Parts I and II together.

SURGERY AND THE GENERALIZED KERVAIRE INVARIANT, I

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[Received 1 November 1982—Revised 24 January 1984]

Introduction

(i) *Synopsis.* The discovery, around 1960, of the 'Kervaire Invariant' for almost framed manifolds of dimension $4k+2$ (see [**12**]) was an important stimulant for the development of surgery theory; but it also led to the theory of the 'generalized Kervaire Invariant' of Browder and Brown [**2**, **3**].

The present paper is an attempt at uniting these two theories, by constructing a non-simply-connected and in other respects updated version of the generalized Kervaire Invariant.

The construction has three surprising aspects. Firstly, it is conceptually satisfying and, in the simply-connected case, clarifies Brown's original theory; for instance, the 'product formula problem' (see [**4**]) evaporates. Most of the new concepts are borrowed from the 'algebraic theory of surgery'; see [**15**]. Secondly, it is computationally satisfying. Thirdly, it has applications to classical surgery theory, especially to the calculation of the symmetric L-groups of [**13**] and [**15**]; and therefore to anything which involves product formulae for surgery obstructions.

A black box description of the theory has been given in [**22**]; in this introduction I shall concentrate on the concepts inside the box.

(ii) *Symmetric forms on (co-)homology groups vs. symmetric forms on chain complexes.* If M is a $2k$-dimensional geometric Poincaré complex, then $H^k(M; Z_2)$ carries a non-degenerate symmetric bilinear form.

Let γ be a spherical fibration on a space X. Brown's generalization of the Kervaire Invariant [**3**] is based on the observation that, under certain conditions on γ, a bundle map $\nu_M \to \gamma$ (with underlying classifying map $f\colon M \to X$) determines a refinement of the symmetric form on $H^k(M; Z_2)$ to a quadratic form with values in Z_4; here ν_M denotes the Spivak normal fibration of M.

So much for the simply-connected theory. (The term 'simply-connected' is a little misleading here; of course X above does not have to be simply-connected—it is just that we pay no special attention to $\pi_1(X)$.) Now let π be a discrete group, let $w\colon \pi \to Z_2$ be a homomorphism, let α be a principal π-bundle on X, and let j be an identification of the two double covers arising from the given data (namely, the orientation cover associated with γ, and $w^{\rightarrow}(\alpha)$). Such sextuples $(\pi, w; X, \gamma; \alpha, j)$ will be the main objects of study.

For a geometric Poincaré complex M^n with bundle map $\nu_M \to \gamma$ and underlying classifying map $f\colon M \to X$ as before (but with n arbitrary), let $C(\tilde{M})$ be the cellular chain complex of the total space of $f^{\leftarrow}(\alpha)$ (the principal π-bundle on M; assuming that M is a CW-space). Now $C(\tilde{M})$ is a chain complex of finitely generated (f.g.) projective left A-modules, and $A = \mathbb{Z}[\pi]$ is regarded as a ring with involution (the w-twisted

A.M.S. (1980) subject classification: 18F25.
Proc. London Math. Soc. (3), 51 (1985), 146–192.

involution,

$$\sum_{g \in \pi} n_g \cdot g \mapsto \sum_{g \in \pi} n_g \cdot (-)^{w(g)} \cdot g^{-1}).$$

What next? Inspection shows that the approach of [3] does not work very well here, even if $n = 2k$. That is, some kind of symmetric bilinear form is defined on $H^k(C(\tilde{M}); A)$, but it has very unpleasant properties in general.

The solution to this dilemma is suggested by recent developments in surgery theory. In [13] and [15], the notion of 'symmetric bilinear form on a chain complex (of f.g. projective left A-modules)' is defined or implicit (details follow below); the notion is homotopy invariant, and it is shown that the dual chain complex of $C(\tilde{M})$, written $C(\tilde{M})^{-*} = \operatorname{Hom}_A(C(\tilde{M}), A)$, carries a non-degenerate symmetric bilinear form. (Again, the dimension n is arbitrary.) So, instead of trying to 'refine' a symmetric bilinear form on some middle-dimensional cohomology group of $C(\tilde{M})$, we shall refine Mishchenko's non-degenerate symmetric bilinear form on the chain complex $C(\tilde{M})^{-*}$.

(iii) *The technical terms.* Let C be a chain complex of f.g. projective left A-modules, with $C_r = 0$ except for a finite number of indices r. Using the involution on A, we can also regard C as a chain complex of right A-modules, written C^t. So $C^t \otimes_A C$ is defined, and is a chain complex of $\mathbb{Z}[Z_2]$-modules; the generator $T \in Z_2$ acts by switching factors, with the usual sign rules.

Define $\mathbb{Z}[Z_2]$-module chain complexes W, $\hat{W}$ by

$$W_r = \begin{cases} \mathbb{Z}[Z_2] & \text{for } r \geqslant 0, \\ 0 & \text{for } r < 0, \end{cases}$$

with $d\colon W_r \to W_{r-1};\ x \mapsto (1 + (-)^r T) \cdot x$ for $r > 0$, and by

$$\hat{W}_r = \mathbb{Z}[Z_2] \quad \text{for all } r,$$

with $d\colon \hat{W}_r \to \hat{W}_{r-1};\ x \mapsto (1 + (-)^r T) \cdot x$.

Write $W \& C$ and $\hat{W} \& C$ for the chain complexes (of abelian groups) $\operatorname{Hom}_{\mathbb{Z}[Z_2]}(W, C^t \otimes_A C)$ and $\operatorname{Hom}_{\mathbb{Z}[Z_2]}(\hat{W}, C^t \otimes_A C)$, respectively. Finally, let

$$Q^n(C) := H_n(W \& C) \quad \text{and} \quad \hat{Q}^n(C) := H_n(\hat{W} \& C).$$

The covariant functors $C \mapsto Q^n(C)$, $C \mapsto \hat{Q}^n(C)$ are homotopy invariant (this is proved in [15], but also in this paper).

Now define a 'symmetric bilinear form of degree n on C' to be

$$\begin{aligned} &\text{either an } n\text{-cycle } \varphi \in W \& C^{-*} \\ &\text{or a class } [\varphi] \in Q^n(C^{-*}) = H_n(W \& C^{-*}). \end{aligned}$$

(There are two different schools of thought here; I find the first definition better to work with, but the second is homotopy invariant.) We now give some 'motivations'.

(a) The n-cycle φ is a $\mathbb{Z}[Z_2]$-module chain map from $\Sigma^n W$ to

$$C^{-*t} \otimes_A C^{-*} = \operatorname{Hom}_A(C, C^{-*}).$$

In particular, the value of φ on $1 \in \mathbb{Z}[Z_2] = W_0$ is an n-cycle in $\operatorname{Hom}_A(C, C^{-*})$, that is, a chain map

$$\varphi_0\colon \Sigma^n C \to C^{-*}$$

which can be considered as a bilinear form of degree n on C.

(b) If V_1, V_2 are vector spaces over a field F (for example, $F = Z_2$), then there is an abelian group isomorphism

$$\begin{aligned}&(\text{symmetric bilinear forms on } V_1 \oplus V_2)\\ &\qquad \cong (\text{symmetric bilinear forms on } V_1)\\ &\qquad\quad \oplus (\text{symmetric bilinear forms on } V_2) \oplus \mathrm{Hom}_F(V_1, V_2).\end{aligned}$$

In generalizing the notion of symmetric bilinear form to (certain) chain complexes over A, this is a property one would like to retain; now it is indeed true that

$$Q^n((C \oplus D)^{-*}) \cong Q^n(C^{-*}) \oplus Q^n(D^{-*}) \oplus H_n(\mathrm{Hom}_A(C, D^{-*})),$$

and there is clearly no simpler definition of 'symmetric bilinear form on a chain complex' which is homotopy invariant and has this property.

(c) It is shown in [13] and [15] that a closed manifold or geometric Poincaré complex M^n, equipped with a principal π-bundle α' and an identification of double covers

$$w^{\rightarrow}(\alpha') \cong (\text{orientation cover of } M^n),$$

gives rise to a pair (C, φ) in which $C = C(\tilde{M})$, and φ is an n-cycle in $W \,\&\, C$ (sufficiently well determined). In the terminology introduced above, φ is a symmetric bilinear form of degree n on C^{-*}; in the case at hand, φ is non-degenerate, that is, the chain map

$$\varphi_0\colon \Sigma^n(C^{-*}) \to C = C^{-*-*}$$

is a chain homotopy equivalence.

A pair (C, φ) as above (with φ non-degenerate) is called an 'n-dimensional (symmetric) algebraic Poincaré complex'. It is treated here as the chain level analogue of a closed manifold or geometric Poincaré complex, just as chain complexes (over A) are treated as the analogues of spaces. 'Bordisms' between symmetric algebraic Poincaré complexes of the same dimension can be defined, etc.

(iv) *Chain bundles.* If spaces and closed manifolds (geometric Poincaré complexes) have analogues in the chain complex world, what about vector bundles (or spherical fibrations)?

By 'chain complex world' is meant the category $\mathscr{C}_A$ of chain complexes C of f.g. projective left A-modules, with $C_r = 0$ except for a finite number of indices r. The conceptual vacuum is filled as follows:

a 'chain bundle' on a chain complex C (in $\mathscr{C}_A$) is a 0-dimensional cycle in $\hat{W} \,\&\, C^{-*}$.

Motivation for this definition is as follows.

(a) The homotopy invariant (and contravariant) functors

$$C \mapsto \hat{Q}^{-n}(C^{-*}), \quad n \in \mathbb{Z},$$

constitute a cohomology theory on $\mathscr{C}_A$; that is, they satisfy the analogues of the Eilenberg–Steenrod axioms in the (co-)homology theory of spaces, except the dimension axiom.

(b) It is shown in [15] that a sextuple $(\pi, w; X, \gamma; \alpha, j)$ as in (ii) above determines a 'characteristic class' in $\hat{Q}^0(C(\tilde{X})^{-*})$ [15, Part II, 9.3]; similarly, a stable automorphism of γ determines a class in $\hat{Q}^1(C(\tilde{X})^{-*})$ [15, Part II, 9.9]. This suggests that the cohomology theory $C \mapsto \{\hat{Q}^{-n}(C^{-*}) \mid n \in \mathbb{Z}\}$ is the chain level analogue of spherical K-theory. So it is reasonable to expect that chain bundles on chain complexes C (that is, representing cycles for elements in $\hat{Q}^0(C^{-*})$) are the chain level analogues of spherical fibrations. And indeed it is possible to refine the 'characteristic class' above to a rule which associates chain bundles to spherical fibrations. (*Warning*: this rule is not additive; the geometric Whitney sum has not much to do with the addition in $\hat{W}\&C^{-*}$.)

(c) The definition of 'chain bundle' is so designed that any symmetric algebraic Poincaré complex (C, φ) carries a 'normal chain bundle' (the chain level analogue of the Spivak normal bundle of a geometric Poincaré complex). For details, see the main text; the idea stems from [**15**, Part II, 9.6].

The non-simply-connected generalized Kervaire Invariant can now be described as follows. Let $(\pi, w; X, \gamma; \alpha, j)$ be a sextuple as in (ii). The chain level image of γ is a chain bundle $c(\gamma)$ on the chain complex $C(\tilde{X})$ (over $A = \mathbb{Z}[\pi]$). Using the dictionary

space $\leftrightarrow$ *chain complex of projective left A-modules,*

geometric Poincaré complex M $\leftrightarrow$ *symmetric algebraic Poincaré complex* (C, φ),

spherical fibration $\leftrightarrow$ *chain bundle,*

Spivak normal fibration of M $\leftrightarrow$ *normal chain bundle of* (C, φ),

one obtains homomorphisms (for $n \in \mathbb{Z}$)

$$\text{flexible signature: } \Omega_n^P(X, \gamma) \to L^n(C(\tilde{X}), c(\gamma)).$$

Here $\Omega_n^P(X, \gamma)$ is the bordism group of geometric Poincaré complexes M^n equipped with a map of spherical fibrations from ν_M to γ; similarly, $L^n(C(\tilde{X}), c(\gamma))$ is the bordism group of n-dimensional algebraic Poincaré complexes (C, φ) over A, equipped with a 'chain bundle map' from the normal chain bundle (on C) to the chain bundle $c(\gamma)$ (on $C(\tilde{X})$).

The relationship with the Wall groups $L_n(\mathbb{Z}[\pi])$ is as follows. Firstly, if the sextuple $(\pi, w; X, \gamma; \alpha, j)$ is such that $X = \varnothing$, then $L^n(C(\tilde{X}), c(\gamma)) \cong L_n(\mathbb{Z}[\pi])$. Secondly, if X is arbitrary again, we may still consider the inclusion $\varnothing \hookrightarrow X$; it induces homomorphisms

$$\text{release: } L_n(\mathbb{Z}[\pi]) \to L^n(C(\tilde{X}), c(\gamma))$$

for $n \in \mathbb{Z}$. Now let $f\colon M^n \to N^n$ be a degree-1 normal map between geometric Poincaré complexes. Suppose that N is equipped with a map of spherical fibrations from ν_N to γ. Then the normal map f induces a similar structure on M, and the equation

$$\sigma^*(M) - \sigma^*(N) = \text{release}(\sigma_*(f))$$

holds. Here σ^* is the flexible signature (in $L^n(C(\tilde{X}), c(\gamma))$), and σ_* is the surgery obstruction (in $L_n(\mathbb{Z}[\pi])$).

(v) *Computations.* Let β be any chain bundle on a chain complex B (in $\mathscr{C}_A$, for a ring with involution A). Write 0 for the only chain bundle on the zero chain complex 0_A in $\mathscr{C}_A$. The inclusion $0_A \hookrightarrow B$ is covered by a unique 'chain bundle map'; so there are induced homomorphisms of algebraic bordism groups

$$\text{release: } L^n(0_A, 0) \to L^n(B, \beta).$$

These algebraic bordism groups are defined just like $L^n(C(\tilde{X}), c(\gamma))$ in the preceding section.

Now $L^n(0_A, 0)$ is naturally isomorphic to the Wall group $L_n(A)$ (see [21] or [15]). Further, $L^n(0_A, 0) \cong L_n(A)$ and $L^n(B, \mathscr{E})$ are the nth homotopy groups of certain spectra $\mathscr{L}^{\cdot}(0_A, 0)$ and $\mathscr{L}^{\cdot}(B, \mathscr{E})$ respectively, and the release homomorphisms are induced by a map of spectra,

$$\text{release: } \mathscr{L}^{\cdot}(0_A, 0) \to \mathscr{L}^{\cdot}(B, \mathscr{E}).$$

Let $\hat{L}^n(B, \mathscr{E})$ be the nth (relative) homotopy group of 'release'. If we succeed in calculating $\hat{L}^n(B, \mathscr{E})$ for all n, then we have largely reduced the calculation of $\{L^n(B, \mathscr{E}) \mid n \in \mathbb{Z}\}$ to that of $\{L^n(0_A, 0) = L_n(A) \mid n \in \mathbb{Z}\}$. The following theorem shows, surprisingly, that the groups $\hat{L}^n(B, \mathscr{E})$ are homological objects and therefore usually easy to compute.

MAIN THEOREM. *There is a natural long exact sequence*

$$\ldots \to \hat{Q}^{n+1}(B) \to \hat{L}^n(B, \mathscr{E}) \to Q^n(B) \to \hat{Q}^n(B) \to \hat{L}^{n-1}(B, \mathscr{E}) \to \ldots \quad (n \in \mathbb{Z}).$$

The proof uses the algebraic surgery techniques of [15]. The standard application is to the case where $B = C(\tilde{X})$, $\mathscr{E} = c(\gamma)$ as in (iv). However, the main theorem has another application (to the more classical surgery theory): let B be the 'classifying chain complex for chain bundles' and $\mathscr{E}$ the 'universal chain bundle' on B. (So the role of B in the chain complex world is similar to that of the spaces BO or BG in topology.) Then $L^n(B, \mathscr{E})$ is the bordism group of symmetric algebraic Poincaré complexes of dimension n (with no particular structure), called $L^n(A)$ in [15]. The groups $L^n(A)$ are useful in obtaining product formulae for surgery obstructions. The main theorem above shows that the relative terms $\hat{L}^n(A) = \hat{L}^n(B, \mathscr{E})$ appearing in the long exact sequence relating $L_*(A)$ and $L^*(A)$ are homological objects. The homological description of $\hat{L}^n(A)$ is made even more explicit by a complete analysis of the 'classifying chain complex for chain bundles' which is obtained in Part II [23], for $A = \mathbb{Z}[\pi]$. (The result: it is as simple as it can be.)

(vi) *The 'ordinary generalized Kervaire Invariant' revisited.* The theory outlined so far has an unoriented version: instead of working with sextuples $(\pi, w; X, \gamma; \alpha, j)$, consider quadruples $(\pi; X, \gamma; \alpha)$ and replace $\mathbb{Z}[\pi]$ by $Z_2[\pi]$. The resulting algebraic bordism groups will be written $L^n(C(\tilde{X}; Z_2), c(\gamma; Z_2))$; here $C(\tilde{X}; Z_2) = C(\tilde{X}) \otimes_{\mathbf{Z}} Z_2$, etc.

Now assume further that $\pi = \{1\}$. In this case the flexible signature can be considered as a mild improvement on the 'generalized Kervaire Invariant' of [3]. That is to say, if $n = 2k$ and the $(k+1)$th Wu class of γ (in $H^{k+1}(X; Z_2)$) is zero, there is a commutative diagram

$$\begin{tikzcd} \Omega^P_{2k}(X, \gamma) \arrow[r] \arrow[d, "\text{fl.sig.}"] & Z_8 \\ L^{2k}(C(X; Z_2), c(\gamma; Z_2)) \arrow[ur] & \end{tikzcd}$$

in which the horizontal arrow is the invariant of [3]. The homomorphism $L^{2k}(C(X; Z_2), c(\gamma; Z_2)) \to Z_8$ is obtained by adapting the methods of [3]. Elements of $L^{2k}(C(X; Z_2), c(\gamma; Z_2))$ are represented by $2k$-dimensional algebraic Poincaré complexes (C, φ) (over the ring with involution $Z_2 = Z_2[\{1\}]$), with a certain structure; and this structure permits one to refine the non-degenerate symmetric bilinear form on $H^k(C; Z_2)$ to a *quadratic form*, with values in Z_4.

(Recall that the invariant of [3] is non-canonical, i.e. depends on a choice; the same is true for its algebraic counterpart, $L^{2k}(C(X; Z_2), c(\gamma; Z_2)) \to Z_8$, and there is a one-one correspondence between the two kinds of choices.)

Summarizing, it seems legitimate in the case at hand to regard the flexible signature itself as 'the' generalized Kervaire Invariant. It is defined for arbitrary n, without conditions on the spherical fibration γ, involves no choices, and looks pretty in product formulae. Finally, the groups $L^n(C(X; Z_2), c(\gamma; Z_2))$ are easy to compute with the help of the main theorem in (v) above. (Remember that the functors $Q^n(-)$, $\hat{Q}^n(-)$ are homotopy invariant, and that any chain complex over Z_2 is homotopy equivalent to its homology.)

(vii) *A Caveat.* Let $(\pi, w; X, \gamma; \alpha, j)$ be a sextuple as in (ii) again, and suppose that γ is the trivial spherical fibration on X. One is tempted to think that the release homomorphisms

$$L_n(\mathbb{Z}[\pi]) \to L^n(C(\tilde{X}), c(\gamma))$$

have particularly attractive properties in that case, such as being split injective; but they are probably not even injective in general.

(Here is an example to meditate upon. Let $X = \mathbb{R}P^\infty$, let $\tilde{X} = S^\infty$, and let γ, $\tilde{\gamma}$ be the trivial spherical fibrations on X and $\tilde{X}$ respectively. Then, for $n = 2, 6, 14, 30$, or $62(?)$, the composite

$$\Omega_n^P(X, \gamma) \xrightarrow{\text{transfer}} \Omega_n^P(\tilde{X}, \tilde{\gamma}) \xrightarrow{\text{Kervaire Invariant}} Z_2$$

is surjective, but

$$L_n(\mathbb{Z}[Z_2]) \xrightarrow{\text{transfer}} L_n(\mathbb{Z}[\{1\}]) \cong Z_2$$

is the zero map.)

Acknowledgments. Having written a first version of this essay without knowing the contents of [**15**], and having obtained a lot of advice from A. Ranicki on how to write the present version, I am very conscious of how much I owe to him. I am further indebted to Brian Sanderson for helpful supervision, to C. T. C. Wall and Charles Thomas for helpful criticism, to J. Eells for generous encouragement, and to the I.H.E.S. for a one-year invitation.

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0. *Conventions*

Since the present paper is inspired mainly by [**15**], I have tried to retain Ranicki's conventions wherever possible. They are listed here for convenience, with a few alterations and additions that appeared necessary.

The symbol 'II' refers to Part II of this paper [**23**].

0.1. The letter A is usually reserved for a *ring with involution*, that is, a ring with 1 equipped with an involutory antiautomorphism

$$\bar{\ }: A \to A;\ a \mapsto \bar{a}.$$

0.2. Unless otherwise specified, 'A-module' will mean *left A-module*. Sometimes, however, it is necessary to shift an A-action from left to right; so if M is a left A-module, let M^t be the right A-module with the same additive group as M, and with A-action

$$M^t \times A \to M^t;\ (x,a) \mapsto \bar{a}x.$$

0.3. The *dual module* M^* of an A-module M is

$$M^* = \mathrm{Hom}_A(M,A)$$

with A acting (on the left) by

$$A \times M \to M;\ (a,f) \mapsto (x \mapsto f(x)\cdot\bar{a}).$$

If M is finitely generated projective, then so is M^*, and the A-module homomorphism $M \to M^{**};\ x \mapsto (f \mapsto \overline{f(x)})$ is an isomorphism.

The dual of an A-module homomorphism $f\colon M \to N$ is the A-module homomorphism $f^*\colon N^* \to M^*;\ g \mapsto g\cdot f$.

0.4. Two chain maps $f, f'\colon C \to D$ (between A-module chain complexes graded over the integers) are *homotopic* if there exists a collection of A-module homomorphisms $\{g_r\colon C_r \to D_{r+1} \mid r \in \mathbb{Z}\}$ so that $f'-f = d_D g + g d_C\colon C \to D$. The collection is called a *chain homotopy* from f to f'.

0.5. Still assuming that C, D are A-module chain complexes, define abelian group chain complexes $C^t \otimes_A D$ and $\mathrm{Hom}_A(C,D)$ by

$$(C^t \otimes_A D)_n = \bigoplus_{p+q=n} C^t_p \otimes_A D_q,$$

$$d(x \otimes y) = x \otimes d_D(y) + (-)^q d_C(x) \otimes y;$$

$$(\mathrm{Hom}_A(C,D))_n = \prod_{q-p=n} \mathrm{Hom}_A(C_p, D_q),$$

$$d(f) = d_D(f) - (-)^n f d_C.$$

In both cases, d_D and d_C denote the differentials on D and C respectively, and d the differential on $C^t \otimes_A D$ or on $\mathrm{Hom}_A(C, D)$.

(The definition of $C^t \otimes_A D$ agrees well with certain geometric constructions, and with that given in [**15**]. Ranicki [**15, 16**] has a slightly different definition for $\mathrm{Hom}_A(C, D)$; I prefer the above because it gives more direct identifications

$$\text{(Group of cycles in } (\mathrm{Hom}_A(C, D))_0) \cong \text{(Group of chain maps } C \to D)$$

and

$$H_0(\mathrm{Hom}_A(C, D)) \cong \text{(Group of chain homotopy classes of chain maps from } C \text{ to } D).)$$

Note that the involution on A does not appear in the definition of $\mathrm{Hom}_A(C, D)$; we can do without it.

0.6. The *dual chain complex* C^{-*} of an A-module chain complex C is given by $(C^{-*})_r := (C_{-r})^* =: C^{-r}$, with differential

$$(-)^r d_C^*\colon C^{-r} \to C^{-r+1}.$$

A chain map $f\colon C \to D$ induces $f\colon D^{-*} \to C^{-*}$; $g \mapsto g \cdot f$.

Given an abelian group (or A-module) G, write (G, n) for the chain complex with $(G, n)_r = G$ if $r = n$ and $(G, n)_r = 0$ otherwise. It is worth noting that the 'obvious' identification $C^{-*} = \mathrm{Hom}_A(C, (A, 0))$ is not a chain map; some sign changes are necessary. Still, the choice of differential in C^{-*} has certain advantages. (It agrees with [**15**, p. 104, bottom], but not with [**15**, p. 98, bottom].)

If C is a chain complex of f.g. projective A-modules, then the chain map

$$C \to C^{-*-*};\ x \to (f \mapsto (-)^r \overline{f(x)})$$

is an isomorphism. If also $C_r = 0$ except for finitely many $r \in \mathbb{Z}$, then the slant chain map

$$\begin{gathered} C^t \otimes_A D \to \mathrm{Hom}_A(C^{-*}, D), \\ x \otimes y \to (f \mapsto \overline{f(x)} \cdot y) \end{gathered}$$

is an isomorphism.

0.7. The *suspension* ΣC of a chain complex

$$C\colon \ldots \to C_{r+1} \to C_r \to C_{r-1} \to \ldots$$

is the chain complex $(\Sigma C)_r = C_{r-1}$, $d_{\Sigma C} = -d_C$. (This differs from the definition in [**15**] by a sign.) So $\Sigma C \cong C \otimes_{\mathbb{Z}} (\mathbb{Z}, 1)$ (cf. 0.6), and

$$H_n(\mathrm{Hom}_A(C, D)) \cong \text{(group of homotopy classes of chain maps from } \Sigma^n C \text{ to } D),$$

if C, D are A-module chain complexes.

Let I stand for the cellular chain complex of the standard 1-simplex, or unit interval. Corresponding to the two endpoints of the 1-simplex, there are two chain maps $i_0, i_1\colon (\mathbb{Z}, 0) \to I$; if $f\colon C \to D$ is a chain map, we define the mapping cylinder of f to be the pushout of the diagram

$$\begin{tikzcd} C \otimes_{\mathbb{Z}} I & & D \\ & C \otimes_{\mathbb{Z}} (\mathbb{Z}, 0) \cong C \arrow[ul, "\mathrm{id} \otimes i_1"] \arrow[ur, "f"'] & \end{tikzcd}$$

and similarly for the mapping cone, Cone(f). So

$$\mathrm{Cone}(f)_r = D_r \oplus C_{r-1},$$

with differential d given by

$$d(x, y) = (d_D(x) + f(y), -d_C(y)).$$

If $f = \mathrm{id}\colon C \to C$, write Cone($C$) instead of Cone(id).

Every chain map $f\colon C \to D$ has an associated *Puppe sequence*

$$\ldots \to \Sigma^{-1}D \to \Sigma^{-1}\mathrm{Cone}(f) \to C \to D \to \mathrm{Cone}(f) \to \Sigma C \to \Sigma D \to \ldots,$$

infinite on both sides.

0.8. $\mathscr{C}_A$ (or simply $\mathscr{C}$) will be the category of chain complexes C, graded over the integers, such that each C_r is a f.g. projective left A-module, and such that $C_r = 0$ except for finitely many $r \in \mathbb{Z}$; the morphisms in $\mathscr{C}_A$ are A-module chain maps.

A morphism $f\colon C \to D$ in $\mathscr{C}_A$ is called a *fibration* if it is surjective, and a *cofibration* if its dual f^* is a fibration.

0.9. If C is a chain complex in $\mathscr{C}_A$, the group Z_2 acts on the abelian group chain complex $C^t \otimes_A C$ (cf. 0.5) by the transposition involution

$$T\colon C^t_p \otimes_A C_q \to C^t_p \otimes_A C_q;\ x \otimes y \mapsto (-)^{pq} y \otimes x.$$

Following [**15**], we shall have to deal with the 'cohomology groups of Z_2 with coefficients in the $\mathbb{Z}[Z_2]$-module chain complex $C^t \otimes_A C$' and the like. Here $\mathbb{Z}[Z_2]$ is the group ring, without any particular involution; we let W be the standard free resolution of the trivial $\mathbb{Z}[Z_2]$-module $\mathbb{Z}$ (viz., $W_r = \mathbb{Z}[Z_2]$ if $r \geqslant 0$, $W_r = 0$ if $r < 0$, with differential $d\colon W_r \to W_{r-1};\ x \mapsto (1 + (-)^r T)\cdot x$ where $r > 0$, T being the generator of Z_2), and write $W \% C$ for the abelian group chain complex

$$\mathrm{Hom}_{\mathbb{Z}[Z_2]}(W, C^t \otimes_A C).$$

Then

$$Q^n(C) := H_n(W \% C)$$

is the $(-n)$*th cohomology group of* Z_2 *with coefficients in* $C^t \otimes_A C$. (See [**10**] if the terminology appears mysterious.)

On replacing the standard resolution W by the standard complete resolution $\hat{W}$ (with $\hat{W}_r = \mathbb{Z}[Z_2]$ for all r, $d\colon \hat{W}_r \to \hat{W}_{r-1};\ x \mapsto (1 + (-)^r T)\cdot x$, for all r) we obtain a chain complex $\hat{W} \% C := \mathrm{Hom}_{\mathbb{Z}[Z_2]}(\hat{W}, C^t \otimes_A C)$ whose homology groups

$$\hat{Q}^n(C) := H_n(\hat{W} \% C)$$

are the *Tate cohomology groups of* Z_2 *with coefficients in* $C^t \otimes_A C$.

0.10. We will need a detailed description of the abelian group chain complexes $\hat{W} \% C$, $W \% C$, and $\hat{W} \% C^{-*}$ $(= W\%(C^{-*}))$.

Case 1: $\hat{W} \% C$. Then

$$(\hat{W} \% C)_n = \prod_s \mathrm{Hom}_{\mathbb{Z}[Z_2]}(\hat{W}_s, (C^t \otimes_A C)_{n+s})$$

$$= \prod_s (C^t \otimes_A C)_{n+s},$$

and the differential d sends a collection

$$\varphi = \{\varphi_s \in (C^t \otimes_A C)_{n+s} \mid s \in \mathbb{Z}\} \in (\hat{W}\& C)_n$$

to $d(\varphi) \in (\hat{W}\& C)_{n-1}$, with $(d(\varphi))_s \in (C^t \otimes_A C)_{n-1+s}$ given by

$$d_{C^t \otimes_A C}(\varphi_s) - (-)^n(\varphi_{s-1} + (-)^s T\varphi_{s-1}).$$

If we use the identification

$$C^t \otimes_A C \cong \operatorname{Hom}_A(C^{-*}, C)$$

of 0.6, then φ can be regarded as a collection

$$\{\varphi_s \in \prod_r \operatorname{Hom}_A(C^{n-r+s}, C_r) \mid s \in \mathbb{Z}\};$$

the differential sends this to $d(\varphi)$, with $(d(\varphi))_s = \prod_r \operatorname{Hom}_A(C^{n-1-r+s}, C_r)$ given by

$$d_C \cdot \varphi_s - (-)^{n+s}\varphi_s \cdot d_{C^{-*}} - (-)^n(\varphi_{s-1} + (-)^s T\varphi_{s-1}).$$

Here Z_2 acts on $\operatorname{Hom}_A(C^{-*}, C)$ by the duality involution

$$T\colon \operatorname{Hom}_A(C^p, C_q) \to \operatorname{Hom}_A(C^q, C_p);\ g \mapsto (-)^{pq} g^*.$$

If $\varphi = \{\varphi_s\}$ is a cycle, $d(\varphi) = 0$, then φ_s is a chain homotopy from 0 to the chain map

$$(-)^n(\varphi_{s-1} + (-)^s T\varphi_{s-1})\colon \Sigma^{n+s-1}(C^{-*}) \to C,$$

for each s.

Case 2: $W \& C$. This is much the same as Case 1, except that we are now dealing with collections $\varphi = \{\varphi_s\}$ such that $\varphi_s = 0$ for $s < 0$. Identifying $C^t \otimes_A C$ with $\operatorname{Hom}_A(C^{-*}, C)$ again, we find that for a *cycle* φ in $(W \& C)_n$, φ_0 is a chain map from $\Sigma^n(C^{-*})$ to C; it is 'self-dual' up to an infinity of higher chain homotopies (the higher chain homotopies are the φ_s, for $s > 0$).

Case 3: $\hat{W} \& C^{-*}$. Here we make the identification $C^{-*t} \otimes_A C^{-*} \cong \operatorname{Hom}_A(C, C^{-*})$ (using 0.6 and the chain isomorphism $C \cong C^{-*-*}$ specified there) and find that the differential maps

$$\varphi = \{\varphi_s \in \prod_r \operatorname{Hom}(C_{r-n-s}, C^{-r}) \mid s \in \mathbb{Z}\} \in (\hat{W} \& C^{-*})_n$$

to $d(\varphi) = \{(d(\varphi))_s \mid s \in \mathbb{Z}\}$, with $(d(\varphi))_s$ given by

$$d_{C^{-*}} \cdot \varphi_s - (-)^{n+s}\varphi_s \cdot d_C - (-)^n(\varphi_{s-1}(-)^s T\varphi_{s-1}).$$

This time we are obliged to let Z_2 act by

$$T\colon \operatorname{Hom}_A(C_p, C^q) \to \operatorname{Hom}_A(C_q, C^p);\ g \mapsto (-)^{pq+p+q} \cdot g^*.$$

0.11. (Taken from [**15**, Part I, §8].) Define the *tensor product* of two rings with involution A, B to be $A \otimes_{\mathbb{Z}} B$, with involution

$$\overline{(a \otimes b)} \mapsto \bar{a} \otimes \bar{b}.$$

If C is an A-module chain complex, and D is a B-module chain complex, then $C \otimes_{\mathbb{Z}} D$ is an $A \otimes_{\mathbb{Z}} B$-module chain complex, since $A \otimes_{\mathbb{Z}} B$ acts on $C \otimes_{\mathbb{Z}} D$ by $(a \otimes b) \cdot (x \otimes y) = ax \otimes by$. (The differential in $C \otimes_{\mathbb{Z}} D$ is defined with sign conventions as in 0.5.) Then $C \otimes_{\mathbb{Z}} D$ is in $\mathscr{C}_{A \otimes_{\mathbb{Z}} B}$ provided C and D are in $\mathscr{C}_A$ and $\mathscr{C}_B$

respectively. Under the same conditions there is an identification

$$C^{-*} \otimes_{\mathbb{Z}} D^{-*} \to (C \otimes_{\mathbb{Z}} D)^{-*},$$
$$f \otimes g \mapsto (x \otimes y \mapsto (-)^{|g||x|} f(x) \otimes g(y))$$

(vertical bars as in $|g|$, $|x|$ denote dimension). There is another identification of $\mathbb{Z}[Z_2]$-module chain complexes

$$(C^t \otimes_A C) \otimes_{\mathbb{Z}} (D^t \otimes_B D) \to (C \otimes_{\mathbb{Z}} D)^t \otimes_{A \otimes_{\mathbb{Z}} B} (C \otimes_{\mathbb{Z}} D),$$
$$(v \otimes w) \otimes (x \otimes y) \mapsto (-)^{|x||w|} (v \otimes x) \otimes (w \otimes y)$$

(there are no conditions; $T \in Z_2$ acts on the left-hand side by

$$T((v \otimes w) \otimes (x \otimes y)) := T(v \otimes w) \otimes T(x \otimes y)).$$

The $\mathbb{Z}[Z_2]$-module chain complex $\hat{W}$ of 0.9 is equipped with a strictly associative diagonal map

$$\Delta\colon \hat{W} \to \hat{W} \bar{\otimes} \hat{W};\ 1_s \mapsto \sum_{r=-\infty}^{+\infty} 1_r \otimes T^r_{s-r} \quad (s \in \mathbb{Z}).$$

(The symbol $\bar{\otimes}$ indicates that infinite chains are allowed; subscripts denote dimensions, and $\hat{W}_r$ is identified with the ring $\mathbb{Z}[Z_2]$. Regarding W as a factor complex of $\hat{W}$ gives a similar diagonal for W.) It can be used to define exterior products, such as the chain map

$$\times : \hat{W} \,\&\, C \otimes_{\mathbb{Z}} \hat{W} \,\&\, D \to \hat{W} \,\&(C \otimes_{\mathbb{Z}} D);\ \varphi \times \theta := (\varphi \otimes \theta) \cdot \Delta$$

(this makes sense if C and D are in $\mathscr{C}_A$ and $\mathscr{C}_B$ respectively). More explicitly, if $\varphi = \{\varphi_s\}$ is an m-chain, and $\theta = \{\theta_s\}$ is an n-chain (as in 0.7), then

$$(\varphi \times \theta)_s = \sum_{r=-\infty}^{+\infty} \varphi_r \otimes T^r \theta_{s-r}.$$

(So $(\varphi \times \theta)_s$ belongs to

$$((C \otimes_{\mathbb{Z}} D)^t \otimes_{A \otimes_{\mathbb{Z}} B} (C \otimes_{\mathbb{Z}} D))_{m+n+s} \cong ((C^t \otimes_A C) \otimes_{\mathbb{Z}} (D^t \otimes_B D))_{m+n+s},$$

but be warned that this last identification involves sign changes.)

Of course, $\hat{W}$ can be replaced by W. Apart from being associative, the exterior product has something like a *unit*: namely the triple $(\mathbb{Z}, (\mathbb{Z}, 0), v)$, in which $\mathbb{Z}$ is regarded as a ring with involution, $(\mathbb{Z}, 0)$ as a chain complex in $\mathscr{C}_{\mathbb{Z}}$, and v is the 0-chain in $\hat{W} \,\&\, (\mathbb{Z}, 0)$ determined by $v_0 = 1 \in \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$.

Again, this works with $\hat{W}$ replaced by W.

0.12. For A-modules M, N a *sesquilinear map* $\lambda\colon M \times N \to A$ is a biadditive map satisfying

$$\lambda(ax, by) = a\lambda(x, y)\bar{b}$$

for $a, b \in A$, $x \in M$, $y \in N$.

Taking (left) adjoints we can identify the abelian group of sesquilinear maps $M \times N \to A$ with $\mathrm{Hom}_A(M, N^*)$; so λ corresponds to

$$\mathrm{ad}(\lambda)\colon x \to (y \mapsto \overline{\lambda(x, y)}).$$

(Right adjoints will be avoided, although their use would save bars.) The *transpose* $T\lambda$

of a sesquilinear map $\lambda\colon M \times N \to A$ is the sesquilinear map

$$N \times M \to A;\ (y, x) \mapsto \overline{\lambda(x, y)}.$$

Under the left adjoint, T corresponds to the usual transposition

$$\mathrm{Hom}_A(M, N^*) \to \mathrm{Hom}_A(N, M^*).$$

If $M = N$, we speak of a *sesquilinear form*, and denote the abelian group of such forms by $\mathrm{Sel}(M)$.

0.13. If F is a covariant or contravariant functor from a category X to a category Y, and f is a morphism in X, I shall occasionally write $f^{\rightarrow}$ or $f^{\leftarrow}$ instead of $F(f)$ (whereas f^* indicates a 'dual' chain map or homomorphism, as in 0.3 and 0.6). If C is a chain complex in $\mathscr{C}_A$, and $\hat{W}$ and W are as in 0.9, then the canonical projection $\hat{W} \to W$ induces a chain map

$$J\colon W \& C \to \hat{W} \& C$$

which in turn induces homomorphisms $Q^n(C) \to \hat{Q}^n(C)$, also denoted by J.

0.14. The *homotopy pullback* of a diagram $X \to Z \leftarrow Y$ of chain maps is the chain complex P with

$$P_n := X_n \otimes Y_n \otimes Z_{n+1}$$

and

$$d\colon (x, y, z) \mapsto (-dx, -dy, dz + f(x) - f(y)).$$

0.15. If Y is a chain complex (of free abelian groups, say), then an 'n-cycle in Y, well-defined up to an infinity of higher homologies' is a diagram of chain maps of the following sort:

$$(\mathbb{Z}, n) \overset{\simeq}{\leftarrow} X \to Y.$$

Here $(\mathbb{Z}, n)$ is defined in 0.6, and X is another chain complex of free abelian groups.

1. *Chain bundles*

The first part of this section presents a cohomology theory, defined on the category $\mathscr{C}_A$ (see 0.8), which is to real or spherical K-theory as chain complexes (for instance those in $\mathscr{C}_A$) are to CW-spaces. The ring with involution A is kept fixed.

1.1. THEOREM. *The contravariant functors*

$$C \mapsto \hat{Q}^n(C^{-*}), \quad \textit{for } n \in \mathbb{Z},$$

(*see* 0.9) *constitute a cohomology theory on* $\mathscr{C}_A$. *That is,*

(i) *if* $f, f'\colon C \to D$ *are homotopic chain maps in* $\mathscr{C}_A$, *then*

$$f^{\leftarrow} = f'^{\leftarrow}\colon \hat{Q}^n(D^{-*}) \to \hat{Q}^n(C^{-*}), \quad \textit{for } n \in \mathbb{Z};$$

(ii) *there is a natural and canonical equivalence between the two functors*

$$C \mapsto \hat{Q}^{n-1}((\Sigma C)^{-*}) \quad \textit{and} \quad C \mapsto \hat{Q}^n(C^{-*}), \quad \textit{for } n \in \mathbb{Z};$$

(iii) *if* $f\colon C \to D$ *is a chain map in* $\mathscr{C}_A$ *with associated Puppe sequence*

$$\dots \to \Sigma^{-1}\mathrm{Cone}(f) \to C \to D \to \mathrm{Cone}(f) \to \Sigma C \to \Sigma D \to \dots,$$

then the sequence of induced homomorphisms

$$\dots \leftarrow \hat{Q}^0((\Sigma^{-1}\mathrm{Cone}(f))^{-*}) \leftarrow \hat{Q}^0(C^{-*}) \leftarrow \hat{Q}^0(D^{-*}) \leftarrow \hat{Q}^0(\mathrm{Cone}(f)^{-*}) \leftarrow \dots$$

is exact.

Proof. Note first that $C \mapsto \hat{Q}^n(C^{-*})$ is a contravariant functor because a chain map $f\colon C \to D$ induces

$$f^* \otimes f^*\colon D^{-*t} \otimes_A D^{-*} \to C^{-*t} \otimes_A C^{-*},$$

and hence a chain map

$$f^{\leftarrow}\colon \hat{W} \& D^{-*} \to \hat{W} \& C^{-*} \quad \text{etc.}$$

To prove (i), suppose that $f, f'\colon C \to D$ are homotopic; then so are $f^*, f'^*\colon D^{-*} \to C^{-*}$. Let g be a homotopy from f^* to f'^*; think of g as a chain map from $D^{-*} \otimes_{\mathbb{Z}} I$ to C^{-*}. Now the chain maps $f^{\leftarrow}, f'^{\leftarrow}\colon \hat{W} \& D^{-*} \to \hat{W} \& C^{-*}$ are also homotopic; the appropriate chain homotopy maps an n-chain φ in $\hat{W} \& D^{-*}$ to the $(n+1)$-chain $g^{\rightarrow}(\varphi \times \omega)$ in $\hat{W} \& C^{-*}$.

(Explanation: choose, once and for all, a 1-chain ω in

$$\hat{W} \& I = \mathrm{Hom}_{\mathbb{Z}[Z_2]}(\hat{W}, I \otimes_{\mathbb{Z}} I)$$

(cf. 0.7) so that $d(\omega) = i_1^{\rightarrow}(v) - i_0^{\rightarrow}(v)$; as in 0.7, i_0 and i_1 are certain chain maps from $(\mathbb{Z}, 0)$ to I, and v is the unit mentioned at the end of 0.11. For $\varphi \in (\hat{W} \& D^{-*})_n$, the exterior product $\varphi \times \omega$ (cf. 0.11) belongs to $(\hat{W} \& (D^{-*} \otimes_{\mathbb{Z}} I))_{n+1}$, whence $g^{\rightarrow}(\varphi \times \omega)$ is in $(\hat{W} \& C^{-*})_{n+1}$ as required.) See [15] for an explicit formula.

For the proof of (ii) and (iii), recall (for example, from [10]) that a module M over a ring with unit is 'coextended' or 'coinduced' if it has the form $M \cong \mathrm{Hom}_{\mathbb{Z}}(R, G)$ (G being any abelian group, R the ring at issue, acting on the left of $\mathrm{Hom}_{\mathbb{Z}}(R, G)$ by $(a, f) \mapsto (x \mapsto f(x \cdot a))$ for $a \in R$, $f \in \mathrm{Hom}(R, G)$). If, for example, $R = \mathbb{Z}[Z_2]$, the Tate cohomology groups of a coinduced module M, written

$$\hat{H}^n(Z_2; M) := H_{-n}(\mathrm{Hom}_{\mathbb{Z}[Z_2]}(\hat{W}, (M, 0)))$$

(see 0.6), are trivial by a simple argument.

Further, let D be a chain complex of $\mathbb{Z}[Z_2]$-modules; call D *coinduced* if $D_r = 0$ except for a finite number of $r \in \mathbb{Z}$, and if D_r is a coinduced module for all r. Then a familiar induction argument [7, Anhang Proposition 2.1] shows that for such a D, we have $H_n(\mathrm{Hom}_{\mathbb{Z}[Z_2]}(\hat{W}, D)) = 0$ for $n \in \mathbb{Z}$.

We now exploit this fact. Suppose that $f\colon C \to D$ is a *cofibration* in $\mathscr{C}_A$ (see 0.8), so that $0 \to C \to D \to D/\mathrm{im}(f) \to 0$ is a short exact sequence in $\mathscr{C}_A$. Then the sequence of induced maps

$$C^{-*} \otimes_A C^{-*} \xleftarrow{i^{\leftarrow}} D^{-*} \otimes_A D^{-*} \xleftarrow{p^{\leftarrow}} (D/\mathrm{im}(f))^{-*} \otimes_A (D/\mathrm{im}(f))^{-*}$$

is not short exact, because $\ker(i^{\leftarrow})/\mathrm{im}(p^{\leftarrow}) \neq 0$ in general. However, $\ker(i^{\leftarrow})/\mathrm{im}(p^{\leftarrow})$ is a *coinduced* chain complex of $\mathbb{Z}[Z_2]$-modules, which by the above just suffices to show that the homology groups of the chain complexes $\hat{W} \& C^{-*}$, $\hat{W} \& D^{-*}$, $\hat{W} \& (D/\mathrm{im}(f))^{-*}$ fit into a long exact sequence.

This proves (iii) (since every morphism in $\mathscr{C}_A$ is 'homotopy equivalent' to a cofibration), or strictly speaking reduces (iii) to (ii). But (ii) is proved by applying the same argument with f equal to the inclusion of C in Cone(C) (see 0.7).

1.2. REMARKS. (a) Theorem 1.1 is of course equivalent to the statement that the *covariant* functors $C \mapsto \hat{Q}^n(C)$ constitute a homology theory. The 'homotopy invariance' part of the statement can be obtained under more general circumstances: for instance, the covariant functors $C \to Q^n(C)$ are also homotopy invariant, by the same argument.

(b) The 'covariant' suspension isomorphism $\hat{Q}^n(C) \to \hat{Q}^{n+1}(\Sigma C)$ has an explicit description. Regard the collapsing map $C \otimes_{\mathbb{Z}} I \to \Sigma C$ as a chain homotopy from the zero map $0\colon C \to \Sigma C$ to itself. Then the proof of part (i) of 1.1 gives a semi-explicit formula for an induced chain homotopy (depending on the choice of a certain 1-chain ω) from the zero map $0\colon \hat{W} \& C \to \hat{W} \& (\Sigma C)$ to itself, that is, a chain map $\mathfrak{S}$ from $\Sigma(\hat{W} \& C)$ to $\hat{W} \& (\Sigma C)$. Passage to homology groups gives the suspension isomorphism. For a suitable choice of ω, the chain map $\mathfrak{S}\colon \Sigma(\hat{W} \& C) \to \hat{W} \& (\Sigma C)$ takes the form $\varphi \mapsto \mathfrak{S}\varphi$; $\mathfrak{S}\varphi)_{s+1} = (-)^s \mu_2(\varphi_s)$ (where $\mu_1, \mu_2\colon C^t \otimes_A C \to C^t \otimes_A C$ are homomorphisms given by $\mu_1(x \otimes y) = (-)^{|x|} x \otimes y$, $\mu_2(x \otimes y) = (-)^{|y|} x \otimes y$); then $\mathfrak{S}$ is a chain isomorphism.

The same formula yields a suspension homomorphism (not an isomorphism in general) $Q^n(C) \to Q^{n+1}(\Sigma C)$.

Finally, similar formulae exist in the contravariant case, but will not be needed.

(The suspension formula in [**15**, p. 106] is simpler, but I suspect the signs are incorrect.)

1.3. PROPOSITION. *The groups $\hat{Q}^n(C^{-*})$, $\hat{Q}^n(C)$ are Z_2-vector spaces (for C in $\mathscr{C}_A$, $n \in \mathbb{Z}$). Further, the cohomology theory $C \mapsto \{\hat{Q}^n(C^{-*})\}$ is periodic of order 2, that is, there exist natural isomorphisms $\hat{Q}^n(C^{-*}) \cong \hat{Q}^{n+2}(C^{-*})$, with $n \in \mathbb{Z}$, commuting with the suspension isomorphisms. Similarly, the homology theory $C \mapsto \{\hat{Q}^n(C)\}$ is periodic.*

Proof. The first statement holds because the chain map $\hat{W} \to \hat{W}$; $x \mapsto 2x$ is null-homotopic. The periodicity isomorphisms come from

$$\Sigma^2 \operatorname{Hom}_{\mathbb{Z}[Z_2]}(\hat{W}, C^t \otimes_A C) \cong \Sigma^2 \operatorname{Hom}_{\mathbb{Z}[Z_2]}(\Sigma^2 \hat{W}, C^t \otimes_A C)$$
$$\cong \operatorname{Hom}_{\mathbb{Z}[Z_2]}(\hat{W}, C^t \otimes_A C),$$

that is, from the periodicity of $\hat{W}$.

It is possible to give a more economical description of the groups $\hat{Q}^n(C^{-*})$, for C in $\mathscr{C}_A$. Thinking of a class in $\hat{Q}^n(C^{-*})$ as being represented by a collection

$$\{z_{p,q} \in \operatorname{Hom}_A(C_p, C^q) \mid p, q \in \mathbb{Z}\} \quad (\text{cf. } 0.10),$$

one should expect that most of the information is located in the terms $z_{p,q}$ with $p = q$; and something quite similar is true. (This is well known, and I am grateful to John Jones for pointing it out to me, when $A = Z_2$.)

1.4. DEFINITION. For a chain complex C in $\mathscr{C}_A$, let

$$V(C) = \ldots \to V(C)_1 \to V(C)_0 \to V(C)_{-1} \to \ldots$$

be the abelian group chain complex given by

$$V(C)_n := \prod_{r\in\mathbb{Z}} \mathrm{Sel}(C_r) \quad \text{(cf. 0.12)},$$

$$d\colon V(C)_n \to V(C)_{n-1};$$

$$\{\lambda_r\} \mapsto \{\mu_r := (-)^n d_C^{\leftarrow}(\lambda_{r-1}) + (-)^{n+r}\lambda_r - T\lambda_r\}.$$

(Here $\lambda_r, \mu_r, d_C^{\leftarrow}(\lambda_{r-1})$ are in $\mathrm{Sel}(C_r)$; $d_C^{\leftarrow}(\lambda_{r-1})(x,y) := \lambda_{r-1}(d_C(x), d_C(y))$.)

A cycle in $V(C)_0$ is then a sequence of sesquilinear forms $\lambda_r\colon C_r \times C_r \to A$ such that, for each $r \in \mathbb{Z}$, the 'symmetrization of λ_r' ($= T\lambda_r - (-)^r\lambda_r$) equals the pullback $d_C(\lambda_{r-1})$.

Note that $C \mapsto V(C)$ is a contravariant functor.

1.5. Preparation. Think of an element in $(\hat{W} \& C^{-*})_n$ as a collection of sesquilinear forms $\{\varphi_{p,q}\colon C_p \times C_q \to A \mid p, q \in \mathbb{Z}\}$ (whose left adjoints are A-homomorphisms $z_{p,q}\colon C_p \to C^q$). Then the differential $d\colon (\hat{W} \& C^{-*})_{n+1} \to (\hat{W} \& C^{-*})_n$ is described by

$$d(\varphi_{p,q}) = \begin{Bmatrix} (-)^n\varphi_{p,q}(-)^{pq}T\varphi_{p,q},\ (-)^q\varphi_{p,q}(\mathrm{id}\times d_C), \\ (-)^{p+q+1}\varphi_{p,q}(d_C\times \mathrm{id}),\ \text{all other components } 0 \end{Bmatrix}.$$

(Here $\varphi_{p,q}$ is to be regarded as a collection with at most a single non-zero component, whereas the right-hand side has three non-zero components at most.)

1.6. Proposition. *The homomorphisms*

$$\mathrm{Eco}_n\colon V(C)_n \to (\hat{W} \& C^{-*})_n;$$

$$\lambda_r \mapsto \begin{cases} \{\lambda_r,\ (-)^r\lambda_r(d_C\times\mathrm{id}),\ \textit{other components } 0\} \\ \{(-)^{r-1}\lambda_r,\ (-)^{r-1}\lambda_r(\mathrm{id}\times d_C),\ \textit{other components } 0\} \end{cases}$$

(*using the top line for even n, the bottom line for odd n*) *constitute a natural chain map* 'Eco'. (*Again, λ_r is regarded as a collection with a single non-zero member at most, etc.*) *It induces isomorphisms in homology,*

$$H_n(V(C)) \cong \hat{Q}^n(C^{-*}), \quad \textit{with } n \in \mathbb{Z},$$

for any C in $\mathscr{C}_A$.

Proof (of the last sentence). We first show that

$$C \mapsto \{H_n(V(C)) \mid n \in \mathbb{Z}\}$$

is a cohomology theory. This means proving the analogues of Theorem 1.1 (i), (ii), (iii).

Suppose first that $0 \to C \to D \to B \to 0$ is a short exact sequence of chain complexes and chain maps in $\mathscr{C}_A$, with induced sequence

$$0 \longleftarrow V(C) \xleftarrow{i^{\leftarrow}} V(D) \xleftarrow{p^{\leftarrow}} V(B) \longleftarrow 0;$$

then we would like to know that $\ker(i^{\leftarrow})/\mathrm{im}(p^{\leftarrow})$ is an acyclic chain complex. Assuming the contrary, let $\lambda = \{\lambda_r \mid r \in \mathbb{Z}\}$ be an n-chain in $\ker(i^{\leftarrow}) \subset V(D)$ such that $d(\lambda)$ belongs to $\mathrm{im}(p^{\leftarrow})$ and the homology class in $H_n(\ker(i^{\leftarrow})/\mathrm{im}(p^{\leftarrow}))$ represented by λ is non-zero. More precisely, within its homology class choose the representative λ so that the number

$$\min\{r \in \mathbb{Z} \mid \lambda_r \neq 0\}$$

is as large as possible. It is easy to increase this integer by 1, giving a contradiction.

It follows that the homology groups of $V(C)$, $V(D)$, and $V(B)$ are related by a long exact sequence. In particular, if $D = C \oplus B$, then $H_*(V(D)) \cong H(V(C))_* \oplus H_*(V(B))$.

Next, suppose that C is a contractible complex in $\mathscr{C}_A$. Then $H_*(V(C)) = 0$. This can be proved by writing C as a direct sum of contractible chain complexes, each concentrated in two adjacent dimensions, and then applying the additivity principle just obtained.

Next, suppose that $C \to D$ is a morphism in $\mathscr{C}_A$ which is a cofibration and a homotopy equivalence. Then the preceding arguments, when combined, show that the induced homomorphism $H_*(V(D)) \to H_*(V(C))$ is an isomorphism. This applies in particular to the cofibrations

$$\mathrm{id}_C \otimes i_0, \mathrm{id}_C \otimes i_1 \colon C \otimes_{\mathbb{Z}} (\mathbb{Z}, 0) \to C \otimes_{\mathbb{Z}} I$$

(cf. 0.7) and shows therefore that the functors $C \mapsto H_n(V(C))$, for $n \in \mathbb{Z}$, are homotopy invariant. It is now clear that these functors constitute a cohomology theory.

The proof is completed by observing that the natural chain map 'Eco' (which by now induces a transformation of cohomology theories) gives an isomorphism between the respective 'coefficients' of the two cohomology theories; in other words, by observing that the last sentence of 1.6 holds for $C = (A, 0)$. (The usual induction argument from [7, Anhang Proposition 2.1] shows that this suffices.)

The second half of the chapter consists mostly of somewhat tedious definitions making the analogy between (geometric) real K-theory and the cohomology theory $C \mapsto \{\hat{Q}^n(C^{-*})\}$ more precise.

1.7. Definition. A *chain bundle* on a chain complex C in $\mathscr{C}_A$ is a 0-dimensional cycle in $\hat{W} \,\&\, C^{-*}$.

1.8. Definition. If $f\colon C \to D$ is a chain map in $\mathscr{C}_A$, c is a chain bundle on C, and d is a chain bundle on D, then a 'chain bundle map from c to d, covering f' is a homology $y \in (\hat{W} \,\&\, C^{-*})_1$ from c to $f^{\leftarrow}(d)$ (so that $c + d(y) = f^{\leftarrow}(d)$ in $\hat{W} \,\&\, C^{-*}$).

Observe that chain bundle maps can be composed. If we take $f = \mathrm{id}\colon C \to C$, then the category of chain bundles on C and chain bundle maps covering id is a groupoid; its components are the elements of $\hat{Q}^0(C^{-*})$.

Sometimes a 'change of rings' has to be allowed: suppose that there is given a homomorphism $A \to A'$ of rings with involution (and unit), making A' into a right module over A. Then

$$C \mapsto A' \otimes_A C$$

is a functor from $\mathscr{C}_A$ to $\mathscr{C}_{A'}$. In the same spirit, a chain bundle c on C determines a chain bundle $A' \otimes_A c$ on $A' \otimes_A C$, and so on. So if C is in $\mathscr{C}_A$ and D is in $\mathscr{C}_{A'}$, we would contemplate chain maps of the form $f\colon A' \otimes_A C \to D$; and if c is a chain bundle on C and d is a chain bundle on D, then we would also contemplate chain bundle maps $A' \otimes_A c \to d$, covering f.

It is of course possible to define higher homotopies between chain bundle maps with the same domain and range.

There are at least two distinct ways (1.9 and 1.11) of making the chain bundles over a fixed chain complex C in $\mathscr{C}_A$ into a simplicial set (see [**6**] for general information on simplicial sets).

1.9. Definition. The 'Kan–Dold simplicial set of chain bundles on C' is the simplicial set $\mathrm{KD}(\hat{W} \& C^{-*})$ obtained by applying the *Kan–Dold functor* KD to the chain complex $\hat{W} \& C^{-*}$ (cf. 1.10).

1.10. Explanation. Let $\mathrm{Cl}(\Delta_n)$ be the cellular chain complex of the standard n-simplex. Given a chain complex E (of abelian groups, graded over the integers), define $\mathrm{KD}(E)$ to be the simplicial abelian group with set of n-simplices

$$\mathrm{KD}(E)_n := (\text{set of chain maps from } \mathrm{Cl}(\Delta_n) \text{ to } E);$$

the face and degeneracy operators are then obvious. Conversely, given a simplicial abelian group G, let NG be the chain complex such that

$$NG_q = \bigcap_{i \neq 0} \ker[d_i\colon G_q \to G_{q-1}]$$

(the d_i being the face operators) and $d\colon NG_q \to NG_{q-1}$ equals the restriction of d_0. Then $\mathrm{KD}(NG) \cong G$, and $N(\mathrm{KD}(E))$ is isomorphic to the subcomplex E^+ of E with

$$E_n^+ = \begin{cases} E_n & \text{if } n > 0, \\ \ker[d\colon E_0 \to E_{-1}] & \text{if } n = 0, \\ 0 & \text{if } n < 0. \end{cases}$$

See also [**6**].

1.11. Definition. For a chain complex C in $\mathscr{C}_A$, the *simplicial set of concordances of chain bundles on* C, written $\mathscr{B}(C)$, has as set of n-simplices

$$\mathscr{B}(C)_n := (\text{set of chain bundles on } C \otimes_{\mathbb{Z}} \mathrm{Cl}(\Delta_n))$$

(with $\mathrm{Cl}(\Delta_n)$ as in 1.10; here $C \otimes_{\mathbb{Z}} \mathrm{Cl}(\Delta_n)$ is regarded as a chain complex in $\mathscr{C}_A$, and, I hope, the simplicial operators are again obvious).

Both $\mathrm{KD}(\hat{W} \& C^{-*})$ and $\mathscr{B}(C)$ are simplicial abelian groups. They are useful because most chain bundles occurring in nature are only well defined up to an infinity of higher homologies, or of higher concordances—just like the (geometric) stable normal bundle of a manifold, or the Spivak normal fibration of a geometric Poincaré complex.

The next proposition is a generalization of Theorem 1.1 (i) (as its proof will make clear).

1.12. Proposition. *There is a natural homomorphism of simplicial abelian groups*

$$\mathrm{Lin}\colon \mathscr{B}(C) \to \mathrm{KD}(\hat{W} \& C^{-*})$$

inducing an isomorphism in homotopy groups ('Lin' stands for linearization).

Proof. First, we require a sequence

$$\{\rho(n) \in (\hat{W} \& \mathrm{Cl}(\Delta_n))_n \mid n = 0, 1, 2, \ldots\}$$

such that

(i) $\rho(0) = \nu$ (see the end of 0.11; $\mathrm{Cl}(\Delta_0)$ has been identified with $(\mathbb{Z}, 0)$),
(ii) $\sum_{i=0}^{n}(-)^i \cdot e_i^{\to}(\rho(n-1)) = d(\rho(n))$ for $n = 1, 2, \ldots$ (e_i: $\Delta_{n-1} \to \Delta_n$ is the inclusion of the ith face, and d is the differential in $\hat{W} \,\&\, \mathrm{Cl}(\Delta_n)$),
(iii) the chain map $\hat{W} \,\&\, \mathrm{Cl}(\Delta_n) \to \hat{W} \,\&\, \mathrm{Cl}(\Delta_0)$ (induced by the map $\Delta_n \to \Delta_0$) maps $\rho(n)$ to 0, for $n > 0$.

It is easy to construct such a sequence by induction, because

$$\ker[\hat{W} \,\&\, \mathrm{Cl}(\Delta_n) \to \hat{W} \,\&\, \mathrm{Cl}(\Delta_0)]$$

is acyclic (by 1.1 (i), or rather its covariant analogue). In defining $\hat{W} \,\&\, \mathrm{Cl}(\Delta_n)$ etc., we have to work temporarily over the ring with involution $\mathbb{Z}$.

Second, we need the evaluation chain map

$$ev\colon (C \otimes_{\mathbb{Z}} \mathrm{Cl}(\Delta_n))^{-*} \otimes_{\mathbb{Z}} \mathrm{Cl}(\Delta_n) \to C^{-*};$$
$$f \otimes y \mapsto (x \mapsto (-)^{|x||y|} f(x \otimes y)).$$

Third, note that for a chain complex E, any n-simplex s in $\mathrm{KD}(E)$ determines an element $s_{\mathrm{top}} \in E_n$ (namely the image under s: $\mathrm{Cl}(\Delta_n) \to E$ of the generator corresponding to the n-cell of Δ_n). If Y is a simplicial set, and f: $Y \to \mathrm{KD}(E)$ is a simplicial map, then the knowledge of $f(s)_{\mathrm{top}}$ for all simplices s in Y suffices to reconstruct the simplicial map.

Now take an element $\mathscr{b}$ in $\mathscr{B}(C)_n$ (that is, in the group of 0-dimensional cycles in $\hat{W} \,\&\, (C \otimes_{\mathbb{Z}} \mathrm{Cl}(\Delta_n))^{-*}$, c.f. 1.11 and 1.7), and put

$$(\mathrm{Lin}(\mathscr{b}))_{\mathrm{top}} := (ev)^{\to}(\mathscr{b} \times \rho(n)) \in (\hat{W} \,\&\, C^{-*})_n,$$

using the exterior product of 0.11.

Checking that Lin induces an isomorphism of homotopy groups is easy using 1.1.

2. *Algebraic bordism theories*

The aim here is to construct, for each chain complex B in $\mathscr{C}_A$ and chain bundle $\mathscr{b}$ on B, associated algebraic bordism spectra $\mathscr{L}^{\cdot}(B, \mathscr{b})$ and $\mathbb{L}^0(B, \mathscr{b})$. ($\mathscr{L}^{\cdot}(B, \mathscr{b})$ is a more sophisticated version of $\mathbb{L}^0(B, \mathscr{b})$, with better algebraic properties.) Inspiration comes from the mock-bundle philosophy of [**5**].

2.1. Definition [**13, 15**]. An 'n dimensional algebraic Poincaré complex (over A)' is a pair (C, φ) consisting of a positive chain complex C in $\mathscr{C}_A$ (that is, $C_r = 0$ for $r < 0$) and an n-dimensional cycle $\varphi \in W \,\&\, C$ so that

$$\varphi_0\colon \Sigma^n(C^{-*}) \to C \quad \text{(cf. 0.10)}$$

is a chain homotopy equivalence.

(In [**15**], such a (C, φ) is called a 'symmetric algebraic Poincaré complex', as distinct from a 'quadratic algebraic Poincaré complex'. The point of view taken here is that a 'quadratic algebraic Poincaré complex' is a 'symmetric algebraic Poincaré complex' with additional structure; more will be said in 2.17.)

2.2. DEFINITION. An n-dimensional (cofibrant) algebraic Poincaré pair

$$(f: C \to D, (\psi, \varphi))$$

consists of two positive chain complexes C, D in $\mathscr{C}_A$, a cofibration $f: C \to D$, an n-chain $\psi \in W \,\&\, D$, and an $(n-1)$-cycle $\varphi \in W \,\&\, C$ so that

(i) $d(\psi) = f^{\to}(\varphi)$ in $(W\& D)_{n-1}$,

(ii) the chain map $\mathrm{pr}\cdot\psi_0$: $\Sigma^n(D^{-*}) \to D/\mathrm{im}(f)$ (explained below) is a homotopy equivalence.

(Here $\mathrm{pr}\cdot\psi_0$ is the composite of the projection chain map pr: $D \to D/\mathrm{im}(f)$ with the homomorphism of graded abelian groups ψ_0: $\Sigma^n(D^{-*}) \to D$; cf. 0.10, Case 2. Although ψ_0 is not a chain map in general, $\mathrm{pr}\cdot\psi_0$ is.)

The condition that $f: C \to D$ be a cofibration is not essential; if it is not satisfied, $\mathrm{pr}\cdot\psi_0$ has to be replaced by a chain map going from $\Sigma^n(D^{-*})$ to Cone(f). See **[15]**.

The 'boundary' (C, φ) of the algebraic Poincaré pair in 2.2 is an $(n-1)$-dimensional algebraic Poincaré complex.

2.3. DEFINITION (of a higher algebraic bordism). Let $2^{\{0,1,\dots,q\}}$ be the category of subsets of $\{0, 1, \dots, q\}$, with inclusion maps as morphisms (so there is at most one morphism between any two objects; if $S' \subset S$, denote this morphism by $j_{S',S}$).

For $S \subset \{0, 1, \dots, q\}$ and $0 \leqslant i < |S|$, let d_iS stand for the 'ith face' of S (d_0S is obtained from S by deleting the least element, d_1S by deleting the next, etc.).

A 'higher bordism of algebraic Poincaré complexes, of dimension n and order q' consists of a covariant functor

$$\mathrm{Fun}: 2^{\{0,1,\dots,q\}} \to \mathscr{C}_A$$

and a function Φ which for each subset $S \subset \{0, 1, \dots, q\}$ picks an $(n-q+|S|-1)$-chain $\Phi(S) \in W \,\&\, \mathrm{Fun}(S)$; here Fun and Φ are subject to certain conditions. They are as follows.

(i) Each Fun(S) is a positive chain complex; $\mathrm{Fun}(\varnothing) = 0$.

(ii) For any ideal in $2^{\{0,1,\dots,q\}}$ (i.e. a collection $\mathscr{J}$ of subsets of $\{0, 1, \dots, q\}$ such that $S \in \mathscr{J}$ and $S' \subset S$ implies $S' \in \mathscr{J}$), the canonical map

$$\mathrm{Fun}(\{0, 1, \dots, q\})^{-*} \to \operatorname*{inv\,lim}_{S\in\mathscr{J}} \{\mathrm{Fun}(S)^{-*}\} \subset \prod_{S\in\mathscr{J}} \mathrm{Fun}(S)^{-*};$$

$$f \mapsto (f\cdot j_{S,\{0,1,\dots,q\}})_{S\in\mathscr{J}}$$

is surjective. (This condition generalizes the cofibration condition in 2.2; it implies, by an induction proof, that $\mathrm{inv\,lim}_{S\in\mathscr{J}}\{\mathrm{Fun}(S)^{-*}\}$ is in $\mathscr{C}_A$ for any ideal $\mathscr{J}$.)

(iii) For $S \subset \{0, 1, \dots, q\}$,

$$\sum_{i=0}^{|S|-1} (-)^i \cdot j^{\to}_{d_iS,S}(\Phi(d_iS)) = (-)^{n-k}\cdot d(\Phi(S))$$

in $W \,\&\, \mathrm{Fun}(S)$ (d is the differential in $W \,\&\, \mathrm{Fun}(S)$).

(iv) For each $S \subset \{0, 1, \dots, q\}$, the chain complexes

$$C := \sum_{i=0}^{|S|-1} j^{\to}_{d_iS,S}(\mathrm{Fun}(d_iS)), \quad D := \mathrm{Fun}(S),$$

the inclusion map $f: C \to D$, the chain $\psi := (-)^{n-k}\Phi(S)$ in $W \,\&\, D$, and the cycle $\varphi := \sum_{i=0}^{|S|-1}(-)^i\Phi(d_iS)$ in $W \,\&\, C$ (in loose notation) together constitute an algebraic

Poincaré pair of dimension $(n-q+|S|-1)$. (This means that the non-degeneracy condition 2.2(ii) holds—everything else is redundant.)

2.4. REMARKS. If (Fun, Φ) is a higher algebraic Poincaré bordism as above, of dimension n and order q, and if S is a subset of $\{0, 1, \ldots, q\}$ with complement S', then the restriction of Fun to 2^S and the corresponding restriction of Φ form a higher algebraic Poincaré bordism of dimension $n-|S'|$ and order $q-|S'|$, written $(\mathrm{Fun}/2^S, \Phi/2^S)$.

An algebraic Poincaré bordism (Fun, Φ) of order 1 such that $\mathrm{Fun}(\{0\}) = 0$ or $\mathrm{Fun}(\{1\}) = 0$ will also be called an algebraic Poincaré pair; this agrees with 2.2 up to sign.

2.5. CONSTRUCTION. Assume that (Fun, Φ) is still as above, and forget all the higher homotopies contained in Φ, retaining only $\Phi(S)_0$ (cf. 0.10, Case 2) for all $S \subset \{0, 1, \ldots, q\}$. Condition 2.3(iii) gives a system of chain maps (one for each $S \subset \{0, 1, \ldots, q\}$)

$$\int_{2^S} \Phi_0\colon \Sigma^{n-q}(\mathrm{Fun}(S)^{-*} \otimes_{\mathbb{Z}} \mathrm{Cl}(\Delta(S))) \to \mathrm{Fun}(S);$$
$$f \otimes [S'] \mapsto j_{S',S}^{\rightarrow} \cdot \Phi(S')_0(f \cdot j_{S',S}^{\rightarrow}).$$

(Explanation: $\Delta(S)$ is the $(|S|-1)$-dimensional simplex spanned by S and $\mathrm{Cl}(\Delta(S))$ is its cellular chain complex, with one $(|S'|-1)$-dimensional generator $[S']$ for each non-empty subset S' of S. The inclusion $j_{S',S}\colon S' \to S$ induces $j_{S',S}^{\rightarrow}\colon \mathrm{Fun}(S') \to \mathrm{Fun}(S)$; starting with $f \in \Sigma^{n-q}(\mathrm{Fun}(S)^{-*})$ and suspending liberally, we find that $f \cdot j_{S',S}^{\rightarrow}$ is in $\Sigma^{n-q+|S'|-1}(\mathrm{Fun}(S')^{-*})$, which is the domain of $\Phi(S')_0$, etc.)

For $S' \subset S \subset \{0, 1, \ldots, q\}$, the chain maps $\int_{2^{S'}} \Phi_0$ and $\int_{2^S} \Phi_0$ are related by a certain commutative diagram.

We will now repeat 2.1 and 2.3, adding '$\mathscr{E}$-structures'; here $\mathscr{E}$ is a chain bundle on a chain complex B in $\mathscr{C}_A$, to be kept fixed until further notice.

2.6. DEFINITION. A $\mathscr{E}$-structure on an n-dimensional algebraic Poincaré complex (C, φ) (over A) consists of a chain map

$$g\colon C \to B \quad \text{(the 'classifying map')}$$

and a homology

$$z \in (\hat{W} \,\&\, C)_{n+1} \quad \text{(the 'clutching homology')}$$

from $\mathfrak{S}^n \cdot (\varphi_0 g^*)^{\rightarrow}(b) \in (\hat{W} \,\&\, C)_n$ to $J(\varphi) \in (\hat{W} \,\&\, C)_n$.

(Explanation: $\varphi_0 g^*$ is a chain map from B^{-*} to $\Sigma^{-n}C$, inducing

$$(\varphi_0 g^*)^{\rightarrow}\colon \hat{W} \,\&\, B^{-*} \to \hat{W} \,\&(\Sigma^{-n}C),$$

and $\mathfrak{S}^n$ is the n-fold iteration of the explicit suspension isomorphism of 1.2(b). See also 0.13.)

The notion of 'normal chain bundle' should help clarify 2.6. Suppose that (C, φ) is an n-dimensional algebraic Poincaré complex over A, and $\mathscr{n} \in \hat{W} \,\&\, C^{-*}$ a chain bundle on C equipped with the additional structure of a homology $z \in (\hat{W} \,\&\, C)_{n+1}$ from $\mathfrak{S}^n \cdot \varphi_0^{\rightarrow}(\mathscr{n}) \in (\hat{W} \,\&\, C)_n$ to $J(\varphi) \in (\hat{W} \,\&\, C)_n$. Then $\mathscr{n}$, or rather the pair $(\mathscr{n}, z)$, is called the 'normal chain bundle of (C, φ)'.

It is easy to see that the normal chain bundle $\mathscr{n}$ of (C, φ) is well defined *up to an infinity of higher homologies in* $\hat{W} \& C^{-*}$ (see 0.15). That is, it is something better than a mere class in $H_0(\hat{W} \& C^{-*}) = \hat{Q}^0(C^{-*})$. In this respect it resembles the normal bundle of a geometric manifold or the Spivak normal fibration of a geometric Poincaré complex, which are also well defined up to an infinity of higher concordances.

A $\mathscr{b}$-structure on (C, φ) consists, then, of a classifying chain map $g\colon C \to B$ and an identification of the 'induced' chain bundle $g^{\leftarrow}(\mathscr{b})$ on C with the normal chain bundle $\mathscr{n}$.

2.7. DEFINITION (in outline only). A $\mathscr{b}$-structure (g, z) on a higher bordism (Fun, Φ) of algebraic Poincaré complexes (of dimension n and order q) consists of

(i) a 'classifying chain map' $g\colon \mathrm{Fun}(\{0, 1, \ldots, q\}) \to B$,
(ii) an appropriate collection (explained below) $z = \{z(S) \mid S \subset \{0, 1, \ldots, q\}\}$ of clutching homologies.

2.8. EXPLANATION of 2.7 (ii). In 2.9 below, a sequence

$$\{\rho(m) \in (\hat{W} \& \mathrm{Cl}(\Delta_m))_m \mid m = 0, 1, 2, \ldots\}$$

satisfying conditions (i), (ii), and (iii) in the proof of 1.12 will be fixed once and for all. If S is any finite ordered set and $m = |S|-1$, then the unique order-preserving bijection $S \to \{0, 1, \ldots, m\}$ gives an identification

$$\hat{W} \& \mathrm{Cl}(\Delta(S)) \cong \hat{W} \& \mathrm{Cl}(\Delta_m);$$

write $\rho(S) \in \hat{W} \& \mathrm{Cl}(\Delta(S))$ for the m-chain corresponding to $\rho(m)$.

The main point is that the clutching homologies in 2.7 (ii) form a set

$$\{z(S) \in (\hat{W} \& \mathrm{Fun}(S))_{n-q+|S|} \mid S \subset \{0, 1, \ldots, q\}\}$$

so that, for every $S \subset \{0, 1, \ldots, q\}$, the equation

$$d(z(S)) + (-)^{n-q} \cdot \sum_{i=0}^{|S|-1} (-)^i \cdot \vec{j}_{d_i S, S}(z(d_i S)) = J(\Phi(S)) - \int_{2^S} \Phi_0(\mathfrak{S}^{n-q} \cdot ((g^*)^{\rightarrow}(\mathscr{b}) \times \rho(S)))$$

holds in $(\hat{W} \& \mathrm{Fun}(S))_{n-q+|S|-1}$.

(The integral sign comes from 2.5, $\times$ is the exterior product of 0.11, d is the differential in $\hat{W} \& \mathrm{Fun}(S)$, and $\mathfrak{S}$ is the suspension of 1.2 (b); I have written $g\colon \mathrm{Fun}(S) \to B$ when I should have written

$$g \cdot j_{S, \{0, 1, \ldots, q\}}\colon \mathrm{Fun}(S) \to \mathrm{Fun}(\{0, 1, \ldots, q\}) \to B,$$

and so the map $(g^*)^{\rightarrow}$ goes from $\hat{W} \& B^{-*}$ to $\hat{W} \& (\mathrm{Fun}(S))^{-*}$.)

2.9. CONVENTIONS. We shall first fix a 1-chain ω in $\hat{W} \& \mathrm{Cl}(\Delta_1) = \hat{W} \& I$ like the one used in the proof of 1.1 (i) and in 1.2 (b); and then a sequence $\{\rho(m) \in (\hat{W} \& \mathrm{Cl}(\Delta_m))_m \mid m = 0, 1, 2, \ldots\}$ as in the proof of 1.12.

(i) Let x, y_0, y_1 be the standard generators of the chain complex $I = \mathrm{Cl}(\Delta_1)$; then $|x| = 1$, $|y_0| = |y_1| = 0$, and $d(x) = y_1 - y_0$. Using notation as in Case 1 of 0.10, put

$$\omega_1 = x \otimes x, \quad \omega_0 = x \otimes y_0 + y_1 \otimes x, \quad \omega_t = 0 \text{ for } t \neq 0, 1.$$

Then $\omega = \{\omega_s \mid s \in \mathbb{Z}\} \in \hat{W}\&\mathrm{Cl}(\Delta_1)$ gives the explicit suspension formula in 1.2(b). Note also that ω belongs to the subcomplex $W\&\mathrm{Cl}(\Delta_1) \subset \hat{W}\&\mathrm{Cl}(\Delta_1)$.

(ii) The map

$$\Delta_m \times \Delta_1 \to \Delta_{m+1};\ ((t_0, t_1, \ldots, t_m), (u_0, u_1)) \mapsto (u_0, u_1 t_0, u_1 t_1, \ldots, u_1 t_m)$$

(in barycentric coordinates; so $t_0 + t_1 + \ldots + t_m = 1 = u_0 + u_1$) induces a chain map of cellular chain complexes

$$p_m\colon \mathrm{Cl}(\Delta_m) \otimes_{\mathbb{Z}} \mathrm{Cl}(\Delta_1) \cong \mathrm{Cl}(\Delta_m \times \Delta_1) \to \mathrm{Cl}(\Delta_{m+1}).$$

Define inductively

$$\rho(0) := v \in \hat{W}\&\mathrm{Cl}(\Delta_0),$$
$$\rho(m+1) := p_m^{\to}(\rho(m) \times \omega) \in \hat{W}\&\mathrm{Cl}(\Delta_{m+1}),$$

where $\times$ denotes the exterior product of 0.11.

2.10. Remark. A $\mathscr{E}$-structure (g, z) on a higher algebraic Poincaré bordism (Fun, Φ) of order q induces a $\mathscr{E}$-structure $(g/2^S, z/2^S)$ on each of the face bordisms $(\mathrm{Fun}/2^S, \Phi/2^S)$ defined in 2.4, with $S \subset \{0, 1, \ldots, q\}$.

In particular, it is clear how to define the notion of a bordism (of order 1) between two algebraic Poincaré complexes of the same dimension, with $\mathscr{E}$-structure as in 2.6. Granting that 'bordant' is an equivalence relation, we can define the corresponding bordism groups. We shall now construct a spectrum whose homotopy groups they are.

The construction contains very few surprises. Recall from **[17]** that a Δ-set (or incomplete simplicial set) is a contravariant functor from the category Δ (whose objects are the standard q-simplices Δ_q for $q = 0, 1, \ldots$, and whose morphisms are the linear maps defined by order-preserving *injective* maps of the vertex sets) to the category of sets.

It is shown in **[17]** that Δ-sets are well behaved if they satisfy the analogue of the Kan condition for simplicial sets, in which case they are called Kan Δ-sets.

Certain set-theoretic precautions are understood in the next definition, and in several others of a similar type in §3. Without such precautions we would have to work with Δ-classes rather than Δ-sets in many places.

2.11. Definition. For $p \in \mathbb{Z}$, let $\mathbb{L}^p(B, \mathscr{E})$ be the Δ-set whose q-simplices are the higher bordisms of algebraic Poincaré complexes (over A) of dimension $q+p$ and order q, equipped with a $\mathscr{E}$-structure; the face operators are as outlined in 2.10.

2.12. Proposition. *The Δ-set $\mathbb{L}^p(B, \mathscr{E})$ satisfies the Kan condition.*

Proof. This is straightforward and left to the reader.

2.13. Construction. *There are natural homotopy equivalences*

$$\varepsilon_p\colon \mathbb{L}^{-p}(B, \mathscr{E}) \to \Lambda\mathbb{L}^{-(p+1)}(B, \mathscr{E}) \quad \textit{for } p \in \mathbb{Z}.$$

(Here Λ denotes the loop space. See 2.15 for the meaning of 'natural'.)

Proof. First, the loop space $\Lambda\mathbb{L}^{-(p+1)}(B, \mathscr{E})$ must be defined. The set $\mathbb{L}^{-(p+1)}(B, \mathscr{E})$ has a canonical 'base point' (that is, a Δ-map from the Δ-set given by the constant one-point functor to $\mathbb{L}^{-(p+1)}(B,\mathscr{E})$); by decree, a q-simplex of $\Lambda\mathbb{L}^{-(p+1)}(B,\mathscr{E})$ is the same as a $(q+1)$-simplex of $\mathbb{L}^{-(p+1)}(B,\mathscr{E})$ whose 0th vertex and 0th face are at the base point. (The 0th face is opposite the 0th vertex.) So

$$(\Lambda\mathbb{L}^{-(p+1)}(B,\mathscr{E}))_q \subset (\mathbb{L}^{-(p+1)}(B,\mathscr{E}))_{q+1},$$

and for $0 \leqslant i \leqslant q$ the face operators d_i are chosen so as to make the diagram

$$\begin{array}{ccc} (\Lambda\mathbb{L}^{-(p+1)}(B,\mathscr{E}))_q & \xrightarrow{d_i} & (\Lambda\mathbb{L}^{-(p+1)}(B,\mathscr{E}))_{q-1} \\ \cap & & \cap \\ (\mathbb{L}^{-(p+1)}(B,\mathscr{E}))_{q+1} & \xrightarrow[d_{i+1}]{} & (\mathbb{L}^{-(p+1)}(B,\mathscr{E}))_q \end{array}$$

commute.

Now let x be a q-simplex in $\mathbb{L}^{-p}(B,\mathscr{E})$, that is, an algebraic Poincaré bordism (Fun,Φ) of dimension $q-p$ and order q, with $\mathscr{E}$-structure (g,z). Then $\varepsilon_p(x)$ has to be an algebraic Poincaré bordism (Fun',Φ') of dimension $q-p$ and order $q+1$, with $\mathscr{E}$-structure (g',z').

Let $e\colon \{0,1,\dots,q\} \hookrightarrow \{0,1,\dots,q+1\}$ send s to $s+1$. For $S \subset \{0,1,\dots,q+1\}$, define

$$\mathrm{Fun}'(S) := \begin{cases} \mathrm{Fun}(e^{-1}(S)) & \text{if } 0 \in S, \\ 0 & \text{otherwise,} \end{cases}$$

$$\Phi'(S) := \Phi(e^{-1}(S)) \quad \text{if } 0 \in S,$$

$$g' := g,$$

and

$$z'(S) := z(e^{-1}(S)) \quad \text{if } 0 \in S.$$

Since all Δ-sets in sight satisfy the Kan condition, their homotopy groups can be defined via the 'pillow construction', which shows that ε_p is a homotopy equivalence.

2.14. Corollary. *Definition* 2.11 *and Construction* 2.13 *define a spectrum*

$$\underline{\mathbb{L}}^0(B,\mathscr{E}) = \{\mathbb{L}^{-p}(B,\mathscr{E}), \varepsilon_p \mid p \in \mathbb{Z}\}.$$

Again, the pillow philosophy shows that $\pi_n(\underline{\mathbb{L}}^0(B,\mathscr{E}))$ is the bordism group of n-dimensional algebraic Poincaré complexes with $\mathscr{E}$-structure (2.12 implies that bordism is an equivalence relation).

2.15. Proposition. *The association $B,\mathscr{E} \mapsto \underline{\mathbb{L}}^0(B,\mathscr{E})$ is functorial. If $\mathscr{E}$ is a chain bundle on B (in $\mathscr{C}_A$), $\mathscr{E}'$ is a chain bundle on B' (in $\mathscr{C}_{A'}$), $A \to A'$ is a homomorphism of rings with involution, and $f\colon A' \otimes_A B \to B'$ is a chain map covered by a chain bundle map (cf. 1.8) from $A' \otimes_A \mathscr{E}$ to $\mathscr{E}'$, then there is an induced map of spectra $\underline{\mathbb{L}}^0(B,\mathscr{E}) \to \underline{\mathbb{L}}^0(B',\mathscr{E}')$.*

Idea of proof. Take a q-simplex in $\mathbb{L}^p(B,\mathscr{E})$, say (Fun,Φ), (g,z), and define a q-simplex (Fun',Φ'), (g',z') in $\mathbb{L}^p(B',\mathscr{E}')$ by letting

$$\mathrm{Fun}' := A' \otimes_A \mathrm{Fun}, \quad \Phi' := A' \otimes_A \Phi, \quad g' := f\cdot(A' \otimes_A g),$$

and for $S \subset \{0,1,\dots,q\}$,

$$z'(S) := A' \otimes_A z(S) + \int_{2^S} \Phi_0'(\mathfrak{S}^p((A' \otimes_A g)^{\frown}(y) \times \rho(S))),$$

y being the chain bundle map (cf. 1.8). (As in 2.8, g is an abbreviation for $g \cdot j\vec{s}_{,\{0,1,\dots,q\}\cdot}$.)

A glance at the homotopy groups shows

2.16. PROPOSITION. *If $A = A'$ in 2.15, and if $f\colon B \to B'$ is a chain homotopy equivalence (covered by a chain bundle map as before), then $\underline{\mathbb{L}}^0(B, \ell) \to \underline{\mathbb{L}}^0(B', \ell')$ is a homotopy equivalence of spectra.*

We conclude with two 'extreme' examples, for which the ring with involution A is fixed again.

2.17. EXAMPLE. Take ℓ to be the trivial bundle on the trivial chain complex $B = 0_A$ in $\mathscr{C}_A$.

A ℓ-structure on an algebraic Poincaré complex (C, φ) is then a homology in $\hat{W} \& C$ from 0 to $J(\varphi)$. It is clear that this is the same as a 'quadratic Poincaré complex structure' in the sense of [**15**]. So $\pi_n(\underline{\mathbb{L}}^0(0_A, 0))$ is isomorphic to the (projective version of the) Wall group $L_n(A)$ for $n \geqslant 0$, and spectra homotopy equivalent to $\underline{\mathbb{L}}^0(0_A, 0)$ have of course been constructed before (by Quinn and Ranicki; see also [**21**, Chapter 17A]).

It is tempting to regard the Wall groups as the bordism groups of 'framed algebraic Poincaré complexes' (see the paragraphs between 2.6 and 2.7), but this can lead to confusion: bear in mind that the trivial chain complex 0_A is the algebraic counterpart of an empty space, not of a contractible space or a $K(\pi, 1)$.

2.18. EXAMPLE. Take ℓ to be the 'universal chain bundle'. (This involves a certain amount of cheating. What I claim is that the functor $C \mapsto \hat{Q}^0(C^{-*})$, when restricted to the category of positive chain complexes in $\mathscr{C}_A$, is 'almost representable'. That is, there exist a positive chain complex B and a chain bundle ℓ on B so that the transformation of functors $H_0(\mathrm{Hom}_A(C, B)) \to \hat{Q}^0(C^{-*})$; $[f] \mapsto [f^{\frown}(\ell)]$ is an isomorphism, with C in $\mathscr{C}_A$. However, we must allow B to be a chain complex of possibly non-finitely generated projective A-modules, and possibly infinitely many of them non-zero; also the notion of chain bundle must be defined with some care. The appendix to this section is devoted to an explicit construction.)

Now a ℓ-structure on an algebraic Poincaré complex is as good as no structure at all; consequently $\pi_n(\underline{\mathbb{L}}^0(B, \ell))$ is isomorphic to the symmetric L-group $L^n(A)$ defined in [**13**] and [**15**].

2.19. REMARK. The construction of $\underline{\mathbb{L}}^0(B, \ell)$ may seem a little arbitrary, since it relies on a peculiar choice made in 2.9. Here is a more convincing alternative.

(i) Choose your own favourite sequence $\{\rho(m)\}$ satisfying the conditions in the proof of 1.12. This will be 'essentially unique' only, but there is no need to be more specific. Define a 'ℓ-structure' (on a higher bordism of algebraic Poincaré complexes) accordingly.

(ii) Construct a Δ-set $\mathbb{L}^0(B, \ell)$ as in 2.11, still using your own favourite sequence $\{\rho(m)\}$.

(iii) Prove that $\mathbb{L}^0(B, \ell)$ is an *infinite loop space*, using Segal's machine [**18**]. For this purpose, let

$$E(1) := \mathbb{L}^0(B, \ell);$$

more generally, for $n \geqslant 0$ let $E(n)$ be the Δ-set whose q-simplices are 'functors' which to each non-empty subset V of $\{1, 2, \ldots, n\}$ associate a q-simplex of $\mathbb{L}^0(B, \ell)$, say $({}^V\mathrm{Fun}, {}^V\Phi)$, $({}^Vg, {}^Vz)$; and to each inclusion $U \subset V$ associate a chain map

$$\lambda_{U,V}\colon {}^U\mathrm{Fun}(\{0, 1, \ldots, q\}) \to {}^V\mathrm{Fun}(\{0, 1, \ldots, q\}),$$

subject to certain very natural conditions. (The conditions are:

$${}^Vg \cdot \lambda_{U,V} = {}^Ug;$$

next, the $\lambda_{U,V}$ should give a direct sum decomposition

$$({}^V\mathrm{Fun}, {}^V\Phi) \cong ({}^U\mathrm{Fun}, {}^U\Phi) \oplus ({}^{V-U}\mathrm{Fun}, {}^{V-U}\Phi)$$

whenever $U \subset V \subset \{1, 2, \ldots, n\}$, and $U \neq \varnothing \neq V - U$; and finally, the projection

$${}^V\mathrm{Fun} \to {}^U\mathrm{Fun}$$

resulting from the previous condition in the case where $U \subset V$ and $U \neq \varnothing \neq V - U$, should send ${}^Vz$ to ${}^Uz$.)

Then the collection $\{E(n) \mid n \geqslant 0\}$ (with obvious structure maps) is a Γ-space in the sense of [**18**], and so $E(1) = \mathbb{L}^0(B, \ell)$ is an infinite loop space. Notice the similarity of the construction above with Segal's construction of the algebraic K-theory spectrum, also in [**18**].

To prove that the infinite loop space structure on $\mathbb{L}^0(B, \ell)$ just defined coincides with that given by 2.13, use

2.20. Lemma. *The map ε_p in 2.13 has a canonical refinement to a map of Γ-spaces.*

(Explanation: observe first that each $\mathbb{L}^p(B, \ell)$ yields a Γ-space, just like $\mathbb{L}^0(B, \ell)$. Secondly, if $\{F(n) \mid n \geqslant 0\}$ is any Γ-space, then so is $\{\Lambda F(n) \mid n \geqslant 0\}$; the structure maps for $\{\Lambda F(n)\}$ are obtained by applying the loop functor Λ to those for $\{F(n)\}$. Hence the lemma makes sense; the proof is easy, and it is also easy to see that it proves precisely what is needed.)

2.21. Notation. (i) The spectrum $\underline{\mathbb{L}}^0(B, \ell)$ and the infinite loop spaces $\mathbb{L}^p(B, \ell)$ have been defined in 2.14 and 2.11 respectively.

(ii) An n-dimensional 'unrestricted algebraic Poincaré complex' (C, φ) consists, by definition, of a chain complex C in $\mathscr{C}_A$ and an n-cycle $\varphi \in W\&C$ so that $\varphi_0\colon \Sigma^n(C^{-*}) \to C$ is a chain homotopy equivalence (C is not required to be positive). The notion is interesting even when $n < 0$. The whole of this section (with the exception of 2.19, which is unsuitable) can be rewritten with 'algebraic Poincaré complexes' replaced by 'unrestricted algebraic Poincaré complexes', etc. The outcome is

for every $p \in \mathbb{Z}$, a Δ-set $\mathscr{L}^p(B, \ell)$ (the 'unrestricted analogue' of $\mathbb{L}^p(B, \ell)$)
and hence a usually non-connective spectrum $\underline{\mathscr{L}}^{\cdot}(B, \ell)$ (the 'unrestricted analogue' of $\underline{\mathbb{L}}^0(B, \ell)$).

(iii) Write $\underline{\mathscr{L}}_{\cdot}(A) = \underline{\mathscr{L}}^{\cdot}(0_A, 0)$, and let

$$\text{release}\colon \underline{\mathscr{L}}_{\cdot}(A) \to \underline{\mathscr{L}}^{\cdot}(B, \ell)$$

be the map of spectra induced by the chain bundle map from $0_A, 0$ to B, ℓ (cf. the introduction). Note that

$$\pi_n(\underline{\mathscr{L}}_{\cdot}(A)) = L_n(A) \quad \text{for } n \in \mathbb{Z},$$

and $L_n(A)$ depends only on the residue of n mod 4. The cofibre of

$$\text{release: } \underline{\mathscr{L}}_{\cdot}(A) \to \underline{\mathscr{L}}^{\cdot}(B, \mathscr{b})$$

is denoted by $\underline{\hat{\mathscr{L}}}^{\cdot}(B, \mathscr{b})$.

(iv) Finally, write

$$L^n(B, \mathscr{b}) := \pi_n(\underline{\mathscr{L}}^{\cdot}(B, \mathscr{b})), \quad \hat{L}^n(B, \mathscr{b}) := \pi_n(\underline{\hat{\mathscr{L}}}^{\cdot}(B, \mathscr{b})).$$

(No special notation is introduced for $\pi_n(\mathbb{L}^0(B, \mathscr{b}))$; but we will see later that, if B is a positive chain complex, the forgetful homomorphisms

$$\pi_n(\mathbb{L}^0(B, \mathscr{b})) \to \pi_n(\underline{\mathscr{L}}^{\cdot}(B, \mathscr{b})) = L^n(B, \mathscr{b})$$

are isomorphisms for $n \geqslant 0$, whereas $\pi_n(\mathbb{L}^0(B, \mathscr{b})) = 0$ for $n < 0$.)

2.22. Remark. Change of K-theory. The whole theory so far has been written in terms of f.g. projective modules over A; there are versions which use stably free A-modules instead, or stably free and based A-modules. As in ordinary L-theory, there is a long exact sequence relating the projective and stably free versions of $L^n(B, \mathscr{b})$, involving the groups $\hat{H}^n(Z_2; K_0(A))$; and another long exact sequence relating the stably free and stably-free-and-based versions, involving the groups $\hat{H}^n(Z_2; K_1(A))$ (or $\hat{H}^n(Z_2; \bar{K}_1(A))$ etc.). Cf. [15].

The relative groups $\hat{L}^n(B, \mathscr{b})$ are not affected at all by a change of K-theory.

2.A. *Appendix: The universal chain bundle*

Let B be any chain complex of projective left A-modules, not necessarily in $\mathscr{C}_A$. Then the sequence of functors

$$C \mapsto H_n(\mathrm{Hom}_A(C, B))$$

(where C is a chain complex in $\mathscr{C}_A$ and $n \in \mathbb{Z}$) constitutes a cohomology theory on $\mathscr{C}_A$, that is, the analogues of Conditions (i), (ii), and (iii) of 1.1 are satisfied.

Conversely, any cohomology theory on $\mathscr{C}_A$ is isomorphic to one obtained in this way (from a suitable chain complex B). This is the analogue in the chain complex world of E. H. Brown's representation theorem, which normally lives in the world of CW-spaces; see [7]. We shall now prove it in detail for the special case of the cohomology theory

$$C \mapsto \{\hat{Q}^n(C^{-*}) \mid n \in \mathbb{Z}\}.$$

If B is an arbitrary chain complex of projective A-modules (not necessarily in $\mathscr{C}_A$), the abelian group chain complex $\hat{W} \& B^{-*}$ can be defined as in 1.5; so an n-chain in $\hat{W} \& B^{-*}$ is a collection of sesquilinear forms

$$\{\varphi_{p,q}\colon B_p \times B_q \to A \mid p, q \in \mathbb{Z}\},$$

and the differential (from $(\hat{W} \& B^{-*})_{n+1}$ to $(\hat{W} \& B^{-*})_n$) is as in 1.5.

Further, a chain complex $V(B)$ can be defined word for word as in 1.4; the chain map

$$\mathrm{Eco}\colon V(B) \to \hat{W} \& B^{-*}$$

of 1.6 is still there, although perhaps not in general a homology equivalence. At any rate, a 0-cycle in $V(B)$ can also be regarded as a chain bundle on B.

2.A.1. DEFINITION. Given a chain bundle $\mathscr{E}$ on B, or just a class $[\mathscr{E}]$ in $\hat{Q}^0(B^{-*})$, there are homomorphisms (called the Wu classes of $\mathscr{E}$, see [15])

$$v_r(\mathscr{E})\colon H_r(B) \to \hat{H}^r(Z_2; A);\ [f] \mapsto [f^{\leftarrow}(\mathscr{E})].$$

(Explanation: $\hat{H}^r(Z_2; A)$ is the rth Tate cohomology group of Z_2 with coefficients in the Z_2-module A; the involution makes A into a Z_2-module; $H_r(B)$ has been identified with $H_0(\mathrm{Hom}_A((A,r),B))$, so that f is a chain map from (A,r) to B; see 0.6 for notation. Also $\hat{H}^r(Z_2; A)$ has been identified with $\hat{Q}^0((A,r)^{-*})$ so that $f^{\leftarrow}(\mathscr{E})$ is a chain bundle on (A,r).)

Now $\hat{H}^r(Z_2; A)$ is a left A-module, with A acting by

$$a\cdot[x] \mapsto [ax\bar{a}]$$

(for $a \in A$ and $x \in \ker[\mathrm{id}-(-)^r\cdot\text{involution}\colon A \to A]$, and

$$[x] \in \frac{\ker[\mathrm{id}-(-)^r\cdot\text{involution}\colon A \to A]}{\mathrm{im}[\mathrm{id}+(-)^r\cdot\text{involution}\colon A \to A]} = \hat{H}^r(Z_2; A)).$$

The Wu classes $v_r(\mathscr{E})$ are A-module homomorphisms.

2.A.2. EXAMPLE. Suppose that $\lambda = \{\lambda_r\colon B_r \times B_r \to A \mid r \in \mathbb{Z}\}$ is a 0-cycle in $V(B)$ (see 1.4), regarded as a chain bundle on B. The Wu classes are then given by

$$v_r(\lambda)\colon H_r(B) \to \hat{H}^r(Z_2; A);\ [y] \mapsto [\lambda_r(y, y)].$$

(Assume that $y \in \ker[d\colon B_r \to B_{r-1}]$; then $\lambda_r(y, y)$ represents an element in $\hat{H}^r(Z_2; A)$, since λ is a cycle.)

2.A.3. CONSTRUCTION. By induction on skeletons, we will construct a positive chain complex B (of free A-modules) and a 0-cycle $\{\lambda_r\colon B_r \times B_r \to A \mid r \in \mathbb{Z}\}$ in $V(B)$ such that, for all $r \geqslant 0$, the Wu class

$$v_r(\lambda)\colon H_r(B) \to \hat{H}^r(Z_2; A)$$

is an isomorphism. This property easily implies that λ, when regarded as a chain bundle on B, has the universal property required in 2.18.

Suppose that the n-skeleton

$$B_{\leqslant n} := B_n \to B_{n-1} \to B_{n-2} \to \dots \to B_1 \to B_0$$

has already been constructed, and that the sesquilinear forms

$$\lambda_r\colon B_r \times B_r \to A$$

have been defined for $r \leqslant n$ in such a way that

$$\lambda_{\leqslant n} := \{\lambda_r \mid 0 \leqslant r \leqslant n\}$$

is a 0-cycle in $V(B_{\leqslant n})$. Suppose further that the Wu class

$$v_r(\lambda_{\leqslant n})\colon H_r(B_{\leqslant n}) \to \hat{H}^r(Z_2; A)$$

is an isomorphism for $r < n$, and a surjection for $r = n$.

Let K be the kernel of $v_n(\lambda_{\leqslant n})$. Note that K is an A-submodule of B_n,

$$K \subset H_n(B_{\leqslant n}) \subset B_n.$$

Choose a free A-module B'_{n+1} and a map

$$d'\colon B'_{n+1} \to B_n$$

such that $\operatorname{im}(d') = K$; choose a sesquilinear form λ'_{n+1} on B'_{n+1} so that

$$T\lambda'_{n+1} - (-)^{n+1}\lambda'_{n+1} = d'^{\leftarrow}(\lambda_n)$$

(this is possible by definition of K).

Further, choose a free A-module B''_{n+1} and a sesquilinear form λ''_{n+1} on B''_{n+1} so that

(i) $T\lambda''_{n+1} - (-)^{n+1}\lambda''_{n+1} = 0$,

(ii) the A-module map

$$B''_{n+1} \to \hat{H}^{n+1}(Z_2; A);\ x \mapsto [\lambda''_{n+1}(x, x)]$$

is surjective.

Now let

$$B_{n+1} := B'_{n+1} \oplus B''_{n+1},$$

$$d = d' \oplus 0\colon B'_{n+1} \oplus B''_{n+1} \to B_n,$$

and

$$\lambda_{n+1} := \lambda'_{n+1} \oplus \lambda''_{n+1}.$$

The induction step (from n to $n+1$) is complete.

2.A.4. Variation on 2.A.3. It is also possible to construct a chain complex B^∞ (of free A-modules, but usually not in $\mathscr{C}_A$) and a chain bundle $\mathscr{E}^\infty$ on B^∞ such that the Wu classes

$$v_r(\mathscr{E}^\infty)\colon H_r(B) \to \hat{H}^r(Z_2; A)$$

are isomorphisms for all $r \in \mathbb{Z}$. Then, for any chain complex C in $\mathscr{C}_A$ (not necessarily positive), the homomorphism

$$H_0(\operatorname{Hom}_A(C, B^\infty)) \to \hat{Q}^0(C^{-*});\ [f] \mapsto [f^{\leftarrow}(\mathscr{E}^\infty)]$$

is an isomorphism.

Therefore a $\mathscr{E}^\infty$-structure on an unrestricted algebraic Poincaré complex (cf. 2.21 (ii)) over A is as good as no structure at all. It follows that $L^n(B^\infty, \mathscr{E}^\infty) = \pi_n(\mathscr{L}^{\cdot}(B^\infty, \mathscr{E}^\infty))$ (cf. 2.21 (iii), (iv)) is the bordism group of n-dimensional unrestricted algebraic Poincaré complexes over A, for $n \in \mathbb{Z}$.

The groups $L^n(B^\infty, \mathscr{E}^\infty)$ are periodic in n, with period 4, almost by definition.

Note that $L^n(B^\infty, \mathscr{E}^\infty)$ can be identified with the direct limit $\varinjlim_k L^{n+4k}(A)$ of the symmetric L-groups under the double skew-suspension maps

$$\bar{S}^2\colon L^{n+4k}(A) \to L^{n+4(k+1)}(A);$$

see [**15**].

3. *Passage from geometry to algebra*

In this section we show that a spherical fibration determines a chain bundle, and that a geometric Poincaré complex determines an algebraic Poincaré complex whose normal chain bundle agrees with the chain bundle determined by the Spivak normal fibration.

3.1. CONVENTIONS (relating to simplicial sets). Very little distinction will be made between a simplicial set X and its geometric realization (which is a CW-space). The cellular chain complex of X is $C(X)$; it is freely generated by the non-degenerate simplices of X.

If π is a (discrete) group, a principal π-bundle on X consists of a simplicial set $\tilde{X}$ with a simplicial π-action which freely permutes the simplices of $\tilde{X}$, and an identification of simplicial sets $\tilde{X}/\pi \cong X$.

Suppose that X and Y are simplicial sets. The acyclic model theorem [9] yields a chain homotopy equivalence

$$C(X \times Y) \to C(X) \otimes_{\mathbb{Z}} C(Y)$$

natural in both variables with respect to simplicial maps. (Note: we are talking about cellular chain complexes.) To be more thorough, the acyclic model theorem yields an 'Eilenberg–Zilber' chain map

$$\mathrm{EZ} = \mathrm{EZ}(X, Y)\colon C(X \times Y) \to \mathrm{Hom}_{\mathbb{Z}}(W, C(X) \otimes_{\mathbb{Z}} C(Y))$$

which

(i) is natural in both variables X and Y,
(ii) agrees with the canonical and obvious chain isomorphism if $X = Y =$ point,
(iii) is Z_2-equivariant.

(The last condition means that the diagram

$$\begin{array}{ccc} C(X \times Y) & \xrightarrow{\mathrm{EZ}} & \mathrm{Hom}_{\mathbb{Z}}(W, C(X) \otimes_{\mathbb{Z}} C(Y)) \\ \Big\downarrow{\scriptstyle \text{switch}} & & \Big\downarrow{\scriptstyle \text{conjugation by } T} \\ C(Y \times X) & \xrightarrow{\mathrm{EZ}} & \mathrm{Hom}_{\mathbb{Z}}(W, C(Y) \otimes_{\mathbb{Z}} C(X)) \end{array}$$

commutes for arbitrary X and Y; here T is the generator of Z_2, which acts on the chain complex W as usual, and 'conjugation by T' sends $f \in \mathrm{Hom}_{\mathbb{Z}}(W, C(X) \otimes_{\mathbb{Z}} C(Y))$ to TfT.) To prove the existence of such an EZ, observe that EZ is equivalent to a natural chain map

$$\mathrm{DIA}\colon C(Y) \to \mathrm{Hom}_{\mathbb{Z}[Z_2]}(W, C(Y) \otimes_{\mathbb{Z}} C(Y))$$

for simplicial sets Y, which agrees with the obvious and canonical chain isomorphism in the case where Y is a point. Indeed, DIA is obtained from EZ by letting $X = Y$ in the description of EZ and exploiting Z_2-equivariance; and EZ is obtained from DIA by substituting $X \times Y$ for Y in the description of DIA and composing with suitable projections. But the existence of DIA is a straightforward consequence of acyclic model theory; see [9], especially [9, Lemma 6.2].

The acyclic model theorem also states that EZ (or DIA) is essentially unique; fix it for the rest of the section.

Evaluating EZ on the standard generator $1 \in W_0 \subset W$ gives

$$\mathrm{EZ}_0\colon C(X \times Y) \to C(X) \otimes_{\mathbb{Z}} C(Y),$$

a natural chain homotopy equivalence.

3.2. CONVENTIONS (concerning spherical fibrations and simplicial sets). Let G be a simplicial monoid (associative, with unit). A 'classifying' simplicial set BG is defined as follows (cf. [6, Definition 3.20] and [11, Definition 10.3]):

$$BG_q = \{(g_0, g_1, \ldots, g_{q-1}) \mid g_i \in G_i\} \quad \text{for } q \geqslant 0,$$

$$d_i(g_0, g_1, \ldots, g_{q-1}) = (g_0, \ldots, g_{q-i-2}, g_{q-i-1} \cdot d_0 g_{q-i}, d_1 g_{q-i-1}, \ldots, d_{i-1} g_{q-1}),$$

$$s_i(g_0, g_1, \ldots, g_{q-1}) = (g_0, \ldots, g_{q-i-1}, 1_{q-i}, s_0 g_{q-i}, \ldots, s_{i-1} g_{q-1}) \quad \text{for } 0 \leqslant i \leqslant q.$$

(Reading instructions: d_i and s_i are the face and degeneracy operators respectively. It is understood that BG_0 is a singleton, and that the expressions for $d_i(g_0, \ldots, g_{q-1})$ and $s_i(g_0, \ldots, g_{q-1})$ are read from the left if $i = 0$ and from the right if $i = q$.)

Next, let EG be the simplicial set given by

$$\begin{aligned} EG_q &= G_q \times BG_q && \text{for } q \geqslant 0, \\ s_i(g, b) &= (s_i g, s_i b) && \text{for } 0 \leqslant i \leqslant q, \\ d_i(g, b) &= (d_i g, d_i b) && \text{for } 0 < i \leqslant q, \\ d_0(g, b) &= (t(b) \cdot d_0 g, d_0 b). && \end{aligned}$$

(Here $t(b)$ is the 'top component' of b; so if $b = (g_0, \ldots, g_{q-1})$, then $t(b) = g_{q-1}$. In the terminology of [6], EG is a twisted cartesian product with base BG and fibre G.)

Let p: $EG \to BG$ be the projection $(g, b) \mapsto b$. If $\pi_0(G)$ is a group, then the geometric realization of p is a quasi-fibration [8]. If moreover G is a Kan simplicial set, then so is BG (this is proved in 3.21 below). Under these conditions it follows easily that $\pi_n(BG) \cong \pi_{n-1}(G)$ for all n, and that EG is contractible. (Even so, EG does not satisfy the Kan condition as a rule.)

Now let $\bar{G}(n)$ be the topological monoid of self-homotopy-equivalences of the pair (D^n, S^{n-1}) (with the compact-open topology, say; D^n is the n-disk). Let $G(n)$ be the singular simplicial set of $\bar{G}(n)$, that is, the standard simplicial approximation. Using the construction BG above, with $G = G(n)$, we make the following definition:

> an n-dimensional spherical fibration on a simplicial set X is a simplicial map from X to $BG(n)$.

Such a spherical fibration γ on X has a geometric realization: let $E(\gamma)$ be the space

$$\coprod_{q \geqslant 0} \Delta_q \times D^n \times X_2 / \sim$$

where $\sim$ is the obvious equivalence relation, i.e. is generated by

$$\begin{aligned} (s_i^{\to}(u), v, x) &\sim (u, v, s_i(x)) && \text{for } q \geqslant i \geqslant 0,\ u \in \Delta_{q+1},\ v \in D^n,\ x \in X_q, \\ (d_i^{\to}(u), v, x) &\sim (u, v, d_i(x)) && \text{for } q \geqslant i > 0,\ u \in \Delta_{q-1},\ v \in D^n,\ x \in X_q, \\ (d_0^{\to}(u), v, x) &\sim (u, t_x(u, v), d_0(x)) && \text{for } q > 0,\ u \in \Delta_{q-1},\ v \in D^n,\ x \in X_q. \end{aligned}$$

(Here t_x: $\Delta_{q-1} \times D^n \to D^n$ is the image of $x \in X_q$ under the composition

$$X_q \longrightarrow BG_q \xrightarrow{\ t\ } G_{q-1}.$$

Notice the formal similarity in the descriptions of $E(\gamma)$ and EG.) Similarly, let $\partial E(\gamma)$ be the space

$$(\coprod_{q \geqslant 0} \Delta_q \times S^{n-1} \times X_q) / \sim,$$

so that $\partial E(\gamma) \subset E(\gamma)$. The diagram

$$\begin{tikzcd} \partial E(\gamma) \arrow[r] \arrow[dr] & E(\gamma) \arrow[d] \\ & X \end{tikzcd}$$

(in which the projection $E(\gamma) \to X$ is obvious) allows one to interpret γ as a pair of quasi-fibrations over X, with fibre pair (D^n, S^{n-1}).

Neither $E(\gamma)$ nor $\partial E(\gamma)$ have canonical CW-structures; however, the Thom space $E(\gamma)/\partial E(\gamma)$ is a CW-space (not a simplicial set) whose cells are in one-one correspondence with those of X. This is extremely convenient.

There are canonical inclusions

$$\ldots \to BG(n-1) \hookrightarrow BG(n) \hookrightarrow BG(n+1) \hookrightarrow \ldots;$$

a simplicial map $X \to BG(\infty) := \bigcup BG(n)$ is called a stable spherical fibration on X.

3.3. Reminder. A Poincaré space (or geometric Poincaré complex) is, for the purposes of this section, a finitely generated simplicial set Y equipped with a fundamental class and satisfying Poincaré duality with arbitrary local coefficients—see [**15**] for details. (So the torsion is allowed to be non-zero.)

Such a Poincaré space Y, of formal dimension n, has a 'Spivak normal fibration', i.e. a stable spherical fibration ν_Y on Y, characterized by the following property: there exists a map of CW-spectra

$$r_Y\colon S^n \to M(Y, \nu_Y) := (\text{Thom spectrum of } \nu_Y)$$

such that, in loose notation,

$$(\text{Thom class of } \nu_Y) \cap h(r_Y) = (\text{fundamental class of } Y).$$

(The Hurewicz image of r_Y in $H_n(M(Y, \nu_Y); \mathbb{Z})$ has been denoted by $h(r_Y)$; here $M(Y, \nu_Y)$ is the formally desuspended Thom space of ν_Y, and the expression 'CW-spectrum' will always mean a spectrum in the sense of Boardman, cf. [**19**]. A 'map' between CW-spectra is defined as in [**19**, Definition 8.12], so is automatically cellular.)

'Characterized', in this context, means more than just 'unique up to (stable) concordance'; it means, for example, that the bordism theory of triples (Y, ν_Y, r_Y) as above can be identified with the bordism theory of Poincaré spaces Y. So we shall often think of a Poincaré space Y as a triple (Y, ν_Y, r_Y), and similarly for geometric Poincaré pairs.

More generally, a 'higher bordism of Poincaré spaces, of dimension n and order q' consists of

(i) a functor $V \mapsto Y(V)$ from the category $2^{\{0,1,\ldots,q\}}$ to the category of finitely generated simplicial sets Y and injective simplicial maps ('cofibrations'),

(ii) a stable spherical fibration ν on $Y(\{0, 1, \ldots, q\})$,

(iii) a compatible collection of maps of CW-spectra

$$r(V)\colon \Delta(V)_+ \wedge S^{n-q} \to M(Y(V), \nu/Y(V)) = \text{Thom spectrum}.$$

The functor in (i) is required to be 'intersection-preserving', that is, for $V_1, V_2 \subset \{0, 1, \ldots, q\}$ we have $Y(V_1 \cap V_2) = Y(V_1) \cap Y(V_2)$ if these spaces are interpreted as subspaces of $Y(V_1 \cup V_2)$; also $Y(\varnothing) = \varnothing$. Finally, each $Y(V)$ is required to be a geometric Poincaré pair (with boundary equal to $\bigcup Y(U)$, where the union ranges over the proper subsets U of V, and relative fundamental class equal to the Hurewicz image of $r(V)$).

For the rest of the section we need: a group π and homomorphism $w\colon \pi \to Z_2$; a finitely generated simplicial set X and a spherical fibration γ on X; a principal π-bundle α on X and an identification of double covers of X,

$$j\colon w^{\to}(\alpha) \cong (\text{orientation cover of } \gamma).$$

Write $C(\tilde{X})$ for the cellular chain complex of the total space of α. Then $C(\tilde{X})$ is a chain complex in $\mathscr{C}_A$, with $A = \mathbb{Z}[\pi]$ (equipped with the involution

$$\sum_{g\in\pi} n_g \cdot g \mapsto \sum_{g\in\pi} (-)^{w(g)} \cdot n_g \cdot g^{-1}).$$

3.4. THEOREM. *The data $(\pi, w;\ X, \gamma;\ \alpha, j)$ determine (up to an infinity of higher homologies—see* 0.15*) a chain bundle $c(\gamma)$ on $C(\tilde{X})$. The construction is functorial.*

3.5. THEOREM. *The geometric bordism spectrum $\underline{\Omega}^P(X,\gamma)$ (details follow) and the algebraic bordism spectrum $\mathbb{L}^0(C(\tilde{X}), c(\gamma))$ are related by a natural map*

$$\underline{\Omega}^P(X,\gamma) \to \mathbb{L}^0(C(\tilde{X}), c(\gamma)).$$

(There is also a map $\underline{\Omega}^P(X,\gamma) \to \underline{\mathscr{L}}^{\cdot}(C(\tilde{X}), c(\gamma))$ obtained by composing with the forgetful map from $\mathbb{L}^0(C(\tilde{X}), c(\gamma))$ to $\underline{\mathscr{L}}^{\cdot}(C(\tilde{X}), c(\gamma))$; both maps are called *flexible signature.*)

Explanation. Let $\Omega_0^P(X,\gamma)$ be the Δ-set (incomplete simplicial set) whose q-simplices are the higher bordisms of Poincaré spaces $\{Y(V), v, r(V) \mid V \subset \{0, 1, \ldots, q\}\}$ of dimension q and order q (as in 3.3), equipped with a simplicial classifying map $g\colon Y(\{0,1,\ldots,q\}) \to$ so that v equals the pullback $g^{\leftarrow}(\gamma)$ (which ought to be written $\gamma \cdot g$). Then $\Omega_0^P(X,\gamma)$ is an infinite loop space (see 3.20 below), and the associated spectrum is $\underline{\Omega}^P(X,\gamma)$.

Most of this section is devoted to proving 3.4 and 3.5.

3.6. DEFINITION. A 'π-space' will mean (in this section at least) a simplicial set Y with a base point (distinguished 0-simplex) and a simplicial π-action which fixes the base point, but freely permutes the other cells (non-degenerate simplices) of Y. For such a Y,

$$\dot{C}(Y) := C(Y)/C(\text{base point})$$

is a chain complex in $\mathscr{C}_A$, provided Y is finitely generated (over π).

3.7. PROPOSITION ('symmetric construction', cf. [15]). *For every π-space Y, there is defined a chain map*

$$\text{Sym}\colon \mathbb{Z}^t \otimes_A \dot{C}(Y) \to W \,\&\, \dot{C}(Y),$$

inducing maps in homology

$${}^t\dot{H}_n(Y/\pi; \mathbb{Z}) := H_n(\mathbb{Z}^t \otimes_A \dot{C}(Y)) \to Q^n(\dot{C}(Y)).$$

It is natural with respect to π-maps.

Note: the w-twisted involution on $A = \mathbb{Z}[\pi]$ is used.

Proof. Take the map DIA: $\dot{C}(Y) \to \mathrm{Hom}_{\mathbb{Z}[Z_2]}(W, \dot{C}(Y) \otimes_{\mathbb{Z}} \dot{C}(Y))$ of 3.1; note that π (and hence A) acts on $\dot{C}(Y)$ as usual, and also on $\mathrm{Hom}_{\mathbb{Z}[Z_2]}(W, \dot{C}(Y) \otimes_{\mathbb{Z}} \dot{C}(Y))$ via the diagonal action on $\dot{C}(Y) \otimes_{\mathbb{Z}} \dot{C}(Y)$. Tensoring with $\mathbb{Z}$ on the left gives

$$\begin{aligned}\mathbb{Z}^t \otimes_A \dot{C}(Y) \to \mathbb{Z}^t &\otimes_A (\mathrm{Hom}_{\mathbb{Z}[Z_2]}(W, \dot{C}(Y) \otimes_{\mathbb{Z}} \dot{C}(Y))) \\ &\cong \mathrm{Hom}_{\mathbb{Z}[Z_2]}(W, \mathbb{Z}^t \otimes_A (\dot{C}(Y) \otimes_{\mathbb{Z}} \dot{C}(Y))) \\ &\cong \mathrm{Hom}_{\mathbb{Z}[Z_2]}(W, \dot{C}(Y)^t \otimes_A \dot{C}(Y)) = W \,\&\, \dot{C}(Y).\end{aligned}$$

3.8. EXAMPLE. Let (Y, ν_Y, r_Y) be a Poincaré space, of formal dimension n; suppose that there is given a principal π-bundle β on Y and an identification of twofold covers,

$$w^{\rightharpoondown}(\beta) \cong (\text{orientation cover of } \nu_Y) \quad (= \text{orientation cover of } Y)$$

(with w as in 3.4). Then $\tilde{Y}_+$ (the total space of β, with an added disjoint base point) is a π-space.

Let $\varphi \in W \,\&\, \dot{C}(\tilde{Y}_+) = W \,\&\, C(\tilde{Y})$ be the image of the fundamental cycle under the chain map Sym. (The fundamental cycle is the cycle determined by r_Y; it represents the fundamental class.)

Then $(C(\tilde{Y}), \varphi)$ is an n-dimensional algebraic Poincaré complex over A—the 'algebraic image' of Y.

3.9. OUTLINE. We are now in a position to obtain a sketch proof of 3.4 and 3.5. It is taken without essential change from [**15**, Part II, §9].

(i) Starting with a string $(\pi, w; X, \gamma; \alpha, j)$ as in 3.4, and assuming that γ is k-dimensional, Ranicki obtains a characteristic class $[c(\gamma)] \in \hat{Q}^0(C(\tilde{X})^{-*})$ by choosing a π-equivariant S-dual $T(\tilde{X}, \tilde{\gamma})^*$ of the Thom π-space $T(\tilde{X}, \tilde{\gamma})$, and applying the symmetric construction

$${}^t\dot{H}_*(T(X, \gamma)^*) \to Q^*(\dot{C}(T(\tilde{X}, \tilde{\gamma})^*))$$

of 3.7 to the dual of the Thom class in ${}^t\dot{H}^k(T(X, \gamma))$. This yields a class $[\psi]$ in $Q^n(\dot{C}(T(\tilde{X}, \tilde{\gamma})^*))$ for some n, to begin with. Observe now that we have a chain homotopy equivalence

$$f\colon \Sigma^n(C(\tilde{X})^{-*}) \xrightarrow{\simeq} \dot{C}(T(\tilde{X}, \tilde{\gamma})^*)$$

by composing S-duality with the Thom isomorphism. Then

$$f^{\rightharpoondown}\colon \hat{Q}^n(\Sigma^n(C(\tilde{X})^{-*})) \to \hat{Q}^n(\dot{C}(T(\tilde{X}, \tilde{\gamma})^*))$$

is an isomorphism, so that we may define the characteristic class $[c(\gamma)] \in \hat{Q}^0(C(\tilde{X})^{-*})$ by the formula

$$J([\psi]) = f^{\rightharpoondown} \cdot \mathfrak{S}^n([c(\gamma)]) \quad \text{in } \hat{Q}^n(\dot{C}(T(\tilde{X}, \tilde{\gamma})^*)).$$

This nearly proves 3.4.

(ii) Suppose next that the string $(\pi, w; X, \gamma; \alpha, j)$ is such that X is an n-dimensional Poincaré space with Spivak normal fibration γ. Let $(C(\tilde{X}), \varphi)$ be the n-dimensional algebraic Poincaré complex constructed from X as in 3.8. We may take, as is well known,

$$T(\tilde{X}, \tilde{\gamma})^* = \tilde{X}_+.$$

It is then easy to check that $[\psi] = [\varphi]$ and $f = \varphi_0$. If we insert this in the formula defining $[c(\gamma)]$, we see that

$$[c(\gamma)] = [n],$$

where n is the normal chain bundle of $(C(\tilde{X}), \varphi)$. So 3.5 is nearly proved.

The reader is advised to omit the rest of the section except 3.16, 3.17, and 3.18, and to regard the arguments above as proofs. It should be realized, however, that they are inadequate in two respects. Firstly, in 3.4 we do need a chain bundle $c(\gamma)$ rather than just a class $[c(\gamma)]$, as is shown in 3.17. Secondly, the argument for 3.5 given above does not survive the generalization from Poincaré spaces to *normal spaces* (see Part II, § 7 of this paper). We begin with the rigorous proof.

3.10. MACHINERY. Let $\aleph$ be a Δ-set. We will regard $\aleph$ as a category (whose objects are the simplices of $\aleph$; a morphism from an n-simplex x to an m-simplex y is an injective order-preserving map $\{0, 1, \ldots, n\} \to \{0, 1, \ldots, m\}$ so that the corresponding face operator sends y to x).

An '$\aleph$-indexed chain complex' is a covariant functor G from $\aleph$ to the category of chain complexes. Given such a G, and given any Δ-subset $\aleph'$ of $\aleph$, define a new (ordinary) chain complex $(\mathrm{Sect}_{\aleph'}; G)$, or $(\mathrm{Sect}_{\aleph'}; G(-))$, by

$$(\mathrm{Sect}_{\aleph'}; G)_n = \prod_{x \in \aleph'} (G(x))_{n+|x|},$$

$$[ds]_x := \mathrm{diff}_x([s]_x) - \sum_{0 \leq i \leq |x|} (-)^{n-i} \cdot j^{\to}_{d_i x, x}([s]_{d_i x}).$$

Here $s \in (\mathrm{Sect}_{\aleph'}; G)_n$; if x is a simplex in $\aleph'$, we write $[s]_x$ for its x-component, and diff_x for the differential in $G(x)$. Finally, $j_{d_i x, x}$ is the inclusion of the ith face as usual, and d is the differential in $(\mathrm{Sect}_{\aleph'}; G)$.)

Alternatively, $(\mathrm{Sect}_{\aleph'}; G)$ can be described as the subcomplex of natural chains in

$$\prod_{x \in \aleph'} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(\Delta_{|x|}), G(x)),$$

where $\mathrm{Cl}(\Delta_{|x|})$ is the cellular chain complex of the standard simplex. (Call a collection $\{f_x \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(\Delta_{|x|})G(x)) \mid x \in \aleph'\}$ *natural* if

$$p^{\to} \cdot f_x = f_y \cdot p^{\to}$$

for every morphism $p\colon x \to y$ in $\aleph'$.)

Note that the construction has 'sheaflike' properties: given two Δ-subsets $\aleph'$ and $\aleph''$ of $\aleph$, there is a pullback square of restriction maps

$$\begin{tikzcd}
(\mathrm{Sect}_{\aleph' \cup \aleph''}; G) \arrow[r] \arrow[d] & (\mathrm{Sect}_{\aleph'}; G) \arrow[d] \\
(\mathrm{Sect}_{\aleph''}; G) \arrow[r] & (\mathrm{Sect}_{\aleph' \cap \aleph''}; G)
\end{tikzcd}$$

Suppose next that G above is a covariant functor, not merely from the category $\aleph$ to the category of chain complexes, but from $\aleph$ to $\mathscr{C}_A$. Suppose also that we have an ordinary chain complex C in $\mathscr{C}_A$ and an A-module chain map

$$f\colon C \to (\mathrm{Sect}_{\aleph}; G(-)).$$

An induced chain map

$$f^{\rightarrow}\colon \hat{W}\,\&\,C \to (\mathrm{Sect}_{\aleph};\hat{W}\,\&\,G(-))$$

is defined as follows. Using the second description of $(\mathrm{Sect}_{\aleph};G(-))$, we see that f is nothing but a natural collection of chain maps

$$f_x\colon C\otimes_{\mathbb{Z}}\mathrm{Cl}(\Delta_{|x|}) \to G(x).$$

Now define $f^{\rightarrow}$ by $[f^{\rightarrow}(\varphi)]_x := f_x^{\rightarrow}(\varphi\times\rho(|x|)) \in \hat{W}\,\&\,G(x)$, for $\varphi\in\hat{W}\,\&\,C$; to make sense of this formula, use the sequence $\{\rho(m)\mid m\geqslant 0\}$ of 2.9.

3.11. Definition. From now on $\aleph$ will be the *moduli space of all attempts at being equivariantly S-dual to the Thom π-spectrum $M(\tilde{X},\tilde{\gamma})$*. In detail, a typical q-simplex y of $\aleph$ shall consist of

an intersection-preserving functor Ftr_y from $2^{\{0,1,\ldots,q\}}$ to the category of finitely generated π-spaces and π-cofibrations, and

a compatible collection of maps of CW-spectra

$$\Delta(V)_+ \to M(\tilde{X}\times_\pi \mathrm{Ftr}_y(V),\gamma)/M(X,\gamma),$$

one for each $V\subset\{0,1,\ldots,q\}$.

This should require a fair amount of explanation.

(i) A π-space U is *finitely generated* if the space or rather simplicial set U/π is finitely generated. *Intersection-preserving* means here that

$$\mathrm{Ftr}_y(V_1\cap V_2)=\mathrm{Ftr}_y(V_1)\cap\mathrm{Ftr}_y(V_2)$$

if these spaces are interpreted as subspaces of $\mathrm{Ftr}_y(V_1\cup V_2)$, and that $\mathrm{Ftr}_y(\varnothing)$ is a point.

(ii) If U is any π-space, then $X=\tilde{X}\times_\pi(\text{base point})$ is contained in $\tilde{X}\times_\pi U$. Note further that

$$M(\tilde{X}\times_\pi U,\gamma)/M(X,\gamma)\cong M(\tilde{X},\tilde{\gamma})\wedge_\pi U,$$

if the cell decompositions are disregarded; here the pullback of γ to $\tilde{X}\times_\pi U$ has also been called γ.

In particular, if y is a q-simplex in $\aleph$ having all faces at the base point, then y consists of a π-space $U=\mathrm{Ftr}_y(\{0,1,\ldots,n\})$ and a map of spectra from S^q to $M(\tilde{X},\tilde{\gamma})\wedge_\pi U$. We may call this an attempt on the part of U at being equivariantly S-dual to $M(\tilde{X},\tilde{\gamma})$.

Having specified $\aleph$, we shall also specify the $\aleph$-indexed chain complex G in 3.10 by letting

$$G(y)=\dot{C}(\mathrm{Ftr}_y(\{0,1,\ldots,q\}))$$

for a q-simplex y in $\aleph$; in other words $G(y)$ is the reduced cellular chain complex of the underlying π-space of y.

We have now collected most of the material necessary to rewrite 3.9 in a rigorous fashion. What follows is a *parametrized* version of 3.9 where $\aleph$ serves as parameter space. So, instead of choosing an equivariant S-dual $T(\tilde{X},\tilde{\gamma})^*$ of the Thom π-space $T(\tilde{X},\tilde{\gamma})=\Sigma^k M(\tilde{X},\tilde{\gamma})$, we shall consider *simultaneously* all attempts at being equivariantly S-dual to $M(\tilde{X},\tilde{\gamma})$. The role of $\dot{C}(T(\tilde{X},\tilde{\gamma})^*)$ in 3.9 will be played by $(\mathrm{Sect}_{\aleph};G(-))$; the next construction shows that the symbols f and ψ in 3.9(i) also have their parametrized counterparts.

3.12. CONSTRUCTION. We shall construct

(i) a chain map $f\colon C(\tilde{X})^{-*} \to (\mathrm{Sect}_{\aleph}; G(-))$,

(ii) a 0-cycle ψ in $(\mathrm{Sect}_{\aleph}; W \,\&\, G(-))$.

For (i), let $\bar{G}$ be the $\aleph$-indexed chain complex such that

$$\bar{G}(y) = \dot{C}(M(X \times_{\pi} \mathrm{Ftr}_y(\{0, 1, \dots, q\}), \gamma)/M(X, \gamma))$$

if y is a q-simplex in $\aleph$. The very definition of $\aleph$ yields a canonical 0-cycle $\bar{z}$ in $(\mathrm{Sect}_{\aleph}; \bar{G}(-))$. Our conventions concerning spherical fibrations (see the end of 3.2) give us a Thom isomorphism on the chain level; composing with the Eilenberg–Zilber map EZ_0 of 3.1, we get a natural homotopy equivalence

$$\bar{G}(y) \xrightarrow{\simeq} C(\tilde{X}) \otimes_A G(y) \cong \mathrm{Hom}_A(C(\tilde{X})^{-*}, G(y))$$

for any simplex y in $\aleph$. Therefore we now have a 0-cycle

$$z \in (\mathrm{Sect}_{\aleph}; \mathrm{Hom}_A(C(\tilde{X})^{-*}, G(-))) \quad \text{(the image of } \bar{z}).$$

Define $f\colon C(\tilde{X})^{-*} \to (\mathrm{Sect}_{\aleph}; G(-))$ by

$$[f(x)]_y := [z]_y(x)$$

for $x \in C(\tilde{X})^{-*}$ and y a simplex in $\aleph$.

In (ii), we put

$$\psi := [\mathrm{Sym} \cdot (\mathbb{Z}^t \otimes_A f)](1_c).$$

Here $\mathbb{Z}^t \otimes_A f$ is the chain map from $\mathbb{Z}^t \otimes_A C(\tilde{X})^{-*}$ to $(\mathrm{Sect}_{\aleph}; \mathbb{Z}^t \otimes_A G(-))$ obtained by tensoring f with $\mathbb{Z}^t$; further, 1_c is the unique 0-cycle in $\mathbb{Z}^t \otimes_A C(\tilde{X})^{-*}$ representing $1 \in H^0(X; \mathbb{Z}) \cong H_0(\mathbb{Z}^t \otimes_A C(\tilde{X})^{-*})$, and Sym denotes the parametrized symmetric construction which is a chain map from $(\mathrm{Sect}_{\aleph}; \mathbb{Z}^t \otimes_A G(-))$ to $(\mathrm{Sect}_{\aleph}; W \,\&\, G(-))$. See 3.7.

3.13. KEY LEMMA. *Both $f\colon C(\tilde{X})^{-*} \to (\mathrm{Sect}_{\aleph}; G(-))$ and the induced chain map $f^{\to}\colon \hat{W} \,\&\, C(\tilde{X})^{-*} \to (\mathrm{Sect}_{\aleph}; \hat{W} \,\&\, G(-))$ are chain homotopy equivalences.*

The proof is deferred; see 3.19. Now let

$$\mathscr{P}(\pi, w; X, \gamma; \alpha, j)$$

be the homotopy pullback (see 0.14) of the diagram of chain maps

$$\begin{array}{ccc} & & \hat{W} \,\&\, C(\tilde{X})^{-*} \\ & & \simeq \Big\downarrow f^{\to} \\ (\mathbb{Z}, 0) & \xrightarrow{1 \mapsto J(\psi)} & (\mathrm{Sect}_{\aleph}; \hat{W} \,\&\, G(-)) \end{array}$$

Then the projections

$$\begin{array}{ccc} \mathscr{P}(\pi, w; X, \gamma; \alpha, j) & \longrightarrow & \hat{W} \,\&\, C(\tilde{X})^{-*} \\ \Big\downarrow \simeq & & \\ (\mathbb{Z}, 0) & & \end{array}$$

constitute a 0-*cycle in* $\hat{W} \,\&\, C(\tilde{X})^{-*}$, *well-defined up to an infinity of higher homologies* (see 0.15). This proves 3.4, since a 0-cycle in $\hat{W} \,\&\, C(\tilde{X})^{-*}$ is a chain bundle on $C(\tilde{X})$.

In the next lemma, an *admissible* 0-cycle in $\mathscr{P}(\pi, w; X, \gamma; \alpha, j)$ means a cycle in the class $1 \in \mathbb{Z} \cong H_0(\mathscr{P}(\pi, w; X, \gamma; \alpha, j))$.

3.14. LEMMA. *Every admissible* 0-*cycle* s *in* $\mathscr{P}(\pi, w; X, \gamma; \alpha, j)$ *determines a* Δ-*map* $(\text{fl.sig.})_s$: $\mathbf{\Omega}_0^P(X, \gamma) \to \mathbb{L}^0(C(\tilde{X}), c_s(\gamma))$. (See the paragraph following 3.5.) *Here* $c_s(\gamma)$ *is the chain bundle on* $C(\tilde{X})$ *determined by* s.

Proof. We begin with a Δ-map i: $\mathbf{\Omega}_0^P(X, \gamma) \to \aleph$. Let x be a q-simplex in $\mathbf{\Omega}_0^P(X, \gamma)$; then x consists of a collection $\{Y(V), v, r(V) \mid V \subset \{0, 1, \ldots, q\}\}$ and a classifying map g: $Y(\{0, 1, \ldots, q\}) \to X$ such that $g^{\leftarrow}(\gamma) = v$. (See the end of 3.3.) Each $Y(V)$ inherits a principal π-bundle from X, with total space $\tilde{Y}(V)$.

Now $i(x)$ is the q-simplex in $\aleph$ such that

$$\mathrm{Ftr}_{i(x)}(V) \to \tilde{Y}(V)_+;$$

for the stable map from $\Delta(V)_+$ to $M(\tilde{X} \times_\pi \mathrm{Ftr}_{i(x)}(V), \gamma)/M(X, \gamma)$ required in 3.11 we take the composition

$$\Delta(V)_+ \xrightarrow{\quad r(V) \quad} M(Y(V), v) \xrightarrow{\text{classifying map} \times \text{id}} M(\tilde{X} \times_\pi \tilde{Y}(V), \gamma).$$

This is a cellular map (as it should be) because X and $Y(V)$ are simplicial sets. The description of i is complete.

Next, observe that the parametrized version of 3.8 produces a canonical 0-cycle

$$\varphi \in (\mathrm{Sect}_{\mathbf{\Omega}_0^P(X, \gamma)}; G(-)).$$

(Here G is short for $G \cdot i$.) All we need now in order to get a map from $\mathbf{\Omega}_0^P(X, \gamma)$ to $\mathbb{L}^0(C(\tilde{X}), c_s(\gamma))$ is a collection of clutching homologies (see 2.7, 2.8, 2.9). In other words, we are searching for a 1-chain

$$z \in (\mathrm{Sect}_{\mathbf{\Omega}_0^P(X, \gamma)}; \hat{W} \,\&\, G(-)).$$

Now I claim that such a 1-chain z can be extracted from the admissible cycle $s \in \mathscr{P}(\pi, w; X, \gamma; \alpha, j)$ in 3.14. Indeed, we constructed $\mathscr{P}(\pi, w; X, \gamma; \alpha, j)$ as a chain homotopy pullback, so our admissible 0-cycle is a triple

$$(1, c_s(\gamma), \bar{z})$$

with $1 \in (\mathbb{Z}, 0)$, $c_s(\gamma) \in \hat{W} \,\&\, C(\tilde{X})^{-*}$, and $\bar{z} \in (\mathrm{Sect}_\aleph; \hat{W} \,\&\, G(-))$. (See 0.14.) The pullback of $\bar{z}$ under i: $\mathbf{\Omega}_0^P(X, \gamma) \to \aleph$ is the required 1-chain z. Inspection shows that it satisfies the equations in 2.8.

3.15. LEMMA. *The* Δ-*map* $(\text{fl.sig.})_s$ *in* 3.14 *has a canonical refinement to a map of infinite loop spaces.*

For the proof, see 3.20. To complete the proof of 3.5, we still have to show that the map of spectra

$$\text{flexible signature: } \underline{\mathbf{\Omega}}^P(X, \gamma) \to \underline{\mathbb{L}}^0(C(\tilde{X}), c(\gamma))$$

obtained from 3.15 does not depend too much on the choice of an admissible 0-cycle s in $\mathscr{P}(\pi, w; X, \gamma; \alpha, j)$. This should follow from the existence of a homotopy equivalence $\mathscr{P}(\pi, w; X, \gamma; \alpha, j) \simeq (\mathbb{Z}, 0)$. But the following argument is easier. Pick another admissible 0-cycle s' in $\mathscr{P}(\pi, w; X, \gamma; \alpha, j)$. Crossing the string of data $(\pi, w; X, \gamma; \alpha, j)$ with the unit interval $[0, 1]$ gives a new string, written

$$(\pi, w; X \times [0, 1], \gamma \times [0, 1]; \alpha, j).$$

Choose an admissible 0-cycle s'' in $\mathscr{P}(\pi, w; X \times [0, 1], \gamma \times [0, 1]; \alpha, j)$ whose image under the restriction map

$$\mathscr{P}(\pi, w; X \times [0, 1], \gamma \times [0, 1]; \alpha, j) \\ \to \mathscr{P}(\pi, w; X \times \{0\}, \gamma; \alpha, j) \oplus \mathscr{P}(\pi, w; X \times \{1\}, \gamma; \alpha, j)$$

is (s, s'). Such an s'' exists. Then the commutative diagram

$$\begin{tikzcd}
\underline{\Omega}^P(X \times \{0\}, \gamma) \arrow[r] \arrow[d] & \underline{\Omega}^P(X \times [0, 1], \gamma \times [0, 1]) \arrow[d] & \underline{\Omega}^P(X \times \{1\}, \gamma) \arrow[l] \arrow[d] \\
\underline{\mathbb{L}}^0(C(\tilde{X}), c_s(\gamma)) \arrow[r] & \underline{\mathbb{L}}^0(C(\tilde{X} \times [0, 1]), c_{s''}(\gamma \times [0, 1])) & \underline{\mathbb{L}}^0(C(\tilde{X}), c_{s'}(\gamma)) \arrow[l]
\end{tikzcd}$$

shows what we want, since all horizontal arrows are homotopy equivalences. The proof of 3.5 is complete; the naturality part is stated separately below.

3.16. Remark. Suppose that there are given two strings $(\pi, w; X, \gamma; \alpha, j)$ and $(\pi', w'; X', \gamma'; \alpha', j')$ as in 3.4, and

(i) a homomorphism $h\colon \pi \to \pi'$ such that $w' \cdot h = w$;
(ii) a simplicial map $g\colon X \to X'$ covered by a map of spherical fibrations from γ to γ';
(iii) an identification $h^{\to}(\alpha) \cong g^{\leftarrow}(\alpha')$ of principal π'-bundles on X, compatible with j and j'.

Such a 'morphism' induces a sufficiently well-defined chain bundle map

$$C(\tilde{X}), c(\gamma) \to C(X'), c(\gamma')$$

(involving a change of rings, cf. 1.8 and sequel); and the diagram

$$\begin{tikzcd}
\underline{\Omega}^P(X, \gamma) \arrow[r] \arrow[d] & \underline{\mathbb{L}}^0(C(\tilde{X}), c(\gamma)) \arrow[d] \\
\underline{\Omega}^P(X', \gamma') \arrow[r] & \underline{\mathbb{L}}^0(C(\tilde{X}'), c(\gamma'))
\end{tikzcd}$$

is sufficiently commutative for all practical purposes.

Geometric transfer maps also have algebraic counterparts: let $(\pi, w; X, \gamma; \alpha, j)$ be a string of data as usual, and suppose that $\pi'' \subset \pi$ is a subgroup of finite index. A second string $(\pi'', w''; X'', \gamma''; \alpha'', j'')$ is then given by

$w'' :=$ $w \cdot$ inclusion,
$X'' :=$ total space of α, modulo action of π'',
$\gamma'' :=$ pullback of γ,
$\alpha'' :=$ principal π''-bundle on X'' derived from α,
$j'' :=$ identification derived from j.

There is a sufficiently commutative diagram of maps of spectra

$$\begin{tikzcd}
\underline{\mathbf{\Omega}}^P(X'',\gamma'') \arrow[r] & \underline{\mathbb{L}}^0(C(\tilde{X}''),c(\gamma'')) \\
\underline{\mathbf{\Omega}}^P(X,\gamma) \arrow[r] \arrow[u, "\begin{array}{l}\text{geometric}\\\text{transfer}\end{array}"'] & \underline{\mathbb{L}}^0(C(X),c(\gamma)) \arrow[u, "\begin{array}{l}\text{algebraic}\\\text{transfer}\end{array}"']
\end{tikzcd}$$

As in 3.5, $\underline{\mathbb{L}}^0(\ldots)$ can be replaced by $\underline{\mathscr{L}}^{\cdot}(\ldots)$.

3.17. Example. Using notation as in 3.4, let $[c(\gamma)] \in \hat{Q}^0(C(\tilde{X})^{-*})$ be the class of $c(\gamma)$. In § 1 (after 1.8) elements of $\hat{Q}^0(C(\tilde{X})^{-*})$ were interpreted as 'isomorphism classes of chain bundles' on $C(\tilde{X})$. Consequently, knowledge of $[c(\gamma)] \in \hat{Q}^0(C(\tilde{X})^{-*})$ suffices to reconstruct the groups $L^n(C(\tilde{X}), c(\gamma))$ (see 2.21 (iv)) 'up to isomorphism'. More cannot be expected, as is shown by the following example.

Let N^n be a smooth closed manifold admitting two stable framings Fr_1, Fr_2 such that the Kervaire invariants of (N, Fr_1) and (N, Fr_2) are defined and distinct. (Such manifolds are known to exist for $n = 2$, 6, 14, 30, 62 (?).) Specify the string $(\pi, w; X, \gamma; \alpha, j)$ as follows (see 3.4): $X = N$ and γ is trivial, $\pi = \{1\}$, etc. Let us work with smooth manifolds instead of Poincaré spaces; we may then replace $\underline{\mathbf{\Omega}}^P(X,\gamma)$ by the Thom spectrum $M(X,\gamma)$. The difference between the two framings Fr_1 and Fr_2 is a map from $N = X$ to the orthogonal group; it can also be regarded as a stable automorphism of the trivial bundle γ on X, written tw. The algebraic counterpart of tw is a chain bundle automorphism τ of $c(\gamma)$ (covering the identity $C(X) \to C(X)$). There is a commutative diagram

$$\begin{tikzcd}
\pi_n(M(X,\gamma)) \arrow[r, "(tw)^{\to}"'] \arrow[d, "\begin{array}{r}\text{flexible}\\\text{signature}\end{array}"'] & \pi_n(M(X,\gamma)) \arrow[d, "\begin{array}{l}\text{flexible}\\\text{signature}\end{array}"] \\
L^n(C(X),c(\gamma)) \arrow[r, "\tau^{\to}"] & L^n(C(X),c(\gamma))
\end{tikzcd}$$

(notation as in 3.4, 3.5, 2.21 (iv)).

Claim. Both $(tw)^{\to}$ and $\tau^{\to}$ are non-trivial group automorphisms. Indeed, let $y \in \pi_n(M(X,\gamma))$ be the bordism class represented by $(\mathrm{id}, \mathrm{Fr}_1)$: $N, \nu_N \to X, \gamma$ (recall that $X = N$, and γ is the trivial bundle). Also, let X' be a one-point space, and let γ' be the trivial bundle on X'; the obvious bundle map $X, \gamma \to X', \gamma'$ then induces a homomorphism

$$?: \pi_n(M(X,\gamma)) \to \pi_n^s = \pi_n(M(X',\gamma')).$$

The elements $?(y)$ and $?((tw)^{\to}(y))$ of π_n^s have distinct Kervaire invariants by construction; hence $(tw)^{\to}$ is not the identity. Practically the same argument shows that

$$\mathrm{fl.sig.}(y) \neq (\mathrm{fl.sig.}(y)),$$

proving that $\tau^{\to}$ is non-trivial.

So it is impossible to give an 'honest' description of $L^n(C(\tilde{X}), c(\gamma))$ in terms of $[c(\gamma)] \in \hat{Q}^0(C(\tilde{X})^{-*})$.

3.18. Remark. If $(C(\tilde{Y}), \varphi)$ is the n-dimensional algebraic Poincaré complex derived (as in 3.8) from an n-dimensional Poincaré space Y, with Spivak normal bundle ν_Y etc., then there is a canonical identification

$$c(\nu_Y) = (\text{normal chain bundle of } (C(\tilde{Y}), \varphi))$$

(see the sequel to 2.6). This is clear from the proof of 3.5: the identity map $Y, \nu_Y \to Y, \nu_Y$ represents an element in $\pi_n(\underline{\Omega}^P(Y, \nu_Y))$; and we know that $(C(\tilde{Y}), \varphi)$ has a preferred $c(\nu_Y)$-structure (g, z), in which g: $C(\tilde{Y}) \to C(\tilde{Y})$ is the identity. In other words, $c(\nu_Y)$ has the property which characterizes the normal chain bundles, as required.

Given a degree-1 map e: $P_1 \to P_2$ between Poincaré spaces of the same formal dimension n (equipped with suitable data, such as principal π-bundles), **[15]** defines the 'symmetric kernel' of e, an n-dimensional algebraic Poincaré complex (C', φ'). If in addition the map e has the attributes of a normal map, then it is easy to see that the normal chain bundle of (C', φ') is 'trivialized' (using 3.18); by 2.17, the algebraic Poincaré complex (C', φ') together with this trivialization defines an element of $L_n(\mathbb{Z}[\pi])$.

Finally, let $(\pi, w; X, \gamma; \alpha, j)$ be a string as usual, and suppose that we are given maps of finite CW-spaces

$$P_1 \xrightarrow{g} P_2 \xrightarrow{h} X$$

and a map of spectra r: $S^n \to M(P_1, (h \cdot g)^{\leftarrow}(\gamma))$ (where $M(\ldots)$ denotes the Thom spectrum) such that the triples $(P_1, (h \cdot g)^{\leftarrow}(\gamma), r)$ and $(P_2, h^{\leftarrow}(\gamma), g^{\rightarrow}(r))$ are Poincaré spaces in the sense of 3.3 and 3.8.

Then g is clearly a normal map of degree 1, so, by what we have just seen, an element $\sigma_*(g)$ in $L_n(\mathbb{Z}[\pi])$ is defined, traditionally called the surgery obstruction. We have

$$\sigma^*(P_1) - \sigma^*(P_2) = \text{release}(\sigma_*(g))$$

in $L^n(C(\tilde{X}), c(\gamma))$, where σ^* denotes the flexible signature.

To prove this, note that the degree-1 map g induces a splitting of the algebraic Poincaré complex of P_1 into two direct summands; one of these is homotopy equivalent to the algebraic Poincaré complex of P_2, the other is the 'symmetric kernel' of g. It follows easily that

$$\sigma^*(P_1) = \sigma^*(P_2) + \text{release}(\sigma_*(g)),$$

which completes the proof. See also 3.17.

Here are the remaining proofs.

3.19. Proof of 3.13 (an application of the equivariant S-duality theory of **[15]**). In **[15]**, a '$CW\pi$-space' is defined to be a CW-space with base point (distinguished 0-cell) and a cellular π-action which leaves the base point fixed, but permutes the other cells freely. (This is slightly more general than the π-spaces in 3.6.)

A *$CW\pi$-spectrum* is defined along the same lines. A $CW\pi$-space or $CW\pi$-spectrum E is called *finite* if E/π has only finitely many cells; in that case the (reduced) cellular chain complex $\dot{C}(E)$ belongs to $\mathscr{C}_A$ (with $A = \mathbb{Z}[\pi]$).

Let $\partial\Delta_n$ be the union of the proper faces of the simplex Δ_n; regard $(\partial\Delta_n)_+$ as a subspectrum of $(\Delta_n)_+$. The following result is implicit in **[15]**.

(i) *Suppose that E and ∂F are finite CW-spectra, and*

$$\partial g\colon (\partial\Delta_n)_+ \to E \wedge_\pi \partial F$$

is a map of CW-spectra (the smash product being defined in the naive way). Then there exists a CWπ-spectrum F containing ∂F, and a map

$$g\colon (\Delta_n)_+ \to E \wedge_\pi F$$

extending ∂g, such that

$$g/\partial g\colon (\Delta_n)_+/(\partial\Delta_n)_+ \to E \wedge_\pi F/\partial F$$

(where $(\Delta_n)_+/(\partial\Delta_n)_+ \simeq S^n$) is an $S\pi$-duality (see [**15**]*).*

We apply this to the study of the Δ-set $\aleph$ defined before 3.11. Every q-simplex y in $\aleph$ stands for a π-space $\mathrm{Ftr}_y(\{0,1,\ldots,q\})$ (as in 3.6) and a map of spectra

$$(\Delta_q)_+ \to M(\tilde{X} \times_\pi \mathrm{Ftr}_y(\{0,1,\ldots,q\}),\gamma) \simeq M(\tilde{X},\gamma) \wedge_\pi \mathrm{Ftr}_y(\{0,1,\ldots,q\}),$$

etc.; collapsing boundaries gives a map of spectra

$$\eta_y\colon (\Delta_q)_+/(\partial\Delta_q) \to M(\tilde{X},\gamma) \wedge_\pi (\mathrm{Ftr}_y(\{0,1,\ldots,q\})/\bigcup \mathrm{Ftr}_y(V))$$

(where $(\Delta_q)_+/(\partial\Delta_q) \simeq S^q$ and where V ranges over the proper subsets of $\{0,1,\ldots,q\}$). Call y *regular* if η_y is an $S\pi$-duality map. Then we have

(ii) *A q-simplex y is regular if and only if the composition*

$$C(\tilde{X})^{-*} \xrightarrow{\quad f \quad} (\mathrm{Sect}_\aleph; G(-)) \xrightarrow{\text{pullback}} (\mathrm{Sect}_{\Delta_q}; G\cdot \mathrm{ch}_y(-))$$

is a chain homotopy equivalence. Here $\mathrm{ch}_y\colon \Delta_q \to \aleph$ *is the characteristic Δ-map associated with y.*

Proof. Let $\partial G(y) \subset G(y)$ be the chain subcomplex generated by the images $j^{\to}_{z,y}(G(z)) \subset G(y)$, where z ranges over the proper faces of y. Then we have to show that the obvious projection

$$p\colon (\mathrm{Sect}_{\Delta_q}; G\cdot \mathrm{ch}_y(-)) \to \Sigma^q(G(y)/\partial G(y))$$

is a chain homotopy equivalence.

Observe that the skeletal filtration of Δ_q induces a filtration of $(\mathrm{Sect}_{\Delta_q}; G\cdot\mathrm{ch}_y(-))$ by subcomplexes. It is not difficult to construct a similar filtration of the homotopy type of $\Sigma^q(G(y)/\partial G(y))$, and to show that p induces chain homotopy equivalences on the successive quotients. This proves (ii).

(iii) *The regular simplices generate $\aleph$ (that is, every simplex in $\aleph$ is a face of some regular simplex).*

Proof. Given a q-simplex y in $\aleph$, say $y = (\mathrm{Ftr}_y,\ldots)$, let $\mathrm{Cone}(y) = (\mathrm{Ftr}_{\mathrm{Cone}(y)},\ldots)$ be a $(q+1)$-dimensional simplex in $\aleph$ such that $d_0(\mathrm{Cone}(y)) = y$, and such that $\mathrm{Ftr}_{\mathrm{Cone}(y)}(V)$ is a contractible π-space whenever $0 \in V \subset \{0,1,\ldots,q+1\}$. Iterating the construction, define simplices

$$\mathrm{Cone}^n(y) = \mathrm{Cone}(\mathrm{Cone}^{n-1}(y))$$

of dimension $q+n$, for all $n > 1$. Using (i) above it is easy to see that for sufficiently large n, there exists a $(q+n)$-simplex x in $\aleph$ which is regular and such that

$$d_i(x) = d_i(\mathrm{Cone}^n(y)) \quad \text{for } 0 \leqslant i \leqslant q+n.$$

Clearly y is a face of x, which proves (iii).

(iv) *Suppose that g_1: $K_1 \to \aleph$ is a map of Δ-sets, with K_1 finite. Then there exists a diagram of Δ-sets and Δ-maps*

$$\begin{tikzcd} K_1 \arrow[rr, "e"] \arrow[dr, "g_1"'] & & K_2 \arrow[dl, "g_2"] \\ & \aleph & \end{tikzcd}$$

with K_2 finite, such that the composition

$$C(\tilde{X})^{-*} \xrightarrow{f} (\mathrm{Sect}_{\aleph}; G(-)) \to (\mathrm{Sect}_{K_2}; G \cdot g_2(-))$$

is a chain homotopy equivalence.

Proof. Suppose first that K_1 is a polyhedron. This means that K_1 can be embedded (as a Δ-set) in a standard simplex Δ_n. Choose such an embedding e: $K_1 \to \Delta_n := K_2$. Since $\aleph$ is a contractible Kan Δ-set, there exists a Δ-map g_2: $\Delta_n \to \aleph$ such that $g_2 \cdot e = g_1$. We may also assume that g_2 maps Δ_n to a regular simplex in $\aleph$ (otherwise replace Δ_n by a standard simplex of greater dimension, using (iii) above). This proves (iv) in the special case where K_1 is a polyhedron.

Now let K_1 be an arbitrary finite Δ-set. Choose a diagram of Δ-sets and Δ-maps

$$\begin{tikzcd} K_1 \arrow[r, hook, "\simeq"] \arrow[dr, "g_1"'] & M \arrow[d, "h"] & \bar{K}_1 \arrow[l, hook', "\simeq"'] \arrow[dl, "\bar{g}_1"] \\ & \aleph & \end{tikzcd}$$

with M and $\bar{K}_1$ finite, such that $\bar{K}_1$ is a polyhedron and such that the restriction maps

$$(\mathrm{Sect}_{K_1}; G \cdot g_1(-)) \leftarrow (\mathrm{Sect}_M; G \cdot h(-)) \to (\mathrm{Sect}_{\bar{K}_1}; G \cdot \bar{g}_1(-))$$

are chain homotopy equivalences. (Such a diagram is easy to construct; for instance, $\bar{K}_1$ can be taken isomorphic to an iterated barycentric subdivision of K_1.) Since $\bar{K}_1$ is a polyhedron, we can find another diagram

$$\begin{tikzcd} \bar{K}_1 \arrow[rr] \arrow[dr, "\bar{g}_1"'] & & \bar{K}_2 \arrow[dl, "\bar{g}_2"] \\ & \aleph & \end{tikzcd}$$

with $\bar{K}_2$ finite, such that the composition

$$C(\tilde{X})^{-*} \xrightarrow{f} (\mathrm{Sect}_{\aleph}; G(-)) \longrightarrow (\mathrm{Sect}_{\bar{K}_2}; G \cdot \bar{g}_2(-))$$

is a chain homotopy equivalence. Let K_2 be the pushout of $M \leftarrow \bar{K}_1 \to \bar{K}_2$; let $g_2\colon K_2 \to \aleph$ be the amalgam of h and $\bar{g}_2$, and let e be the composition $K_1 \to M \to K_2$. The proof of (iv) is complete.

We can now finish the proof of 3.13 using standard limit arguments. Let $\mathscr{K}$ be the category whose objects are Δ-maps $g\colon K \to \aleph$ with K finite, and with the property that the composite

$$C(\tilde{X})^{-*} \xrightarrow{f} (\mathrm{Sect}_{\aleph}; G(-)) \longrightarrow (\mathrm{Sect}_K; G \cdot g(-))$$

is a chain homotopy equivalence. A morphism from $g\colon K \to \aleph$ to $g'\colon K' \to \aleph$ shall be a Δ-map $h\colon K' \to K$ such that $g \cdot h = g'$. By (iv), the category $\mathscr{K}$ is a small left filtering in the sense of [**1**]; by (iv) again, we may write

$$(\mathrm{Sect}_{\aleph}; G(-)) \cong \varprojlim_{g\colon K \to \aleph} (\mathrm{Sect}_K; G \cdot g(-)),$$

where the inverse limit is taken over $\mathscr{K}$.

Note that all chain maps in this inverse system are homotopy equivalences, by definition of $\mathscr{K}$. So the following implies 3.13.

(v) *The injection*

$$\varprojlim_{g\colon K \to \aleph} (\mathrm{Sect}_K; G \cdot g(-)) \to \underset{g\colon K \to \aleph}{\underleftarrow{\mathrm{holim}}} (\mathrm{Sect}_K; G \cdot g(-))$$

is a chain homotopy equivalence. (*Both limits are taken over* $\mathscr{K}$.)

Explanation and proof. The homotopy inverse limit $\underleftarrow{\mathrm{holim}}$ is defined as in [**1**], mutatis mutandis.

For an object $g\colon K \to \aleph$ of $\mathscr{K}$, let G_g be the $\aleph$-indexed chain complex such that

$$G_g(y) = \prod_{x \in g^{-1}(y)} G(y)$$

whenever y is a simplex in $\aleph$. Then

$$(\mathrm{Sect}_K; G \cdot g(-)) \cong (\mathrm{Sect}_{\aleph}; G_g(-)).$$

Therefore

$$\underleftarrow{\mathrm{holim}} (\mathrm{Sect}_K; G \cdot g(-)) \cong \underleftarrow{\mathrm{holim}} (\mathrm{Sect}_{\aleph}; G_g(-)) \cong (\mathrm{Sect}_{\aleph}; \underleftarrow{\mathrm{holim}}\, G_g(-)),$$

and similarly

$$\varprojlim (\mathrm{Sect}_K; G \cdot g(-)) \cong (\mathrm{Sect}_{\aleph}; \varprojlim G_g(-)).$$

It is now sufficient to show that for each simplex y in $\aleph$, the injection

$$\varprojlim G_g(y) \cong G(y) \to \underleftarrow{\mathrm{holim}}\, G_g(y)$$

is a chain homotopy equivalence. But this is obvious.

We have now shown that f in 3.13 is a homotopy equivalence; the proof for $f^{\to}$ is similar because the functor $C \mapsto \hat{W} \& C$ on $\mathscr{C}_A$ is essentially linear, as is shown in 1.1.

3.20. Proof of 3.15. In 2.19 we saw that $\mathbb{L}^0(C(\tilde{X}), c(\gamma))$ is the underlying space $E(1)$ of a Γ-space $\{E(n)\text{etc.} \mid n \geqslant 0\}$. Much the same argument, with direct sums replaced by disjoint unions, makes $\mathbf{\Omega}_0^p(X,\gamma)$ the underlying space $F(1)$ of a Γ-space $\{F(n)\,\text{etc.} \mid n \geqslant 0\}$.

The symmetric group on n letters $S(n)$ acts on $F(n)$; this action is free for all n (on the complement of the base point), and in particular for $n = 2$. We shall use this fact below.

Our problem is to refine the map $F(1) \to E(1)$ in 3.15 to a family of maps $F(n) \to E(n)$, where $n \geqslant 0$, commuting with the various structure maps which are part of a Γ-space.

It is a good idea to think of a simplex in $E(n)$ as an affair with three levels: the first level involves nothing more complicated than chain complexes (and chain maps), the second nothing more complicated than algebraic Poincaré complexes, and the third involves everything (i.e. algebraic Poincaré complexes with $c(\gamma)$-structure).

(i) With regard to the first and second levels, it is clear what the maps $F(n) \to E(n)$ ought to do (if they are to commute with the Γ-space structure maps).

(ii) It follows that there is essentially just one reasonable map from $F(2)$ to $E(2)$ which is compatible with the given map from $F(1)$ to $E(1)$ (and with the Γ-space structure maps relating $F(1)$ and $F(2)$ on one hand, and $E(1)$, $E(2)$ on the other).

(iii) It follows also that the remaining maps $F(n) \to E(n)$, with $n > 2$, are determined (by induction on n) once the map $F(2) \to E(2)$ has been fixed.

To prove (ii), note that if ${}^1C$, ${}^2C$ are chain complexes in $\mathscr{C}_A$, then the projection

$$\hat{W}\,\&\,({}^1C \oplus {}^2C) \to \hat{W}\,\&\,{}^1C \oplus \hat{W}\,\&\,{}^2C$$

is surjective, and is a chain homotopy equivalence (though not an isomorphism, which accounts for the word 'essentially' in (ii)). To prove (iii), note that if ${}^1C, {}^2C, \ldots, {}^nC$ are chain complexes in $\mathscr{C}_A$, then an element in $\hat{W}\,\&\,(\bigoplus_i {}^iC)$ is determined by its projections to $\hat{W}\,\&\,(\bigoplus_{i \neq k} {}^iC)$ for $k \in \{1, 2, \ldots, n\}$, provided $n > 2$. The proof of 3.15 is complete.

3.21. PROOF of the fact that BG is Kan provided G is a Kan simplicial monoid and $\pi_0(G)$ is a group; see 3.2. Suppose first that we are given a Δ-map $f\colon \partial\Delta_n \to BG$. Then f extends over Δ_n if and only if a certain obstruction in $\pi_{n-2}(G)$ vanishes. For the desired extension corresponds to an element $(g_0, g_1, \ldots, g_{n-1})$ in BG_n; here $g_0, g_1, \ldots, g_{n-2}$ are prescribed, and the $d_i g_{n-1}$ are also prescribed for $0 \leqslant i \leqslant n-1$, because that much information is contained in f. So we are looking for a simplex in G_{n-1} with prescribed boundary (namely g_{n-1}), which amounts to showing that an obstruction in $\pi_{n-2}(G)$ vanishes.

It follows easily that any Δ-map $\mathrm{Horn}_i(\Delta_n) \to BG$ can be extended over Δ_n (extend over the missing ith face first, and then over the whole simplex).

3.A. *Appendix: Chain bundles and sliding forms*

Let $(\pi, w; X, \gamma; \alpha, j)$ be the usual string of data, and let $c(\gamma)$ be the chain bundle on $C(\tilde{X})$ mentioned in 3.4. The geometric description of $c(\gamma)$ given below is inspired by [**14**] rather than [**15**]. We assume that γ is a vector bundle.

In this appendix, define a γ-structure on a smooth manifold N^n (with tangent bundle τ_N) to consist of a classifying map $e\colon N \to X$ and a stable trivialization of $\tau_N \oplus e^{\leftarrow}(\gamma)$.

For $x \in \mathbb{R}$, write $[x] := \max\{z \in \mathbb{Z} \mid z \leqslant x\}$.

3.A.1. DEFINITION. By a filtered γ-thickening of X is meant a sequence $\{P^n \mid n = 0, 1, \ldots\}$ of compact smooth manifolds with boundary (the superscripts indicate the dimension, but simultaneously serve as labels), with γ-structure, such that:

(i) each P^n comes equipped with a map

$$e_n\colon P^n \to ([\tfrac{1}{2}n]\text{-skeleton of } X)$$

which is a homotopy equivalence, and the composite

$$P^n \xrightarrow{e_n} ([\tfrac{1}{2}n]\text{-skeleton of } X) \hookrightarrow X$$

equals the classifying map for the γ-structure on P^n;

(ii) P^n is contained in ∂P^{n+1}, as a smooth codimension-0 submanifold-with-γ-structure. (In particular, the diagrams

$$\begin{tikzcd}
P^n \arrow[r, "e_n"'] \arrow[d, phantom, "\cap"] & {[\tfrac{1}{2}n]\text{-skeleton of } X} \arrow[dd, hook] \\
\partial P^{n+1} \arrow[d, phantom, "\cap"] & \\
P^{n+1} \arrow[r, "e_{n+1}"'] & {[n+\tfrac{1}{2}]\text{-skeleton of } X}
\end{tikzcd}$$

are strictly commutative.)

3.A.2. PROPOSITION. (i) *A filtered γ-thickening of X exists and is unique up to an infinity of higher concordances.*

(ii) *Any filtered γ-thickening of X determines a 0-dimensional cycle in the chain complex $V(C(\tilde{X}))$ of* 1.4 *(with $C(\tilde{X})$ as in* 3.4*). This cycle may be regarded as a chain bundle $c(\gamma)_{\mathrm{new}}$ via* 1.6*; it is well determined up to an infinity of higher concordances, by* (i).

Proof. (i) Existence is clear. The uniqueness half follows from 3.A.3 below (which is equally clear).

(ii) Let $\mathbb{Z}[\alpha]$ be the coefficient sheaf over X whose stalk over $p \in X$ is the free abelian group generated by the points in the fibre (over p) of the principal π-bundle α; the stalk is then a free $\mathbb{Z}[\alpha]$-module on one generator. Denote the induced sheaves over P^n, for $n = 0, 1, \ldots$, by $\mathbb{Z}[\alpha]$ also.

The maps e_{2n}, e_{2n-1} of 3.A.1 give an identification

$$H_n(P^{2n}, P^{2n-1}; \mathbb{Z}[\alpha]) \xrightarrow{\cong} C(\tilde{X})_n,$$

with $P^{2n-1} \subset \partial P^{2n} \subset P^{2n}$. On the other hand, $H_n(P^{2n}, P^{2n-1}; \mathbb{Z}[\alpha])$ carries a sesquilinear form λ_n: its left adjoint is the $\mathbb{Z}[\pi]$-module homomorphism obtained by composing (explanation follows)

$$\begin{tikzcd}
H_n(P^{2n}, P^{2n-1}) \cong H_n(P^{2n}, \underline{P}^{2n-2}) \arrow[r] & H_n(P^{2n}, \underline{P}^{2n-1}) \arrow[d, phantom, "\wr\|"] \\
(\text{dual module of } H_n(P^{2n}, P^{2n-1})) \cong H^n(P^{2n}, P^{2n-1}) &
\end{tikzcd}$$

(Explanation: the coefficients are $\mathbb{Z}[\alpha]$ throughout; $\underline{P}^n$ is the closed complement of P^n in ∂P^{n+1}, for all n; the isomorphism in the top row is induced by the inclusion $\underline{P}^{2n-2} \hookrightarrow P^{2n-1}$, which is a homology equivalence by 3.A.1 (i); the other homomorphism in the top row is induced by the inclusion $\underline{P}^{2n-2} \hookrightarrow \underline{P}^{2n-1}$, and the vertical isomorphism is Poincaré duality.)

This sliding procedure is due to Quinn (see [**14**]). Combining these two observations, we obtain a sequence of sesquilinear forms

$$\{\lambda_n\colon C(\tilde{X})_n \times C(\tilde{X})_n \to \mathbb{Z}[\pi] \mid n = 0, 1, \ldots\};$$

inspection shows that the sequence is a cycle in $V(C(\tilde{X}))_0$ (see also [**20**]).

3.A.3. Lemma. *Let X and γ be the same as ever, let X' be a CW-subcomplex of X, and γ' the restriction of γ to X'. Then any filtered γ'-thickening of X' (say $\{P'^n \mid n = 0, 1, \ldots\}$) can be extended to a filtered γ-thickening of X (say $\{P^n \mid n = 0, 1, \ldots\}$), in the sense that P'^n is contained in P^n as a codimension-0 submanifold-with-γ-structure (for all n).*

Observe that under these conditions the 0-cycles

$$c(\gamma')_{\text{new}} \in V(C(\tilde{X}')), \quad c(\gamma)_{\text{new}} \in V(C(\tilde{X}))$$

(constructed as in 3.A.2 (ii) from $\{P'^n\}$ and $\{P^n\}$ respectively) are such that

$$c(\gamma')_{\text{new}} = i^{-}(c(\gamma)_{\text{new}}),$$

if $i\colon X' \to X$ is the inclusion.

Applying 3.A.3 to the inclusion $X \times \{0, 1\} \to X \times [0, 1]$ proves that filtered thickenings are 'unique up to concordance'; similarly for higher concordances, which proves 3.A.2 (i).

3.A.4. Theorem. *We have $c(\gamma)_{\text{new}} = c(\gamma)$, up to an infinity of higher concordances.*

The proof will be given in II, §4.A. (Strictly speaking, it is first necessary to extend 3.4 from simplicial sets X to CW-spaces X, but that causes no serious problems.)

References

1. A. Bousfield and D. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics 304 (Springer, Berlin, 1972).
2. W. Browder, 'The Kervaire invariant of framed manifolds and its generalisations', *Ann. of Math.*, 90 (1969), 157–186.
3. E. H. Brown, Jr., 'Generalisations of the Kervaire invariant', *Ann. of Math.*, 95 (1972), 368–383.
4. E. H. Brown, Jr., 'A product formula for generalisations of the Kervaire invariant', *Trans. Amer. Math. Soc.*, 216 (1976), 295–311.
5. S. Buoncristiano, C. Rourke, and B. Sanderson, *A geometric approach to homology theory*, London Mathematical Society Lecture Note Series 18 (Cambridge University Press, 1976).
6. E. B. Curtis, 'Simplicial homotopy theory', *Adv. in Math.*, 6 (1971), 107–209.
7. A. Dold, *Halbexakte Homotopiefunktoren*, Lecture Notes in Mathematics 12 (Springer, Berlin, 1970).
8. A. Dold and R. Thom, 'Quasifaserungen und unendliche symmetrische Produkte', *Ann. of Math.*, 67 (1958), 239–281.
9. S. Eilenberg and S. MacLane, 'Acyclic models', *Amer. J. Math.*, 75 (1953), 189–199.
10. K. W. Gruenberg, *Cohomological topics in group theory*, Lecture Notes in Mathematics 143 (Springer, Berlin, 1970).
11. D. Kan, 'On homotopy theory and c.s.s. groups', *Ann. of Math.*, 68 (1958), 38–53.
12. M. Kervaire, 'A manifold which does not admit any differentiable structure', *Comment. Math. Helv.*, 34 (1960), 257–270.
13. A. S. Mishchenko, 'Homotopy invariants of non-simply-connected manifolds III. Higher signatures', *Izv. Akad. Nauk. SSSR Ser. Mat.*, 35 (1971), 1316–1355.

14. F. Quinn, 'Open book decompositions, and the bordism of automorphisms', *Topology* 18 (1979), 55–73.

15. A. Ranicki, 'The algebraic theory of surgery I, II', *Proc. London Math. Soc.* (3), 40 (1980), 87–283.

16. A. Ranicki, *Exact sequences in the algebraic theory of surgery*, Mathematics Notes (Princeton University Press, 1981).

17. C. Rourke and B. Sanderson, 'Δ-sets I. Homotopy theory', *Quart. J. Math. Oxford* (2), 22 (1971), 321–338.

18. G. Segal, 'Categories and cohomology theories', *Topology*, 13 (1974), 293–312.

19. R. M. Switzer, *Algebraic topology—homotopy and homology* (Springer, Berlin, 1975).

20. C. T. C. Wall, 'On the axiomatic foundations of the theory of hermitian forms', *Proc. Cambridge Philos. Soc.*, 67 (1970), 243–250.

21. C. T. C. Wall, *Surgery on compact manifolds* (Academic Press, London, 1970).

22. M. Weiss, 'Surgery and bordism invariants', *Bull. Amer. Math. Soc.*, 9 (1983), 223–226.

23. M. Weiss, 'Surgery and the generalized Kervaire invariant, II', *Proc. London Math. Soc.* (3), 51 (1985), 193–230.

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SURGERY AND THE GENERALIZED KERVAIRE INVARIANT, II

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[Received 4 March 1983—Revised 24 January 1984]

4. *Algebraic Thom complexes and algebraic thickenings*

This and the following section (on algebraic surgery) are so close in spirit to [**11**, Part I, §4] as to be almost superfluous. Here are two remarks for justification:

we need at least a glimpse of the theory of algebraic surgery on algebraic Poincaré complexes 'with $\mathscr{B}$-structure' (see [**14**]—hereafter denoted 'I'—Definition 2.6);

I attempt to save formulae by giving a categorical description of the algebraic thickening construction.

Recall from I, 2.21 (iii), (iv) that there is a long exact sequence

$$\ldots \longrightarrow \hat{L}^{n+1}(B,\mathscr{B}) \longrightarrow L_n(A) \xrightarrow{\text{release}} L^n(B,\mathscr{B}) \longrightarrow \hat{L}^n(B,\mathscr{B}) \longrightarrow \ldots$$

for any chain complex B in $\mathscr{C}_A$ and chain bundle $\mathscr{B}$ on B. Our main result in this section is an expression of the relative group $\hat{L}^n(B,\mathscr{B})$ as a bordism group of *single but degenerate objects*, as opposed to the standard description in terms of non-degenerate pairs.

Most of this section is written in terms of *unrestricted* (UR for short) algebraic Poincaré complexes, higher bordisms, etc.; see I, 2.21 (ii).

4.1. Definition. An *n-dimensional UR symmetric chain complex* is a pair (C,φ) in which C is a chain complex in $\mathscr{C}_A$ and φ is an n-dimensional cycle in $W\&C$.

4.2. Definition. Let $(f\colon C\to D,(\psi,\varphi))$ be an n-dimensional UR algebraic Poincaré pair (over A; see I, 2.2). Write $\psi^?$ for the image of ψ under the map

$$W\&D \to W\&(D/\mathrm{im}(f))$$

induced by the projection $D\to D/\mathrm{im}(f)$. Then $(D/\mathrm{im}(f),\ \psi^?)$ is an n-dimensional UR symmetric chain complex, called the *algebraic Thom complex* of the pair $(f\colon C\to D,(\psi,\varphi))$.

The passage from an algebraic Poincaré pair to its algebraic Thom complex has a geometric analogue, namely the passage from a geometric Poincaré pair $(N,\partial N)$ to its *Thom complex* $N/\partial N$. (If N is equipped with a principal π-bundle, one would be interested in the Thom π-complex $\tilde{N}/\partial\tilde{N}$ instead, where $\tilde{N}$ is the total space. See [**11**] for a detailed description of this analogy.)

4.3. The theme (of this section). *The passage from an UR algebraic Poincaré pair to its algebraic Thom complex is reversible.*

A.M.S. (1980) subject classification: 18F25.
Proc. London Math. Soc. (3), 51 (1985), 193–230.

We shall first state this with more precision (in 4.6, after some preparatory definitions), then prove it, and then list some variants. The appendix 4.A contains a proof of I, 3.A.4, as a first application of the theory.

4.4. DEFINITION. An n-dimensional UR symmetric pair $(f\colon C \to D, (\psi, \varphi))$ consists of a cofibration $f\colon C \to D$ in $\mathscr{C}_A$, an n-chain $\psi \in W\&D$ and an $(n-1)$-cycle $\varphi \in W\&C$ so that

$$f^{\to}(\varphi) = d(\psi) \quad \text{in } (W\&D)_{n-1}$$

(or, if you prefer, $f^{\to}(\varphi) = -d(\psi)$; compare I, 2.4).

4.5. DEFINITION. Let $(E, \psi^?)$ be an n-dimensional UR symmetric chain complex. An UR symmetric pair over $(E, \psi^?)$ consists of

(i) an n-dimensional UR symmetric pair $(f\colon C \to D, (\psi, \varphi))$,

(ii) a chain map $p\colon D \to E$ which is such that the sequence of chain maps

$$0 \longrightarrow C \xrightarrow{f} D \xrightarrow{p} E \longrightarrow 0$$

is short exact, and such that $p^{\to}(\psi) = \psi^?$ in $W\&E$.

For a fixed n-dimensional UR symmetric chain complex $(E, \psi^?)$, the UR symmetric pairs over $(E, \psi^?)$ form a category $\mathscr{P}\downarrow(E, \psi^?)$ in the following way. Let

$$P_1 = (f\colon C \to D, (\psi, \varphi)), \quad p\colon D \to E,$$

and

$$P_2 = (f'\colon C' \to D', (\psi', \varphi')), \quad p'\colon D' \to E$$

be two UR symmetric pairs over $(E, \psi^?)$. The set of morphisms in $\mathscr{P}\downarrow(E, \psi^?)$ from P_1 to P_2 is to be a certain subset of the set

$$\mathscr{F}(p, p') := \{\text{fibre homotopy classes of chain maps } g\colon D \to D' \text{ so that } p' \cdot g = p\}.$$

(A fibre homotopy connecting two maps $g_1, g_2\colon D \to D'$ so that $p' \cdot g_1 = p = p' \cdot g_2$ is a homotopy which factors through $\ker(p') \subset D'$.) Let

$$\begin{tikzcd} D \arrow[rr, "g"] \arrow[dr, "p"'] & & D' \arrow[dl, "p'"] \\ & E & \end{tikzcd}$$

represent an element $[g]$ in $\mathscr{F}(p, p')$; then

$$(\psi' - g^{\to}(\psi), \varphi' - g^{\to}(\varphi))$$

represents a homology class defect$([g])$ in $H_n(\ker(p'^{\to})/\mathrm{im}(f'^{\to}))$, with

$$f'^{\to}\colon W\&C' \to W\&D', \quad p'^{\to}\colon W\&D' \to W\&E.$$

We regard $[g] \in \mathscr{F}(p, p')$ as a morphism from P_1 to P_2 precisely if defect$([g]) = 0$.

The promised reformulation of 4.3 now reads as follows:

4.6. THEOREM. (i) *The category $\mathscr{P}\downarrow(E, \psi^?)$ has an initial object* (*that is, an object admitting precisely one morphism to any other object*).

(ii) *An object in this category, say* $(f\colon C \to D, (\psi, \varphi))$, $p\colon D \to E$, *is initial if and only if* $(f\colon C \to D, (\psi, \varphi))$ *is an UR algebraic Poincaré pair.*

Proof of (i). It helps to consider a small classification problem first. Fix a short exact sequence of chain maps

$$0 \longrightarrow C \xrightarrow{f} D \xrightarrow{p} E \longrightarrow 0$$

in $\mathscr{C}_A$ (with E as in 4.6). We wish to classify the various ways in which this can be enhanced to an UR symmetric pair over $(E, \psi^?)$; that is, we wish to classify, up to a suitable notion of equivalence, pairs (ψ, φ) such that

$$(f\colon C \to D, (\psi, \varphi)), \quad p\colon D \to E$$

is an UR symmetric pair over $(E, \psi^?)$.

(Regard two such pairs (ψ, φ) and (ψ', φ') as equivalent if the identity map $D \to D$ is a morphism in $\mathscr{P} \downarrow (E, \psi^?)$ from $(f\colon C \to D, (\psi, \varphi))$, $p\colon D \to E$ to $(f\colon C \to D, (\psi', \varphi'))$, $p\colon D \to E$.)

4.7. Lemma. *The set of equivalence classes of such pairs* (ψ, φ) *is non-empty if and only if a certain obstruction in* $H_{n-1}(E^t \otimes_A C)$ *vanishes; and in that case, the group* $H_n(E^t \otimes_A C)$ *acts on this set in a sharply transitive manner.*

Proof. We use the diagram

$$\begin{array}{ccccc} W\&C & & & & \\ \big\downarrow{\scriptstyle f^{\to}} & & & & \\ \ker(p^{\to}) & \longrightarrow & W\&D & \xrightarrow{p^{\to}} & W\&E \\ \big\downarrow & & & & \\ \ker(p^{\to})/\mathrm{im}(f^{\to}) & & & & \end{array}$$

in which both row and column are short exact. Let $[\psi^?] \in Q^n(E) = H_n(W\&E)$ be the class of $\psi^?$. It is clear that the short exact sequence $0 \to C \to D \to E \to 0$ can be enhanced to an UR symmetric pair over $(E, \psi^?)$ if and only if the class

$$\partial[\psi^?] \in H_{n-1}(\ker(p^{\to}))$$

comes from a class in $Q^{n-1}(C) = H_{n-1}(W\&C)$, which is the case if and only if the image of $\partial[\psi^?]$ in $H_{n-1}(\ker(p^{\to})/\mathrm{im}(f^{\to}))$ is zero. A similar argument shows that the group $H_n(\ker(p^{\to})/\mathrm{im}(f^{\to}))$ acts in a sharply transitive way on the set of equivalence classes of 'enhancements'.

It remains to be seen that the chain complexes $\ker(p^{\to})/\mathrm{im}(f^{\to})$ and $E^t \otimes_A C$ are homotopy equivalent. A related idea was used in the proof of I, 1.1 (iii):

$$\ker(p^{\to})/\mathrm{im}(f^{\to}) \cong \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, G);$$

here $G := \ker(p \otimes p)/\mathrm{im}(f \otimes f)$, with

$$f \otimes f\colon C^t \otimes_A C \to D^t \otimes_A D, \quad p \otimes p\colon D^t \otimes_A D \to E^t \otimes_A E.$$

But now G is canonically isomorphic, as a $\mathbb{Z}[Z_2]$-chain complex, to the 'coinduced' chain complex $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[Z_2], E^t \otimes_A C)$ (obtained from $E^t \otimes_A C$ by applying $\mathrm{Hom}(\mathbb{Z}[Z_2], -)$ in each dimension). Therefore

$$\ker(p^{\rightarrow})/\mathrm{im}(f^{\rightarrow}) \cong \mathrm{Hom}_{\mathbb{Z}}(W, E^t \otimes_A C) \simeq E^t \otimes_A C,$$

which proves the lemma.

We shall use Lemma 4.7 to give a simpler description of the category $\mathscr{P}\downarrow(E, \psi^?)$. Let $\mathscr{A}(E, \psi^?)$ be the following category: an object of $\mathscr{A}(E, \psi^?)$ is a triple $(F, j, [h])$ in which F denotes a chain complex in $\mathscr{C}_A$, j: $E \to F$ is a cofibration, and $[h]$ is an equivalence class of chain homotopies from

$$j \cdot \psi_0^?: \Sigma^n(E^{-*}) \to F$$

to 0. (Call h, h' equivalent if the difference chain map

$$h - h': \Sigma^{n+1}(E^{-*}) \to F$$

is nullhomotopic.) Given two such objects $(F, j, [h])$, $(F', j', [h'])$, a morphism from the first to the second is a cofibre homotopy class of chain maps g: $F \to F'$ making the diagram

$$\begin{tikzcd} & F \arrow[dd, "g"] \\ E \arrow[ur, "j"] \arrow[dr, "j'"'] & \\ & F' \end{tikzcd}$$

commutative, and so that $g^{\rightarrow}([h]) = [h']$.

4.8. Lemma. *The categories $\mathscr{A}(E, \psi^?)$ and $\mathscr{P}\downarrow(E, \psi^?)$ are equivalent.*

Proof. This is little more than a reformulation of 4.7. Given an object $(F, j, [h])$ in $\mathscr{A}(E, \psi^?)$, there is the short exact sequence in $\mathscr{C}_A$,

$$0 \to \Sigma^{-1}F \to \Sigma^{-1}(\mathrm{Cone}(j)) \to E \to 0.$$

By Lemma 4.7 the obstruction to 'enhancing' this short exact sequence to an UR symmetric pair over $(E, \psi^?)$ is a certain class in

$$H_{n-1}(E^t \otimes_A \Sigma^{-1}F) = H_{n-1}(\mathrm{Hom}_A(E^{-*}, \Sigma^{-1}F));$$

and inspection shows that this class is represented by the chain map

$$\Sigma^{-1}(j \cdot \psi_0^?): \Sigma^{n-1}(E^{-*}) \to \Sigma^{-1}F.$$

But this is nullhomotopic; in fact, $[h]$ gives us a preferred class of nullhomotopies. It is easy to deduce that every object in $\mathscr{A}(E, \psi^?)$ gives rise to an object in $\mathscr{P}\downarrow(E, \psi^?)$, well defined up to isomorphism (in $\mathscr{P}\downarrow(E, \psi^?)$); the construction can be extended to morphisms, and yields the required equivalence of categories.

We return to the proof of 4.6(i). We are now reduced to showing that the category $\mathscr{A}(E, \psi^?)$ of 4.8 has an initial object $(F, j, [h])$. But this is clear: put

$$F := (\text{Cone of } \psi_0^?: \Sigma^n(E^{-*}) \to E), \quad \text{etc.}$$

Proof of 4.6(ii). One direction is straightforward, since we have an explicit description of an initial object

$$(f^{\text{in}}: C^{\text{in}} \to D^{\text{in}}, (\psi^{\text{in}}, \varphi^{\text{in}})), \quad p^{\text{in}}: D^{\text{in}} \to E,$$

from which it can be seen that $(f^{\text{in}}: C^{\text{in}} \to D^{\text{in}}, (\psi^{\text{in}}, \varphi^{\text{in}}))$ is an UR algebraic Poincaré pair. (See also 4.9 below.) Conversely, let

$$Y := (f: C \to D, (\psi, \varphi)), \quad p: D \to E$$

be an object in $\mathscr{P}\downarrow(E, \psi^?)$ such that $(f: C \to D, (\psi, \varphi))$ is an UR algebraic Poincaré pair. Then the unique morphism from the initial object to Y induces a degree-1 map of UR algebraic Poincaré pairs

$$\begin{array}{c}(f^{\text{in}}: C^{\text{in}} \to D^{\text{in}}, (\psi^{\text{in}}, \varphi^{\text{in}}))\\ \Big\downarrow g\\ (f: C \to D, (\psi, \varphi))\end{array}$$

('degree-1' means that $(\psi - g^{\to}(\psi^{\text{in}}), \varphi - g^{\to}(\varphi^{\text{in}}))$ represents the zero class in $H_n(f^{\to})$, with $f^{\to}: W\&C \to W\&D$). We know further that the induced chain map

$$E \cong D^{\text{im}}/\text{im}(f^{\text{in}}) \to D/\text{im}(f) \cong E$$

is a chain isomorphism. It follows easily that the unique morphism in question is an isomorphism in $\mathscr{P}\downarrow(E, \psi^?)$.

4.9. Definition and Description. The UR algebraic Poincaré pair in 4.6 is called the *algebraic Poincaré thickening* of $(E, \psi^?)$.

An explicit description is as follows. Let C and D be the mapping cones of

$$\Sigma^{-1}\psi_0^?: \Sigma^{n-1}(E^{-*}) \to \Sigma^{-1}E$$

and

$$\Sigma^{-1}\psi_0^? \oplus \text{id}: \Sigma^{n-1}(E^{-*}) \oplus \Sigma^{-1}E \to \Sigma^{-1}E$$

respectively. Thus $C \subset D$, and $D/C \cong E$. Define an n-chain $\psi \in W\&D$ and an $(n-1)$-cycle $\varphi \in W\&C$ by letting (explanation follows)

(i)

$$\psi_0 = \begin{pmatrix} 0 & 0 & -\text{id} \\ 0 & 0 & 0 \\ (-)^{pq-1}\cdot\text{id} & (-)^q\cdot T\psi_1^? & \psi_0^? \end{pmatrix},$$

(ii) for $s > 0$,

$$\psi_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & (-)^{q-s}\cdot T\psi_{s+1}^? & \psi_s^? \end{pmatrix},$$

(iii)

$$\varphi_0 = \begin{pmatrix} 0 & (-)^{p-1}\cdot\text{id} \\ (-)^{pq-p-1}\cdot\text{id} & (-)^{q-1}\cdot T\psi_1^? \end{pmatrix},$$

(iv) for $s > 0$,

$$\varphi_s = \begin{pmatrix} 0 & 0 \\ 0 & (-)^{q-1-s} \cdot T\psi^?_{s+1} \end{pmatrix}.$$

(Explanation: $\psi^?_s$ and $T\psi^?_s$ are considered as chains in $\mathrm{Hom}_A(E^{-*}, E)$. Matrices (i) and (ii) describe

$$\begin{array}{ccc} \psi_s\colon D^p & \longrightarrow & D_q \\ \wr\| & & \wr\| \\ (E^*_{n-p} \oplus E_{p+1} \oplus E_p)^* & & E^*_{n-q} \oplus E_{q+1} \oplus E_q \end{array};$$

it is understood that, for fixed s, p and q range over all pairs so that $p+q = n+s$. Similarly, matrices (iii) and (iv) describe

$$\begin{array}{ccc} \varphi_s\colon C^p & \longrightarrow & C_q \\ \wr\| & & \wr\| \\ (E^*_{n-p} \oplus E_{p+1})^* & & E^*_{n-q} \oplus E_{q+1} \end{array}$$

with $p+q = n-1+s$.)

Now $(C \to D, (\psi, \varphi))$ is an UR algebraic Poincaré pair, and the projection $p\colon D \to E$ (with kernel C) is such that

$$p^{\to}(\psi) = \psi^? \in (W \,\&\, E)_n.$$

Hence $(C \to D, (\psi, \varphi))$ is 'the' algebraic Poincaré thickening of $(E, \psi^?)$.

4.10. First variation on 4.3. *The relative structureless case.* Let (Fun, Φ) be a higher bordism of UR algebraic Poincaré complexes, of dimension n and order 2 (that is, modelled on the 2-simplex; see I, 2.3), with the property that

$$\mathrm{Fun}(\{0,1\}) = 0.$$

(Such a thing is commonly called a triad; it is just an UR algebraic Poincaré pair whose boundary is split into two halves.) 'Collapsing the edge $\{0,2\}$', that is, collapsing one half of the boundary, gives an UR symmetric pair (cf. 4.4)

$$(\mathrm{Fun}(\{1,2\})/\mathrm{Fun}(\{2\}) \to \mathrm{Fun}(\{0,1,2\})/\mathrm{Fun}(\{0,2\}), (\Phi^?(\{0,1,2\}), \Phi^?(\{1,2\})))$$

(in shorthand notation; certain cofibrations have been written as inclusions, and $\Phi^?(\{0,1,2\})$, $\Phi^?(\{1,2\})$ are the images of $\Phi(\{0,1,2\})$ and $\Phi(\{1,2\})$ respectively).

This collapsing process, or the passage from the triad (Fun, Φ) *to an UR symmetric pair (of the same dimension n), is reversible. That is, any UR symmetric pair can be obtained from an UR algebraic Poincaré triad by collapsing, as above; and the triad is essentially determined by the UR symmetric pair.*

The UR algebraic Poincaré triad is called the *algebraic Poincaré thickening* of the UR symmetric pair.

4.11. Remark. Given an UR symmetric pair $(C \to D, (\psi^?, \varphi^?))$, any algebraic Poincaré thickening (in the sense of 4.9) of the boundary UR symmetric complex $(C, \varphi^?)$ can be extended, in an essentially unique way, to an algebraic Poincaré thickening (in the sense of 4.10) of $(C \to D, (\psi^?, \varphi^?))$.

The proofs of 4.10 and 4.11 are similar to that of 4.3.

For the rest of the section, fix a chain complex B in $\mathscr{C}_A$ and a chain bundle $\mathscr{E}$ on B.

We will work towards an analogue of 4.3 for an UR algebraic Poincaré pair $(f\colon C \to D, (\psi, \varphi))$ with a $\mathscr{E}$-structure (g, z) (see I, 2.7; pairs are regarded as bordisms of order 1, as in I, 2.4)—under the very restrictive assumption that *the classifying map g vanishes on the boundary*. (This means that $g \cdot f\colon C \to B$ is zero.)

There are two reasons for making such an assumption. One is technical: quite simply, we wish to collapse the boundary of our UR algebraic Poincaré pair, and the classifying map g would be in the way if it did not vanish on the boundary. The second and more important reason is that an n-dimensional UR algebraic Poincaré pair $(f\colon C \to D, (\psi, \varphi))$ as above (with a $\mathscr{E}$-structure (g, z) such that $g \cdot f = 0$) is a *typical representative of an element in $\hat{L}^n(B, \mathscr{E})$*. See I, 2.21 (iii), (iv), and I, 2.17.

To begin with, here is a stimulating definition:

4.12. DEFINITION. A *normal structure* on an n-dimensional UR symmetric complex (C, φ) is a pair (c, z) in which

c denotes a chain bundle on C,

z is a 'clutching homology' on $(\hat{W} \& C)_{n+1}$ from $\mathfrak{S}^n \cdot \varphi_0^{\to}(c) \in (\hat{W} \& C)_n$ to $J(\varphi) \in (\hat{W} \& C)_n$.

(Explanation: φ_0 is regarded as a chain map from C^{-*} to $\Sigma^{-n}C$, inducing $\varphi_0^{\to}\colon \hat{W} \& C^{-*} \to \hat{W} \& (\Sigma^{-n}C)$; $\mathfrak{S}^n$ is the n-fold iteration of the explicit suspension isomorphism of I, 1.2(b).)

4.13. REMARKS. (i) It will be shown in § 7 that 'UR symmetric chain complexes with normal structure' are the algebraic counterparts of '(geometric) normal spaces' (see [**10**] or § 7 of this paper)—just as (UR) algebraic Poincaré complexes are the algebraic counterparts of Poincaré spaces.

(ii) An n-dimensional UR algebraic Poincaré complex (C, φ) can always be regarded as an n-dimensional UR symmetric complex (C, φ) with normal structure (c, z). Indeed, c and z exist and are essentially unique because $\varphi_0\colon C^{-*} \to \Sigma^{-n}C$ is a chain homotopy equivalence. Note that c is the normal chain bundle of the UR algebraic Poincaré complex. (See the sequel to I, 2.6.)

4.14. DEFINITION. A *normal $\mathscr{E}$-structure* on an n-dimensional UR symmetric complex (C, φ) is a pair (g, z) in which

g is a chain map from C to B,

$z \in (\hat{W} \& C)_{n+1}$ is a clutching homology' from $\mathfrak{S}^n \cdot \varphi_0^{\to}(g^{\leftarrow}(\mathscr{E})) \in (\hat{W} \& C)_n$ to $J(\varphi) \in (\hat{W} \& C)_n$.

4.15. REMARKS. (i) A normal $\mathscr{E}$-structure (g, z) on (C, φ) induces a normal structure $(g^{\leftarrow}(\mathscr{E}), z)$ on (C, φ).

(ii) If (C, φ) happens to be an UR algebraic Poincaré complex, then a normal $\mathscr{E}$-structure on (C, φ) (in the sense of 4.14) is the same as a $\mathscr{E}$-structure on (C, φ) (in the sense of I, 2.6).

(iii) Define a *higher bordism* (Fun, Φ) *of UR symmetric complexes* (of dimension n and order q) like a higher bordism of UR algebraic Poincaré complexes, dropping only the non-degeneracy condition I, 2.3(iv). In view of the similarity between 4.14 and I, 2.6, it is clear how to define a normal $\mathscr{E}$-structure (g, z) on such a higher bordism of UR symmetric complexes (namely, by imitating I, 2.7). We will only need UR symmetric pairs with normal $\mathscr{E}$-structure.

(iv) It is also possible to define a normal structure (cf. 4.12) on an UR symmetric pair or on a higher bordism of UR symmetric complexes (but this notion will not be used much).

4.16. SECOND VARIATION ON 4.3. *The absolute case, with $\mathscr{E}$-structure.* Let $(f\colon C \to D, (\psi, \varphi))$ be an UR algebraic Poincaré pair with $\mathscr{E}$-structure (g, z). Suppose that $g \cdot f = 0$.

Then the algebraic Thom complex of the pair (the UR symmetric complex $(D/\mathrm{im}(f), \psi^?)$) carries a canonical normal $\mathscr{E}$-structure $(g^?, z^?)$. *The passage from*

the UR algebraic Poincaré pair $(f\colon C \to D, (\psi, \varphi))$ with $\mathscr{E}$-structure (g, z) such that $g \cdot f = 0$

to

the UR symmetric complex $(D/\mathrm{im}(f), \psi^?)$ with normal $\mathscr{E}$-structure $(g^?, z^?)$

is reversible.

4.17. THIRD VARIATION ON 4.3. *The relative case, with $\mathscr{E}$-structure.* Let (Fun, Φ) be an UR algebraic Poincaré triad (i.e. a higher bordism of order 2 such that $\mathrm{Fun}(\{0,1\}) = 0$). Suppose that a $\mathscr{E}$-structure (g, z) on (Fun, Φ) is given so that the composite

$$\mathrm{Fun}(\{0,2\}) \longrightarrow \mathrm{Fun}(\{0,1,2\}) \xrightarrow{g} B$$

is zero. Then the UR symmetric pair

$$(\mathrm{Fun}(\{1,2\})/\mathrm{Fun}(\{2\}) \to \mathrm{Fun}(\{0,1,2\})/\mathrm{Fun}(\{0,2\}), (\Phi^?(\{0,1,2\}), \Phi^?(\{1,2\})))$$

(compare 4.10) carries a canonical normal $\mathscr{E}$-structure. *The passage from*

the UR algebraic Poincaré triad (Fun, Φ) *with $\mathscr{E}$-structure* (g, z) *such that the classifying map g vanishes on* $\mathrm{Fun}(\{0,2\})$

to

an UR symmetric pair with normal $\mathscr{E}$-structure

is reversible.

Proof of 4.16. Recall first that, in order to be able to speak of a $\mathscr{E}$-structure (g, z), we must consider the UR algebraic Poincaré pair as a bordism of order 1 (as in I, 2.3 and 2.7); also, that $z = \{z(S) \mid S \subset \{0,1\}\}$ is a collection. Let

(i) $z^?$ be the image of $z(\{0,1\})$ under the map $p^{\to}\colon \hat{W} \,\&\, D \to \hat{W} \,\&\, (D/\mathrm{im}(f))$ where $p\colon D \to D/\mathrm{im}(f)$ denotes the projection; and define $g^?\colon D/\mathrm{im}(f) \to B$ so that

(ii) $g = g^? \cdot p$.

Then I, 2.9 implies that $(g^?, z^?)$ is a normal $\mathscr{E}$-structure on $(D/\mathrm{im}(f), \psi^?)$.

To see that 'the passage ... is reversible', note that $(f\colon C \to D, (\psi, \varphi))$ can be recovered as the algebraic Poincaré thickening of $(D/\mathrm{im}(f), \psi^?)$, according to I, 4.3. It is not hard to see that there exists an essentially unique $\mathscr{E}$-structure (g, z) on $(f\colon C \to D, (\psi, \varphi))$ satisfying equations (i) and (ii) just above—in other words, (g, z) can be recovered from $(g^?, z^?)$. (If in trouble, remember I, 1.1 (iii).)

The proof of 4.17 is similar.

4.18. COROLLARY. *The following notions are interchangeable:*
(a) *n-dimensional UR symmetric complex with normal structure (see* 4.12*);*
(b) *n-dimensional UR algebraic Poincaré pair $(C \to D, (\psi, \varphi))$ whose boundary (C, φ) is equipped with a* 0*-structure.*

(Explanation: '0' is the unique chain bundle on the trivial chain complex 0_A in $\mathscr{C}_A$, discussed in I, 2.19. Note that (C, φ), with its 0-structure, represents an element in the Wall group $L_{n-1}(A)$. The corollary is useful in understanding (geometric) normal spaces, obstructions to Poincaré transversality, etc. **[10]**; more in §7.)

Proof. Apply 4.16 with $\mathscr{E}$ equal to the universal chain bundle $\mathscr{E}^\infty$ of I, 2.A.4, and $B = B^\infty$.

(A normal structure on an UR symmetric complex is practically the same as a normal $\mathscr{E}^\infty$-structure; similarly, a 0-structure on the boundary (C, φ) of an UR algebraic Poincaré pair $(C \to D, (\psi, \varphi))$ is as good as a $\mathscr{E}^\infty$-structure (g, z) on $(C \to D, (\psi, \varphi))$ such that g vanishes on the boundary C.)

4.19. COROLLARY. *The relative group $\hat{L}^n(B, \mathscr{E})$ is isomorphic to the bordism group of n-dimensional UR symmetric complexes with normal $\mathscr{E}$-structure.*

4.A. *Appendix: Chain bundles and sliding forms again*

The proof of I, 3.A.4 to be given here begins with yet another variation on 4.3. This time a different 'model' is required: the 3-disk D^3_+, regarded as a CW-complex with one 3-cell (whose closure is D^3_+), two 2-cells (with closures D^2_+, D^2_-), two 1-cells (with closures D^1_+, D^1_-), and two 0-cells D^0_+, D^0_-. The two 0-cells are positively oriented; the remaining cells are oriented so that the inclusions $D^0_+ \hookrightarrow D^1_+$, $D^1_+ \hookrightarrow D^2_+$, $D^2_+ \hookrightarrow D^3_+$ and $D^0_- \hookrightarrow D^1_-$, $D^1_- \hookrightarrow D^2_-$ are orientation-preserving.

4.A.1. FOURTH VARIATION ON 4.3. Let (Fun, Φ) be an n-dimensional bordism of UR algebraic Poincaré complexes modelled on D^3_+ (just as the higher bordisms of I, 2.3 were modelled on the standard simplex Δ_q; in particular, Fun is now a functor from the category of faces, that is, closures of cells, of D^3_+, to $\mathscr{C}_A$).

Assume that $\mathrm{Fun}(D^0_-) = 0$, and $n > 1$. Then

$$(\mathrm{Fun}(D^1_+)/\mathrm{Fun}(D^0_+) \to \mathrm{Fun}(D^2_+)/\mathrm{Fun}(D^1_-), (\Phi^?(D^2_+), \Phi^?(D^1_+)))$$

is an $(n-1)$-dimensional UR symmetric pair, and

$$(\mathrm{Fun}(D^2_+)/(\mathrm{Fun}(D^1_+) \oplus \mathrm{Fun}(D^1_-)) \to \mathrm{Fun}(D^3_+)/\mathrm{Fun}(D^2_-), (\Phi^{??}(D^3), \Phi^{??}(D^2)))$$

is an n-dimensional UR symmetric pair.

These two UR symmetric pairs are related as follows: the UR symmetric chain complex obtained by collapsing the boundary of the first is identified with the boundary of the second. *The passage from*

the n-dimensional bordism of UR algebraic Poincaré complexes (Fun, Φ), *modelled on D^3_+ and such that* $\mathrm{Fun}(D^0_-) = 0$

to

the two related UR symmetric pairs above

is reversible.

The proof of 4.A.1 can be modelled on that of 4.3, but it is also amusing to derive it from 4.10 and 4.11.

4.A.2. OUTLINE (of the proof of I, 3.A.4). We keep the notation of I, 3.A. Fix a filtered thickening $\{P^n \mid n = 0, 1, \ldots\}$ of X. Each P^n is a manifold with boundary, so it gives rise to an algebraic Poincaré pair

$$(C^{(n)} \hookrightarrow D^{(n)}, (\psi^{(n)}, \varphi^{(n)}))$$

of dimension n, endowed with a chain map (which is a homotopy equivalence)

$$D^{(n)} \xrightarrow{\varepsilon^{(n)}} C(\tilde{X}_{[\frac{1}{2}n]})$$

(corresponding to $e_n\colon P^n \overset{\simeq}{\rightarrow} X_{[\frac{1}{2}n]} := [\frac{1}{2}n]$-skeleton of X).

More important to us than the algebraic Poincaré pair above is its algebraic Thom complex, the n-dimensional (UR) symmetric complex

(i) $$(D^{(n)}/C^{(n)}, \psi^{(n)?}).$$

There is another n-dimensional UR symmetric complex about, namely

(ii) $$(\Sigma^n((C(\tilde{X}_{[\frac{1}{2}n]}))^{-*}), \mathfrak{S}^n(c(\gamma)_{\text{new}})).$$

(Explanation: it is understood that

$$c(\gamma)_{\text{new}} \in \hat{W} \,\&\, C(\tilde{X})^{-*}$$

is the chain bundle derived as in I, 3.A, from the filtered thickening P^n above and no other. I have also written $c(\gamma)_{\text{new}}$ for the image of $c(\gamma)_{\text{new}}$ in $\hat{W} \,\&\, C(\tilde{X}_{[\frac{1}{2}n]})^{-*}$, so that the n-fold suspension

$$\mathfrak{S}^n(c(\gamma)_{\text{new}})$$

is an n-dimensional cycle in $\hat{W} \,\&\, \Sigma^n(C(\tilde{X}_{[\frac{1}{2}n]})^{-*})$; it cannot help lying in the subcomplex

$$W \,\&\, \Sigma^n(C(\tilde{X}_{[\frac{1}{2}n]})^{-*}) \subset \hat{W} \,\&\, \Sigma^n(C(\tilde{X}_{[\frac{1}{2}n]})^{-*}),$$

so that (ii) is indeed an UR symmetric complex.)

The idea of the proof is to show that the chain homotopy equivalence

$$g^{(n)}\colon \Sigma^n(C(\tilde{X}_{[\frac{1}{2}n]})^{-*}) \overset{\simeq}{\rightarrow} D^{(n)}/C^{(n)}$$

obtained by composing the map

$$\Sigma^n(C(\tilde{X}_{[\frac{1}{2}n]})^{-*}) \xrightarrow{\varepsilon^{(n)\leftarrow}} \Sigma^n((D^{(n)})^{-*})$$

with the Poincaré duality chain equivalence

$$\Sigma^n((D^{(n)})^{-*}) \overset{\simeq}{\rightarrow} D^{(n)}/C^{(n)}$$

is such that

(iii) $$g^{(n)\rightarrow}(\mathfrak{S}^n(c(\gamma)_{\text{new}})) = \psi^{(n)?}$$

in $W \,\&\, (D^{(n)}/C^{(n)})$, up to an infinity of higher homologies. In other words, the UR symmetric complexes (i) and (ii) are more or less identical.

For large n, $C(\tilde{X}_{[\frac{1}{2}n]}) = C(\tilde{X})$, and we also have by definition of $c(\gamma)$ the equation

(iv) $$g^{(n)\rightarrow}(\mathfrak{S}^n(c(\gamma))) = J(\psi^{(n)?})$$

in $\hat{W}\&(D^{(n)}/C^{(n)})$, up to an infinity of higher homologies.

Putting (iii) and (iv) together proves I, 3.A.4. (The method of proof is so 'natural' that the infinity of higher homologies which identifies $c(\gamma)$ and $c(\gamma)_{\text{new}}$ is sufficiently independent of the choice of filtered thickening; see I, 3.A.2 (i) and 3.A.3.)

The outline is over; we are left with equation (iii). This can be proved by induction on n. If n is even, the induction step from n to $n+1$ is trivial; for $n = 2q-1$, it is contained in Lemma 4.A.2 below, for which the assumptions are as follows.

Let $(E \to F, (\lambda, \chi))$ be a $(2q)$-dimensional UR symmetric pair (in $\mathscr{C}_A$, with $q > 0$) such that

(a) the $(2q-1)$-dimensional UR symmetric complex (E, χ) is the suspension of a $(2q-2)$-dimensional UR symmetric complex (G, μ) (such that $E = \Sigma G$, $\chi = \mathfrak{S}(\mu)$);

(b) $H_i(F) = 0$ for $i \neq q$.

Then $H^q(F) \cong (H_q(F))^*$ (coefficients A) is a f.g. projective A-module. Moreover, $H^q(F)$ carries two sesquilinear forms, β_1 and β_2.

Description of β_1: choose a chain map (in $\mathscr{C}_A$)

$$f\colon F \to H_*(F)$$

such that the induced map in homology is the identity (regarding $H_*(F)$ as a chain complex in $\mathscr{C}_A$). Then the $2q$-chain

$$f^{\rightarrow}(\lambda) \in W\& H_*(F)$$

is nothing but an element in $H_q(F)^t \otimes_A H_q(F)$ (which is well defined!), and so can be regarded as a sesquilinear form β_1 on $(H_q(F))^* = H^q(F)$.

Description of β_2: β_2 is a 'sliding form'. Condition (a) just above gives us a $(2q-1)$-dimensional UR symmetric pair

$$(G \to \text{Cone}(G), (\text{Cone}(\mu), \mu))$$

(with 'Cone(μ)' equal to the image of μ under the map induced by the projection $G \otimes_{\mathbb{Z}} I \to \text{Cone}(G)$; see the text between I, 2.13 and 2.14).

The UR symmetric complex obtained by collapsing the boundary of the UR symmetric pair

$$(G \to \text{Cone}(G), (\text{Cone}(\mu), \mu))$$

is equal to the boundary of the UR symmetric pair

$$(E \to F, (\lambda, \chi)).$$

So 4.A.1 can be applied.

Let (Fun, Φ) be the resulting (UR) algebraic Poincaré bordism modelled on D^3_+, with $\text{Fun}(D^0_-) = 0$. We can define a sliding form β_2 on

$$H_q(\text{Fun}(D^3_+)/\text{Fun}(D^2_+)) \cong H^q(\text{Fun}(D^3_+)/\text{Fun}(D^2_-)) \cong H^q(F)$$

by imitating the construction in the proof of I, 3.A.2 (ii). (P^{2q} corresponds to $\text{Fun}(D^3_+)$, P^{2q-1} to $\text{Fun}(D^2_+)$; P^{2q-2} to $\text{Fun}(D^1_+)$; $\underline{P}^{2q-1}$ to $\text{Fun}(D^2_-)$, $\underline{P}^{2q-2}$ to $\text{Fun}(D^1_-)$; and $\partial(P^{2q-2}) = \partial(\underline{P}^{2q-2})$ to $\text{Fun}(D^0)$.)

4.A.2. LEMMA. $\beta_1 = \beta_2$.

To prove the lemma, it suffices (by a naturality argument) to consider the special case where the map $E \hookrightarrow F$ is an isomorphism, and both E and F are concentrated in dimension q; details are left to the reader.

5. *Algebraic surgery*

5.1. DEFINITION. Let (Fun, Φ) be an n-dimensional UR algebraic Poincaré bordism (over A), of order 1 (see I, 2.3), and let

$$x \in H_{k+1}(\mathrm{Fun}(\{0\}) \hookrightarrow \mathrm{Fun}(\{0,1\})).$$

Call $((\mathrm{Fun}, \Phi), x)$ an *elementary bordism of index* $k+1$ if

$$H_i(\mathrm{Fun}(\{0\}) \hookrightarrow \mathrm{Fun}(\{0,1\})) = \begin{cases} 0 & \text{if } i \neq k+1, \\ \text{the free } A\text{-module spanned by } x, & \text{if } i = k+1. \end{cases}$$

Motivation. Suppose that we are given an n-dimensional bordism of manifolds (of order 1) which is 'elementary of index $k+1$' (that is, which possesses a Morse function having exactly one critical point, and that of order $k+1$). Suppose also that this geometric bordism is equipped with the usual data—principal π-bundle, etc., as in I, 3.8. Then the algebraic Poincaré bordism (Fun, Φ) derived from the given geometric bordism (by the method of I, 3.8) is elementary of index $k+1$ for a suitable choice of $x \in H_{k+1}(\mathrm{Fun}(\{0\}) \hookrightarrow \mathrm{Fun}(\{0,1\}))$.

5.2. DEFINITION. Let (C, φ) be an UR algebraic Poincaré complex and $y \in H_k(C)$. Say that y can be *killed by algebraic surgery* if there exists an elementary (UR algebraic Poincaré) bordism $((\mathrm{Fun}, \Phi), x)$, as in 5.1, so that

$$\mathrm{Fun}(\{0\}) = C, \quad \Phi(\{0\}) = \varphi;$$
$$\partial x = y \quad \text{in } H_k(\mathrm{Fun}(\{0\})) = H_k(C).$$

For the next definition, let B be a chain complex in $\mathscr{C}_A$, and let $\mathscr{E}$ be a chain bundle on B.

5.3. DEFINITION. Let (C, φ) be an UR algebraic Poincaré complex with $\mathscr{E}$-structure (g, z) (cf. I, 2.6), and let

$$y' \in H_{k+1}(g\colon C \to B).$$

Say that y' can be *killed by algebraic surgery* if there exists an elementary UR algebraic Poincaré bordism $((\mathrm{Fun}, \Phi), x)$ of index $k+1$, and a $\mathscr{E}$-structure $(\bar{g}, \bar{z})$ on (Fun, Φ) such that

$\mathrm{Fun}(\{0\}) = C$, $\Phi(\{0\}) = \varphi$, and the $\mathscr{E}$-structure $(\bar{g}, \bar{z})$ on (Fun, Φ) extends the $\mathscr{E}$-structure (g, z) on (C, φ);

under the homomorphism

$$H_{k+1}(\mathrm{Fun}(\{0\}) \hookrightarrow \mathrm{Fun}(\{0,1\})) \to H_{k+1}(\mathrm{Fun}(\{0\}) = C \to B)$$

induced by the chain map $\bar{g}\colon \mathrm{Fun}(\{0,1\}) \to B$, x maps to y'.

5.4. PROPOSITION. (i) *In* 5.2, *y can be killed by algebraic surgery if and only if a certain obstruction*

$$\mathrm{ob}(y) \in H^{n-2k}(Z_2; A)$$

vanishes.

(ii) *In* 5.3, *y′ can be killed by algebraic surgery if and only if a certain obstruction*

$$\mathrm{ob}(y') \in H_{2k-n}(Z_2; A)$$

vanishes.

Proof of (i). The Poincaré dual of $y \in H_k(C)$ is a cohomology class in $H^{n-k}(C)$ (coefficients A), which can be represented by a chain map (in $\mathscr{C}_A$)

$$f_y\colon C \to (A, n-k) \quad \text{(see I, 0.6).}$$

Let

$$\mathrm{ob}(y) := f_y^{\to}(\varphi) \in Q^n((A, n-k)) \cong H^{n-2k}(Z_2; A).$$

Let D be the mapping cylinder of f_y. Suppose that $\mathrm{ob}(y) = 0$. Then there exists an $(n+1)$-chain $\psi \in W \& D \simeq W \& (A, n-k)$ such that

$$(C \to D, (\psi, \varphi))$$

is an $(n+1)$-dimensional UR symmetric pair (whose boundary symmetric complex (C, φ) happens to be an UR algebraic Poincaré complex). The algebraic Poincaré thickening (in the sense of 4.10) of $(C \to D, (\psi, \varphi))$ is an UR algebraic Poincaré triad (Fun, Φ) of dimension $n+1$. Since (C, φ) was already an UR algebraic complex, (Fun, Φ) can also be regarded as a bordism of order 1 (from (C, φ) to something else). It has the properties required in 5.1, 5.2. The converse is similar.

Proof of 5.4 (ii). The obstruction $\mathrm{ob}(y')$ is somewhat harder to define in this case. The following definition helps:

5.5. DEFINITION. For a chain complex E in $\mathscr{C}_A$, let $Q_n(E)$ be the nth relative homology group of the forgetful map

$$J\colon W \& E \to \hat{W} \& E.$$

(So there is a long exact sequence

$$\ldots \to \hat{Q}^{n+1}(E) \to Q_n(E) \to Q^n(E) \to \hat{Q}^n(E) \to Q_{n-1}(E) \to \ldots, \quad n \in \mathbb{Z};$$

see [11] for more details.)

5.6. EXAMPLE. Let (C, φ) be an n-dimensional UR algebraic Poincaré complex with $\mathscr{B}$-structure (g, z) (for example, the one in 5.3). Let E be the mapping cone of the usual map (cf. I, 2.6)

$$\varphi_0 \cdot g^*\colon \Sigma^n(B^{-*}) \to C.$$

The inclusion $j\colon C \to E$ gives a class $j^{\to}(\varphi) \in Q^n(E)$. It is not hard to see that the $\mathscr{B}$-structure on (C, φ) determines a canonical lifting $\chi \in Q_n(E)$ of $j^{\to}(\varphi) \in Q^n(E)$ ('upwards' the long exact sequence in 5.5).

We return to the proof of 5.4 (ii). Now

$$y' \in H_{k+1}(g\colon C \to B) \quad \text{(from 5.4 (ii) and 5.3)}$$

corresponds to an element in

$$H^{n-k}(g^*\colon \Sigma^n(B^{-*}) \to \Sigma^n(C^{-*}))$$

(cohomology with coefficients A throughout). Since

$$\varphi_0\colon \Sigma^n(C^{-*}) \to C$$

is a chain homotopy equivalence, we can also think of this as an element in

$$H^{n-k}(\varphi_0 \cdot g^*\colon \Sigma^n(B^{-*}) \to C) = H^{n-k}(E)$$

(where E denotes the mapping cone of $\varphi_0 \cdot g^*$).

Represent this element in $H^{n-k}(E)$ by an A-module chain map

$$f_{y'}\colon E \to (A, n-k).$$

Let $\chi \in Q_n(E)$ be the 'lifting' described in 5.6, and put

$$\mathrm{ob}(y') := f_{y'}^{\to}(\chi) \in Q_n((A, n-k)) \cong H_{2k-n}(Z_2; A).$$

The rest of the proof is left to the reader.

5.7. Corollary. *Let $\mathscr{E}$ be a chain bundle on a positive chain complex B. Then the forgetful homomorphisms* (see I, 2.21)

$$\pi_n(\mathbb{L}^0(B, \mathscr{E})) \to \pi_n(\mathscr{L}^{\cdot}(B, \mathscr{E})) = L^n(B, \mathscr{E})$$

are isomorphisms for $n \geqslant 0$. For $n \leqslant -3$, $L^n(B, \mathscr{E})$ is isomorphic to $L_n(A)$, the Wall group of A.

Proof. To see that $\pi_n(\mathbb{L}^0(B, \mathscr{E})) \to \pi_n(\mathscr{L}^{\cdot}(B, \mathscr{E}))$ is surjective, take an n-dimensional UR algebraic Poincaré complex (C, φ) with $\mathscr{E}$-structure (g, z). Then 5.4 (ii) allows us to kill the homology groups of C in negative dimensions, because

for $k < 0$, every element $y \in H_k(C)$ lifts to $y' \in H_{k+1}(g\colon C \to B)$, since B is positive;

$\mathrm{ob}(y') \in H_{2k-n}(Z_2; A)$ is zero since $H_i(Z_2; A) = 0$ for $i < 0$ (and since we assume that $n \geqslant 0$).

Hence (C, φ), with its $\mathscr{E}$-structure, is bordant to a (restricted!) algebraic Poincaré complex with $\mathscr{E}$-structure, as required.

The proof of injectivity is similar (admittedly, it uses a somewhat relativized version of 5.4 (ii)).

For $n \leqslant -3$, performing surgery below the middle dimension using 5.4 (ii) shows that

$$L^n(B, \mathscr{E}) \cong L^n(0_A, 0) \cong L_n(A)$$

(see I, 2.19).

6. *A homological description for $\hat{L}^n(B, \mathscr{E})$*

6.1. Theorem (see I, 2.21). *For any chain complex B in $\mathscr{C}_A$ and chain bundle $\mathscr{E}$ on B, there is a long exact sequence*

$$\dots \to \hat{Q}^{n+1}(B) \to \hat{L}^n(B, \mathscr{E}) \to Q^n(B) \to \hat{Q}^n(B) \to \hat{L}^{n-1}(B, \mathscr{E}) \to \dots \quad (n \in \mathbb{Z}).$$

6.2. Addendum. The homomorphisms $Q^n(B) \to \hat{Q}^n(B)$ in 6.1 are not in general identical with J (of I, 0.13); instead they have the form $J - \mathrm{ind}(\mathscr{E})$, where

$$\mathrm{ind}(\mathscr{E})\colon Q^n(B) \to \hat{Q}^n(B)$$

sends $[\varphi]$ to $[\mathfrak{S}^n \cdot \varphi_0^{\to}(\mathscr{E})]$. (Again, φ_0 is regarded as a chain map from B^{-*} to $\Sigma^{-n}B$.)

We give an outline of the proof of 6.1. It is rather easy to define, for $n \in \mathbb{Z}$, abelian groups $Q_n(B, \mathscr{E})$, depending on B and $\mathscr{E}$, which by construction fit into a long exact sequence

$$\dots \xrightarrow{J\text{-ind}(\mathscr{E})} \hat{Q}^{n+1}(B) \longrightarrow Q_n(B, \mathscr{E}) \longrightarrow Q^n(B) \xrightarrow{J\text{-ind}(\mathscr{E})} \hat{Q}^n(B) \longrightarrow \dots.$$

Here $Q_n(B, \mathscr{E})$ is the group of suitable equivalence classes of pairs

$$(\varphi, z) \in (W \,\&\, B)_n \times (\hat{W} \,\&\, B)_{n+1}$$

so that (B, φ) is an UR symmetric complex with normal structure $(\mathscr{E}, z)$ (see 4.12). This amounts to saying that

$$\varphi_0^{\rightarrow}(\mathfrak{S}^n(\mathscr{E})) + d(z) = J(\varphi) \quad \text{in } \hat{W} \,\&\, B.$$

According to 4.19 we can interpret $\hat{L}^n(B, \mathscr{E})$ as the bordism group of n-dimensional UR symmetric complexes (C, φ) with normal $\mathscr{E}$-structure (g, z). Given such a (C, φ) with normal $\mathscr{E}$-structure (g, z), one finds that $(g^{\rightarrow}(\varphi), g^{\rightarrow}(z))$ represents an element in $Q_n(B, \mathscr{E})$. Conversely, given an element $[(\varphi, z)]$ in $Q_n(B, \mathscr{E})$, the n-dimensional UR symmetric complex (B, φ) with normal $\mathscr{E}$-structure (id, z) represents an element in $\hat{L}^n(B, \mathscr{E})$. So there is an isomorphism $\hat{L}^n(B, \mathscr{E}) \cong Q_n(B, \mathscr{E})$, and the proof is complete.

The details are as follows. For the definition of $Q_n(B, \mathscr{E})$, take two pairs (φ, z) and (φ', z') in $(W \,\&\, B)_n \times (\hat{W} \,\&\, B)_{n+1}$ such that (B, φ) and (B, φ') are n-dimensional UR symmetric complexes with normal structures $(\mathscr{E}, z)$ and $(\mathscr{E}, z')$ respectively. Call (φ, z) and (φ', z') *equivalent* if there exists a pair $(\psi, y) \in (W \,\&\, B)_{n+1} \times (\hat{W} \,\&\, B)_{n+2}$ so that

$$\varphi + d(\psi) = \varphi' \quad \text{in } W \,\&\, B;$$

$$d(y) + z' - z = J(\psi) - \mathfrak{S}^n \cdot \psi_0^{\rightarrow}(\mathscr{E} \times \omega) \quad \text{in } \hat{W} \,\&\, B.$$

(Explanation: ψ_0 is regarded as a chain homotopy from φ_0 to φ_0', or as a chain map from $B^{-*} \otimes_{\mathbb{Z}} I$ to $\Sigma^{-n}B$; therefore $\mathfrak{S}^n \cdot \psi_0^{\rightarrow}(\mathscr{E} \times \omega)$ is a homology from $\mathfrak{S}^n \cdot \varphi_0^{\rightarrow}(\mathscr{E})$ to $\mathfrak{S}^n \cdot \varphi_0'(\mathscr{E})$. See the proof of I, 1.1 (i) and I, 2.9.)

6.3. Proposition and Definition. *The set of equivalence classes, written $Q_n(B, \mathscr{E})$, is an abelian group.*

Proof. Let $\mathrm{pr}_1, \mathrm{pr}_2 \colon B \oplus B \to B$ be the two projections, and $g = \mathrm{pr}_1 + \mathrm{pr}_2$. Given two elements $[(\varphi, z)]$ and $[(\varphi', z')] \in Q_n(B, \mathscr{E})$, choose $z'' \in (\hat{W} \,\&\, (B \oplus B))_{n+1}$ so that

(i) (g, z'') is a normal $\mathscr{E}$-structure on the UR symmetric complex $(B \oplus B,\ \varphi \oplus \varphi')$;

(ii) $$\mathrm{pr}_1^{\rightarrow} \colon \hat{W} \,\&\, (B \oplus B) \to \hat{W} \,\&\, B \text{ sends } z'' \text{ to } z,$$
$$\mathrm{pr}_2^{\rightarrow} \colon \hat{W} \,\&\, (B \oplus B) \to \hat{W} \,\&\, B \text{ sends } z'' \text{ to } z'.$$

Such a z'' exists and is 'essentially unique'. Put

$$[(\varphi, z)] + [(\varphi', z')] := [(\varphi + \varphi', g^{\rightarrow}(z''))].$$

6.4. Proposition. *There is a long exact sequence* (cf. 6.2)

$$\dots \xrightarrow{J\text{-ind}(\mathscr{E})} \hat{Q}^{n+1}(B) \longrightarrow Q_n(B, \mathscr{E}) \longrightarrow Q^n(B)$$
$$\xrightarrow{J\text{-ind}(\mathscr{E})} \hat{Q}^n(B) \longrightarrow \dots \quad (n \in \mathbb{Z}).$$

Proof. Go from $\hat{Q}^{n+1}(B)$ to $Q_n(B, \mathscr{E})$ by $[z] \mapsto [(0, z)]$ (for an $(n+1)$-cycle $z \in \hat{W} \& B$), and from $Q_n(B, \mathscr{E})$ to $Q^n(B)$ by $[(\varphi, z)] \mapsto [\varphi]$. Exactness is almost obvious.

6.5. REMARK. If $\mathscr{E} = 0$, $Q_n(B, \mathscr{E})$ equals $Q_n(B)$ (of 5.5), and the long exact sequence in 6.4 is the usual one.

For the next proposition, interpret $\hat{L}^n(B, \mathscr{E})$ as the bordism group of n-dimensional UR symmetric complexes with normal $\mathscr{E}$-structure (g, z) (as in 4.19).

6.6. PROPOSITION. *The homomorphism*

$$\hat{L}^n(B, \mathscr{E}) \to Q_n(B, \mathscr{E})$$

which sends the bordism class of the n-dimensional UR symmetric complex (C, φ) with normal $\mathscr{E}$-structure (g, z) to $[(g^{\to}(\varphi), g^{\to}(z))] \in Q_n(B, \mathscr{E})$ is an isomorphism.

Proof. The inverse homomorphism

$$Q_n(B, \mathscr{E}) \to \hat{L}^n(B, \mathscr{E})$$

sends $[(\varphi, z)] \in Q_n(B, \mathscr{E})$ to the bordism class of the n-dimensional UR symmetric complex (B, φ) with normal $\mathscr{E}$-structure (id, z). Clearly the composite $Q_n(B, \mathscr{E}) \to \hat{L}^n(B, \mathscr{E}) \to Q_n(B, \mathscr{E})$ is the identity. Given an UR symmetric complex (C, φ) with normal $\mathscr{E}$-structure (g, z), the mapping cylinder of $g\colon C \to B$ can be equipped with suitable data so as to constitute a bordism between (C, φ) (with normal $\mathscr{E}$-structure (g, z)) and $(B, g^{\to}(\varphi))$ (with normal $\mathscr{E}$-structure $(\mathrm{id}, g^{\to}(z))$). This shows that the composite

$$\hat{L}^n(B, \mathscr{E}) \to Q_n(B, \mathscr{E}) \to \hat{L}^n(B, \mathscr{E})$$

is the identity also, which proves 6.6.

Finally, combining 6.4 and 6.6 proves 6.1.

6.7. EXAMPLE. Take $B, \mathscr{E}$ as in I, 2.20 and 2.A.3; so $\mathscr{E}$ is universal for chain bundles on positive chain complexes in $\mathscr{C}_A$. Then $L^n(B, \mathscr{E}) \cong L^n(A)$ for $n \in \mathbb{Z}$, where $L^n(A)$ is the symmetric L-group of [**11**] (as introduced by Mishchenko). (For $n \geqslant 0$, this is clear from 5.7 and the discussion in I, 2.20; for $n < 0$, take it as a definition of $L^n(A)$. It agrees with the definition in [**11**, Part I, §6].)

Write $\hat{L}^n(A) := \hat{L}^n(B, \mathscr{E})$, so that there is a long exact sequence

$$\dots \to L_n(A) \to L^n(A) \to \hat{L}^n(A) \to L_{n-1}(A) \to \dots \quad (n \in \mathbb{Z}).$$

From 6.1 we obtain another long exact sequence

$$\dots \to \hat{Q}^{n+1}(B) \to \hat{L}^n(A) \to Q^n(B) \to \hat{Q}^n(B) \to \dots \quad (n \in \mathbb{Z}),$$

showing that the groups $\hat{L}^n(A)$ are homological objects.

(This requires explanation, since 6.1 is valid for chain complexes in $\mathscr{C}_A$—and B is usually not in $\mathscr{C}_A$. Put $Q^n(B) := H_n(\mathrm{Hom}^b_{\mathbb{Z}[Z_2]}(W, B^t \otimes_A B))$, where the superscript 'b' stands for the subcomplex of bounded chains in $\mathrm{Hom}_{\mathbb{Z}[Z_2]}(W, B^t \otimes_A B)$; that is, chains which vanish on W_s for all but finitely many $s \in \mathbb{Z}$. Proceed similarly for $\hat{Q}^n(B)$. With these conventions 6.1 can be generalized to cover the case at hand, that is, the case of an arbitrary chain complex of projective left A-modules, equipped with a chain bundle as defined in I, 2.A. The proof uses a direct limit argument.)

In order to apply 6.1, we need means for computing groups of type $Q^n(B)$, $\hat{Q}^n(B)$ or $Q_n(B, \mathscr{E}) \cong \hat{L}^n(B, \mathscr{E})$. Often this is elementary, at least if B is sufficiently well understood. Some times the chain homotopy invariance of the functors $Q^n(-)$, $\hat{Q}^n(-)$ can be exploited, as in 6.8 below; if that is not sufficient, there is a spectral sequence for computing $Q^n(B)$ (or $\hat{Q}^n(B)$), based on the filtration of W (or $\hat{W}$) by skeletons.

6.8. Example. Assume that A is such that every left A-module is projective. Then every chain complex of left A-modules is homotopy equivalent to one with zero differential (its homology), and hence the computation of groups such as $Q^n(B)$, $\hat{Q}^n(B)$, $Q_n(B, \mathscr{E})$ is usually a trivial matter. For instance, if $A = Z_2$, one has (cf. 6.7)

$$\hat{L}^n(A) \cong \begin{cases} Z_2 & \text{if } n \geqslant -1, \\ 0 & \text{if } n < -1. \end{cases}$$

7. *Spherical fibrations, normal spaces, and L-theory*

Recall from [10] or [11] that a normal space of formal dimension n consists of a finitely generated simplicial set Y, a spherical fibration $\nu_Y\colon Y \to BG(\infty)$, and a map of spectra

$$\rho_Y\colon S^n \to M(Y, \nu_Y),$$

where $M(Y, \nu_Y)$ is the Thom spectrum. Call ν_Y the normal bundle, and call

$$(\text{Thom class of } \nu_Y \cap h(\rho_Y)) \in H_n(Y; \mathbb{Z}^{tw})$$

the fundamental class; here $\mathbb{Z}^{tw}$ is the twisted integer coefficients, the twisting being given by the first Stiefel–Whitney class of ν_Y.

Example. Every Poincaré space of formal dimension n is a normal space of formal dimension n, according to I, 3.3.

Now let $(\pi, w; X, \gamma; \alpha, j)$ be a string as in I, 3.4. Let $M(X, \gamma)$ be the Thom spectrum. The homotopy group $\pi_n(M(X, \gamma))$ can be identified with the bordism group of formally n-dimensional normal spaces (Y, ν_Y, ρ_Y) equipped with a classifying map $g\colon Y \to X$ such that $g^{*}(\gamma) = \nu_Y$. (Proof: any quadruple (Y, ν_Y, ρ_Y, g) as above yields $g^{\to}(\rho_Y)\colon S^n \to M(X, \gamma)$; conversely, any $\rho\colon S^n \to M(X, \gamma)$ yields a quadruple $(X, \gamma, \rho, \mathrm{id}_X)$. Moreover, the quadruples (Y, ν_Y, ρ_Y, g) and $(X, \gamma, g^{\to}(\rho_Y), \mathrm{id}_X)$ are bordant: the mapping cylinder of g is a bordism between the two.)

7.1. Theorem. *There is a canonical map of spectra*

$$M(X, \gamma) \to \underline{\hat{\mathscr{L}}}^{\cdot}(C(\tilde{X}), c(\gamma))$$

(cf. I, 2.21 (iii) *and* 3.4) *which fits into a commutative square*

$$\begin{array}{ccc} \underline{\Omega}^P(X, \gamma) & \longrightarrow & \underline{\mathscr{L}}^{\cdot}(C(\tilde{X}), c(\gamma)) \\ \downarrow & & \downarrow \\ M(X, \gamma) & \longrightarrow & \underline{\hat{\mathscr{L}}}^{\cdot}(C(\tilde{X}), c(\gamma)) \end{array}$$

and therefore results in a map of long exact sequences (see comment below)

$$\begin{array}{ccccccccc} \cdots \to & \pi_{n+1}(M(X,\gamma)) & \to & L_n(\mathbb{Z}[\pi_1(X)]) & \to & \Omega_n^P(X,\gamma) & \to & \pi_n(M(X,\gamma)) & \to \cdots \\ & \downarrow & & \downarrow{\scriptstyle (a)} & & \downarrow{\scriptstyle (b)} & & \downarrow & \\ \cdots \to & \hat{L}^{n+1}(C(\tilde{X}), c(\gamma)) & \to & L_n(\mathbb{Z}[\pi]) & \to & L^n(C(\tilde{X}), c(\gamma)) & \to & \hat{L}^n(C(\tilde{X}), c(\gamma)) & \to \cdots \end{array}$$

(for $n \geq 5$, with $\Omega_n^P(X,\gamma) = \pi_n(\Omega^P(X,\gamma))$).

Proof and comment. We have just interpreted $M(X,\gamma)$ in terms of normal spaces mapping to X, γ and in §§4 and 6 we interpreted $\hat{\mathscr{L}}^{\cdot}(C(\tilde{X}), c(\gamma))$ in terms of symmetric chain complexes with $c(\gamma)$-structure. So 7.1 (apart from the long exact sequences) is a perfect analogue of I, 3.5, and the same proof applies—just substitute 'normal spaces' for 'Poincaré spaces' everywhere.

The maps from $\pi_n(M(X,\gamma))$ to $\hat{L}^n(C(\tilde{X}), c(\gamma))$ are flexible versions of the hyperquadratic signature maps

$$\hat{\sigma}^*\colon \pi_n(M(X,\gamma)) \to \hat{L}^n(\mathbb{Z}[\pi])$$

constructed in [**12**, §7.4].

The upper long exact sequence in 7.1 was announced by Quinn in [**10**].

All L-theoretic constructions in 7.1 are meant to be based on free modules rather than projective ones, so that, for example, $L_n(\mathbb{Z}[\pi])$ means $L_n^h(\mathbb{Z}[\pi])$. The vertical arrows (a) in 7.1 are induced by the sufficiently well defined homomorphism $h\colon \pi_1(X) \to \pi$ corresponding to the principal π-bundle α, and (b) is the flexible signature.

Warning. If γ is a vector bundle, or if for any other reason transversality arguments can be applied to γ, then transversality gives a splitting of the upper long exact sequence in 7.1. But there is absolutely no way in general of getting a compatible splitting for the lower long exact sequence. The case in which $\pi = \{1\}$ and X is a point is instructive.

8. *An injectivity criterion for the release map*

In order to complement §6, we will examine here the release homomorphisms $L_n(A) \to L^n(B, \mathscr{E})$ for a chain complex B in $\mathscr{C}_A$ with chain bundle $\mathscr{E}$; cf. I, 2.21.

8.1. THEOREM. *Take $B, \mathscr{E}$ as above, and let c be another chain bundle on B; suppose that*

$$[\mathscr{E}] - [c] \in \hat{Q}^0(B^{-*})$$

belongs to the image of the composite homomorphism

$$Q^n(\Sigma^n(B^{-*})) \xrightarrow{J} \hat{Q}^n(\Sigma^n(B^{-*})) \xrightarrow{\cong} \hat{Q}^0(B^{-*}).$$

Then there exists an isomorphism

$$\hat{L}^{n+1}(B, \mathscr{E}) \cong \hat{L}^{n+1}(B, c)$$

making the triangle

$$\begin{tikzcd}[column sep=tiny] \hat{L}^{n+1}(B,\mathscr{b}) \cong \hat{L}^{n+1}(B,c) \arrow[d, "\partial"'] \arrow[d, "\partial"] \\ L_n(A) \end{tikzcd}$$

commute. So the release homomorphisms $L_n(A) \to L^n(B,\mathscr{b})$ *and* $L_n(A) \to L^n(B,c)$ *have the same kernel.*

Next, keep the assumptions of 8.1, but assume also that $A = \mathbb{Z}[\pi]$ is a group ring (with the w-twisted involution for some $w\colon \pi \to Z_2$). Let $\pi'' \subset \pi$ be a subgroup of finite index and let $A'' = \mathbb{Z}[\pi'']$ be the corresponding ring with involution. Write B'' for the chain complex in $\mathscr{C}_{A''}$ obtained by regarding B as an A''-module chain complex; write $\mathscr{b}''$ and c'' for the chain bundles on B'' obtained from $\mathscr{b}$ and c respectively. Transfer (see I, 3.16) gives maps of spectra $\underline{\mathscr{L}}_{\cdot}(A) \to \underline{\mathscr{L}}_{\cdot}(A'')$, $\underline{\mathscr{L}}^{\cdot}(B,\mathscr{b}) \to \underline{\mathscr{L}}^{\cdot}(B'',\mathscr{b}'')$, and $\hat{\underline{\mathscr{L}}}^{\cdot}(B,\mathscr{b}) \to \hat{\underline{\mathscr{L}}}^{\cdot}(B'',\mathscr{b}'')$, whose cofibres we denote by

$$\underline{\mathscr{L}}_{\cdot}(A\uparrow A''),\quad \underline{\mathscr{L}}^{\cdot}(B\uparrow B'',\ \mathscr{b}\uparrow\mathscr{b}''),\quad \text{and}\quad \hat{\underline{\mathscr{L}}}^{\cdot}(B\uparrow B'',\ \mathscr{b}\uparrow\mathscr{b}'')$$

respectively. The nth homotopy groups of these cofibres are written

$$L_n(A\uparrow A''),\quad L^n(B\uparrow B'',\ \mathscr{b}\uparrow\mathscr{b}''),\quad \text{and}\quad \hat{L}^n(B\uparrow B'',\ \mathscr{b}\uparrow\mathscr{b}'').$$

8.2. THEOREM. *With the hypotheses of* 8.1, *there exists an isomorphism*

$$\hat{L}^{n+1}(B\uparrow B'',\ \mathscr{b}\uparrow\mathscr{b}'') \cong \hat{L}^{n+1}(B\uparrow B'',\ c\uparrow c'')$$

making the triangle

$$\begin{tikzcd}[column sep=tiny] \hat{L}^{n+1}(B\uparrow B'',\ \mathscr{b}\uparrow\mathscr{b}'') \cong \hat{L}^{n+1}(B\uparrow B'',\ c\uparrow c'') \arrow[d, "\partial"'] \arrow[d, "\partial"] \\ L_n(A\uparrow A'') \end{tikzcd}$$

commute. So the release homomorphisms

$$L_n(A\uparrow A'') \to L^n(B\uparrow B'',\ \mathscr{b}\uparrow\mathscr{b}'') \quad \textit{and} \quad L_n(A\uparrow A'') \to L^n(B\uparrow B'',\ c\uparrow c'')$$

have the same kernel.

8.3. COMMENT. There are two reasons for stating 8.1. Firstly, suppose that (C,φ) is an n-dimensional algebraic Poincaré complex over A, with normal chain bundle ν; then certain geometric analogies suggest that the release map $L_n(A) \to L^n(C,\nu)$ ought to be injective. This is confirmed by 8.1.

Secondly, suppose that $A = Z_2$; then the simply-connected theory of [**1**] and [**2**] suggests that the release homomorphism $L_{2k}(A) \to L^{2k}(B,\mathscr{b})$ ought to be injective if and only if the Wu class

$$v_{k+1}(\mathscr{b})\colon H_{k+1}(B) \to Z_2$$

vanishes (see I, 2.A.1). This is true (see the argument just below for the 'if' part; for the 'only if' part, reduce to the case where B is concentrated in dimension $k+1$ and compute, using 6.1). Unfortunately, it does not generalize well to arbitrary A. However, 8.1 is a reasonable substitute in many cases. We now give a standard application. Taking A arbitrary, and assuming that $\mathscr{b}$ in 8.1 is the object of study, suppose that c can be so chosen that

the hypothesis of 8.1 is satisfied;

the class $[c] \in \hat{Q}^0(B^{-*})$ belongs to the kernel of the restriction homomorphism $\hat{Q}^0(B^{-*}) \to \hat{Q}^0((B_{\leqslant [\frac{1}{2}n]+1})^{-*})$ induced by the inclusion of the skeleton $B_{\leqslant [\frac{1}{2}n]+1} \hookrightarrow B$ (with notation as in I, 2.A.3).

Then the release maps $L_n(A) \to L^n(B, \mathscr{b})$ and, if applicable,

$$L_n(A \uparrow A'') \to L^n(B \uparrow B'', \mathscr{b} \uparrow \mathscr{b}'')$$

are *injective*.

Indeed, by 8.1 and 8.2 we may replace $\mathscr{b}$ by c; now the hypothesis on c means that there exists a chain bundle $\bar{c}$ on $B/B_{\leqslant [\frac{1}{2}n]+1}$ and a chain bundle map from c to $\bar{c}$ covering the projection $B \to B/B_{\leqslant [\frac{1}{2}n]+1}$. Hence we could have assumed from the outset that B is $([\frac{1}{2}n]+1)$-connected, and $\mathscr{b} = c = \bar{c}$. But then the homomorphisms $L_n(A) \to L^n(B, \mathscr{b})$ and $L_n(A \uparrow A'') \to L^n(B \uparrow B'', \mathscr{b} \uparrow \mathscr{b}'')$ are isomorphisms. (The proof uses surgery below the middle dimension.)

Proof of 8.1. We may assume that

$$c = \mathscr{b} + \mathfrak{S}^{-n}(J(h)),$$

where h is an n-cycle in $W \& \Sigma^n(B^{-*})$. Let (φ, z) be a typical representative of $Q_{n+1}(B, \mathscr{b}) \cong \hat{L}^{n+1}(B, \mathscr{b})$ (using notation as in 6.3). Then

$$(\varphi + \varphi_0^{\to}(\mathfrak{S}h), z)$$

is a typical representative of $Q_{n+1}(B, c) \cong \hat{L}^{n+1}(B, c)$; remember that φ_0 is a chain map from $\Sigma^{n+1}(B^{-*})$ to B. This gives a bijection between $\hat{L}^{n+1}(B, \mathscr{b})$ and $\hat{L}^{n+1}(B, c)$; going back to 6.3 and using the fact that

$$Q^{n+1}(B \oplus B) \cong Q^{n+1}(B) \oplus Q^{n+1}(B) \oplus H_{n+1}(B^t \otimes_A B),$$

we find that it is a group isomorphism.

To prove that the isomorphism commutes with the boundary maps to $L_n(A)$, we use the original definition of $\hat{L}^{n+1}(B, \mathscr{b})$ and $\hat{L}^{n+1}(B, c)$ in terms of algebraic Poincaré pairs. So let $(f\colon C \to D, (\bar{\varphi}, \phi))$ be an UR algebraic Poincaré pair of dimension $n+1$, with a $\mathscr{b}$-structure $(g, \bar{z})$ such that $g \cdot f\colon C \to B$ is zero. (Then $\bar{\varphi} \in (W \& D)_{n+1}$ and $\phi \in (W \& C)_n$, etc.) Now

$$j = \bar{\varphi}_0 \cdot g\colon \Sigma^{n+1}(B^{-*}) \to D$$

happens to be a chain map (because g vanishes on the boundary); and

$$(f\colon C \to D, (\bar{\varphi} + j^{\to}(\mathfrak{S}h), \phi))$$

is an UR algebraic Poincaré pair of dimension $n+1$, with a c-structure $(g, \bar{z})$ such that $g \cdot f = 0$. If we let $\varphi := g^{\to}(\bar{\varphi})$ and $z := g^{\to}(\bar{z}(\{0, 1\}))$, then the first algebraic Poincaré pair above corresponds to $[(\varphi, z)] \in Q_{n+1}(B, \mathscr{b})$, and the second to

$$[(\varphi + \varphi_0^{\to}(\mathfrak{S}h), z)] \in Q_{n+1}(B, c).$$

But the boundaries of the two algebraic Poincaré pairs (with additional structure) are identical; so they represent the same element in $L_n(A)$, as required. (I am obliged to A. Ranicki for help with the proof.)

Proof of 8.2. This is identical with the proof of 8.1, except that it calls for a more categorical point of view. We are dealing with certain A-modules (mostly the chain modules B_n and their duals); but we usually regard them as A''-modules only, and moreover adopt the policy of regarding A''-module homomorphisms between them as 'negligible' if they preserve the A-module structure.

For instance, the group $\hat{L}^{n+1}(B \uparrow B'', \mathscr{E} \uparrow \mathscr{E}'')$, which we might also call

$$Q_{n+1}(B \uparrow B'', \mathscr{E} \uparrow \mathscr{E}''),$$

has a description in terms of equivalence classes of pairs (φ, z), with $\varphi \in (W \& B'')_{n+1}$ and $z \in (\hat{W} \& B'')_{n+2}$; however, instead of requiring that

$$d(\varphi) = 0 \qquad \text{in } W \& B''$$

and

$$d(z) = J(\varphi) - \mathfrak{S}^{n+1}(\varphi_0^{\to}(\mathscr{E}'')) \qquad \text{in } \hat{W} \& B''$$

(as we should in defining $Q_{n+1}(B'', \mathscr{E}'')$), we merely ask that

$$d(\varphi) \equiv 0$$

and

$$d(z) \equiv J(\varphi) - \mathfrak{S}^{n+1}(\varphi_0^{\to}(\mathscr{E}'')),$$

where $\equiv$ indicates that the difference between the left-hand and right-hand terms belongs to the 'negligible' subcomplexes $W \& B \subset W \& B''$ or $\hat{W} \& B \subset \hat{W} \& B''$. (We have, for instance, $W \& B \subset W \& B''$ because

$$B^t \otimes_A B \cong \mathrm{Hom}_A(B^{-*}, B) \subset \mathrm{Hom}_{A''}(B''^{-*}, B'') \cong B''^t \otimes_{A''} B''.)$$

The details are left to the reader.

8.4. Remark. If a version of the theory is used where projective class and/or torsion matters, then 8.2 must be formulated with greater care; see I, 2.22. However, this affects $L_n(A \uparrow A'')$ only, not the relative groups $\hat{L}^{n+1}(B \uparrow B'', \mathscr{E} \uparrow \mathscr{E}'')$.

9. *Products and Whitney sums*

9.1. Definition. If $\mathscr{E}$ is a chain bundle on a chain complex B in $\mathscr{C}_A$ and $\mathscr{E}'$ is a chain bundle on a chain complex B' in $\mathscr{C}_{A'}$, then $\mathscr{E} \times \mathscr{E}'$ is a chain bundle on the chain complex $B \otimes_{\mathbb{Z}} B'$ in $\mathscr{C}_{A \otimes_{\mathbb{Z}} A'}$, called the exterior product of $\mathscr{E}$ and $\mathscr{E}'$ (cf. I, 0.11).

9.2. Proposition. *There are multiplication maps*

$$\underline{\mathscr{L}}^{\cdot}(B, \mathscr{E}) \wedge \underline{\mathscr{L}}^{\cdot}(B', \mathscr{E}') \to \underline{\mathscr{L}}^{\cdot}(B \otimes_{\mathbb{Z}} B', \mathscr{E} \times \mathscr{E}')$$

and

$$\hat{\underline{\mathscr{L}}}^{\cdot}(B, \mathscr{E}) \wedge \hat{\underline{\mathscr{L}}}^{\cdot}(B', \mathscr{E}') \to \hat{\underline{\mathscr{L}}}^{\cdot}(B \otimes_{\mathbb{Z}} B', \mathscr{E} \times \mathscr{E}'),$$

inducing multiplication homomorphisms

$$L^m(B, \mathscr{E}) \otimes L^n(B', \mathscr{E}') \to L^{m+n}(B \otimes_{\mathbb{Z}} B', \mathscr{E} \times \mathscr{E}')$$

and

$$\hat{L}^m(B, \mathscr{E}) \otimes \hat{L}^n(B', \mathscr{E}') \to \hat{L}^{m+n}(B \otimes_{\mathbb{Z}} B', \mathscr{E} \times \mathscr{E}').$$

Proof. Let (C, φ) be an m-dimensional UR symmetric chain complex over A with normal $\mathscr{E}$-structure (g, z) (cf. 4.14); and let (C', φ') be an n-dimensional UR symmetric chain complex over A' with normal $\mathscr{E}'$-structure (g', z'). Then $(C \otimes_{\mathbb{Z}} C', \varphi \times \varphi')$ is an

$(m+n)$-dimensional UR symmetric chain complex over $A \otimes_{\mathbb{Z}} A'$, with normal $\mathscr{E} \times \mathscr{E}'$-structure

$$(g \otimes g', \varphi \times z' + (-)^n \cdot z \times (\varphi' - d(z'))).$$

Passage to bordism classes defines the multiplication

$$\hat{L}^m(B, \mathscr{E}) \otimes \hat{L}^n(B', \mathscr{E}') \to \hat{L}^{m+n}(B \otimes_{\mathbb{Z}} B', \mathscr{E} \times \mathscr{E}').$$

If in addition (C, φ) and (C', φ') are both UR algebraic Poincaré complexes, then so is $(C \otimes_{\mathbb{Z}} C', \varphi \times \varphi')$, which explains the multiplication

$$L^m(B, \mathscr{E}) \otimes L^n(B', \mathscr{E}') \to L^{m+n}(B \otimes_{\mathbb{Z}} B', \mathscr{E} \times \mathscr{E}').$$

The rest of the proof is unpleasant and left to the reader.

9.3. PROPOSITION. *Under the association*

$$\text{spherical fibration} \mapsto \text{chain bundle}$$

of I, 3.4, *exterior Whitney sums (explanation below) correspond to exterior products of chain bundles.*

Proof. The passage from spherical fibrations to chain bundles in I, 3.4 (and previously in [11]) was based on equivariant S-duality; 9.3 is an application of the principle that (equivariant) S-duality commutes with smash products. Details are again left to the reader. To get a good definition of Whitney sums, return to I, 3.2 and use an identification $D^n \times D^m \cong D^{n+m}$.

9.4. PROPOSITION. *The diagram of maps of spectra*

$$\begin{array}{ccc}
\underline{\Omega}^P(X, \gamma) \wedge \underline{\Omega}^P(X', \gamma') & \longrightarrow & \underline{\Omega}^P(X \times X', \gamma \times \gamma') \\
\downarrow & & \downarrow \\
\underline{\mathscr{L}}^{\cdot}(C(\tilde{X}), c(\gamma)) \wedge \underline{\mathscr{L}}^{\cdot}(C(\tilde{X}'), c(\gamma')) & \longrightarrow & \underline{\mathscr{L}}^{\cdot}(C(\tilde{X} \times \tilde{X}'), c(\gamma \times \gamma')) \\
 & & \simeq \underline{\mathscr{L}}^{\cdot}(C(\tilde{X}) \otimes_{\mathbb{Z}} C(\tilde{X}'), c(\gamma) \times c(\gamma'))
\end{array}$$

commutes.

(Here γ and γ' are spherical fibrations on simplicial sets X and X', equipped with certain data; $\gamma \times \gamma'$ on $X \times X'$ is the exterior Whitney sum.)

There is a similar commutative diagram in which $\underline{\Omega}^P(X, \gamma)$ and $\underline{\mathscr{L}}^{\cdot}(C(\tilde{X}), c(\gamma))$ are replaced by $M(X, \gamma)$ and $\hat{\underline{\mathscr{L}}}^{\cdot}(C(\tilde{X}), c(\gamma))$, respectively (similarly for X' and γ'). See 7.1. The proof of 9.4 is left to the reader.

The analogue of 9.3 for internal Whitney sums looks as follows. Given strings of data $(\pi, w; X, \gamma; \alpha, j)$ and $(\pi', w'; X, \gamma'; \alpha', j')$ as in I, 3.4, their internal Whitney sum is the string $(\pi \times \pi', w \times w'; X, \gamma \oplus \gamma'; \alpha \times_X \alpha', j \times j')$. The associated chain bundles $c(\gamma)$, $c(\gamma')$, and $c(\gamma \oplus \gamma')$ (which will also be denoted, for greater precision, by $c(\gamma; \pi)$, $c(\gamma'; \pi')$, and $c(\gamma \oplus \gamma'; \pi \times \pi')$ respectively) are related by a 'Cartan formula':

9.5. COROLLARY. $c(\gamma \oplus \gamma'; \pi \times \pi') = c(\gamma; \pi) \cup c(\gamma'; \pi')$.

(Explanation: $c(\gamma;\pi) \cup c(\gamma';\pi')$ is, by definition, the pullback of $c(\gamma;\pi) \times c(\gamma';\pi')$ under the composition

$$C(\tilde{X}) \xrightarrow{\text{Eilenberg–Zilber}} C(\tilde{X}) \otimes_{\mathbb{Z}} C(\tilde{X}) \xrightarrow{\quad\text{proj.}\quad} C(\tilde{X}/\pi') \otimes_{\mathbb{Z}} C(\tilde{X}/\pi)$$

in which $\tilde{X}$ denotes the total space of $\alpha \times_X \alpha'$.)

Next, let $(\pi, w; X, \gamma; \alpha, j)$ be a string as usual, and assume for simplicity that $w\colon \pi \to Z_2$ is trivial. Write this string as the internal Whitney sum of the two strings

$$(\pi, w; X, \text{trivial spherical fibration}; \alpha, \text{trivial})$$

and

$$(\{1\}, \text{trivial}; X, \gamma; \text{trivial}, j).$$

In other words, write

$$\gamma = (\text{trivial spherical fibration} \oplus \gamma);$$

let the first Whitney summand (trivial spherical fibration) carry the weight of the data, and equip the second summand with the trivial data. Then 9.5 implies

9.6. Corollary ('Separation principle').

$$c(\gamma;\pi) = c(\text{trivial spherical fibration};\pi) \cup c(\gamma;\{1\}).$$

Both 9.5 and 9.6 have to be interpreted in the usual woolly way, namely 'up to an infinity of higher homologies'; but at any rate, 9.6 shows that $c(\gamma;\pi)$ is determined in a sense by $c(\gamma;\{1\})$.

The situation is similar for stable fibre homotopy equivalences of spherical fibrations. Suppose that we are given two strings

$$(\pi, w; X, \gamma_1; \alpha, j_1) \quad \text{and} \quad (\pi, w; X, \gamma_2; \alpha, j_2)$$

as in I, 3.4 (with $w = 0$), and an orientation-preserving stable fibre homotopy equivalence

$$\mu\colon \gamma_1 \cong \gamma_2\,.$$

(With our restrictive notion of spherical fibration, it is best to assume that μ comes in the shape of a stable spherical fibration on $X \times [0, 1]$ which restricts to γ_1 on $X \times \{0\}$ and to γ_2 on $X \times \{1\}$.)

9.7. Addendum to 9.6. *The 'chain bundle isomorphism'* (cf. I, 1.8) $c(\gamma_1;\pi) \cong c(\gamma_2;\pi)$ *induced by* μ (cf. I, 1.12) *is determined by the chain bundle isomorphism*

$$c(\gamma_1;\{1\}) \cong c(\gamma_2;\{1\})$$

(also induced by μ*).*

These trivial algebraic observations have a non-trivial geometric consequence. Let $f\colon \mathrm{BSO} \to \prod_{k\geq 0} K(Z_2, 2k)$ be a map in the homotopy class $(v_0, v_2, v_4, \ldots)$, where the v_i are the Wu classes.

Define a *pseudo-surgery problem* over (π, w) (with $w = 0$ as before) to consist of

a degree-1 map $\bar{p}$ from a compact smooth oriented manifold N^n with boundary ∂N to a finite (simple, if you wish) oriented geometric Poincaré pair $(X^n, \partial X)$, restricting to a (simple) homotopy equivalence of the boundaries;

a principal π-bundle on X;
a map (not just a homotopy class)

$$g\colon X \to \prod_{k\geq 0} K(Z_2, 2k);$$

and a homotopy from

$$N \xrightarrow{\nu_N} \mathrm{BSO} \xrightarrow{f} \prod K(Z_2, 2k)$$

to

$$N \xrightarrow{\bar{p}} X \xrightarrow{g} \prod K(Z_2, 2k).$$

An ordinary surgery problem (as in [13]) gives rise to a pseudo one: take

$$g := f \cdot (\text{classifying map for the vector bundle on } X).$$

9.8. THEOREM. *There is a canonical factorization* (*broken arrow*)

$$\begin{tikzcd}
\begin{array}{l}\text{Bordism group of } n\text{-dimensional}\\ \text{ordinary surgery problems}\end{array} \arrow[d] \arrow[dr] & \\
& \begin{array}{l}\text{Bordism group of } n\text{-dimensional}\\ \text{pseudo-surgery problems}\end{array} \arrow[dl, dashed] \\
L_n(\mathbb{Z}[\pi]) &
\end{tikzcd}$$

As an example, consider an ordinary surgery problem (consisting of a degree-1 map $g\colon N^n \to X^n$ as before, a vector bundle γ on X, and a stable trivialization of $g^{\leftarrow}(\gamma) \oplus \tau_N$). Let us alter the stable trivialization by a map $f\colon N \to \mathrm{SO}$. Suppose that

$$f^{\leftarrow}(\delta(v_i)) = 0 \quad \text{in } H^{i-1}(N; Z_2) \text{ for all } i > 0,$$

where $\delta(v_i) \in H^{i-1}(\mathrm{SO}; Z_2)$ is the cohomology desuspension of the Wu class

$$v_i \in H^i(\mathrm{BSO}; Z_2).$$

Then the change of framing f does not affect the surgery obstruction, by 9.8.

Proof of 9.8. Write $A = \mathbb{Z}[\pi]$. The pseudo-surgery problem described just before 9.8 gives rise to a degree-1 map of algebraic Poincaré pairs

$$p\colon (C(\partial\tilde{N}) \to C(\tilde{N}), (\psi, \varphi)) \to (C(\partial\tilde{X}) \to C(\tilde{X}), (\lambda, \eta)),$$

where 'degree-1' means that $p^{\rightarrow}(\psi, \varphi) = (\lambda, \eta)$, strictly. Specifying a map

$$X \to \prod K(Z_2, 2k)$$

is another way of specifying a chain bundle $c_{\mathbb{Z}}$ (over $\mathbb{Z}$!) on $C(X) = \mathbb{Z} \otimes_A C(\tilde{X})$. (It is easy to see that $\hat{Q}^0(C(X)^{-*}) \cong \prod_{k\geq 0} H^{2k}(X; Z_2)$, for instance by applying I, 2.A to the ring with involution $\mathbb{Z}$.)

The remaining data give an isomorphism of chain bundles (over $\mathbb{Z}$!)

$$\mathrm{Is}_{\mathbb{Z}}\colon p^{\leftarrow}(c_{\mathbb{Z}}) \cong (\text{normal chain bundle of } C(N)),$$

with $C(N) = \mathbb{Z} \otimes_A C(\tilde{N})$. (Of course, algebraic Poincaré pairs, too, have normal chain bundles.)

What we really need in order to get an element in $L_n(A)$ is the chain level analogue of a surgery problem, i.e. apart from the degree-1 map p of algebraic Poincaré pairs (which restricts to a chain homotopy equivalence of the boundaries),

a chain bundle c_A (over A!) on $C(\tilde{X})$, and

an identification of chain bundles (over A!),

$$\mathrm{Is}_A\colon p^{\leftarrow}(c_A) \cong (\text{normal chain bundle of } C(\tilde{N})).$$

Now 9.6 suggests that c_A can be defined using $c_{\mathbb{Z}}$; similarly, 9.7 suggests that Is_A can be defined using $\mathrm{Is}_{\mathbb{Z}}$.

In detail, let $\mathscr{E}$ be the chain bundle (over A) on $C(\tilde{X})$ determined, as in I, 3.4, by the trivial vector bundle on X; put

$$c_A := \mathscr{E} \cup c_{\mathbb{Z}},$$

and let Is_A be the identification

$$\begin{aligned} p^{\leftarrow}(c_A) &= p^{\leftarrow}(\mathscr{E} \cup c_{\mathbb{Z}}) = p^{\leftarrow}(\mathscr{E}) \cup p^{\leftarrow}(c_{\mathbb{Z}}) \\ &\cong p^{\leftarrow}(\mathscr{E}) \cup (\text{normal chain bundle over } \mathbb{Z} \text{ of } C(N)) \\ &\cong (\text{normal chain bundle over } A \text{ of } C(\tilde{N})). \end{aligned}$$

(The last in this sequence of identifications stems from 9.6, taking into account the first sentence of I, § 3; and the previous one is induced by $\mathrm{Is}_{\mathbb{Z}}$.)

9.9. Remark. In the twisted case, that is, when $w\colon \pi \to Z_2$ is non-trivial, 9.8 remains valid with no essential change, except that the classifying map for the normal bundle ν_N,

$$N \to \mathrm{BSO}$$

has to be replaced by the classifying map for

$$\gamma_N \oplus w\text{-twisted line bundle},$$

which still goes from N to BSO. Moreover, there is no harm in replacing the smooth manifold with boundary N by a geometric Poincaré pair.

Let X be a finitely generated simplicial set, let α be a principal π-bundle on X, and let $K_{\mathrm{osf}}(X)$ be the group of stable fibre homotopy equivalence classes of orientable spherical fibrations.

9.10. Proposition. *The diagonal maps* $X \to X \times X$ *and* $\pi \to \pi \times \pi$ *make* $\hat{Q}^0(C(\tilde{X})^{-*})$ *into a (commutative, associative) ring. The rule* $\gamma \mapsto [c(\gamma)]$ *defines a multiplicative map from* $K_{\mathrm{osf}}(X)$ *to* $\hat{Q}^0(C(\tilde{X})^{-*})$, *so it transforms Whitney sums into products.* (*See also* 10.13.)

Explanation and proof. $C(\tilde{X})$ is in $\mathscr{C}_{\mathbb{Z}[\pi]}$, and $\mathbb{Z}[\pi]$ carries the involution coming from the trivial homomorphism $w\colon \pi \to Z_2$. The ring structure on $\hat{Q}^0(C(\tilde{X})^{-*})$ is obtained as follows. Given chain bundles $\mathscr{E}$ and $\mathscr{E}'$ on $C(\tilde{X})$, note that $\mathscr{E} \times \mathscr{E}'$ is a chain bundle on $C(\tilde{X}) \otimes_{\mathbb{Z}} C(\tilde{X})$, regarded as a chain complex over

$$\mathbb{Z}[\pi] \otimes_{\mathbb{Z}} \mathbb{Z}[\pi] \cong \mathbb{Z}[\pi \times \pi].$$

Although the diagonal subgroup $\pi \subset \pi \times \pi$ will not in general have finite index, an ad hoc transfer argument shows that

$C(\tilde{X}) \otimes_{\mathbb{Z}} C(\tilde{X})$ can be thought of as a chain complex of $\mathbb{Z}[\pi]$-modules (free, but not necessarily finitely generated);

$\mathscr{E} \times \mathscr{E}'$ determines a chain bundle $\mathrm{tr}(\mathscr{E} \times \mathscr{E}')$ (over $\mathbb{Z}[\pi]$!) on the said chain complex.

(To see this, write $D := C(\tilde{X}) \otimes_{\mathbb{Z}} C(\tilde{X})$, and let

$$f\colon \mathbb{Z}[\pi \times \pi] \to \mathbb{Z}[\pi]$$

be the homomorphism of free abelian groups which sends the generator

$$(x, y) \in \pi \times \pi \subset \mathbb{Z}[\pi \times \pi] \text{ to } x \in \pi \subset \mathbb{Z}[\pi] \text{ if } x = y,$$

and to 0 otherwise. Define 'chain bundles' as at the beginning of I, 2.A; if

$$\mathscr{E} \times \mathscr{E}' = \{\varphi_{p,q}\colon D_p \times D_q \to \mathbb{Z}[\pi \times \pi] \mid p, q \in \mathbb{Z}\},$$

put

$$\mathrm{tr}(\mathscr{E} \times \mathscr{E}') := \{f \cdot \varphi_{p,q}\colon D_p \times D_q \to \mathbb{Z}[\pi] \mid p, q \in \mathbb{Z}\}.)$$

The Eilenberg–Zilber diagonal $EZ_0\colon C(\tilde{X}) \to C(\tilde{X}) \otimes_{\mathbb{Z}} C(\tilde{X})$ is a chain map over $\mathbb{Z}[\pi]$; the ring structure on $\hat{Q}^0(C(\tilde{X})^{-*})$ is given by

$$[\mathscr{E}] \cdot [\mathscr{E}'] = (EZ)_0^{\sim}[\mathrm{tr}(\mathscr{E} \times \mathscr{E}')].$$

Now let γ be an orientable spherical fibration on X; choose an orientation j. Then $(\pi, w; X, \gamma; \alpha, j)$ is a string of data as in I, 3.4, with $w = 0$. Therefore $[c(\gamma)]$ in $\hat{Q}^0(C(\tilde{X})^{-*})$ is defined. Choosing a different orientation does not affect the result.

There is a version of 9.10 which covers the non-orientable case: the appropriate ring to consider is then

$$\bigoplus_w \hat{Q}^0({}^wC(\tilde{X})^{-*})$$

(where w ranges over all homomorphisms from π to Z_2, and the superscript w in $\hat{Q}^0({}^wC(\tilde{X})^{-*})$ indicates which involution on $\mathbb{Z}[\pi]$ is used to define $\hat{Q}^0(C(\tilde{X})^{-*})$.)
The principal π-bundle α on X would be fixed, however, as in 9.10. See also 10.13.

10. *Classification of chain bundles over a group ring*

10.1. THEOREM. *Let R be either $\mathbb{Z}$ or Z_2; let $A = R[\pi]$ be the group ring, equipped with the w-twisted involution for some $w\colon \pi \to Z_2$. The cohomology theory*

$$C \mapsto \{\hat{Q}^{-n}(C^{-*}) \mid n \in \mathbb{Z}\}$$

on $\mathscr{C}_A$ is then an ordinary cohomology theory, that is, there are canonical natural isomorphisms

$$\hat{Q}^{-n}(C^{-*}) \cong \prod_{k \in \mathbb{Z}} H^{k+n}(C; \hat{H}^k(Z_2; A))$$

for $n \in \mathbb{Z}$, commuting with the suspension isomorphisms.

(Note that the groups $\hat{H}^k(Z_2; A) \cong \hat{Q}^0((A, k)^{-*})$ are the coefficients of the cohomology theory and therefore carry a left A-module structure, made explicit in I, 2.A.)

Assume now that $A = Z_2[\pi]$. Let PR be a projective resolution of the left A-module $\hat{H}^0(Z_2; A)$ (so that $H_0(PR)$ is canonically identified with $\hat{H}^0(Z_2; A)$). Theorem 10.1 follows (in the case where $A = Z_2[\pi]$) from

10.2. ROUGH STATEMENT. There exists a (somehow distinguished) chain bundle $\mathscr{d}$ on PR so that the 0th Wu class defined in I, 2.A,

$$v_0(\mathscr{d})\colon H_0(PR) \to \hat{H}^0(Z_2; A),$$

agrees with the canonical identification. ('Chain bundle' has to be interpreted here as in I, 2.A.)

Proof of the implication 10.2 $\Rightarrow$ 10.1, *for* $A = Z_2[\pi]$. It suffices to specify a natural isomorphism

$$\prod_{k\in\mathbb{Z}} H^k(C; \hat{H}^k(Z_2; A)) \cong \hat{Q}^0(C^{-*})$$

for C in $\mathscr{C}_A$, since $\hat{Q}^n(C^{-*}) \cong \hat{Q}^0((\Sigma^n C)^{-*})$. Since A has characteristic 2,

$$\hat{H}^k(Z_2; A) \cong \hat{H}^0(Z_2; A) \quad \text{for } k \in \mathbb{Z};$$

further, the cohomology theory at issue is now periodic with period 1 (not merely 2; see I, 1.3). To be more precise, if B is a possibly huge chain complex of projective left A-modules, if $\hat{W} \,\&\, B^{-*}$ is defined as at the beginning of I, 2.A, and if

$$\varphi = \{\varphi_{p,q}\colon B_p \times B_q \to A \mid p, q \in \mathbb{Z}\}$$

is an n-cycle in $\hat{W} \,\&\, B^{-*}$ for some $n \in \mathbb{Z}$, then φ can be viewed as an n-cycle in $\hat{W} \,\&\, B^{-*}$ for all $n \in \mathbb{Z}$. Equivalently, if $\mathscr{b}$ is a chain bundle on B, then $\mathfrak{S}^{-n}\mathscr{b}$ can be regarded as a chain bundle on $\Sigma^n B$.

Now let

$$B^\infty := \bigoplus_{k\in\mathbb{Z}} \Sigma^k PR;$$

supposing that 10.2 holds, let

$$\mathscr{b}^\infty := \bigoplus_{k\in\mathbb{Z}} \mathfrak{S}^{-k}\mathscr{d}.$$

Since all the Wu classes

$$v_k(\mathscr{b}^\infty)\colon H_k(B^\infty) \to \hat{H}^k(Z_2; A)$$

are now isomorphisms, we recognize in $\mathscr{b}^\infty$, B^∞ the 'universal chain bundle' of I, 2.A.4. Therefore or otherwise, the natural homomorphism

$$\prod_{k\in\mathbb{Z}} H^k(C; \hat{H}^k(Z_2; A)) \cong H_0(\mathrm{Hom}_A(C, B^\infty)) \to \hat{Q}^0(C^{-*}),$$

$$[f] \mapsto [f^{\leftarrow}(\mathscr{b}^\infty)]$$

is a natural isomorphism, for C in $\mathscr{C}_A$. (Of course, in order to make it canonical we have to specify a canonical $\mathscr{d}$ in 10.2.)

The proof of 10.2 proceeds by obstruction theory. Write $\mathscr{D}_A$ for the category of projective left A-module chain complexes C which are positive, such as PR in 10.2.

(For the moment, A can be any ring with involution.) For C in $\mathscr{D}_A$, let

$$\mathscr{P}_k\hat{Q}^0(C^{-*}) := \operatorname{im}[\hat{Q}^0((C_{\leqslant k+1})^{-*}) \to \hat{Q}^0((C_{\leqslant k})^{-*})].$$

(Here $C_{\leqslant k}$ is the k-skeleton of C.)

So the homotopy-invariant functor

(i) $$C \mapsto \mathscr{P}_k\hat{Q}^0(C^{-*})$$

is the 'kth Postnikov base' of the (homotopy-invariant) functor

(ii) $$C \mapsto \hat{Q}^0(C^{-*});$$

see [4] for a completely analogous topological definition. Or in other words, passing from (ii) to (i) amounts to 'killing' the coefficient groups of the functor (ii) in dimension greater than k, that is, the groups $\hat{Q}^0((A, n)^{-*})$ for $n > k$. (Cf. I, 0.6.) There is a commutative diagram of natural forgetful maps

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & \mathscr{P}_{k+1}\hat{Q}^0(C^{-*}) \arrow[d] \\
\hat{Q}^0(C^{-*}) \arrow[ur, dotted] \arrow[r] \arrow[dr, dotted] & \mathscr{P}_k\hat{Q}^0(C^{-*}) \arrow[d] \\
 & \mathscr{P}_{k-1}\hat{Q}^0(C^{-*}) \arrow[d] \\
 & \vdots
\end{tikzcd}$$

and there are natural homomorphisms

$$\text{ob}: \mathscr{P}_{k-1}\hat{Q}^0(C^{-*}) \to H^{k+1}(C; \hat{H}^k(Z_2; A))$$

so that the sequence

$$\mathscr{P}_k\hat{Q}^0(C^{-*}) \longrightarrow \mathscr{P}_{k-1}\hat{Q}^0(C^{-*}) \xrightarrow{\text{ob}} H^{k+1}(C; \hat{H}^k(Z_2; A))$$

is exact. (Again, $\hat{H}^k(Z_2; A) \cong \hat{Q}^0((A, k)^{-*})$ plays the role of 'coefficient group'.)

10.3. Description of ob: $\mathscr{P}_{k-1}\hat{Q}^0(C^{-*}) \to H^{k+1}(C; \hat{H}^k(Z_2; A))$. For y in

$$\mathscr{P}_{k-1}\hat{Q}^0(C^{-*}) = \operatorname{im}[\hat{Q}^0((C_{\leqslant k})^{-*}) \to \hat{Q}^0((C_{\leqslant k-1})^{-*})],$$

let $\bar{y} \in \hat{Q}^0((C_{\leqslant k})^{-*})$ be a lifting; treat the differential $d: C_{k+1} \to C_k$ as a chain map from (C_{k+1}, k) to $C_{\leqslant k}$. Then

$$d^{-}(\bar{y}) \in \hat{Q}^0((C_{k+1}, k)^{-*}) \cong \operatorname{Hom}_A(C_{k+1}, \hat{H}^k(Z_2; A))$$

represents an element $\text{ob}(y) \in H^{k+1}(C; \hat{H}^k(Z_2; A))$, independent of the choice of lifting $\bar{y}$.

10.4. Definition. For $k > 0$ and C in $\mathscr{D}_A$, let

$$\text{socle}: \mathscr{P}_{k-1}\hat{Q}^0(C^{-*}) \to H^0(C; \hat{H}^0(Z_2; A)) \cong \mathscr{P}_0\hat{Q}^0(C^{-*})$$

be the forgetful map. (Note that C is positive.)

Let C, C' be chain complexes in $\mathscr{D}_A, \mathscr{D}_{A'}$ respectively, where A and A' are arbitrary rings with involution. Then $C \otimes_{\mathbb{Z}} C'$ is in $\mathscr{D}_{A \otimes_{\mathbb{Z}} A'}$; if $y \in \mathscr{P}_{k-1}\hat{Q}^0(C^{-*})$ and $y' \in \mathscr{P}_{k-1}\hat{Q}^0(C'^{-*})$, the exterior product $y \times y'$ is an element of $\mathscr{P}_{k-1}\hat{Q}^0((C \otimes_{\mathbb{Z}} C')^{-*})$.

10.5. LEMMA.

$$\operatorname{socle}(y \times y') = \operatorname{socle}(y) \times \operatorname{socle}(y');$$
$$\operatorname{ob}(y \times y') = \operatorname{ob}(y) \times \operatorname{socle}(y') + \operatorname{socle}(y) \times \operatorname{ob}(y').$$

To make sense of these formulae, note that there are 'exterior multiplication maps'

$$\hat{H}^i(Z_2; A) \times \hat{H}^j(Z_2; A') \to \hat{H}^{i+j}(Z_2; A \otimes_{\mathbb{Z}} A')$$

derived from the diagonal map $\hat{W} \to \hat{W} \otimes_{\mathbb{Z}} \hat{W}$ (in fact, A and A' could be replaced by arbitrary $\mathbb{Z}[Z_2]$-modules). Consequently, there are exterior cohomology products

$$H^p(C; \hat{H}^i(Z_2; A)) \times H^q(C'; \hat{H}^j(Z_2; A')) \to H^{p+q}(C \otimes_{\mathbb{Z}} C'; \hat{H}^{i+j}(Z_2; A \otimes_{\mathbb{Z}} A')).$$

The lemma is easy to verify.

Now let $A = Z_2[\pi]$ again, and let PR be as in 10.2. The ring A is equipped with a diagonal homomorphism

$$Z_2[\pi] = A \to A \otimes_{\mathbb{Z}} A \cong Z_2[\pi \times \pi]$$

corresponding to the diagonal inclusion $\pi \to \pi \times \pi$. Therefore $PR \otimes_{\mathbb{Z}} PR$ can be regarded as a left A-module chain complex (in $\mathscr{D}_A$).

10.6. OBSERVATION. PR is equipped with a canonical (homotopy commutative, homotopy associative) diagonal chain map

$$PR \to PR \otimes_{\mathbb{Z}} PR$$

of A-module chain complexes. Therefore $\hat{Q}^0(PR^{-*})$ and $\mathscr{P}_k\hat{Q}^0(PR^{-*})$ are rings, for $k \geq 0$.

Proof. Since PR is a projective resolution of $\hat{H}^0(Z_2; A)$ and $PR \otimes_{\mathbb{Z}} PR$ is a projective resolution of $\hat{H}^0(Z_2; A) \otimes_{\mathbb{Z}} \hat{H}^0(Z_2; A)$, specifying such a chain map is equivalent to specifying a 'diagonal map' of left A-modules

$$\hat{H}^0(Z_2; A) \to \hat{H}^0(Z_2; A) \otimes_{\mathbb{Z}} \hat{H}^0(Z_2; A).$$

A further reduction is possible. There is a functor FR from the category of π-sets to that of $Z_2[\pi]$-modules: to every π-set S it associates the Z_2-vector space generated by S, with π-action induced from the π-action on S. We have

$$\hat{H}^0(Z_2; A) \cong FR(\operatorname{tor}_2(\pi)),$$

where

$$\operatorname{tor}_2(\pi) = \{x \in \pi \mid x^2 = 1\},$$

and where π acts on $\operatorname{tor}_2(\pi)$ by conjugation. Similarly,

$$\hat{H}^0(Z_2; A) \otimes_{\mathbb{Z}} \hat{H}^0(Z_2; A) = FR(\operatorname{tor}_2(\pi) \times \operatorname{tor}_2(\pi)).$$

So all we need is a π-map

$$\operatorname{tor}_2(\pi) \to \operatorname{tor}_2(\pi) \times \operatorname{tor}_2(\pi);$$

we take the diagonal map.

The ring structures on $\hat{Q}^0(PR^{-*})$ and $\mathscr{P}_k\hat{Q}^0(PR^{-*})$ are defined as in 9.10.

10.7. LEMMA. *Suppose that* $y \in \mathscr{P}_k\hat{Q}^0(PR^{-*})$ *satisfies*

$$\mathrm{socle}(y) = 0.$$

Then y *is nilpotent.*

Proof. If $\mathrm{socle}(y) = 0$, then y can be represented by a chain bundle

$$\{\varphi_{p,q}\colon PR_p \otimes_{\mathbb{Z}} PR_q \to A \mid p, q \leqslant k+1\}$$

on $PR_{\leqslant k+1}$ so that $\varphi_{0,0} = 0$. The definition of the multiplication in $\mathscr{P}_k\hat{Q}^0(PR^{-*})$ implies then that y^n can be represented by a chain bundle

$$\{\psi_{p,q}\colon PR_p \otimes_{\mathbb{Z}} PR_q \to A \mid p, q \leqslant k+1\}$$

on $PR_{\leqslant k+1}$ so that $\psi_{p,q} = 0$ whenever $p+q < n$. So $y^n = 0$ for $n > 2k$.

10.8. LEMMA. *For any* $y \in \mathscr{P}_k\hat{Q}^0(PR^{-*})$, y^{2^n} *is idempotent if* n *is sufficiently large.*

Proof. Write $y^2 = y+h$. Then $\mathrm{socle}(h) = 0$, since $\mathrm{socle}(y^2) = \mathrm{socle}(y)$. By 10.7, h is nilpotent; choose n large enough so that $h^{2^n} = 0$. Then

$$(y^{2^n})^2 = (y^2)^{2^n} = y^{2^n} + h^{2^n} = y^{2^n},$$

as required. (Note that we are in characteristic 2, so $y \mapsto y^{2^n}$ is a ring endomorphism.)

It is now possible to reformulate 10.2, as follows. Firstly, it is not necessary in 10.2 to construct a chain bundle on PR; a class in $\hat{Q}^0(PR^{-*})$ will do just as well. Secondly, although the map

$$\hat{Q}^0(PR^{-*}) \to \varprojlim_k \mathscr{P}_k\hat{Q}^0(PR^{-*})$$

need not be an isomorphism, it is clear that an element in $\varprojlim \mathscr{P}_k\hat{Q}^0(PR^{-*})$ is quite sufficient for the application to 10.1 (in the case where $A = Z_2[\pi]$). The next proposition exhibits such an element.

10.9. PROPOSITION. *For every* $k \geqslant 0$, *there is a unique element* y_k *in* $\mathscr{P}_k\hat{Q}^0(PR^{-*})$ *such that*

(i) $\mathrm{socle}(y_k) \in H^0(PR; \hat{H}^0(Z_2; A)) \cong \mathrm{Hom}_A(\hat{H}^0(Z_2; A), \hat{H}^0(Z_2; A))$ *is the identity,*
(ii) y_k *is idempotent.*

Proof. Clearly y_0 exists and is unique. Suppose that y_{k-1} has already been constructed. Then $(y_{k-1})^2 = y_{k-1}$; now 10.5 implies that

$$\mathrm{ob}(y_{k-1}) \in H^{k+1}(PR; \hat{H}^k(Z_2; A))$$

is divisible by 2, and hence equal to 0. So there exists an element z in $\mathscr{P}_k\hat{Q}^0(PR^{-*})$ which lifts y_{k-1}. Put

$$y_k := z^{2^n} \quad \text{for sufficiently large } n,$$

so that y_k is idempotent (10.8). Clearly y_k satisfies conditions (i) and (ii). To prove uniqueness, suppose that $y'_k \in \mathscr{P}_k\hat{Q}^0(PR^{-*})$ also satisfies (i) and (ii). Then $y_k - y'_k$ is idempotent, since we are in characteristic 2, but also nilpotent by 10.7. Therefore $y_k = y'_k$.

The proof of 10.1 for $R = Z_2$ is complete. (The argument fails for $R = \mathbb{Z}$, even when $w\colon \pi \to Z_2$ is trivial. The point is that 10.6 becomes false: if PR is a projective resolution of $\hat{H}^0(Z_2; A)$ over $\mathbb{Z}[\pi]$, then it is also a projective resolution of $\hat{H}^0(Z_2; A)$ over $\mathbb{Z}$; therefore $PR \otimes_{\mathbb{Z}} PR$ will not be a projective resolution over $\mathbb{Z}$ or over $\mathbb{Z}[\pi]$, since $H_1(PR \otimes_{\mathbb{Z}} PR) \neq 0$.)

Proof of 10.1 *for* $R = \mathbb{Z}$. Write $A = \mathbb{Z}[\pi]$. Indicate reduction mod 2 by a double prime; thus $A'' = Z_2[\pi]$ and $C'' = C \otimes_{\mathbb{Z}} Z_2$ for C in $\mathscr{C}_A$.

Five cohomology theories on $\mathscr{C}_A$ will be needed, namely

$$TH_1\colon C \mapsto \{\hat{Q}^{-n}(C''^{-*}) \mid n \in \mathbb{Z}\},$$
$$TH_2\colon C \mapsto \{\prod_{k \in \mathbb{Z}} H^{k+n}(C; \hat{H}^k(Z_2; A'')) \mid n \in \mathbb{Z}\},$$
$$TH_3\colon C \mapsto \{\hat{Q}^{-n}(C^{-*}) \mid n \in \mathbb{Z}\},$$
$$TH_4\colon C \mapsto \{\prod_{k \in \mathbb{Z}} H^{k+n}(C; \hat{H}^k(Z_2; A)) \mid n \in \mathbb{Z}\},$$
$$TH_5\colon C \mapsto \{\prod_{k \in \mathbb{Z}} H^{k+n}(C; \hat{H}^{k-1}(Z_2; A)) \mid n \in \mathbb{Z}\}.$$

The canonical direct sum decomposition

$$\hat{H}^k(Z_2; A'') \cong \hat{H}^k(Z_2; A) \oplus \hat{H}^{k-1}(Z_2; A), \quad \text{for } k \in \mathbb{Z},$$

gives a canonical isomorphism

(i) $$TH_2 \cong TH_4 \oplus TH_5.$$

That part of 10.1 which has been proved gives an identification

(ii) $$TH_1 \cong TH_2.$$

Let

(iii) $$TH_3 \oplus TH_5 \to TH_1 \cong TH_2$$

be the map of cohomology theories which on the first direct summand is the obvious reduction mod 2, $TH_3 \to TH_1$; and which on the second summand is the inclusion $TH_5 \hookrightarrow TH_2 \cong TH_1$ of (i) just above. The map (iii) is also an isomorphism of cohomology theories, because it induces an isomorphism on coefficient groups. Combining (i), (ii), and (iii), we obtain a commutative diagram

$$\begin{tikzcd}
 & TH_5 \arrow[dl, hook'] \arrow[dr, hook] & \\
TH_3 \oplus TH_5 \arrow[dr, "(\mathrm{iii})"', "\cong"] & & TH_4 \oplus TH_5 \arrow[dl, "(\mathrm{ii}),\,(\mathrm{i})", "\cong"'] \\
 & TH_1 &
\end{tikzcd}$$

showing that $TH_3 \cong TH_1/TH_5 \cong TH_4$. So there is a canonical isomorphism $TH_3 \cong TH_4$, as required.

10.10. REMARK. The preceding proof shows that TH_3 and TH_4 are isomorphic by showing that both are direct summands of TH_1, with common complement $TH_5 \subset TH_1$. If $w: \pi \to Z_2$ is trivial, then it is easy to see that TH_3 and TH_4 are in fact identical as direct summands of TH_1.

10.11. PROPOSITION. *The isomorphism in* 10.1 *is compatible with ring structures if* $R = Z_2$.

Explanation. Write $A = Z_2[\pi]$. For any space X with principal π-bundle α, define a ring structure on $\hat{Q}^0(C(\tilde{X})''^{-*})$ as in 9.10, where $C(\tilde{X})'' = C(\tilde{X}) \otimes_{\mathbb{Z}} Z_2$. Under the isomorphism

$$\hat{Q}^0(C(\tilde{X})''^{-*}) \cong \prod_{k \in \mathbb{Z}} H^k(C(\tilde{X}); \hat{H}^k(Z_2; A))$$

of 10.1, this corresponds to the 'ordinary ring structure' on

$$\prod_{k \in \mathbb{Z}} H^k(C(\tilde{X}); \hat{H}^k(Z_2; A)).$$

To understand what 'ordinary ring structure' means, note that pointwise multiplication μ makes $\hat{H}^0(Z_2; A)$, the set of functions from $\mathrm{tor}_2(\pi)$ to Z_2, into an A-algebra; that is,

$$\mu: \hat{H}^0(Z_2; A) \otimes_{\mathbb{Z}} \hat{H}^0(Z_2; A) \to \hat{H}^0(Z_2; A)$$

is an A-module chain map (with the diagonal A-action on $\hat{H}^0(Z_2; A) \otimes_{\mathbb{Z}} \hat{H}^0(Z_2; A)$; see the text preceding 10.6). Therefore $\{\hat{H}^k(Z_2; A) \mid k \in \mathbb{Z}\}$ is a graded A-algebra, since $\hat{H}^k(Z_2; A) \cong \hat{H}^0(Z_2; A)$.

Sketch proof of 10.11. The cohomology theory under consideration is periodic with period 1, and so 10.11 can be reduced to the claim below.

Let PR be as in 10.2, and let $y_k \in \mathscr{P}_k \hat{Q}^0(PR^{-*})$ be as in 10.9. Then $PR \otimes_{\mathbb{Z}} PR$ is, a priori, an $A \otimes_{\mathbb{Z}} A$-module chain complex, and $y_k \times y_k \in \mathscr{P}_k \hat{Q}^0((PR \otimes_{\mathbb{Z}} PR)^{-*})$ has to be interpreted accordingly.

Using the 'ad hoc transfer' tr associated with the diagonal inclusion $\pi \subset \pi \times \pi$, regard $PR \otimes_{\mathbb{Z}} PR$ as an A-module chain complex (see 9.10 and its proof); $\mathrm{tr}(y_k \times y_k)$ is then an element of

$$\mathscr{P}_k \hat{Q}^0_A((PR \otimes_{\mathbb{Z}} PR)^{-*}),$$

(the subscript A indicates that everything takes place over A, not $A \otimes_{\mathbb{Z}} A$).

Now let $\mu_{\mathrm{res}}: PR \otimes_{\mathbb{Z}} PR \to PR$ be the chain map of A-module chain complexes whose induced homomorphism in 0-dimensional homology is the multiplication

$$\begin{array}{ccc} \mu: \hat{H}^0(Z_2; A) \otimes_{\mathbb{Z}} \hat{H}^0(Z_2; A) & \longrightarrow & \hat{H}^0(Z_2; A) \\ \wr\| & & \wr\| \\ H_0(PR \otimes_{\mathbb{Z}} PR) & & H_0(PR) \end{array}$$

CLAIM. $\mu_{\mathrm{res}}^{\leftarrow}(y_k) = \mathrm{tr}(y_k \times y_k)$ *in* $\mathscr{P}_k \hat{Q}^0_A((PR \otimes_{\mathbb{Z}} PR)^{-*})$.

(To see how the claim implies 10.11, suppose that

$$\left.\begin{array}{l} u \in H^i(C(\tilde{X})''; \hat{H}^i(Z_2; A)) \\ v \in H^j(C(\tilde{X})''; \hat{H}^j(Z_2; A)) \end{array}\right\} \subset \prod_{k \in \mathbb{Z}} H^k(C(\tilde{X})''; \hat{H}^k(Z_2; A)) \cong \hat{Q}^0(C(\tilde{X})''^{-*});$$

write down the two definitions of $u \cdot v$, and compare them.)

Proof of claim. Clearly

$$\mathrm{socle}(\overleftarrow{\mu}_{\mathrm{res}}(y_k)) = \mathrm{socle}(\mathrm{tr}(y_k \times y_k)).$$

(cf. 10.4). Further, it is possible to define a homotopy commutative, homotopy associative diagonal chain map of A-module chain complexes

$$PR \otimes_{\mathbb{Z}} PR \to (PR \otimes_{\mathbb{Z}} PR) \otimes_{\mathbb{Z}} (PR \otimes_{\mathbb{Z}} PR)$$

(imitating 10.6), and thereby a ring structure on $\mathscr{P}_k\hat{Q}^0_A((PR \otimes_{\mathbb{Z}} PR)^{-*})$, such that

(i) $\overleftarrow{\mu}_{\mathrm{res}}\colon \mathscr{P}_k\hat{Q}^0(PR^{-*}) \to \mathscr{P}_k\hat{Q}^0_A((PR \otimes_{\mathbb{Z}} PR)^{-*})$ is a ring homomorphism,

(ii) $\mathrm{tr}(y_k \times y_k) \in \mathscr{P}_k\hat{Q}^0_A((PR \otimes_{\mathbb{Z}} PR)^{-*})$ is idempotent.

Summarizing, we see that $\overleftarrow{\mu}_{\mathrm{res}}(y_k)$ and $\mathrm{tr}(y_k \times y_k)$ are both idempotent, and have the same socle; so they are equal, by the argument used in the proof of 10.10.

10.12. Remark. The analogue of 10.11 for $A = \mathbb{Z}[\pi]$ makes sense and is correct if $w\colon \pi \to Z_2$ is trivial. This follows from 10.10.

Let X be a finite CW-space with principal π-bundle α, and write $A = \mathbb{Z}[\pi]$ (equipped with the w-twisted involution for some $w\colon \pi \to Z_2$), $A'' = Z_2[\pi]$, etc. The group homomorphism $\{1\} \to \pi$ induces

(i) a map $Z_2 = Z_2[\{1\}] \to Z_2[\pi] = A''$ of group rings,

(ii) by (i), a map $Z_2 \cong \hat{H}^k(Z_2; Z_2) \to \hat{H}^k(Z_2; A'')$,

(iii) by (ii), a map

$$\Gamma''_X\colon \prod_{k\in\mathbb{Z}} H^k(X; Z_2) \to \prod_{k\in\mathbb{Z}} H^k(C(\tilde{X}); \hat{H}^k(Z_2; A'')) \cong \hat{Q}^0(C(\tilde{X})''^{-*}).$$

Since (i) is canonically split as a map of $\mathbb{Z}[Z_2]$-modules, (ii) and (iii) are also split. Working over $A = \mathbb{Z}[\pi]$, one obtains similarly a split inclusion

$$\Gamma_X\colon \prod_{k\in\mathbb{Z}} H^{2k}(X; Z_2) \to \hat{Q}^0(C(\tilde{X})^{-*}).$$

10.13. Proposition. *Suppose that π, w, X, α form part of a string of data $(\pi, w; X, \gamma; \alpha, j)$ as in* I, 3.4. *Then*

$$[c(\gamma)] = \Gamma_X(v_0, v_2, v_4, \dots)$$

in $\hat{Q}^0(C(\tilde{X})^{-*})$; here $v_i \in H^i(X; Z_2)$ is the ith Wu class of γ. The corresponding formula over $A'' = Z_2[\pi]$ is

$$[c''(\gamma)] = \Gamma''_X(v_0, v_1, v_2, v_3, \dots)$$

in $\hat{Q}^0(C(\tilde{X})''^{-*})$. (In this case, w and j can be omitted from the string.)

Sketch proof. Over A'', the formula is correct for the trivial spherical fibration γ, and hence correct for arbitrary γ because of 9.6. To get the formula over A, note that $[c(\gamma)] \in \hat{Q}^0(C(\tilde{X})^{-*})$ maps to $[c''(\gamma)] \in \hat{Q}^0(C(\tilde{X})''^{-*})$ under the reduction mod 2 map $\hat{Q}^0(C(\tilde{X})^{-*}) \to \hat{Q}^0(C(\tilde{X})''^{-*})$. On the other hand, the identification

$$\hat{Q}^0(C(\tilde{X})^{-*}) \cong \prod_{k\in\mathbb{Z}} H^k(C(\tilde{X}); \hat{H}^k(Z_2; A))$$

was defined as the composite

$$\begin{tikzcd} \hat{Q}^0(C(\tilde{X})^{-*}) \arrow[r] & \hat{Q}^0(C(\tilde{X})^{\prime\prime -*}) \cong \prod H^k(C(\tilde{X}); \hat{H}^k(Z_2; A'')) \arrow[d, "f"'] \\ & \prod H^k(C(\tilde{X}); \hat{H}^k(Z_2; A)) \end{tikzcd}$$

where f is induced by the projections $\hat{H}^k(Z_2; A'') \to \hat{H}^k(Z_2; A)$. Hence the only element in $\hat{Q}^0(C(\tilde{X})^{-*})$ which maps to $\Gamma''_X(v_0, v_1, v_2, \ldots) \in \hat{Q}^0(C(\tilde{X})^{\prime\prime -*})$ is

$$f(\Gamma''_X(v_0, v_1, v_2, \ldots)) = \Gamma_X(v_0, v_2, v_4, \ldots).$$

This completes the proof.

11. *Miscellany*

This section contains two distinct illustrations of the theory. The first is related to the 'generalized Kervaire invariants' of [1] and [2], and the second is a not-so-new proof of Browder's theorem [1] on the Kervaire invariant (which sheds light on the results of § 10, but not on Browder's theorem).

We shall work with CW-spaces instead of simplicial sets in this section; see the remark after I, 3.A.4.

Generalized Kervaire invariants. Here the ring with involution is fixed: $A = Z_2$, to be regarded as the group ring $Z_2[\{1\}]$ of the trivial group. If X is a finite CW-space, its algebraic counterpart for the time being is $C(X) \otimes_{\mathbb{Z}} Z_2$; any spherical fibration γ on X determines a chain bundle on $C(X) \otimes_{\mathbb{Z}} Z_2$. No orientation is needed.

Most computational problems evaporate upon observing that every chain complex in $\mathscr{C}_A$ is homotopy equivalent to one with zero differential (its homology). For example, using the fact that the functors $\hat{Q}^n(-)$ commute with direct sums, we obtain directly (i.e. without using 10.1):

11.1. PROPOSITION. *There is a natural isomorphism*

$$\hat{Q}^n(C^{-*}) \cong \prod_{k \in \mathbb{Z}} H^{k-n}(C; A) \quad \text{for } C \text{ in } \mathscr{C}_A.$$

The next proposition, proved in [11], is a special case of 10.13 (but has been used implicitly in the proof of 10.13); it can also be deduced from I, 3.A.

11.2. PROPOSITION. *We have* $[c(\gamma)] = (v_0, v_1, v_2, v_3, \ldots)$ *in*

$$\hat{Q}^0((C(X) \otimes_{\mathbb{Z}} Z_2)^{-*}) \cong \prod_{k \in \mathbb{Z}} H^k(X; Z_2)$$

if γ *is a spherical fibration on* X *with Wu classes* $v_i \in H^i(X; Z_2)$.

11.3. PROPOSITION. (i)

$$L^n(0_A, 0) = L_n(A) \cong \begin{cases} Z_2 & (\textit{Kervaire invariant}) \quad \textit{if } n = 2k, \\ 0 & \qquad\qquad\qquad\qquad\quad \textit{if } n = 2k+1. \end{cases}$$

(ii) *Suppose that the* $(k+1)$*th Wu class of* γ *(in* $H^{k+1}(X; Z_2)$*) is zero (with* X *and* γ *as in* 11.2*). Then*

$$\text{release: } L_{2k}(A) \to L^{2k}(C(X) \otimes_{\mathbb{Z}} Z_2, c(\gamma))$$

is injective.

Proof. (i) is easy; (ii) follows from 8.3. The converse of (ii) also holds.

(*Warning*: the groups $L^{2k}(C(X) \otimes_{\mathbb{Z}} Z_2, c(\gamma))$ are not in general Z_2-vector spaces, even though we are working with $A = Z_2$.)

In view of 11.3 (ii), we could call the homomorphism

$$\text{flexible signature: } \Omega^P_{2k}(X, \gamma) \to L^{2k}(C(X) \otimes_{\mathbb{Z}} Z_2, c(\gamma))$$

a 'generalized Kervaire invariant', at least if the $(k+1)$th Wu class of γ vanishes. (See 7.1 for notation.)

For clarification, suppose that d is any ('abstract') chain bundle on a chain complex D in $\mathscr{C}_A$; and that, for some $k \geqslant 0$, the '$(k+1)$th Wu class of d' is zero. (This makes sense by 11.1.)

Given a $2k$-dimensional algebraic Poincaré complex (C, φ) over A, with d-structure, can we imitate [2] to obtain a 'quadratic form with values in Z_4' on $H^k(C; A)$?

The answer is 'yes' (certain choices are, however, necessary, just as in [2]). The following examples constitute a sketch proof.

Example 1. Take $D = (Z_2, k)$ (that is, $D_k = Z_2$, $D_r = 0$ for $r \neq k$); of the two chain bundles on D, let $d(k)$ be the non-trivial one. Certainly the $(k+1)$th Wu class of $d(k)$ is zero. If (C, φ) is a $2k$-dimensional algebraic Poincaré complex in $\mathscr{C}_A$, then $H^k(C; A)$ carries a non-degenerate symmetric bilinear form. It is not hard (but highly amusing) to see that a $d(k)$-structure on (C, φ) (as in I, 2.6) determines an enhancement of the bilinear form to a quadratic form—with values in a group isomorphic to Z_4, as required.

Example 2. Let m be a large integer; put

$$D^u = \bigoplus_{\substack{-m<r<m \\ r \neq k+1}} (Z_2, r) \quad \text{and} \quad d^u = \bigoplus_{\substack{-m<r<m \\ r \neq k+1}} d(r)$$

(in the notation of the previous example). If (C, φ) is a $2k$-dimensional algebraic Poincaré complex with d^u-structure, then $H^k(C; A)$ carries a quadratic form with values in a group isomorphic to Z_4 (same proof as before).

Example 3. The general case. Let d be a chain bundle on D whose $(k+1)$th Wu class is zero; then there exists a chain map $D \to D^u$ (see Example 2) covered by a chain bundle map $d \to d^u$. (N.B. m is large.) Choose such a chain map; then any d-structure on an algebraic Poincaré complex over A determines a d^u-structure, and we obtain the desired quadratic forms from Example 2.

(There is a one-to-one correspondence between the choices used here—that is, homotopy classes of chain bundle maps from d to d^u—and the choices used in [2], if $D = C(X) \otimes_{\mathbb{Z}} Z_2$ and $d = c(\gamma)$ for a space X with spherical fibration γ.)

Browder's theorem on the Kervaire invariant. The theorem in question states that the Kervaire invariant $\pi^s_{2k} \to Z_2$ (defined for arbitrary $k > 0$, but interesting only when k is odd) is zero if k is not of the form $2^p - 1$ for some integer $p > 0$.

Let $X = \mathbb{R}P^m$, let γ be the trivial vector bundle on X, let $X'' = S^m$ be the standard twofold cover of X, and let γ'' be the trivial vector bundle on X''; take m large.

Transfer gives a map of bordism groups

$$\pi_n(M(X, \gamma)) \to \pi_n(M(X'', \gamma'')) = \pi^s_n.$$

(We assume that $n < m$; on the left is the bordism group of 'framed n-manifolds with twofold covers', and on the right is the bordism group of framed n-manifolds. The transfer assigns to a 'framed manifold with twofold cover' its twofold cover.)

The celebrated theorem of Kahn and Priddy [9] implies that this transfer is surjective for $1 \leqslant n < m$. (It is also known that the Kahn–Priddy theorem has the Browder theorem above among its corollaries; see [7] and [8]. That is why I have apostrophized the argument below as 'not-so-new'.)

Put $\pi = Z_2$; then $A = \mathbb{Z}[\pi]$ is a ring with involution (coming from the trivial homomorphism $w\colon \pi \to Z_2$, not from the identity). Let α be the non-trivial principal π-bundle on X, and let $j\colon w^{\to}(\alpha) \cong$ (orientation cover of γ) be the standard identification.

Apply I, 3.4: the result is a chain bundle $c(\gamma)$ on

$$C(\tilde{X}) = \ldots \xrightarrow{1-T} \mathbb{Z}[Z_2] \xrightarrow{1+T} \mathbb{Z}[Z_2] \xrightarrow{1-T} \mathbb{Z}[Z_2]$$

whose homology class we wish to describe explicitly. Recall the isomorphism $\hat{Q}^0(C(\tilde{X})^{-*}) \cong H_0(V(C(\tilde{X})))$ of I, 1.6.

11.4. Proposition. *The class* $[c(\gamma)] \in \hat{Q}^0(C(\tilde{X})^{-*}) \cong H_0(V(C(\tilde{X})))$ *is represented by the* 0*-cycle* $\{\lambda_r\} \in V(C(\tilde{X}))$ *with*

$$\lambda_r = \begin{cases} (1) & \text{if } r = 0, \\ (1+(-)^r T) & \text{if } r = 2^p \text{ for some } p \geqslant 0, \\ (0) & \text{otherwise.} \end{cases}$$

(Sesquilinear forms on $C(\tilde{X})_r$ are identified, for $r \geqslant 0$, with 1×1-matrices with coefficients in A; apart from that, the notation of I, 1.4 has been used.)

Theoretically, 11.4 can be verified using 10.1 and 10.13. However, the isomorphism in 10.1 is very mysterious. A geometric proof of 11.4 is given below (after 11.7).

11.5. Corollary. *The algebraic transfer*

$$L^n(C(\tilde{X}), c(\gamma)) \to L^n(C(X''), c(\gamma''))$$

is zero if $n = 2k < m$, *for* k *odd with* $k \neq 2^p - 1$.

Explanation and proof. The transfer is associated with the inclusion

$$\{1\} \hookrightarrow Z_2 = \pi.$$

Using, for example, 6.1, one finds, for $m > n > 0$, that

$$L^n(C(X''), c(\gamma'')) = \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 \bmod 4 \text{ ('signature/8')}, \\ Z_2 & \text{if } n \equiv 1 \text{ or } 3 \bmod 4 \text{ ('Hopf invariant')}, \\ Z_2 & \text{if } n \equiv 2 \bmod 4 \text{ ('Kervaire invariant')}. \end{cases}$$

For the proof of 11.5, apply 8.2 and 8.3: there is a commutative diagram

$$\begin{tikzcd} L_n(\mathbb{Z}[\{1\}] \arrow[r] & L^n(C(X''), c(\gamma'')) \\ L_n(\mathbb{Z}[\pi]) \arrow[r] \arrow[u, "\text{transfer}"'] & L^n(C(\tilde{X}), c(\gamma)) \arrow[u, "\text{transfer}"'] \end{tikzcd}$$

If $n = 2 \bmod 4$, the top horizontal arrow is isomorphic and the left vertical arrow is zero (by [**13**, Chapter 13A]). If, moreover, $n \neq 2(2^p - 1)$, then 8.2 and 8.3 imply that the transfer on the right is also zero.

11.6. COROLLARY. *If $n = 2k$ is as in* 11.5, *then the Kervaire invariant $\pi_{2k}^s \to Z_2$ is zero.*

Proof. There is a commutative diagram

$$\begin{tikzcd} \pi_{2k}^s = \pi_{2k}(M(X'', \gamma'')) \arrow[r] & L^{2k}(C(X''), c(\gamma'')) \cong Z_2 \\ \pi_{2k}(M(X, \gamma)) \arrow[r] \arrow[u, "\substack{\text{geometric} \\ \text{transfer}}"] & L^{2k}(C(\tilde{X}), c(\gamma)) \arrow[u, "\substack{\text{algebraic} \\ \text{transfer}}"'] \end{tikzcd}$$

in which
the left vertical arrow is surjective (Kahn–Priddy),
the right vertical arrow is zero (by 11.5),
so that the horizontal arrow (which is the Kervaire invariant) is also zero.

To conclude the chapter, here is the geometric proof of 11.4. The idea and construction are based on I, § 3.A; so we shall produce a sequence $\{P^n \mid n \geqslant 0\}$ of framed manifolds (each P^n having the homotopy type of the $[\frac{1}{2}n]$-skeleton of $\mathbb{R}P^m$, etc.) such that the sesquilinear sliding forms λ_r (cf. I, 3.A.2 (ii)) are as specified in 11.4. We use the standard CW-structure on $\mathbb{R}P^m$.

Suppose that $P^0, P^1, \ldots, P^{2k}$ have already been constructed so that the sliding forms $\lambda_0, \lambda_1, \ldots, \lambda_k$ are the required ones. (Assume that $m \geqslant 2k > 2$, otherwise there is little to prove.)

CLAIM. *Let z be a generator of $\pi_k(P^{2k}) \cong \pi_k(\mathbb{R}P^k) \cong \mathbb{Z}$. Because of Hirsch's immersion theorem* [**6**, **5**], *and because P^{2k} is framed, z determines a regular homotopy class of immersions $i_k\colon S^k \to P^{2k}$. The self-intersection number $\mu(i_k)$ of this immersion equals $1 + T$ if $k = 2^p - 1$ for some integer $p > 1$, and 0 otherwise. (It belongs to $\mathbb{Z}[Z_2]$ if k is even, to $Z_2[Z_2]$ if k is odd.)*

Assuming the truth of the claim, we can easily construct framed manifolds P^{2k+1}, P^{2k+2}, etc. giving the correct sliding form λ_{k+1}.

To prove the claim, we can suppose that k is odd. (For even k, there is nothing to prove because $\mu(i_k)$ is algebraically determined by the sliding form λ_k: we have $2\mu(i_k) = \lambda_k(1-T, 1-T)$, since i_k represents the element $1-T$ in $C(\tilde{X})_k = \mathbb{Z}[Z_2]$.)

Since k is odd, there are just two regular homotopy classes of immersions $S^k \to P^{2k}$ homotopic to i_k; if i_k has self-intersection number $a \cdot 1 + b \cdot T$ (with $a, b \in Z_2$), then the immersions in the class not containing i_k have self-intersection number $(a+1) \cdot 1 + b \cdot T$.

Let i_k': $S^k \to P^{2k}$ be an immersion which factors as follows:

$$S^k \xrightarrow{\text{double cover}} \mathbb{R}P^k \xrightarrow{\quad g \quad} P^{2k},$$

where g is an immersion and also a homotopy equivalence. Arguing exactly as in [3], one finds that the self-intersection number of i_k' is $0 \cdot 1 + 1 \cdot T \in Z_2[Z_2]$ if $k = 2^p - 1$, and 0 otherwise.

Let f: $P^{2k} \to \mathbb{R}^{2k}$ be the codimension-0 immersion determined, up to regular homotopy, by the framing of P^{2k}. Then the immersion $f \cdot i_k'$: $S^k \to \mathbb{R}^{2k}$ has self-intersection number $1 \in Z_2$ if $k = 2^p - 1$, and 0 otherwise (also by [3]). From the definition of i_k, it is clear that $f \cdot i_k$: $S^k \to \mathbb{R}^{2k}$ has self-intersection number 0 for all k, being regularly homotopic to the standard embedding. Therefore i_k and i_k' are regularly homotopic if and only if $k \neq 2^p - 1$.

Putting these observations together proves the claim.

References

1. W. Browder, 'The Kervaire invariant of framed manifolds and its generalisations', *Ann. of Math.*, 90 (1969), 157–186.
2. E. H. Brown, Jr., 'Generalisations of the Kervaire invariant', *Ann. of Math.*, 95 (1972), 368–383.
3. E. H. Brown, Jr., 'A remark concerning immersions of S^n in $\mathbb{R}^{2n}$', *Quart. J. Math. Oxford* (2), 24 (1973), 559–560.
4. A. Dold, *Halbexakte Homotopiefunktoren*, Lecture Notes in Mathematics 12 (Springer, Berlin, 1966).
5. A. Haefliger and V. Poenaru, 'La classification des immersions combinatoires', Publications Mathématiques 23 (Institut des Hautes Études Scientifiques, Paris, 1964), pp. 75–91.
6. M. Hirsch, 'Immersions of manifolds', *Trans. Amer. Math. Soc.*, 93 (1959), 242–276.
7. J. Jones and E. Rees, 'Kervaire's invariant for framed manifolds', Proceedings of Symposia 32 (American Mathematical Society, Providence, R.I., 1978), pp. 141–147.
8. J. Jones and E. Rees, 'A note on the Kervaire invariant', *Bull. London Math. Soc.*, 7 (1975), 279–282.
9. D. Kahn and S. Priddy, 'Applications of the transfer to stable homotopy theory', *Bull. Amer. Math. Soc.*, 78 (1972), 981–987.
10. F. Quinn, 'Surgery on Poincaré and normal spaces', *Bull. Amer. Math. Soc.*, 78 (1972), 262–267.
11. A. Ranicki, 'The algebraic theory of surgery I, II', *Proc. London Math. Soc.*, (3), 40 (1980), 87–283.
12. A. Ranicki, *Exact sequences in the algebraic theory of surgery*, Mathematics Notes (Princeton University Press, 1981).
13. C. T. C. Wall, *Surgery on compact manifolds* (Academic Press, London, 1970).
14. M. Weiss, 'Surgery and the generalized Kervaire invariant, I', *Proc. London Math. Soc.* (3), 51 (1985), 146–192.

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