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THERE EXIST FINITELY PRESENTED GROUPS WITH INFINITE OOZE

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For certain problems in geometric topology calculations of nonsimply connected surgery obstructions [W] are necessary. In many cases the surgery problem can be related to one that is between closed manifolds. Such obstructions are severely restricted; for instance, in the finite fundamental group oriented case, the projective surgery obstruction is detected by signature, arf invariant, and codimension-one arf invariants [H] [TW2]. For certain infinite fundamental groups the question of which obstructions can arise is related to the Novikov higher signature conjecture.

More precisely, there are formulae for the surgery obstructions of normal invariants $f\colon M \to G/\mathrm{Top}$, M closed and oriented, in terms of characteristic classes of M and f. For instance, upon localizing at 2, [TW1 (1.7)] prove that the surgery obstruction of f, $\sigma(f)$, satisfies

$$\sigma(f)_{(2)} = A_*\Big(g_*\big(\nu^*(\mathfrak{L}) \cup f^*(l) + v^*(\mathfrak{L}) \cup f^*(k) + \delta^*\big(\nu^*(V\mathrm{Sq}^1 V) \cup f^*(k)\big)\big) \cap [M]\Big) \in L_n(\pi_1 M) \otimes Z_{(2)}$$

where

- $\mathfrak{L}$ is the Morgan-Sullivan class [MS],
- l, k are classes in $H^{4*}(G/\mathrm{Top}; Z_{(2)})$ and $H^{4^*+2}(G/\mathrm{Top}; Z_2)$, describing $(G/\mathrm{Top})_{(2)}$ as a product of Eilenberg-Maclane spaces,
- v is the classifying map of the normal bundle $v : M \to \mathrm{BSTop}$,
- g: $M \to K(\pi_1 M, 1)$ classifies the fundamental group, and

$$A_* : \oplus H_{n-4*}(\pi; Z_{(2)}) \oplus H_{n-4*+2}(\pi; Z_2) \to L_n^s(\pi) \otimes Z_{(2)}$$

is some "assembly" homomorphism. Regard A_* as a sum of homomorphisms

$$\mathcal{I}_i : H_i(\pi; Z_{(2)}) \to L_i^s(\pi) \otimes Z_{(2)}$$

and

$$\mathcal{K}_{i-2} : H_{i-2}(\pi; Z_2) \to L_i^s(\pi) \otimes Z_{(2)} \;;$$

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i.e.

$$A_* = \sum_n K_{*+4n+2} + \mathcal{I}_{*+4n}.$$

We say that π has n-dimensional ooze if either $\mathcal{I}_n$ or $\mathcal{K}_n$ is nontrivial and $\mathcal{I}_i$ and $\mathcal{K}_i$ vanish for $i > n$.

Remark. There are corresponding definitions and, sometimes stronger, results available for other L-groups, e.g. L^h and L^p. For this paper, the type of L-theory considered is irrelevant.

One can show (folklore) that if π has n-dimensional ooze then for some surgery problem between closed manifolds with fundamental group π all simply connected surgery problems associated to (generalized) submanifolds of codimension less than n vanish, yet the nonsimply connected surgery obstruction is nontrivial. (In this connection, recall that by the characteristic variety theorem, the normal invariant, and therefore the surgery obstruction, is determined by the simply connected surgery obstructions of generalized submanifolds.)

In this paper we show the existence of a finitely presented group with infinite ooze. This solves a problem proposed by Bruce Williams at the 1981 Topology Conference in London, Ontario. The author would also like to thank Sylvain Cappell for encouragement and helpful conversations.

THEOREM. *There exists a finitely presented group with infinite ooze generated by four elements.*

Proof. At various points we will have to deal with the L-theory of infinitely generated groups and similar matters. The following lemma will be used without reference.

LEMMA 1.

$$L_n(G) \approx \varinjlim_{\substack{\pi \text{ finitely generated} \\ \pi \subset G}} L_n(\pi)$$

Proof. Consider the obvious map of $\varinjlim L_n(\pi) \to L_n(G)$. An element of $L_n(G)$ is represented by a matrix with coefficients in ZG. This is in the image of $L_n(Z\pi)$ where π is the subgroup generated by the elements of G occurring in the matrix. Similarly, to describe, say, a subkernel only requires a finite number of elements of G, i.e. a somewhat larger finitely generated subgroup of G. $\square$

Also, versions of the calculations of $L_n(Z \times_\alpha G)$ and $L_n(H_1 *_G H_2)$ are needed for G not finitely presented. The original results, see [Sh] for $Z \times G$ (α other than 1 can be treated by a similar technique) and [C1] for $H_1 *_G H_2$, were proved geometrically and therefore require all groups to be finitely presented. Now purely algebraic proofs of these results exist, mainly through the work of Andrew Ranicki, which eliminate all finiteness requirements on G and H_i. For $Z \times_\alpha G$ see [R]. The case of amalgamated free products has yet to be published.

Given the above, we will work with exact sequences of L-groups just as if all groups were finitely presented, and will be satisfied with references in the literature to these cases.

PROPOSITION 2. *For* $0 \leqslant n \leqslant \infty$, $A_* : \oplus H_{*+4k}(Z^n; Z_{(2)}) + H_{*+4k+2}(Z^n; Z_2) \to L_*(Z^n) \otimes Z_{(2)}$ *is an isomorphism.*

Proof. For $n = \infty$, the result follows from the case of n finite and the lemma above. The finite case is an induction using Shaneson's product formula [Sh] which in this context means that

$$\mathcal{S}^s_n(Z \times G) = \mathcal{S}^s_n(G) \oplus \mathcal{S}^h_{n-1}(G)$$

$$\mathcal{K}^s_n(Z \times G) = \mathcal{K}^s_n(G) \oplus \mathcal{K}^h_{n-1}(G)$$

(identifying $H_n(Z \times G)$ with $H_n(G) \oplus H_{n-1}(G)$) the case $n = 0$ being forced by direct homology and surgery calculations (together with the existence of Kervaire and Milnor manifolds, so that $\mathcal{K}_0$ and $\mathcal{S}_0$ are surjective). $\square$

Recall the wreath construction. $G \wr H$ is by definition $G \times_\alpha H^{|G|}$ where G acts on the product of $|G|$ copies of H by permutation of coordinates.

COROLLARY 3. *For* $Z \wr Z$, A_* *is an isomorphism.*

Proof. $\mathrm{Wh}(Z^\infty) = 0$, so apply the argument of [C3, Lemma 1].[1]

The value of $Z \wr Z$ is that it contains Z^∞ and is generated by two elements.

$$Z \wr Z = \langle x, t \mid [x, t^i x t^{-i}] = 1, -\infty < i < \infty \rangle$$

(The subgroup generated by $\{t^i x t^{-i}\}$ is isomorphic to Z^∞). We leave the following to the reader:

PROPOSITION 4. (a) $\tilde{K}_0(Z \wr Z) = Wh(Z \wr Z) = 0$

(b) $H_i(Z \wr Z; Z) = \begin{cases} Z, & i = 0, 1 \\ Z^\infty, & i > 1 \end{cases}$ $\square$

Let A denote a finitely presented acyclic group; for instance

$$\begin{aligned} A &= F_2 *_{F_4} F_2 \\ &= \langle x, y, x', y' \mid [x^3, y^3] = x'[x', y'][x^4, y^4] \\ &\quad = y'[x'^2, y'^2]\, x[x, y] = [x'^3, y'^3]\, y[x^2, y^2] = [x'^4, y'^4] \rangle \end{aligned}$$

[1] A_* is compatible with the Mayer-Vietoris sequence.

That this is acyclic is trivial. Since $Z \subset A$ (any inclusion will do), $Z \wr Z \subset Z \wr A \subset A *_Z Z \wr A$. This last group will be denoted by B;

$$B = A *_Z Z \wr A$$

Note that

PROPOSITION 5. $H_i(B; Z) = 0$, $i > 0$.

$\tilde{L}_i(Z \wr Z) \to \tilde{L}_i(B)$ is trivial, at least after tensoring with $Z_{(2)}$. □

The second statement follows from the first and the commutative diagram

$$\begin{array}{ccc} \bigoplus_{i \equiv n(4)} \tilde{H}_i(Z \wr Z; Z_{(2)}) \oplus \tilde{H}_{i+2}(Z \wr Z; Z_2) & \xrightarrow[\approx]{A_*} & \tilde{L}_n(Z \wr Z) \otimes Z_{(2)} \\ \downarrow & & \downarrow \\ 0 = \bigoplus_{i \equiv n(4)} \tilde{H}_i(B; Z_{(2)}) \oplus \tilde{H}_{i+2}(B; Z_2) & \xrightarrow{A_*} & \tilde{L}_n(B) \otimes Z_{(2)} \end{array}$$

We wish to embed B in a finitely presented group C. B is finitely generated. G. Higman classified finitely generated subgroups of finitely presented groups. A group is recursively presented if there is an algorithm to calculate the relators. (Note that there is no demand that there is an algorithm to decide whether or not a word is a relation; because of this Higman's theorem can be used to show the existence of finitely presented groups with unsolvable word problem [LS, p. 215].) Higman's theorem states that a finitely generated group is embeddable in a finitely presented group (in fact two-generators) iff it is recursively presented ([LS, p. 215]). Clearly B is recursively presented, so let $C \supset B$ be finitely presented. Notice that $G = C *_{Z \wr Z} C$ is finitely presented for if $C = \langle c_i \mid r_j \rangle$, write $Z \wr Z = \langle t, x \mid [t^i x t^{-i}, x] = 1 \rangle$; then

$$C *_{Z \wr Z} C = \langle c_i, c_j' \mid r_j, r_j', t(c_i) = t(c_i'),\ x(c_i) = x(c_i') \rangle$$

Claim. G has infinite ooze; in fact for all $i > 2$, $\mathcal{K}_i$ and $\mathcal{I}_i$ have image with infinite rank.

Proof. Let Φ denote the quadrad

$$\begin{array}{ccc} Z[Z \wr Z] & \to & Z[C] \\ \downarrow & & \downarrow \\ Z[C] & \to & Z[G] \end{array}$$

and $\hat{L}_n(G) = \operatorname{cok} L_n(\Phi) \to L_n(G)$. (This map actually splits and $L_n(\Phi)$ is 2-primary.) There are Mayer-Vietoris sequences in homology and L-theory [C2, Theorem 5 (i)],

$$\begin{tikzcd}
{} \arrow[r] & \begin{array}{c}\oplus H_{n+1}(G; Z_{(2)}) \\ \oplus H_{n-1}(G; Z_2)\end{array} \arrow[r] \arrow[d, "\sum \mathcal{I}_{n+1} + \mathcal{K}_{n-1}"] & \begin{array}{c}\bigoplus\limits_{n \equiv i\ (4)} H_n(Z \wr Z; Z_{(2)}) \\ \oplus H_{n-2}(Z \wr Z; Z_2)\end{array} \arrow[r] \arrow[d, "\sum \mathcal{I}_n + \mathcal{K}_{n-2}"] & \begin{array}{c}\oplus H_n(C, \mathbb{Z}_{(2)}) \oplus H_n(C, \mathbb{Z}_{(2)}) \\ \oplus H_{n-2}(C, \mathbb{Z}_2) \oplus H_{n-2}(C, \mathbb{Z}_2)\end{array} \arrow[d] \\
& \hat{L}_{i+1}(G) \otimes Z_{(2)} \arrow[r] & L_i(Z \wr Z) \otimes Z_{(2)} \arrow[r] & L_i(C) \oplus \hat{L}_i(C) \otimes Z_{(2)}
\end{tikzcd}$$

The rightmost horizontal arrows vanish by Proposition 5. The result follows from Proposition 4, Corollary 3, and diagram chase. □

References

C1 S. Cappell, *Mayer-Vietoris sequences in Hermitian K-theory*, Springer LNM 343.

C2 ———, *Unitary nilpotent groups and Hermitian K-theory*, BAMS **80** (1974) 1117–1122.

C3 ———, *On homotopy invariance of higher signatures*, Inventiones Math. **33** (1976) 171–179.

H I. Hambleton, *Projective surgery obstructions for closed manifolds*, 1980 Preprint, McMaster.

LS Lyndon and Schupp, *Combinatorial group theory* 1978, Springer.

MS J. Morgan and D. Sullivan, *The transversality characteristic class and linking cycles in surgery theory*, Ann. of Math. **99** (1974) 463–544.

R A. Ranicki, *Algebraic L-theory III*; *twisted laurent extensions*, Springer, LNM 342.

TW1 L. Taylor and B. Williams, *Surgery spaces*; *structure and formulae*, Springer LNM 741.

TW2 ——— and ———, *Surgery on closed manifolds*, 1980 Preprint, Notre Dame.

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