SURGERY AND THE GENERALIZED KERVAIRE INVARIANT, I

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Introduction

(i) Synopsis. The discovery, around 1960, of the 'Kervaire Invariant' for almost framed manifolds of dimension 4k + 2 (see [12]) was an important stimulant for the development of surgery theory; but it also led to the theory of the 'generalized Kervaire Invariant' of Browder and Brown [2, 3].

The present paper is an attempt at uniting these two theories, by constructing a non-simply-connected and in other respects updated version of the generalized Kervaire Invariant.

The construction has three surprising aspects. Firstly, it is conceptually satisfying and, in the simply-connected case, clarifies Brown's original theory; for instance, the 'product formula problem' (see [4]) evaporates. Most of the new concepts are borrowed from the 'algebraic theory of surgery'; see [15]. Secondly, it is computationally satisfying. Thirdly, it has applications to classical surgery theory, especially to the calculation of the symmetric L-groups of [13] and [15]; and therefore to anything which involves product formulae for surgery obstructions.

A black box description of the theory has been given in [22]; in this introduction I shall concentrate on the concepts inside the box.

(ii) Symmetric forms on (co-)homology groups vs. symmetric forms on chain complexes. If M is a 2k-dimensional geometric Poincaré complex, then $H^k(M; Z_2)$ carries a nondegenerate symmetric bilinear form.

Let γ be a spherical fibration on a space X. Brown's generalization of the Kervaire Invariant [3] is based on the observation that, under certain conditions on γ , a bundle map $\nu_M \rightarrow \gamma$ (with underlying classifying map $f: M \rightarrow X$) determines a refinement of the symmetric form on $H^k(M; \mathbb{Z}_2)$ to a quadratic form with values in \mathbb{Z}_4 ; here ν_M denotes the Spivak normal fibration of M.

So much for the simply-connected theory. (The term 'simply-connected' is a little misleading here; of course X above does not have to be simply-connected—it is just that we pay no special attention to $\pi_1(X)$.) Now let π be a discrete group, let $w: \pi \to Z_2$ be a homomorphism, let α be a principal π -bundle on X, and let j be an identification of the two double covers arising from the given data (namely, the orientation cover associated with γ , and $w^{\neg}(\alpha)$). Such sextuples $(\pi, w; X, \gamma; \alpha, j)$ will be the main objects of study.

For a geometric Poincaré complex M^n with bundle map $v_M \to \gamma$ and underlying classifying map $f: M \to X$ as before (but with *n* arbitrary), let $C(\tilde{M})$ be the cellular chain complex of the total space of $f^{-}(\alpha)$ (the principal π -bundle on M; assuming that M is a CW-space). Now $C(\tilde{M})$ is a chain complex of finitely generated (f.g.) projective left A-modules, and $A = \mathbb{Z}[\pi]$ is regarded as a ring with involution (the w-twisted

involution.

$$\sum_{g\in\pi} n_g \cdot g \mapsto \sum_{g\in\pi} n_g \cdot (-)^{w(g)} \cdot g^{-1}).$$

What next? Inspection shows that the approach of [3] does not work very well here, even if n = 2k. That is, some kind of symmetric bilinear form is defined on $H^{k}(C(\tilde{M}); A)$, but it has very unpleasant properties in general.

The solution to this dilemma is suggested by recent developments in surgery theory. In [13] and [15], the notion of 'symmetric bilinear form on a chain complex (of f.g. projective left A-modules)' is defined or implicit (details follow below); the notion is homotopy invariant, and it is shown that the dual chain complex of $C(\tilde{M})$, written $C(\tilde{M})^{-*} = \operatorname{Hom}_{A}(C(\tilde{M}), A)$, carries a non-degenerate symmetric bilinear form. (Again, the dimension n is arbitrary.) So, instead of trying to 'refine' a symmetric bilinear form on some middle-dimensional cohomology group of $C(\tilde{M})$, we shall refine Mishchenko's non-degenerate symmetric bilinear form on the chain complex $C(\tilde{M})^{-*}$.

(iii) The technical terms. Let C be a chain complex of f.g. projective left A-modules, with $C_r = 0$ except for a finite number of indices r. Using the involution on A, we can also regard C as a chain complex of right A-modules, written C'. So $C' \otimes_A C$ is defined, and is a chain complex of $\mathbb{Z}[\mathbb{Z}_2]$ -modules; the generator $T \in \mathbb{Z}_2$ acts by switching factors, with the usual sign rules.

Define $\mathbb{Z}[\mathbb{Z}_2]$ -module chain complexes W, \hat{W} by

$$W_r = \begin{cases} \mathbb{Z}[\mathbb{Z}_2] & \text{for } r \ge 0, \\ 0 & \text{for } r < 0, \end{cases}$$

with d: $W_r \rightarrow W_{r-1}$; $x \mapsto (1+(-)^r T) \cdot x$ for r > 0, and by

$$\widehat{W}_r = \mathbb{Z}[\mathbb{Z}_2]$$
 for all r ,

with $d: \hat{W}_r \to \hat{W}_{r-1}; x \mapsto (1+(-)^r T) \cdot x$. Write W & C and $\hat{W} \& C$ for the chain complexes (of abelian groups) $\operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C' \otimes_A C)$ and $\operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\hat{W}, C' \otimes_A C)$, respectively. Finally, let

$$Q^{n}(C) := H_{n}(W \& C) \text{ and } \hat{Q}^{n}(C) := H_{n}(\hat{W} \& C).$$

The covariant functors $C \mapsto Q^n(C)$, $C \mapsto \hat{Q}^n(C)$ are homotopy invariant (this is proved in [15], but also in this paper).

Now define a 'symmetric bilinear form of degree n on C' to be

either an *n*-cycle
$$\varphi \in W \& C^{-*}$$

or a class $[\varphi] \in Q^n(C^{-*}) = H_n(W \& C^{-*}).$

(There are two different schools of thought here; I find the first definition better to work with, but the second is homotopy invariant.) We now give some 'motivations'.

(a) The *n*-cycle φ is a $\mathbb{Z}[\mathbb{Z}_2]$ -module chain map from $\Sigma^n W$ to

$$C^{-*'} \otimes_A C^{-*} = \operatorname{Hom}_A(C, C^{-*})$$

In particular, the value of φ on $1 \in \mathbb{Z}[\mathbb{Z}_2] = W_0$ is an *n*-cycle in Hom₄(C, C^{-*}), that is, a chain map

$$\varphi_0: \Sigma^n C \to C^{-*}$$

which can be considered as a bilinear form of degree n on C.

(b) If V_1 , V_2 are vector spaces over a field F (for example, $F = Z_2$), then there is an abelian group isomorphism

(symmetric bilinear forms on $V_1 \oplus V_2$)

 \cong (symmetric bilinear forms on V_1)

 \oplus (symmetric bilinear forms on V_2) \oplus Hom_F(V_1 , V_2).

In generalizing the notion of symmetric bilinear form to (certain) chain complexes over A, this is a property one would like to retain; now it is indeed true that

 $Q^{n}((C \oplus D)^{-*}) \cong Q^{n}(C^{-*}) \oplus Q^{n}(D^{-*}) \oplus H_{n}(\operatorname{Hom}_{A}(C, D^{-*})),$

and there is clearly no simpler definition of 'symmetric bilinear form on a chain complex' which is homotopy invariant and has this property.

(c) It is shown in [13] and [15] that a closed manifold or geometric Poincaré complex M^n , equipped with a principal π -bundle α' and an identification of double covers

 $w^{\neg}(\alpha') \cong$ (orientation cover of M^n),

gives rise to a pair (C, φ) in which $C = C(\tilde{M})$, and φ is an *n*-cycle in W & C (sufficiently well determined). In the terminology introduced above, φ is a symmetric bilinear form of degree *n* on C^{-*} ; in the case at hand, φ is non-degenerate, that is, the chain map

$$\varphi_0: \Sigma^n(C^{-*}) \to C = C^{-*-*}$$

is a chain homotopy equivalence.

A pair (C, φ) as above (with φ non-degenerate) is called an '*n*-dimensional (symmetric) algebraic Poincaré complex'. It is treated here as the chain level analogue of a closed manifold or geometric Poincaré complex, just as chain complexes (over A) are treated as the analogues of spaces. 'Bordisms' between symmetric algebraic Poincaré complexes of the same dimension can be defined, etc.

(iv) *Chain bundles*. If spaces and closed manifolds (geometric Poincaré complexes) have analogues in the chain complex world, what about vector bundles (or spherical fibrations)?

By 'chain complex world' is meant the category \mathscr{C}_A of chain complexes C of f.g. projective left A-modules, with $C_r = 0$ except for a finite number of indices r. The conceptual vacuum is filled as follows:

a 'chain bundle' on a chain complex C (in \mathscr{C}_A) is a 0-dimensional cycle in $\hat{W} \& C^{-*}$.

Motivation for this definition is as follows.

(a) The homotopy invariant (and contravariant) functors

$$C \mapsto \hat{Q}^{-n}(C^{-*}), \quad n \in \mathbb{Z},$$

constitute a cohomology theory on \mathscr{C}_A ; that is, they satisfy the analogues of the Eilenberg-Steenrod axioms in the (co-)homology theory of spaces, except the dimension axiom.

(b) It is shown in [15] that a sextuple $(\pi, w; X, \gamma; \alpha, j)$ as in (ii) above determines a 'characteristic class' in $\hat{Q}^0(C(\tilde{X})^{-*})$ [15, Part II, 9.3]; similarly, a stable automorphism of γ determines a class in $\hat{Q}^1(C(\tilde{X})^{-*})$ [15, Part II, 9.9]. This suggests that the cohomology theory $C \mapsto \{\hat{Q}^{-n}(C^{-*}) \mid n \in \mathbb{Z}\}$ is the chain level analogue of spherical *K*-theory. So it is reasonable to expect that chain bundles on chain complexes *C* (that is, representing cycles for elements in $\hat{Q}^0(C^{-*})$ are the chain level analogues of spherical fibrations. And indeed it is possible to refine the 'characteristic class' above to a rule which associates chain bundles to spherical fibrations. (*Warning*: this rule is not additive; the geometric Whitney sum has not much to do with the addition in $\hat{W} \& C^{-*}$.)

(c) The definition of 'chain bundle' is so designed that any symmetric algebraic Poincaré complex (C, φ) carries a 'normal chain bundle' (the chain level analogue of the Spivak normal bundle of a geometric Poincaré complex). For details, see the main text; the idea stems from [15, Part II, 9.6].

The non-simply-connected generalized Kervaire Invariant can now be described as follows. Let $(\pi, w; X, \gamma; \alpha, j)$ be a sextuple as in (ii). The chain level image of γ is a chain bundle $c(\gamma)$ on the chain complex $C(\tilde{X})$ (over $A = \mathbb{Z}[\pi]$). Using the dictionary

space \leftrightarrow chain complex of projective left A-modules,

geometric Poincaré complex $M \leftrightarrow$ symmetric algebraic Poincaré complex (C, φ) ,

spherical fibration \leftrightarrow chain bundle,

Spivak normal fibration of $M \leftrightarrow$ normal chain bundle of (C, φ) ,

one obtains homomorphisms (for $n \in \mathbb{Z}$)

flexible signature: $\Omega_n^P(X, \gamma) \to L^n(C(\tilde{X}), c(\gamma)).$

Here $\Omega_n^P(X, \gamma)$ is the bordism group of geometric Poincaré complexes M^n equipped with a map of spherical fibrations from v_M to γ ; similarly, $L^n(C(\tilde{X}), c(\gamma))$ is the bordism group of *n*-dimensional algebraic Poincaré complexes (C, φ) over A, equipped with a 'chain bundle map' from the normal chain bundle (on C) to the chain bundle $c(\gamma)$ (on $C(\tilde{X})$).

The relationship with the Wall groups $L_n(\mathbb{Z}[\pi])$ is as follows. Firstly, if the sextuple $(\pi, w; X, \gamma; \alpha, j)$ is such that $X = \emptyset$, then $L^n(C(\tilde{X}), c(\gamma)) \cong L_n(\mathbb{Z}[\pi])$. Secondly, if X is arbitrary again, we may still consider the inclusion $\emptyset \subset X$; it induces homomorphisms

release:
$$L_n(\mathbb{Z}[\pi]) \to L^n(C(\tilde{X}), c(\gamma))$$

for $n \in \mathbb{Z}$. Now let $f: M^n \to N^n$ be a degree-1 normal map between geometric Poincaré complexes. Suppose that N is equipped with a map of spherical fibrations from v_N to γ . Then the normal map f induces a similar structure on M, and the equation

$$\sigma^*(M) - \sigma^*(N) = \text{release}(\sigma_*(f))$$

holds. Here σ^* is the flexible signature (in $L^n(C(\tilde{X}), c(\gamma))$), and σ_* is the surgery obstruction (in $L_n(\mathbb{Z}[\pi])$).

(v) Computations. Let ℓ be any chain bundle on a chain complex B (in \mathscr{C}_A , for a ring with involution A). Write 0 for the only chain bundle on the zero chain complex 0_A in \mathscr{C}_A . The inclusion $0_A \hookrightarrow B$ is covered by a unique 'chain bundle map'; so there are induced homomorphisms of algebraic bordism groups

release:
$$L^{n}(0_{A}, 0) \rightarrow L^{n}(B, \ell)$$
.

These algebraic bordism groups are defined just like $L^n(C(\tilde{X}), c(\gamma))$ in the preceding section.

Now $L^n(0_A, 0)$ is naturally isomorphic to the Wall group $L_n(A)$ (see [21] or [15]). Further, $L^n(0_A, 0) \cong L_n(A)$ and $L^n(B, \ell)$ are the *n*th homotopy groups of certain spectra $\mathscr{L}^{(0)}(0_A, 0)$ and $\mathscr{L}^{(0)}(B, \ell)$ respectively, and the release homomorphisms are induced by a map of spectra,

release:
$$\mathscr{L}^{:}(0_{A}, 0) \rightarrow \mathscr{L}^{:}(B, \mathscr{E}).$$

Let $\hat{L}^n(B, \ell)$ be the *n*th (relative) homotopy group of 'release'. If we succeed in calculating $\hat{L}^n(B, \ell)$ for all *n*, then we have largely reduced the calculation of $\{L^n(B, \ell) \mid n \in \mathbb{Z}\}$ to that of $\{L^n(0_A, 0) = L_n(A) \mid n \in \mathbb{Z}\}$. The following theorem shows, surprisingly, that the groups $\hat{L}^n(B, \ell)$ are homological objects and therefore usually easy to compute.

MAIN THEOREM. There is a natural long exact sequence

$$\dots \to \hat{Q}^{n+1}(B) \to \hat{L}^n(B, \ell) \to Q^n(B) \to \hat{Q}^n(B) \to \hat{L}^{n-1}(B, \ell) \to \dots \quad (n \in \mathbb{Z}).$$

The proof uses the algebraic surgery techniques of [15]. The standard application is to the case where $B = C(\tilde{X})$, $\ell = c(\gamma)$ as in (iv). However, the main theorem has another application (to the more classical surgery theory): let B be the 'classifying chain complex for chain bundles' and ℓ the 'universal chain bundle' on B. (So the role of B in the chain complex world is similar to that of the spaces BO or BG in topology.) Then $L^n(B, \ell)$ is the bordism group of symmetric algebraic Poincaré complexes of dimension n (with no particular structure), called $L^n(A)$ in [15]. The groups $L^n(A)$ are useful in obtaining product formulae for surgery obstructions. The main theorem above shows that the relative terms $\hat{L}^n(A) = \hat{L}^n(B, \ell)$ appearing in the long exact sequence relating $L_*(A)$ and $L^*(A)$ are homological objects. The homological description of $\hat{L}^n(A)$ is made even more explicit by a complete analysis of the 'classifying chain complex for chain bundles' which is obtained in Part II [23], for $A = \mathbb{Z}[\pi]$. (The result: it is as simple as it can be.)

(vi) The 'ordinary generalized Kervaire Invariant' revisited. The theory outlined so far has an unoriented version: instead of working with sextuples $(\pi, w; X, \gamma; \alpha, j)$, consider quadruples $(\pi; X, \gamma; \alpha)$ and replace $\mathbb{Z}[\pi]$ by $Z_2[\pi]$. The resulting algebraic bordism groups will be written $L^n(C(\tilde{X}; Z_2), c(\gamma; Z_2))$; here $C(\tilde{X}; Z_2) = C(\tilde{X}) \otimes_{\mathbb{Z}} Z_2$, etc.

Now assume further that $\pi = \{1\}$. In this case the flexible signature can be considered as a mild improvement on the 'generalized Kervaire Invariant' of [3]. That is to say, if n = 2k and the (k+1)th Wu class of γ (in $H^{k+1}(X; \mathbb{Z}_2)$) is zero, there is a commutative diagram



in which the horizontal arrow is the invariant of [3]. The homomorphism $L^{2k}(C(X; Z_2), c(\gamma; Z_2)) \rightarrow Z_8$ is obtained by adapting the methods of [3]. Elements of $L^{2k}(C(X; Z_2), c(\gamma; Z_2))$ are represented by 2k-dimensional algebraic Poincaré complexes (C, φ) (over the ring with involution $Z_2 = Z_2[\{1\}]$), with a certain

structure; and this structure permits one to refine the non-degenerate symmetric bilinear form on $H^k(C; \mathbb{Z}_2)$ to a quadratic form, with values in \mathbb{Z}_4 .

(Recall that the invariant of [3] is non-canonical, i.e. depends on a choice; the same is true for its algebraic counterpart, $L^{2k}(C(X; Z_2), c(\gamma; Z_2)) \rightarrow Z_8$, and there is a one-one correspondence between the two kinds of choices.)

Summarizing, it seems legitimate in the case at hand to regard the flexible signature itself as 'the' generalized Kervaire Invariant. It is defined for arbitrary n, without conditions on the spherical fibration γ , involves no choices, and looks pretty in product formulae. Finally, the groups $L^n(C(X; Z_2), c(\gamma; Z_2))$ are easy to compute with the help of the main theorem in (v) above. (Remember that the functors $Q^n(-)$, $\hat{Q}^n(-)$ are homotopy invariant, and that any chain complex over Z_2 is homotopy equivalent to its homology.)

(vii) A Caveat. Let $(\pi, w; X, \gamma; \alpha, j)$ be a sextuple as in (ii) again, and suppose that γ is the trivial spherical fibration on X. One is tempted to think that the release homomorphisms

$$L_n(\mathbb{Z}[\pi]) \to L^n(C(\tilde{X}), c(\gamma))$$

have particularly attractive properties in that case, such as being split injective; but they are probably not even injective in general.

(Here is an example to meditate upon. Let $X = \mathbb{R}P^{\infty}$, let $\tilde{X} = S^{\infty}$, and let γ , $\tilde{\gamma}$ be the trivial spherical fibrations on X and \tilde{X} respectively. Then, for n = 2, 6, 14, 30, or 62(?), the composite

 $\Omega_n^P(X,\gamma) \xrightarrow{\text{transfer}} \Omega_n^P(\tilde{X},\tilde{\gamma}) \xrightarrow{\text{Kervaire Invariant}} Z_2$

is surjective, but

 $L_n(\mathbb{Z}[Z_2]) \xrightarrow{\text{transfer}} L_n(\mathbb{Z}[\{1\}]) \cong Z_2$

is the zero map.)

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0. Conventions

Since the present paper is inspired mainly by [15], I have tried to retain Ranicki's conventions wherever possible. They are listed here for convenience, with a few alterations and additions that appeared necessary.

The symbol 'II' refers to Part II of this paper [23].

0.1. The letter A is usually reserved for a *ring with involution*, that is, a ring with 1 equipped with an involutory antiautomorphism

$$f: A \rightarrow A; a \mapsto \overline{a}.$$

0.2. Unless otherwise specified, 'A-module' will mean left A-module. Sometimes, however, it is necessary to shift an A-action from left to right; so if M is a left A-module, let M' be the right A-module with the same additive group as M, and with A-action

$$M' \times A \rightarrow M'; (x, a) \mapsto \bar{a}x.$$

0.3. The dual module M^* of an A-module M is

$$M^* = \operatorname{Hom}_A(M, A)$$

with A acting (on the left) by

$$A \times M \to M; (a, f) \mapsto (x \mapsto f(x) \cdot \overline{a}).$$

If M is finitely generated projective, then so is M^* , and the A-module homomorphism $M \to M^{**}$; $x \mapsto (f \mapsto \overline{f(x)})$ is an isomorphism.

The dual of an A-module homomorphism $f: M \to N$ is the A-module homomorphism $f^*: N^* \to M^*; g \mapsto g \cdot f$.

0.4. Two chain maps $f, f': C \to D$ (between A-module chain complexes graded over the integers) are homotopic if there exists a collection of A-module homomorphisms $\{g_r: C_r \to D_{r+1} | r \in \mathbb{Z}\}$ so that $f'-f = d_Dg + gd_C: C \to D$. The collection is called a *chain homotopy* from f to f'.

0.5. Still assuming that C, D are A-module chain complexes, define abelian group chain complexes $C' \otimes_A D$ and $\operatorname{Hom}_A(C, D)$ by

$$(C^{t} \otimes_{A} D)_{n} = \bigoplus_{p+q=n} C_{p}^{t} \otimes_{A} D_{q},$$

$$d(x \otimes y) = x \otimes d_{D}(y) + (-)^{q} d_{C}(x) \otimes y;$$

$$(\operatorname{Hom}_{A}(C, D))_{n} = \prod_{q-p=n} \operatorname{Hom}_{A}(C_{p}, D_{q}),$$

$$d(f) = d_{D}(f) - (-)^{n} f d_{C}.$$

In both cases, d_D and d_C denote the differentials on D and C respectively, and d the differential on $C^t \otimes_A D$ or on $\text{Hom}_A(C, D)$.

(The definition of $C^t \otimes_A D$ agrees well with certain geometric constructions, and with that given in [15]. Ranicki [15, 16] has a slightly different definition for Hom_A(C, D); I prefer the above because it gives more direct identifications

(Group of cycles in
$$(\text{Hom}_A(C, D))_0) \cong (\text{Group of chain maps } C \to D)$$

and

 $H_0(\text{Hom}_A(C, D)) \cong (\text{Group of chain homotopy classes of chain maps from } C \text{ to } D).)$

Note that the involution on A does not appear in the definition of $\text{Hom}_A(C, D)$; we can do without it.

0.6. The dual chain complex C^{-*} of an A-module chain complex C is given by $(C^{-*})_r := (C_{-r})^* =: C^{-r}$, with differential

$$(-)^r d_C^* \colon C^{-r} \to C^{-r+1}.$$

A chain map $f: C \to D$ induces $f: D^{-*} \to C^{-*}; g \mapsto g \cdot f$.

Given an abelian group (or A-module) G, write (G, n) for the chain complex with $(G, n)_r = G$ if r = n and $(G, n)_r = 0$ otherwise. It is worth noting that the 'obvious' identification $C^{-*} = \text{Hom}_A(C, (A, 0))$ is not a chain map; some sign changes are necessary. Still, the choice of differential in C^{-*} has certain advantages. (It agrees with [15, p. 104, bottom], but not with [15, p. 98, bottom].)

If C is a chain complex of f.g. projective A-modules, then the chain map

$$C \to C^{-*-*}; x \to (f \mapsto (-)^r f(x))$$

is an isomorphism. If also $C_r \doteq 0$ except for finitely many $r \in \mathbb{Z}$, then the slant chain map

$$C^{t} \otimes_{A} D \to \operatorname{Hom}_{A}(C^{-*}, D),$$
$$x \otimes y \to (f \mapsto \overline{f(x)} \cdot y)$$

is an isomorphism.

0.7. The suspension ΣC of a chain complex

$$C: \ldots \to C_{r+1} \to C_r \to C_{r-1} \to \ldots$$

is the chain complex $(\Sigma C)_r = C_{r-1}$, $d_{\Sigma C} = -d_C$. (This differs from the definition in [15] by a sign.) So $\Sigma C \cong C \otimes_{\mathbb{Z}} (\mathbb{Z}, 1)$ (cf. 0.6), and

 $H_n(\operatorname{Hom}_A(C, D)) \cong ($ group of homotopy classes of chain maps from $\Sigma^n C$ to D),

if C, D are A-module chain complexes.

Let I stand for the cellular chain complex of the standard 1-simplex, or unit interval. Corresponding to the two endpoints of the 1-simplex, there are two chain maps $i_0, i_1: (\mathbb{Z}, 0) \to I$; if $f: C \to D$ is a chain map, we define the mapping cylinder of f to be the pushout of the diagram



and similarly for the mapping cone, Cone(f). So

 $\operatorname{Cone}(f)_r = D_r \oplus C_{r-1},$

with differential d given by

 $d(x, y) = (d_D(x) + f(y), -d_C(y)).$

If $f = id: C \rightarrow C$, write Cone(C) instead of Cone(id).

Every chain map $f: C \rightarrow D$ has an associated Puppe sequence

 $\dots \to \Sigma^{-1}D \to \Sigma^{-1}\operatorname{Cone}(f) \to C \to D \to \operatorname{Cone}(f) \to \Sigma C \to \Sigma D \to \dots,$

infinite on both sides.

0.8. \mathscr{C}_A (or simply \mathscr{C}) will be the category of chain complexes C, graded over the integers, such that each C_r is a f.g. projective left A-module, and such that $C_r = 0$ except for finitely many $r \in \mathbb{Z}$; the morphisms in \mathscr{C}_A are A-module chain maps.

A morphism $f: C \to D$ in \mathscr{C}_A is called a *fibration* if it is surjective, and a *cofibration* if its dual f^* is a fibration.

0.9. If C is a chain complex in \mathscr{C}_A , the group Z_2 acts on the abelian group chain complex $C' \otimes_A C$ (cf. 0.5) by the transposition involution

$$T: C_p^{\iota} \otimes_A C_q \to C_p^{\iota} \otimes_A C_q; x \otimes y \mapsto (-)^{pq} y \otimes x.$$

Following [15], we shall have to deal with the 'cohomology groups of Z_2 with coefficients in the $\mathbb{Z}[Z_2]$ -module chain complex $C' \otimes_A C$ ' and the like. Here $\mathbb{Z}[Z_2]$ is the group ring, without any particular involution; we let W be the standard free resolution of the trivial $\mathbb{Z}[Z_2]$ -module \mathbb{Z} (viz., $W_r = \mathbb{Z}[Z_2]$ if $r \ge 0$, $W_r = 0$ if r < 0, with differential $d: W_r \to W_{r-1}$; $x \mapsto (1 + (-)^r T) \cdot x$ where r > 0, T being the generator of Z_2), and write W & C for the abelian group chain complex

$$\operatorname{Hom}_{\mathbb{Z}[Z]}(W, C^{t} \otimes_{A} C).$$

Then

$$Q^n(C) := H_n(W\& C)$$

is the (-n)th cohomology group of Z_2 with coefficients in $C' \otimes_A C$. (See [10] if the terminology appears mysterious.)

On replacing the standard resolution \hat{W} by the standard complete resolution \hat{W} (with $\hat{W}_r = \mathbb{Z}[\mathbb{Z}_2]$ for all $r, d: \hat{W}_r \to \hat{W}_{r-1}; x \mapsto (1 + (-)^r T) \cdot x$, for all r) we obtain a chain complex $\hat{W} \& C := \text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\hat{W}, C' \otimes_A C)$ whose homology groups

$$\widehat{Q}^n(C) := H_n(\widehat{W}\& C)$$

are the Tate cohomology groups of Z_2 with coefficients in $C' \otimes_A C$.

0.10. We will need a detailed description of the abelian group chain complexes $\hat{W} \& C$, W & C, and $\hat{W} \& C^{-*} (= W \& (C^{-*}))$.

Case 1: \hat{W} & C. Then

$$(\widehat{W}\& C)_n = \prod_s \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}_s, (C^t \otimes_A C)_{n+s})$$
$$= \prod_s (C^t \otimes_A C)_{n+s},$$

and the differential d sends a collection

$$\varphi = \{\varphi_s \in (C^t \otimes_A C)_{n+s} | s \in \mathbb{Z}\} \in (\tilde{W}\& C)_n$$

to $d(\varphi) \in (\tilde{W}\& C)_{n-1}$, with $(d(\varphi))_s \in (C^t \otimes_A C)_{n-1+s}$ given by

$$d_{C^{\iota} \otimes AC}(\varphi_{s}) - (-)^{n}(\varphi_{s-1} + (-)^{s}T\varphi_{s-1}).$$

If we use the identification

$$C' \otimes_A C \cong \operatorname{Hom}_A(C^{-*}, C)$$

of 0.6, then φ can be regarded as a collection

$$\{\varphi_s \in \prod_r \operatorname{Hom}_{\mathcal{A}}(C^{n-r+s}, C_r) | s \in \mathbb{Z}\};$$

the differential sends this to $d(\varphi)$, with $(d(\varphi))_s = \prod_r \operatorname{Hom}_A(C^{n-1-r+s}, C_r)$ given by

$$d_{C} \cdot \varphi_{s} - (-)^{n+s} \varphi_{s} \cdot d_{C-*} - (-)^{n} (\varphi_{s-1} + (-)^{s} T \varphi_{s-1}).$$

Here Z_2 acts on Hom_A(C^{-*} , C) by the duality involution

$$T: \operatorname{Hom}_{A}(C^{p}, C_{q}) \to \operatorname{Hom}_{A}(C^{q}, C_{p}); g \mapsto (-)^{pq}g^{*}.$$

If $\varphi = \{\varphi_s\}$ is a cycle, $d(\varphi) = 0$, then φ_s is a chain homotopy from 0 to the chain map

$$(-)^{n}(\varphi_{s-1}+(-)^{s}T\varphi_{s-1}): \Sigma^{n+s-1}(C^{-*}) \to C,$$

for each s.

Case 2: W & C. This is much the same as Case 1, except that we are now dealing with collections $\varphi = \{\varphi_s\}$ such that $\varphi_s = 0$ for s < 0. Identifying $C' \otimes_A C$ with $\operatorname{Hom}_A(C^{-*}, C)$ again, we find that for a cycle φ in $(W \& C)_n$, φ_0 is a chain map from $\Sigma^n(C^{-*})$ to C; it is 'self-dual' up to an infinity of higher chain homotopies (the higher chain homotopies are the φ_s , for s > 0).

Case 3: $\hat{W} \& C^{-*}$. Here we make the identification $C^{-*t} \otimes_A C^{-*} \cong \operatorname{Hom}_A(C, C^{-*})$ (using 0.6 and the chain isomorphism $C \cong C^{-*-*}$ specified there) and find that the differential maps

$$\varphi = \{\varphi_s \in \prod_r \operatorname{Hom}(C_{r-n-s}, C^{-r}) \mid s \in \mathbb{Z}\} \in (\widehat{W} \& C^{-*})_n$$

to $d(\varphi) = \{(d(\varphi))_s | s \in \mathbb{Z}\}$, with $(d(\varphi))_s$ given by

$$d_{C^{-*}} \cdot \varphi_s - (-)^{n+s} \varphi_s \cdot d_C - (-)^n (\varphi_{s-1}(-)^s T \varphi_{s-1}).$$

This time we are obliged to let Z_2 act by

$$T: \operatorname{Hom}_{A}(C_{p}, C^{q}) \to \operatorname{Hom}_{A}(C_{q}, C^{p}); g \mapsto (-)^{pq+p+q} \cdot g^{*}.$$

0.11. (Taken from [15, Part I, §8].) Define the *tensor product* of two rings with involution A, B to be $A \otimes_{\mathbb{Z}} B$, with involution

$$(a \otimes b) \mapsto a \otimes b.$$

If C is an A-module chain complex, and D is a B-module chain complex, then $C \otimes_{\mathbb{Z}} D$ is an $A \otimes_{\mathbb{Z}} B$ -module chain complex, since $A \otimes_{\mathbb{Z}} B$ acts on $C \otimes_{\mathbb{Z}} D$ by $(a \otimes b) \cdot (x \otimes y) = ax \otimes by$. (The differential in $C \otimes_{\mathbb{Z}} D$ is defined with sign conventions as in 0.5.) Then $C \otimes_{\mathbb{Z}} D$ is in $\mathscr{C}_{A \otimes_{\mathbb{Z}} B}$ provided C and D are in \mathscr{C}_A and \mathscr{C}_B

respectively. Under the same conditions there is an identification

$$C^{-*} \otimes_{\mathbb{Z}} D^{-*} \to (C \otimes_{\mathbb{Z}} D)^{-*},$$

$$f \otimes g \mapsto (x \otimes y \mapsto (-)^{|g||x|} f(x) \otimes g(y))$$

(vertical bars as in |g|, |x| denote dimension). There is another identification of $\mathbb{Z}[\mathbb{Z}_2]$ -module chain complexes

$$(C^{\iota} \otimes_{A} C) \otimes_{\mathbb{Z}} (D^{\iota} \otimes_{B} D) \to (C \otimes_{\mathbb{Z}} D)^{\iota} \otimes_{A \otimes_{\mathbb{Z}} B} (C \otimes_{\mathbb{Z}} D),$$
$$(v \otimes w) \otimes (x \otimes y) \mapsto (-)^{|x| ||w|} (v \otimes x) \otimes (w \otimes y)$$

(there are no conditions; $T \in \mathbb{Z}_2$ acts on the left-hand side by

 $T((v \otimes w) \otimes (x \otimes y)) := T(v \otimes w) \otimes T(x \otimes y)).$

The $\mathbb{Z}[\mathbb{Z}_2]$ -module chain complex \hat{W} of 0.9 is equipped with a strictly associative diagonal map

$$\Delta \colon \hat{W} \to \hat{W} \bar{\otimes} \hat{W}; \ \mathbf{1}_s \mapsto \sum_{r=-\infty}^{+\infty} \mathbf{1}_r \otimes T_{s-r}^r \quad (s \in \mathbb{Z}).$$

(The symbol $\overline{\otimes}$ indicates that infinite chains are allowed; subscripts denote dimensions, and \hat{W} , is identified with the ring $\mathbb{Z}[Z_2]$. Regarding W as a factor complex of \hat{W} gives a similar diagonal for W.) It can be used to define exterior products, such as the chain map

$$\times : \widehat{W} \& C \otimes_{\mathbb{Z}} \widehat{W} \& D \to \widehat{W} \& (C \otimes_{\mathbb{Z}} D); \ \varphi \times \theta := (\varphi \otimes \theta) \cdot \Delta$$

(this makes sense if C and D are in \mathscr{C}_A and \mathscr{C}_B respectively). More explicitly, if $\varphi = \{\varphi_s\}$ is an *m*-chain, and $\theta = \{\theta_s\}$ is an *n*-chain (as in 0.7), then

$$(\varphi \times \theta)_s = \sum_{r=-\infty}^{+\infty} \varphi_r \otimes T^r \theta_{s-r}.$$

(So $(\varphi \times \theta)_s$ belongs to

$$((C \otimes_{\mathbb{Z}} D)^{t} \otimes_{A \otimes_{\mathbb{Z}} B} (C \otimes_{\mathbb{Z}} D))_{m+n+s} \cong ((C^{t} \otimes_{A} C) \otimes_{\mathbb{Z}} (D^{t} \otimes_{B} D))_{m+n+s}$$

but be warned that this last identification involves sign changes.)

Of course, \hat{W} can be replaced by W. Apart from being associative, the exterior product has something like a *unit*: namely the triple $(\mathbb{Z}, (\mathbb{Z}, 0), \nu)$, in which \mathbb{Z} is regarded as a ring with involution, $(\mathbb{Z}, 0)$ as a chain complex in $\mathscr{C}_{\mathbb{Z}}$, and ν is the 0-chain in \hat{W} & $(\mathbb{Z}, 0)$ determined by $\nu_0 = 1 \in \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$.

Again, this works with \hat{W} replaced by W.

0.12. For A-modules M, N a sesquilinear map $\lambda: M \times N \to A$ is a biadditive map satisfying

$$\lambda(ax, by) = a\lambda(x, y)b$$

for $a, b \in A$, $x \in M$, $y \in N$.

Taking (left) adjoints we can identify the abelian group of sesquilinear maps $M \times N \rightarrow A$ with Hom_A (M, N^*) ; so λ corresponds to

$$ad(\lambda): x \to (y \mapsto \lambda(x, y)).$$

(Right adjoints will be avoided, although their use would save bars.) The transpose $T\lambda$

of a sesquilinear map $\lambda: M \times N \rightarrow A$ is the sesquilinear map

$$N \times M \rightarrow A$$
; $(y, x) \mapsto \lambda(x, y)$.

Under the left adjoint, T corresponds to the usual transposition

$$\operatorname{Hom}_{A}(M, N^{*}) \to \operatorname{Hom}_{A}(N, M^{*}).$$

If M = N, we speak of a sesquilinear form, and denote the abelian group of such forms by Sel(M).

0.13. If F is a covariant or contravariant functor from a category X to a category Y, and f is a morphism in X, I shall occasionally write f^{\rightarrow} or f^{\leftarrow} instead of F(f) (whereas f^* indicates a 'dual' chain map or homomorphism, as in 0.3 and 0.6). If C is a chain complex in \mathscr{C}_A , and \hat{W} and W are as in 0.9, then the canonical projection $\hat{W} \rightarrow W$ induces a chain map

$$J: W\&C \rightarrow \widehat{W}\&C$$

which in turn induces homomorphisms $Q^n(C) \to \hat{Q}^n(C)$, also denoted by J.

0.14. The homotopy pullback of a diagram $X \to Z \leftarrow Y$ of chain maps is the chain complex P with

and

$$P_n := X_n \otimes Y_n \otimes Z_{n+1}$$

$$d: (x, y, z) \mapsto (-dx, -dy, dz + f(x) - f(y)).$$

0.15. If Y is a chain complex (of free abelian groups, say), then an 'n-cycle in Y, well-defined up to an infinity of higher homologies' is a diagram of chain maps of the following sort:

$$(\mathbb{Z},n) \stackrel{\simeq}{\leftarrow} X \to Y.$$

Here (\mathbb{Z}, n) is defined in 0.6, and X is another chain complex of free abelian groups.

1. Chain bundles

The first part of this section presents a cohomology theory, defined on the category \mathscr{C}_A (see 0.8), which is to real or spherical K-theory as chain complexes (for instance those in \mathscr{C}_A) are to CW-spaces. The ring with involution A is kept fixed.

1.1. THEOREM. The contravariant functors

$$C \mapsto \hat{Q}^n(C^{-*}), \text{ for } n \in \mathbb{Z},$$

(see 0.9) constitute a cohomology theory on \mathscr{C}_A . That is,

(i) if $f, f': C \to D$ are homotopic chain maps in \mathscr{C}_A , then

$$f^{\leftarrow} = f'^{\leftarrow} \colon \hat{Q}^n(D^{-*}) \to \hat{Q}^n(C^{-*}), \quad for \ n \in \mathbb{Z};$$

(ii) there is a natural and canonical equivalence between the two functors

$$C \mapsto \hat{Q}^{n-1}((\Sigma C)^{-*})$$
 and $C \mapsto \hat{Q}^n(C^{-*})$, for $n \in \mathbb{Z}$;

(iii) if $f: C \to D$ is a chain map in \mathscr{C}_A with associated Puppe sequence

$$\dots \rightarrow \Sigma^{-1} \operatorname{Cone}(f) \rightarrow C \rightarrow D \rightarrow \operatorname{Cone}(f) \rightarrow \Sigma C \rightarrow \Sigma D \rightarrow \dots,$$

then the sequence of induced homomorphisms

$$\dots \leftarrow \hat{Q}^{0}((\Sigma^{-1}\operatorname{Cone}(f))^{-*}) \leftarrow \hat{Q}^{0}(C^{-*}) \leftarrow \hat{Q}^{0}(D^{-*}) \leftarrow \hat{Q}^{0}(\operatorname{Cone}(f)^{-*}) \leftarrow \dots$$

is exact.

Proof. Note first that $C \mapsto \hat{Q}^n(C^{-*})$ is a contravariant functor because a chain map $f: C \to D$ induces

$$f^* \otimes f^* \colon D^{-*\iota} \otimes_A D^{-*} \to C^{-*\iota} \otimes_A C^{-*},$$

and hence a chain map

$$f^-: \widehat{W} \& D^{-*} \to \widehat{W} \& C^{-*}$$
 etc.

To prove (i), suppose that $f, f': C \to D$ are homotopic; then so are f^*, f'^* : $D^{-*} \to C^{-*}$. Let g be a homotopy from f^* to f'^* ; think of g as a chain map from $D^{-*} \otimes_{\mathbb{Z}} I$ to C^{-*} . Now the chain maps $f^-, f'^-: \hat{W} \& D^{-*} \to \hat{W} \& C^{-*}$ are also homotopic; the appropriate chain homotopy maps an n-chain φ in $\hat{W} \& D^{-*}$ to the (n+1)-chain $g^-(\varphi \times \omega)$ in $\hat{W} \& C^{-*}$.

(Explanation: choose, once and for all, a 1-chain ω in

$$\hat{W} \& I = \operatorname{Hom}_{\mathbb{Z}[Z_2]}(\hat{W}, I \otimes_{\mathbb{Z}} I)$$

(cf. 0.7) so that $d(\omega) = i_1^-(v) - i_0^-(v)$; as in 0.7, i_0 and i_1 are certain chain maps from $(\mathbb{Z}, 0)$ to *I*, and *v* is the unit mentioned at the end of 0.11. For $\varphi \in (\widehat{W} \& D^{-*})_n$, the exterior product $\varphi \times \omega$ (cf. 0.11) belongs to $(\widehat{W} \& (D^{-*} \otimes_{\mathbb{Z}} I))_{n+1}$, whence $g^-(\varphi \times \omega)$ is in $(\widehat{W} \& C^{-*})_{n+1}$ as required.) See [15] for an explicit formula.

For the proof of (ii) and (iii), recall (for example, from [10]) that a module M over a ring with unit is 'coextended' or 'coinduced' if it has the form $M \cong \operatorname{Hom}_{\mathbb{Z}}(R, G)$ (G being any abelian group, R the ring at issue, acting on the left of $\operatorname{Hom}_{\mathbb{Z}}(R, G)$ by $(a, f) \mapsto (x \mapsto f(x \cdot a))$ for $a \in R$, $f \in \operatorname{Hom}(R, G)$). If, for example, $R = \mathbb{Z}[\mathbb{Z}_2]$, the Tate cohomology groups of a coinduced module M, written

$$\hat{H}^{n}(\mathbb{Z}_{2}; M) := H_{-n}(\operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_{2}]}(\hat{W}, (M, 0)))$$

(see 0.6), are trivial by a simple argument.

Further, let D be a chain complex of $\mathbb{Z}[\mathbb{Z}_2]$ -modules; call D coinduced if $D_r = 0$ except for a finite number of $r \in \mathbb{Z}$, and if D_r is a coinduced module for all r. Then a familiar induction argument [7, Anhang Proposition 2.1] shows that for such a D, we have $H_n(\text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}, D)) = 0$ for $n \in \mathbb{Z}$.

We now exploit this fact. Suppose that $f: C \to D$ is a *cofibration* in \mathscr{C}_A (see 0.8), so that $0 \to C \to D \to D/\text{im}(f) \to 0$ is a short exact sequence in \mathscr{C}_A . Then the sequence of induced maps

$$C^{-*} \otimes_A C^{-*} \xleftarrow{i^-} D^{-*} \otimes_A D^{-*} \xleftarrow{p^-} (D/\operatorname{im}(f))^{-*} \otimes_A (D/\operatorname{im}(f))^{-*}$$

is not short exact, because ker(i^-)/im(p^-) $\neq 0$ in general. However, ker(i^-)/im(p^-) is a *coinduced* chain complex of $\mathbb{Z}[\mathbb{Z}_2]$ -modules, which by the above just suffices to show that the homology groups of the chain complexes $\hat{W} \& C^{-*}$, $\hat{W} \& D^{-*}$, $\hat{W} \& (D/im(f))^{-*}$ fit into a long exact sequence. This proves (iii) (since every morphism in \mathscr{C}_A is 'homotopy equivalent' to a cofibration), or strictly speaking reduces (iii) to (ii). But (ii) is proved by applying the same argument with f equal to the inclusion of C in Cone(C) (see 0.7).

1.2. REMARKS. (a) Theorem 1.1 is of course equivalent to the statement that the *covariant* functors $C \mapsto \hat{Q}^n(C)$ constitute a homology theory. The 'homotopy invariance' part of the statement can be obtained under more general circumstances: for instance, the covariant functors $C \to Q^n(C)$ are also homotopy invariant, by the same argument.

(b) The 'covariant' suspension isomorphism $\hat{Q}^n(C) \to \hat{Q}^{n+1}(\Sigma C)$ has an explicit description. Regard the collapsing map $C \otimes_{\mathbb{Z}} I \to \Sigma C$ as a chain homotopy from the zero map 0: $C \to \Sigma C$ to itself. Then the proof of part (i) of 1.1 gives a semi-explicit formula for an induced chain homotopy (depending on the choice of a certain 1-chain ω) from the zero map 0: $\hat{W} \& C \to \hat{W} \& (\Sigma C)$ to itself, that is, a chain map \mathfrak{S} from $\Sigma(\hat{W} \& C)$ to $\hat{W} \& (\Sigma C)$. Passage to homology groups gives the suspension isomorphism. For a suitable choice of ω , the chain map $\mathfrak{S}: \Sigma(\hat{W} \& C) \to \hat{W} \& (\Sigma C)$ takes the form $\varphi \mapsto \mathfrak{S}\varphi; \mathfrak{S}\varphi)_{s+1} = (-)^{s}\mu_2(\varphi_s)$ (where $\mu_1, \mu_2: C^t \otimes_A C \to C^t \otimes_A C$ are homomorphisms given by $\mu_1(x \otimes y) = (-)^{|x|} x \otimes y, \ \mu_2(x \otimes y) = (-)^{|y|} x \otimes y$); then \mathfrak{S} is a chain isomorphism.

The same formula yields a suspension homomorphism (not an isomorphism in general) $Q^{n}(C) \rightarrow Q^{n+1}(\Sigma C)$.

Finally, similar formulae exist in the contravariant case, but will not be needed.

(The suspension formula in [15, p. 106] is simpler, but I suspect the signs are incorrect.)

1.3. PROPOSITION. The groups $\hat{Q}^n(C^{-*})$, $\hat{Q}^n(C)$ are \mathbb{Z}_2 -vector spaces (for C in \mathscr{C}_A , $n \in \mathbb{Z}$). Further, the cohomology theory $C \mapsto \{\hat{Q}^n(C^{-*})\}$ is periodic of order 2, that is, there exist natural isomorphisms $\hat{Q}^n(C^{-*}) \cong \hat{Q}^{n+2}(C^{-*})$, with $n \in \mathbb{Z}$, commuting with the suspension isomorphisms. Similarly, the homology theory $C \mapsto \{\hat{Q}^n(C)\}$ is periodic.

Proof. The first statement holds because the chain map $\hat{W} \to \hat{W}$; $x \mapsto 2x$ is null-homotopic. The periodicity isomorphisms come from

$$\Sigma^{2} \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_{2}]}(\hat{W}, C^{t} \otimes_{A} C) \cong \Sigma^{2} \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_{2}]}(\Sigma^{2} \hat{W}, C^{t} \otimes_{A} C)$$
$$\cong \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_{2}]}(\hat{W}, C^{t} \otimes_{A} C),$$

that is, from the periodicity of \hat{W} .

It is possible to give a more economical description of the groups $\hat{Q}^n(C^{-*})$, for C in \mathscr{C}_A . Thinking of a class in $\hat{Q}^n(C^{-*})$ as being represented by a collection

$$\{z_{p,q} \in \text{Hom}_A(C_p, C^q) | p, q \in \mathbb{Z}\}$$
 (cf. 0.10),

one should expect that most of the information is located in the terms $z_{p,q}$ with p = q; and something quite similar is true. (This is well known, and I am grateful to John Jones for pointing it out to me, when $A = Z_2$.)

1.4. DEFINITION. For a chain complex C in \mathscr{C}_A , let

$$V(C) = \dots \rightarrow V(C)_1 \rightarrow V(C)_0 \rightarrow V(C)_{-1} \rightarrow \dots$$

be the abelian group chain complex given by

$$V(C)_n := \prod_{r \in \mathbb{Z}} \operatorname{Sel}(C_r) \quad (\text{cf. 0.12}),$$

$$d: V(C)_n \to V(C)_{n-1};$$

$$\{\lambda_r\} \mapsto \{\mu_r := (-)^n d_C^{-}(\lambda_{r-1}) + (-)^{n+r} \lambda_r - T \lambda_r\}.$$

(Here $\lambda_r, \mu_r, d_C^-(\lambda_{r-1})$ are in Sel(C_r); $d_C^-(\lambda_{r-1})(x, y) := \lambda_{r-1}(d_C(x), d_C(y))$.)

A cycle in $V(C)_0$ is then a sequence of sesquilinear forms $\lambda_r: C_r \times C_r \to A$ such that, for each $r \in \mathbb{Z}$, the 'symmetrization of λ_r ' $(= T\lambda_r - (-)^r\lambda_r)$ equals the pullback $d_C(\lambda_{r-1})$.

Note that $C \mapsto V(C)$ is a contravariant functor.

1.5. PREPARATION. Think of an element in $(\hat{W} \& C^{-*})_n$ as a collection of sesquilinear forms $\{\varphi_{p,q}: C_p \times C_q \to A \mid p, q \in \mathbb{Z}\}$ (whose left adjoints are A-homomorphisms $z_{p,q}: C_p \to C^q$). Then the differential $d: (\hat{W} \& C^{-*})_{n+1} \to (\hat{W} \& C^{-*})_n$ is described by

$$d(\varphi_{p,q}) = \begin{cases} (-)^n \varphi_{p,q}(-)^{pq} T \varphi_{p,q}, (-)^q \varphi_{p,q}(\operatorname{id} \times d_C), \\ (-)^{p+q+1} \varphi_{p,q}(d_C \times \operatorname{id}), \text{ all other components } 0 \end{cases}.$$

(Here $\varphi_{p,q}$ is to be regarded as a collection with at most a single non-zero component, whereas the right-hand side has three non-zero components at most.)

1.6. PROPOSITION. The homomorphisms

$$\begin{aligned} & \operatorname{Eco}_n: \ V(C)_n \to (\widehat{W} \& C^{-*})_n; \\ & \lambda_r \mapsto \begin{cases} \{\lambda_r, \ (-)^r \lambda_r(d_C \times \mathrm{id}), \ other \ components \ 0\} \\ \{(-)^{r-1}\lambda_r, \ (-)^{r-1}\lambda_r(\mathrm{id} \times d_C), \ other \ components \ 0\} \end{cases} \end{aligned}$$

(using the top line for even n, the bottom line for odd n) constitute a natural chain map 'Eco'. (Again, λ_r is regarded as a collection with a single non-zero member at most, etc.) It induces isomorphisms in homology,

$$H_n(V(C)) \cong \hat{Q}^n(C^{-*}), \text{ with } n \in \mathbb{Z},$$

for any C in \mathcal{C}_A .

Proof (of the last sentence). We first show that

$$C \mapsto \{H_n(V(C)) \mid n \in \mathbb{Z}\}$$

is a cohomology theory. This means proving the analogues of Theorem 1.1 (i), (ii), (iii).

Suppose first that $0 \to C \to D \to B \to 0$ is a short exact sequence of chain complexes and chain maps in \mathscr{C}_A , with induced sequence

$$0 \longleftarrow V(C) \xleftarrow{i^{\leftarrow}} V(D) \xleftarrow{p^{\leftarrow}} V(B) \longleftarrow 0;$$

then we would like to know that $\ker(i^-)/\operatorname{im}(p^-)$ is an acyclic chain complex. Assuming the contrary, let $\lambda = \{\lambda_r \mid r \in \mathbb{Z}\}$ be an *n*-chain in $\ker(i^-) \subset V(D)$ such that $d(\lambda)$ belongs to $\operatorname{im}(p^-)$ and the homology class in $H_n(\ker(i^-)/\operatorname{im}(p^-))$ represented by λ is non-zero. More precisely, within its homology class choose the representative λ so that the number

$$\min\{r \in \mathbb{Z} \mid \lambda_r \neq 0\}$$

is as large as possible. It is easy to increase this integer by 1, giving a contradiction.

It follows that the homology groups of V(C), V(D), and V(B) are related by a long exact sequence. In particular, if $D = C \oplus B$, then $H_{\star}(V(D)) \cong H(V(C))_{\star} \oplus H_{\star}(V(B))$.

Next, suppose that C is a contractible complex in \mathscr{C}_{A} . Then $H_{*}(V(C)) = 0$. This can be proved by writing C as a direct sum of contractible chain complexes, each concentrated in two adjacent dimensions, and then applying the additivity principle iust obtained.

Next, suppose that $C \to D$ is a morphism in \mathscr{C}_A which is a cofibration and a homotopy equivalence. Then the preceding arguments, when combined, show that the induced homomorphism $H_*(V(D)) \to H_*(V(C))$ is an isomorphism. This applies in particular to the cofibrations

$$\operatorname{id}_C \otimes i_0, \operatorname{id}_C \otimes i_1 \colon C \otimes_{\mathbb{Z}} (\mathbb{Z}, 0) \to C \otimes_{\mathbb{Z}} I$$

(cf. 0.7) and shows therefore that the functors $C \mapsto H_n(V(C))$, for $n \in \mathbb{Z}$, are homotopy invariant. It is now clear that these functors constitute a cohomology theory.

The proof is completed by observing that the natural chain map 'Eco' (which by now induces a transformation of cohomology theories) gives an isomorphism between the respective 'coefficients' of the two cohomology theories; in other words, by observing that the last sentence of 1.6 holds for C = (A, 0). (The usual induction argument from [7, Anhang Proposition 2.1] shows that this suffices.)

The second half of the chapter consists mostly of somewhat tedious definitions making the analogy between (geometric) real K-theory and the cohomology theory $C \mapsto \{\hat{Q}^n(C^{-*})\}$ more precise.

1.7. DEFINITION. A chain bundle on a chain complex C in \mathscr{C}_A is a 0-dimensional cycle in $\widehat{W} \& C^{-*}$.

1.8. DEFINITION. If $f: C \to D$ is a chain map in \mathscr{C}_A , c is a chain bundle on C, and d is a chain bundle on D, then a 'chain bundle map from c to d, covering f' is a homology $y \in (\hat{W} \& C^{-*})_1$ from c to $f^{-}(d)$ (so that $c + d(y) = f^{-}(d)$ in $\hat{W} \& C^{-*}$).

Observe that chain bundle maps can be composed. If we take $f = id: C \rightarrow C$, then the category of chain bundles on C and chain bundle maps covering id is a groupoid; its components are the elements of $\hat{Q}^0(C^{-*})$.

Sometimes a 'change of rings' has to be allowed: suppose that there is given a homomorphism $A \rightarrow A'$ of rings with involution (and unit), making A' into a right module over A. Then

$$C \mapsto A' \otimes_{A} C$$

is a functor from \mathscr{C}_A to $\mathscr{C}_{A'}$. In the same spirit, a chain bundle c on C determines a chain bundle $A' \otimes_A c$ on $A' \otimes_A C$, and so on. So if C is in \mathscr{C}_A and D is in $\mathscr{C}_{A'}$, we would contemplate chain maps of the form $f: A' \otimes_A C \to D$; and if c is a chain bundle on C and d is a chain bundle on D, then we would also contemplate chain bundle maps $A' \otimes_A c \to d$, covering f.

It is of course possible to define higher homotopies between chain bundle maps with the same domain and range. 5388.3.51 Κ

There are at least two distinct ways (1.9 and 1.11) of making the chain bundles over a fixed chain complex C in \mathscr{C}_A into a simplicial set (see [6] for general information on simplicial sets).

1.9. DEFINITION. The 'Kan-Dold simplicial set of chain bundles on C' is the simplicial set $KD(\hat{W} \& C^{-*})$ obtained by applying the Kan-Dold functor KD to the chain complex $\hat{W} \& C^{-*}$ (cf. 1.10).

1.10. EXPLANATION. Let $Cl(\Delta_n)$ be the cellular chain complex of the standard *n*-simplex. Given a chain complex *E* (of abelian groups, graded over the integers), define KD(E) to be the simplicial abelian group with set of *n*-simplices

$$KD(E)_n :=$$
(set of chain maps from $Cl(\Delta_n)$ to E);

the face and degeneracy operators are then obvious. Conversely, given a simplicial abelian group G, let NG be the chain complex such that

$$NG_q = \bigcap_{i \neq 0} \ker[d_i: G_q \to G_{q-1}]$$

(the d_i being the face operators) and $d: NG_q \rightarrow NG_{q-1}$ equals the restriction of d_0 . Then $KD(NG) \cong G$, and N(KD(E)) is isomorphic to the subcomplex E^+ of E with

$$E_n^+ = \begin{cases} E_n & \text{if } n > 0, \\ \ker[d: E_0 \to E_{-1}] & \text{if } n = 0, \\ 0 & \text{if } n < 0. \end{cases}$$

See also [6].

1.11. DEFINITION. For a chain complex C in \mathcal{C}_A , the simplicial set of concordances of chain bundles on C, written $\mathcal{B}(C)$, has as set of n-simplices

 $\mathscr{B}(C)_n := (\text{set of chain bundles on } C \otimes_{\mathbb{Z}} \operatorname{Cl}(\Delta_n))$

(with $Cl(\Delta_n)$ as in 1.10; here $C \otimes_{\mathbb{Z}} Cl(\Delta_n)$ is regarded as a chain complex in \mathscr{C}_A , and, I hope, the simplicial operators are again obvious).

Both $KD(\hat{W} \& C^{-*})$ and $\mathscr{B}(C)$ are simplicial abelian groups. They are useful because most chain bundles occurring in nature are only well defined up to an infinity of higher homologies, or of higher concordances—just like the (geometric) stable normal bundle of a manifold, or the Spivak normal fibration of a geometric Poincaré complex.

The next proposition is a generalization of Theorem 1.1 (i) (as its proof will make clear).

1.12. PROPOSITION. There is a natural homomorphism of simplicial abelian groups

Lin:
$$\mathscr{B}(C) \to \mathrm{KD}(\widehat{W} \& C^{-*})$$

inducing an isomorphism in homotopy groups ('Lin' stands for linearization).

Proof. First, we require a sequence

$$\{\rho(n) \in (\widehat{W}\& \operatorname{Cl}(\Delta_n))_n \mid n = 0, 1, 2, ...\}$$

such that

- (i) $\rho(0) = v$ (see the end of 0.11; $Cl(\Delta_0)$ has been identified with $(\mathbb{Z}, 0)$),
- (ii) $\sum_{i=0}^{n} (-)^{i} \cdot e_{i}^{\rightarrow}(\rho(n-1)) = d(\rho(n))$ for $n = 1, 2, ..., (e_{i}: \Delta_{n-1} \rightarrow \Delta_{n}$ is the inclusion of the *i*th face, and *d* is the differential in $\widehat{W} \& \operatorname{Cl}(\Delta_{n})$,
- (iii) the chain map $\hat{W} \& \operatorname{Cl}(\Delta_n) \to \hat{W} \& \operatorname{Cl}(\Delta_0)$ (induced by the map $\Delta_n \to \Delta_0$) maps $\rho(n)$ to 0, for n > 0.

It is easy to construct such a sequence by induction, because

$$\ker[\widehat{W} \& \operatorname{Cl}(\Delta_n) \to \widehat{W} \& \operatorname{Cl}(\Delta_0)]$$

is acyclic (by 1.1 (i), or rather its covariant analogue). In defining $\hat{W} \& Cl(\Delta_n)$ etc., we have to work temporarily over the ring with involution \mathbb{Z} .

Second, we need the evaluation chain map

$$ev: (C \otimes_{\mathbb{Z}} \operatorname{Cl}(\Delta_n))^{-*} \otimes_{\mathbb{Z}} \operatorname{Cl}(\Delta_n) \to C^{-*};$$

$$f \otimes y \mapsto (x \mapsto (-)^{|x||y|} f(x \otimes y)).$$

Third, note that for a chain complex E, any *n*-simplex s in KD(E) determines an element $s_{top} \in E_n$ (namely the image under $s: Cl(\Delta_n) \to E$ of the generator corresponding to the *n*-cell of Δ_n). If Y is a simplicial set, and $f: Y \to KD(E)$ is a simplicial map, then the knowledge of $f(s)_{top}$ for all simplices s in Y suffices to reconstruct the simplicial map.

Now take an element ℓ in $\mathscr{B}(C)_n$ (that is, in the group of 0-dimensional cycles in $\widehat{W} \& (C \otimes_{\mathbb{Z}} \operatorname{Cl}(\Delta_n))^{-*}$, c.f. 1.11 and 1.7), and put

$$(\operatorname{Lin}(\mathscr{E}))_{\operatorname{top}} := (ev)^{\rightarrow}(\mathscr{E} \times \rho(n)) \in (\widehat{W} \& C^{-*})_n,$$

using the exterior product of 0.11.

Checking that Lin induces an isomorphism of homotopy groups is easy using 1.1.

2. Algebraic bordism theories

The aim here is to construct, for each chain complex B in \mathscr{C}_A and chain bundle ℓ on B, associated algebraic bordism spectra $\mathscr{L}^{:}(B, \ell)$ and $\mathbb{L}^{0}(B, \ell)$. ($\mathscr{L}^{:}(B, \ell)$ is a more sophisticated version of $\mathbb{L}^{0}(B, \ell)$, with better algebraic properties.) Inspiration comes from the mock-bundle philosophy of [5].

2.1. DEFINITION [13, 15]. An 'n dimensional algebraic Poincaré complex (over A)' is a pair (C, φ) consisting of a positive chain complex C in \mathscr{C}_A (that is, $C_r = 0$ for r < 0) and an n-dimensional cycle $\varphi \in W \& C$ so that

$$\varphi_0: \Sigma^n(C^{-*}) \to C \quad (\text{cf. 0.10})$$

is a chain homotopy equivalence.

(In [15], such a (C, φ) is called a 'symmetric algebraic Poincaré complex', as distinct from a 'quadratic algebraic Poincaré complex'. The point of view taken here is that a 'quadratic algebraic Poincaré complex' is a 'symmetric algebraic Poincaré complex' with additional structure; more will be said in 2.17.) 2.2. DEFINITION. An *n*-dimensional (cofibrant) algebraic Poincaré pair

$$(f: C \rightarrow D, (\psi, \varphi))$$

consists of two positive chain complexes C, D in \mathscr{C}_A , a cofibration $f: C \to D$, an *n*-chain $\psi \in W \& D$, and an (n-1)-cycle $\varphi \in W \& C$ so that

- (i) $d(\psi) = f^{-}(\varphi)$ in $(W\& D)_{n-1}$,
- (ii) the chain map $\operatorname{pr} \cdot \psi_0 \colon \Sigma^n(D^{-*}) \to D/\operatorname{im}(f)$ (explained below) is a homotopy equivalence.

(Here $\operatorname{pr} \cdot \psi_0$ is the composite of the projection chain map $\operatorname{pr} \colon D \to D/\operatorname{im}(f)$ with the homomorphism of graded abelian groups $\psi_0 \colon \Sigma^n(D^{-*}) \to D$; cf. 0.10, Case 2. Although ψ_0 is not a chain map in general, $\operatorname{pr} \cdot \psi_0$ is.)

The condition that $f: C \to D$ be a cofibration is not essential; if it is not satisfied, pr $\cdot \psi_0$ has to be replaced by a chain map going from $\Sigma^n(D^{-*})$ to Cone(f). See [15].

The 'boundary' (C, φ) of the algebraic Poincaré pair in 2.2 is an (n-1)-dimensional algebraic Poincaré complex.

2.3. DEFINITION (of a higher algebraic bordism). Let $2^{\{0, 1, \dots, q\}}$ be the category of subsets of $\{0, 1, \dots, q\}$, with inclusion maps as morphisms (so there is at most one morphism between any two objects; if $S' \subset S$, denote this morphism by $j_{S',S}$).

For $S \subset \{0, 1, ..., q\}$ and $0 \le i < |S|$, let d_iS stand for the 'ith face' of S (d_0S is obtained from S by deleting the least element, d_1S by deleting the next, etc.).

A 'higher bordism of algebraic Poincaré complexes, of dimension n and order q' consists of a covariant functor

Fun:
$$2^{\{0, 1, \dots, q\}} \rightarrow \mathscr{C}_A$$

and a function Φ which for each subset $S \subset \{0, 1, ..., q\}$ picks an (n-q+|S|-1)-chain $\Phi(S) \in W \& \operatorname{Fun}(S)$; here Fun and Φ are subject to certain conditions. They are as follows.

(i) Each Fun(S) is a positive chain complex; $Fun(\emptyset) = 0$.

(ii) For any ideal in $2^{\{0, 1, \dots, q\}}$ (i.e. a collection \mathscr{J} of subsets of $\{0, 1, \dots, q\}$ such that $S \in \mathscr{J}$ and $S' \subset S$ implies $S' \in \mathscr{J}$), the canonical map

$$\operatorname{Fun}(\{0, 1, \dots, q\})^{-*} \to \operatorname{inv}\lim_{S \in \mathscr{J}} \left\{ \operatorname{Fun}(S)^{-*} \right\} \subset \prod_{S \in \mathscr{J}} \operatorname{Fun}(S)^{-*};$$

$$f \mapsto (f \cdot j_{S,\{0,1,\ldots,q\}})_{S \in \mathscr{J}}$$

is surjective. (This condition generalizes the cofibration condition in 2.2; it implies, by an induction proof, that $\operatorname{inv} \lim_{S \in \mathcal{J}} \{\operatorname{Fun}(S)^{-*}\}$ is in \mathscr{C}_A for any ideal \mathcal{J} .)

(iii) For $S \subset \{0, 1, ..., q\}$,

$$\sum_{i=0}^{|S|-1} (-)^i \cdot j_{d_i S, S}^{-1}(\Phi(d_i S)) = (-)^{n-k} \cdot d(\Phi(S))$$

in W& Fun(S) (d is the differential in W& Fun(S)).

(iv) For each $S \subset \{0, 1, ..., q\}$, the chain complexes

$$C := \sum_{i=0}^{|S|-1} j_{d_i S, S}(\operatorname{Fun}(d_i S)), \quad D := \operatorname{Fun}(S),$$

the inclusion map $f: C \to D$, the chain $\psi := (-)^{n-k} \Phi(S)$ in W & D, and the cycle $\varphi := \sum_{i=0}^{|S|-1} (-)^i \Phi(d_i S)$ in W & C (in loose notation) together constitute an algebraic

Poincaré pair of dimension (n-q+|S|-1). (This means that the non-degeneracy condition 2.2(ii) holds—everything else is redundant.)

2.4. REMARKS. If (Fun, Φ) is a higher algebraic Poincaré bordism as above, of dimension *n* and order *q*, and if *S* is a subset of $\{0, 1, ..., q\}$ with complement *S'*, then the restriction of Fun to 2^{S} and the corresponding restriction of Φ form a higher algebraic Poincaré bordism of dimension n-|S'| and order q-|S'|, written (Fun/ $2^{S}, \Phi/2^{S}$).

An algebraic Poincaré bordism (Fun, Φ) of order 1 such that Fun({0}) = 0 or Fun({1}) = 0 will also be called an algebraic Poincaré pair; this agrees with 2.2 up to sign.

2.5. CONSTRUCTION. Assume that (Fun, Φ) is still as above, and forget all the higher homotopies contained in Φ , retaining only $\Phi(S)_0$ (cf. 0.10, Case 2) for all $S \subset \{0, 1, ..., q\}$. Condition 2.3(iii) gives a system of chain maps (one for each $S \subset \{0, 1, ..., q\}$)

$$\int_{2^{s}} \Phi_{0} \colon \Sigma^{n-q}(\operatorname{Fun}(S)^{-*} \otimes_{\mathbb{Z}} \operatorname{Cl}(\Delta(S))) \to \operatorname{Fun}(S);$$
$$f \otimes [S'] \mapsto j_{S',S}^{-} \cdot \Phi(S')_{0}(f \cdot j_{S',S}^{-}).$$

(Explanation: $\Delta(S)$ is the (|S|-1)-dimensional simplex spanned by S and $\operatorname{Cl}(\Delta(S))$ is its cellular chain complex, with one (|S'|-1)-dimensional generator [S'] for each non-empty subset S' of S. The inclusion $j_{S',S}: S' \to S$ induces $j_{\overline{S'},S}: \operatorname{Fun}(S') \to \operatorname{Fun}(S)$; starting with $f \in \Sigma^{n-q}(\operatorname{Fun}(S)^{-*})$ and suspending liberally, we find that $f \cdot j_{\overline{S'},S}$ is in $\Sigma^{n-q+|S'|-1}(\operatorname{Fun}(S')^{-*})$, which is the domain of $\Phi(S')_0$, etc.) For $S' \subset S \subset \{0, 1, ..., q\}$, the chain maps $\int_{2^S} \Phi_0$ and $\int_{2^S} \Phi_0$ are related by a certain

For $S \subset S \subset \{0, 1, ..., q\}$, the chain maps $\int_{2^s} \Psi_0$ and $\int_{2^s} \Psi_0$ are related by a certain commutative diagram.

We will now repeat 2.1 and 2.3, adding ' \mathcal{E} -structures'; here \mathcal{E} is a chain bundle on a chain complex B in \mathcal{C}_A , to be kept fixed until further notice.

2.6. DEFINITION. A ℓ -structure on an *n*-dimensional algebraic Poincaré complex (C, φ) (over A) consists of a chain map

 $g: C \rightarrow B$ (the 'classifying map')

and a homology

 $z \in (\widehat{W} \& C)_{n+1}$ (the 'clutching homology')

from \mathfrak{S}^{n} . $(\varphi_0 g^*)^{\neg}(b) \in (\hat{W} \& C)_n$ to $J(\varphi) \in (\hat{W} \& C)_n$. (Explanation: $\varphi_0 g^*$ is a chain map from B^{-*} to $\Sigma^{-n}C$, inducing

$$(\varphi_0 g^*)^{\rightarrow} : \widehat{W} \& B^{-*} \to \widehat{W} \& (\Sigma^{-n} C),$$

and \mathfrak{S}^n is the *n*-fold iteration of the explicit suspension isomorphism of 1.2(b). See also 0.13.)

The notion of 'normal chain bundle' should help clarify 2.6. Suppose that (C, φ) is an *n*-dimensional algebraic Poincaré complex over A, and $n \in \hat{W} \& C^{-*}$ a chain bundle on C equipped with the additional structure of a homology $z \in (\hat{W} \& C)_{n+1}$ from $\mathfrak{S}^n \cdot \varphi_0^-(n) \in (\hat{W} \& C)_n$ to $J(\varphi) \in (\hat{W} \& C)_n$. Then n, or rather the pair (n, z), is called the 'normal chain bundle of (C, φ) '. It is easy to see that the normal chain bundle n of (C, φ) is well defined up to an infinity of higher homologies in $\hat{W} \& C^{-*}$ (see 0.15). That is, it is something better than a mere class in $H_0(\hat{W} \& C^{-*}) = \hat{Q}^0(C^{-*})$. In this respect it resembles the normal bundle of a geometric manifold or the Spivak normal fibration of a geometric Poincaré complex, which are also well defined up to an infinity of higher concordances.

A l-structure on (C, φ) consists, then, of a classifying chain map $g: C \to B$ and an identification of the 'induced' chain bundle $g^{-}(\ell)$ on C with the normal chain bundle n.

2.7. DEFINITION (in outline only). A ℓ -structure (g, z) on a higher bordism (Fun, Φ) of algebraic Poincaré complexes (of dimension *n* and order *q*) consists of

- (i) a 'classifying chain map' g: Fun($\{0, 1, ..., q\}$) $\rightarrow B$,
- (ii) an appropriate collection (explained below) $z = \{z(S) | S \subset \{0, 1, ..., q\}\}$ of clutching homologies.
- 2.8. EXPLANATION of 2.7 (ii). In 2.9 below, a sequence

$$\{\rho(m) \in (\widehat{W} \& \operatorname{Cl}(\Delta_m))_m \mid m = 0, 1, 2, ...\}$$

satisfying conditions (i), (ii), and (iii) in the proof of 1.12 will be fixed once and for all. If S is any finite ordered set and m = |S| - 1, then the unique order-preserving bijection $S \rightarrow \{0, 1, ..., m\}$ gives an identification

$$\widehat{W}$$
 & Cl($\Delta(S)$) \cong \widehat{W} & Cl(Δ_m);

write $\rho(S) \in \widehat{W} \& \operatorname{Cl}(\Delta(S))$ for the *m*-chain corresponding to $\rho(m)$.

The main point is that the clutching homologies in 2.7(ii) form a set

$$\{z(S) \in (W \& \operatorname{Fun}(S))_{n-q+|S|} \mid S \subset \{0, 1, \dots, q\}\}$$

so that, for every $S \subset \{0, 1, ..., q\}$, the equation

$$d(z(S)) + (-)^{n-q} \cdot \sum_{i=0}^{|S|-1} (-)^{i} \cdot j_{d,S,S}(z(d,S))$$

= $J(\Phi(S)) - \int_{2^{S}} \Phi_{0}(\mathfrak{S}^{n-q} \cdot ((g^{*})^{\neg}(\ell) \times \rho(S)))$

holds in $(\widehat{W} \& \operatorname{Fun}(S))_{n-q+|S|-1}$.

(The integral sign comes from 2.5, \times is the exterior product of 0.11, *d* is the differential in \hat{W} & Fun(S), and \mathfrak{S} is the suspension of 1.2(b); I have written $g: \operatorname{Fun}(S) \to B$ when I should have written

 $g \cdot j_{S,\{0,1,\ldots,q\}}$: Fun(S) \rightarrow Fun($\{0,1,\ldots,q\}$) \rightarrow B,

and so the map $(g^*)^{\neg}$ goes from $\hat{W} \& B^{-*}$ to $\hat{W} \& (\operatorname{Fun}(S))^{-*}$.)

2.9. CONVENTIONS. We shall first fix a 1-chain ω in $\hat{W} \& \operatorname{Cl}(\Delta_1) = \hat{W} \& I$ like the one used in the proof of 1.1 (i) and in 1.2 (b); and then a sequence $\{\rho(m) \in (\hat{W} \& \operatorname{Cl}(\Delta_m))_m | m = 0, 1, 2, ...\}$ as in the proof of 1.12.

(i) Let x, y_0 , y_1 be the standard generators of the chain complex $I = Cl(\Delta_1)$; then |x| = 1, $|y_0| = |y_1| = 0$, and $d(x) = y_1 - y_0$. Using notation as in Case 1 of 0.10, put

$$\omega_1 = x \otimes x, \quad \omega_0 = x \otimes y_0 + y_1 \otimes x, \quad \omega_t = 0 \text{ for } t \neq 0, 1.$$

Then $\omega = \{\omega_s | s \in \mathbb{Z}\} \in \hat{W} \& \operatorname{Cl}(\Delta_1)$ gives the explicit suspension formula in 1.2(b). Note also that ω belongs to the subcomplex $W \& \operatorname{Cl}(\Delta_1) \subset \hat{W} \& \operatorname{Cl}(\Delta_1)$.

(ii) The map

$$\Delta_m \times \Delta_1 \rightarrow \Delta_{m+1}; ((t_0, t_1, \dots, t_m), (u_0, u_1)) \mapsto (u_0, u_1 t_0, u_1 t_1, \dots, u_1 t_m)$$

(in barycentric coordinates; so $t_0 + t_1 + ... + t_m = 1 = u_0 + u_1$) induces a chain map of cellular chain complexes

$$p_m: \operatorname{Cl}(\Delta_m) \otimes_{\mathbb{Z}} \operatorname{Cl}(\Delta_1) \cong \operatorname{Cl}(\Delta_m \times \Delta_1) \to \operatorname{Cl}(\Delta_{m+1}).$$

Define inductively

$$\rho(0) := v \in \widehat{W} \& \operatorname{Cl}(\Delta_0),$$

$$\rho(m+1) := p_m^{\rightarrow}(\rho(m) \times \omega) \in \widehat{W} \& \operatorname{Cl}(\Delta_{m+1}),$$

where \times denotes the exterior product of 0.11.

2.10. REMARK. A ℓ -structure (g, z) on a higher algebraic Poincaré bordism (Fun, Φ) of order q induces a ℓ -structure $(g/2^s, z/2^s)$ on each of the face bordisms (Fun/ $2^s, \Phi/2^s$) defined in 2.4, with $S \subset \{0, 1, ..., q\}$.

In particular, it is clear how to define the notion of a bordism (of order 1) between two algebraic Poincaré complexes of the same dimension, with ℓ -structure as in 2.6. Granting that 'bordant' is an equivalence relation, we can define the corresponding bordism groups. We shall now construct a spectrum whose homotopy groups they are.

The construction contains very few surprises. Recall from [17] that a Δ -set (or incomplete simplicial set) is a contravariant functor from the category Δ (whose objects are the standard q-simplices Δ_q for q = 0, 1, ..., and whose morphisms are the linear maps defined by order-preserving *injective* maps of the vertex sets) to the category of sets.

It is shown in [17] that Δ -sets are well behaved if they satisfy the analogue of the Kan condition for simplicial sets, in which case they are called Kan Δ -sets.

Certain set-theoretic precautions are understood in the next definition, and in several others of a similar type in §3. Without such precautions we would have to work with Δ -classes rather than Δ -sets in many places.

2.11. DEFINITION. For $p \in \mathbb{Z}$, let $\mathbb{L}^{p}(B, \ell)$ be the Δ -set whose q-simplices are the higher bordisms of algebraic Poincaré complexes (over A) of dimension q+p and order q, equipped with a ℓ -structure; the face operators are as outlined in 2.10.

2.12. PROPOSITION. The Δ -set $\mathbb{L}^{p}(B, \ell)$ satisfies the Kan condition.

Proof. This is straightforward and left to the reader.

2.13. CONSTRUCTION. There are natural homotopy equivalences

$$\varepsilon_p \colon \mathbb{L}^{-p}(B, \mathscr{E}) \to \Lambda \mathbb{L}^{-(p+1)}(B, \mathscr{E}) \quad \text{for } p \in \mathbb{Z}.$$

(Here Λ denotes the loop space. See 2.15 for the meaning of 'natural'.)

Proof. First, the loop space $\Lambda \mathbb{L}^{-(p+1)}(B, \ell)$ must be defined. The set $\mathbb{L}^{-(p+1)}(B, \ell)$ has a canonical 'base point' (that is, a Δ -map from the Δ -set given by the constant one-

point functor to $\mathbb{L}^{-(p+1)}(B, \mathcal{E})$; by decree, a *q*-simplex of $\Lambda \mathbb{L}^{-(p+1)}(B, \mathcal{E})$ is the same as a (q+1)-simplex of $\mathbb{L}^{-(p+1)}(B, \mathcal{E})$ whose 0th vertex and 0th face are at the base point. (The 0th face is opposite the 0th vertex.) So

$$(\Lambda \mathbb{L}^{-(p+1)}(B, \mathscr{E}))_{q} \subset (\mathbb{L}^{-(p+1)}(B, \mathscr{E}))_{q+1},$$

and for $0 \le i \le q$ the face operators d_i are chosen so as to make the diagram

$$(\Lambda \mathbb{L}^{-(p+1)}(B, \mathscr{E}))_q \xrightarrow{d_i} (\Lambda \mathbb{L}^{-(p+1)}(B, \mathscr{E}))_{q-1}$$
$$\cap$$
$$(\mathbb{L}^{-(p+1)}(B, \mathscr{E}))_{q+1} \xrightarrow{d_{i+1}} (\mathbb{L}^{-(p+1)}(B, \mathscr{E}))_q$$

commute.

Now let x be a q-simplex in $\mathbb{L}^{-p}(B, \ell)$, that is, an algebraic Poincaré bordism (Fun, Φ) of dimension q-p and order q, with ℓ -structure (g, z). Then $\varepsilon_p(x)$ has to be an algebraic Poincaré bordism (Fun', Φ') of dimension q-p and order q+1, with ℓ -structure (g', z').

Let $e: \{0, 1, ..., q\} \longrightarrow \{0, 1, ..., q+1\}$ send s to s+1. For $S \subset \{0, 1, ..., q+1\}$, define

$$\operatorname{Fun}'(S) := \begin{cases} \operatorname{Fun}(e^{-1}(S)) & \text{if } 0 \in S, \\ 0 & \text{otherwise}, \end{cases}$$
$$\Phi'(S) := \Phi(e^{-1}(S)) & \text{if } 0 \in S, \\ g' := g, \\ z'(S) := z(e^{-1}(S)) & \text{if } 0 \in S. \end{cases}$$

and

Since all Δ -sets in sight satisfy the Kan condition, their homotopy groups can be defined via the 'pillow construction', which shows that ε_p is a homotopy equivalence.

2.14. COROLLARY. Definition 2.11 and Construction 2.13 define a spectrum

 $\underline{\mathbb{L}}^{0}(B, \mathscr{E}) = \{ \mathbb{L}^{-p}(B, \mathscr{E}), \varepsilon_{p} \mid p \in \mathbb{Z} \}.$

Again, the pillow philosophy shows that $\pi_n(\underline{\mathbb{L}}^0(B, \ell))$ is the bordism group of *n*-dimensional algebraic Poincaré complexes with ℓ -structure (2.12 implies that bordism is an equivalence relation).

2.15. PROPOSITION. The association $B, \ell \mapsto \underline{\mathbb{L}}^0(B, \ell)$ is functorial. If ℓ is a chain bundle on B (in \mathscr{C}_A), ℓ' is a chain bundle on B' (in $\mathscr{C}_{A'}$), $A \to A'$ is a homomorphism of rings with involution, and $f: A' \otimes_A B \to B'$ is a chain map covered by a chain bundle map (cf. 1.8) from $A' \otimes_A \ell$ to ℓ' , then there is an induced map of spectra $\underline{\mathbb{L}}^0(B, \ell) \to \underline{\mathbb{L}}^0(B', \ell')$.

Idea of proof. Take a q-simplex in $\mathbb{L}^{p}(B, \delta)$, say (Fun, Φ), (g, z), and define a q-simplex (Fun', Φ'), (g', z') in $\mathbb{L}^{p}(B', \delta')$ by letting

 $\operatorname{Fun}' := A' \otimes_A \operatorname{Fun}, \quad \Phi' := A' \otimes_A \Phi, \quad g' := f \cdot (A' \otimes_A g),$

and for $S \subset \{0, 1, ..., q\}$,

$$z'(S) := A' \otimes_{A} z(S) + \int_{2^{S}} \Phi'_{0}(\mathfrak{S}^{p}((A' \otimes_{A} g)^{-}(y) \times \rho(S))),$$

A glance at the homotopy groups shows

2.16. PROPOSITION. If A = A' in 2.15, and if $f: B \to B'$ is a chain homotopy equivalence (covered by a chain bundle map as before), then $\mathbb{L}^{0}(B, \ell) \to \mathbb{L}^{0}(B', \ell')$ is a homotopy equivalence of spectra.

We conclude with two 'extreme' examples, for which the ring with involution A is fixed again.

2.17. EXAMPLE. Take δ to be the trivial bundle on the trivial chain complex $B = 0_A$ in \mathscr{C}_A .

A &lphi-structure on an algebraic Poincaré complex (C, φ) is then a homology in $\hat{W} \& C$ from 0 to $J(\varphi)$. It is clear that this is the same as a 'quadratic Poincaré complex structure' in the sense of [15]. So $\pi_n(\underline{\mathbb{L}}^0(0_A, 0))$ is isomorphic to the (projective version of the) Wall group $L_n(A)$ for $n \ge 0$, and spectra homotopy equivalent to $\underline{\mathbb{L}}^0(0_A, 0)$ have of course been constructed before (by Quinn and Ranicki; see also [21, Chapter 17A]).

It is tempting to regard the Wall groups as the bordism groups of 'framed algebraic Poincaré complexes' (see the paragraphs between 2.6 and 2.7), but this can lead to confusion: bear in mind that the trivial chain complex 0_A is the algebraic counterpart of an empty space, not of a contractible space or a $K(\pi, 1)$.

2.18. EXAMPLE. Take ℓ to be the 'universal chain bundle'. (This involves a certain amount of cheating. What I claim is that the functor $C \mapsto \hat{Q}^0(C^{-*})$, when restricted to the category of positive chain complexes in \mathscr{C}_A , is 'almost representable'. That is, there exist a positive chain complex B and a chain bundle ℓ on B so that the transformation of functors $H_0(\operatorname{Hom}_A(C, B)) \to \hat{Q}^0(C^{-*})$; $[f] \mapsto [f^+(\ell)]$ is an isomorphism, with C in \mathscr{C}_A . However, we must allow B to be a chain complex of possibly non-finitely generated projective A-modules, and possibly infinitely many of them non-zero; also the notion of chain bundle must be defined with some care. The appendix to this section is devoted to an explicit construction.)

Now a l-structure on an algebraic Poincaré complex is as good as no structure at all; consequently $\pi_n(\mathbb{L}^0(B, \ell))$ is isomorphic to the symmetric *L*-group $L^n(A)$ defined in [13] and [15].

2.19. REMARK. The construction of $\underline{\mathbb{L}}^{0}(B, \ell)$ may seem a little arbitrary, since it relies on a peculiar choice made in 2.9. Here is a more convincing alternative.

(i) Choose your own favourite sequence $\{\rho(m)\}$ satisfying the conditions in the proof of 1.12. This will be 'essentially unique' only, but there is no need to be more specific. Define a ' δ -structure' (on a higher bordism of algebraic Poincaré complexes) accordingly.

(ii) Construct a Δ -set $\mathbb{L}^{0}(B, \ell)$ as in 2.11, still using your own favourite sequence $\{\rho(m)\}$.

(iii) Prove that $\mathbb{L}^{0}(B, \ell)$ is an *infinite loop space*, using Segal's machine [18]. For this purpose, let

$$E(1) := \mathbb{L}^{0}(B, \mathscr{E});$$

more generally, for $n \ge 0$ let E(n) be the Δ -set whose q-simplices are 'functors' which to each non-empty subset V of $\{1, 2, ..., n\}$ associate a q-simplex of $\mathbb{L}^{0}(B, \ell)$, say (^VFun, ^V Φ), (^Vg, ^Vz); and to each inclusion $U \subset V$ associate a chain map

$$\lambda_{U,V}: {}^{U}$$
Fun({0, 1, ..., q}) $\rightarrow {}^{V}$ Fun({0, 1, ..., q}),

subject to certain very natural conditions. (The conditions are:

$$^{V}g\cdot\lambda_{U,V}={}^{U}g;$$

,next, the $\lambda_{U,V}$ should give a direct sum decomposition

$$({}^{\nu}\mathsf{Fun}, {}^{\nu}\Phi) \cong ({}^{\nu}\mathsf{Fun}, {}^{\nu}\Phi) \oplus ({}^{\nu-\nu}\mathsf{Fun}, {}^{\nu-\nu}\Phi)$$

whenever $U \subset V \subset \{1, 2, ..., n\}$, and $U \neq \emptyset \neq V - U$; and finally, the projection

^{ν}Fun $\rightarrow {}^{\upsilon}$ Fun

resulting from the previous condition in the case where $U \subset V$ and $U \neq \emptyset \neq V - U$, should send v_z to v_z .)

Then the collection $\{E(n) \mid n \ge 0\}$ (with obvious structure maps) is a Γ -space in the sense of [18], and so $E(1) = \mathbb{L}^{0}(B, \ell)$ is an infinite loop space. Notice the similarity of the construction above with Segal's construction of the algebraic K-theory spectrum, also in [18].

To prove that the infinite loop space structure on $\mathbb{L}^{0}(B, \ell)$ just defined coincides with that given by 2.13, use

2.20. LEMMA. The map ε_p in 2.13 has a canonical refinement to a map of Γ -spaces.

(Explanation: observe first that each $\mathbb{L}^{p}(B, \delta)$ yields a Γ -space, just like $\mathbb{L}^{0}(B, \delta)$. Secondly, if $\{F(n) \mid n \ge 0\}$ is any Γ -space, then so is $\{\Lambda F(n) \mid n \ge 0\}$; the structure maps for $\{\Lambda F(n)\}$ are obtained by applying the loop functor Λ to those for $\{F(n)\}$. Hence the lemma makes sense; the proof is easy, and it is also easy to see that it proves precisely what is needed.)

2.21. NOTATION. (i) The spectrum $\mathbb{L}^{0}(B, \ell)$ and the infinite loop spaces $\mathbb{L}^{p}(B, \ell)$ have been defined in 2.14 and 2.11 respectively.

(ii) An *n*-dimensional 'unrestricted algebraic Poincaré complex' (C, φ) consists, by definition, of a chain complex C in \mathscr{C}_A and an *n*-cycle $\varphi \in W \& C$ so that $\varphi_0: \Sigma^n(C^{-*}) \to C$ is a chain homotopy equivalence (C is not required to be positive). The notion is interesting even when n < 0. The whole of this section (with the exception of 2.19, which is unsuitable) can be rewritten with 'algebraic Poincaré complexes' replaced by 'unrestricted algebraic Poincaré complexes', etc. The outcome is

for every $p \in \mathbb{Z}$, a Δ -set $\mathscr{L}^{p}(B, \ell)$ (the 'unrestricted analogue' of $\mathbb{L}^{p}(B, \ell)$)

and hence a usually non-connective spectrum $\mathcal{L}^{(B,\ell)}(\mathbf{h})$ (the 'unrestricted analogue' of $\mathbb{L}^{0}(B,\ell)$).

(iii) Write $\mathcal{L}(A) = \mathcal{L}(0_A, 0)$, and let

release:
$$\mathscr{L}(A) \to \mathscr{L}(B, \ell)$$

be the map of spectra induced by the chain bundle map from 0_A , 0 to B, ℓ (cf. the introduction). Note that

$$\pi_n(\mathscr{L}(A)) = L_n(A) \quad \text{for } n \in \mathbb{Z},$$

and $L_n(A)$ depends only on the residue of $n \mod 4$. The cofibre of

release: $\mathscr{L}(A) \to \mathscr{L}(B, \ell)$

is denoted by $\hat{\mathscr{L}}^{:}(B, \mathscr{E})$.

(iv) Finally, write

 $L^{n}(B, \ell) := \pi_{n}(\underline{\mathscr{L}}^{:}(B, \ell)), \quad \hat{L}^{n}(B, \ell) := \pi_{n}(\underline{\hat{\mathscr{L}}}^{:}(B, \ell)).$

(No special notation is introduced for $\pi_n(\underline{\mathbb{L}}^0(B, \ell))$; but we will see later that, if B is a positive chain complex, the forgetful homomorphisms

$$\pi_n(\underline{\mathbb{L}}^0(B, \mathscr{E})) \to \pi_n(\mathscr{L}^1(B, \mathscr{E})) = L^n(B, \mathscr{E})$$

are isomorphisms for $n \ge 0$, whereas $\pi_n(\underline{\mathbb{L}}^0(B, \ell)) = 0$ for n < 0.)

2.22. REMARK. Change of K-theory. The whole theory so far has been written in terms of f.g. projective modules over A; there are versions which use stably free A-modules instead, or stably free and based A-modules. As in ordinary L-theory, there is a long exact sequence relating the projective and stably free versions of $L^n(B, \delta)$, involving the groups $\hat{H}^n(Z_2; K_0(A))$; and another long exact sequence relating the stably free and stably-free-and-based versions, involving the groups $\hat{H}^n(Z_2; K_1(A))$ (or $\hat{H}^n(Z_2; \bar{K}_1(A))$ etc.). Cf. [15].

The relative groups $\hat{L}^{n}(B, \ell)$ are not affected at all by a change of K-theory.

2.A. Appendix: The universal chain bundle

Let B be any chain complex of projective left A-modules, not necessarily in \mathscr{C}_A . Then the sequence of functors

$$C \mapsto H_n(\operatorname{Hom}_A(C, B))$$

(where C is a chain complex in \mathscr{C}_A and $n \in \mathbb{Z}$) constitutes a cohomology theory on \mathscr{C}_A , that is, the analogues of Conditions (i), (ii), and (iii) of 1.1 are satisfied.

Conversely, any cohomology theory on \mathscr{C}_A is isomorphic to one obtained in this way (from a suitable chain complex *B*). This is the analogue in the chain complex world of E. H. Brown's representation theorem, which normally lives in the world of *CW*-spaces; see [7]. We shall now prove it in detail for the special case of the cohomology theory

$$C \mapsto \{ \hat{Q}^n(C^{-*}) \mid n \in \mathbb{Z} \}.$$

If B is an arbitrary chain complex of projective A-modules (not necessarily in \mathscr{C}_A), the abelian group chain complex $\hat{W} \& B^{-*}$ can be defined as in 1.5; so an n-chain in $\hat{W} \& B^{-*}$ is a collection of sequilinear forms

$$\{\varphi_{p,q}: B_p \times B_q \to A \mid p, q \in \mathbb{Z}\},\$$

and the differential (from $(\hat{W} \& B^{-*})_{n+1}$ to $(\hat{W} \& B^{-*})_n$) is as in 1.5.

Further, a chain complex V(B) can be defined word for word as in 1.4; the chain map

Eco:
$$V(B) \rightarrow \hat{W} \& B^{-*}$$

of 1.6 is still there, although perhaps not in general a homology equivalence. At any rate, a 0-cycle in V(B) can also be regarded as a chain bundle on B.

2.A.1. DEFINITION. Given a chain bundle ℓ on B, or just a class $\lfloor \ell \rfloor$ in $\hat{Q}^0(B^{-*})$, there are homomorphisms (called the Wu classes of ℓ , see $\lfloor 15 \rfloor$)

$$v_r(\mathscr{E}): H_r(B) \to \widehat{H}^r(\mathbb{Z}_2; A); [f] \mapsto [f^-(\mathscr{E})].$$

(Explanation: $\hat{H}'(Z_2; A)$ is the rth Tate cohomology group of Z_2 with coefficients in the Z_2 -module A; the involution makes A into a Z_2 -module; $H_r(B)$ has been identified with $H_0(\text{Hom}_A((A, r), B))$, so that f is a chain map from (A, r) to B; see 0.6 for notation. Also $\hat{H}'(Z_2; A)$ has been identified with $\hat{Q}^0((A, r)^{-*})$ so that $f^-(\ell)$ is a chain bundle on (A, r).)

Now $\hat{H}^r(Z_2; A)$ is a left A-module, with A acting by

 $a \cdot [x] \mapsto [ax\bar{a}]$

(for $a \in A$ and $x \in \text{ker}[\text{id} - (-)^r \cdot \text{involution}: A \to A]$, and

$$[x] \in \frac{\ker[\operatorname{id} - (-)^r \cdot \operatorname{involution}: A \to A]}{\operatorname{im}[\operatorname{id} + (-)^r \cdot \operatorname{involution}: A \to A]} = \widehat{H}^r(Z_2; A)).$$

The Wu classes $v_r(\ell)$ are A-module homomorphisms.

2.A.2. EXAMPLE. Suppose that $\lambda = \{\lambda_r : B_r \times B_r \to A \mid r \in \mathbb{Z}\}$ is a 0-cycle in V(B) (see 1.4), regarded as a chain bundle on B. The Wu classes are then given by

$$v_r(\lambda): H_r(B) \to H^r(\mathbb{Z}_2; A); [y] \mapsto [\lambda_r(y, y)].$$

(Assume that $y \in \text{ker}[d: B_r \to B_{r-1}]$; then $\lambda_r(y, y)$ represents an element in $\hat{H}^r(Z_2; A)$, since λ is a cycle.)

2.A.3. CONSTRUCTION. By induction on skeletons, we will construct a positive chain complex B (of free A-modules) and a 0-cycle $\{\lambda_r: B_r \times B_r \to A \mid r \in \mathbb{Z}\}$ in V(B) such that, for all $r \ge 0$, the Wu class

$$v_r(\lambda): H_r(B) \to \hat{H}^r(Z_2; A)$$

is an isomorphism. This property easily implies that λ , when regarded as a chain bundle on *B*, has the universal property required in 2.18.

Suppose that the *n*-skeleton

$$B_{\leq n} := B_n \to B_{n-1} \to B_{n-2} \to \dots \to B_1 \to B_0$$

has already been constructed, and that the sesquilinear forms

$$\lambda_r: B_r \times B_r \to A$$

have been defined for $r \leq n$ in such a way that

$$\lambda_{\leq n} := \{\lambda_r \mid 0 \leq r \leq n\}$$

is a 0-cycle in $V(B_{\leq n})$. Suppose further that the Wu class

$$v_r(\lambda_{\leq n}): H_r(B_{\leq n}) \to \tilde{H}^r(\mathbb{Z}_2; A)$$

is an isomorphism for r < n, and a surjection for r = n.

Let K be the kernel of $v_n(\lambda_{\leq n})$. Note that K is an A-submodule of B_n ,

$$K \subset H_n(B_{\leq n}) \subset B_n.$$

Choose a free A-module B'_{n+1} and a map

$$d'\colon B'_{n+1}\to B_n$$

such that im(d') = K; choose a sesquilinear form λ'_{n+1} on B'_{n+1} so that

$$T\lambda'_{n+1} - (-)^{n+1}\lambda'_{n+1} = d' - (\lambda_n)$$

(this is possible by definition of K).

Further, choose a free A-module B''_{n+1} and a sesquilinear form λ''_{n+1} on B''_{n+1} so that

- (i) $T\lambda_{n+1}'' (-)^{n+1}\lambda_{n+1}'' = 0$, (ii) the A module map
- (ii) the A-module map

$$B_{n+1}'' \to \widehat{H}^{n+1}(Z_2; A); x \mapsto [\lambda_{n+1}''(x, x)]$$

is surjective.

Now let

$$B_{n+1} := B'_{n+1} \oplus B''_{n+1},$$

$$d = d' \oplus 0: B'_{n+1} \oplus B''_{n+1} \to B_n.$$

and

$$\lambda_{n+1} := \lambda'_{n+1} \oplus \lambda''_{n+1}.$$

The induction step (from n to n+1) is complete.

2.A.4. VARIATION ON 2.A.3. It is also possible to construct a chain complex B^{∞} (of free *A*-modules, but usually not in \mathscr{C}_A) and a chain bundle \mathscr{E}^{∞} on B^{∞} such that the Wu classes

$$v_r(\mathscr{E}^\infty)$$
: $H_r(B) \to \widehat{H}^r(\mathbb{Z}_2; A)$

are isomorphisms for all $r \in \mathbb{Z}$. Then, for any chain complex C in \mathscr{C}_A (not necessarily positive), the homomorphism

$$H_0(\operatorname{Hom}_A(C, B^{\infty})) \to \widehat{Q}^0(C^{-*}); [f] \mapsto [f^{-}(\mathscr{E}^{\infty})]$$

is an isomorphism.

Therefore a ℓ^{∞} -structure on an unrestricted algebraic Poincaré complex (cf. 2.21 (ii)) over A is as good as no structure at all. It follows that $L^n(B^{\infty}, \ell^{\infty}) = \pi_n(\underline{\mathscr{L}}^{:}(B^{\infty}, \ell^{\infty}))$ (cf. 2.21 (iii), (iv)) is the bordism group of *n*-dimensional unrestricted algebraic Poincaré complexes over A, for $n \in \mathbb{Z}$.

The groups $L^n(B^{\infty}, \ell^{\infty})$ are periodic in *n*, with period 4, almost by definition.

Note that $L^n(B^{\infty}, \ell^{\infty})$ can be identified with the direct limit $\lim_{k \to \infty} L^{n+4k}(A)$ of the symmetric L-groups under the double skew-suspension maps

$$\overline{S}^{2} \colon L^{n+4k}(A) \to L^{n+4(k+1)}(A);$$

see [15].

3. Passage from geometry to algebra

In this section we show that a spherical fibration determines a chain bundle, and that a geometric Poincaré complex determines an algebraic Poincaré complex whose normal chain bundle agrees with the chain bundle determined by the Spivak normal fibration. 3.1. CONVENTIONS (relating to simplicial sets). Very little distinction will be made between a simplicial set X and its geometric realization (which is a CW-space). The cellular chain complex of X is C(X); it is freely generated by the non-degenerate simplices of X.

If π is a (discrete) group, a principal π -bundle on X consists of a simplicial set \tilde{X} with a simplicial π -action which freely permutes the simplices of \tilde{X} , and an identification of simplicial sets $\tilde{X}/\pi \cong X$.

Suppose that X and Y are simplicial sets. The acyclic model theorem [9] yields a chain homotopy equivalence

$$C(X \times Y) \to C(X) \otimes_{\mathbb{Z}} C(Y)$$

natural in both variables with respect to simplicial maps. (Note: we are talking about cellular chain complexes.) To be more thorough, the acyclic model theorem yields an 'Eilenberg-Zilber' chain map

$$\mathsf{EZ} = \mathsf{EZ}(X, Y) \colon C(X \times Y) \to \operatorname{Hom}_{\mathbb{Z}}(W, C(X) \otimes_{\mathbb{Z}} C(Y))$$

which

(i) is natural in both variables X and Y,

(ii) agrees with the canonical and obvious chain isomorphism if X = Y = point,

(iii) is Z_2 -equivariant.

(The last condition means that the diagram

commutes for arbitrary X and Y; here T is the generator of Z_2 , which acts on the chain complex W as usual, and 'conjugation by T' sends $f \in \text{Hom}_{\mathbb{Z}}(W, C(X) \otimes_{\mathbb{Z}} C(Y))$ to TfT.) To prove the existence of such an EZ, observe that EZ is equivalent to a natural chain map

DIA:
$$C(Y) \rightarrow \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C(Y) \otimes_{\mathbb{Z}} C(Y))$$

for simplicial sets Y, which agrees with the obvious and canonical chain isomorphism in the case where Y is a point. Indeed, DIA is obtained from EZ by letting X = Y in the description of EZ and exploiting Z_2 -equivariance; and EZ is obtained from DIA by substituting $X \times Y$ for Y in the description of DIA and composing with suitable projections. But the existence of DIA is a straightforward consequence of acyclic model theory; see [9], especially [9, Lemma 6.2].

The acyclic model theorem also states that EZ (or DIA) is essentially unique; fix it for the rest of the section.

Evaluating EZ on the standard generator $1 \in W_0 \subset W$ gives

$$EZ_0: C(X \times Y) \rightarrow C(X) \otimes_{\mathbb{Z}} C(Y),$$

a natural chain homotopy equivalence.

3.2. CONVENTIONS (concerning spherical fibrations and simplicial sets). Let G be a simplicial monoid (associative, with unit). A 'classifying' simplicial set BG is defined as

follows (cf. [6, Definition 3.20] and [11, Definition 10.3]):

$$BG_{q} = \{(g_{0}, g_{1}, \dots, g_{q-1}) | g_{i} \in G_{i}\} \text{ for } q \ge 0,$$

$$d_{i}(g_{0}, g_{1}, \dots, g_{q-1}) = (g_{0}, \dots, g_{q-i-2}, g_{q-i-1} \cdot d_{0}g_{q-i}, d_{1}g_{q-i-1}, \dots, d_{i-1}g_{q-1}),$$

$$s_{i}(g_{0}, g_{1}, \dots, g_{q-1}) = (g_{0}, \dots, g_{q-i-1}, 1_{q-i}, s_{0}g_{q-i}, \dots, s_{i-1}g_{q-1}) \text{ for } 0 \le i \le q$$

(Reading instructions: d_i and s_i are the face and degeneracy operators respectively. It is understood that BG_0 is a singleton, and that the expressions for $d_i(g_0, \ldots, g_{q-1})$ and $s_i(g_0, \ldots, g_{q-1})$ are read from the left if i = 0 and from the right if i = q.)

Next, let EG be the simplicial set given by

$$\begin{split} EG_q &= G_q \times BG_q & \text{for } q \ge 0, \\ s_i(g,b) &= (s_ig,s_ib) & \text{for } 0 \le i \le q, \\ d_i(g,b) &= (d_ig,d_ib) & \text{for } 0 < i \le q, \\ d_0(g,b) &= (t(b) \cdot d_0g,d_0b). \end{split}$$

(Here t(b) is the 'top component' of b; so if $b = (g_0, ..., g_{q-1})$, then $t(b) = g_{q-1}$. In the terminology of [6], EG is a twisted cartesian product with base BG and fibre G.)

Let $p: EG \to BG$ be the projection $(g, b) \mapsto b$. If $\pi_0(G)$ is a group, then the geometric realization of p is a quasi-fibration [8]. If moreover G is a Kan simplicial set, then so is BG (this is proved in 3.21 below). Under these conditions it follows easily that $\pi_n(BG) \cong \pi_{n-1}(G)$ for all n, and that EG is contractible. (Even so, EG does not satisfy the Kan condition as a rule.)

Now let $\overline{G}(n)$ be the topological monoid of self-homotopy-equivalences of the pair (D^n, S^{n-1}) (with the compact-open topology, say; D^n is the *n*-disk). Let G(n) be the singular simplicial set of $\overline{G}(n)$, that is, the standard simplicial approximation. Using the construction BG above, with G = G(n), we make the following definition:

an *n*-dimensional spherical fibration on a simplicial set X is a simplicial map from X to BG(n).

Such a spherical fibration γ on X has a geometric realization: let $E(\gamma)$ be the space

$$\coprod_{q \ge 0} \Delta_q \times D^n \times X_2 / \sim$$

where \sim is the obvious equivalence relation, i.e. is generated by

$$(s_i^{\rightarrow}(u), v, x) \sim (u, v, s_i(x)) \qquad \text{for } q \ge i \ge 0, \ u \in \Delta_{q+1}, \ v \in D^n, \ x \in X_q,$$
$$(d_i^{\rightarrow}(u), v, x) \sim (u, v, d_i(x)) \qquad \text{for } q \ge i \ge 0, \ u \in \Delta_{q-1}, \ v \in D^n, \ x \in X_q.$$

$$a_i(u), v, x) \sim (u, v, a_i(x)) \qquad \text{ for } q \ge i > 0, \ u \in \Delta_{q-1}, \ v \in D^*, \ x \in X_q,$$

$$(d_0^{-1}(u), v, x) \sim (u, t_x(u, v), d_0(x))$$
 for $q > 0, u \in \Delta_{q-1}, v \in D^n, x \in X_q$.

(Here $t_x: \Delta_{q-1} \times D^n \to D^n$ is the image of $x \in X_q$ under the composition

$$X_q \longrightarrow BG_q \xrightarrow{t} G_{q-1}.$$

Notice the formal similarity in the descriptions of $E(\gamma)$ and EG.) Similarly, let $\partial E(\gamma)$ be the space

$$(\coprod_{q \ge 0} \Delta_q \times S^{n-1} \times X_q) / \sim,$$

so that $\partial E(\gamma) \subset E(\gamma)$. The diagram



(in which the projection $E(\gamma) \to X$ is obvious) allows one to interpret γ as a pair of quasi-fibrations over X, with fibre pair (D^n, S^{n-1}) .

Neither $E(\gamma)$ nor $\partial E(\gamma)$ have canonical *CW*-structures; however, the Thom space $E(\gamma)/\partial E(\gamma)$ is a *CW*-space (not a simplicial set) whose cells are in one-one correspondence with those of X. This is extremely convenient.

There are canonical inclusions

 $\dots \rightarrow BG(n-1) \subset BG(n) \subset BG(n+1) \subset \dots;$

a simplicial map $X \to BG(\infty) := \bigcup BG(n)$ is called a stable spherical fibration on X.

3.3. REMINDER. A Poincaré space (or geometric Poincaré complex) is, for the purposes of this section, a finitely generated simplicial set Y equipped with a fundamental class and satisfying Poincaré duality with arbitrary local coefficients—see [15] for details. (So the torsion is allowed to be non-zero.)

Such a Poincaré space Y, of formal dimension n, has a 'Spivak normal fibration', i.e. a stable spherical fibration v_Y on Y, characterized by the following property: there exists a map of CW-spectra

$$r_{\gamma}: S^n \to M(Y, v_{\gamma}) := (\text{Thom spectrum of } v_{\gamma})$$

such that, in loose notation,

(Thom class of $v_y \cap h(r_y) = ($ fundamental class of Y).

(The Hurewicz image of r_Y in $H_n(M(Y, v_Y); \mathbb{Z})$ has been denoted by $h(r_Y)$; here $M(Y, v_Y)$ is the formally desuspended Thom space of v_Y , and the expression 'CW-spectrum' will always mean a spectrum in the sense of Boardman, cf. [19]. A 'map' between CW-spectra is defined as in [19, Definition 8.12], so is automatically cellular.)

'Characterized', in this context, means more than just 'unique up to (stable) concordance'; it means, for example, that the bordism theory of triples (Y, v_Y, r_Y) as above can be identified with the bordism theory of Poincaré spaces Y. So we shall often think of a Poincaré space Y as a triple (Y, v_Y, r_Y) , and similarly for geometric Poincaré pairs.

More generally, a 'higher bordism of Poincaré spaces, of dimension n and order q' consists of

- (i) a functor $V \mapsto Y(V)$ from the category $2^{\{0, 1, \dots, q\}}$ to the category of finitely generated simplicial sets Y and injective simplicial maps ('cofibrations'),
- (ii) a stable spherical fibration v on $Y(\{0, 1, ..., q\})$,
- (iii) a compatible collection of maps of CW-spectra

 $r(V): \Delta(V)_+ \wedge S^{n-q} \rightarrow M(Y(V), v/Y(V)) =$ Thom spectrum.

The functor in (i) is required to be 'intersection-preserving', that is, for $V_1, V_2 \subset \{0, 1, ..., q\}$ we have $Y(V_1 \cap V_2) = Y(V_1) \cap Y(V_2)$ if these spaces are interpreted as subspaces of $Y(V_1 \cup V_2)$; also $Y(\emptyset) = \emptyset$. Finally, each Y(V) is required to be a

geometric Poincaré pair (with boundary equal to $\bigcup Y(U)$, where the union ranges over the proper subsets U of V, and relative fundamental class equal to the Hurewicz image of r(V)).

For the rest of the section we need: a group π and homomorphism w: $\pi \to Z_2$; a finitely generated simplicial set X and a spherical fibration γ on X; a principal π -bundle α on X and an identification of double covers of X,

$$j: w^{\rightarrow}(\alpha) \cong (\text{orientation cover of } \gamma).$$

Write $C(\tilde{X})$ for the cellular chain complex of the total space of α . Then $C(\tilde{X})$ is a chain complex in \mathcal{C}_A , with $A = \mathbb{Z}[\pi]$ (equipped with the involution

$$\sum_{g \in \pi} n_g \cdot g \mapsto \sum_{g \in \pi} (-)^{w(g)} \cdot n_g \cdot g^{-1}).$$

3.4. THEOREM. The data $(\pi, w; X, \gamma; \alpha, j)$ determine (up to an infinity of higher homologies—see 0.15) a chain bundle $c(\gamma)$ on $C(\tilde{X})$. The construction is functorial.

3.5. THEOREM. The geometric bordism spectrum $\underline{\Omega}^{\mathbf{p}}(X, \gamma)$ (details follow) and the algebraic bordism spectrum $\underline{\mathbb{L}}^{0}(C(\tilde{X}), c(\gamma))$ are related by a natural map

$$\underline{\Omega}^{P}(X,\gamma) \to \underline{\mathbb{L}}^{0}(C(\widetilde{X}),c(\gamma)).$$

(There is also a map $\underline{\Omega}^{P}(X,\gamma) \to \underline{\mathscr{L}}^{\circ}(C(\tilde{X}), c(\gamma))$ obtained by composing with the forgetful map from $\underline{\mathbb{L}}^{0}(C(\tilde{X}), c(\gamma))$ to $\underline{\mathscr{L}}^{\circ}(C(\tilde{X}), c(\gamma))$; both maps are called *flexible signature*.)

Explanation. Let $\Omega_0^p(X, \gamma)$ be the Δ -set (incomplete simplicial set) whose q-simplices are the higher bordisms of Poincaré spaces $\{Y(V), v, r(V) \mid V \subset \{0, 1, ..., q\}\}$ of dimension q and order q (as in 3.3), equipped with a simplicial classifying map $g: Y(\{0, 1, ..., q\}) \rightarrow$ so that v equals the pullback $g^-(\gamma)$ (which ought to be written $\gamma \cdot g$). Then $\Omega_0^p(X, \gamma)$ is an infinite loop space (see 3.20 below), and the associated spectrum is $\Omega^P(X, \gamma)$.

Most of this section is devoted to proving 3.4 and 3.5.

3.6. DEFINITION. A ' π -space' will mean (in this section at least) a simplicial set Y with a base point (distinguished 0-simplex) and a simplicial π -action which fixes the base point, but freely permutes the other cells (non-degenerate simplices) of Y. For such a Y,

$$\dot{C}(Y) := C(Y)/C$$
(base point)

is a chain complex in \mathscr{C}_A , provided Y is finitely generated (over π).

3.7. PROPOSITION ('symmetric construction', cf. [15]). For every π -space Y, there is defined a chain map

Sym:
$$\mathbb{Z}' \otimes_A \dot{C}(Y) \to W \& \dot{C}(Y),$$

inducing maps in homology

$${}^{t}\dot{H}_{n}(Y/\pi;\mathbb{Z}) := H_{n}(\mathbb{Z}^{t}\otimes_{A}\dot{C}(Y)) \rightarrow Q^{n}(\dot{C}(Y)).$$

It is natural with respect to π -maps. 5388.3.51

Note: the w-twisted involution on $A = \mathbb{Z}[\pi]$ is used.

Proof. Take the map DIA: $\dot{C}(Y) \rightarrow \text{Hom}_{\mathbb{Z}[Z_2]}(W, \dot{C}(Y) \otimes_{\mathbb{Z}} \dot{C}(Y))$ of 3.1; note that π (and hence A) acts on $\dot{C}(Y)$ as usual, and also on $\text{Hom}_{\mathbb{Z}[Z_2]}(W, \dot{C}(Y) \otimes_{\mathbb{Z}} \dot{C}(Y))$ via the diagonal action on $\dot{C}(Y) \otimes_{\mathbb{Z}} \dot{C}(Y)$. Tensoring with \mathbb{Z} on the left gives

$$\mathbb{Z}' \otimes_{A} \dot{C}(Y) \to \mathbb{Z}' \otimes_{A} (\operatorname{Hom}_{\mathbb{Z}[Z_{2}]}(W, \dot{C}(Y) \otimes_{\mathbb{Z}} \dot{C}(Y)))$$

$$\cong \operatorname{Hom}_{\mathbb{Z}[Z_{2}]}(W, \mathbb{Z}' \otimes_{A} (\dot{C}(Y) \otimes_{\mathbb{Z}} \dot{C}(Y)))$$

$$\cong \operatorname{Hom}_{\mathbb{Z}[Z_{2}]}(W, \dot{C}(Y)' \otimes_{A} \dot{C}(Y)) = W \& \dot{C}(Y).$$

3.8. EXAMPLE. Let (Y, v_Y, r_Y) be a Poincaré space, of formal dimension *n*; suppose that there is given a principal π -bundle β on Y and an identification of twofold covers,

 $w^{-}(\beta) \cong$ (orientation cover of v_{Y}) (= orientation cover of Y)

(with w as in 3.4). Then \tilde{Y}_+ (the total space of β , with an added disjoint base point) is a π -space.

Let $\varphi \in W \& \dot{C}(\tilde{Y}_+) = W \& C(\tilde{Y})$ be the image of the fundamental cycle under the chain map Sym. (The fundamental cycle is the cycle determined by r_Y ; it represents the fundamental class.)

Then $(C(\tilde{Y}), \varphi)$ is an *n*-dimensional algebraic Poincaré complex over A—the 'algebraic image' of Y.

3.9. OUTLINE. We are now in a position to obtain a sketch proof of 3.4 and 3.5. It is taken without essential change from [15, Part II, \S 9].

(i) Starting with a string $(\pi, w; X, \gamma; \alpha, j)$ as in 3.4, and assuming that γ is k-dimensional, Ranicki obtains a characteristic class $[c(\gamma)] \in \hat{Q}^0(C(\tilde{X})^{-*})$ by choosing a π -equivariant S-dual $T(\tilde{X}, \tilde{\gamma})^*$ of the Thom π -space $T(\tilde{X}, \tilde{\gamma})$, and applying the symmetric construction

$${}^{\prime}\dot{H}_{*}(T(X,\gamma)^{*}) \rightarrow Q^{*}(\dot{C}(T(\tilde{X},\tilde{\gamma})^{*}))$$

of 3.7 to the dual of the Thom class in $'\dot{H}^k(T(X,\gamma))$. This yields a class $[\psi]$ in $Q^n(\dot{C}(T(\tilde{X},\tilde{\gamma})^*))$ for some *n*, to begin with. Observe now that we have a chain homotopy equivalence

$$f \colon \Sigma^n(C(\tilde{X})^{-*}) \xrightarrow{\simeq} \dot{C}(T(\tilde{X},\tilde{\gamma})^*)$$

by composing S-duality with the Thom isomorphism. Then

$$f^{\neg}: \hat{Q}^{n}(\Sigma^{n}(C(\tilde{X})^{-*})) \to \hat{Q}^{n}(\dot{C}(T(\tilde{X},\tilde{\gamma})^{*}))$$

is an isomorphism, so that we may define the characteristic class $[c(\gamma)] \in \hat{Q}^0(C(\tilde{X})^{-*})$ by the formula

$$J(\llbracket \psi \rrbracket) = f^{\neg} \cdot \mathfrak{S}^n(\llbracket c(\gamma) \rrbracket) \quad \text{in} \ \hat{Q}^n(\dot{C}(T(\tilde{X}, \tilde{\gamma})^*)).$$

This nearly proves 3.4.

(ii) Suppose next that the string $(\pi, w; X, \gamma; \alpha, j)$ is such that X is an *n*-dimensional Poincaré space with Spivak normal fibration γ . Let $(C(\tilde{X}), \varphi)$ be the *n*-dimensional algebraic Poincaré complex constructed from X as in 3.8. We may take, as is well known,

$$T(\tilde{X},\tilde{\gamma})^* = \tilde{X}_+.$$

It is then easy to check that $[\psi] = [\varphi]$ and $f = \varphi_0$. If we insert this in the formula defining $[c(\gamma)]$, we see that

$$[c(\gamma)] = [n],$$

where *n* is the normal chain bundle of $(C(\tilde{X}), \varphi)$. So 3.5 is nearly proved.

The reader is advised to omit the rest of the section except 3.16, 3.17, and 3.18, and to regard the arguments above as proofs. It should be realized, however, that they are inadequate in two respects. Firstly, in 3.4 we do need a chain bundle $c(\gamma)$ rather than just a class $[c(\gamma)]$, as is shown in 3.17. Secondly, the argument for 3.5 given above does not survive the generalization from Poincaré spaces to *normal spaces* (see Part II, §7 of this paper). We begin with the rigorous proof.

3.10. MACHINERY. Let \aleph be a Δ -set. We will regard \aleph as a category (whose objects are the simplices of \aleph ; a morphism from an *n*-simplex x to an *m*-simplex y is an injective order-preserving map $\{0, 1, ..., n\} \rightarrow \{0, 1, ..., m\}$ so that the corresponding face operator sends y to x).

An 'N-indexed chain complex' is a covariant functor G from N to the category of chain complexes. Given such a G, and given any Δ -subset N' of N, define a new (ordinary) chain complex (Sect_{N'}; G), or (Sect_{N'}; G(-)), by

$$(\operatorname{Sect}_{\aleph'}; G)_n = \prod_{x \in \aleph'} (G(x))_{n+|x|},$$
$$[ds]_x := \operatorname{diff}_x([s]_x) - \sum_{0 \le i \le |x|} (-)^{n-i} \cdot j_{d_ix,x}([s]_{d_ix}).$$

Here $s \in (\text{Sect}_{\aleph'}; G)_n$; if x is a simplex in \aleph' , we write $[s]_x$ for its x-component, and diff_x for the differential in G(x). Finally, $j_{d_ix,x}$ is the inclusion of the *i*th face as usual, and d is the differential in $(\text{Sect}_{\aleph'}; G)$.)

Alternatively, $(Sect_{\aleph'}; G)$ can be described as the subcomplex of natural chains in

$$\prod_{x\in\mathcal{N}'}\operatorname{Hom}_{\mathbb{Z}}(\operatorname{Cl}(\Delta_{|x|}),G(x)),$$

where $Cl(\Delta_{|x|})$ is the cellular chain complex of the standard simplex. (Call a collection $\{f_x \in Hom_{\mathbb{Z}}(Cl(\Delta_{|x|})G(x)) | x \in \mathbb{N}'\}$ natural if

$$p^{\neg} \cdot f_x = f_y \cdot p^{\neg}$$

for every morphism $p: x \to y$ in \aleph' .)

Note that the construction has 'sheaflike' properties: given two Δ -subsets N' and N'' of N, there is a pullback square of restriction maps



Suppose next that G above is a covariant functor, not merely from the category \aleph to the category of chain complexes, but from \aleph to \mathscr{C}_A . Suppose also that we have an ordinary chain complex C in \mathscr{C}_A and an A-module chain map

$$f: C \rightarrow (\operatorname{Sect}_{\aleph}; G(-)).$$

An induced chain map

$$f^{\rightarrow}: \widehat{W} \& C \to (\operatorname{Sect}_{\aleph}; \widehat{W} \& G(-))$$

is defined as follows. Using the second description of $(Sect_{\aleph}; G(-))$, we see that f is nothing but a natural collection of chain maps

$$f_x: C \otimes_{\mathbb{Z}} \operatorname{Cl}(\Delta_{|x|}) \to G(x).$$

Now define $f \to by [f \to (\varphi)]_x := f_x \to (\varphi \times \rho(|x|)) \in \hat{W} \& G(x)$, for $\varphi \in \hat{W} \& C$; to make sense of this formula, use the sequence $\{\rho(m) \mid m \ge 0\}$ of 2.9.

3.11. DEFINITION. From now on \aleph will be the moduli space of all attempts at being equivariantly S-dual to the Thom π -spectrum $M(\tilde{X}, \tilde{\gamma})$. In detail, a typical q-simplex y of \aleph shall consist of

an intersection-preserving functor Ftr_y from $2^{\{0, 1, \dots, q\}}$ to the category of finitely generated π -spaces and π -cofibrations, and

a compatible collection of maps of CW-spectra

$$\Delta(V)_+ \to M(X \times_{\pi} \operatorname{Ftr}_{y}(V), \gamma) / M(X, \gamma),$$

one for each $V \subset \{0, 1, \dots, q\}$.

This should require a fair amount of explanation.

(i) A π -space U is finitely generated if the space or rather simplicial set U/π is finitely generated. Intersection-preserving means here that

$$\operatorname{Ftr}_{v}(V_{1} \cap V_{2}) = \operatorname{Ftr}_{v}(V_{1}) \cap \operatorname{Ftr}_{v}(V_{2})$$

if these spaces are interpreted as subspaces of $\operatorname{Ftr}_{y}(V_1 \cup V_2)$, and that $\operatorname{Ftr}_{y}(\emptyset)$ is a point.

(ii) If U is any π -space, then $X = \tilde{X} \times_{\pi}$ (base point) is contained in $\tilde{X} \times_{\pi} U$. Note further that

$$M(\tilde{X} \times_{\pi} U, \gamma) / M(X, \gamma) \cong M(\tilde{X}, \tilde{\gamma}) \wedge_{\pi} U,$$

if the cell decompositions are disregarded; here the pullback of γ to $\tilde{X} \times_{\pi} U$ has also been called γ .

In particular, if y is a q-simplex in \aleph having all faces at the base point, then y consists of a π -space $U = \operatorname{Ftr}_{y}(\{0, 1, ..., n\})$ and a map of spectra from S^q to $M(\tilde{X}, \tilde{\gamma}) \wedge_{\pi} U$. We may call this an attempt on the part of U at being equivariantly S-dual to $M(\tilde{X}, \tilde{\gamma})$.

Having specified \aleph , we shall also specify the \aleph -indexed chain complex G in 3.10 by letting

$$G(y) = C(\operatorname{Ftr}_{y}(\{0, 1, \dots, q\}))$$

for a q-simplex y in \aleph ; in other words G(y) is the reduced cellular chain complex of the underlying π -space of y.

We have now collected most of the material necessary to rewrite 3.9 in a rigorous fashion. What follows is a *parametrized* version of 3.9 where \aleph serves as parameter space. So, instead of choosing an equivariant S-dual $T(\tilde{X}, \tilde{\gamma})^*$ of the Thom π -space $T(\tilde{X}, \tilde{\gamma}) = \Sigma^k M(\tilde{X}, \tilde{\gamma})$, we shall consider *simultaneously* all attempts at being equivariantly S-dual to $M(\tilde{X}, \tilde{\gamma})$. The role of $\dot{C}(T(\tilde{X}, \tilde{\gamma})^*)$ in 3.9 will be played by (Sect_{\aleph}; G(-)); the next construction shows that the symbols f and ψ in 3.9(i) also have their parametrized counterparts.

- 3.12. CONSTRUCTION. We shall construct
- (i) a chain map $f: C(\tilde{X})^{-*} \to (Sect_{\aleph}; G(-)),$
- (ii) a 0-cycle ψ in (Sect₈; W & G(-)).

For (i), let \overline{G} be the \aleph -indexed chain complex such that

$$\overline{G}(y) = C(M(X \times_{\pi} \operatorname{Ftr}_{v}(\{0, 1, \dots, q\}), \gamma) / M(X, \gamma))$$

if y is a q-simplex in \aleph . The very definition of \aleph yields a canonical 0-cycle \bar{z} in (Sect_{\aleph}; $\bar{G}(-)$). Our conventions concerning spherical fibrations (see the end of 3.2) give us a Thom isomorphism on the chain level; composing with the Eilenberg–Zilber map EZ₀ of 3.1, we get a natural homotopy equivalence

 $\bar{G}(y) \xrightarrow{\simeq} C(\tilde{X}) \otimes_{A} G(y) \cong \operatorname{Hom}_{A}(C(\tilde{X})^{-*}, G(y))$

for any simplex y in \aleph . Therefore we now have a 0-cycle

 $z \in (\text{Sect}_{\aleph}; \text{Hom}_{A}(C(\tilde{X})^{-*}, G(-)))$ (the image of \bar{z}).

Define $f: C(\tilde{X})^{-*} \rightarrow (\operatorname{Sect}_{\aleph}; G(-))$ by

$$[f(x)]_y := [z]_y(x)$$

for $x \in C(\tilde{X})^{-*}$ and y a simplex in \aleph .

In (ii), we put

$$\psi := [\operatorname{Sym} \cdot (\mathbb{Z}' \otimes_A f)](1_c).$$

Here $\mathbb{Z}^{t} \otimes_{A} f$ is the chain map from $\mathbb{Z}^{t} \otimes_{A} C(\tilde{X})^{-*}$ to $(\text{Sect}_{\aleph}; \mathbb{Z}^{t} \otimes_{A} G(-))$ obtained by tensoring f with \mathbb{Z}^{t} ; further, 1_{c} is the unique 0-cycle in $\mathbb{Z}^{t} \otimes_{A} C(\tilde{X})^{-*}$ representing $1 \in H^{0}(X; \mathbb{Z}) \cong H_{0}(\mathbb{Z}^{t} \otimes_{A} C(\tilde{X})^{-*})$, and Sym denotes the parametrized symmetric construction which is a chain map from $(\text{Sect}_{\aleph}; \mathbb{Z}^{t} \otimes_{A} G(-))$ to $(\text{Sect}_{\aleph}; W \& G(-))$. See 3.7.

3.13. Key LEMMA. Both $f: C(\tilde{X})^{-*} \to (\text{Sect}_{\aleph}; G(-))$ and the induced chain map $f^{-}: \hat{W} \& C(\tilde{X})^{-*} \to (\text{Sect}_{\aleph}; \hat{W} \& G(-))$ are chain homotopy equivalences.

The proof is deferred; see 3.19. Now let

$$\mathscr{P}(\pi,w;X,\gamma;\alpha,j)$$

be the homotopy pullback (see 0.14) of the diagram of chain maps

$$\widehat{W} \& C(\widehat{X})^{-*}$$

$$\simeq \int f^{-}$$

$$(\mathbb{Z}, 0) \xrightarrow{1 \mapsto J(\psi)} (\operatorname{Sect}_{\aleph}; \widehat{W} \& G(-))$$

Then the projections

$$\mathcal{P}(\pi, w; X, \gamma; \alpha, j) \longrightarrow \widehat{W} \& C(\overline{X})^{-*}$$

$$\downarrow \simeq$$

$$(\mathbb{Z}, 0)$$

constitute a 0-cycle in $\hat{W} \& C(\tilde{X})^{-*}$, well-defined up to an infinity of higher homologies (see 0.15). This proves 3.4, since a 0-cycle in $\hat{W} \& C(\tilde{X})^{-*}$ is a chain bundle on $C(\tilde{X})$.

In the next lemma, an *admissible* 0-cycle in $\mathscr{P}(\pi, w; X, \gamma; \alpha, j)$ means a cycle in the class $1 \in \mathbb{Z} \cong H_0(\mathscr{P}(\pi, w; X, \gamma; \alpha, j))$.

3.14. LEMMA. Every admissible 0-cycle s in $\mathscr{P}(\pi, w; X, \gamma; \alpha, j)$ determines a Δ -map (fl.sig.)_s: $\Omega_0^P(X, \gamma) \to \mathbb{L}^0(C(\tilde{X}), c_s(\gamma))$. (See the paragraph following 3.5.) Here $c_s(\gamma)$ is the chain bundle on $C(\tilde{X})$ determined by s.

Proof. We begin with a Δ -map $i: \Omega_0^p(X, \gamma) \to \aleph$. Let x be a q-simplex in $\Omega_0^p(X, \gamma)$; then x consists of a collection $\{Y(V), v, r(V) | V \subset \{0, 1, ..., q\}\}$ and a classifying map $g: Y(\{0, 1, ..., q\}) \to X$ such that $g^-(\gamma) = v$. (See the end of 3.3.) Each Y(V) inherits a principal π -bundle from X, with total space $\tilde{Y}(V)$.

Now i(x) is the q-simplex in \aleph such that

$$\operatorname{Ftr}_{i(x)}(V) \to \widetilde{Y}(V)_+;$$

for the stable map from $\Delta(V)_+$ to $M(\tilde{X} \times_{\pi} \operatorname{Ftr}_{i(x)}(V), \gamma)/M(X, \gamma)$ required in 3.11 we take the composition

$$\Delta(V)_+ \xrightarrow{r(V)} M(Y(V), v) \xrightarrow{\text{classifying map} \times \text{id}} M(\tilde{X} \times_{\pi} \tilde{Y}(V), \gamma).$$

This is a cellular map (as it should be) because X and Y(V) are simplicial sets. The description of *i* is complete.

Next, observe that the parametrized version of 3.8 produces a canonical 0-cycle

$$\varphi \in (\operatorname{Sect}_{\Omega^p_0(X,\gamma)}; G(-)).$$

(Here G is short for $G \cdot i$.) All we need now in order to get a map from $\Omega_0^P(X, \gamma)$ to $\mathbb{L}^0(C(\tilde{X}), c_s(\gamma))$ is a collection of clutching homologies (see 2.7, 2.8, 2.9). In other words, we are searching for a 1-chain

 $z \in (\operatorname{Sect}_{\Omega_0^p(X,\gamma)}; \widehat{W} \& G(-)).$

Now I claim that such a 1-chain z can be extracted from the admissible cycle $s \in \mathscr{P}(\pi, w; X, \gamma; \alpha, j)$ in 3.14. Indeed, we constructed $\mathscr{P}(\pi, w; X, \gamma; \alpha, j)$ as a chain homotopy pullback, so our admissible 0-cycle is a triple

 $(1, c_s(\gamma), \bar{z})$

with $1 \in (\mathbb{Z}, 0)$, $c_s(\gamma) \in \hat{W} \& C(\tilde{X})^{-*}$, and $\bar{z} \in (\text{Sect}_{\aleph}; \hat{W} \& G(-))$. (See 0.14.) The pullback of \bar{z} under $i: \Omega_0^p(X, \gamma) \to \aleph$ is the required 1-chain z. Inspection shows that it satisfies the equations in 2.8.

3.15. LEMMA. The Δ -map (fl.sig.)_s in 3.14 has a canonical refinement to a map of infinite loop spaces.

For the proof, see 3.20. To complete the proof of 3.5, we still have to show that the map of spectra

flexible signature: $\Omega^{P}(X, \gamma) \rightarrow \mathbb{L}^{0}(C(\tilde{X}), c(\gamma))$

obtained from 3.15 does not depend too much on the choice of an admissible 0-cycle s in $\mathcal{P}(\pi, w; X, \gamma; \alpha, j)$. This should follow from the existence of a homotopy equival-

ence $\mathscr{P}(\pi, w; X, \gamma; \alpha, j) \simeq (\mathbb{Z}, 0)$. But the following argument is easier. Pick another admissible 0-cycle s' in $\mathscr{P}(\pi, w; X, \gamma; \alpha, j)$. Crossing the string of data $(\pi, w; X, \gamma; \alpha, j)$ with the unit interval [0, 1] gives a new string, written

$$(\pi, w; X \times [0, 1], \gamma \times [0, 1]; \alpha, j).$$

Choose an admissible 0-cycle s" in $\mathscr{P}(\pi, w; X \times [0, 1], \gamma \times [0, 1]; \alpha, j)$ whose image under the restriction map

$$\mathcal{P}(\pi, w; X \times [0, 1], \gamma \times [0, 1]; \alpha, j)$$

$$\rightarrow \mathcal{P}(\pi, w; X \times \{0\}, \gamma; \alpha, j) \oplus \mathcal{P}(\pi, w; X \times \{1\}, \gamma; \alpha, j)$$

is (s, s'). Such an s'' exists. Then the commutative diagram

shows what we want, since all horizontal arrows are homotopy equivalences. The proof of 3.5 is complete; the naturality part is stated separately below.

3.16. REMARK. Suppose that there are given two strings $(\pi, w; X, \gamma; \alpha, j)$ and $(\pi', w'; X', \gamma'; \alpha', j')$ as in 3.4, and

- (i) a homomorphism $h: \pi \to \pi'$ such that $w' \cdot h = w$;
- (ii) a simplicial map $g: X \to X'$ covered by a map of spherical fibrations from γ to γ' ;
- (iii) an identification $h^{-}(\alpha) \cong g^{-}(\alpha')$ of principal π' -bundles on X, compatible with j and j'.

Such a 'morphism' induces a sufficiently well-defined chain bundle map

$$C(\tilde{X}), c(\gamma) \rightarrow C(X'), c(\gamma')$$

(involving a change of rings, cf. 1.8 and sequel); and the diagram

is sufficiently commutative for all practical purposes.

Geometric transfer maps also have algebraic counterparts: let $(\pi, w; X, \gamma; \alpha, j)$ be a string of data as usual, and suppose that $\pi'' \subset \pi$ is a subgroup of finite index. A second string $(\pi'', w''; X'', \gamma''; \alpha'', j'')$ is then given by

 $w'' := w \cdot inclusion,$ $X'' := total space of \alpha, modulo action of \pi'',$ $\gamma'' := pullback of \gamma,$ $\alpha'' := principal \pi''-bundle on X'' derived from \alpha,$ j'' := identification derived from j. There is a sufficiently commutative diagram of maps of spectra



As in 3.5, $\mathbb{L}^{0}(...)$ can be replaced by $\mathcal{L}^{:}(...)$.

3.17. EXAMPLE. Using notation as in 3.4, let $[c(\gamma)] \in \hat{Q}^0(C(\tilde{X})^{-*})$ be the class of $c(\gamma)$. In §1 (after 1.8) elements of $\hat{Q}^0(C(\tilde{X})^{-*})$ were interpreted as 'isomorphism classes of chain bundles' on $C(\tilde{X})$. Consequently, knowledge of $[c(\gamma)] \in \hat{Q}^0(C(\tilde{X})^{-*})$ suffices to reconstruct the groups $L^n(C(\tilde{X}), c(\gamma))$ (see 2.21 (iv)) 'up to isomorphism'. More cannot be expected, as is shown by the following example.

Let N^n be a smooth closed manifold admitting two stable framings Fr_1 , Fr_2 such that the Kervaire invariants of (N, Fr_1) and (N, Fr_2) are defined and distinct. (Such manifolds are known to exist for n = 2, 6, 14, 30, 62 (?).) Specify the string $(\pi, w; X, \gamma; \alpha, j)$ as follows (see 3.4): X = N and γ is trivial, $\pi = \{1\}$, etc. Let us work with smooth manifolds instead of Poincaré spaces; we may then replace $\Omega^P(X, \gamma)$ by the Thom spectrum $M(X, \gamma)$. The difference between the two framings Fr_1 and Fr_2 is a map from N = X to the orthogonal group; it can also be regarded as a stable automorphism of the trivial bundle γ on X, written tw. The algebraic counterpart of tw is a chain bundle automorphism τ of $c(\gamma)$ (covering the identity $C(X) \to C(X)$). There is a commutative diagram

$$\pi_n(M(X,\gamma)) \xrightarrow{(tw)} \pi_n(M(X,\gamma))$$
flexible
signature
$$L^n(C(X), c(\gamma)) \xrightarrow{\tau^-} L^n(C(X), c(\gamma))$$

(notation as in 3.4, 3.5, 2.21 (iv)).

Claim. Both $(tw)^{\neg}$ and τ^{\neg} are non-trivial group automorphisms. Indeed, let $y \in \pi_n(M(X, \gamma))$ be the bordism class represented by (id, Fr_1) : $N, v_N \to X, \gamma$ (recall that X = N, and γ is the trivial bundle). Also, let X' be a one-point space, and let γ' be the trivial bundle on X'; the obvious bundle map $X, \gamma \to X', \gamma'$ then induces a homomorphism

?:
$$\pi_n(M(X,\gamma)) \rightarrow \pi_n^s = \pi_n(M(X',\gamma')).$$

The elements ?(y) and $?((tw)^{\rightarrow}(y))$ of π_n^s have distinct Kervaire invariants by construction; hence $(tw)^{\rightarrow}$ is not the identity. Practically the same argument shows that

 $fl.sig.(y) \neq (fl.sig.(y)),$

proving that τ^{\rightarrow} is non-trivial.

So it is impossible to give an 'honest' description of $L^{n}(C(\tilde{X}), c(\gamma))$ in terms of $[c(\gamma)] \in \hat{Q}^{0}(C(\tilde{X})^{-*})$.

3.18. REMARK. If $(C(\tilde{Y}), \varphi)$ is the *n*-dimensional algebraic Poincaré complex derived (as in 3.8) from an *n*-dimensional Poincaré space Y, with Spivak normal

bundle v_{y} etc., then there is a canonical identification

 $c(v_{Y}) = (\text{normal chain bundle of } (C(\tilde{Y}), \varphi))$

(see the sequel to 2.6). This is clear from the proof of 3.5: the identity map $Y, v_Y \to Y, v_Y$ represents an element in $\pi_n(\Omega^P(Y, v_Y))$; and we know that $(C(\tilde{Y}), \varphi)$ has a preferred $c(v_Y)$ -structure (g, z), in which $g: C(\tilde{Y}) \to C(\tilde{Y})$ is the identity. In other words, $c(v_Y)$ has the property which characterizes the normal chain bundles, as required.

Given a degree-1 map $e: P_1 \rightarrow P_2$ between Poincaré spaces of the same formal dimension *n* (equipped with suitable data, such as principal π -bundles), [15] defines the 'symmetric kernel' of *e*, an *n*-dimensional algebraic Poincaré complex (C', φ'). If in addition the map *e* has the attributes of a normal map, then it is easy to see that the normal chain bundle of (C', φ') is 'trivialized' (using 3.18); by 2.17, the algebraic Poincaré complex (C', φ') together with this trivialization defines an element of $L_n(\mathbb{Z}[\pi])$.

Finally, let $(\pi, w; X, \gamma; \alpha, j)$ be a string as usual, and suppose that we are given maps of finite CW-spaces

$$P_1 \xrightarrow{g} P_2 \xrightarrow{h} X$$

and a map of spectra $r: S^n \to M(P_1, (h \cdot g)^-(\gamma))$ (where M(...) denotes the Thom spectrum) such that the triples $(P_1, (h \cdot g)^-(\gamma), r)$ and $(P_2, h^-(\gamma), g^-(r))$ are Poincaré spaces in the sense of 3.3 and 3.8.

Then g is clearly a normal map of degree 1, so, by what we have just seen, an element $\sigma_*(g)$ in $L_n(\mathbb{Z}[\pi])$ is defined, traditionally called the surgery obstruction. We have

$$\sigma^*(P_1) - \sigma^*(P_2) = \text{release}(\sigma_*(g))$$

in $L^{n}(C(\tilde{X}), c(\gamma))$, where σ^{*} denotes the flexible signature.

To prove this, note that the degree-1 map g induces a splitting of the algebraic Poincaré complex of P_1 into two direct summands; one of these is homotopy equivalent to the algebraic Poincaré complex of P_2 , the other is the 'symmetric kernel' of g. It follows easily that

$$\sigma^*(P_1) = \sigma^*(P_2) + \text{release}(\sigma_*(g)),$$

which completes the proof. See also 3.17.

Here are the remaining proofs.

3.19. PROOF OF 3.13 (an application of the equivariant S-duality theory of [15]). In [15], a ' $CW\pi$ -space' is defined to be a CW-space with base point (distinguished 0-cell) and a cellular π -action which leaves the base point fixed, but permutes the other cells freely. (This is slightly more general than the π -spaces in 3.6.)

A $CW\pi$ -spectrum is defined along the same lines. A $CW\pi$ -space or $CW\pi$ -spectrum E is called *finite* if E/π has only finitely many cells; in that case the (reduced) cellular chain complex $\dot{C}(E)$ belongs to \mathscr{C}_A (with $A = \mathbb{Z}[\pi]$).

Let $\partial \Delta_n$ be the union of the proper faces of the simplex Δ_n ; regard $(\partial \Delta_n)_+$ as a subspectrum of $(\Delta_n)_+$. The following result is implicit in [15].

(i) Suppose that E and ∂F are finite CW-spectra, and

$$\partial g: (\partial \Delta_n)_+ \to E \wedge_\pi \partial F$$

is a map of CW-spectra (the smash product being defined in the naive way). Then there exists a $CW\pi$ -spectrum F containing ∂F , and a map

$$g: (\Delta_n)_+ \to E \wedge_\pi F$$

extending ∂g , such that

$$g/\partial g: (\Delta_n)_+/(\partial \Delta_n)_+ \to E \wedge_\pi F/\partial F$$

(where $(\Delta_n)_+/(\partial \Delta_n)_+ \simeq S^n$) is an $S\pi$ -duality (see [15]).

We apply this to the study of the Δ -set \aleph defined before 3.11. Every q-simplex y in \aleph stands for a π -space $Ftr_{v}(\{0, 1, ..., q\})$ (as in 3.6) and a map of spectra

 $(\Delta_{q})_{+} \rightarrow M(\tilde{X} \times_{\pi} \operatorname{Ftr}_{v}(\{0, 1, \dots, q\}), \gamma) \simeq M(\tilde{X}, \gamma) \wedge_{\pi} \operatorname{Ftr}_{v}(\{0, 1, \dots, q\}),$

etc.; collapsing boundaries gives a map of spectra

$$\eta_{y}: (\Delta_{q})_{+}/(\partial \Delta_{q}) \to M(\tilde{X}, \gamma) \wedge_{\pi}(\operatorname{Ftr}_{y}(\{0, 1, \dots, q\})/\bigcup \operatorname{Ftr}_{y}(V))$$

(where $(\Delta_q)_+/(\partial \Delta_q) \simeq S^q$ and where V ranges over the proper subsets of $\{0, 1, ..., q\}$). Call y regular if η_y is an $S\pi$ -duality map. Then we have

(ii) A q-simplex y is regular if and only if the composition

$$C(\widetilde{X})^{-*} \xrightarrow{f} (\operatorname{Sect}_{\aleph}; G(-)) \xrightarrow{\operatorname{pullback}} (\operatorname{Sect}_{\Delta_q}; G \cdot \operatorname{ch}_{\nu}(-))$$

is a chain homotopy equivalence. Here $ch_y: \Delta_q \to \aleph$ is the characteristic Δ -map associated with y.

Proof. Let $\partial G(y) \subset G(y)$ be the chain subcomplex generated by the images $j_{z,y}^{\rightarrow}(G(z)) \subset G(y)$, where z ranges over the proper faces of y. Then we have to show that the obvious projection

$$p: (\operatorname{Sect}_{\Delta_q}; G \cdot \operatorname{ch}_{v}(-)) \to \Sigma^{q}(G(y)/\partial G(y))$$

is a chain homotopy equivalence.

Observe that the skeletal filtration of Δ_q induces a filtration of $(\text{Sect}_{\Delta_q}; G \cdot \text{ch}_y(-))$ by subcomplexes. It is not difficult to construct a similar filtration of the homotopy type of $\Sigma^q(G(y)/\partial G(y))$, and to show that p induces chain homotopy equivalences on the successive quotients. This proves (ii).

(iii) The regular simplices generate \aleph (that is, every simplex in \aleph is a face of some regular simplex).

Proof. Given a q-simplex y in \aleph , say $y = (Ftr_y, ...)$, let $Cone(y) = (Ftr_{Cone(y)}, ...)$ be a (q+1)-dimensional simplex in \aleph such that $d_0(Cone(y)) = y$, and such that $Ftr_{Cone(y)}(V)$ is a contractible π -space whenever $0 \in V \subset \{0, 1, ..., q+1\}$. Iterating the construction, define simplices

$$\operatorname{Cone}^{n}(y) = \operatorname{Cone}(\operatorname{Cone}^{n-1}(y))$$

of dimension q + n, for all n > 1. Using (i) above it is easy to see that for sufficiently

large n, there exists a (q+n)-simplex x in \aleph which is regular and such that

$$d_i(x) = d_i(\text{Cone}^n(y)) \text{ for } 0 \le i \le q+n.$$

Clearly y is a face of x, which proves (iii).

(iv) Suppose that $g_1: K_1 \to \aleph$ is a map of Δ -sets, with K_1 finite. Then there exists a diagram of Δ -sets and Δ -maps



with K_2 finite, such that the composition

$$C(\tilde{X})^{-*} \xrightarrow{f} (\operatorname{Sect}_{\aleph}; G(-)) \to (\operatorname{Sect}_{K_2}; G \cdot g_2(-))$$

is a chain homotopy equivalence.

Proof. Suppose first that K_1 is a polyhedron. This means that K_1 can be embedded (as a Δ -set) in a standard simplex Δ_n . Choose such an embedding $e: K_1 \to \Delta_n := K_2$. Since \aleph is a contractible Kan Δ -set, there exists a Δ -map $g_2: \Delta_n \to \aleph$ such that $g_2 \cdot e = g_1$. We may also assume that g_2 maps Δ_n to a regular simplex in \aleph (otherwise replace Δ_n by a standard simplex of greater dimension, using (iii) above). This proves (iv) in the special case where K_1 is a polyhedron.

Now let K_1 be an arbitrary finite Δ -set. Choose a diagram of Δ -sets and Δ -maps



with M and \bar{K}_1 finite, such that \bar{K}_1 is a polyhedron and such that the restriction maps

$$(\operatorname{Sect}_{K_1}; G \cdot g_1(-)) \leftarrow (\operatorname{Sect}_M; G \cdot h(-)) \rightarrow (\operatorname{Sect}_{\bar{K}_1}; G \cdot \bar{g}_1(-))$$

are chain homotopy equivalences. (Such a diagram is easy to construct; for instance, \bar{K}_1 can be taken isomorphic to an iterated barycentric subdivision of K_1 .) Since \bar{K}_1 is a polyhedron, we can find another diagram



with \bar{K}_2 finite, such that the composition

$$C(\tilde{X})^{-*} \xrightarrow{f} (\operatorname{Sect}_{\aleph}; G(-)) \longrightarrow (\operatorname{Sect}_{\tilde{K}_2}; G \cdot \bar{g}_2(-))$$

is a chain homotopy equivalence. Let K_2 be the pushout of $M \leftarrow \overline{K}_1 \rightarrow \overline{K}_2$; let $g_2: K_2 \rightarrow \aleph$ be the amalgam of h and \overline{g}_2 , and let e be the composition $K_1 \rightarrow M \rightarrow K_2$. The proof of (iv) is complete.

We can now finish the proof of 3.13 using standard limit arguments. Let \mathscr{K} be the category whose objects are Δ -maps $g: K \to \aleph$ with K finite, and with the property that the composite

$$C(\tilde{X})^{-*} \xrightarrow{f} (\operatorname{Sect}_{\aleph}; G(-)) \longrightarrow (\operatorname{Sect}_{K}; G \cdot g(-))$$

is a chain homotopy equivalence. A morphism from $g: K \to \aleph$ to $g': K' \to \aleph$ shall be a Δ -map $h: K' \to K$ such that $g \cdot h = g'$. By (iv), the category \mathscr{K} is a small left filtering in the sense of [1]; by (iv) again, we may write

$$(\operatorname{Sect}_{\aleph}; G(-)) \cong \lim_{\substack{g: K \to \aleph}} (\operatorname{Sect}_{K}; G \cdot g(-)),$$

where the inverse limit is taken over \mathcal{K} .

Note that all chain maps in this inverse system are homotopy equivalences, by definition of \mathcal{K} . So the following implies 3.13.

(v) The injection

$$\lim_{g:K\to\infty} (\operatorname{Sect}_K; G \cdot g(-)) \to \operatorname{holim}_{g:K\to\infty} (\operatorname{Sect}_K; G \cdot g(-))$$

is a chain homotopy equivalence. (Both limits are taken over \mathcal{K} .)

Explanation and proof. The homotopy inverse limit holim is defined as in [1], mutatis mutandis.

For an object $g: K \to \aleph$ of \mathscr{K} , let G_g be the \aleph -indexed chain complex such that

$$G_g(y) = \prod_{x \in g^{-1}(y)} G(y)$$

whenever y is a simplex in \aleph . Then

$$(\operatorname{Sect}_K; G \cdot g(-)) \cong (\operatorname{Sect}_{\aleph}; G_q(-)).$$

Therefore

$$\operatorname{holim}(\operatorname{Sect}_{\kappa}; G \cdot g(-)) \cong \operatorname{holim}(\operatorname{Sect}_{\aleph}; G_g(-)) \cong (\operatorname{Sect}_{\aleph}; \operatorname{holim} G_g(-)),$$

and similarly

$$\lim \left(\operatorname{Sect}_{K}; G \cdot g(-)\right) \cong \left(\operatorname{Sect}_{\aleph}; \lim G_{g}(-)\right).$$

It is now sufficient to show that for each simplex y in \aleph , the injection

$$\lim_{x \to g} G_g(y) \cong G(y) \to \operatorname{holim} G_g(y)$$

is a chain homotopy equivalence. But this is obvious.

We have now shown that f in 3.13 is a homotopy equivalence; the proof for $f \stackrel{\neg}{}$ is similar because the functor $C \mapsto \widehat{W} \& C$ on \mathscr{C}_A is essentially linear, as is shown in 1.1.

3.20. PROOF OF 3.15. In 2.19 we saw that $\mathbb{L}^{0}(C(\tilde{X}), c(\gamma))$ is the underlying space E(1) of a Γ -space $\{E(n)$ etc. $| n \ge 0\}$. Much the same argument, with direct sums replaced

by disjoint unions, makes $\Omega_0^P(X, \gamma)$ the underlying space F(1) of a Γ -space $\{F(n)$ etc. $| n \ge 0\}$.

The symmetric group on *n* letters S(n) acts on F(n); this action is free for all *n* (on the complement of the base point), and in particular for n = 2. We shall use this fact below.

Our problem is to refine the map $F(1) \rightarrow E(1)$ in 3.15 to a family of maps $F(n) \rightarrow E(n)$, where $n \ge 0$, commuting with the various structure maps which are part of a Γ -space.

It is a good idea to think of a simplex in E(n) as an affair with three levels: the first level involves nothing more complicated than chain complexes (and chain maps), the second nothing more complicated than algebraic Poincaré complexes, and the third involves everything (i.e. algebraic Poincaré complexes with $c(\gamma)$ -structure).

(i) With regard to the first and second levels, it is clear what the maps $F(n) \rightarrow E(n)$ ought to do (if they are to commute with the Γ -space structure maps).

(ii) It follows that there is essentially just one reasonable map from F(2) to E(2) which is compatible with the given map from F(1) to E(1) (and with the Γ -space structure maps relating F(1) and F(2) on one hand, and E(1), E(2) on the other).

(iii) It follows also that the remaining maps $F(n) \rightarrow E(n)$, with n > 2, are determined (by induction on n) once the map $F(2) \rightarrow E(2)$ has been fixed.

To prove (ii), note that if ${}^{1}C$, ${}^{2}C$ are chain complexes in \mathscr{C}_{A} , then the projection

$$\hat{W}\&({}^{1}C\oplus {}^{2}C)\to \hat{W}\&{}^{1}C\oplus \hat{W}\&{}^{2}C$$

is surjective, and is a chain homotopy equivalence (though not an isomorphism, which accounts for the word 'essentially' in (ii)). To prove (iii), note that if ${}^{1}C, {}^{2}C, ..., {}^{n}C$ are chain complexes in \mathscr{C}_{A} , then an element in $\widehat{W} \& (\bigoplus_{i} {}^{i}C)$ is determined by its projections to $\widehat{W} \& (\bigoplus_{i \neq k} {}^{i}C)$ for $k \in \{1, 2, ..., n\}$, provided n > 2. The proof of 3.15 is complete.

3.21. PROOF of the fact that BG is Kan provided G is a Kan simplicial monoid and $\pi_0(G)$ is a group; see 3.2. Suppose first that we are given a Δ -map $f: \partial \Delta_n \to BG$. Then f extends over Δ_n if and only if a certain obstruction in $\pi_{n-2}(G)$ vanishes. For the desired extension corresponds to an element $(g_0, g_1, \dots, g_{n-1})$ in BG_n ; here g_0, g_1, \dots, g_{n-2} are prescribed, and the d_ig_{n-1} are also prescribed for $0 \leq i \leq n-1$, because that much information is contained in f. So we are looking for a simplex in G_{n-1} with prescribed boundary (namely g_{n-1}), which amounts to showing that an obstruction in $\pi_{n-2}(G)$ vanishes.

It follows easily that any Δ -map Horn_i $(\Delta_n) \rightarrow BG$ can be extended over Δ_n (extend over the missing *i*th face first, and then over the whole simplex).

3.A. Appendix: Chain bundles and sliding forms

Let $(\pi, w; X, \gamma; \alpha, j)$ be the usual string of data, and let $c(\gamma)$ be the chain bundle on $C(\tilde{X})$ mentioned in 3.4. The geometric description of $c(\gamma)$ given below is inspired by [14] rather than [15]. We assume that γ is a vector bundle.

In this appendix, define a γ -structure on a smooth manifold N^n (with tangent bundle τ_N) to consist of a classifying map $e: N \to X$ and a stable trivialization of $\tau_N \oplus e^{-(\gamma)}$.

For $x \in \mathbb{R}$, write $[x] := \max\{z \in \mathbb{Z} \mid z \leq x\}$.

3.A.1. DEFINITION. By a filtered γ -thickening of X is meant a sequence $\{P^n \mid n = 0, 1, ...\}$ of compact smooth manifolds with boundary (the superscripts indicate the dimension, but simultaneously serve as labels), with γ -structure, such that:

(i) each P^n comes equipped with a map

$$e_n: P^n \to ([\frac{1}{2}n]$$
-skeleton of X)

which is a homotopy equivalence, and the composite

$$P^n \xrightarrow{e_n} ([\frac{1}{2}n] \text{-skeleton of } X) \subset X$$

equals the classifying map for the γ -structure on P^n ;

(ii) P^n is contained in ∂P^{n+1} , as a smooth codimension-0 submanifold-with- γ -structure. (In particular, the diagrams

are strictly commutative.)

3.A.2. PROPOSITION. (i) A filtered γ -thickening of X exists and is unique up to an infinity of higher concordances.

(ii) Any filtered γ -thickening of X determines a 0-dimensional cycle in the chain complex $V(C(\tilde{X}))$ of 1.4 (with $C(\tilde{X})$ as in 3.4). This cycle may be regarded as a chain bundle $c(\gamma)_{new}$ via 1.6; it is well determined up to an infinity of higher concordances, by (i).

Proof. (i) Existence is clear. The uniqueness half follows from 3.A.3 below (which is equally clear).

(ii) Let $\mathbb{Z}[\alpha]$ be the coefficient sheaf over X whose stalk over $p \in X$ is the free abelian group generated by the points in the fibre (over p) of the principal π -bundle α ; the stalk is then a free $\mathbb{Z}[\alpha]$ -module on one generator. Denote the induced sheaves over P^n , for n = 0, 1, ..., by $\mathbb{Z}[\alpha]$ also.

The maps e_{2n} , e_{2n-1} of 3.A.1 give an identification

$$H_n(P^{2n}, P^{2n-1}; \mathbb{Z}[\alpha]) \stackrel{\cong}{\Rightarrow} C(\tilde{X})_n,$$

with $P^{2n-1} \subset \partial P^{2n} \subset P^{2n}$. On the other hand, $H_n(P^{2n}, P^{2n-1}; \mathbb{Z}[\alpha])$ carries a sesquilinear form λ_n : its left adjoint is the $\mathbb{Z}[\pi]$ -module homomorphism obtained by composing (explanation follows)

$$H_n(P^{2n}, P^{2n-1}) \cong H_n(P^{2n}, \underline{P}^{2n-2}) \longrightarrow H_n(P^{2n}, \underline{P}^{2n-1})$$

$$\exists \parallel$$

(dual module of $H_n(P^{2n}, P^{2n-1})) \cong H^n(P^{2n}, P^{2n-1})$

(Explanation: the coefficients are $\mathbb{Z}[\alpha]$ throughout; \underline{P}^n is the closed complement of P^n in ∂P^{n+1} , for all *n*; the isomorphism in the top row is induced by the inclusion $\underline{P}^{2n-2} \longrightarrow P^{2n-1}$, which is a homology equivalence by 3.A.1 (i); the other homomorphism in the top row is induced by the inclusion $\underline{P}^{2n-2} \longrightarrow \underline{P}^{2n-1}$, and the vertical isomorphism is Poincaré duality.)

This sliding procedure is due to Quinn (see [14]). Combining these two observations, we obtain a sequence of sesquilinear forms

$$\{\lambda_n: C(\tilde{X})_n \times C(\tilde{X})_n \to \mathbb{Z}[\pi] \mid n = 0, 1, \ldots\};\$$

inspection shows that the sequence is a cycle in $V(C(\tilde{X}))_0$ (see also [20]).

3.A.3. LEMMA. Let X and γ be the same as ever, let X' be a CW-subcomplex of X, and γ' the restriction of γ to X'. Then any filtered γ' -thickening of X' (say $\{P'' | n = 0, 1, ...\}$) can be extended to a filtered γ -thickening of X (say $\{P'' | n = 0, 1, ...\}$), in the sense that P''' is contained in P'' as a codimension-0 submanifold-with- γ -structure (for all n).

Observe that under these conditions the 0-cycles

 $c(\gamma')_{new} \in V(C(\tilde{X}')), \ c(\gamma)_{new} \in V(C(\tilde{X}))$

(constructed as in 3.A.2 (ii) from $\{P'^n\}$ and $\{P^n\}$ respectively) are such that

$$c(\gamma')_{\rm new} = i^{-}(c(\gamma)_{\rm new}),$$

if $i: X' \to X$ is the inclusion.

Applying 3.A.3 to the inclusion $X \times \{0, 1\} \rightarrow X \times [0, 1]$ proves that filtered thickenings are 'unique up to concordance'; similarly for higher concordances, which proves 3.A.2(i).

3.A.4. THEOREM. We have $c(\gamma)_{new} = c(\gamma)$, up to an infinity of higher concordances.

The proof will be given in II, §4.A. (Strictly speaking, it is first necessary to extend 3.4 from simplicial sets X to CW-spaces X, but that causes no serious problems.)

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