polynomial in the Pontrjagin classes). Although this is not enough to conclude that oriented homotopy equivalent manifolds are cobordant (rationally) over their fundamental group, it is close. To make this precise recall the notion of a Witt space, introduced by Siegel (9). A pseudomanifold is a Witt space if the link of every odd-codimensional simplex has vanishing middle-dimensional intersection homology with middle perversity (see refs. 10 and 11). Siegel shows that the bordism of such spaces describes, away from 2, a cycle theory for KO_*, so that rationally, via the Pontrjagin character, the higher signatures are the only obstruction to Witt cobordism over $B\pi$, so we assume a Witt cobordism of between our manifolds.

Now, Witt spaces were also introduced by Cheeger, essentially as the spaces for which the L^2-cohomological signature operator is formally self-adjoint, and he established Hodge theory (12) and an index theorem (13) for them. Consequently, one can deduce that the difference of η_ρ for the manifolds here is the "reduced" ρ-signature of the cobounding Witt space, in the sense of intersection homology. This immediately leads to the rationality statement in *Theorem 2*. To compute this signature, consider the space obtained by gluing the boundary components together by the given homotopy equivalence. Although this is not a Witt space, a little thought shows that it can naturally be given the structure of an algebraic Poincaré complex (APC) whose signature equals that of the Witt coboundary. (Use a homology theory that is ordinary near the codimension one stratum and intersection homological in a neighborhood of the lower strata.) If every APC were algebraically cobordant to a smooth manifold, cobordism invariance of signature would imply the vanishing of the ρ-signature by an application of the index theorem (and the vanishing of Chern classes for flat bundles). Since cobordism classes of APCs are isomorphic to the algebraic L-theory (see ref. 14), *Theorem 1* results from

The publication costs of this article were defrayed in part by page charge payment. This article must therefore be hereby marked "*advertisement*" in accordance with 18 U.S.C. §1734 solely to indicate this fact.

the surjectivity of the L-theory assembly map. This is known in the second two cases, proven in the references cited. For the nonpositively curved case, one factors the ρ-signature through the K-theory of the C^*-algebra $C^*[\pi]$ and applies the corresponding surjectivity statement there, due to Kasparov.

Penultimately, the statement about homotopy of representations is a consequence of the constancy of signature over families of finite-dimensional nondegenerate quadratic forms. The final statement about fiber homotopy invariance comes about by "resolving" the fundamental group of the base to lie in the Cappell–Waldhausen class and applying the homology surgery of Cappell and Shaneson (15) to produce a manifold with the better fundamental group and homology h-cobordant to the given base, which enables one to reduce to the final case of *Theorem 1*.

I would like to thank Jonathan Rosenberg for many stimulating conversations and Jeff Cheeger for explaining some of his work to me. I also acknowledge the support of a National Science Foundation grant, a Presidential Young Investigator Award, and a Sloan Foundation fellowship.

1. Atiyah, M., Patodi, V. & Singer, I. (1975) *Math. Proc. Cambridge Philos. Soc.* **77**, 43–69.
2. Atiyah, M., Patodi, V. & Singer, I. (1976) *Math. Proc. Cambridge Philos. Soc.* **78**, 405–432.
3. Atiyah, M., Patodi, V. & Singer, I. (1976) *Math. Proc. Cambridge Philos. Soc.* **79**, 315–330.
4. Neumann, W. (1978) *Proc. Symp. Pure Math.* **32**, 181–188.
5. Kasparov, G. (1983) *Operator K-Theory and Its Applications: Elliptic Operators, Group Representations, Higher Signatures, C* Extensions*, Proceedings of the International Congress of Mathematics, eds. Gleason, A. & Gleason, M., pp. 987–999.
6. Yamasaki, M. (1988) *Michigan Math. J.*, in press.
7. Farrell, F. & Hsiang, W. C. (1983) *Am. J. Math.* **105**, 641–672.
8. Cappell, S. (1976) *Invent. Math.* **33**, 171–179.
9. Siegel, P. (1983) *Am. J. Math.*, 1067–1105.
10. Goresky, M. & MacPherson, R. (1983) *Topology* **19**, 135–162.
11. Goresky, M. & MacPherson, R. (1983) *Invent. Math.* **72**, 77–130.
12. Cheeger, J. (1980) *Proc. Symp. Pure Math.* **36**, 91–146.
13. Cheeger, J. (1983) *J. Differ. Geom.* **18**, 575–657.
14. Ranicki, A. (1973) *Proc. London Math. Soc.* **27**, 101–125.
15. Cappell, S. & Shaneson, J. (1974) *Ann. Math.* **99**, 277–348.