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CONSTRUCTING HOMOTOPY EQUIVALENCES

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IN THIS paper we study the following construction of homotopy equivalences: Take a codimension one separating submanifold N^{n-1} of M^n, cut along N and glue the pieces together by a homeomorphism of N homotopic to the identity. Aside from the question of which homotopy equivalences can be so obtained, we will study qualitative questions such as stability, type of submanifold, etc. Relations to $\Sigma\Omega$, the oozing problem in surgery theory, and Kervaire classes will be discussed.

INTRODUCTION

It is well known that in dimension at least five that every smooth homotopy sphere can be obtained by cutting the standard sphere along the equator and pasting the hemispheres together by a diffeomorphism homotopic to the identity. Here we study to what extent similar statements hold for other homotopy equivalences between manifolds. Throughout we will work in the topological category.

Roughly speaking, a homotopy equivalence is cut-pastable (CP) if after composing with a homeomorphism it can be obtained by cutting along a codimension-one submanifold and glueing the pieces together by a homeomorphism homotopic to the identity. (If general homeomorphisms are allowed see [12].) A more precise definition will be given in §1. A homotopy equivalence is specially cut-pastable (SCP) if the codimension-one submanifold can be taken to have the same fundamental group as the ambient manifold. The object of this paper is to study and classify those homotopy equivalences which are CP, SCP or the result of a sequence of such operations. Related problems are to find the "most efficient" fundamental group for the submanifold, (a bound on) the number of CP's necessary in a sequence, and behavior under homology equivalences.

We start with a classification of SCP homotopy equivalences.

THEOREM 3.5. *Let* $h\colon M' \to M^n$ $(n \geq 5)$ *be a homotopy equivalence, then* h *is* SCP iff:
(1) *h is a simple homotopy equivalence*
(2) $\nu(h)\colon M \to G/\mathrm{Top}$ *the normal invariant of* h*, lifts to* $\Sigma\Omega(G/\mathrm{Top})$ and
(3) $\nu(h)^*(k_2) = 0$ where $k_2 \in H^2(G/\mathrm{Top}; \mathbb{Z}_2)$ *is the Kervaire class.*

Actually, the first two conditions are necessary for h to be CP. This suggests that there is a basic interplay between $\pi_1(N)$ and $\nu(h)^*(k_2)$. (Recall N is the codimension-one submanifold CP along.)

Definition. A homotopy equivalence is *twisted* if $\nu(h)^*(k_2) \neq 0$ lies in the image of $H^2(\pi_1 M; Z_2) \to H^2(M; Z_2)$. Otherwise, h is *untwisted.* Note that if $H^2(\pi_1 M; Z_2) = 0$ then all homotopy equivalences are untwisted.

A geometric interpretation of twistedness can be given as follows: If $h\colon M' \to M$ is CP, then the submanifold N CP along can be taken to divide M into two components M_+ and M_- for each of which $\ker \pi_1 M_\pm \to \pi_1 M$ has order at most two, see §2. This leads quite naturally to five normal forms for the quadruple $(\pi_1 M, \pi_1 M_+, \pi_1 M_-, \pi_1 N)$ (with given homeomorphisms induced by the geometric inclusions). The most interesting normal form occurs when h is twisted. h is twisted iff $\pi_1 M_\pm \to \pi_1 M$ are nontrivial Z_2 extensions. Extensions $1 \to Z_2 \to E \to \pi_1 M \to 1$ are classified by elements of $H^2(\pi_1 M; Z_2)$. If the homotopy equivalence is twisted, the preimage of $v(h)^*(k_2)$ in $H^2(\pi_1 M; Z_2)$ corresponds to exactly the extension $1 \to Z_2 \to \pi_1 M_\pm \to \pi_1 M \to 1$.

For untwisted homotopy equivalences there is a quite satisfactory theorem:

THEOREM 4.14 (+4.15). *Let h: $M' \to M^n$ ($n \geq 5$) be an untwisted homotopy equivalence between manifolds of dimension at least five (which restricts to a homeomorphism of any, perhaps empty, boundary). Suppose $\pi_1 M$ is*

(a) *finite with 2-sylow subgroup a product of elementary abelian groups and dihedral groups (or just abelian × dihedral if* dim $M \geq 6$).

(b) *abelian* dim $M > \max(5, (\text{rank } \pi_1 M) + 3)$; *or*

(c) *such that $H_*(\pi_1 M; Z_{(2)}) = 0$ for $* > \dim M - 2$* (e.g. *$\pi_1 M$ a classical knot group or a surface group*);

then h is CP iff

(1) *h is a simple homotopy equivalence; and*

(2) *$\nu(h)$ lifts to* $\Sigma\Omega(G/\text{Top})$.

In the untwisted case the property of being CP is actually a normal cobordism invariant, but in the twisted case we find (§4A) another obstruction. This requires a technique of ambient surgery on homeomorphisms homotopic to the identity, which extends the familiar handle trading ideas of other codimension one contexts. This leads to an example (§4D) of a non-CP homotopy equivalence satisfying (1) and (2) of 4.14.

In [38] we show by example that these results fail in dimension four both PL and topologically even after taking connect sums with $S^2 \times S^2$.

For working out specific examples, the above theorems often suffice. For example, for highly connected manifolds we have:

THEOREM 5.1. *Let M^{2n} be $n-1$* connected $n > 8$. *Then h: $M' \to M$ is* CP iff *h is* SCP. *If n is not a multiple of four this is always the case. On the other hand, if $n = 4k$ and M^{8k} is not a sphere there always exists a non*-CP *homotopy equivalence to M. Moreover, if the quadratic form $H^{4k}(M; Q) \otimes H^{4k}(M; Q) \to Q$ is*

(a) *definite, then h is* CP iff *h is homotopic to a homeomorphism*

(b) *indefinite, then every homotopy equivalence is the result of a sequence of* CP's.

In particular, there are non-CP homotopy equivalences to $S^{4k} \times S^{4k}$ which are the result of a sequence of CP's. A class of spaces for which this does not occur includes various quotients of spheres by group actions.

THEOREM 5.4 (+5.5). *Let M^n be a homotopy real, complex or quaternionic projective space or a homotopy lens space with the order of π_1 squarefree, n at least five. Let h: $M' \to M$ be a homotopy equivalence, then the following are equivalent:*

(1) *h is* CP,

(2) *h is* SCP,

(3) *h is the result of a sequence of* CPs (SCP's),

(4) *h is a simple homotopy equivalence, and the first half of the splitting invariants (see* [26] *for* CP^n [14] *for* RP^n, *and* [11] *for lens spaces. The case of quaternionic projective space is analogous and easier) of M and M' coincide,*

(5) *h is a simple homotopy equivalence and $v(h)|_{M^{[n/2]}}$ is nullhomotopic, where M^k is the k-skeleton of M^n, and*

(6) *h is the result of cutting and pasting along the boundary of the regular neighborhood of* $M^{[n/2]} \subset M^n$.

Surprisingly, if the fundamental group of the lens space has a square factor, the conclusion of Theorem 6.5 can fail:

THEOREM 5.6. *For* $L_{p^2}^{8k+1}$ *with* $8k+1$ *less than* $2p+1$, *condition* (1) *does not imply condition* (5) *in the above theorem; however, for an arbitrary manifold* (*of dimension at least five*) (5) *implies* (1) *and* (2).

For further calculations and qualitative results see §5. The organization is as follows:
§1. Preliminaries.
§2. Smoothing Pre-Cut-Pastes into Cut-Pastes.
§3. Special Cut-Pastes.
§4. Which homotopy equivalences are Cut-Pastable?
§5. Calculations.

The results here form part of the author's thesis submitted June 1982 to N.Y.U. I would like to thank my advisor Sylvain Cappell for his unfailing encouragement and many perspicacious comments.

Some of these results have been announced in [39].

§1. DEFINITIONS AND PRELIMINARIES

Let M^n be a compact manifold, perhaps with boundary, * a base point in M, and N^{n-1} a codimension-one properly embedded compact submanifold of $M - *$. If h is a homeomorphism $h: N \to N$, the manifold obtained by cutting M along N and glueing M_+, (the component of $M - N$ containing *), to M_- with h, i.e. $h(n_-) = n_+$ will be denoted by $M_+ \cup_{h^{-1}} M_-$ or $M(N, h)$. Thus, for any N, $M(N, 1_N) = M$, and if h is homotopic to the identity $M(N,h)$ has the same homotopy type as M. A *cut-paste* (abbreviated CP) is a triple (N, h, H) consisting of a separating codimension-one submanifold, N, a homeomorphism, h, from N to itself, which restricts to the identity relative to the boundary and a homotopy H from h to the identity. Given a CP (N, h, H) there is a canonical homotopy equivalence (rel ∂) $\bar{H}: M(N,h) \to M$ given by

$$\bar{H}: M_+ \cup_1 N \times I \cup_{h^{-1}} M_- \xrightarrow{1_{M_+} \cup H \cup 1_{M_-}} M.$$

A homotopy equivalence $f: M' \to M$ is *cut-pastable* if there is a CP (N, h, H) and a homeomorphism $F: M' \to M(N, h)$ such that $f \sim \bar{H} \circ F$, i.e.

$$\begin{array}{ccc} M' & \xrightarrow{f} & M \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle \bar{H}} & \\ & M(N,h) & \end{array}$$

commutes up to homotopy (rel ∂). A CP homotopy equivalence between manifolds with boundary can be taken to have $\partial N = \emptyset$ by Fig. 1. A homotopy equivalence is *specially cut-pastable* (SCP) if the CP (N, h, H) can be taken to have $\pi_1 N \to \pi_1 M$ an isomorphism. The problem is to classify those homotopy equivalences which can be obtained by cutting and pasting, special cutting and pasting, and sequences of such operations.

Remark. There is no loss in generality in assuming N connected since we can pipe the

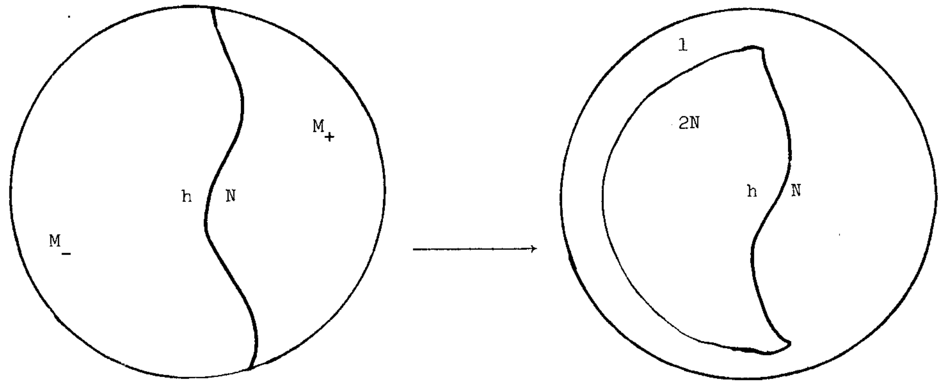


Fig. 1. Replaces a CP by one which has the submanifold in Int M.

components, homeomorphisms, and homotopies. Similarly, there is no loss of generality in assuming N separating rather than merely two-sided since we can push a copy of a 2-sided N off itself to obtain a separating submanifold and take (h,H) to be the identity on the second copy of N.

The useful notion in what follows is that of a *pre-cut-paste* (abbreviated PCP), which captures the homotopy data underlying a CP. A PCP is a pair (N,n), consisting of a codimension-one submanifold N of Int M, and a map $n\colon N \to \Omega(G/\mathrm{Top})$. A CP gives rise to a PCP as follows. Let $\hat{n}\colon \Sigma N \to G/\mathrm{Top}$ be the normal invariant of $H\colon N \times I \operatorname{rel} \partial \to N \times I \operatorname{rel} \partial$ and the map $N \to \Omega(G/\mathrm{Top})$ is the adjoint. Two PCP's, (N_1, n_1) and (N_2, n_2) are cobordant if there is a manifold $P \subset M \times I$ and a map $p\colon P \to \Omega(G/\mathrm{Top})$ such that $\partial P = N_1 \times 0 \cup N_2 \times 1$ and $P|_{N_1 \times 0} = n_1$, $P|_{N_2 \times 1} = n_2$.

Proposition 1. PCP *cobordism in* M^n, $n \geq 5$ *is in a one–one correspondence with* $[M,_*\colon \Sigma\,\Omega(G/\mathrm{Top})$, *cone point*]. (*For M with boundary, with* $[(M, \partial M \cup {*})\colon (\Sigma\,\Omega(G/\mathrm{Top})$, *the two cone points*)].)

Proof. Let $c_N\colon M \to \Sigma N$ be the collapse map. From a PCP (N,n) $c_N \circ \Sigma n\colon M \to \Sigma\,\Omega(G/\mathrm{Top})$ is the desired map.

From a map $M \to \Sigma\,\Omega(G/\mathrm{Top})$, the transverse inverse image of $\Omega(G/\mathrm{Top})$ is a codimension-one separating submanifold equipped, by restriction, with a map to $\Omega(G/\mathrm{Top})$.

Clearly these operations are inverse to each other and behave correctly under the equivalence relations above. □

Remarks. (1) That $\Omega(G/\mathrm{Top}) \subset \Sigma\,\Omega(G/\mathrm{Top})$ is not an inclusion of manifolds presents no problem in applying transversality. All that is necessary is a "normal structure". Alternatively one can take manifold-pairs that homotopically approximate this inclusion.

(2) We have applied topological transversality which is justified for $n \geq 6$ by [13] and for $n = 5$ by Freedman's amazing construction of a simply connected topological four-manifold with intersection form E_8 and [36]. Actually the following elementary consequence of PL transversality and triangulation theory[13] would suffice for $n = 5$:

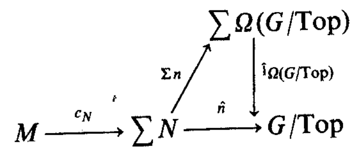
Lemma. *Topological transversality holds for five dimensional manifolds and maps into simply connected spaces.* □

Given a PCP (N,n) one can construct a normal invariant $M \to G/\mathrm{Top}$ as $c_N \circ \hat{n}$ ($\hat{n}$ is the adjoint of n). It is clear that the normal invariant associated to the PCP associated to the

CP (N, h, H) is the same as the normal invariant associated to $\bar{H}$: $M(N, h) \to M$. A homotopy theoretic version of this construct then is useful.

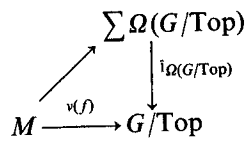
PROPOSITION 2. *The map from* PCP's *to normal invariants above is just composition*
$[M : \Sigma\,\Omega(G/\mathrm{Top})] \xrightarrow{i_{\Omega(G/\mathrm{Top})*}} [M : G/\mathrm{Top}]$.

Proof. This just commutativity of:



□

COROLLARY 3. *If f*: $M' \to M$ *is a* CP *homotopy equivalence then there is a lift:*



($\nu(f)$ *is the normal invariant of f.*) □

Much of the rest of the paper is devoted to studying the extent to which the converse to Corollary 3 holds. The following proposition gives another essential necessary condition:

PROPOSITION 4. *All* CP *homotopy equivalences are simple.*

Proof. By topological invariance of Whitehead torsion[13] we just must calculate $\tau(\bar{H})$ for the CP (N, h, H):

$$\tau(\bar{H}) = \tau(1_{M_+}) + \tau(1_{M_-}) - \tau(H) \quad \text{see [9]}$$

$$= -\tau(H) = 0$$

as H is homotopic to a homeomorphism. □

§2. SMOOTHING PRE-CUT-PASTES INTO CUT-PASTES

In this section we describe precisely which PCP's are PCP-cobordant to ones arising from CP's. We use ambient surgery to choose normal forms for every PCP-cobordism class. There are five essentially different normal forms which can be distinguished for the most part by cohomological invariants and give rise to obstructions in surgery obstruction groups of three classes of groups which are extensions of $\pi_1 M$.

Let $f: M \to G/\mathrm{Top}$ be a normal map. The 2-dimensional Kervaire invariant of f, $k_2(f) \in H^2(M; Z_2)$ is the pullback of the unique nontrivial element of $H^2(G/\mathrm{Top}; Z_2)$. For M with boundary we will be interested in both $k_2(f) \in H^2(M; Z_2)$ and $k_2^\partial(f) \in H^2(M, \partial M; Z_2)$. When $k_2(f) \neq 0$ the normal form of all the lifts of f to $\Sigma\,\Omega(G/\mathrm{Top})$ will coincide and can be calculated directly from $k_2(f)$. When $k_2(f) = 0$ there are several normal forms and the normal form depends on a further lifting problem (see §3).

A *normal form* is a sextuple $(G, H_1, H_2, h_1, h_2, h_3)$ consisting of three groups and three homeomorphism $h_1: G \to H_1$, $h_2: G \to H_2$, $h_3: G \to Z_2$, of concern here are the following:

(I) $(\pi, \pi, \pi, 1, 1, h)$ h arbitrary
(II) $(\pi \times Z_2, \pi \times Z_2, \pi, 1, p_1, p_2)$ p_i projection to ith coordinate
(III) $(\pi \times Z_2 \times Z_2, \pi \times Z_2, \pi \times Z_2, p_1 \times p_2, p_1 \times p_3, p_2 + p_3)$
(IV) $(\pi \times Z_2, \pi, \pi, p_1, p_1, p_2)$
(V) $(E \times Z_2, E, E, p_1, p_1 + \epsilon, p_2)$

$$1 \to Z_2 \xrightarrow{\epsilon} E \to \pi \text{ a nontrivial } Z_2 \text{ extension.}$$

PROPOSITION 1. *Any* PCP *is cobordant to a* (N, n) *such that* $(\pi_1 N, \pi_1 M_\pm, \pi_1 M_\mp, i_{\pm *}, i_{\mp *}, n_*)$ *is in one of the five normal forms described above with* $\pi = \pi_1 M$.

Proof. In order to describe PCP's the following is useful:

LEMMA. *Let* $N \subset M$. *There is some* $n: N \to \Omega G/\text{Top}$ *such that* $f: M \to G/\text{Top}$ *comes from* (N, n) iff $f|_{M-N}$ *is nullhomotopic.*

Proof. This is direct from the Baratt–Puppe sequence [40 (III. 6.13)].

When describing PCP cobordisms we will only describe the change in the submanifold and check that the normal invariant remains trivial on all components of the complement. It is easy to see how the map to $\Omega(G/\text{Top})$ changes. When there is no confusion the modified submanifold will also be called N.

Step I. Making N connected. Pipe together two components. This changes the homotopy type of the complement as follows. One component is slightly smaller so the normal invariant is certainly nullhomotopic on that component. Also two components are now joined by an arc. The obstruction to extending the old nullhomotopies is in $\pi_1(G/\text{Top}) = 0$. Repeating this finitely many times yields N connected.

Step II. Making $\pi_1 N \to \pi_1 M_+$ and $\pi_1 N \to \pi_1 M_-$ both onto. As $\pi_1 M_\pm$ are finitely generated there are finitely many circles which generate these groups. As before N piped with the boundaries of tubular neighborhoods of these circles will work.

Step III. Making $\pi_1 N \xrightarrow{i_{\pm *} \times n_*} \pi_1 M_\pm \times Z_2$ (both) injective. Recall:

LEMMA. *If* $\phi: G \to H$ *is a surjection of finitely presented groups then* ker ϕ *is finitely normally generated,* see [6].

First we make $\pi_1 N \to \pi_1 M_+ \times Z_2$ injective and then do the same for $\pi_1 N \to \pi_1 M_- \times Z_2$. To apply the above lemma use the fact that $\pi_1 N \to \pi_1 M_+ \times Z_2$ has image isomorphic to either $\pi_1 M_+$ or $\pi_1 M_+ \times Z_2$. Thus $\pi_1 N \to \pi_1 M_+ \times Z_2$ has finitely normally generated kernel $= \langle c_1 \dots c_k \rangle$. Let C_i denote a circle representing c_i. Each C_i bounds a disk D_i in M_+ and we can assume $D_i \cap D_j = \emptyset$ for $i \neq j$. As $n_*[C_i] = 0$ we can extend $n: N \to \Omega(G/\text{Top})$ to a map $\bar{n}: N \cup \bigcup_i D_i \to \Omega(G/\text{Top})$, and therefore to a regular neighborhood of the latter complex. This provides a PCP-cobordism between (N, n) and $(N', \bar{n}|_{\partial N'})$ where N' is the result of doing surgery on C_i. Note that after these surgeries either $\pi_1 N \to \pi_1 M_+$ or

$\pi_1 N \to \pi_1 M_+ \times \mathbb{Z}_2$ is an isomorphism and $\pi_1 N \to \pi_1 M_-$ is still onto. Since the former property remains after surgeries on circles, we can perform surgeries as above to guarantee the same for $\pi_1 N \to \pi_1 M_- \times Z_2$.

At this point we check that $(\pi_1 N, \pi_1 M_+, \pi_1 M_-, i_{+*}, n_*)$ is one of the forms I–V.

Case I: $\pi_1 N \to \pi_1 M_\pm$ are isomorphisms, then we are clearly in Form I.

Case II: $\pi_1 N \to \pi_1 M_+$ an isomorphism as is $\pi_1 N \to \pi_1 M_- \times Z_2$, then clearly we are in Form II.

Case III: $\pi_1 N \to \pi_1 M_\pm \times Z_2$ are isomorphisms. Let $u_\pm$ be the unique nontrivial elements of $\ker \pi_1 N \to \pi_1 M_\pm$. Again there are two cases: $u_+ = u_-$ and $u_+ \neq u_-$. In the first case we are in Form IV. Suppose now $u_+ \neq u_-$. Write $\pi_1 N = \pi_1 M_+ \times Z_2$ where $n_*|_{\pi_1 M_+}$ is trivial and Z_2 is generated by u_+. Van Kampen's theorem implies $\pi_1 M = \pi_1 N / \langle u_+, u_- \rangle = \pi_1 M_+ / (u_+ - u_-)$ so letting $\varepsilon: Z_2 \to \pi_1 M_+$ by sending generator to $u_+ - u_-$ we see that $\pi_1 M_+$ is a Z_2 extension of $\pi_1 M$ and that we are in Forms III or V depending on whether or not the extension is trivial. □

Remark. We will soon see the need to distinguish Forms III and V which at the moment must seen artificial.

PROPOSITION 2. *If a* PCP *corresponding to the normal invariant f is in Forms* I, II or III *then* $k_2(f) = 0$. *Conversely, if* $k_2(f) = 0$ *then the normal form for any* PCP *lifting f is of Type* I, II or III.

Proof. In Cases I–III we have $H_1(N) \to H_1(M_+) \oplus H_1(M_-)$ injective. Consider

$$\begin{array}{ccccccc}
H_2(M_+) \oplus H_2(M_-) & \longrightarrow & H_2(M) & \longrightarrow & H_1(N) & \longrightarrow & H_1(M_+) \oplus H_1(M_-) \\
\downarrow & & \swarrow & & \searrow & & \searrow \\
0 & \longrightarrow & H_2\left(\sum \Omega(G/\mathrm{Top})\right) & \xrightarrow[\approx]{} & H_1(\Omega(G/\mathrm{Top})) & \longrightarrow & 0 \\
 & & \downarrow \approx & & & & \\
 & & H_2(G/\mathrm{Top}) & & & &
\end{array}$$

(Z_2 coefficients understood)

A straightforward diagram chase shows $H_2(M) \xrightarrow{f_*} H_2(G/\mathrm{Top})$ is trivial, so that $k_2(f) = 0$. A similar argument shows the converse using the fact that in Types IV and V $\ker H_1(N) \to H_1(M_+) \oplus H_1(M_-)$ is mapped isomorphically $H_1(\Omega(G/\mathrm{Top}))$. (For Form V this uses the observation that for $Z_2 \to E \to \pi$ a nontrivial extension $\epsilon_*: H_1(Z_2) \to H_1(E)$ is trivial.) □

We postpone further discussion of $k_2(f) = 0$ until the next section, where it will be shown, for closed manifolds, that this together with the existence of a PCP is equivalent to the homotopy equivalence being SCP. Suppose then $k_2(f) \neq 0$. We need some criterion to distinguish when the normal form will be Type IV and when Type V, and if the latter, how to calculate the extension.

First we digress to discuss Z_2 extensions of groups. Usually Z_2 extensions of π are classified by elements of $H^2(\pi; Z_2)$. This is equivalent to subgroups of index (at most) two in $H_2(\pi; Z_2)$ by examining the kernel of the Kronecker pairing. To get a handle on this subgroup, consider the Serre spectral sequence of the fibration:

$$\begin{array}{ccc} K(E,1) & \longrightarrow & K(\pi,1) \\ & & \downarrow \\ & & K(Z_2,2) \end{array}$$

which yields the exact sequence:

$$H_2(E;Z_2)\to H_2(\pi;Z_2)\to Z_2\to H_1(E;Z_2)\to H_1(\pi;Z_2)\to 0.$$

Then, the relevant subgroup is just the image of $H_2(E;\mathbb{Z}_2)$.

THEOREM 3. *Let*

$$\begin{array}{ccc} & & \sum\Omega(G/\mathrm{Top}) \\ & \overset{\bar f}{\nearrow} & \downarrow \\ M & \xrightarrow{\ f\ } & G/\mathrm{Top} \end{array}$$

$k_2(f)\neq 0$. *Then the normal form of the* PCP *corresponds to* $\bar f$ *is either Type* IV *or* V. *It is*:

(a) *Type* IV iff $f_*\colon \pi_2 M\to\pi_2(G/\mathrm{Top})$ *is nontrivial.*

(b) *Type* V iff $k_2(f)\in \operatorname{Im} H^2(\pi;Z_2)$.

Moreover, the extension of π *is that determined by the preimage in* $H^2(\pi;Z_2)$.

Proof. The first part of this theorem was Proposition 2. To see the next statement one shows that $f_*\colon \pi_2 M\to\pi_2(G/\mathrm{Top})$ is nontrivial iff there is an $S^2\subset M$ on which f is essential such that $S^2\cap N$ is a circle. This circle would be an element of $\mathrm{Ker}_+\cap\mathrm{Ker}_-$. Conversely, glueing $D^2_+\subset M_+$ and $D^2_-\subset M_-$ along $u_+=u_-$ gives an element of $\pi_2 M$ on which f is essential. The last statement follows from the following calculation. Suppose the PCP is of Type V, then we have:

$$\begin{array}{ccccccc} H_2(M_+)\oplus H_2(M_-) & \longrightarrow & H_2(M) & \xrightarrow{\ \partial\ } & H_1(N) & \longrightarrow & H_1(M_+)\oplus H_1(M_-) \\ \downarrow & & \downarrow & & & \searrow & \downarrow \\ H_2(C(G/\mathrm{Top}))\oplus H_2(C(G/\mathrm{Top})) & \longrightarrow & H_2\left(\sum\Omega G/\mathrm{Top}\right) & \overset{\approx}{\rightarrow} & & & H_1(\Omega G/\mathrm{Top})\to 0 \\ \| & & \downarrow\approx & & & & \\ 0 & & H_2(G/\mathrm{Top}) & & & & \end{array}$$

Therefore, $H_2(M)\to H_2(G/\mathrm{Top})$ is exactly the boundary map $H_2(M)\to H_1(N)$. Thus $\ker k_2(f)=\operatorname{Im} H_2(M_+)\oplus H_2(M_-)$. Let us examine the image of this kernal in $H_2(\pi)$.

$$\begin{array}{ccc} H_2(M_+)\oplus H_2(M_-) & \longrightarrow & H_2(M) \\ \downarrow & & \downarrow \\ H_2(E)\oplus H_2(E) & \longrightarrow & H_2(\pi). \end{array}$$

Thus, the kernel has the image classification of the extension. By the (Hopf) exact sequence

$$0 \longrightarrow H^2(\pi;\mathbb{Z}_2) \longrightarrow H^2(M;\mathbb{Z}_2) \longrightarrow \mathrm{Hom}(\pi_2 M:\mathbb{Z}_2)$$

we are done.

If the normal form of $v(h)$ is of Type V, h will be called *twisted*; otherwise h is *untwisted*.

Although it is not yet essential, the following proposition will be useful later:

PROPOSITION 4. *If* $h: M' \to M$ (dim $M \geq 5$) *is a* CP *homotopy equivalence, then it can be obtained by cutting and pasting along a submanifold in normal form.* (*Moreover, this can be arranged without changing the underlying* PCP *cobordism class.*)

Proof. Proposition 1 is a homotopy version of this, and implies, by surgery theoretic arguments, below that h is normally cobordant to a CP homotopy equivalence, CP in normal form.

In order to get the more precise result that the CP for h itself can be taken in normal form, more powerful tools are necessary. We will now repeat the proof of Proposition 1 with ambient surgery on CP's replacing the ambient surgery on PCP's performed in that proof.

Let (N,h,H) be a CP. Note that if $G: N \times I \to N \times I$ is a homeomorphism with $G|_{N\times 0} = 1|_{N\times 0}$, then $(N, h\circ G|_{N\times 1}, H\circ G)$ is another CP with $M(N,h) = M(N, h\circ G|_{N\times 1})$ and underlying PCP's PCP bordant. Thus, there is no loss in generality in composing with such pseudoisotopes when necessary.

Steps I and II of the proof of Proposition 1 can be repeated with no difficulty. In order to complete Step III the circles C_i on which we would like to do surgery must have $h|_{C_i} = 1|_{C_i}$ and $H|_{C_i} = 1|_{C_i}$. Then we can replace N by N surgered along the circles, $\bar{N}$, with h and H being the identity outside of $N - \cup C_i$. (Actually, we must also arrange that on a neighborhood of the circles h and H are the identity which *a priori* leads to an additional obstruction in $\pi_1(0) = Z_2$ (and then in $\pi_2(0) = 0$); but since h is homotopic to the identity this obstruction is easily seen to vanish, see [8].)

First homotop H rel ∂ so that:

$$H|_{H^{-1}(C_i\times I)}: H^{-1}(C_i \times I) \to C_i \times I$$

is a (rel ∂) homotopy equivalence. The Browder splitting theorem [5] indentifies the obstruction to doing this with the surgery obstruction of $\theta(H|_{H^{-1}(C_i\times I)}: H^{-1}(C_i\times I) \to C_i \times I) \in L_2(Z) = L_2(0) = Z_2$. This vanishes since we can identify θ with n_*: $\pi_1 N \to \pi_1 \Omega(G/\mathrm{Top})$ and n_* vanishes on each of the C_i.

It of course, follows that these $H^{-1}(C_i \times I)$ are abstractly twisted and tangled cylinders. Let G be a pseudoisotopy

$$G: N \times I \to N \times I$$
$$G|_{N\times 0} = 1|_{N\times 0}$$
$$G: H^{-1}(C_i \times I) \to C_i \times I.$$

Then $(N, h\circ (G|_{N\times 1})^{-1}, H\circ G^{-1})$ is a CP which we can surger since h and N are the identity on C_i. The existence of G is guaranteed by the Straightening Lemma. □

Straightening Lemma. Let $N \subset M$ be a codimension at least 3 embedding (dim $M > 5$; for dim $M = 4$, the "stable " version of what follows is claimed). Let $c: N \times I \to M \times I$ be a proper embedding $e|_{N\times 0}$ coinciding with the "standard" embedding and $e(N \times 1) \subset M \times 1$. Then there is a pseudoisotopy $G: M \times I \to M \times I$ sending e to the product embedding $(N \subset M) \times I$.

Proof. We will construct G in stages. Let $K \subset K'$ and $L \subset L'$ be concentric regular neighborhoods of $(N \subset M) \times I$ and $e(N \times I)$ which agree on $M \times 0$. One uses the s-cobordism theorem to build homeomorphisms from $L \to K$ and $M \times I - L' \to M \times I - K'$, which are the identity on the portion lying on $M \times 0$. There is no problem extending the homeomorphism to the "annular" region $L' - L \to K' - K$ (see Fig. 2, construct the Pseudoisotopy on the shaded regions first.)

Remark. One can use the straightening lemma to give a proof of the Zeeman Unknotting Theorem[41].

At this point we can define the obstruction to smoothing, up to cobordism, a PCP (N, n) coming from a normal invariant $f: M^n \to G/\mathrm{Top}$ with $k_2(f) \neq 0$ into an actual CP. Let $\bar{f}$ be the lift of f to $\Sigma\Omega(G/\mathrm{Top})$ corresponding to (N, n). Define

$$\Phi(\bar{f}) = \theta(\hat{n}) \in L_n^s(\pi_1 N)$$

for (N, n) in normal form.

PROPOSITION 5. *$\Phi(\bar{f})$ is well defined*; i.e. *it does not depend on which representative in normal form of the* PCP *cobordism class is chosen. Moreover, $\Phi(\bar{f}) \in \ker L_n^s(\pi_1 N) \to L_n^s(\pi_1 M)$ and vanishes if $\bar{f}$ comes from a* CP.

Note. As $k_2(f) \neq 0$ we can identify, *a priori*, $\pi_1 N$ by Theorem 3.

Proof. The difficult case is when the normal form is of Type V. Let (N_1, n_1) and (N_2, n_2) be two representatives in normal form for $\bar{f}$ and (P, p) a PCP-cobordism between them. Without loss of generality (P, p) is also in normal form. We thus have a diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z_2 \oplus Z_2 & \longrightarrow & \pi_1 N_i & \longrightarrow & \pi_1 M & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & Z_2 \oplus Z_2 & \longrightarrow & \pi_1 P & \longrightarrow & \pi_1 M \times I & \longrightarrow & 1. \end{array}$$

To show that $\pi_1 N_i \to \pi_1 P$ is an isomorphism it suffices to show that the maps on the $Z_2 \oplus Z_2$'s are injective. This is clear since the generators are characterized by being elements of $\pi_1 P$ dying on one of the two sides but being nonzero on p_* and the images

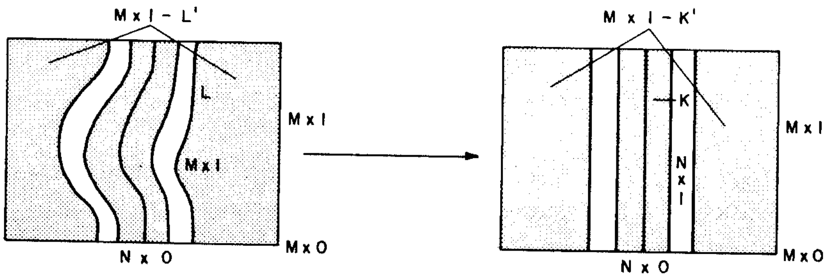


Fig. 2.

of the generators of the $Z_2 \oplus Z_2$'s corresponding to $\pi_1 N_i$ have these properties so generator goes to generator.

The proof for Type IV goes through the same way but we give an alternate description of Φ below which will make this trivial.

The next statement follows from the fact that the addition formula for surgery obstructions implies that the image of $\theta(\hat{n})$ in $L_n^s(\pi_1 M)$ is the same as that of $\theta(f)$ which is zero since f comes from a simple homotopy equivalence.

The last statement follows from Proposition 4, or directly to handle dim $M = 5$. □

Remark. The penultimate statement shows that there is no obstruction to smoothing Type I CP into a SCP, see Theorem 7.

One can give an *a priori* definition of the obstruction to smoothing PCP's for Types II and IV, leaving only Type III to the next section. Let (N, n) be an *arbitrary* PCP

$$\Phi(\bar{f}) = (i_* \times n_*)\theta(\hat{n}) \in L_n(\pi \times Z_2)(\pi = \pi_1 M)$$

$$(i_* \times n_*)_*: L_n(\pi_1 N) \to L_n(\pi_1 M \times Z_2).$$

PROPOSITION 6. *$\phi(\bar{f})$ is well defined. For Type* IV PCP's $\phi(\bar{f}) = \Phi(\bar{f})$.

Proof. Surgery obstructions in $L_n(G)$ only depend on the element of $\Omega_n(K(G, 1) \times G/\text{Top}, * \times G/\text{Top})$. All manifolds and cobordisms are equipped with such maps so the element in cobordism of $K(\pi \times Z_2, 1) \times G/\text{Top}$ is well defined. The second statement follows trivially from the first. □

We close the section with:

THEOREM 7. *Let $h: M' \to M$ be a simple homotopy equivalence with normal invariant $f: M \to G/\text{Top}$ lifting to $\bar{f}: M \to G/\text{Top}$, and with $k_2(f) \neq 0$. Then there is cut-paste in the* PCP *cobordism class of $\bar{f}$* iff $\Phi(\bar{f}) = 0$.

Remarks. (1) In the next section we will extend the definition of Φ to all PCP's so that this theorem will hold in general.

(2) There is no claim that the homotopy equivalence obtained by cutting and pasting with a "smoothed" PCP in a class with $\Phi = 0$ is h. It will only be normally cobordant to h. There is a further obstruction which will be discussed in §4 where the question of "what are the CP homotopy equivalences?" will be studied.

Proof. The necessity of the vanishing of Φ was already proven. Conversely, $\Phi = 0$ implies that for (N, n) in normal form, $\theta(\hat{n}) = 0$. Thus there is a simple homotopy equivalence $H: \bar{N}, \partial \to N \times I, \partial$ with normal invariant n. By the s-cobordism theorem $\bar{N}$ is abstractly $N \times I$ and comparing the identifications given by the s-cobordism theorem and by H we obtain the pasting map. It is clearly homotopic to the identity by H. □

§3. SPECIAL CUT-PASTES

In this section we classify specially cut-pastable simple homotopy equivalences. For closed manifolds, the result is surprisingly simple; the existence of a PCP together with $k_2(f) = 0$ is necessary and sufficient. The same techniques solve the problem of smoothing PCP's when $k_2(f) = 0$.

There is a unique (up to homotopy) essential map $\Omega(G/\text{Top}) \to K(Z_2, 1)$. Let cyl denote the mapping cylinder of this map. Define:

$$\mathrm{BSCP} = \mathrm{cyl} \cup_{\Omega(G/\mathrm{Top})} \mathrm{cyl}$$

$$\mathrm{BLI} = \mathrm{cyl} \cup_{\Omega(G/\mathrm{Top})} C(\Omega(G/\mathrm{Top}))$$

$$\mathrm{BRI} = C(\Omega(G/\mathrm{Top})) \cup_{\Omega(G/\mathrm{Top})} \mathrm{cyl}$$

where CX denotes the cone of X. (Notice that BLI and BRI are exactly the same space.) We will be concerned with the following tower of spaces

$$\begin{array}{ccccc} & & \mathrm{BSCP} & & \\ & \swarrow & & \searrow & \\ \mathrm{BLI} & & & & \mathrm{BRI} \\ & \searrow & & \swarrow & \\ & & \sum \Omega(G/\mathrm{Top}) & & \\ & & \downarrow & & \\ & & G/\mathrm{Top} & & \end{array}$$

where the top diamond of maps are defined by collapsing appropriate $K(Z_2, 1)$'s to cone points. The notation BSCP, BLI, BRI is intended to suggest classifying spaces for special cut-pastes, PCP's with $\pi_1 N \to$Left, Right sides injective, respectively.

Theorem 1. *Let $h: M', \partial M' \to M^n$, $M, n \geq 5$, be a homotopy equivalence restricting to a homeomorphism $h|_{\partial M'}: \partial M' \to \partial M$. Then h is* SCP *iff:*

(1) *h is a simple homotopy equivalence, and*

(2) *there is a lift*

$$\begin{array}{ccc} & & \mathrm{BSCP}, K(Z_2,1) \cup K(Z_2,1) \\ & \nearrow & \downarrow \\ M, \partial & \xrightarrow{\nu(h)} & G/\mathrm{Top}, \end{array}$$

where $\nu(h)$ in the normal invariant of h.

Proof. The necessity of (1) is Proposition 1.4. To show the necessity of (2) we show that the lift of $\nu(h)$ to $\Sigma\Omega(G/\mathrm{Top})$ defined in §1 lifts to BSCP given a SCP. Let (N, h, H) be the data for a SCP and (N, n) the associated PCP. The lift to $\Sigma\Omega(G/\mathrm{Top})$ rel $* \cup *$ is given by $c_N \circ \Sigma \hat{n}$. The map to BSCP is defined thus: There is an onto map

$$I \times \Omega(G/\mathrm{Top}) \longrightarrow \mathrm{BSCP}$$

given by crushing $\partial I \times \Omega(G/\mathrm{Top})$ to $\partial I \times K(Z_2, 1)$. Now there is a map $1_I \times \hat{n}: I \times N \to I \times \Omega(G/\mathrm{Top}) \to \mathrm{BSCP}$. Now we just have to extend this over $M_+ \cup M_-$ sending both sides to $K(Z_2, 1)$. This is no problem since we just have to be able to solve:

$$\begin{array}{ccc} \pi_1 N & \xrightarrow{n_*} & K(Z_2, 1) \\ \downarrow & \nearrow & \\ \pi_1 M_\pm & & \end{array}$$

which is clearly possible.

Conversely, let $\overline{\nu(h)}$: $M_1 \to$ BSCP be a lift of $\nu(h)$. We will get a Type I PCP by taking the transverse inverse image of $\Omega(G/\text{Top}) \subset$ BSCP. The quick way of doing this is to observe:

$$\text{BSCP} = K(Z_2, 1) \times \sum \Omega(G/\text{Top})^2$$

where $(G/\text{Top})^2$ is the second connective cover of (G/Top), and then take the transverse inverse image of $\Omega(G/\text{Top})^2 \subset \Sigma\,\Omega(G/\text{Top})^2$ and do surgery as in the proof of Proposition 2.1 to get a Type I PCP. The splitting BSCP $= K(Z_2, 1) \times \Sigma\,\Omega(G/\text{Top})^2$ is immediate from a splitting $G/\text{Top} \approx (G/\text{Top})^2 \times K(Z_2, 2)$. At the prime 2 there is such a splitting as G/Top is a product of Eilenberg–Maclane spaces[13, 26], and at the odd primes both sides are identical. Since all this is compatible over the rationals, there is such a splitting.

Now let (N, n) be a Type I PCP. As in the proof of Proposition 2.7 we can describe $M'' = M_+ \cup \bar{N} \cup M_-$ which is an SCP homotopy equivalence normally cobordant (§1) to $M' \xrightarrow{h} M$. As $L_{n+1}(\pi_1 N) \to L_{n+1}(\pi_1 M)$ is onto we can, by taking the action of $L_{n+1}(\pi_1 N)$ on $h\text{Top}(\bar{N}, \partial)$, arrange that the surgery obstruction of the normal cobordism between $M_+ \cup \bar{N} \cup M_-$ and M' be trivial, and therefore (M', h) coincides with $(M_+ \cup \bar{N} \cup M_-$, natural map to $M)$. As in Proposition 2.7 $\bar{N}$ is abstractly $N \times I$ or, when $\dim N = 4$, after first taking $\# \, k(S^2 \times S^2 \times I)$ so we get a map

$$M_+ \cup_1 (N \times I) \cup_{g^{-1}} M_- \xrightarrow{1 \cup G \cup 1} M$$

equivalent to $h: M' \to M$. □

Proposition 2. *A* PCP *cobordism class* $f: M \to \Sigma\,\Omega(G/\text{Top})$ *contains a Type* I *representative* iff *it lifts to* BSCP.

Remark. The proof of Theorem 1 shows that it suffices that two obstructions vanish for a PCP to represent a CP for a fixed simple homotopy equivalence h. The first is some version of Φ; we will complete the definition of Φ in this section. The second is an element of $\text{cok}[L_{n+1}(\pi_1 N) \to L_{n+1}(\pi_1 M)]$ where $\pi_1 N$ is determined by the normal form. We will briefly discuss this invariant in §4.

Proposition 3. *A* PCP *cobordism class* $\bar{f}: M \to \Sigma\,\Omega(G/\text{Top})$ *contains only* Type III *normal forms* iff $\bar{f}$ *does not lift to* BLI $\cup$ BRI *and* $k_2(f) = 0$.

The proof of this is an ambient surgery argument which we omit. □

Now define

$$\Phi(\bar{f}) = \begin{cases} \Phi(\bar{f}) \text{ as defined in 2 if } k_2(f) \neq 0 \\ \theta(\hat{n})\ (N, n) \text{ in normal form if } \bar{f} \text{ does not lift} \\ \qquad\qquad\qquad\qquad \text{to BLI} \cup \text{BRI} \\ \phi(f) \text{ otherwise.} \end{cases}$$

The proof of Proposition 2.5 shows that Φ is well defined and Theorem 2.7 can now be extended to the following:

Proposition 4. *A* PCP $\bar{f}: M \to \Sigma\,\Omega(G/\text{Top})$ *is smoothable into a* CP iff $\Phi(\bar{f}) = 0$.

Remark. Until this point, all the theorems and proofs apply equally well to the

Topological, Piecewise linear and smooth categories. (The proof of Theorem 1 has to be slightly changed though.) It is starting here that facts peculiar to the topological category begin being important.

For closed manifolds the lifting problem

$$\begin{array}{ccc} & & \mathrm{BSCP} \\ & \nearrow & \downarrow \\ & & \Sigma\Omega(G/\mathrm{Top}) \\ & & \downarrow \\ M & \longrightarrow & G/\mathrm{Top} \end{array}$$

to BSCP can be analyzed in terms of that to $\Sigma\Omega(G/\mathrm{Top})$. (For manifolds with boundary it is a relative lifting problem which turns out to be much more difficult.)

THEOREM 5. *A homotopy equivalence between closed manifolds* $h: M' \to M$ *is* SCP *iff*
(1) h *is a simple homotopy equivalence*
(2) $\nu(h)$ *lifts to* $\Sigma\Omega(G/\mathrm{Top})$
(3) $k_2(\nu(h)) = 0$.

Proof. Necessity was already done.

For sufficiency, it is convenient to observe that there is a map $\Sigma\Omega(G/\mathrm{Top})^2 \to \mathrm{BSCP}$ which is a 2-fold cover. It is not hard to see that a PCP has a representative (N, n) with $n_*: \pi_1 N \to Z_2$ trivial iff there is a lift to $\Sigma\Omega(G/\mathrm{Top})^2$. We now observe that if (N, n) is any PCP for a normal invariant f with $k_2(f) = 0$ then there is an $\bar{n}$ such that $(N, \bar{n})$ is a PCP for f and $\bar{n}_* = 0$ and hence a lift to $\Sigma\Omega(G/\mathrm{Top})^2$ proving the theorem. $\bar{n}$ is produced by Theorem 1 as follows: Regard G/Top as $(G/\mathrm{Top})^2 \times K(Z_2, 2)$ and $\Omega(G/\mathrm{Top})$ as $\Omega(G/\mathrm{Top})^2 \times K(Z_2, 1)$ and let $\bar{n}$ agree with n as a map to $\Omega(G/\mathrm{Top})^2$ but instead be trivial on the $K(Z_2, 1)$ coordinate. It is straightforward that this new PCP also gives f as $f_2: M \to K(Z_2, 2)$ is nullhomotopic. □

Remarks. It is only in the last line that the closedness of M is used. For M with boundary we thus see that the vanishing of $k_2^{\partial}(f)$ together with the existence of a PCP is sufficient. However, it is easy to give examples showing that $k_2^{\partial}(f) \neq 0$ even for normal invariants of SCP homotopy equivalences.

COROLLARY 6. *For a closed manifold* M^n, *with* $n \geq 5$ *and* $Sq^2: H^2(M; Z_2) \to H^4(M; Z_2)$ *injective then the following are equivalent for a homotopy equivalence* $h: M' \to M$:
(1) h *is* CP
(2) h *is* SCP
(3) h *is a simple homotopy equivalence and* $\nu(h)$ *lifts to* $\Sigma\Omega(G/\mathrm{Top})$.

Proof. We only have to show that (3) implies (2). Thus we only have to show that (3) implies $k_2(f) = 0$, but this is clear as $Sq^2: H^2(\Sigma\Omega(G/\mathrm{Top}); Z_2) \to H^4(\Sigma\Omega(G/\mathrm{Top}); Z_2)$ vanishes by the commutative diagram

$$\begin{array}{ccc} H^2(\Sigma\Omega(G/\mathrm{Top}); Z_2) & \xrightarrow{Sq^2} & H^4(\Sigma\Omega(G/\mathrm{Top}); Z_2) \\ \approx \uparrow \delta & & \approx \uparrow \delta \\ H^1(\Omega(G/\mathrm{Top}); Z_2) & \xrightarrow{Sq^2 = 0} & H^3(\Omega(G/\mathrm{Top}); Z_2) \end{array}$$

□

§4. WHICH HOMOTOPY EQUIVALENCES ARE CUT-PASTABLE?

4A. *The invariant* $\psi(\bar{f}, h)$

The first problem we will discuss is the obstruction to obtaining a particular (simple) homotopy equivalence $h: M' \to M$ with normal invariant $f: M \to G/\mathrm{Top}$ by a CP within a given PCP cobordism class $\bar{f}$ lifting f to $\Sigma\,\Omega(G/\mathrm{Top})$. Using the fact that for all the normal forms (N, n) except Type V, $L_{n+1}(\pi_1 N) \to L_{n+1}(\pi_1 M)$ is onto, in these cases the only obstruction is $\Phi(\bar{f})$, the smoothing obstruction. We therefore assume that the normal form is of Type V (see the proof of Theorem 2.1). In this case of twisted h, the vanishing of Φ seems only to imply that $h: M' \to M$ is normally cobordant to a CP homotopy equivalence. The first goal of this section is to describe precisely the obstruction to obtaining exactly h. Recall the surgery exact sequence of Wall and Sullivan[32].

$$\left[\sum M : G/\mathrm{Top}\right] \xrightarrow{\theta} L_{n+1}(\pi_1 M) \xrightarrow{\alpha} h\mathrm{Top}(M) \to [M : G/\mathrm{Top}] \xrightarrow{\theta} L_n(\pi_1 M).$$

Let $\bar{f}$ be a lifting of f to $\Sigma\Omega(G/\mathrm{Top})$ with $0 \neq k^2(f) \in \mathrm{Im}\, H^2(\pi_1 M; Z_2)$ the preimage corresponding to $Z_2 \to E \to \pi$, and $h: M' \to M$ a simple homotopy equivalence with normal invariant f. Suppose further that $\Phi(\bar{f}) = 0$. We define

$$\psi(\bar{f}, h) \in L_{n+1}(\pi_1 M)/(\mathrm{Im}\,\theta([\Sigma\, M : G/\mathrm{Top}]) \oplus L_{n+1}(E))$$

as follows: As $\Phi(\bar{f}) = 0$ we can smooth $\bar{f}$ to a CP (N, g, G) with $\bar{G}: M(N, g) \to M$ normally cobordant to h. $\psi(\bar{f}, h)$ is the surgery obstruction of this normal cobordism rel ∂ reduced to this quotient.

THEOREM 1. *$\psi(\bar{f}, h)$ is well defined. Moreover, $\psi(\bar{f}, h) = 0$* iff *h is cut pastable by a* CP *in the cobordism class of $\bar{f}$.*

Proof. The key point of the well definedness of ψ was already done. According to Proposition 2.4 we can assume that the CP used to define ψ is in normal form. Suppose (N_1, g_1, G_1), (N_2, g_2, G_2) are two normal form CP's and (P, p) is a PCP cobordism between their underlying PCP's also in normal form. Naturality implies that the surgery obstruction of the normal cobordism between $M(N_1, g_1)$ and $M(N_-, g_2)$ is the image of $\theta(\hat{p})$ under the map $L_{n+1}(\pi_1 P) \to L_{n+1}(\pi_1 M)$ which factors through $L_{n+1}(E)$. Thus the indeterminacy is killed by taking the quotient $\mathrm{Im}[\Sigma\, M : G/\mathrm{Top}] \oplus L_{n+1}(E)$.

By definition, if h is CP within the class of $\bar{f}$, $\psi(\bar{f}, h) = 0$. Conversely, we take a CP (N, g, G) in normal form. As $\psi(\bar{f}, h) = 0$ we can let $L_{n+1}(E)$ act on $h\mathrm{Top}(N \times I, \partial)$ so that h is normally cobordant to a CP homotopy equivalence by normal cobordism with surgery obstruction in $\mathrm{Im}\,[\Sigma M : G/\mathrm{Top}]$ but this image acts trivially on $H\mathrm{Top}(M)$, so h is in fact CP. □

Remarks. (1) The realization theorem for surgery obstructions implies that $\psi(\bar{f}, -)$ is onto its range.

(2) In many cases of interest the cokernel of $L_{n+1}(E) \to L_{n+1}(\pi)$ is trivial. For example, this is the case if π is cyclic. This is trivially always the case if $H^2(\pi, Z_2) = 0$, e.g. $\pi = Z$ or π finite of odd order. The most elementary example where it is nonzero is $\pi = Z \oplus Z$, E the (unique) nontrivial extension. In this case:

$$\mathrm{cok}\, L_4(E) \to L_4(Z \oplus Z) = Z_2.$$

(This Z_2 is always in the image of $[\Sigma M : G/\mathrm{Top}]$ however.)

COROLLARY 2. *A simple homotopy equivalence* $h: M' \to M$ *is normally cobordant to a* CP *homotopy equivalence* iff *there is a lift* $\overline{v(h)}$ *of the normal invariant* $v(h)$ *to* $\Sigma\Omega(G/\mathrm{Top})$ *such that* $\Phi(\overline{v(h)}) = 0$. *h is itself* CP iff *in addition* $k_2(v(h)) = 0$, $v(h)$ *is nontrivial on* $\pi_2 M$, *or for the above lift* $\psi(h, \overline{v(h)}) = 0$. □

Corollary 2 is an effective summary of the classification of CP homotopy equivalences that the methods described so far can produce. It is undesirable that secondary, and sometimes tertiary obstructions arise. We would like to have just a single computable obstruction. Corollary 3.6 gives a case where the existence of a PCP implies the existence of a CP. We will try to extend this to other cases. Much success can be achieved in the untwisted case.

Even in the twisted case one can often show that the existence of a PCP implies that the homotopy equivalence is the result of a sequence of CP's.

4B. ***Avoiding*** Φ ***for Type II and IV PCP's (untwisted h)***

In this subsection we develop two techniques for changing the lift of a normal invariant to $\Sigma\Omega(G/\mathrm{Top})$. The first is based on the idea of the old result that a simply connected Poincaré complex has the homotopy type of a (topological) manifold iff it possesses a normal invariant. The second technique roughly speaking trades low dimensional Kervaire classes for higher ones until one can show that Φ vanishes.

Since we are dealing only with Type II and IV PCP's we can use the *a priori* definition ϕ given in §2 for Φ.

Notation: $\pi = \pi_1 M^n$
$h: M' \to M$ a homotopy equivalence
$f: M \to G/\mathrm{Top}$ its normal invariant, $\bar{f}$ a lift of f to $\Sigma\Omega(G/\mathrm{Top})$,
(N, n) a PCP corresponding to h, or f, or $\bar{f}$ of Type II or IV
$\ker_n(\pi) = \ker(L_n^s(\pi \times Z_2) \to L_N^s(\pi))$
$\phi(\bar{f}) = \Phi(\bar{f}) \in \ker_n(\pi)$.

If M is simply connected then we are concerned with $\ker_n(0) = Z, 0, 0, Z_2$, $n \equiv 0, 1, 2, 3$ mod 4.

PROPOSITION 3. *If* $\pi_1 M^n = 0$ *then* $\phi(\bar{f}) = 0$ *unless* $n \equiv 3$ mod 4. *For* $n \equiv 3$ mod 4 *there is a* PCP (N, n) *on* S^n *such that* $\phi(N, n) \neq 0$.

Proof. If $n \equiv 0$ mod 4, $L_0(Z_2) = Z \oplus Z$ and the surgery obstruction is calculable by signatures and signatures of 2-fold covers. Since the obstructions arising here are from closed manifolds, the vanishing of the simply connected obstruction, i.e. signatures, implies the vanishing of signatures of two-fold covers and hence the obstruction in $L_0(Z_2)$ vanishes.

For $n = 4k + 3$, $L_3(Z_2) = Z_2$ and we have an isomorphism $L_3(Z) \to L_3(Z_2)$ induced by the group homeomorphism. Let (N, n) be $(S^1 \times S^{4k+1}, k_1 p_1 + k_{4k+1} p_2)$, where $k_{4i+1}: S^{4i+1} \to \Omega(G/\mathrm{Top})$ are induced by Kervaire problems and p_i is the ith projection (N, n) is a PCP for S^{4k+3} which by [25] is easily seen to have $\theta(N, n) \neq 0$. □

COROLLARY 4. *For* M^n, $n \geq 5$, *closed and simply connected* $h: M' \to M$ *is* CP iff $v(h)$ *lifts to* $\Sigma\Omega(G/\mathrm{Top})$.

Proof. We just have to show this for M^{4n+3}. Let (N, n) be a PCP of Type IV for (N, n). (Theorem 2.3 guarantees that we do not have Type V PCP's ever occurring.) If $\theta(N, n) \neq 0$, we can connect sum with the above PCP on S^{4n+3}. It does not change the normal invariants, but ϕ is clearly additive so that on the connected sum ϕ vanishes. Corollary 2 now applies. □

This corollary holds even if M has boundary, see §4C.

Using [33, Theorem 12], Corollary 4 holds for π of odd order. We will be able to prove this in more generality later, e.g. if $H_*(\pi; Z_{(2)}) = 0$ Corollary 4 holds.

Definition. $\ker_n^{cl}(\pi)$ is the subgroup of $\ker L_n^S(\pi \times Z_2) \to L_n^S(\pi)$ realizable by problems on closed topological manifolds.

Example. $\ker_n^{cl}(0) = 0$, $n \not\equiv 3(4)$, $\ker_n^{cl}(0) = Z_2$, $n \equiv 3(4)$. Observe that $\ker_n^{cl}(\pi)$ need not be isomorphic to $\ker_{n+4}^{cl}(\pi)$, but by taking products with CP^2, is canonically a subgroup of it.

PROPOSITION. $\ker_n^{cl}(Z \times \pi) \approx \ker_n^{cl}(\pi) \oplus \ker_{n-1}^{h,cl}(\pi)$.

Proof. This is straightforward consequence of [32].

COROLLARY 6. *For M^n closed orientable with $\pi_1 M^n = Z$, $n \geq 5$, $h: M' \to M$ is* CP iff $\nu(h)$ *lifts to* $\Sigma\Omega(G/\mathrm{Top})$.

Addendum. The same result holds if $\pi_1 M$ is $\pi_1(S^3 - K)$ for K a knot.

Proof. Proposition 5 shows that $\ker_n^{cl}(Z) = 0$ unless $n \equiv 0, 3 \bmod 4$. For $n \equiv 3 \bmod 4$, $\ker_n^{cl}(0) \to \ker_n^{cl}(Z)$ is an isomorphism so we can argue as in Corollary 4. For $n \equiv 0 \bmod 4$ consider

$$\beta: S^1 \times S^{n-1} \xrightarrow{\alpha P_2} \sum \Omega(G/\mathrm{Top})$$

where α is the element of $\pi_{n-1}(\Sigma\Omega(G/\mathrm{Top}))$ constructed in Proposition 3. The normal invariant of β is trivial and $\phi(\beta) \neq 0$. Let C be the generating circle of $\pi_1 M$ assumed without loss of generality not to intersect N for some representative (N, n) for $\bar{f}$; then if $\bar{f}: M \to \Sigma\Omega(G/\mathrm{Top})$ is a lift with $\phi(\bar{f}) \neq 0$ the composite

$$M \to M^n \cup_C S^1 \times S^{n-1} \xrightarrow{\bar{f} \cup \beta} \Sigma\Omega(G/\mathrm{Top})$$

is another lift with ϕ vanishing.

To prove the addendum it suffices to show for π a classical knot group $\ker_n^{cl}(Z) \to \ker_n^{cl}(\pi)$ is an isomorphism where $Z \to \pi$ is the meridional inclusion. Injectivity is trivial; the composition with the map induced by abelianization $\pi \to Z$ induces a splitting. For surjectivity observe that

$$\Omega_n(K(Z,1) \times K(Z_2,1) \times G/\mathrm{Top}) \to \Omega_n(K(\pi,1) \times K(Z_2,1) \times G/\mathrm{Top})$$

is onto (in fact an isomorphism by the Atiayah–Hirzebruch spectral sequence). As surgery obstructions for closed manifolds factor through bordism (see [28, 32]) we are done. □

Using [7] we can prove the same result for free groups. For surface groups we do not have as general results as $H^2(\pi; Z_2) \neq 0$. If we exclude, by hypothesis, Type V CP's by the same methods but more detailed calculation one can prove the same characterization.

COROLLARY 7. *If M is closed orientable with $\pi_1 M$ a surface group $h: M' \to M$ an untwisted homotopy equivalence then h is* CP iff $\nu(h)$ *lifts to* $\Sigma\Omega(G/\mathrm{Top})$.

This is the apparent limit that these techniques can be used to prove. All these results can be proven more easily and in more generality using the next method, that of PCP replication. The above technique however will be used in §4C in an important way.

Let (N, n) be a Type II or IV PCP. Suppose $n': N \to \Omega(G/\mathrm{Top})$ is another map with $n_* = n'_*: \pi_1 N \to \Omega(G/\mathrm{Top})$ and $\theta(\hat{n}') = 0 \in L_n(\pi_1 N)$. We now form a new PCP as follows: $\pi_1(M_+) = \pi_1(M)$. Let N_+ be a copy of N pushed off into M_+. The new PCP is $(N_+ \cup N, n' \cup (n - n'))$ where $(n' - n)$ is intended in the loop multiplication sense. It is evident that these two PCP's have the same normal invariant.

PROPOSITION 8. $\Phi(N_+ \cup N, n' \cup (n - n')) = 0$.

Proof. Using the intrinsic definition ϕ of Φ it is clearly additive over components. Clearly $\phi(N_+, n') = 0$. Notice now that $(n - n')_*: \pi_1 N \to \pi_1(\Omega(G/\mathrm{Top}))$ is trivial, so $\phi(N, n - n') = 0$. (One can do surgery on u_-, in the notation in §2, and then the two nonintersecting PCP's both can individually be smoothed, see Fig. 3). □

The rest of this subsection is devoted to giving conditions on G so that for any homeomorphism $h: G \to Z_2$ and any closed manifold N with $\pi_1 N = G$ there is a map $n': N \to \Omega(G/\mathrm{Top})$ with $n'_* = h$ and $\theta(\hat{n}') = 0$. If this is solvable for G and any $k - 1$-manifold and homeomorphism, G will be said to be *k-amenable*. Observe we have shown that $G \times Z_2$ k-amenable implies that the analogue of Corollary 7 holds for M k-dimensional ($k > 4$) and $\pi_1 M = G$. G is *amenable* if it is k-amenable for all $k > 4$. After elementary observations a sufficient condition for k-amenability is given in terms of the surgery characteristic class studied in the "Oozing problem".

THEOREM 9. *If $H_*(\pi; Z_{(2)}) = 0$, then π and $\pi \times Z_2$ are amenable.*

Proof. First observe that the trivial group is amenable since by modifying any n with $n_* = h$ on the top cell we can arrange $\theta(\hat{n}) = 0$. Similarly $\pi = Z_2$ is amenable; we have at most to modify n in the tubular neighborhood of a circle generating π_1. Now let $\pi_1 N$ be as in the hypothesis. There is a map $N \to \Omega(G/\mathrm{Top})$ realizing h by the description of

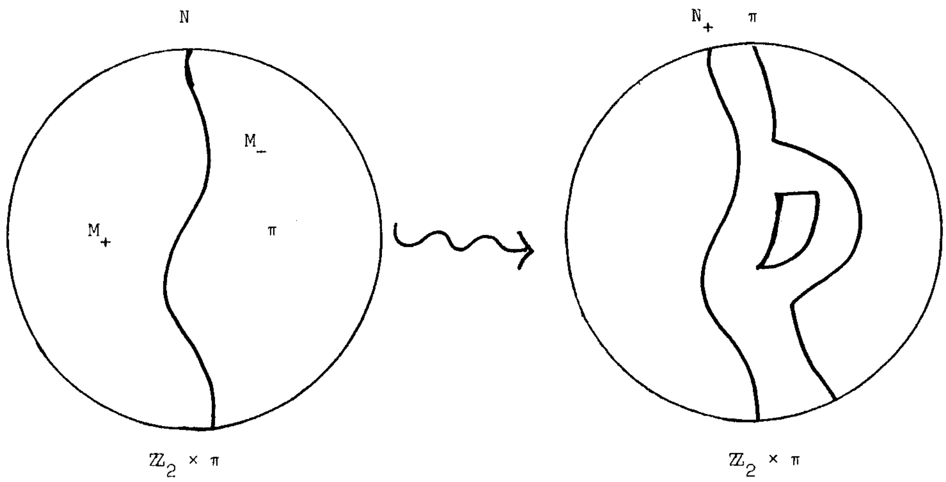


Fig. 3. PCP Replication I.

$\Omega(G/\mathrm{Top})_{(2)}$ as a product of Eilenberg–Maclane spaces. The surgery obstructions factor through

$$\begin{array}{ccc} \Omega_{n-1}(K(Z_2,1)\times\Omega(G/\mathrm{Top})) & \longrightarrow & \Omega_{n-1}(K(\pi_1 N,1)\times\Omega(G/\mathrm{Top})) \\ \downarrow & & \downarrow \\ L_n(Z_2) & \longrightarrow & L_n(\pi_1 N) \end{array}$$

Since the top line is a $Z_{(2)}$ isomorphism by multiplying n by an *odd* number l, $\theta(ln) \in \mathrm{Im}\, L_n(Z_2)$ so that by further modification on the top two cells we can make $\theta = 0$. As l is odd, $(ln)_* = n_* = h$. □

This implies Corollary 4 and the remarks following it. We also have the following general fact:

PROPOSITION 10. *Φ is up to indeterminacy (for Types* II *and* IV *lifts) of order* 2.

Proof. Let $n'': N \to \Omega(G/\mathrm{Top})$ just be the projection of n to $K(Z_2, 1)$, see §3. Then Φ (replicated PCP) is of order two. □

This 2-adic nature of Φ allows us to study only the surgery obstruction in $L_n(\pi)\otimes Z_{(2)}$. (Multiply as in the proof of Theorem 9.) If $f: (M,\partial)\to G/\mathrm{Top}$ is a normal invariant there is a simple formula[28, 33] for $\theta(f)_{(2)} \in L_n(\pi)\otimes Z_{(2)}$. There are homeomorphisms

$$\mathscr{I}_n: H_n(\pi; Z_{(2)}) \longrightarrow L_n(\pi)\otimes Z_{(2)}$$

$$\kappa_n: H_n(\pi; Z_2) \longrightarrow L_{n+2}(\pi)\otimes Z_{(2)}$$

and classes

$V =$ Total Wu Class $\in H^*(\mathrm{BSTop}; Z_2)$

$\mathscr{L} \in H^{4i}(\mathrm{BSTop}; Z_{(2)})$, the Morgan–Sullivan class[20]

$l \in H^{4i}(G/\mathrm{Top}; Z_{(2)})$, $k \in H^{4i+2}(G/\mathrm{Top}; Z_2)$ (Milnor and Kervaire classes)

such that if

$f: (M,\partial)\to G/\mathrm{Top}$ is a normal invariant

$g: M \to K(\pi, 1)$ classifies $\pi_1 M$

$\nu: M \to \mathrm{BSTop}$ classifies the normal bundle,

then

$$\theta(f)_{(2)} = A_* g_*((\nu^*(\mathscr{L})\cup f^*(l) + \nu^*(\mathscr{L})\cup f^*(k)) + \delta^*(\nu^*(VSq^1V)\cup f^*(k))\cap [M, \partial M]) \qquad (*)$$

where δ is the Bockstein and

$$A_*: \oplus H_{n-4i}(\pi; Z_{(2)}) \oplus H_{n-4i-2}(\pi; Z_2) \xrightarrow{\oplus \mathscr{I}_{n-4i} \oplus \kappa_{n-4i-2}} \oplus L_{n-4i}(\pi) \otimes Z_{(2)} \xrightarrow{\text{sum}} L_n(\pi) \otimes Z_{(2)}.$$

This generalizes the formula of [20] for the simply connected case.

THEOREM 11. *If* $\mathscr{K}^s_{k-2}(\pi) = 0$, *then* π *is* k-*amenable.*

Proof. The formula (*) above shows that $\theta(f)_{(2)}$ only depends on the graded class $Q(f) = v^*(\mathscr{L}) \cup f^*(l+k) + \delta^*(VSq^1V) \cup f^*(k))$. Since the k and l classes describe a splitting of $(G/\mathrm{Top})_{(2)}$ (as can be seen by calculating Kervaire and Milnor problems) we can arrange for $f^*(k)$ and $f^*(l)$ to be arbitrary elements of order two in $H^{4i+2}(\Sigma N; Z_2)$ and $H^{4i}(\Sigma N; Z_{(2)})$. The homeomorphism $\pi \to Z_2$ determines $f^*(k_2)$ and conversely. The problem then is: "Given M arbitrary and $f^*(k_2) \in H^2(M; Z_2)$ is there an $\bar{f}: M \to G/\mathrm{Top}$ with $\bar{f}^*(k_2) = f^*(k_2)$ and $\theta(\bar{f}) = 0$!"

If we examine (*) we see that only the even dimensional grades of $Q(f)$ matter. (In fact, the vanishing of the odd dimensional Wu classes implies that only even grades occur.) We can write

$$Q(f)_{4m} = f^*(l_{4m}) + C(f^*(k_i), i < 4m, f^*(l_j), j < 4m, \mathscr{L}(M), V(M))$$

$$Q(f)_{4m+2} = f^*(k_{4m+2}) + D(f^*(k_i), i < 4m+2, f^*(l_j), j < 4m+2, \mathscr{L}(M), V(M)).$$

Since $f^*(k_{4m+2})$ and $f^*(l_{4m})$ are arbitrary two-torsion for $m > 0$, we can arrange by setting f inductively that

$$f^*(l_{4m}) = -C(f^*(k_i), i < 4m, f^*(l_j), j < 4m, \mathscr{L}(M), V(M)), m > 0$$

$$f^*(k_{4m+2}) = -D(f^*(k_i), i < 4m+2, f^*(l_j), j < 4m, \mathscr{L}(M), V(M)), m > 0$$

$f^*(k_2)$ is initial data.

For this f, (*) gives

$$\theta(f)_{(2)} = \mathscr{K}_{k-2}(g_*(f^*(k_2) \cap [M, \partial M])) \in L_k(\pi_1 M)_{(2)}. \qquad (**)$$

By hypothesis $\mathscr{K}_{k-2} = 0$ so $\theta(f)_{(2)} = 0$. Multiplying f by an odd number we get $\theta(f) = 0$ and $f^*(k_2)$ as desired.

There is a large recent literature regarding the $\mathscr{I}$ and κ classes for various finite groups in various L groups (i.e. L^{cl}, L^h, L^s, L', etc.)[10, 19, 29, 30]. Using Theorem 11 and the comments before Theorem 9 it is easy to translate the results of [29] into results about the sufficiency of the existence of PCP's when h is untwisted for the existence of CP's. For example:

COROLLARY 12. Any abelian group G is amenable above a certain dimension (depending on G). In particular G is k-amenable for $k > \max(5, \dim_Q G \otimes Q + 3)$.

COROLLARY 13. All finite groups with abelian 2-sylow subgroups are k-amenable, $k > 5$. Addendum: The same is true if the 2-sylow subgroup is a product of dihedral groups and abelian groups.

Proofs. If π has abelian 2-sylow subgroup then [29] have shown that $\mathcal{K}_i^s(\pi)=0$, $i>3$, $\mathcal{K}_i^h(\pi)=0$, $i>2$, $\mathcal{K}_i^p(\pi)=0$, $i>1$. This implies Corollary 12, and Corollary 13 if G is finite. Let $G=Z^k\times\pi$, π finite. [Shaneson] and [Ranicki] imply

$$\kappa_i^s(Z\times\pi)=\kappa_i^s(\pi)+\kappa_{i-1}^s(\pi)$$

$$\kappa_i^h(Z\times\pi)=\kappa_i^h(\pi)+\kappa_{i-1}^p(\pi)$$

so for $i>4$, $\kappa_i^h(Z\times\pi)=0$ and $i>5$, $\kappa_i^s(Z^2\times\pi)=0$, etc.

For $Z_2\times D_2k_1\times\ldots\times D_2k_n$, κ_i^s vanishes for $i>1$, since Quillen has shown[21]

$$\oplus\ H_*(E;Z_2)\to H_*(D_2k_n;Z_2)$$

is onto. The Elementary abelian subgroups E of D_2k_m: Kunneth formula implies the same holds true for the product, naturality implies the conclusion since κ_i^s (elementary abelian groups) $=0$, $i>1$. (This is a modification of the proof in [29] for dihedral groups.)

Similarly for $G=Z_2\times$ Abelian $\times D_2k_1\times\ldots\times D_2k_n$, the homology is generated by abelian subgroups so again naturality, together with [29] suffice to show $\kappa_1^s(G)=0$, $i>3$. □

Certainly more results of this type can be proven in the same way and any new results on the oozing problem can be translated into this context. The following theorem, which closes this subsection, summarizes and extends the results of this subsection:

THEOREM 14. *Let $h\colon M'\to M$, $\dim M>5$, be a simple homotopy equivalence* rel ∂ *between manifolds. Suppose $\nu(h)$ lifts to $\Sigma\Omega(G/\mathrm{Top})$ and that $k_2(\nu(h))\notin H^2(\pi_1M;\mathbb{Z}_2)$. (If M is closed we can allow $k_2(\nu(h))=0$. In* §4C *we will extend this to manifolds with boundary.) Then if M is orientable and any of the following hold:*

(1) *π_1M finite with abelian 2-sylow subgroup or 2-sylow subgroup being a product of dihedral groups*

(2) *π_1M abelian,* $\dim M>\max(5,\dim_Q\pi_1M\otimes Q+3)$ *or*

(3) $H_*(\pi_1M;Z_{(2)})=0$ *for* $*>n$, $\dim M>n+4$;

h is then cut-pastable.

Proof. We just have to prove the sufficiency of (3). (Corollaries 6 and 7 and Theorem 9 are special cases of this.) According to [29] there is a commutative diagram

$$\begin{array}{ccc}
H_i(Z_2;Z_2)\otimes H_{k-i}(\pi;Z_{(2)}) & \longrightarrow & H_k(Z_2\times\pi;Z_2)\\
\Big\downarrow{\scriptstyle \kappa_i\otimes\mathcal{I}^{k-i}} & & \Big\downarrow{\scriptstyle \kappa_k}\\
L_{i+2}(Z_2)_{(2)}\otimes L^{k-i}(\pi)_{(2)} & \longrightarrow & L_{1+2}(Z_2\times\pi).
\end{array}$$

(For the definition and properties of the L^k and $\mathcal{I}^k$ see [23].) For $k>n+2$ either $i>2$ or $k-i>n$; in either case $\mathcal{K}_i\otimes\mathcal{I}^{k-i}=0$. Since the sum of these (over i) generate $H_k(Z_2\times\pi;Z_2)$, $\mathcal{K}_k=0$ so above dimension $n+4$, $\pi\times Z_2$ is amenable. □

Remarks. (1) Using [29] nonorientable results can also be obtained in the same way using a modified formula (**) and their modified $\mathcal{K}$-classes.

(2) Unfortunately, not all finitely presented groups have finite ooze[37], and for these, all our techniques fail.

4C. *Remarks on the remaining cases*

First we complete the discussion of $k_2(f) = 0$. The remaining case is that of Type III PCP's. We exploit the particular form of the induced map n_* to give a modified version of PCP replication.

PROPOSITION 15. *Theorem* 14 *holds even if M has boundary.*

Proof. All the groups π listed there have $\pi \times Z_2 \times Z_2$ k-amenable in the range given. (For (3) replace Z_2 by $Z_2 \times Z_2$ in the proof, and use the fact that $\kappa_1{}^s(Z_2 \times Z_2) = 0$, $i > 1$.) We show that if $\pi \times Z_2 \times Z_2$ is k-amenable, then we can avoid the obstruction ϕ for k-dimensional problems. Let $N_\pm$ denote copies of N pushed off into $M_\pm$. Let $n_1: N \to \Omega(G/\mathrm{Top})$ be a map inducing $p_2: \pi \times Z_2 \times Z_2 \to Z_2$ and $n_2: N_- \to \Omega(G/\mathrm{Top})$ be a map inducing $p_3: \pi \times Z_2 \times Z_2 \to Z_2$; both having vanishing surgery obstruction. The new PCP is $(N_+ \cup N \cup N_-, n_1 \cup (n - n_1 - n_2) \cup n_2)$. Clearly this has the correct normal invariant. Observe that

$$\ker(\pi_1 N_+ \to \pi_1 M_+) \subset \ker(\pi_1 N_+ \to \pi_1(\Omega(G/\mathrm{Top})))$$

$$\ker(\pi_1 N_- \to \pi_1 M_-) \subset \ker(\pi_1 N_- \to \pi_1(\Omega(G/\mathrm{Top})))$$

so that we can do surgery as in §2 on the two kernels $\pi_1 N_\pm \to \pi_1 M_\pm$, to get $\bar{N}_\pm$, $\bar{n}_\pm$. Notice that these surgeries do not intersect N. Let $C_\pm$ denote the region bounded between N and $\bar{N}_\pm$. $(n - n_1 - n_2)_*$: $\pi_1 N \to \pi_1(\Omega(G/\mathrm{Top}))$ is trivial so there is no obstruction to doing surgery on $\ker(\pi_1 N \to \pi_1 C_\pm)$. (Do the surgeries in the "shadows" of the surgeries of $N_\pm$.) Since $\pi_1 C_\pm \xrightarrow{\approx} \pi M_\pm$, we have succeeded in killing $\ker \pi_1 N \to \pi_1 M_\pm$, i.e. $\ker \pi_1 N \to \pi_1 M$. Let $(\bar{N}, \bar{n})$ be the resulting PCP component. The surgery obstructions of $(\bar{N}_+ \cup \bar{N} \cup \bar{N}_-, \bar{n}_+ \cup \bar{n} \cup \bar{n}_-)$ vanish for each component. For $(\bar{N}_+, \bar{n}_+)$ and $(\bar{N}_-, \bar{n}_-)$ this follows from the fact that $\theta(\hat{n}_1) = \theta(\hat{n}_2) = 0$. Now $\theta((n - \hat{n}_1 - n_2)) = (\hat{n})$ so the image of $\theta(\hat{n})$ in $L_n(\pi_1 M)$ is zero, but this is just, as $\pi_1 \bar{N} \to \pi_1 M$ is an isomorphism, $\theta(\hat{\bar{n}}) = 0$ (see Fig. 4). □

Our most powerful tool available, PCP replication, does not seem to work at all for Type V PCP's to get a new lift which is smoothable into a CP, let alone arranging that for the new lift $\bar{f}$, $\phi(\bar{f}, h) = 0$. Before we give any general results it is useful to study the simplest case in detail, that of cyclic groups.

THEOREM 16. *If $\pi_1 M$ is cyclic the conclusion of Theorem* 14 *holds even for twisted h.*

Proof. The only nontrivial Z_2 extension of Z_n is Z_{2n} and $L_k{}^s(Z_{2n}) \to L_k{}^s(Z_n)$ is always onto, so we need only deal with Φ; $\psi \equiv 0$. $\Phi(\bar{f}) \in \ker L_k(Z_2 \times Z_{2n}) \to L_k(Z_n)$. Replication shows that up to indeterminacy of lift $\Phi(\bar{f}) \in \ker L_k(Z_{2n}) \to L_k(Z_n)$: For let (N, n) be a PCP in normal form. Let $n': N_+ \to \Omega(G/\mathrm{Top})$ be a map with $n'_*: \pi_1 N_+ \to \pi_1(\Omega(G/\mathrm{Top}))$ trivial and $\theta(\hat{n}') = \theta(n)$. This can always be found using Theorem 11 and the calculations of [29]. Now $\theta(N_+ \cup N,\ n' \cup (n - n'))$ is a new PCP representing the same normal invariant. $\theta(n - n') = 0$. We can do surgery on $\ker \pi_1 N_+ \to \pi_1 M_+$ and it is easy to see that

$$\Phi(N'_+ \cup N, n' \cup (n - n')) = s_* p_* \Phi(N, n)$$

where

$$p_*: L_n(Z_2 \times E) \to L_n(E)$$

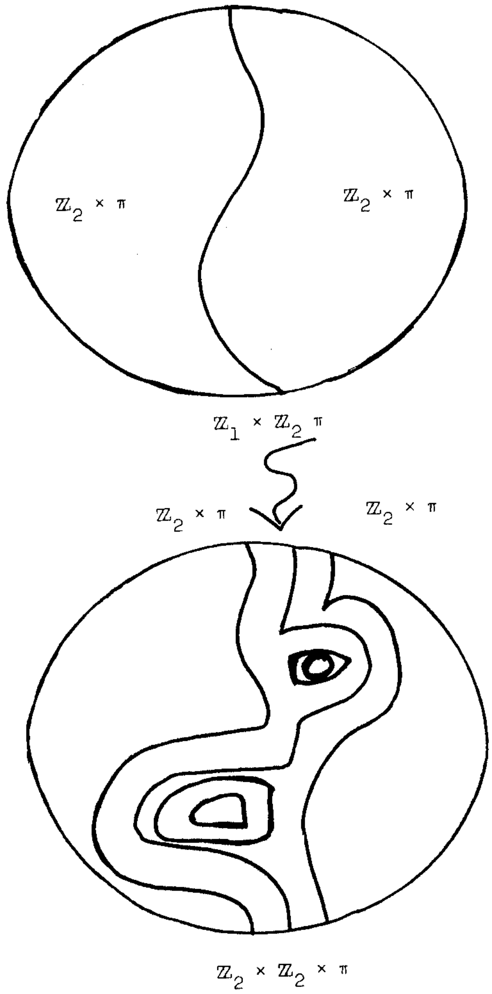


Fig. 4. PCP Replication II.

and

$$s_*: L_n(E) \to L_n(Z_2 \times E).$$

For cyclic groups $\mathscr{K}_i^s = 0, i > 1$, $\mathscr{I}_j^s = 0, j > 0$, and $\mathscr{K}_1^s$ is $1-1$. It follows easily that obstructions for problems between closed manifolds in $L_m(Z_{2n})$ can be detected by their image in $L_m(Z_2)$. In particular $\phi(\bar{f}) = 0$ for $\bar{f}$ corresponding to the second PCP. □

Remark. The PCP cobordism trick often lets one replace $\Phi(\bar{f})$ up to indeterminacy with an element of $\ker L_n(E) \to L_n(\pi)$. For example for $\pi = Z_2 \times Z_2$ there are two choices for E, $Z_2 \times Z_4$ and D_4. $\ker L_n^{cl}(D_4) \to L_n^{cl}(Z_1 \times Z_2) = 0$ and $\ker L_n^{cl}(Z_2 Z_4) \to L_n^{cl}(Z_2 \times Z_2) = 0$ unless $n \equiv 0(4)$ when it is $Z_2 \in \operatorname{Im} L_0(Z \times Z)$ according to [19]. We can kill this element by the modification technique of §4B. ■

In general, let $k_2(f) = \operatorname{Im} \alpha(f) \in H^2(\pi_1 M; Z_2)$ and E be the Z_2 extension of $\pi_1 M$ corresponding to $\phi(f)$. Then we have:

PROPOSITION 17. *Let $h: M' \to M^n$, $n \geq 5$, be a simple homotopy equivalence with $f = \nu(h)$ lifting to $\Sigma\Omega(G/\mathrm{Top})$, then if $\kappa_{n-2}^s((Z_2 \times E_{\alpha(f)}) = 0$, then h is the result of a sequence two cut-pastes.*

Proof. This hypothesis lets one do PCP replication as in §4B. However, we cannot kill $\pi_1 N_+ \to \pi_1 M$ by doing surgeries not intersecting N. First cutting and pasting along N and *then* doing the surgeries on N_+ we can do another cut-paste along $\bar{N}_+$ to get exactly $h: M' \to M$ (no normal cobordism problems because $L_{n+1}(\pi_1 \bar{N}_+) \to L_{n+1}(\pi_1 M)$ is onto). □

This easily leads to a characterization of which homotopy equivalences are the result of a sequence of CP's for many fundamental groups, i.e. those for which $\kappa^s_{n-2}(\mathbb{Z}_2 \times E) = 0$ for all $\mathbb{Z}_2$ extensions of π_1. We leave this to the reader. One can do slightly better by "stabilizing" slightly:

PROPOSITION 19. *Let $h: M' \to M^n$, $n \geq 5$, be a simple homotopy equivalence and $\pi_1 M$ as in Theorem* 14; *then if $v(h)$ lifts to $\Sigma\Omega(G/\mathrm{Top})$*

$$h \# 1_{S^2 \times S^{n-2}}: M' \# S^2 \times S^{n-2} \to M \# S^2 \times S^{n-2}$$

is the result of a sequnece of CP's.

Proof. $v(h \# 1_{S^1 \times S^{n-2}}) \sim v(h) \# k_2 p_1 + k_2 p_1$ and both summands are nontrivial on π_2 so the theory of §4B applies. □

4D. ***An example***

We apply the theory of §4A to produce a simple homotopy equivalence which has a PCP but is nonetheless not CP. Thus some hypothesis is necessary to gain the conclusion of Theorem 14, but to what extent the results of §3, §4B and §4C can be strengthened is yet to be seen.

LEMMA 1. *Let $E \to Z \oplus Z$ be the Z_2 extension, then*

$$\mathrm{cok}\,(L_0(E) \to L_0(Z \oplus Z)) = Z_2.$$

Proof. We write $E = Z \times_\alpha (Z \times Z_2)$ thinking of the following presentation,

$$E = \langle s, t, u \mid u^2 = 1, sts^{-1} = tu, sus^{-1} = u, tut^{-1} = u \rangle;$$

where $\alpha: Z \times Z_2$ has $\alpha(t) = tu, \alpha(u) = u$. There is an exact Mayer–Vietoris sequence[7].

$$\begin{array}{ccccccccccc}
L_0(Z \times Z_2) & \xrightarrow{1-\alpha_*} & L_0(Z \times Z_2) & \longrightarrow & L_0(E) & \xrightarrow{\partial} & L_3(Z \times Z_2) & \xrightarrow{1-\alpha_*} & L_3(Z \times Z_2) & \longrightarrow & \dots \\
\searrow & & \downarrow & & \swarrow & & \searrow & & \downarrow & & \\
L_0(Z) & \xrightarrow{0} & L_0(Z) & \longrightarrow & L_0(Z \times Z) & \longrightarrow & L_3(Z) & \xrightarrow{0} & L_3(Z) & \longrightarrow & \dots
\end{array}$$

$L_3(Z \times Z_2) \approx L_3(Z_2) \times L_2(Z_2)$. Let $Z = \langle g \rangle$, then $i_{1,2}: Z \to Z \times Z_2$ given by $i_1(g) = t$, $i_2(g) = tu$ gives a basis $i_{1,2*}(L_3(Z) = Z_2)$ for $L_3(Z \times Z_2)$. (This uses the fact[32] that $L_3(Z) \to L_3(Z_2)$ is onto.) To show that cok $= Z_2$ we have to show that the codimension 2 arf invariant is not in the image of $L_0(E)$. If it were, some preimage of arf $\in L_3(Z)$ in $L_3(Z \times Z_2)$ goes to 0 in $L_3(Z \times Z_2)$ under $1 - \alpha_*$. Note that $\alpha i_1 = i_2$ and $\alpha i_2 = i_1$, so $1 - \alpha_*$ in the above basis is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. The two preimages of $L_3(Z)$ in $L_3(Z \times Z_2)$ are the basis elements neither of which goes to zero. □

COROLLARY 2. $\operatorname{cok}(L_7(Z^5 \times E) \to L_7(Z^5 \times Z^2)) \supset Z_2^{10}$.

Now let M^6 be the boundary of a regular neighborhood of the 2-skeleton of T^7. Note that

$$H_1(M^6) \to H_1(T^7),\ H_2(M^6) \to H_2(T^7),\ H_3(M^6) \to 0$$

are all isomorphisms. Let $k_2 \in H^2(M^6; Z_2)$ be the pullback of the class in $H^2(Z^7, Z_2)$ corresponding to $Z^5 \times E \to Z^7$. Let $f: M^6 \to (Z_2, 2) \times (G/\mathrm{Top})_2$ be given by k_2 on the first coordinate and the constant map on the second (using the splitting given by the Kervaire classes).

LEMMA 3. *$\theta(f) \in L_6(Z^7)$ is trivial.*

Proof. M is stably parallelizable to $\theta(f)_{(2)} = g_*(k_2 \cap [M])$

$$(A_*\colon \oplus\, H_{n+4i}(Z^6; Z_{(2)}) \oplus H_{n+4i+2}(Z^6; Z_2) \to L_n(Z^6)_{(2)} \text{ is } 1-1.)$$

Thus the image in $L_6(Z^3) = 0$. We only need calculate surgery obstruction as codimension 4 arf invariants, i.e. $\theta(f|_{g^{-1}(T^2 \subset T^6)} \subset L_2(Z^2)$. However we can homotop g so that for any torus $T^2 \subset T^6, g(M) \cap T^2 = \emptyset$ so $\theta(f) = 0$.

LEMMA 4. *$f: M \to G/\mathrm{Top}$ lifts to $\Sigma\Omega(G/\mathrm{Top})$.*

Proof. As in §1, we only have to show that f factors through a suspension. We have

$$\begin{array}{ccccccccc} M & \longrightarrow & T^7 & \longrightarrow & T^2 & \longrightarrow & K(Z_2, 2) & \longrightarrow & G/\mathrm{Top} \\ & & & & \downarrow & & \uparrow & & \\ & & & & T^2/(S^1 \vee S^1) & \xrightarrow{\ \simeq\ } & S^2 & & \end{array}$$

so f factors through S^2. □

LEMMA 5. *The elements of $L_7(Z^7)$ which act trivially on $h\mathrm{Top}(M^6)$ are in the image of $L_7(E)$.*

Proof. Apply to $[\Sigma M: G/\mathrm{Top}] \to L_7(Z^7)$ the proof of Lemma 3. □

LEMMA 6. *Let h be a simple homotopy equivalence $h: M' \to M$ with normal invariant f. Then $\psi(h, \bar{f})$ can take on at most two distinct values for different lifts $\bar{f}$.*

Proof. Let (N, n) be a PCP in normal form. The Eilenberg obstructions to homotoping two lifts lie in

$$H^4(M; \pi_4(F)) \approx H_2(M; Z_2) \approx H_2(\pi; Z_2)$$
$$H^5(M; \pi_5(F)) \approx H_1(M; \pi_5(F)) \approx H_1(\pi; \pi_5(F))$$
$$H^6(M; \pi_6(F)) \approx \pi_6(F)$$

where F is the fiber of $\Sigma\Omega(G/\mathrm{Top}) \to G/\mathrm{Top}$. Thus any PCP is PCP cobordant to (N, n)

in the complement of a surface (perhaps intersecting N), a circle and a point. Note that $H_2(M_+; Z_2) \to H_2(M; Z_2)$ has index 2 by the exact sequence of a group extension. Therefore modulo what can be obtained by modifying the PCP in the neighborhood of a surface Σ and a circle C not intersecting N, there is at most one other PCP. Thus we have to show that these modifications do not change $\psi(\bar{f}, h)$. Note that $[\Sigma \times D^4, \partial; G/\mathrm{Top}]$ and $[C \times D^5, \partial; G/\mathrm{Top}]$ both map injectively into $[M: G/\mathrm{Top}]$ and their ranges only coincide with the image of $[D^6, \partial: G/\mathrm{Top}]$. Thus without loss of generality we can assume that the modifications as normal invariants are trivial. These modifications can be regarded as PCP's. We observe that they can be smoothed, for the obstruction lies at worst in $\ker L_6^{cl}(\pi_1\Sigma^2 \times Z_2 \times Z_2) \to L_6(\pi_1\Sigma^2)$. Since $\kappa_i^s(Z_2 \times Z_2) = 0$ for $i > 1$ this vanishes. Since $h\mathrm{Top}(\Sigma^2 \times D^4, \partial) = \{1\}$ and $h\mathrm{Top}(C \times D^5, \partial) = \{1\}$ these cut-pastes do not change which element of $h\mathrm{Top}(M)$ is produced, so in particular $\psi(\bar{f}, h)$ is invariant. □

THEOREM. *There is a non-*CP *simple homotopy equivalence* $h: M' \to M$ *for which there exists a* PCP.

Proof. Let h_1 be given by smoothing out f using surgery theory and Lemma 3. Suppose h_1 is CP. Let $L_7(Z^7)$ act on $h_1 \in h\mathrm{Top}(M^6)$ and take h_2, h_3 such that the cobordisms between h_1 and h_2, and h_2 and h_3 are not in the image of $L_7(E)$. Corollary 2 permits this. Lemma 5 guarantees that for one of h_2 or h_3, $\psi(\bar{f}, h_i) \neq 0$ for all lifts. Theorem 1 implies that it is not CP.

□

Remark. Using more advanced ideas another less computational example can be given.

§5. CALCULATIONS

In this section we apply the theory of §1–§4 to the problem of determining what are the CP homotopy equivalences to a fixed manifold. This section is divided into several short mostly independent subsections. The examples computed show a fairly wide range of different phenomena in the subject.

5A. $n - 1$ *connected* $2n$ *manifolds* ($n > 2$)

Recall the quadratic form of an $(n-1)$ connected $2n$ manifold is given by intersection, on middle dimensional homology.

THEOREM 1. *Let* M, M' *be closed* $n - 1$ *connected* $2n$ *manifolds,* $f: M' \to M$ *a homotopy equivalence, then* f *is* CP *iff* f *is* SCP. *Moreover, if*

(1) $n \not\equiv 0 \bmod 4$, f *is* CP *and thus* SCP

(2) $n \equiv 0 \bmod 4$, *then unless* M *is a sphere there is a non* CP *homotopy equivalence to* M. *Furthermore, if the quadratic form of* M *is* (a) *definite, then* f *is* CP *iff* f *is homotopy to a homeomorphism* (b) *indefinite and* $n \neq 4, 8$ *then every homotopy equivalence is the result of a sequence of* CP's.

Proof. The first statement follows from Corollary 3.6. For n odd every homotopy equivalence is homotopic to a homeomorphism so there is nothing to show. For n even and the cases in the theorem, Adams' solution to the Hopf invariant problem shows that the quadratic form of M is even. Let $(G/\mathrm{Top})^k$ be the kth connective cover of G/Top, i.e. $\pi_i((G/\mathrm{Top})^k) \to \pi_i(G/\mathrm{Top})$ is an isomorphism for $i > k$, $\pi_i((G/\mathrm{Top})^k) = 0$, $i < k$. There is a lift $M \xrightarrow{\nu(f)} (G/\mathrm{Top})$ to $M \xrightarrow{g} (G/\mathrm{Top})^{n-1}$ and a commutative diagram:

$$\begin{array}{ccccc} & & \sum\Omega((G/\mathrm{Top})^{n-1}) & \longrightarrow & \sum\Omega(G/\mathrm{Top}) \\ & & \downarrow & & \downarrow \\ M & \xrightarrow{g} & (G/\mathrm{Top})^{n-1} & \longrightarrow & G/\mathrm{Top}. \end{array}$$

There is a map $(G/\mathrm{Top})^{n-1} \to K(L_n(0), n)$ inducing an isomorphism on $H_n(-, Z)$. We show that g lifts to $\Sigma\Omega((G/\mathrm{Top})^{n-1})$ iff the square of the pullback of this cohomology class $g^*(i_n)$ vanishes. This shows that $n \equiv 2 \bmod 4$ there is no obstruction and reduces the rest of Theorem 1 to a number theoretic fact. The necessity of the square vanishing is trivial, since the cup squares vanish in all suspensions and this class factors through a suspension. Conversely, the primary and in our case only obstruction to solving the lifting problem

$$\begin{array}{ccc} & & \sum\Omega(G/\mathrm{Top})^{n-1} \\ & & \downarrow \\ M^{2n} & \longrightarrow & (G/\mathrm{Top})^{n-1} \end{array}$$

is cup square.

If the quadratic form is definite there is no class whose square vanishes, so only the trivial normal invariant is CP. If the quadratic form is indefinite, we must show that there is a basis for $H_n(M^{2n}; Z)$ with elements whose squares vanish. It is well known (see [35]) that all indefinite even unimodular quadratic forms are sums of

$$E_8 = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \quad \text{and } U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Suppose the quadratic form for M is $kE_8 \oplus l\,U (l \geq 1)$. Let $x_1 \ldots x_n$, u_1, u_2 be a basis for $kE_8 \oplus lU$ where $x_1 \ldots x_n$ is a basis for $E_8 \oplus (l-1)U$. Let $x_i^2 = 2a_i$. Then $x_1 - a_1u_1 - u_2, \ldots, x_n - a_nu_1 - u_2$, u_1, u_2 is a basis of isotropic vectors as desired. □

COROLLARY 2. *If M^8 or M^{16} is highly connected and the quadratic form is indefinite, then every homotopy equivalence is the result of a sequence of cut-pastes* iff *the quadratic form is even. If the form is odd then "half" of the homotopy equivalences are the result of such sequences.*

Proof. If the form is even the above proof still works. If it is odd any normal invariant whose cohomology class has odd square is not the result of sequence since reducing mod 2 squaring is a homeomorphism and these are not in the kernel. Conversely it is easy to see that $\{\mathbf{x} \in Z^n | \mathbf{x} \cdot \mathbf{x} \equiv 0(2)\}$ is a sublattice of index 2 on which the quadratic form is even indefinite unimodular, so there is a basis of isotropic vectors. □

Example. Let M^8 be a Milnor manifold corresponding to E_8. Then only homeomorphisms to $M^8 \# M^8$ are the result of a cut-paste. On the other hand, for $M^8 \# -M^8$ every homotopy equivalence is the result of a sequence of at most two cut-pastes.

5B. ***The various projective spaces***

The key facts that we use are the following:

$$[M: G/\mathrm{Top}] \longrightarrow [M: (G/\mathrm{Top})_{(2)}] = [M: G/\mathrm{Top}]_{(2)} \text{ is injective (1).}$$
$$\text{If } x \in H^i(M^n; Z \text{ or } Z_2) \text{ and } x^2 = 0 \text{ then } x = 0 \text{ or } 2i > n;\ (2)$$

for M real, complex, or quaternionic projective spaces. (1) is the fact that $[M:G/\mathrm{Top}]$ has no odd torsion which follows from obstruction theory and (2) is a consequence of the polynomial algebra structure of $H^*(M)$.

THEOREM 3. *If $M^n, n \geq 5$, satisfies* (1) *and* (2) *then a simple homotopy equivalence $h: M' \to M$ is* CP *iff $\nu(h)|_{M^{[n/2]}}$ is nullhomotopic.*

Proof. If $\nu(h)|_{M^{[n/2]}}$ is nullhomotopic then $\nu(h)^*(k_2) = 0$ so it suffices to show that $\nu(h)$ lifts to $\Sigma\Omega(G/\mathrm{Top})$, so that by Theorem 3.5 h is SCP. This is clear since by hypothesis $\nu(h)$ factors through $M^n/N^{[n/2]}$ which has the homotopy type of a suspension. (In general an $[n/2]$ connected n-complex is of the homotopy type of a suspension. One can prove this using the Freudenthal suspension theorem to "desuspend" the attaching maps for each cell) and there is a commutative diagram:

$$\begin{array}{ccc} & & \sum \Omega(G/\mathrm{Top}) \\ & \overset{\Sigma\hat{g}}{\nearrow} & \downarrow \\ \sum X & \xrightarrow{\ g\ } & G/\mathrm{Top}. \end{array}$$

Conversely we have a diagram

$$\begin{array}{ccccc} \sum \Omega(G/\mathrm{Top}) & \longrightarrow & \sum \Omega(G/\mathrm{Top})_{(2)} & & \\ \uparrow{\scriptstyle \bar{g}} & & \downarrow & & \\ M \xrightarrow{\ g\ } (G/\mathrm{Top}) & \longrightarrow & (G/\mathrm{Top})_{(2)} & \approx & \Pi(Z_2, 4n+2) \times \Pi K(Z_{(2)}, 4n). \end{array}$$

Thus we must show that the pullbacks of the k_i and $l_i, i < n/2$ (which determine the splitting) vanish so the composite $M^{[n/2]} \to (G/\mathrm{Top})_{(2)}$ is nullhomotopic and hence, by (1) $M^{[n/2]} \to G/\mathrm{Top}$ is nullhomotopic. Since the map to $(G/\mathrm{Top})_{(2)}$ factors through a suspension $f^*(k_i)^2 = f^*(l_i)^2 = 0$ for all i, so by (2) for $i \leq n/2$ these pullbacks are trivial. □

Topological homotopy projective spaces have an easy classification. For Quaternionic and Complex Projective Spaces, there are a number of splitting invariants (see [26]), defined by taking a homotopy equivalence $h: P^{n'} \to P^n$ and calculating signatures and arf invariants of subproblems along subprojective spaces, which determine the homotopy type of P^n. For RP^n the classification is much the same except that for RP^{4k+3} there is an additional integral Browder–Livesay invariant (see [14]). From the definitions it is clear that $h: P^{n'} \to P^n$ has $\nu(h)|_{(p^n)^k} = *$ iff the splitting invariants corresponding to subproblems of dimension less than or equal k vanish. Thus, we have:

COROLLARY 4. *Let $M^n, n > 5$, be a homotopy real, complex, or quaternionic projective space $h: M' \to M$ a homotopy equivalence; then the following are equivalent*:

(1) *h is* CP

(2) *h is* SCP
(3) *h is the result of a sequence of cut-pastes*
(4) *The first half of the splitting invariants of M' and M^n coincide*
(5) $v(h)|_{M^{[n/2]}}$ *is nullhomotopic.*

Compare this to the behavior of highly connected manifolds. In the case of complex projective space, [23] describes the cut-paste for the top splitting invariant.

5C. *Lens spaces*

If the conclusion to Theorem 3 holds we say that M is *stable*, i.e. h is CP iff $v(h)|_{M^{[n/2]}}$ is nullhomotopic. The reason for this terminology is that M being stable means that cutting and pasting M causes the least change in M that one could expect. Thus, all the projective spaces are stable. Our first theorem is that:

THEOREM 5. *L_k^{2n+1}, $(2n+1) \geqslant 5$, k squarefree, is stable.*

Proof. We have to show that if $f\colon L_k^{2n+1} \to G/\mathrm{Top}$ lifts to $\Sigma\Omega(G/\mathrm{Top})$ then $f|_{L_k^n}$ is nullhomotopic. We show this by examining the localizations. If $k = p^a b$, $(b,p) = 1$, then the Atiyah–Hirzebruch spectral sequence shows $[L_{p^a}^{2n+1}\colon G/\mathrm{Top}]_{(p)} = [L_k^{2n+1}\colon G/\mathrm{Top}]_{(p)}$. This reduces us to the case of prime powers. If $p = 2$, then §5B completes the proof. If p is odd we have to show that if $f_{(p)}\colon L_p^{2k+1} \to (G/\mathrm{Top})_{(p)}$ lifts, then $f_{(p)}|L_p{}^k$ is nullhomotopic. For odd primes $(G/\mathrm{Top})_{(p)} \approx BO_{(p)}$, see [26, 27]. Also $BU_{(p)} \approx BO_{(p)} \times \Omega^2 BO_{(p)}$ so it suffices to show the same thing for BU. We show that if $u \in KU(L_p^{2k+1})$ has $u^2 = 0$ then the image of u in $KU(L_p{}^k)$ is trivial, since in any cohomology theory squares vanish in a suspension. Recall (see [22]) that

$$\begin{aligned} KU(L_k^{2n+1}) &\approx Z[x]/\langle (x-1)^n,\quad x^k - 1\rangle \\ &\approx Z[Z_k]/I_n \text{ where } I_n = \langle (g-1)^n \rangle. \end{aligned}$$

It suffices to deal with the case of $L_p^{2n+1} \subset L_p^{4n+1}$, i.e. to show $u^2 \in I_{2n}$ implies $u \in I_n$. If k is p^a we will see that this is true precisely when $a = 1$ (or 0 trivially). There is a diagram generalizing the Rim diagram (see [18])

$$\begin{array}{ccc} Z[Z_{p^a}] & \xrightarrow{\alpha_1} & Z[\eta], \ \eta \text{ a primitive } p^a\text{th root of unity} \\ \Big\downarrow{\scriptstyle \alpha_2} & & \Big\downarrow \\ Z[Z_{p^{a-1}}] & \longrightarrow & Z_p[x]/(x-1)^{p^{a-1}} = Z_p[Z_{p^{a-1}}]. \end{array}$$

Let $A_m \subset Z[\eta]$ be the image of I_m. (It is the principal ideal $\langle (n-1)^m \rangle$.) Note that $Z[\eta]$ is a Dedekind domain so $\bar{u}^2 \in A_m$ implies that $\bar{u} \in A_m$. If $a = 1$ then $u^2 \in I_{2n}$ implies the image of u in $Z[Z_{p^0}] = Z$ is trivial. Thus $\bar{u} = \alpha(\eta - 1)^n$. Let $v \in Z[Z_p]$ have an image α. Then it is clear that $u = v(g-1)^n$. If $a > 1$ let u be such that $\alpha_1(u) = (\eta - 1)^{p^{a-1}}$ and $\alpha_2(u) = -(g-1)^{p^{a+1}}$. (Such a u exists since the image of these elements in $Z_p[x]/(x-1)^{p^{a-1}}$ is trivial.) Note that $u^2 \in I_{2p^{a-1}}$. One can calculate that $u \notin I_{p^{a-1}}$ □

THEOREM 6. *$L_{p^2}^{8k+1}$ is unstable for $8k + 1 < 2p + 1$.*

Proof. It is easy to see that $[L_p^{2n+1}\colon G/\mathrm{Top}] \to [L_p^{2n+1}\colon (G/\mathrm{Top})_{(p)}]$ is an isomorphism and similarly for suspensions of loop spaces. We wish to find the range through which $(G/\mathrm{Top})_{(p)}$ is a product of Eilenberg–Maclane spaces. In other words, we must find the first n for which $h_{4k}\colon \pi_{4k}(G/\mathrm{Top}) \to H_{4k}(G/\mathrm{Top})$ has image a multiple of p.

LEMMA. Im h_{4k} *is divisible exactly by the largest odd divisor of* $(2k-1)!$

Proof of Lemma. That it is an odd number because $(G/\mathrm{Top})_{(2)}$ is a product of Eilenberg–Maclane spaces. Now we have at the odds

$$\begin{array}{ccc}
\pi_{4k}(G/\mathrm{Top}) & \longrightarrow & H_{4k}(G/\mathrm{Top}) \\
\approx\downarrow & & \downarrow\approx \\
\pi_{4k}(BO) & \longrightarrow & H_{4k}(BO) \\
\approx\downarrow & & \downarrow \text{split injective} \\
Z \approx \pi_{4k}(BU) & \xrightarrow[\text{by } (2k-1)!]{\text{multiplication}} & H_{4k}(BU)
\end{array}$$

The bottom line comes from identifying $\pi_{4k}(BU) \to H_{4k}(BU)$ with the Chern class and applying Hirzebruch's version Bott periodicity (see [1]). Thus if $p = 2k - 1$ until dimension $4k - 1 = 2(p+1) - 1 = 2p + 1$, G/Top is a product of Eilenberg–Maclane spaces. $H^*(L_p^{8k+1}; Z_{(p)}) = \mathbb{Z}_{(p)}[u_2]/(u_2^{4k+1} = 0)$. Let $f: L_{p^2}^{2n+1} \to G/\mathrm{Top}$ correspond to pU_2^{2k+1}. Since $[L_p^{2n+1}; G/\mathrm{Top}] \to L_{2n+1}(\mathbb{Z}_p)$ is trivial f is the normal invariant of a simple homotopy equivalence. It suffices to show that $\bar{f}: L_2^{8k+1} \to K(Z_{(p)}, 4k)$ given by the above cohomology class lifts to $\Sigma K(Z_{(p)}, 4k-1)$.

Through dimension $8k+1$ the fiber of $\Sigma K(Z_{(p)}, 4k-1) \to K(Z_{(p)}, 4k)$ looks like $K(Z_{(p)}, 8k+1)$ with k-invariant cup square, see [2]. Since the cup square vanishes we can make all our lifts and thus the simple homotopy equivalence with normal invariant f is an unstable SCP homotopy equivalence.

Remark. For $p = 2$ we can remove the hypothesis on k since $(G/\mathrm{Top})_{(2)}$ is a product of Eilenberg–Maclane spaces.

5D. *Products with spheres*

THEOREM 7. *Let* $h: N \to M \times S^k$ *be a simple homotopy equivalence,* $k > 2$, *with* $\pi_1(M)$ *on the list in Theorem* 4.4 *if* $k = 2$. *Then h is the result of a sequence of cut-pastes* iff *the image of* $\nu(h)$ *in* $[M: G/\mathrm{Top}]$ *is the sum of elements which lift to* $\Sigma\Omega(G/\mathrm{Top})$.

Proof. The necessity of this condition is obvious. Conversely there is a split exact sequence

$$[M \times D^n, \partial: G/\mathrm{Top}] \to [M \times S^n: G/\mathrm{Top}] \to [M: G/\mathrm{Top}] \to 0.$$

We can get the image of $[M \times D^n, \partial: G/\mathrm{Top}]$ by cutting and pasting under the hypothesis of the theorem. We must show that the image of $[M: G/\mathrm{Top}] \to [M \times S^n: G/\mathrm{Top}]$ can be arrived at by cutting and pasting (or a sequence). Suppose $M \to G/\mathrm{Top}$ has a lift to $\Sigma\Omega(G/\mathrm{Top})$ then taking any PCP for the lift and crossing with S^n kills the surgery obstruction and we can smooth the PCP. This yields the result. □

COROLLARY 8. *For* $M \times S^k$ *as in Theorem* 7 *to have all simple homotopy equivalences the result of several cut-pastes it is necessary and sufficient that* $[M: G/\mathrm{Top}]$ *be generated by the image of* $[M: \Sigma\Omega(G/\mathrm{Top})]$.

Example. Any product of spheres other than $T^k \times S^2$ is covered by Corollary 8. For this case the smoothing obstruction is zero anyway. So for any product of spheres sequences of CP's provide all the homotopy equivalences.

Remark. One can use PCP replication to show that the obstruction Φ always lies in the indeterminancy of lifts in product situations—even $\times\ \mathrm{CP}^2$ kills Φ.

5E. *Rational calculations*

If one is merely interested in knowing whether or not the CP homotopy equivalences generate a subgroup of finite index in $h\mathrm{Top}(M)$ in many cases one can give a purely cohomological criterion. One can work simply after tensoring with the rationals. This also gives a simple way of showing that one or many CP's are different notions for many manifolds.

THEOREM 9. *Let M^n be a manifold with $\pi_1 M$ poly Z of rank less than $n/2$, then the* CP *homotopy equivalences generate a subgroup of finite index in* $h\mathrm{Top}(M)$ iff ker $\oplus H_{4*}(M;Q) \to \oplus H_{4*}(\pi_1 M;Q)$ *is generated by elements whose squares vanish.*

Proof. We do the simply connected case: the general case is no more difficult. Use a basis of $\oplus H^{4*}(M;Q)$ whose squares vanish to modify the Pontrjagin classes one at a time. Of course this is not possible, but by changing the top Pontrjagin class in keeping with the Hirzebruch signature theorem it is possible once we show that for $x \subset H^{4*}(M;Q)$, $x^2 = 0$ implies x lifts to $\Sigma K(Q, 4*-1)$. This is true since S^{4*} rationally is just the fiber of $K(Q, 4*) \to K(Q, 8*)$ given by cup square. □

5F. *Homology propagation*

Let Σ^n, $n > 5$, be a homology n-sphere, then $\Sigma^n \to S^n$ induces an isomorphism $[\Sigma^n : G/\mathrm{Top}] \to [S^n : G/\mathrm{Top}]$ and thus every simple homotopy Σ^n is SCP. The question we study is whether or not the questions of whether all simple homotopy equivalences are SCP or the result of several SCP's propagate through homology in some sense.

THEOREM 10. *Let $M' \to M^n$, $n > 5$, be a tangential homology equivalence. Then if every simple homotopy equivalence to M is* SCP *the same can be said for M'. However if every simple homotopy equivalence to M is the result of several* (S)CP's *it need not follow that the same is true for M'.*

Remark. This result shows from some problems it is more conceptually natural to work with a single CP despite the results of the previous subsections which show the computational advantage of allowing sequences.

Proof. Let $f: M' \to M$ be a tangential map, then there is a commutative diagram:

$$\begin{array}{ccc} [M : G/\mathrm{Top}] & \longrightarrow & L_n(\pi_1 M) \\ \downarrow & & \uparrow \\ [M' : G/\mathrm{Top}] & \longrightarrow & L_n(\pi_1 M') \end{array}$$

This follows from the formulae of [28, 33]. In fact the images in $[K(\pi_1 M, 1); G/\mathrm{Top}]$ and $[K(\pi_1 M', 1); G/\mathrm{Top}]$ replacing $L_n(\pi_1 M)$ and $L_n(\pi_1 M')$ still results in a commutative diagram since $f^*(\mathscr{L}(M)) = \mathscr{L}(M')$, $f^*(V(M)) = V(M')$ where $\mathscr{L}$ is the Morgan–Sullivan class, V the Wu class, and $f^*(\Delta(M)) = \Delta(M')$ where Δ is Sullivan's K-theoretic orientation, see[27]. Now f_*: $[M : G/\mathrm{Top}] \to [M' : G/\mathrm{Top}]$ is an isomorphism. The image has surgery obstruction which vanishes only if it had surgery obstruction in $L_n(\pi_1 M)$. For these there is a lift to $\Sigma\Omega(G/\mathrm{Top})$.

To show that the problem of whether the simple homotopy equivalences to M are all the result of a sequence of (S)CP's does not propagate through homology we give an example. Let $M^{24} = (S^3)^8 \# M^{24}_{E_8}$ where $M^{24}_{E_8}$ is a Milnor manifold with quadratic form E_8. Let Σ^3 denote the homology sphere obtained by glueing two trefoil knot complements together longitude to meridian. It is an irreducible sufficiently large manifold. $[(\Sigma^3)^8: G/\mathrm{Top}] \to L_0((\pi_1\Sigma)^8)$ is an isomorphism. Let $M' = (\Sigma^3)^8 \# M^{24}_{E_8}$. $M' \to M$ is a tangential homology equivalence. For M everything is the result of several cut-pastes. (This can be seen by the methods of §5A and §5D.) For M' only the identity is CP. This is a consequence of the above result on L-groups and the fact that E_8 is positive definite.

5G. *Stability*

Suppose $h: M' \to M^n (n \geq 5)$ is the result of a sequence of CP's, can we give a bound on the length of the sequence?

Definition. $K(M) = \max_{\substack{\{h \text{ is the result of} \\ \text{a sequence of CP's}\}}}$ (min a length of a sequence of CP's yielding h)

Example. If M is stable, $K(M) = 1$.

Example. $K((S^4)^m) = m$.

Proof. $K((S^4)^m) \leq m$ follows from §6D.

Using the usual basis for the cohomology algebra $H^*((S^4)^m)$, let $h: M \to (S^4)^m$ have $l_4(h) = x_1 + \ldots + x_m\, H^4((S^4)^m; Z)$. Suppose h is the result of k CP's, $k < m$, then

$$l_4(h) = u_1 + \ldots + u_k, \; u_i^2 = 0,$$

where the u_i are l_4 of the various CP's. Note that

$$(u_1 + \ldots + u_k)^{k+1} = 0.$$

However $(x_1 + \ldots + x_m)^m = m \in H^4((S^4)^m; Z)$. □

Several considerations make the following plausible:

Conjecture. There is a function f such that

$$K(M^n) \leq f(n).$$

Perhaps, $f(n) = n/2$ would suffice.

It is quite easy to use the methods of this paper to produce large classes of manifolds (e.g. for large classes of π_i) for which $K(M^n) < \infty$. In [39], we announced too optimistically that this holds for all M^n $n \geq 5$. This should be viewed as an open problem.

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