SURGERY AND THE GENERALIZED KERVAIRE INVARIANT, II

MICHAEL WEISS

[Received 4 March 1983—Revised 24 January 1984]

4. *Algebraic Thom complexes and algebraic thickenings*

This and the following section (on algebraic surgery) are so close in spirit to [**11**, Part I, § 4] as to be almost superfluous. Here are two remarks for justification:

we need at least a glimpse of the theory of algebraic surgery on algebraic Poincaré complexes 'with $\mathscr{B}$-structure' (see [**14**]—hereafter denoted 'I'—Definition 2.6);

I attempt to save formulae by giving a categorical description of the algebraic thickening construction.

Recall from I, 2.21 (iii), (iv) that there is a long exact sequence

$$\ldots \longrightarrow \hat{L}^{n+1}(B,\mathscr{B}) \longrightarrow L_n(A) \xrightarrow{\text{release}} L^n(B,\mathscr{B}) \longrightarrow \hat{L}^n(B,\mathscr{B}) \longrightarrow \ldots$$

for any chain complex B in $\mathscr{C}_A$ and chain bundle $\mathscr{B}$ on B. Our main result in this section is an expression of the relative group $\hat{L}^n(B,\mathscr{B})$ as a bordism group of *single but degenerate objects*, as opposed to the standard description in terms of non-degenerate pairs.

Most of this section is written in terms of *unrestricted* (UR for short) algebraic Poincaré complexes, higher bordisms, etc.; see I, 2.21 (ii).

4.1. DEFINITION. An *n-dimensional UR symmetric chain complex* is a pair (C,φ) in which C is a chain complex in $\mathscr{C}_A$ and φ is an n-dimensional cycle in $W\&C$.

4.2. DEFINITION. Let $(f\colon C \to D,(\psi,\varphi))$ be an n-dimensional UR algebraic Poincaré pair (over A; see I, 2.2). Write $\psi^?$ for the image of ψ under the map

$$W\&D \to W\&(D/\mathrm{im}(f))$$

induced by the projection $D \to D/\mathrm{im}(f)$. Then $(D/\mathrm{im}(f),\ \psi^?)$ is an n-dimensional UR symmetric chain complex, called the *algebraic Thom complex* of the pair $(f\colon C \to D,(\psi,\varphi))$.

The passage from an algebraic Poincaré pair to its algebraic Thom complex has a geometric analogue, namely the passage from a geometric Poincaré pair $(N,\partial N)$ to its *Thom complex* $N/\partial N$. (If N is equipped with a principal π-bundle, one would be interested in the Thom π-complex $\tilde{N}/\partial\tilde{N}$ instead, where $\tilde{N}$ is the total space. See [**11**] for a detailed description of this analogy.)

4.3. THE THEME (of this section). *The passage from an UR algebraic Poincaré pair to its algebraic Thom complex is reversible.*

A.M.S. (1980) subject classification: 18F25.
Proc. London Math. Soc. (3), 51 (1985), 193–230.

We shall first state this with more precision (in 4.6, after some preparatory definitions), then prove it, and then list some variants. The appendix 4.A contains a proof of I, 3.A.4, as a first application of the theory.

4.4. DEFINITION. An n-dimensional UR symmetric pair $(f\colon C \to D, (\psi, \varphi))$ consists of a cofibration $f\colon C \to D$ in $\mathscr{C}_A$, an n-chain $\psi \in W\&D$ and an $(n-1)$-cycle $\varphi \in W\&C$ so that

$$f^{\to}(\varphi) = d(\psi) \quad \text{in } (W\&D)_{n-1}$$

(or, if you prefer, $f^{\to}(\varphi) = -d(\psi)$; compare I, 2.4).

4.5. DEFINITION. Let $(E, \psi^?)$ be an n-dimensional UR symmetric chain complex. An UR symmetric pair over $(E, \psi^?)$ consists of

(i) an n-dimensional UR symmetric pair $(f\colon C \to D, (\psi, \varphi))$,

(ii) a chain map $p\colon D \to E$ which is such that the sequence of chain maps

$$0 \longrightarrow C \xrightarrow{f} D \xrightarrow{p} E \longrightarrow 0$$

is short exact, and such that $p^{\to}(\psi) = \psi^?$ in $W\&E$.

For a fixed n-dimensional UR symmetric chain complex $(E, \psi^?)$, the UR symmetric pairs over $(E, \psi^?)$ form a category $\mathscr{P}\downarrow(E, \psi^?)$ in the following way. Let

$$P_1 = (f\colon C \to D, (\psi, \varphi)), \quad p\colon D \to E,$$

and

$$P_2 = (f'\colon C' \to D', (\psi', \varphi')), \quad p'\colon D' \to E$$

be two UR symmetric pairs over $(E, \psi^?)$. The set of morphisms in $\mathscr{P}\downarrow(E, \psi^?)$ from P_1 to P_2 is to be a certain subset of the set

$$\mathscr{F}(p, p') := \{\text{fibre homotopy classes of chain maps } g\colon D \to D' \text{ so that } p' \cdot g = p\}.$$

(A fibre homotopy connecting two maps $g_1, g_2\colon D \to D'$ so that $p' \cdot g_1 = p = p' \cdot g_2$ is a homotopy which factors through $\ker(p') \subset D'$.) Let

$$\begin{tikzcd} D \arrow[rr, "g"] \arrow[dr, "p"'] & & D' \arrow[dl, "p'"] \\ & E & \end{tikzcd}$$

represent an element $[g]$ in $\mathscr{F}(p, p')$; then

$$(\psi' - g^{\to}(\psi), \varphi' - g^{\to}(\varphi))$$

represents a homology class defect($[g]$) in $H_n(\ker(p'^{\to})/\operatorname{im}(f'^{\to}))$, with

$$f'^{\to}\colon W\&C' \to W\&D', \quad p'^{\to}\colon W\&D' \to W\&E.$$

We regard $[g] \in \mathscr{F}(p, p')$ as a morphism from P_1 to P_2 precisely if defect($[g]$) $= 0$.

The promised reformulation of 4.3 now reads as follows:

4.6. THEOREM. (i) *The category $\mathscr{P}\downarrow(E, \psi^?)$ has an initial object (that is, an object admitting precisely one morphism to any other object).*

(ii) *An object in this category, say* $(f\colon C \to D, (\psi, \varphi))$, $p\colon D \to E$, *is initial if and only if* $(f\colon C \to D, (\psi, \varphi))$ *is an UR algebraic Poincaré pair.*

Proof of (i). It helps to consider a small classification problem first. Fix a short exact sequence of chain maps

$$0 \longrightarrow C \xrightarrow{f} D \xrightarrow{p} E \longrightarrow 0$$

in $\mathscr{C}_A$ (with E as in 4.6). We wish to classify the various ways in which this can be enhanced to an UR symmetric pair over $(E, \psi^?)$; that is, we wish to classify, up to a suitable notion of equivalence, pairs (ψ, φ) such that

$$(f\colon C \to D, (\psi, \varphi)), \quad p\colon D \to E$$

is an UR symmetric pair over $(E, \psi^?)$.

(Regard two such pairs (ψ, φ) and (ψ', φ') as equivalent if the identity map $D \to D$ is a morphism in $\mathscr{P} \downarrow (E, \psi^?)$ from $(f\colon C \to D, (\psi, \varphi))$, $p\colon D \to E$ to $(f\colon C \to D, (\psi', \varphi'))$, $p\colon D \to E$.)

4.7. Lemma. *The set of equivalence classes of such pairs* (ψ, φ) *is non-empty if and only if a certain obstruction in* $H_{n-1}(E^t \otimes_A C)$ *vanishes; and in that case, the group* $H_n(E^t \otimes_A C)$ *acts on this set in a sharply transitive manner.*

Proof. We use the diagram

$$\begin{tikzcd} W\&C \arrow[d, "f^{\to}"] & & \\ \ker(p^{\to}) \arrow[r] \arrow[d] & W\&D \arrow[r, "p^{\to}"] & W\&E \\ \ker(p^{\to})/\mathrm{im}(f^{\to}) & & \end{tikzcd}$$

in which both row and column are short exact. Let $[\psi^?] \in Q^n(E) = H_n(W\&E)$ be the class of $\psi^?$. It is clear that the short exact sequence $0 \to C \to D \to E \to 0$ can be enhanced to an UR symmetric pair over $(E, \psi^?)$ if and only if the class

$$\partial[\psi^?] \in H_{n-1}(\ker(p^{\to}))$$

comes from a class in $Q^{n-1}(C) = H_{n-1}(W\&C)$, which is the case if and only if the image of $\partial[\psi^?]$ in $H_{n-1}(\ker(p^{\to})/\mathrm{im}(f^{\to}))$ is zero. A similar argument shows that the group $H_n(\ker(p^{\to})/\mathrm{im}(f^{\to}))$ acts in a sharply transitive way on the set of equivalence classes of 'enhancements'.

It remains to be seen that the chain complexes $\ker(p^{\to})/\mathrm{im}(f^{\to})$ and $E^t \otimes_A C$ are homotopy equivalent. A related idea was used in the proof of I, 1.1 (iii):

$$\ker(p^{\to})/\mathrm{im}(f^{\to}) \cong \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, G);$$

here $G := \ker(p \otimes p)/\mathrm{im}(f \otimes f)$, with

$$f \otimes f\colon C^t \otimes_A C \to D^t \otimes_A D, \quad p \otimes p\colon D^t \otimes_A D \to E^t \otimes_A E.$$

But now G is canonically isomorphic, as a $\mathbb{Z}[Z_2]$-chain complex, to the 'coinduced' chain complex $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[Z_2], E^t \otimes_A C)$ (obtained from $E^t \otimes_A C$ by applying $\mathrm{Hom}(\mathbb{Z}[Z_2], -)$ in each dimension). Therefore

$$\ker(p^{\rightharpoonup})/\mathrm{im}(f^{\rightharpoonup}) \cong \mathrm{Hom}_{\mathbb{Z}}(W, E^t \otimes_A C) \simeq E^t \otimes_A C,$$

which proves the lemma.

We shall use Lemma 4.7 to give a simpler description of the category $\mathscr{P}\downarrow(E, \psi^?)$. Let $\mathscr{A}(E, \psi^?)$ be the following category: an object of $\mathscr{A}(E, \psi^?)$ is a triple $(F, j, [h])$ in which F denotes a chain complex in $\mathscr{C}_A$, $j\colon E \to F$ is a cofibration, and $[h]$ is an equivalence class of chain homotopies from

$$j \cdot \psi^?_0\colon \Sigma^n(E^{-*}) \to F$$

to 0. (Call h, h' equivalent if the difference chain map

$$h - h'\colon \Sigma^{n+1}(E^{-*}) \to F$$

is nullhomotopic.) Given two such objects $(F, j, [h])$, $(F', j', [h'])$, a morphism from the first to the second is a cofibre homotopy class of chain maps $g\colon F \to F'$ making the diagram

$$\begin{tikzcd} & F \arrow[dd, "g"] \\ E \arrow[ur, "j"] \arrow[dr, "j'"'] & \\ & F' \end{tikzcd}$$

commutative, and so that $g^{\rightharpoonup}([h]) = [h']$.

4.8. LEMMA. *The categories $\mathscr{A}(E, \psi^?)$ and $\mathscr{P}\downarrow(E, \psi^?)$ are equivalent.*

Proof. This is little more than a reformulation of 4.7. Given an object $(F, j, [h])$ in $\mathscr{A}(E, \psi^?)$, there is the short exact sequence in $\mathscr{C}_A$,

$$0 \to \Sigma^{-1}F \to \Sigma^{-1}(\mathrm{Cone}(j)) \to E \to 0.$$

By Lemma 4.7 the obstruction to 'enhancing' this short exact sequence to an UR symmetric pair over $(E, \psi^?)$ is a certain class in

$$H_{n-1}(E^t \otimes_A \Sigma^{-1}F) = H_{n-1}(\mathrm{Hom}_A(E^{-*}, \Sigma^{-1}F));$$

and inspection shows that this class is represented by the chain map

$$\Sigma^{-1}(j \cdot \psi^?_0)\colon \Sigma^{n-1}(E^{-*}) \to \Sigma^{-1}F.$$

But this is nullhomotopic; in fact, $[h]$ gives us a preferred class of nullhomotopies. It is easy to deduce that every object in $\mathscr{A}(E, \psi^?)$ gives rise to an object in $\mathscr{P}\downarrow(E, \psi^?)$, well defined up to isomorphism (in $\mathscr{P}\downarrow(E, \psi^?)$); the construction can be extended to morphisms, and yields the required equivalence of categories.

We return to the proof of 4.6(i). We are now reduced to showing that the category $\mathscr{A}(E, \psi^?)$ of 4.8 has an initial object $(F, j, [h])$. But this is clear: put

$$F := (\text{Cone of } \psi^?_0\colon \Sigma^n(E^{-*}) \to E), \quad \text{etc.}$$

Proof of 4.6(ii). One direction is straightforward, since we have an explicit description of an initial object

$$(f^{\mathrm{in}}\colon C^{\mathrm{in}} \to D^{\mathrm{in}}, (\psi^{\mathrm{in}}, \varphi^{\mathrm{in}})), \quad p^{\mathrm{in}}\colon D^{\mathrm{in}} \to E,$$

from which it can be seen that $(f^{\mathrm{in}}\colon C^{\mathrm{in}} \to D^{\mathrm{in}}, (\psi^{\mathrm{in}}, \varphi^{\mathrm{in}}))$ is an UR algebraic Poincaré pair. (See also 4.9 below.) Conversely, let

$$Y := (f\colon C \to D, (\psi, \varphi)), \quad p\colon D \to E$$

be an object in $\mathscr{P}\downarrow(E, \psi^?)$ such that $(f\colon C \to D, (\psi, \varphi))$ is an UR algebraic Poincaré pair. Then the unique morphism from the initial object to Y induces a degree-1 map of UR algebraic Poincaré pairs

$$\begin{array}{c} (f^{\mathrm{in}}\colon C^{\mathrm{in}} \to D^{\mathrm{in}}, (\psi^{\mathrm{in}}, \varphi^{\mathrm{in}})) \\ \Big\downarrow g \\ (f\colon C \to D, (\psi, \varphi)) \end{array}$$

('degree-1' means that $(\psi - g^{\to}(\psi^{\mathrm{in}}), \varphi - g^{\to}(\varphi^{\mathrm{in}}))$ represents the zero class in $H_n(f^{\to})$, with $f^{\to}\colon W\&C \to W\&D$). We know further that the induced chain map

$$E \cong D^{\mathrm{im}}/\mathrm{im}(f^{\mathrm{in}}) \to D/\mathrm{im}(f) \cong E$$

is a chain isomorphism. It follows easily that the unique morphism in question is an isomorphism in $\mathscr{P}\downarrow(E, \psi^?)$.

4.9. Definition and Description. The UR algebraic Poincaré pair in 4.6 is called the *algebraic Poincaré thickening* of $(E, \psi^?)$.

An explicit description is as follows. Let C and D be the mapping cones of

$$\Sigma^{-1}\psi_0^?\colon \Sigma^{n-1}(E^{-*}) \to \Sigma^{-1}E$$

and

$$\Sigma^{-1}\psi_0^? \oplus \mathrm{id}\colon \Sigma^{n-1}(E^{-*}) \oplus \Sigma^{-1}E \to \Sigma^{-1}E$$

respectively. Thus $C \subset D$, and $D/C \cong E$. Define an n-chain $\psi \in W\&D$ and an $(n-1)$-cycle $\varphi \in W\&C$ by letting (explanation follows)

(i)

$$\psi_0 = \begin{pmatrix} 0 & 0 & -\mathrm{id} \\ 0 & 0 & 0 \\ (-)^{pq-1}\cdot\mathrm{id} & (-)^q\cdot T\psi_1^? & \psi_0^? \end{pmatrix},$$

(ii) for $s > 0$,

$$\psi_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & (-)^{q-s}\cdot T\psi_{s+1}^? & \psi_s^? \end{pmatrix},$$

(iii)

$$\varphi_0 = \begin{pmatrix} 0 & (-)^{p-1}\cdot\mathrm{id} \\ (-)^{pq-p-1}\cdot\mathrm{id} & (-)^{q-1}\cdot T\psi_1^? \end{pmatrix},$$

(iv) for $s > 0$,

$$\varphi_s = \begin{pmatrix} 0 & 0 \\ 0 & (-)^{q-1-s} \cdot T\psi^{?}_{s+1} \end{pmatrix}.$$

(Explanation: $\psi^{?}_s$ and $T\psi^{?}_s$ are considered as chains in $\mathrm{Hom}_A(E^{-*}, E)$. Matrices (i) and (ii) describe

$$\begin{array}{ccc} \psi_s\colon D^p & \longrightarrow & D_q \\ \cong\| & & \cong\| \\ (E^*_{n-p} \oplus E_{p+1} \oplus E_p)^* & & E^*_{n-q} \oplus E_{q+1} \oplus E_q \end{array};$$

it is understood that, for fixed s, p and q range over all pairs so that $p+q = n+s$. Similarly, matrices (iii) and (iv) describe

$$\begin{array}{ccc} \varphi_s\colon C^p & \longrightarrow & C_q \\ \cong\| & & \cong\| \\ (E^*_{n-p} \oplus E_{p+1})^* & & E^*_{n-q} \oplus E_{q+1} \end{array}$$

with $p+q = n-1+s$.)

Now $(C \to D, (\psi, \varphi))$ is an UR algebraic Poincaré pair, and the projection $p\colon D \to E$ (with kernel C) is such that

$$p^{\to}(\psi) = \psi^{?} \in (W \& E)_n.$$

Hence $(C \to D, (\psi, \varphi))$ is 'the' algebraic Poincaré thickening of $(E, \psi^{?})$.

4.10. First variation on 4.3. *The relative structureless case.* Let (Fun, Φ) be a higher bordism of UR algebraic Poincaré complexes, of dimension n and order 2 (that is, modelled on the 2-simplex; see I, 2.3), with the property that

$$\mathrm{Fun}(\{0,1\}) = 0.$$

(Such a thing is commonly called a triad; it is just an UR algebraic Poincaré pair whose boundary is split into two halves.) 'Collapsing the edge $\{0,2\}$', that is, collapsing one half of the boundary, gives an UR symmetric pair (cf. 4.4)

$$(\mathrm{Fun}(\{1,2\})/\mathrm{Fun}(\{2\}) \to \mathrm{Fun}(\{0,1,2\})/\mathrm{Fun}(\{0,2\}), (\Phi^{?}(\{0,1,2\}), \Phi^{?}(\{1,2\})))$$

(in shorthand notation; certain cofibrations have been written as inclusions, and $\Phi^{?}(\{0,1,2\})$, $\Phi^{?}(\{1,2\})$ are the images of $\Phi(\{0,1,2\})$ and $\Phi(\{1,2\})$ respectively).

This collapsing process, or the passage from the triad (Fun, Φ) *to an UR symmetric pair* (*of the same dimension n*), *is reversible. That is, any UR symmetric pair can be obtained from an UR algebraic Poincaré triad by collapsing, as above; and the triad is essentially determined by the UR symmetric pair.*

The UR algebraic Poincaré triad is called the *algebraic Poincaré thickening* of the UR symmetric pair.

4.11. Remark. Given an UR symmetric pair $(C \to D, (\psi^{?}, \varphi^{?}))$, any algebraic Poincaré thickening (in the sense of 4.9) of the boundary UR symmetric complex $(C, \varphi^{?})$ can be extended, in an essentially unique way, to an algebraic Poincaré thickening (in the sense of 4.10) of $(C \to D, (\psi^{?}, \varphi^{?}))$.

The proofs of 4.10 and 4.11 are similar to that of 4.3.

For the rest of the section, fix a chain complex B in $\mathscr{C}_A$ and a chain bundle $\mathscr{E}$ on B.

We will work towards an analogue of 4.3 for an UR algebraic Poincaré pair $(f\colon C \to D, (\psi, \varphi))$ with a $\mathscr{E}$-structure (g, z) (see I, 2.7; pairs are regarded as bordisms of order 1, as in I, 2.4)—under the very restrictive assumption that *the classifying map g vanishes on the boundary*. (This means that $g \cdot f\colon C \to B$ is zero.)

There are two reasons for making such an assumption. One is technical: quite simply, we wish to collapse the boundary of our UR algebraic Poincaré pair, and the classifying map g would be in the way if it did not vanish on the boundary. The second and more important reason is that an n-dimensional UR algebraic Poincaré pair $(f\colon C \to D, (\psi, \varphi))$ as above (with a $\mathscr{E}$-structure (g, z) such that $g \cdot f = 0$) is a *typical representative of an element in $\hat{L}^n(B, \mathscr{E})$*. See I, 2.21 (iii), (iv), and I, 2.17.

To begin with, here is a stimulating definition:

4.12. Definition. A *normal structure* on an n-dimensional UR symmetric complex (C, φ) is a pair (c, z) in which

c denotes a chain bundle on C,

z is a 'clutching homology' on $(\hat{W} \& C)_{n+1}$ from $\mathfrak{S}^n \cdot \varphi_0^{\to}(c) \in (\hat{W} \& C)_n$ to $J(\varphi) \in (\hat{W} \& C)_n$.

(Explanation: φ_0 is regarded as a chain map from C^{-*} to $\Sigma^{-n}C$, inducing $\varphi_0^{\to}\colon \hat{W} \& C^{-*} \to \hat{W} \& (\Sigma^{-n}C)$; $\mathfrak{S}^n$ is the n-fold iteration of the explicit suspension isomorphism of I, 1.2(b).)

4.13. Remarks. (i) It will be shown in § 7 that 'UR symmetric chain complexes with normal structure' are the algebraic counterparts of '(geometric) normal spaces' (see [**10**] or § 7 of this paper)—just as (UR) algebraic Poincaré complexes are the algebraic counterparts of Poincaré spaces.

(ii) An n-dimensional UR algebraic Poincaré complex (C, φ) can always be regarded as an n-dimensional UR symmetric complex (C, φ) with normal structure (c, z). Indeed, c and z exist and are essentially unique because $\varphi_0\colon C^{-*} \to \Sigma^{-n}C$ is a chain homotopy equivalence. Note that c is the normal chain bundle of the UR algebraic Poincaré complex. (See the sequel to I, 2.6.)

4.14. Definition. A *normal $\mathscr{E}$-structure* on an n-dimensional UR symmetric complex (C, φ) is a pair (g, z) in which

g is a chain map from C to B,

$z \in (\hat{W} \& C)_{n+1}$ is a clutching homology' from $\mathfrak{S}^n \cdot \varphi_0^{\to}(g^{\leftarrow}(\mathscr{E})) \in (\hat{W} \& C)_n$ to $J(\varphi) \in (\hat{W} \& C)_n$.

4.15. Remarks. (i) A normal $\mathscr{E}$-structure (g, z) on (C, φ) induces a normal structure $(g^{\leftarrow}(\mathscr{E}), z)$ on (C, φ).

(ii) If (C, φ) happens to be an UR algebraic Poincaré complex, then a normal $\mathscr{E}$-structure on (C, φ) (in the sense of 4.14) is the same as a $\mathscr{E}$-structure on (C, φ) (in the sense of I, 2.6).

(iii) Define a *higher bordism* (Fun, Φ) *of UR symmetric complexes* (of dimension n and order q) like a higher bordism of UR algebraic Poincaré complexes, dropping only the non-degeneracy condition I, 2.3(iv). In view of the similarity between 4.14 and I, 2.6, it is clear how to define a normal $\mathscr{E}$-structure (g, z) on such a higher bordism of UR symmetric complexes (namely, by imitating I, 2.7). We will only need UR symmetric pairs with normal $\mathscr{E}$-structure.

(iv) It is also possible to define a normal structure (cf. 4.12) on an UR symmetric pair or on a higher bordism of UR symmetric complexes (but this notion will not be used much).

4.16. SECOND VARIATION ON 4.3. *The absolute case, with $\mathscr{B}$-structure.* Let $(f\colon C \to D, (\psi, \varphi))$ be an UR algebraic Poincaré pair with $\mathscr{B}$-structure (g, z). Suppose that $g \cdot f = 0$.

Then the algebraic Thom complex of the pair (the UR symmetric complex $(D/\mathrm{im}(f), \psi^?)$) carries a canonical normal $\mathscr{B}$-structure $(g^?, z^?)$. *The passage from*

the UR algebraic Poincaré pair $(f\colon C \to D, (\psi, \varphi))$ with $\mathscr{B}$-structure (g, z) such that $g \cdot f = 0$

to

the UR symmetric complex $(D/\mathrm{im}(f), \psi^?)$ with normal $\mathscr{B}$-structure $(g^?, z^?)$

is reversible.

4.17. THIRD VARIATION ON 4.3. *The relative case, with $\mathscr{B}$-structure.* Let (Fun, Φ) be an UR algebraic Poincaré triad (i.e. a higher bordism of order 2 such that $\mathrm{Fun}(\{0,1\}) = 0$). Suppose that a $\mathscr{B}$-structure (g, z) on (Fun, Φ) is given so that the composite

$$\mathrm{Fun}(\{0,2\}) \longrightarrow \mathrm{Fun}(\{0,1,2\}) \xrightarrow{g} B$$

is zero. Then the UR symmetric pair

$$(\mathrm{Fun}(\{1,2\})/\mathrm{Fun}(\{2\}) \to \mathrm{Fun}(\{0,1,2\})/\mathrm{Fun}(\{0,2\}), (\Phi^?(\{0,1,2\}), \Phi^?(\{1,2\})))$$

(compare 4.10) carries a canonical normal $\mathscr{B}$-structure. *The passage from*

the UR algebraic Poincaré triad (Fun, Φ) *with $\mathscr{B}$-structure* (g, z) *such that the classifying map g vanishes on* $\mathrm{Fun}(\{0,2\})$

to

an UR symmetric pair with normal $\mathscr{B}$-structure

is reversible.

Proof of 4.16. Recall first that, in order to be able to speak of a $\mathscr{B}$-structure (g, z), we must consider the UR algebraic Poincaré pair as a bordism of order 1 (as in I, 2.3 and 2.7); also, that $z = \{z(S) \mid S \subset \{0,1\}\}$ is a collection. Let

(i) $z^?$ be the image of $z(\{0,1\})$ under the map $p^{\to}\colon \hat{W} \,\&\, D \to \hat{W} \,\&\, (D/\mathrm{im}(f))$ where $p\colon D \to D/\mathrm{im}(f)$ denotes the projection; and define $g^?\colon D/\mathrm{im}(f) \to B$ so that

(ii) $g = g^? \cdot p$.

Then I, 2.9 implies that $(g^?, z^?)$ is a normal $\mathscr{B}$-structure on $(D/\mathrm{im}(f), \psi^?)$.

To see that 'the passage ... is reversible', note that $(f\colon C \to D, (\psi, \varphi))$ can be recovered as the algebraic Poincaré thickening of $(D/\mathrm{im}(f), \psi^?)$, according to I, 4.3. It is not hard to see that there exists an essentially unique $\mathscr{B}$-structure (g, z) on $(f\colon C \to D, (\psi, \varphi))$ satisfying equations (i) and (ii) just above—in other words, (g, z) can be recovered from $(g^?, z^?)$. (If in trouble, remember I, 1.1 (iii).)

The proof of 4.17 is similar.

4.18. COROLLARY. *The following notions are interchangeable:*
(a) *n-dimensional UR symmetric complex with normal structure (see* 4.12*);*
(b) *n-dimensional UR algebraic Poincaré pair $(C \to D, (\psi, \varphi))$ whose boundary (C, φ) is equipped with a* 0*-structure.*

(Explanation: '0' is the unique chain bundle on the trivial chain complex 0_A in $\mathscr{C}_A$, discussed in I, 2.19. Note that (C, φ), with its 0-structure, represents an element in the Wall group $L_{n-1}(A)$. The corollary is useful in understanding (geometric) normal spaces, obstructions to Poincaré transversality, etc. [10]; more in §7.)

Proof. Apply 4.16 with $\mathscr{E}$ equal to the universal chain bundle $\mathscr{E}^\infty$ of I, 2.A.4, and $B = B^\infty$.

(A normal structure on an UR symmetric complex is practically the same as a normal $\mathscr{E}^\infty$-structure; similarly, a 0-structure on the boundary (C, φ) of an UR algebraic Poincaré pair $(C \to D, (\psi, \varphi))$ is as good as a $\mathscr{E}^\infty$-structure (g, z) on $(C \to D, (\psi, \varphi))$ such that g vanishes on the boundary C.)

4.19. COROLLARY. *The relative group $\hat{L}^n(B, \mathscr{E})$ is isomorphic to the bordism group of n-dimensional UR symmetric complexes with normal $\mathscr{E}$-structure.*

4.A. *Appendix: Chain bundles and sliding forms again*

The proof of I, 3.A.4 to be given here begins with yet another variation on 4.3. This time a different 'model' is required: the 3-disk D^3_+, regarded as a CW-complex with one 3-cell (whose closure is D^3_+), two 2-cells (with closures D^2_+, D^2_-), two 1-cells (with closures D^1_+, D^1_-), and two 0-cells D^0_+, D^0_-. The two 0-cells are positively oriented; the remaining cells are oriented so that the inclusions $D^0_+ \hookrightarrow D^1_+$, $D^1_+ \hookrightarrow D^2_+$, $D^2_+ \hookrightarrow D^3_+$ and $D^0_- \hookrightarrow D^1_-$, $D^1_- \hookrightarrow D^2_-$ are orientation-preserving.

4.A.1. FOURTH VARIATION ON 4.3. Let (Fun, Φ) be an n-dimensional bordism of UR algebraic Poincaré complexes modelled on D^3_+ (just as the higher bordisms of I, 2.3 were modelled on the standard simplex Δ_q; in particular, Fun is now a functor from the category of faces, that is, closures of cells, of D^3_+, to $\mathscr{C}_A$).

Assume that $\mathrm{Fun}(D^0_-) = 0$, and $n > 1$. Then

$$(\mathrm{Fun}(D^1_+)/\mathrm{Fun}(D^0_+) \to \mathrm{Fun}(D^2_+)/\mathrm{Fun}(D^1_-), (\Phi^?(D^2_+), \Phi^?(D^1_+)))$$

is an $(n-1)$-dimensional UR symmetric pair, and

$$(\mathrm{Fun}(D^2_+)/(\mathrm{Fun}(D^1_+) \oplus \mathrm{Fun}(D^1_-)) \to \mathrm{Fun}(D^3_+)/\mathrm{Fun}(D^2_-), (\Phi^{??}(D^3), \Phi^{??}(D^2)))$$

is an n-dimensional UR symmetric pair.

These two UR symmetric pairs are related as follows: the UR symmetric chain complex obtained by collapsing the boundary of the first is identified with the boundary of the second. *The passage from*

> *the n-dimensional bordism of UR algebraic Poincaré complexes* (Fun, Φ), *modelled on D^3_+ and such that* $\mathrm{Fun}(D^0_-) = 0$

to

> *the two related UR symmetric pairs above*

is reversible.

The proof of 4.A.1 can be modelled on that of 4.3, but it is also amusing to derive it from 4.10 and 4.11.

4.A.2. OUTLINE (of the proof of I, 3.A.4). We keep the notation of I, 3.A. Fix a filtered thickening $\{P^n \mid n = 0, 1, \ldots\}$ of X. Each P^n is a manifold with boundary, so it gives rise to an algebraic Poincaré pair

$$(C^{(n)} \hookrightarrow D^{(n)}, (\psi^{(n)}, \varphi^{(n)}))$$

of dimension n, endowed with a chain map (which is a homotopy equivalence)

$$D^{(n)} \xrightarrow{\varepsilon^{(n)}} C(\tilde{X}_{[\frac{1}{2}n]})$$

(corresponding to $e_n\colon P^n \overset{\simeq}{\to} X_{[\frac{1}{2}n]} := [\frac{1}{2}n]$-skeleton of X).

More important to us than the algebraic Poincaré pair above is its algebraic Thom complex, the n-dimensional (UR) symmetric complex

(i) $$(D^{(n)}/C^{(n)}, \psi^{(n)?}).$$

There is another n-dimensional UR symmetric complex about, namely

(ii) $$(\Sigma^n((C(\tilde{X}_{[\frac{1}{2}n]}))^{-*}), \mathfrak{S}^n(c(\gamma)_{\text{new}})).$$

(Explanation: it is understood that

$$c(\gamma)_{\text{new}} \in \hat{W} \,\&\, C(\tilde{X})^{-*}$$

is the chain bundle derived as in I, 3.A, from the filtered thickening P^n above and no other. I have also written $c(\gamma)_{\text{new}}$ for the image of $c(\gamma)_{\text{new}}$ in $\hat{W} \,\&\, C(\tilde{X}_{[\frac{1}{2}n]})^{-*}$, so that the n-fold suspension

$$\mathfrak{S}^n(c(\gamma)_{\text{new}})$$

is an n-dimensional cycle in $\hat{W} \,\&\, \Sigma^n(C(\tilde{X}_{[\frac{1}{2}n]})^{-*})$; it cannot help lying in the subcomplex

$$W \,\&\, \Sigma^n(C(\tilde{X}_{[\frac{1}{2}n]})^{-*}) \subset \hat{W} \,\&\, \Sigma^n(C(\tilde{X}_{[\frac{1}{2}n]})^{-*}),$$

so that (ii) is indeed an UR symmetric complex.)

The idea of the proof is to show that the chain homotopy equivalence

$$g^{(n)}\colon \Sigma^n(C(\tilde{X}_{[\frac{1}{2}n]})^{-*}) \overset{\simeq}{\to} D^{(n)}/C^{(n)}$$

obtained by composing the map

$$\Sigma^n(C(\tilde{X}_{[\frac{1}{2}n]})^{-*}) \xrightarrow{\varepsilon^{(n)\leftarrow}} \Sigma^n((D^{(n)})^{-*})$$

with the Poincaré duality chain equivalence

$$\Sigma^n((D^{(n)})^{-*}) \overset{\simeq}{\to} D^{(n)}/C^{(n)}$$

is such that

(iii) $$g^{(n)\to}(\mathfrak{S}^n(c(\gamma)_{\text{new}})) = \psi^{(n)?}$$

in $W \,\&\, (D^{(n)}/C^{(n)})$, up to an infinity of higher homologies. In other words, the UR symmetric complexes (i) and (ii) are more or less identical.

For large n, $C(\tilde{X}_{[\frac{1}{2}n]}) = C(\tilde{X})$, and we also have by definition of $c(\gamma)$ the equation

(iv) $$g^{(n)\to}(\mathfrak{S}^n(c(\gamma))) = J(\psi^{(n)?})$$

in $\hat{W}\&(D^{(n)}/C^{(n)})$, up to an infinity of higher homologies.

Putting (iii) and (iv) together proves I, 3.A.4. (The method of proof is so 'natural' that the infinity of higher homologies which identifies $c(\gamma)$ and $c(\gamma)_{\text{new}}$ is sufficiently independent of the choice of filtered thickening; see I, 3.A.2 (i) and 3.A.3.)

The outline is over; we are left with equation (iii). This can be proved by induction on n. If n is even, the induction step from n to $n+1$ is trivial; for $n = 2q-1$, it is contained in Lemma 4.A.2 below, for which the assumptions are as follows.

Let $(E \to F, (\lambda, \chi))$ be a $(2q)$-dimensional UR symmetric pair (in $\mathscr{C}_A$, with $q > 0$) such that

(a) the $(2q-1)$-dimensional UR symmetric complex (E, χ) is the suspension of a $(2q-2)$-dimensional UR symmetric complex (G, μ) (such that $E = \Sigma G$, $\chi = \mathfrak{S}(\mu)$);

(b) $H_i(F) = 0$ for $i \neq q$.

Then $H^q(F) \cong (H_q(F))^*$ (coefficients A) is a f.g. projective A-module. Moreover, $H^q(F)$ carries two sesquilinear forms, β_1 and β_2.

Description of β_1: choose a chain map (in $\mathscr{C}_A$)

$$f\colon F \to H_*(F)$$

such that the induced map in homology is the identity (regarding $H_*(F)$ as a chain complex in $\mathscr{C}_A$). Then the $2q$-chain

$$f^{\to}(\lambda) \in W\& H_*(F)$$

is nothing but an element in $H_q(F)^t \otimes_A H_q(F)$ (which is well defined!), and so can be regarded as a sesquilinear form β_1 on $(H_q(F))^* = H^q(F)$.

Description of β_2: β_2 is a 'sliding form'. Condition (a) just above gives us a $(2q-1)$-dimensional UR symmetric pair

$$(G \to \operatorname{Cone}(G), (\operatorname{Cone}(\mu), \mu))$$

(with 'Cone(μ)' equal to the image of μ under the map induced by the projection $G \otimes_{\mathbb{Z}} I \to \operatorname{Cone}(G)$; see the text between I, 2.13 and 2.14).

The UR symmetric complex obtained by collapsing the boundary of the UR symmetric pair

$$(G \to \operatorname{Cone}(G), (\operatorname{Cone}(\mu), \mu))$$

is equal to the boundary of the UR symmetric pair

$$(E \to F, (\lambda, \chi)).$$

So 4.A.1 can be applied.

Let (Fun, Φ) be the resulting (UR) algebraic Poincaré bordism modelled on D^3_+, with $\mathrm{Fun}(D^0_-) = 0$. We can define a sliding form β_2 on

$$H_q(\mathrm{Fun}(D^3_+)/\mathrm{Fun}(D^2_+)) \cong H^q(\mathrm{Fun}(D^3_+)/\mathrm{Fun}(D^2_-)) \cong H^q(F)$$

by imitating the construction in the proof of I, 3.A.2 (ii). (P^{2q} corresponds to $\mathrm{Fun}(D^3_+)$, P^{2q-1} to $\mathrm{Fun}(D^2_+)$; P^{2q-2} to $\mathrm{Fun}(D^1_+)$; $\underline{P}^{2q-1}$ to $\mathrm{Fun}(D^2_-)$, $\underline{P}^{2q-2}$ to $\mathrm{Fun}(D^1_-)$; and $\partial(P^{2q-2}) = \partial(\underline{P}^{2q-2})$ to $\mathrm{Fun}(D^0)$.)

4.A.2. LEMMA. $\beta_1 = \beta_2$.

To prove the lemma, it suffices (by a naturality argument) to consider the special case where the map $E \hookrightarrow F$ is an isomorphism, and both E and F are concentrated in dimension q; details are left to the reader.

5. *Algebraic surgery*

5.1. DEFINITION. Let (Fun, Φ) be an n-dimensional UR algebraic Poincaré bordism (over A), of order 1 (see I, 2.3), and let

$$x \in H_{k+1}(\mathrm{Fun}(\{0\}) \hookrightarrow \mathrm{Fun}(\{0,1\})).$$

Call $((\mathrm{Fun}, \Phi), x)$ an *elementary bordism of index* $k+1$ if

$$H_i(\mathrm{Fun}(\{0\}) \hookrightarrow \mathrm{Fun}(\{0,1\})) = \begin{cases} 0 & \text{if } i \neq k+1, \\ \text{the free } A\text{-module spanned by } x, & \text{if } i = k+1. \end{cases}$$

Motivation. Suppose that we are given an n-dimensional bordism of manifolds (of order 1) which is 'elementary of index $k+1$' (that is, which possesses a Morse function having exactly one critical point, and that of order $k+1$). Suppose also that this geometric bordism is equipped with the usual data—principal π-bundle, etc., as in I, 3.8. Then the algebraic Poincaré bordism (Fun, Φ) derived from the given geometric bordism (by the method of I, 3.8) is elementary of index $k+1$ for a suitable choice of $x \in H_{k+1}(\mathrm{Fun}(\{0\}) \hookrightarrow \mathrm{Fun}(\{0,1\}))$.

5.2. DEFINITION. Let (C, φ) be an UR algebraic Poincaré complex and $y \in H_k(C)$. Say that y can be *killed by algebraic surgery* if there exists an elementary (UR algebraic Poincaré) bordism $((\mathrm{Fun}, \Phi), x)$, as in 5.1, so that

$$\mathrm{Fun}(\{0\}) = C, \quad \Phi(\{0\}) = \varphi;$$
$$\partial x = y \quad \text{in } H_k(\mathrm{Fun}(\{0\})) = H_k(C).$$

For the next definition, let B be a chain complex in $\mathscr{C}_A$, and let $\mathscr{E}$ be a chain bundle on B.

5.3. DEFINITION. Let (C, φ) be an UR algebraic Poincaré complex with $\mathscr{E}$-structure (g, z) (cf. I, 2.6), and let

$$y' \in H_{k+1}(g\colon C \to B).$$

Say that y' can be *killed by algebraic surgery* if there exists an elementary UR algebraic Poincaré bordism $((\mathrm{Fun}, \Phi), x)$ of index $k+1$, and a $\mathscr{E}$-structure $(\bar{g}, \bar{z})$ on (Fun, Φ) such that

$\mathrm{Fun}(\{0\}) = C$, $\Phi(\{0\}) = \varphi$, and the $\mathscr{E}$-structure $(\bar{g}, \bar{z})$ on (Fun, Φ) extends the $\mathscr{E}$-structure (g, z) on (C, φ);

under the homomorphism

$$H_{k+1}(\mathrm{Fun}(\{0\}) \hookrightarrow \mathrm{Fun}(\{0,1\})) \to H_{k+1}(\mathrm{Fun}(\{0\}) = C \to B)$$

induced by the chain map $\bar{g}\colon \mathrm{Fun}(\{0,1\}) \to B$, x maps to y'.

5.4. PROPOSITION. (i) *In* 5.2, *y can be killed by algebraic surgery if and only if a certain obstruction*

$$\mathrm{ob}(y) \in H^{n-2k}(Z_2; A)$$

vanishes.

(ii) *In* 5.3, *y′ can be killed by algebraic surgery if and only if a certain obstruction*

$$\mathrm{ob}(y') \in H_{2k-n}(Z_2; A)$$

vanishes.

Proof of (i). The Poincaré dual of $y \in H_k(C)$ is a cohomology class in $H^{n-k}(C)$ (coefficients A), which can be represented by a chain map (in $\mathscr{C}_A$)

$$f_y\colon C \to (A, n-k) \quad \text{(see I, 0.6).}$$

Let

$$\mathrm{ob}(y) := f_y^{\to}(\varphi) \in Q^n((A, n-k)) \cong H^{n-2k}(Z_2; A).$$

Let D be the mapping cylinder of f_y. Suppose that $\mathrm{ob}(y) = 0$. Then there exists an $(n+1)$-chain $\psi \in W \& D \simeq W \& (A, n-k)$ such that

$$(C \to D, (\psi, \varphi))$$

is an $(n+1)$-dimensional UR symmetric pair (whose boundary symmetric complex (C, φ) happens to be an UR algebraic Poincaré complex). The algebraic Poincaré thickening (in the sense of 4.10) of $(C \to D, (\psi, \varphi))$ is an UR algebraic Poincaré triad (Fun, Φ) of dimension $n+1$. Since (C, φ) was already an UR algebraic complex, (Fun, Φ) can also be regarded as a bordism of order 1 (from (C, φ) to something else). It has the properties required in 5.1, 5.2. The converse is similar.

Proof of 5.4 (ii). The obstruction $\mathrm{ob}(y')$ is somewhat harder to define in this case. The following definition helps:

5.5. DEFINITION. For a chain complex E in $\mathscr{C}_A$, let $Q_n(E)$ be the nth relative homology group of the forgetful map

$$J\colon W \& E \to \hat{W} \& E.$$

(So there is a long exact sequence

$$\ldots \to \hat{Q}^{n+1}(E) \to Q_n(E) \to Q^n(E) \to \hat{Q}^n(E) \to Q_{n-1}(E) \to \ldots, \quad n \in \mathbb{Z};$$

see [11] for more details.)

5.6. EXAMPLE. Let (C, φ) be an n-dimensional UR algebraic Poincaré complex with $\mathscr{B}$-structure (g, z) (for example, the one in 5.3). Let E be the mapping cone of the usual map (cf. I, 2.6)

$$\varphi_0 \cdot g^*\colon \Sigma^n(B^{-*}) \to C.$$

The inclusion $j\colon C \to E$ gives a class $j^{\to}(\varphi) \in Q^n(E)$. It is not hard to see that the $\mathscr{B}$-structure on (C, φ) determines a canonical lifting $\chi \in Q_n(E)$ of $j^{\to}(\varphi) \in Q^n(E)$ ('upwards' the long exact sequence in 5.5).

We return to the proof of 5.4 (ii). Now

$$y' \in H_{k+1}(g\colon C \to B) \quad \text{(from 5.4 (ii) and 5.3)}$$

corresponds to an element in

$$H^{n-k}(g^*: \Sigma^n(B^{-*}) \to \Sigma^n(C^{-*}))$$

(cohomology with coefficients A throughout). Since

$$\varphi_0: \Sigma^n(C^{-*}) \to C$$

is a chain homotopy equivalence, we can also think of this as an element in

$$H^{n-k}(\varphi_0 \cdot g^*: \Sigma^n(B^{-*}) \to C) = H^{n-k}(E)$$

(where E denotes the mapping cone of $\varphi_0 \cdot g^*$).

Represent this element in $H^{n-k}(E)$ by an A-module chain map

$$f_{y'}: E \to (A, n-k).$$

Let $\chi \in Q_n(E)$ be the 'lifting' described in 5.6, and put

$$\mathrm{ob}(y') := f_{y'}^{\to}(\chi) \in Q_n((A, n-k)) \cong H_{2k-n}(Z_2; A).$$

The rest of the proof is left to the reader.

5.7. Corollary. *Let $\mathscr{E}$ be a chain bundle on a positive chain complex B. Then the forgetful homomorphisms* (see I, 2.21)

$$\pi_n(\mathbb{L}^0(B, \mathscr{E})) \to \pi_n(\mathscr{L}^{\cdot}(B, \mathscr{E})) = L^n(B, \mathscr{E})$$

are isomorphisms for $n \geqslant 0$. For $n \leqslant -3$, $L^n(B, \mathscr{E})$ is isomorphic to $L_n(A)$, the Wall group of A.

Proof. To see that $\pi_n(\mathbb{L}^0(B, \mathscr{E})) \to \pi_n(\mathscr{L}^{\cdot}(B, \mathscr{E}))$ is surjective, take an n-dimensional UR algebraic Poincaré complex (C, φ) with $\mathscr{E}$-structure (g, z). Then 5.4 (ii) allows us to kill the homology groups of C in negative dimensions, because

for $k < 0$, every element $y \in H_k(C)$ lifts to $y' \in H_{k+1}(g: C \to B)$, since B is positive;

$\mathrm{ob}(y') \in H_{2k-n}(Z_2; A)$ is zero since $H_i(Z_2; A) = 0$ for $i < 0$ (and since we assume that $n \geqslant 0$).

Hence (C, φ), with its $\mathscr{E}$-structure, is bordant to a (restricted!) algebraic Poincaré complex with $\mathscr{E}$-structure, as required.

The proof of injectivity is similar (admittedly, it uses a somewhat relativized version of 5.4 (ii)).

For $n \leqslant -3$, performing surgery below the middle dimension using 5.4 (ii) shows that

$$L^n(B, \mathscr{E}) \cong L^n(0_A, 0) \cong L_n(A)$$

(see I, 2.19).

6. *A homological description for $\hat{L}^n(B, \mathscr{E})$*

6.1. Theorem (see I, 2.21). *For any chain complex B in $\mathscr{C}_A$ and chain bundle $\mathscr{E}$ on B, there is a long exact sequence*

$$\ldots \to \hat{Q}^{n+1}(B) \to \hat{L}^n(B, \mathscr{E}) \to Q^n(B) \to \hat{Q}^n(B) \to \hat{L}^{n-1}(B, \mathscr{E}) \to \ldots \quad (n \in \mathbb{Z}).$$

6.2. Addendum. The homomorphisms $Q^n(B) \to \hat{Q}^n(B)$ in 6.1 are not in general identical with J (of I, 0.13); instead they have the form $J - \mathrm{ind}(\mathscr{E})$, where

$$\mathrm{ind}(\mathscr{E}): Q^n(B) \to \hat{Q}^n(B)$$

sends $[\varphi]$ to $[\mathfrak{S}^n \cdot \varphi_0^{\to}(\mathscr{E})]$. (Again, φ_0 is regarded as a chain map from B^{-*} to $\Sigma^{-n}B$.)

We give an outline of the proof of 6.1. It is rather easy to define, for $n \in \mathbb{Z}$, abelian groups $Q_n(B, \mathscr{E})$, depending on B and $\mathscr{E}$, which by construction fit into a long exact sequence

$$\ldots \xrightarrow{J\text{-ind}(\mathscr{E})} \hat{Q}^{n+1}(B) \longrightarrow Q_n(B, \mathscr{E}) \longrightarrow Q^n(B) \xrightarrow{J\text{-ind}(\mathscr{E})} \hat{Q}^n(B) \longrightarrow \ldots.$$

Here $Q_n(B, \mathscr{E})$ is the group of suitable equivalence classes of pairs

$$(\varphi, z) \in (W \& B)_n \times (\hat{W} \& B)_{n+1}$$

so that (B, φ) is an UR symmetric complex with normal structure $(\mathscr{E}, z)$ (see 4.12). This amounts to saying that

$$\varphi_0^{\rightarrow}(\mathfrak{S}^n(\mathscr{E})) + d(z) = J(\varphi) \quad \text{in } \hat{W} \& B.$$

According to 4.19 we can interpret $\hat{L}^n(B, \mathscr{E})$ as the bordism group of n-dimensional UR symmetric complexes (C, φ) with normal $\mathscr{E}$-structure (g, z). Given such a (C, φ) with normal $\mathscr{E}$-structure (g, z), one finds that $(g^{\rightarrow}(\varphi), g^{\rightarrow}(z))$ represents an element in $Q_n(B, \mathscr{E})$. Conversely, given an element $[(\varphi, z)]$ in $Q_n(B, \mathscr{E})$, the n-dimensional UR symmetric complex (B, φ) with normal $\mathscr{E}$-structure (id, z) represents an element in $\hat{L}^n(B, \mathscr{E})$. So there is an isomorphism $\hat{L}^n(B, \mathscr{E}) \cong Q_n(B, \mathscr{E})$, and the proof is complete.

The details are as follows. For the definition of $Q_n(B, \mathscr{E})$, take two pairs (φ, z) and (φ', z') in $(W \& B)_n \times (\hat{W} \& B)_{n+1}$ such that (B, φ) and (B, φ') are n-dimensional UR symmetric complexes with normal structures $(\mathscr{E}, z)$ and $(\mathscr{E}, z')$ respectively. Call (φ, z) and (φ', z') *equivalent* if there exists a pair $(\psi, y) \in (W \& B)_{n+1} \times (\hat{W} \& B)_{n+2}$ so that

$$\varphi + d(\psi) = \varphi' \quad \text{in } W \& B;$$

$$d(y) + z' - z = J(\psi) - \mathfrak{S}^n \cdot \psi_0^{\rightarrow}(\mathscr{E} \times \omega) \quad \text{in } \hat{W} \& B.$$

(Explanation: ψ_0 is regarded as a chain homotopy from φ_0 to φ_0', or as a chain map from $B^{-*} \otimes_{\mathbb{Z}} I$ to $\Sigma^{-n} B$; therefore $\mathfrak{S}^n \cdot \psi_0^{\rightarrow}(\mathscr{E} \times \omega)$ is a homology from $\mathfrak{S}^n \cdot \varphi_0^{\rightarrow}(\mathscr{E})$ to $\mathfrak{S}^n \cdot \varphi_0'(\mathscr{E})$. See the proof of I, 1.1 (i) and I, 2.9.)

6.3. Proposition and Definition. *The set of equivalence classes, written $Q_n(B, \mathscr{E})$, is an abelian group.*

Proof. Let $\mathrm{pr}_1, \mathrm{pr}_2 \colon B \oplus B \to B$ be the two projections, and $g = \mathrm{pr}_1 + \mathrm{pr}_2$. Given two elements $[(\varphi, z)]$ and $[(\varphi', z')] \in Q_n(B, \mathscr{E})$, choose $z'' \in (\hat{W} \& (B \oplus B))_{n+1}$ so that

(i) (g, z'') is a normal $\mathscr{E}$-structure on the UR symmetric complex $(B \oplus B,\ \varphi \oplus \varphi')$;

(ii) $\mathrm{pr}_1^{\rightarrow} \colon \hat{W} \& (B \oplus B) \to \hat{W} \& B$ sends z'' to z,

$\mathrm{pr}_2^{\rightarrow} \colon \hat{W} \& (B \oplus B) \to \hat{W} \& B$ sends z'' to z'.

Such a z'' exists and is 'essentially unique'. Put

$$[(\varphi, z)] + [(\varphi', z')] := [(\varphi + \varphi', g^{\rightarrow}(z''))].$$

6.4. Proposition. *There is a long exact sequence* (cf. 6.2)

$$\ldots \xrightarrow{J\text{-ind}(\mathscr{E})} \hat{Q}^{n+1}(B) \longrightarrow Q_n(B, \mathscr{E}) \longrightarrow Q^n(B)$$

$$\xrightarrow{J\text{-ind}(\mathscr{E})} \hat{Q}^n(B) \longrightarrow \ldots \quad (n \in \mathbb{Z}).$$

Proof. Go from $\hat{Q}^{n+1}(B)$ to $Q_n(B, \mathscr{E})$ by $[z] \mapsto [(0, z)]$ (for an $(n+1)$-cycle $z \in \hat{W} \& B$), and from $Q_n(B, \mathscr{E})$ to $Q^n(B)$ by $[(\varphi, z)] \mapsto [\varphi]$. Exactness is almost obvious.

6.5. REMARK. If $\mathscr{E} = 0$, $Q_n(B, \mathscr{E})$ equals $Q_n(B)$ (of 5.5), and the long exact sequence in 6.4 is the usual one.

For the next proposition, interpret $\hat{L}^n(B, \mathscr{E})$ as the bordism group of n-dimensional UR symmetric complexes with normal $\mathscr{E}$-structure (g, z) (as in 4.19).

6.6. PROPOSITION. *The homomorphism*

$$\hat{L}^n(B, \mathscr{E}) \to Q_n(B, \mathscr{E})$$

which sends the bordism class of the n-dimensional UR symmetric complex (C, φ) with normal $\mathscr{E}$-structure (g, z) to $[(g^{\to}(\varphi), g^{\to}(z))] \in Q_n(B, \mathscr{E})$ is an isomorphism.

Proof. The inverse homomorphism

$$Q_n(B, \mathscr{E}) \to \hat{L}^n(B, \mathscr{E})$$

sends $[(\varphi, z)] \in Q_n(B, \mathscr{E})$ to the bordism class of the n-dimensional UR symmetric complex (B, φ) with normal $\mathscr{E}$-structure (id, z). Clearly the composite $Q_n(B, \mathscr{E}) \to \hat{L}^n(B, \mathscr{E}) \to Q_n(B, \mathscr{E})$ is the identity. Given an UR symmetric complex (C, φ) with normal $\mathscr{E}$-structure (g, z), the mapping cylinder of $g\colon C \to B$ can be equipped with suitable data so as to constitute a bordism between (C, φ) (with normal $\mathscr{E}$-structure (g, z)) and $(B, g^{\to}(\varphi))$ (with normal $\mathscr{E}$-structure $(\mathrm{id}, g^{\to}(z))$). This shows that the composite

$$\hat{L}^n(B, \mathscr{E}) \to Q_n(B, \mathscr{E}) \to \hat{L}^n(B, \mathscr{E})$$

is the identity also, which proves 6.6.

Finally, combining 6.4 and 6.6 proves 6.1.

6.7. EXAMPLE. Take $B, \mathscr{E}$ as in I, 2.20 and 2.A.3; so $\mathscr{E}$ is universal for chain bundles on positive chain complexes in $\mathscr{C}_A$. Then $L^n(B, \mathscr{E}) \cong L^n(A)$ for $n \in \mathbb{Z}$, where $L^n(A)$ is the symmetric L-group of [**11**] (as introduced by Mishchenko). (For $n \geqslant 0$, this is clear from 5.7 and the discussion in I, 2.20; for $n < 0$, take it as a definition of $L^n(A)$. It agrees with the definition in [**11**, Part I, §6].)

Write $\hat{L}^n(A) := \hat{L}^n(B, \mathscr{E})$, so that there is a long exact sequence

$$\ldots \to L_n(A) \to L^n(A) \to \hat{L}^n(A) \to L_{n-1}(A) \to \ldots \quad (n \in \mathbb{Z}).$$

From 6.1 we obtain another long exact sequence

$$\ldots \to \hat{Q}^{n+1}(B) \to \hat{L}^n(A) \to Q^n(B) \to \hat{Q}^n(B) \to \ldots \quad (n \in \mathbb{Z}),$$

showing that the groups $\hat{L}^n(A)$ are homological objects.

(This requires explanation, since 6.1 is valid for chain complexes in $\mathscr{C}_A$—and B is usually not in $\mathscr{C}_A$. Put $Q^n(B) := H_n(\mathrm{Hom}^b_{\mathbb{Z}[Z_2]}(W, B^t \otimes_A B))$, where the superscript 'b' stands for the subcomplex of bounded chains in $\mathrm{Hom}_{\mathbb{Z}[Z_2]}(W, B^t \otimes_A B)$; that is, chains which vanish on W_s for all but finitely many $s \in \mathbb{Z}$. Proceed similarly for $\hat{Q}^n(B)$. With these conventions 6.1 can be generalized to cover the case at hand, that is, the case of an arbitrary chain complex of projective left A-modules, equipped with a chain bundle as defined in I, 2.A. The proof uses a direct limit argument.)

In order to apply 6.1, we need means for computing groups of type $Q^n(B)$, $\hat{Q}^n(B)$ or $Q_n(B, \mathscr{E}) \cong \hat{L}^n(B, \mathscr{E})$. Often this is elementary, at least if B is sufficiently well understood. Some times the chain homotopy invariance of the functors $Q^n(-)$, $\hat{Q}^n(-)$ can be exploited, as in 6.8 below; if that is not sufficient, there is a spectral sequence for computing $Q^n(B)$ (or $\hat{Q}^n(B)$), based on the filtration of W (or $\hat{W}$) by skeletons.

6.8. EXAMPLE. Assume that A is such that every left A-module is projective. Then every chain complex of left A-modules is homotopy equivalent to one with zero differential (its homology), and hence the computation of groups such as $Q^n(B)$, $\hat{Q}^n(B)$, $Q_n(B, \mathscr{E})$ is usually a trivial matter. For instance, if $A = Z_2$, one has (cf. 6.7)

$$\hat{L}^n(A) \cong \begin{cases} Z_2 & \text{if } n \geqslant -1, \\ 0 & \text{if } n < -1. \end{cases}$$

7. *Spherical fibrations, normal spaces, and L-theory*

Recall from [10] or [11] that a normal space of formal dimension n consists of a finitely generated simplicial set Y, a spherical fibration $\nu_Y\colon Y \to BG(\infty)$, and a map of spectra

$$\rho_Y\colon S^n \to M(Y, \nu_Y),$$

where $M(Y, \nu_Y)$ is the Thom spectrum. Call ν_Y the normal bundle, and call

$$(\text{Thom class of } \nu_Y \cap h(\rho_Y)) \in H_n(Y; \mathbb{Z}^{tw})$$

the fundamental class; here $\mathbb{Z}^{tw}$ is the twisted integer coefficients, the twisting being given by the first Stiefel–Whitney class of ν_Y.

EXAMPLE. Every Poincaré space of formal dimension n is a normal space of formal dimension n, according to I, 3.3.

Now let $(\pi, w; X, \gamma; \alpha, j)$ be a string as in I, 3.4. Let $M(X, \gamma)$ be the Thom spectrum. The homotopy group $\pi_n(M(X, \gamma))$ can be identified with the bordism group of formally n-dimensional normal spaces (Y, ν_Y, ρ_Y) equipped with a classifying map $g\colon Y \to X$ such that $g^{\leftarrow}(\gamma) = \nu_Y$. (Proof: any quadruple (Y, ν_Y, ρ_Y, g) as above yields $g^{\rightarrow}(\rho_Y)\colon S^n \to M(X, \gamma)$; conversely, any $\rho\colon S^n \to M(X, \gamma)$ yields a quadruple $(X, \gamma, \rho, \mathrm{id}_X)$. Moreover, the quadruples (Y, ν_Y, ρ_Y, g) and $(X, \gamma, g^{\rightarrow}(\rho_Y), \mathrm{id}_X)$ are bordant: the mapping cylinder of g is a bordism between the two.)

7.1. THEOREM. *There is a canonical map of spectra*

$$M(X, \gamma) \to \hat{\underline{\mathscr{L}}}^{\cdot}(C(\tilde{X}), c(\gamma))$$

(cf. I, 2.21 (iii) *and* 3.4) *which fits into a commutative square*

$$\begin{array}{ccc} \underline{\Omega}^P(X, \gamma) & \longrightarrow & \underline{\mathscr{L}}^{\cdot}(C(\tilde{X}), c(\gamma)) \\ \downarrow & & \downarrow \\ M(X, \gamma) & \longrightarrow & \hat{\underline{\mathscr{L}}}^{\cdot}(C(\tilde{X}), c(\gamma)) \end{array}$$

and therefore results in a map of long exact sequences (see comment below)

$$\begin{array}{ccccccccc} \dots \to & \pi_{n+1}(M(X,\gamma)) & \to & L_n(\mathbb{Z}[\pi_1(X)]) & \to & \Omega_n^P(X,\gamma) & \to & \pi_n(M(X,\gamma)) & \to \dots \\ & \downarrow & & \downarrow{\scriptstyle (a)} & & \downarrow{\scriptstyle (b)} & & \downarrow & \\ \dots \to & \hat{L}^{n+1}(C(\tilde{X}), c(\gamma)) & \to & L_n(\mathbb{Z}[\pi]) & \to & L^n(C(\tilde{X}), c(\gamma)) & \to & \hat{L}^n(C(\tilde{X}), c(\gamma)) & \to \dots \end{array}$$

(for $n \geqslant 5$, with $\Omega_n^P(X,\gamma) = \pi_n(\underline{\Omega}^P(X,\gamma))$).

Proof and comment. We have just interpreted $M(X,\gamma)$ in terms of normal spaces mapping to X, γ and in §§4 and 6 we interpreted $\hat{\mathscr{L}}^{\cdot}(C(\tilde{X}), c(\gamma))$ in terms of symmetric chain complexes with $c(\gamma)$-structure. So 7.1 (apart from the long exact sequences) is a perfect analogue of I, 3.5, and the same proof applies—just substitute 'normal spaces' for 'Poincaré spaces' everywhere.

The maps from $\pi_n(M(X,\gamma))$ to $\hat{L}^n(C(\tilde{X}), c(\gamma))$ are flexible versions of the hyperquadratic signature maps

$$\hat{\sigma}^*\colon \pi_n(M(X,\gamma)) \to \hat{L}^n(\mathbb{Z}[\pi])$$

constructed in [**12**, §7.4].

The upper long exact sequence in 7.1 was announced by Quinn in [**10**].

All L-theoretic constructions in 7.1 are meant to be based on free modules rather than projective ones, so that, for example, $L_n(\mathbb{Z}[\pi])$ means $L_n^h(\mathbb{Z}[\pi])$. The vertical arrows (a) in 7.1 are induced by the sufficiently well defined homomorphism $h\colon \pi_1(X) \to \pi$ corresponding to the principal π-bundle α, and (b) is the flexible signature.

Warning. If γ is a vector bundle, or if for any other reason transversality arguments can be applied to γ, then transversality gives a splitting of the upper long exact sequence in 7.1. But there is absolutely no way in general of getting a compatible splitting for the lower long exact sequence. The case in which $\pi = \{1\}$ and X is a point is instructive.

8. *An injectivity criterion for the release map*

In order to complement §6, we will examine here the release homomorphisms $L_n(A) \to L^n(B, \mathscr{E})$ for a chain complex B in $\mathscr{C}_A$ with chain bundle $\mathscr{E}$; cf. I, 2.21.

8.1. THEOREM. *Take $B, \mathscr{E}$ as above, and let c be another chain bundle on B; suppose that*

$$[\mathscr{E}] - [c] \in \hat{Q}^0(B^{-*})$$

belongs to the image of the composite homomorphism

$$Q^n(\Sigma^n(B^{-*})) \xrightarrow{J} \hat{Q}^n(\Sigma^n(B^{-*})) \xrightarrow{\cong} \hat{Q}^0(B^{-*}).$$

Then there exists an isomorphism

$$\hat{L}^{n+1}(B, \mathscr{E}) \cong \hat{L}^{n+1}(B, c)$$

making the triangle

$$\begin{tikzcd}[column sep=tiny] \hat{L}^{n+1}(B,\mathscr{b}) \arrow[dr, "\partial"'] & \cong & \hat{L}^{n+1}(B,c) \arrow[dl, "\partial"] \\ & L_n(A) & \end{tikzcd}$$

commute. So the release homomorphisms $L_n(A) \to L^n(B,\mathscr{b})$ *and* $L_n(A) \to L^n(B,c)$ *have the same kernel.*

Next, keep the assumptions of 8.1, but assume also that $A = \mathbb{Z}[\pi]$ is a group ring (with the w-twisted involution for some $w\colon \pi \to Z_2$). Let $\pi'' \subset \pi$ be a subgroup of finite index and let $A'' = \mathbb{Z}[\pi'']$ be the corresponding ring with involution. Write B'' for the chain complex in $\mathscr{C}_{A''}$ obtained by regarding B as an A''-module chain complex; write $\mathscr{b}''$ and c'' for the chain bundles on B'' obtained from $\mathscr{b}$ and c respectively. Transfer (see I, 3.16) gives maps of spectra $\underline{\mathscr{L}}_{\cdot}(A) \to \underline{\mathscr{L}}_{\cdot}(A'')$, $\underline{\mathscr{L}}^{\cdot}(B,\mathscr{b}) \to \underline{\mathscr{L}}^{\cdot}(B'',\mathscr{b}'')$, and $\underline{\hat{\mathscr{L}}}^{\cdot}(B,\mathscr{b}) \to \underline{\hat{\mathscr{L}}}^{\cdot}(B'',\mathscr{b}'')$, whose cofibres we denote by

$$\underline{\mathscr{L}}_{\cdot}(A \uparrow A''), \quad \underline{\mathscr{L}}^{\cdot}(B \uparrow B'', \mathscr{b} \uparrow \mathscr{b}''), \quad \text{and} \quad \underline{\hat{\mathscr{L}}}^{\cdot}(B \uparrow B'', \mathscr{b} \uparrow \mathscr{b}'')$$

respectively. The nth homotopy groups of these cofibres are written

$$L_n(A \uparrow A''), \quad L^n(B \uparrow B'', \mathscr{b} \uparrow \mathscr{b}''), \quad \text{and} \quad \hat{L}^n(B \uparrow B'', \mathscr{b} \uparrow \mathscr{b}'').$$

8.2. THEOREM. *With the hypotheses of* 8.1, *there exists an isomorphism*

$$\hat{L}^{n+1}(B \uparrow B'', \mathscr{b} \uparrow \mathscr{b}'') \cong \hat{L}^{n+1}(B \uparrow B'', c \uparrow c'')$$

making the triangle

$$\begin{tikzcd}[column sep=tiny] \hat{L}^{n+1}(B \uparrow B'', \mathscr{b} \uparrow \mathscr{b}'') \arrow[dr, "\partial"'] & \cong & \hat{L}^{n+1}(B \uparrow B'', c \uparrow c'') \arrow[dl, "\partial"] \\ & L_n(A \uparrow A'') & \end{tikzcd}$$

commute. So the release homomorphisms

$$L_n(A \uparrow A'') \to L^n(B \uparrow B'', \mathscr{b} \uparrow \mathscr{b}'') \quad \textit{and} \quad L_n(A \uparrow A'') \to L^n(B \uparrow B'', c \uparrow c'')$$

have the same kernel.

8.3. COMMENT. There are two reasons for stating 8.1. Firstly, suppose that (C, φ) is an n-dimensional algebraic Poincaré complex over A, with normal chain bundle $\mathscr{n}$; then certain geometric analogies suggest that the release map $L_n(A) \to L^n(C, \mathscr{n})$ ought to be injective. This is confirmed by 8.1.

Secondly, suppose that $A = Z_2$; then the simply-connected theory of [1] and [2] suggests that the release homomorphism $L_{2k}(A) \to L^{2k}(B,\mathscr{b})$ ought to be injective if and only if the Wu class

$$v_{k+1}(\mathscr{b})\colon H_{k+1}(B) \to Z_2$$

vanishes (see I, 2.A.1). This is true (see the argument just below for the 'if' part; for the 'only if' part, reduce to the case where B is concentrated in dimension $k+1$ and compute, using 6.1). Unfortunately, it does not generalize well to arbitrary A. However, 8.1 is a reasonable substitute in many cases. We now give a standard application. Taking A arbitrary, and assuming that $\mathscr{b}$ in 8.1 is the object of study, suppose that c can be so chosen that

the hypothesis of 8.1 is satisfied;

the class $[c] \in \hat{Q}^0(B^{-*})$ belongs to the kernel of the restriction homomorphism $\hat{Q}^0(B^{-*}) \to \hat{Q}^0((B_{\leqslant[\frac{1}{2}n]+1})^{-*})$ induced by the inclusion of the skeleton $B_{\leqslant[\frac{1}{2}n]+1} \hookrightarrow B$ (with notation as in I, 2.A.3).

Then the release maps $L_n(A) \to L^n(B, \mathscr{b})$ and, if applicable,

$$L_n(A \uparrow A'') \to L^n(B \uparrow B'', \mathscr{b} \uparrow \mathscr{b}'')$$

are *injective*.

Indeed, by 8.1 and 8.2 we may replace $\mathscr{b}$ by c; now the hypothesis on c means that there exists a chain bundle $\bar{c}$ on $B/B_{\leqslant[\frac{1}{2}n]+1}$ and a chain bundle map from c to $\bar{c}$ covering the projection $B \to B/B_{\leqslant[\frac{1}{2}n]+1}$. Hence we could have assumed from the outset that B is $([\frac{1}{2}n]+1)$-connected, and $\mathscr{b} = c = \bar{c}$. But then the homomorphisms $L_n(A) \to L^n(B, \mathscr{b})$ and $L_n(A \uparrow A'') \to L^n(B \uparrow B'', \mathscr{b} \uparrow \mathscr{b}'')$ are isomorphisms. (The proof uses surgery below the middle dimension.)

Proof of 8.1. We may assume that

$$c = \mathscr{b} + \mathfrak{S}^{-n}(J(h)),$$

where h is an n-cycle in $W \& \Sigma^n(B^{-*})$. Let (φ, z) be a typical representative of $Q_{n+1}(B, \mathscr{b}) \cong \hat{L}^{n+1}(B, \mathscr{b})$ (using notation as in 6.3). Then

$$(\varphi + \varphi_0^{\to}(\mathfrak{S}h), z)$$

is a typical representative of $Q_{n+1}(B, c) \cong \hat{L}^{n+1}(B, c)$; remember that φ_0 is a chain map from $\Sigma^{n+1}(B^{-*})$ to B. This gives a bijection between $\hat{L}^{n+1}(B, \mathscr{b})$ and $\hat{L}^{n+1}(B, c)$; going back to 6.3 and using the fact that

$$Q^{n+1}(B \oplus B) \cong Q^{n+1}(B) \oplus Q^{n+1}(B) \oplus H_{n+1}(B^t \otimes_A B),$$

we find that it is a group isomorphism.

To prove that the isomorphism commutes with the boundary maps to $L_n(A)$, we use the original definition of $\hat{L}^{n+1}(B, \mathscr{b})$ and $\hat{L}^{n+1}(B, c)$ in terms of algebraic Poincaré pairs. So let $(f\colon C \to D, (\bar{\varphi}, \phi))$ be an UR algebraic Poincaré pair of dimension $n+1$, with a $\mathscr{b}$-structure $(g, \bar{z})$ such that $g \cdot f\colon C \to B$ is zero. (Then $\bar{\varphi} \in (W \& D)_{n+1}$ and $\phi \in (W \& C)_n$, etc.) Now

$$j = \bar{\varphi}_0 \cdot g\colon \Sigma^{n+1}(B^{-*}) \to D$$

happens to be a chain map (because g vanishes on the boundary); and

$$(f\colon C \to D, (\bar{\varphi} + j^{\to}(\mathfrak{S}h), \phi))$$

is an UR algebraic Poincaré pair of dimension $n+1$, with a c-structure $(g, \bar{z})$ such that $g \cdot f = 0$. If we let $\varphi := g^{\to}(\bar{\varphi})$ and $z := g^{\to}(\bar{z}(\{0,1\}))$, then the first algebraic Poincaré pair above corresponds to $[(\varphi, z)] \in Q_{n+1}(B, \mathscr{b})$, and the second to

$$[(\varphi + \varphi_0^{\to}(\mathfrak{S}h), z)] \in Q_{n+1}(B, c).$$

But the boundaries of the two algebraic Poincaré pairs (with additional structure) are identical; so they represent the same element in $L_n(A)$, as required. (I am obliged to A. Ranicki for help with the proof.)

Proof of 8.2. This is identical with the proof of 8.1, except that it calls for a more categorical point of view. We are dealing with certain A-modules (mostly the chain modules B_n and their duals); but we usually regard them as A''-modules only, and moreover adopt the policy of regarding A''-module homomorphisms between them as 'negligible' if they preserve the A-module structure.

For instance, the group $\hat{L}^{n+1}(B \uparrow B'', \mathscr{E} \uparrow \mathscr{E}'')$, which we might also call

$$Q_{n+1}(B \uparrow B'', \mathscr{E} \uparrow \mathscr{E}''),$$

has a description in terms of equivalence classes of pairs (φ, z), with $\varphi \in (W \& B'')_{n+1}$ and $z \in (\hat{W} \& B'')_{n+2}$; however, instead of requiring that

$$d(\varphi) = 0 \qquad \text{in } W \& B''$$

and

$$d(z) = J(\varphi) - \mathfrak{S}^{n+1}(\varphi_0^{\to}(\mathscr{E}'')) \qquad \text{in } \hat{W} \& B''$$

(as we should in defining $Q_{n+1}(B'', \mathscr{E}'')$), we merely ask that

$$d(\varphi) \equiv 0$$

and

$$d(z) \equiv J(\varphi) - \mathfrak{S}^{n+1}(\varphi_0^{\to}(\mathscr{E}'')),$$

where $\equiv$ indicates that the difference between the left-hand and right-hand terms belongs to the 'negligible' subcomplexes $W \& B \subset W \& B''$ or $\hat{W} \& B \subset \hat{W} \& B''$. (We have, for instance, $W \& B \subset W \& B''$ because

$$B^t \otimes_A B \cong \mathrm{Hom}_A(B^{-*}, B) \subset \mathrm{Hom}_{A''}(B''^{-*}, B'') \cong B''^t \otimes_{A''} B''.)$$

The details are left to the reader.

8.4. Remark. If a version of the theory is used where projective class and/or torsion matters, then 8.2 must be formulated with greater care; see I, 2.22. However, this affects $L_n(A \uparrow A'')$ only, not the relative groups $\hat{L}^{n+1}(B \uparrow B'', \mathscr{E} \uparrow \mathscr{E}'')$.

9. *Products and Whitney sums*

9.1. Definition. If $\mathscr{E}$ is a chain bundle on a chain complex B in $\mathscr{C}_A$ and $\mathscr{E}'$ is a chain bundle on a chain complex B' in $\mathscr{C}_{A'}$, then $\mathscr{E} \times \mathscr{E}'$ is a chain bundle on the chain complex $B \otimes_{\mathbb{Z}} B'$ in $\mathscr{C}_{A \otimes_{\mathbb{Z}} A'}$, called the exterior product of $\mathscr{E}$ and $\mathscr{E}'$ (cf. I, 0.11).

9.2. Proposition. *There are multiplication maps*

$$\underline{\mathscr{L}}^{\cdot}(B, \mathscr{E}) \wedge \underline{\mathscr{L}}^{\cdot}(B', \mathscr{E}') \to \underline{\mathscr{L}}^{\cdot}(B \otimes_{\mathbb{Z}} B', \mathscr{E} \times \mathscr{E}')$$

and

$$\hat{\underline{\mathscr{L}}}^{\cdot}(B, \mathscr{E}) \wedge \hat{\underline{\mathscr{L}}}^{\cdot}(B', \mathscr{E}') \to \hat{\underline{\mathscr{L}}}^{\cdot}(B \otimes_{\mathbb{Z}} B', \mathscr{E} \times \mathscr{E}'),$$

inducing multiplication homomorphisms

$$L^m(B, \mathscr{E}) \otimes L^n(B', \mathscr{E}') \to L^{m+n}(B \otimes_{\mathbb{Z}} B', \mathscr{E} \times \mathscr{E}')$$

and

$$\hat{L}^m(B, \mathscr{E}) \otimes \hat{L}^n(B', \mathscr{E}') \to \hat{L}^{m+n}(B \otimes_{\mathbb{Z}} B', \mathscr{E} \times \mathscr{E}').$$

Proof. Let (C, φ) be an m-dimensional UR symmetric chain complex over A with normal $\mathscr{E}$-structure (g, z) (cf. 4.14); and let (C', φ') be an n-dimensional UR symmetric chain complex over A' with normal $\mathscr{E}'$-structure (g', z'). Then $(C \otimes_{\mathbb{Z}} C', \varphi \times \varphi')$ is an

$(m+n)$-dimensional UR symmetric chain complex over $A \otimes_{\mathbb{Z}} A'$, with normal $\mathscr{E} \times \mathscr{E}'$-structure

$$(g \otimes g', \varphi \times z' + (-)^n \cdot z \times (\varphi' - d(z'))).$$

Passage to bordism classes defines the multiplication

$$\hat{L}^m(B, \mathscr{E}) \otimes \hat{L}^n(B', \mathscr{E}') \to \hat{L}^{m+n}(B \otimes_{\mathbb{Z}} B', \mathscr{E} \times \mathscr{E}').$$

If in addition (C, φ) and (C', φ') are both UR algebraic Poincaré complexes, then so is $(C \otimes_{\mathbb{Z}} C', \varphi \times \varphi')$, which explains the multiplication

$$L^m(B, \mathscr{E}) \otimes L^n(B', \mathscr{E}') \to L^{m+n}(B \otimes_{\mathbb{Z}} B', \mathscr{E} \times \mathscr{E}').$$

The rest of the proof is unpleasant and left to the reader.

9.3. Proposition. *Under the association*

spherical fibration $\mapsto$ chain bundle

of I, 3.4, *exterior Whitney sums (explanation below) correspond to exterior products of chain bundles.*

Proof. The passage from spherical fibrations to chain bundles in I, 3.4 (and previously in [11]) was based on equivariant S-duality; 9.3 is an application of the principle that (equivariant) S-duality commutes with smash products. Details are again left to the reader. To get a good definition of Whitney sums, return to I, 3.2 and use an identification $D^n \times D^m \cong D^{n+m}$.

9.4. Proposition. *The diagram of maps of spectra*

$$\begin{array}{ccc} \underline{\Omega}^P(X, \gamma) \wedge \underline{\Omega}^P(X', \gamma') & \longrightarrow & \underline{\Omega}^P(X \times X', \gamma \times \gamma') \\ \downarrow & & \downarrow \\ \underline{\mathscr{L}}^{\cdot}(C(\tilde{X}), c(\gamma)) \wedge \underline{\mathscr{L}}^{\cdot}(C(\tilde{X}'), c(\gamma')) & \longrightarrow & \underline{\mathscr{L}}^{\cdot}(C(\tilde{X} \times \tilde{X}'), c(\gamma \times \gamma')) \\ & & \simeq \underline{\mathscr{L}}^{\cdot}(C(\tilde{X}) \otimes_{\mathbb{Z}} C(\tilde{X}'), c(\gamma) \times c(\gamma')) \end{array}$$

commutes.

(Here γ and γ' are spherical fibrations on simplicial sets X and X', equipped with certain data; $\gamma \times \gamma'$ on $X \times X'$ is the exterior Whitney sum.)

There is a similar commutative diagram in which $\underline{\Omega}^P(X, \gamma)$ and $\underline{\mathscr{L}}^{\cdot}(C(\tilde{X}), c(\gamma))$ are replaced by $M(X, \gamma)$ and $\underline{\hat{\mathscr{L}}}^{\cdot}(C(\tilde{X}), c(\gamma))$, respectively (similarly for X' and γ'). See 7.1. The proof of 9.4 is left to the reader.

The analogue of 9.3 for internal Whitney sums looks as follows. Given strings of data $(\pi, w; X, \gamma; \alpha, j)$ and $(\pi', w'; X, \gamma'; \alpha', j')$ as in I, 3.4, their internal Whitney sum is the string $(\pi \times \pi', w \times w'; X, \gamma \oplus \gamma'; \alpha \times_X \alpha', j \times j')$. The associated chain bundles $c(\gamma)$, $c(\gamma')$, and $c(\gamma \oplus \gamma')$ (which will also be denoted, for greater precision, by $c(\gamma; \pi)$, $c(\gamma'; \pi')$, and $c(\gamma \oplus \gamma'; \pi \times \pi')$ respectively) are related by a 'Cartan formula':

9.5. Corollary. $c(\gamma \oplus \gamma'; \pi \times \pi') = c(\gamma; \pi) \cup c(\gamma'; \pi')$.

(Explanation: $c(\gamma;\pi)\cup c(\gamma';\pi')$ is, by definition, the pullback of $c(\gamma;\pi)\times c(\gamma';\pi')$ under the composition

$$C(\tilde{X}) \xrightarrow{\text{Eilenberg–Zilber}} C(\tilde{X})\otimes_{\mathbb{Z}} C(\tilde{X}) \xrightarrow{\quad\text{proj.}\quad} C(\tilde{X}/\pi')\otimes_{\mathbb{Z}} C(\tilde{X}/\pi)$$

in which $\tilde{X}$ denotes the total space of $\alpha\times_X\alpha'$.)

Next, let $(\pi, w; X, \gamma; \alpha, j)$ be a string as usual, and assume for simplicity that $w\colon \pi \to Z_2$ is trivial. Write this string as the internal Whitney sum of the two strings

$$(\pi, w; X, \text{ trivial spherical fibration; } \alpha, \text{ trivial})$$

and

$$(\{1\}, \text{trivial}; X, \gamma; \text{trivial}, j).$$

In other words, write

$$\gamma = (\text{trivial spherical fibration} \oplus \gamma);$$

let the first Whitney summand (trivial spherical fibration) carry the weight of the data, and equip the second summand with the trivial data. Then 9.5 implies

9.6. COROLLARY ('Separation principle').

$$c(\gamma;\pi) = c(\text{trivial spherical fibration};\pi)\cup c(\gamma;\{1\}).$$

Both 9.5 and 9.6 have to be interpreted in the usual woolly way, namely 'up to an infinity of higher homologies'; but at any rate, 9.6 shows that $c(\gamma;\pi)$ is determined in a sense by $c(\gamma;\{1\})$.

The situation is similar for stable fibre homotopy equivalences of spherical fibrations. Suppose that we are given two strings

$$(\pi, w; X, \gamma_1; \alpha, j_1) \quad\text{and}\quad (\pi, w; X, \gamma_2; \alpha, j_2)$$

as in I, 3.4 (with $w = 0$), and an orientation-preserving stable fibre homotopy equivalence

$$\mu\colon \gamma_1 \cong \gamma_2 .$$

(With our restrictive notion of spherical fibration, it is best to assume that μ comes in the shape of a stable spherical fibration on $X\times[0,1]$ which restricts to γ_1 on $X\times\{0\}$ and to γ_2 on $X\times\{1\}$.)

9.7. ADDENDUM TO 9.6. *The 'chain bundle isomorphism'* (cf. I, 1.8) $c(\gamma_1;\pi)\cong c(\gamma_2;\pi)$ *induced by* μ (cf. I, 1.12) *is determined by the chain bundle isomorphism*

$$c(\gamma_1;\{1\})\cong c(\gamma_2;\{1\})$$

(also induced by μ*).*

These trivial algebraic observations have a non-trivial geometric consequence. Let $f\colon \mathrm{BSO} \to \prod_{k\geq 0} K(Z_2, 2k)$ be a map in the homotopy class $(v_0, v_2, v_4, \ldots)$, where the v_i are the Wu classes.

Define a *pseudo-surgery problem* over (π, w) (with $w = 0$ as before) to consist of

a degree-1 map $\bar{p}$ from a compact smooth oriented manifold N^n with boundary ∂N to a finite (simple, if you wish) oriented geometric Poincaré pair $(X^n, \partial X)$, restricting to a (simple) homotopy equivalence of the boundaries;

a principal π-bundle on X;
a map (not just a homotopy class)

$$g\colon X \to \prod_{k\geq 0} K(Z_2, 2k);$$

and a homotopy from

$$N \xrightarrow{\nu_N} \mathrm{BSO} \xrightarrow{f} \prod K(Z_2, 2k)$$

to

$$N \xrightarrow{\bar{p}} X \xrightarrow{g} \prod K(Z_2, 2k).$$

An ordinary surgery problem (as in [13]) gives rise to a pseudo one: take

$$g := f \cdot (\text{classifying map for the vector bundle on } X).$$

9.8. THEOREM. *There is a canonical factorization* (*broken arrow*)

$$\begin{tikzcd}
\text{Bordism group of } n\text{-dimensional ordinary surgery problems} \arrow[d] \arrow[dr] & \\
& \text{Bordism group of } n\text{-dimensional pseudo-surgery problems} \arrow[dl, dashed] \\
L_n(\mathbb{Z}[\pi]) &
\end{tikzcd}$$

As an example, consider an ordinary surgery problem (consisting of a degree-1 map $g\colon N^n \to X^n$ as before, a vector bundle γ on X, and a stable trivialization of $g^{\leftarrow}(\gamma) \oplus \tau_N$). Let us alter the stable trivialization by a map $f\colon N \to \mathrm{SO}$. Suppose that

$$f^{\leftarrow}(\delta(v_i)) = 0 \quad \text{in } H^{i-1}(N; Z_2) \text{ for all } i > 0,$$

where $\delta(v_i) \in H^{i-1}(\mathrm{SO}; Z_2)$ is the cohomology desuspension of the Wu class

$$v_i \in H^i(\mathrm{BSO}; Z_2).$$

Then the change of framing f does not affect the surgery obstruction, by 9.8.

Proof of 9.8. Write $A = \mathbb{Z}[\pi]$. The pseudo-surgery problem described just before 9.8 gives rise to a degree-1 map of algebraic Poincaré pairs

$$p\colon (C(\partial\tilde{N}) \to C(\tilde{N}), (\psi, \varphi)) \to (C(\partial\tilde{X}) \to C(\tilde{X}), (\lambda, \eta)),$$

where 'degree-1' means that $p^{\rightarrow}(\psi, \varphi) = (\lambda, \eta)$, strictly. Specifying a map

$$X \to \prod K(Z_2, 2k)$$

is another way of specifying a chain bundle $c_{\mathbb{Z}}$ (over $\mathbb{Z}$!) on $C(X) = \mathbb{Z} \otimes_A C(\tilde{X})$. (It is easy to see that $\hat{Q}^0(C(X)^{-*}) \cong \prod_{k\geq 0} H^{2k}(X; Z_2)$, for instance by applying I, 2.A to the ring with involution $\mathbb{Z}$.)

The remaining data give an isomorphism of chain bundles (over $\mathbb{Z}$!)

$$\mathrm{Is}_{\mathbb{Z}}\colon p^{\leftarrow}(c_{\mathbb{Z}}) \cong (\text{normal chain bundle of } C(N)),$$

with $C(N) = \mathbb{Z} \otimes_A C(\tilde{N})$. (Of course, algebraic Poincaré pairs, too, have normal chain bundles.)

What we really need in order to get an element in $L_n(A)$ is the chain level analogue of a surgery problem, i.e. apart from the degree-1 map p of algebraic Poincaré pairs (which restricts to a chain homotopy equivalence of the boundaries),

a chain bundle c_A (over $A!$) on $C(\tilde{X})$, and

an identification of chain bundles (over $A!$),

$$\mathrm{Is}_A\colon p^{\leftarrow}(c_A) \cong (\text{normal chain bundle of } C(\tilde{N})).$$

Now 9.6 suggests that c_A can be defined using $c_{\mathbb{Z}}$; similarly, 9.7 suggests that Is_A can be defined using $\mathrm{Is}_{\mathbb{Z}}$.

In detail, let $\mathscr{E}$ be the chain bundle (over A) on $C(\tilde{X})$ determined, as in I, 3.4, by the trivial vector bundle on X; put

$$c_A := \mathscr{E} \cup c_{\mathbb{Z}},$$

and let Is_A be the identification

$$\begin{aligned} p^{\leftarrow}(c_A) = p^{\leftarrow}(\mathscr{E} \cup c_{\mathbb{Z}}) &= p^{\leftarrow}(\mathscr{E}) \cup p^{\leftarrow}(c_{\mathbb{Z}}) \\ &\cong p^{\leftarrow}(\mathscr{E}) \cup (\text{normal chain bundle over } \mathbb{Z} \text{ of } C(N)) \\ &\cong (\text{normal chain bundle over } A \text{ of } C(\tilde{N})). \end{aligned}$$

(The last in this sequence of identifications stems from 9.6, taking into account the first sentence of I, § 3; and the previous one is induced by $\mathrm{Is}_{\mathbb{Z}}$.)

9.9. Remark. In the twisted case, that is, when $w\colon \pi \to Z_2$ is non-trivial, 9.8 remains valid with no essential change, except that the classifying map for the normal bundle ν_N,

$$N \to \mathrm{BSO}$$

has to be replaced by the classifying map for

$$\gamma_N \oplus w\text{-twisted line bundle},$$

which still goes from N to BSO. Moreover, there is no harm in replacing the smooth manifold with boundary N by a geometric Poincaré pair.

Let X be a finitely generated simplicial set, let α be a principal π-bundle on X, and let $K_{\mathrm{osf}}(X)$ be the group of stable fibre homotopy equivalence classes of orientable spherical fibrations.

9.10. Proposition. *The diagonal maps* $X \to X \times X$ *and* $\pi \to \pi \times \pi$ *make* $\hat{Q}^0(C(\tilde{X})^{-*})$ *into a (commutative, associative) ring. The rule* $\gamma \mapsto [c(\gamma)]$ *defines a multiplicative map from* $K_{\mathrm{osf}}(X)$ *to* $\hat{Q}^0(C(\tilde{X})^{-*})$, *so it transforms Whitney sums into products. (See also* 10.13.)

Explanation and proof. $C(\tilde{X})$ is in $\mathscr{C}_{\mathbb{Z}[\pi]}$, and $\mathbb{Z}[\pi]$ carries the involution coming from the trivial homomorphism $w\colon \pi \to Z_2$. The ring structure on $\hat{Q}^0(C(\tilde{X})^{-*})$ is obtained as follows. Given chain bundles $\mathscr{E}$ and $\mathscr{E}'$ on $C(\tilde{X})$, note that $\mathscr{E} \times \mathscr{E}'$ is a chain bundle on $C(\tilde{X}) \otimes_{\mathbb{Z}} C(\tilde{X})$, regarded as a chain complex over

$$\mathbb{Z}[\pi] \otimes_{\mathbb{Z}} \mathbb{Z}[\pi] \cong \mathbb{Z}[\pi \times \pi].$$

Although the diagonal subgroup $\pi \subset \pi \times \pi$ will not in general have finite index, an ad hoc transfer argument shows that

$C(\tilde{X}) \otimes_{\mathbb{Z}} C(\tilde{X})$ can be thought of as a chain complex of $\mathbb{Z}[\pi]$-modules (free, but not necessarily finitely generated);

$\mathscr{E} \times \mathscr{E}'$ determines a chain bundle $\mathrm{tr}(\mathscr{E} \times \mathscr{E}')$ (over $\mathbb{Z}[\pi]$!) on the said chain complex. (To see this, write $D := C(\tilde{X}) \otimes_{\mathbb{Z}} C(\tilde{X})$, and let

$$f\colon \mathbb{Z}[\pi \times \pi] \to \mathbb{Z}[\pi]$$

be the homomorphism of free abelian groups which sends the generator

$$(x, y) \in \pi \times \pi \subset \mathbb{Z}[\pi \times \pi] \text{ to } x \in \pi \subset \mathbb{Z}[\pi] \text{ if } x = y,$$

and to 0 otherwise. Define 'chain bundles' as at the beginning of I, 2.A; if

$$\mathscr{E} \times \mathscr{E}' = \{\varphi_{p,q}\colon D_p \times D_q \to \mathbb{Z}[\pi \times \pi] \mid p, q \in \mathbb{Z}\},$$

put

$$\mathrm{tr}(\mathscr{E} \times \mathscr{E}') := \{f \cdot \varphi_{p,q}\colon D_p \times D_q \to \mathbb{Z}[\pi] \mid p, q \in \mathbb{Z}\}.)$$

The Eilenberg–Zilber diagonal $EZ_0\colon C(\tilde{X}) \to C(\tilde{X}) \otimes_{\mathbb{Z}} C(\tilde{X})$ is a chain map over $\mathbb{Z}[\pi]$; the ring structure on $\hat{Q}^0(C(\tilde{X})^{-*})$ is given by

$$[\mathscr{E}] \cdot [\mathscr{E}'] = (EZ)_0^{-}[\mathrm{tr}(\mathscr{E} \times \mathscr{E}')].$$

Now let γ be an orientable spherical fibration on X; choose an orientation j. Then $(\pi, w; X, \gamma; \alpha, j)$ is a string of data as in I, 3.4, with $w = 0$. Therefore $[\mathscr{c}(\gamma)]$ in $\hat{Q}^0(C(\tilde{X})^{-*})$ is defined. Choosing a different orientation does not affect the result.

There is a version of 9.10 which covers the non-orientable case: the appropriate ring to consider is then

$$\bigoplus_{w} \hat{Q}^0({}^{w}C(\tilde{X})^{-*})$$

(where w ranges over all homomorphisms from π to Z_2, and the superscript w in $\hat{Q}^0({}^{w}C(\tilde{X})^{-*})$ indicates which involution on $\mathbb{Z}[\pi]$ is used to define $\hat{Q}^0(C(\tilde{X})^{-*})$.)

The principal π-bundle α on X would be fixed, however, as in 9.10. See also 10.13.

10. *Classification of chain bundles over a group ring*

10.1. THEOREM. *Let R be either $\mathbb{Z}$ or Z_2; let $A = R[\pi]$ be the group ring, equipped with the w-twisted involution for some $w\colon \pi \to Z_2$. The cohomology theory*

$$C \mapsto \{\hat{Q}^{-n}(C^{-*}) \mid n \in \mathbb{Z}\}$$

on $\mathscr{C}_A$ is then an ordinary cohomology theory, that is, there are canonical natural isomorphisms

$$\hat{Q}^{-n}(C^{-*}) \cong \prod_{k \in \mathbb{Z}} H^{k+n}(C; \hat{H}^k(Z_2; A))$$

for $n \in \mathbb{Z}$, commuting with the suspension isomorphisms.

(Note that the groups $\hat{H}^k(Z_2; A) \cong \hat{Q}^0((A, k)^{-*})$ are the coefficients of the cohomology theory and therefore carry a left A-module structure, made explicit in I, 2.A.)

Assume now that $A = Z_2[\pi]$. Let PR be a projective resolution of the left A-module $\hat{H}^0(Z_2; A)$ (so that $H_0(PR)$ is canonically identified with $\hat{H}^0(Z_2; A)$). Theorem 10.1 follows (in the case where $A = Z_2[\pi]$) from

10.2. ROUGH STATEMENT. There exists a (somehow distinguished) chain bundle $\mathscr{d}$ on PR so that the 0th Wu class defined in I, 2.A,

$$v_0(\mathscr{d}): H_0(PR) \to \hat{H}^0(Z_2; A),$$

agrees with the canonical identification. ('Chain bundle' has to be interpreted here as in I, 2.A.)

Proof of the implication 10.2 ⇒ 10.1, *for* $A = Z_2[\pi]$. It suffices to specify a natural isomorphism

$$\prod_{k \in \mathbb{Z}} H^k(C; \hat{H}^k(Z_2; A)) \cong \hat{Q}^0(C^{-*})$$

for C in $\mathscr{C}_A$, since $\hat{Q}^n(C^{-*}) \cong \hat{Q}^0((\Sigma^n C)^{-*})$. Since A has characteristic 2,

$$\hat{H}^k(Z_2; A) \cong \hat{H}^0(Z_2; A) \quad \text{for } k \in \mathbb{Z};$$

further, the cohomology theory at issue is now periodic with period 1 (not merely 2; see I, 1.3). To be more precise, if B is a possibly huge chain complex of projective left A-modules, if $\hat{W} \& B^{-*}$ is defined as at the beginning of I, 2.A, and if

$$\varphi = \{\varphi_{p,q}: B_p \times B_q \to A \mid p, q \in \mathbb{Z}\}$$

is an n-cycle in $\hat{W} \& B^{-*}$ for some $n \in \mathbb{Z}$, then φ can be viewed as an n-cycle in $\hat{W} \& B^{-*}$ for all $n \in \mathbb{Z}$. Equivalently, if $\mathscr{b}$ is a chain bundle on B, then $\mathfrak{S}^{-n}\mathscr{b}$ can be regarded as a chain bundle on $\Sigma^n B$.

Now let

$$B^\infty := \bigoplus_{k \in \mathbb{Z}} \Sigma^k PR;$$

supposing that 10.2 holds, let

$$\mathscr{b}^\infty := \bigoplus_{k \in \mathbb{Z}} \mathfrak{S}^{-k}\mathscr{d}.$$

Since all the Wu classes

$$v_k(\mathscr{b}^\infty): H_k(B^\infty) \to \hat{H}^k(Z_2; A)$$

are now isomorphisms, we recognize in $\mathscr{b}^\infty$, B^∞ the 'universal chain bundle' of I, 2.A.4. Therefore or otherwise, the natural homomorphism

$$\prod_{k \in \mathbb{Z}} H^k(C; \hat{H}^k(Z_2; A)) \cong H_0(\operatorname{Hom}_A(C, B^\infty)) \to \hat{Q}^0(C^{-*}),$$

$$[f] \mapsto [f^{\leftarrow}(\mathscr{b}^\infty)]$$

is a natural isomorphism, for C in $\mathscr{C}_A$. (Of course, in order to make it canonical we have to specify a canonical $\mathscr{d}$ in 10.2.)

The proof of 10.2 proceeds by obstruction theory. Write $\mathscr{D}_A$ for the category of projective left A-module chain complexes C which are positive, such as PR in 10.2.

(For the moment, A can be any ring with involution.) For C in $\mathscr{D}_A$, let

$$\mathscr{P}_k\hat{Q}^0(C^{-*}) := \operatorname{im}[\hat{Q}^0((C_{\leqslant k+1})^{-*}) \to \hat{Q}^0((C_{\leqslant k})^{-*})].$$

(Here $C_{\leqslant k}$ is the k-skeleton of C.)

So the homotopy-invariant functor

$$\text{(i)} \qquad C \mapsto \mathscr{P}_k\hat{Q}^0(C^{-*})$$

is the 'kth Postnikov base' of the (homotopy-invariant) functor

$$\text{(ii)} \qquad C \mapsto \hat{Q}^0(C^{-*});$$

see [4] for a completely analogous topological definition. Or in other words, passing from (ii) to (i) amounts to 'killing' the coefficient groups of the functor (ii) in dimension greater than k, that is, the groups $\hat{Q}^0((A,n)^{-*})$ for $n > k$. (Cf. I, 0.6.) There is a commutative diagram of natural forgetful maps

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & \mathscr{P}_{k+1}\hat{Q}^0(C^{-*}) \arrow[d] \\
\hat{Q}^0(C^{-*}) \arrow[ur, dotted] \arrow[r] \arrow[dr, dotted] & \mathscr{P}_k\hat{Q}^0(C^{-*}) \arrow[d] \\
 & \mathscr{P}_{k-1}\hat{Q}^0(C^{-*}) \arrow[d] \\
 & \vdots
\end{tikzcd}$$

and there are natural homomorphisms

$$\text{ob}\colon \mathscr{P}_{k-1}\hat{Q}^0(C^{-*}) \to H^{k+1}(C; \hat{H}^k(Z_2; A))$$

so that the sequence

$$\mathscr{P}_k\hat{Q}^0(C^{-*}) \longrightarrow \mathscr{P}_{k-1}\hat{Q}^0(C^{-*}) \xrightarrow{\text{ob}} H^{k+1}(C; \hat{H}^k(Z_2; A))$$

is exact. (Again, $\hat{H}^k(Z_2; A) \cong \hat{Q}^0((A,k)^{-*})$ plays the role of 'coefficient group'.)

10.3. Description of ob: $\mathscr{P}_{k-1}\hat{Q}^0(C^{-*}) \to H^{k+1}(C; \hat{H}^k(Z_2; A))$. For y in

$$\mathscr{P}_{k-1}\hat{Q}^0(C^{-*}) = \operatorname{im}[\hat{Q}^0((C_{\leqslant k})^{-*}) \to \hat{Q}^0((C_{\leqslant k-1})^{-*})],$$

let $\bar{y} \in \hat{Q}^0((C_{\leqslant k})^{-*})$ be a lifting; treat the differential $d\colon C_{k+1} \to C_k$ as a chain map from (C_{k+1}, k) to $C_{\leqslant k}$. Then

$$d^{-}(\bar{y}) \in \hat{Q}^0((C_{k+1}, k)^{-*}) \cong \operatorname{Hom}_A(C_{k+1}, \hat{H}^k(Z_2; A))$$

represents an element $\text{ob}(y) \in H^{k+1}(C; \hat{H}^k(Z_2; A))$, independent of the choice of lifting $\bar{y}$.

10.4. Definition. For $k > 0$ and C in $\mathscr{D}_A$, let

$$\text{socle}\colon \mathscr{P}_{k-1}\hat{Q}^0(C^{-*}) \to H^0(C; \hat{H}^0(Z_2; A)) \cong \mathscr{P}_0\hat{Q}^0(C^{-*})$$

be the forgetful map. (Note that C is positive.)

Let C, C' be chain complexes in $\mathscr{D}_A, \mathscr{D}_{A'}$ respectively, where A and A' are arbitrary rings with involution. Then $C \otimes_{\mathbb{Z}} C'$ is in $\mathscr{D}_{A \otimes_{\mathbb{Z}} A'}$; if $y \in \mathscr{P}_{k-1}\hat{Q}^0(C^{-*})$ and $y' \in \mathscr{P}_{k-1}\hat{Q}^0(C'^{-*})$, the exterior product $y \times y'$ is an element of $\mathscr{P}_{k-1}\hat{Q}^0((C \otimes_{\mathbb{Z}} C')^{-*})$.

10.5. LEMMA.

$$\operatorname{socle}(y \times y') = \operatorname{socle}(y) \times \operatorname{socle}(y');$$
$$\operatorname{ob}(y \times y') = \operatorname{ob}(y) \times \operatorname{socle}(y') + \operatorname{socle}(y) \times \operatorname{ob}(y').$$

To make sense of these formulae, note that there are 'exterior multiplication maps'

$$\hat{H}^i(Z_2; A) \times \hat{H}^j(Z_2; A') \to \hat{H}^{i+j}(Z_2; A \otimes_{\mathbb{Z}} A')$$

derived from the diagonal map $\hat{W} \to \hat{W} \otimes_{\mathbb{Z}} \hat{W}$ (in fact, A and A' could be replaced by arbitrary $\mathbb{Z}[Z_2]$-modules). Consequently, there are exterior cohomology products

$$H^p(C; \hat{H}^i(Z_2; A)) \times H^q(C'; \hat{H}^j(Z_2; A')) \to H^{p+q}(C \otimes_{\mathbb{Z}} C'; \hat{H}^{i+j}(Z_2; A \otimes_{\mathbb{Z}} A')).$$

The lemma is easy to verify.

Now let $A = Z_2[\pi]$ again, and let PR be as in 10.2. The ring A is equipped with a diagonal homomorphism

$$Z_2[\pi] = A \to A \otimes_{\mathbb{Z}} A \cong Z_2[\pi \times \pi]$$

corresponding to the diagonal inclusion $\pi \to \pi \times \pi$. Therefore $PR \otimes_{\mathbb{Z}} PR$ can be regarded as a left A-module chain complex (in $\mathscr{D}_A$).

10.6. OBSERVATION. PR is equipped with a canonical (homotopy commutative, homotopy associative) diagonal chain map

$$PR \to PR \otimes_{\mathbb{Z}} PR$$

of A-module chain complexes. Therefore $\hat{Q}^0(PR^{-*})$ and $\mathscr{P}_k\hat{Q}^0(PR^{-*})$ are rings, for $k \geqslant 0$.

Proof. Since PR is a projective resolution of $\hat{H}^0(Z_2; A)$ and $PR \otimes_{\mathbb{Z}} PR$ is a projective resolution of $\hat{H}^0(Z_2; A) \otimes_{\mathbb{Z}} \hat{H}^0(Z_2; A)$, specifying such a chain map is equivalent to specifying a 'diagonal map' of left A-modules

$$\hat{H}^0(Z_2; A) \to \hat{H}^0(Z_2; A) \otimes_{\mathbb{Z}} \hat{H}^0(Z_2; A).$$

A further reduction is possible. There is a functor FR from the category of π-sets to that of $Z_2[\pi]$-modules: to every π-set S it associates the Z_2-vector space generated by S, with π-action induced from the π-action on S. We have

$$\hat{H}^0(Z_2; A) \cong FR(\operatorname{tor}_2(\pi)),$$

where

$$\operatorname{tor}_2(\pi) = \{x \in \pi \mid x^2 = 1\},$$

and where π acts on $\operatorname{tor}_2(\pi)$ by conjugation. Similarly,

$$\hat{H}^0(Z_2; A) \otimes_{\mathbb{Z}} \hat{H}^0(Z_2; A) = FR(\operatorname{tor}_2(\pi) \times \operatorname{tor}_2(\pi)).$$

So all we need is a π-map

$$\operatorname{tor}_2(\pi) \to \operatorname{tor}_2(\pi) \times \operatorname{tor}_2(\pi);$$

we take the diagonal map.

The ring structures on $\hat{Q}^0(PR^{-*})$ and $\mathscr{P}_k\hat{Q}^0(PR^{-*})$ are defined as in 9.10.

10.7. LEMMA. *Suppose that* $y \in \mathscr{P}_k\hat{Q}^0(PR^{-*})$ *satisfies*

$$\mathrm{socle}(y) = 0.$$

Then y *is nilpotent.*

Proof. If $\mathrm{socle}(y) = 0$, then y can be represented by a chain bundle

$$\{\varphi_{p,q}\colon PR_p \otimes_{\mathbb{Z}} PR_q \to A \mid p, q \leqslant k+1\}$$

on $PR_{\leqslant k+1}$ so that $\varphi_{0,0} = 0$. The definition of the multiplication in $\mathscr{P}_k\hat{Q}^0(PR^{-*})$ implies then that y^n can be represented by a chain bundle

$$\{\psi_{p,q}\colon PR_p \otimes_{\mathbb{Z}} PR_q \to A \mid p, q \leqslant k+1\}$$

on $PR_{\leqslant k+1}$ so that $\psi_{p,q} = 0$ whenever $p+q < n$. So $y^n = 0$ for $n > 2k$.

10.8. LEMMA. *For any* $y \in \mathscr{P}_k\hat{Q}^0(PR^{-*})$, y^{2^n} *is idempotent if* n *is sufficiently large.*

Proof. Write $y^2 = y+h$. Then $\mathrm{socle}(h) = 0$, since $\mathrm{socle}(y^2) = \mathrm{socle}(y)$. By 10.7, h is nilpotent; choose n large enough so that $h^{2^n} = 0$. Then

$$(y^{2^n})^2 = (y^2)^{2^n} = y^{2^n} + h^{2^n} = y^{2^n},$$

as required. (Note that we are in characteristic 2, so $y \mapsto y^{2^n}$ is a ring endomorphism.)

It is now possible to reformulate 10.2, as follows. Firstly, it is not necessary in 10.2 to construct a chain bundle on PR; a class in $\hat{Q}^0(PR^{-*})$ will do just as well. Secondly, although the map

$$\hat{Q}^0(PR^{-*}) \to \varprojlim_k \mathscr{P}_k\hat{Q}^0(PR^{-*})$$

need not be an isomorphism, it is clear that an element in $\varprojlim \mathscr{P}_k\hat{Q}^0(PR^{-*})$ is quite sufficient for the application to 10.1 (in the case where $A = Z_2[\pi]$). The next proposition exhibits such an element.

10.9. PROPOSITION. *For every* $k \geqslant 0$, *there is a unique element* y_k *in* $\mathscr{P}_k\hat{Q}^0(PR^{-*})$ *such that*

(i) $\mathrm{socle}(y_k) \in H^0(PR; \hat{H}^0(Z_2; A)) \cong \mathrm{Hom}_A(\hat{H}^0(Z_2; A), \hat{H}^0(Z_2; A))$ *is the identity*,

(ii) y_k *is idempotent.*

Proof. Clearly y_0 exists and is unique. Suppose that y_{k-1} has already been constructed. Then $(y_{k-1})^2 = y_{k-1}$; now 10.5 implies that

$$\mathrm{ob}(y_{k-1}) \in H^{k+1}(PR; \hat{H}^k(Z_2; A))$$

is divisible by 2, and hence equal to 0. So there exists an element z in $\mathscr{P}_k\hat{Q}^0(PR^{-*})$ which lifts y_{k-1}. Put

$$y_k := z^{2^n} \quad \text{for sufficiently large } n,$$

so that y_k is idempotent (10.8). Clearly y_k satisfies conditions (i) and (ii). To prove uniqueness, suppose that $y'_k \in \mathscr{P}_k\hat{Q}^0(PR^{-*})$ also satisfies (i) and (ii). Then $y_k - y'_k$ is idempotent, since we are in characteristic 2, but also nilpotent by 10.7. Therefore $y_k = y'_k$.

The proof of 10.1 for $R = Z_2$ is complete. (The argument fails for $R = \mathbb{Z}$, even when $w: \pi \to Z_2$ is trivial. The point is that 10.6 becomes false: if PR is a projective resolution of $\hat{H}^0(Z_2; A)$ over $\mathbb{Z}[\pi]$, then it is also a projective resolution of $\hat{H}^0(Z_2; A)$ over $\mathbb{Z}$; therefore $PR \otimes_{\mathbb{Z}} PR$ will not be a projective resolution over $\mathbb{Z}$ or over $\mathbb{Z}[\pi]$, since $H_1(PR \otimes_{\mathbb{Z}} PR) \neq 0$.)

Proof of 10.1 *for* $R = \mathbb{Z}$. Write $A = \mathbb{Z}[\pi]$. Indicate reduction mod 2 by a double prime; thus $A'' = Z_2[\pi]$ and $C'' = C \otimes_{\mathbb{Z}} Z_2$ for C in $\mathscr{C}_A$.

Five cohomology theories on $\mathscr{C}_A$ will be needed, namely

$$TH_1: C \mapsto \{\hat{Q}^{-n}(C''^{-*}) \mid n \in \mathbb{Z}\},$$
$$TH_2: C \mapsto \{\prod_{k \in \mathbb{Z}} H^{k+n}(C; \hat{H}^k(Z_2; A'')) \mid n \in \mathbb{Z}\},$$
$$TH_3: C \mapsto \{\hat{Q}^{-n}(C^{-*}) \mid n \in \mathbb{Z}\},$$
$$TH_4: C \mapsto \{\prod_{k \in \mathbb{Z}} H^{k+n}(C; \hat{H}^k(Z_2; A)) \mid n \in \mathbb{Z}\},$$
$$TH_5: C \mapsto \{\prod_{k \in \mathbb{Z}} H^{k+n}(C; \hat{H}^{k-1}(Z_2; A)) \mid n \in \mathbb{Z}\}.$$

The canonical direct sum decomposition

$$\hat{H}^k(Z_2; A'') \cong \hat{H}^k(Z_2; A) \oplus \hat{H}^{k-1}(Z_2; A), \quad \text{for } k \in \mathbb{Z},$$

gives a canonical isomorphism

(i) $$TH_2 \cong TH_4 \oplus TH_5.$$

That part of 10.1 which has been proved gives an identification

(ii) $$TH_1 \cong TH_2.$$

Let

(iii) $$TH_3 \oplus TH_5 \to TH_1 \cong TH_2$$

be the map of cohomology theories which on the first direct summand is the obvious reduction mod 2, $TH_3 \to TH_1$; and which on the second summand is the inclusion $TH_5 \hookrightarrow TH_2 \cong TH_1$ of (i) just above. The map (iii) is also an isomorphism of cohomology theories, because it induces an isomorphism on coefficient groups. Combining (i), (ii), and (iii), we obtain a commutative diagram

$$\begin{tikzcd}
 & TH_5 \arrow[dl, hook'] \arrow[dr, hook] & \\
TH_3 \oplus TH_5 \arrow[dr, "(iii)"', "\cong"] & & TH_4 \oplus TH_5 \arrow[dl, "(ii),\,(i)", "\cong"'] \\
 & TH_1 &
\end{tikzcd}$$

showing that $TH_3 \cong TH_1/TH_5 \cong TH_4$. So there is a canonical isomorphism $TH_3 \cong TH_4$, as required.

10.10. REMARK. The preceding proof shows that TH_3 and TH_4 are isomorphic by showing that both are direct summands of TH_1, with common complement $TH_5 \subset TH_1$. If $w: \pi \to Z_2$ is trivial, then it is easy to see that TH_3 and TH_4 are in fact identical as direct summands of TH_1.

10.11. PROPOSITION. *The isomorphism in* 10.1 *is compatible with ring structures if* $R = Z_2$.

Explanation. Write $A = Z_2[\pi]$. For any space X with principal π-bundle α, define a ring structure on $\hat{Q}^0(C(\tilde{X})''^{-*})$ as in 9.10, where $C(\tilde{X})'' = C(\tilde{X}) \otimes_{\mathbb{Z}} Z_2$. Under the isomorphism

$$\hat{Q}^0(C(\tilde{X})''^{-*}) \cong \prod_{k \in \mathbb{Z}} H^k(C(\tilde{X}); \hat{H}^k(Z_2; A))$$

of 10.1, this corresponds to the 'ordinary ring structure' on

$$\prod_{k \in \mathbb{Z}} H^k(C(\tilde{X}); \hat{H}^k(Z_2; A)).$$

To understand what 'ordinary ring structure' means, note that pointwise multiplication μ makes $\hat{H}^0(Z_2; A)$, the set of functions from $\mathrm{tor}_2(\pi)$ to Z_2, into an A-algebra; that is,

$$\mu: \hat{H}^0(Z_2; A) \otimes_{\mathbb{Z}} \hat{H}^0(Z_2; A) \to \hat{H}^0(Z_2; A)$$

is an A-module chain map (with the diagonal A-action on $\hat{H}^0(Z_2; A) \otimes_{\mathbb{Z}} \hat{H}^0(Z_2; A)$; see the text preceding 10.6). Therefore $\{\hat{H}^k(Z_2; A) \mid k \in \mathbb{Z}\}$ is a graded A-algebra, since $\hat{H}^k(Z_2; A) \cong \hat{H}^0(Z_2; A)$.

Sketch proof of 10.11. The cohomology theory under consideration is periodic with period 1, and so 10.11 can be reduced to the claim below.

Let PR be as in 10.2, and let $y_k \in \mathscr{P}_k\hat{Q}^0(PR^{-*})$ be as in 10.9. Then $PR \otimes_{\mathbb{Z}} PR$ is, a priori, an $A \otimes_{\mathbb{Z}} A$-module chain complex, and $y_k \times y_k \in \mathscr{P}_k\hat{Q}^0((PR \otimes_{\mathbb{Z}} PR)^{-*})$ has to be interpreted accordingly.

Using the 'ad hoc transfer' tr associated with the diagonal inclusion $\pi \subset \pi \times \pi$, regard $PR \otimes_{\mathbb{Z}} PR$ as an A-module chain complex (see 9.10 and its proof); $\mathrm{tr}(y_k \times y_k)$ is then an element of

$$\mathscr{P}_k\hat{Q}^0_A((PR \otimes_{\mathbb{Z}} PR)^{-*}),$$

(the subscript A indicates that everything takes place over A, not $A \otimes_{\mathbb{Z}} A$).

Now let $\mu_{\mathrm{res}}: PR \otimes_{\mathbb{Z}} PR \to PR$ be the chain map of A-module chain complexes whose induced homomorphism in 0-dimensional homology is the multiplication

$$\begin{array}{ccc} \mu: \hat{H}^0(Z_2; A) \otimes_{\mathbb{Z}} \hat{H}^0(Z_2; A) & \longrightarrow & \hat{H}^0(Z_2; A) \\ \wr\| & & \wr\| \\ H_0(PR \otimes_{\mathbb{Z}} PR) & & H_0(PR) \end{array}$$

CLAIM. $\mu_{\mathrm{res}}^{\leftarrow}(y_k) = \mathrm{tr}(y_k \times y_k)$ *in* $\mathscr{P}_k\hat{Q}^0_A((PR \otimes_{\mathbb{Z}} PR)^{-*})$.

(To see how the claim implies 10.11, suppose that

$$\left.\begin{array}{l} u \in H^i(C(\tilde{X})''; \hat{H}^i(Z_2; A)) \\ v \in H^j(C(\tilde{X})''; \hat{H}^j(Z_2; A)) \end{array}\right\} \subset \prod_{k \in \mathbb{Z}} H^k(C(\tilde{X})''; \hat{H}^k(Z_2; A)) \cong \hat{Q}^0(C(\tilde{X})''^{-*});$$

write down the two definitions of $u \cdot v$, and compare them.)

Proof of claim. Clearly

$$\operatorname{socle}(\overleftarrow{\mu}_{\text{res}}(y_k)) = \operatorname{socle}(\operatorname{tr}(y_k \times y_k)).$$

(cf. 10.4). Further, it is possible to define a homotopy commutative, homotopy associative diagonal chain map of A-module chain complexes

$$PR \otimes_{\mathbb{Z}} PR \to (PR \otimes_{\mathbb{Z}} PR) \otimes_{\mathbb{Z}} (PR \otimes_{\mathbb{Z}} PR)$$

(imitating 10.6), and thereby a ring structure on $\mathscr{P}_k \hat{Q}^0_A((PR \otimes_{\mathbb{Z}} PR)^{-*})$, such that

(i) $\overleftarrow{\mu}_{\text{res}}\colon \mathscr{P}_k \hat{Q}^0(PR^{-*}) \to \mathscr{P}_k \hat{Q}^0_A((PR \otimes_{\mathbb{Z}} PR)^{-*})$ is a ring homomorphism,

(ii) $\operatorname{tr}(y_k \times y_k) \in \mathscr{P}_k \hat{Q}^0_A((PR \otimes_{\mathbb{Z}} PR)^{-*})$ is idempotent.

Summarizing, we see that $\overleftarrow{\mu}_{\text{res}}(y_k)$ and $\operatorname{tr}(y_k \times y_k)$ are both idempotent, and have the same socle; so they are equal, by the argument used in the proof of 10.10.

10.12. REMARK. The analogue of 10.11 for $A = \mathbb{Z}[\pi]$ makes sense and is correct if $w\colon \pi \to Z_2$ is trivial. This follows from 10.10.

Let X be a finite CW-space with principal π-bundle α, and write $A = \mathbb{Z}[\pi]$ (equipped with the w-twisted involution for some $w\colon \pi \to Z_2$), $A'' = Z_2[\pi]$, etc. The group homomorphism $\{1\} \to \pi$ induces

(i) a map $Z_2 = Z_2[\{1\}] \to Z_2[\pi] = A''$ of group rings,

(ii) by (i), a map $Z_2 \cong \hat{H}^k(Z_2; Z_2) \to \hat{H}^k(Z_2; A'')$,

(iii) by (ii), a map

$$\Gamma''_X\colon \prod_{k \in \mathbb{Z}} H^k(X; Z_2) \to \prod_{k \in \mathbb{Z}} H^k(C(\tilde{X}); \hat{H}^k(Z_2; A'')) \cong \hat{Q}^0(C(\tilde{X})''^{-*}).$$

Since (i) is canonically split as a map of $\mathbb{Z}[Z_2]$-modules, (ii) and (iii) are also split. Working over $A = \mathbb{Z}[\pi]$, one obtains similarly a split inclusion

$$\Gamma_X\colon \prod_{k \in \mathbb{Z}} H^{2k}(X; Z_2) \to \hat{Q}^0(C(\tilde{X})^{-*}).$$

10.13. PROPOSITION. *Suppose that π, w, X, α form part of a string of data $(\pi, w; X, \gamma; \alpha, j)$ as in* I, 3.4. *Then*

$$[c(\gamma)] = \Gamma_X(v_0, v_2, v_4, \ldots)$$

in $\hat{Q}^0(C(\tilde{X})^{-*})$; here $v_i \in H^i(X; Z_2)$ is the ith Wu class of γ. The corresponding formula over $A'' = Z_2[\pi]$ is

$$[c''(\gamma)] = \Gamma''_X(v_0, v_1, v_2, v_3, \ldots)$$

in $\hat{Q}^0(C(\tilde{X})''^{-*})$. (In this case, w and j can be omitted from the string.)

Sketch proof. Over A'', the formula is correct for the trivial spherical fibration γ, and hence correct for arbitrary γ because of 9.6. To get the formula over A, note that $[c(\gamma)] \in \hat{Q}^0(C(\tilde{X})^{-*})$ maps to $[c''(\gamma)] \in \hat{Q}^0(C(\tilde{X})''^{-*})$ under the reduction mod 2 map $\hat{Q}^0(C(\tilde{X})^{-*}) \to \hat{Q}^0(C(\tilde{X})''^{-*})$. On the other hand, the identification

$$\hat{Q}^0(C(\tilde{X})^{-*}) \cong \prod_{k \in \mathbb{Z}} H^k(C(\tilde{X}); \hat{H}^k(Z_2; A))$$

was defined as the composite

$$\begin{tikzcd} \hat{Q}^0(C(\tilde{X})^{-*}) \arrow[r] & \hat{Q}^0(C(\tilde{X})''^{-*}) \cong \prod H^k(C(\tilde{X}); \hat{H}^k(Z_2; A'')) \arrow[d, "f"] \\ & \prod H^k(C(\tilde{X}); \hat{H}^k(Z_2; A)) \end{tikzcd}$$

where f is induced by the projections $\hat{H}^k(Z_2; A'') \to \hat{H}^k(Z_2; A)$. Hence the only element in $\hat{Q}^0(C(\tilde{X})^{-*})$ which maps to $\Gamma''_X(v_0, v_1, v_2, \ldots) \in \hat{Q}^0(C(\tilde{X})''^{-*})$ is

$$f(\Gamma''_X(v_0, v_1, v_2, \ldots)) = \Gamma_X(v_0, v_2, v_4, \ldots).$$

This completes the proof.

11. *Miscellany*

This section contains two distinct illustrations of the theory. The first is related to the 'generalized Kervaire invariants' of [1] and [2], and the second is a not-so-new proof of Browder's theorem [1] on the Kervaire invariant (which sheds light on the results of § 10, but not on Browder's theorem).

We shall work with CW-spaces instead of simplicial sets in this section; see the remark after I, 3.A.4.

Generalized Kervaire invariants. Here the ring with involution is fixed: $A = Z_2$, to be regarded as the group ring $Z_2[\{1\}]$ of the trivial group. If X is a finite CW-space, its algebraic counterpart for the time being is $C(X) \otimes_{\mathbb{Z}} Z_2$; any spherical fibration γ on X determines a chain bundle on $C(X) \otimes_{\mathbb{Z}} Z_2$. No orientation is needed.

Most computational problems evaporate upon observing that every chain complex in $\mathscr{C}_A$ is homotopy equivalent to one with zero differential (its homology). For example, using the fact that the functors $\hat{Q}^n(-)$ commute with direct sums, we obtain directly (i.e. without using 10.1):

11.1. PROPOSITION. *There is a natural isomorphism*

$$\hat{Q}^n(C^{-*}) \cong \prod_{k \in \mathbb{Z}} H^{k-n}(C; A) \quad \textit{for } C \textit{ in } \mathscr{C}_A.$$

The next proposition, proved in [11], is a special case of 10.13 (but has been used implicitly in the proof of 10.13); it can also be deduced from I, 3.A.

11.2. PROPOSITION. *We have* $[c(\gamma)] = (v_0, v_1, v_2, v_3, \ldots)$ *in*

$$\hat{Q}^0((C(X) \otimes_{\mathbb{Z}} Z_2)^{-*}) \cong \prod_{k \in \mathbb{Z}} H^k(X; Z_2)$$

if γ *is a spherical fibration on* X *with Wu classes* $v_i \in H^i(X; Z_2)$.

11.3. PROPOSITION. (i)

$$L^n(0_A, 0) = L_n(A) \cong \begin{cases} Z_2 & (\textit{Kervaire invariant}) \quad \textit{if } n = 2k, \\ 0 & \qquad\qquad\qquad\qquad\quad \textit{if } n = 2k+1. \end{cases}$$

(ii) *Suppose that the* $(k+1)$*th Wu class of* γ *(in* $H^{k+1}(X; Z_2)$*) is zero (with* X *and* γ *as in* 11.2). *Then*

$$\text{release: } L_{2k}(A) \to L^{2k}(C(X) \otimes_{\mathbb{Z}} Z_2, c(\gamma))$$

is injective.

Proof. (i) is easy; (ii) follows from 8.3. The converse of (ii) also holds.

(*Warning*: the groups $L^{2k}(C(X) \otimes_{\mathbb{Z}} Z_2, c(\gamma))$ are not in general Z_2-vector spaces, even though we are working with $A = Z_2$.)

In view of 11.3 (ii), we could call the homomorphism

$$\text{flexible signature: } \Omega^P_{2k}(X, \gamma) \to L^{2k}(C(X) \otimes_{\mathbb{Z}} Z_2, c(\gamma))$$

a 'generalized Kervaire invariant', at least if the $(k+1)$th Wu class of γ vanishes. (See 7.1 for notation.)

For clarification, suppose that d is any ('abstract') chain bundle on a chain complex D in $\mathscr{C}_A$; and that, for some $k \geqslant 0$, the '$(k+1)$th Wu class of d' is zero. (This makes sense by 11.1.)

Given a $2k$-dimensional algebraic Poincaré complex (C, φ) over A, with d-structure, can we imitate [2] to obtain a 'quadratic form with values in Z_4' on $H^k(C; A)$?

The answer is 'yes' (certain choices are, however, necessary, just as in [2]). The following examples constitute a sketch proof.

Example 1. Take $D = (Z_2, k)$ (that is, $D_k = Z_2$, $D_r = 0$ for $r \neq k$); of the two chain bundles on D, let $d(k)$ be the non-trivial one. Certainly the $(k+1)$th Wu class of $d(k)$ is zero. If (C, φ) is a $2k$-dimensional algebraic Poincaré complex in $\mathscr{C}_A$, then $H^k(C; A)$ carries a non-degenerate symmetric bilinear form. It is not hard (but highly amusing) to see that a $d(k)$-structure on (C, φ) (as in I, 2.6) determines an enhancement of the bilinear form to a quadratic form—with values in a group isomorphic to Z_4, as required.

Example 2. Let m be a large integer; put

$$D^u = \bigoplus_{\substack{-m<r<m \\ r \neq k+1}} (Z_2, r) \quad \text{and} \quad d^u = \bigoplus_{\substack{-m<r<m \\ r \neq k+1}} d(r)$$

(in the notation of the previous example). If (C, φ) is a $2k$-dimensional algebraic Poincaré complex with d^u-structure, then $H^k(C; A)$ carries a quadratic form with values in a group isomorphic to Z_4 (same proof as before).

Example 3. The general case. Let d be a chain bundle on D whose $(k+1)$th Wu class is zero; then there exists a chain map $D \to D^u$ (see Example 2) covered by a chain bundle map $d \to d^u$. (N.B. m is large.) Choose such a chain map; then any d-structure on an algebraic Poincaré complex over A determines a d^u-structure, and we obtain the desired quadratic forms from Example 2.

(There is a one-to-one correspondence between the choices used here—that is, homotopy classes of chain bundle maps from d to d^u—and the choices used in [2], if $D = C(X) \otimes_{\mathbb{Z}} Z_2$ and $d = c(\gamma)$ for a space X with spherical fibration γ.)

Browder's theorem on the Kervaire invariant. The theorem in question states that the Kervaire invariant $\pi^s_{2k} \to Z_2$ (defined for arbitrary $k > 0$, but interesting only when k is odd) is zero if k is not of the form $2^p - 1$ for some integer $p > 0$.

Let $X = \mathbb{R}P^m$, let γ be the trivial vector bundle on X, let $X'' = S^m$ be the standard twofold cover of X, and let γ'' be the trivial vector bundle on X''; take m large.

Transfer gives a map of bordism groups

$$\pi_n(M(X, \gamma)) \to \pi_n(M(X'', \gamma'')) = \pi^s_n.$$

(We assume that $n < m$; on the left is the bordism group of 'framed n-manifolds with twofold covers', and on the right is the bordism group of framed n-manifolds. The transfer assigns to a 'framed manifold with twofold cover' its twofold cover.)

The celebrated theorem of Kahn and Priddy [9] implies that this transfer is surjective for $1 \leqslant n < m$. (It is also known that the Kahn–Priddy theorem has the Browder theorem above among its corollaries; see [7] and [8]. That is why I have apostrophized the argument below as 'not-so-new'.)

Put $\pi = Z_2$; then $A = \mathbb{Z}[\pi]$ is a ring with involution (coming from the trivial homomorphism $w\colon \pi \to Z_2$, not from the identity). Let α be the non-trivial principal π-bundle on X, and let $j\colon w^{\rightarrow}(\alpha) \cong$ (orientation cover of γ) be the standard identification.

Apply I, 3.4: the result is a chain bundle $c(\gamma)$ on

$$C(\tilde{X}) = \dots \xrightarrow{1-T} \mathbb{Z}[Z_2] \xrightarrow{1+T} \mathbb{Z}[Z_2] \xrightarrow{1-T} \mathbb{Z}[Z_2]$$

whose homology class we wish to describe explicitly. Recall the isomorphism $\hat{Q}^0(C(\tilde{X})^{-*}) \cong H_0(V(C(\tilde{X})))$ of I, 1.6.

11.4. Proposition. *The class* $[c(\gamma)] \in \hat{Q}^0(C(\tilde{X})^{-*}) \cong H_0(V(C(\tilde{X})))$ *is represented by the* 0-*cycle* $\{\lambda_r\} \in V(C(\tilde{X}))$ *with*

$$\lambda_r = \begin{cases} (1) & \text{if } r = 0, \\ (1+(-)^r T) & \text{if } r = 2^p \text{ for some } p \geqslant 0, \\ (0) & \text{otherwise.} \end{cases}$$

(Sesquilinear forms on $C(\tilde{X})_r$ are identified, for $r \geqslant 0$, with 1×1-matrices with coefficients in A; apart from that, the notation of I, 1.4 has been used.)

Theoretically, 11.4 can be verified using 10.1 and 10.13. However, the isomorphism in 10.1 is very mysterious. A geometric proof of 11.4 is given below (after 11.7).

11.5. Corollary. *The algebraic transfer*

$$L^n(C(\tilde{X}), c(\gamma)) \to L^n(C(X''), c(\gamma''))$$

is zero if $n = 2k < m$, *for* k *odd with* $k \neq 2^p - 1$.

Explanation and proof. The transfer is associated with the inclusion

$$\{1\} \hookrightarrow Z_2 = \pi.$$

Using, for example, 6.1, one finds, for $m > n > 0$, that

$$L^n(C(X''), c(\gamma'')) = \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 \bmod 4 \text{ ('signature/8')}, \\ Z_2 & \text{if } n \equiv 1 \text{ or } 3 \bmod 4 \text{ ('Hopf invariant')}, \\ Z_2 & \text{if } n \equiv 2 \bmod 4 \text{ ('Kervaire invariant')}. \end{cases}$$

For the proof of 11.5, apply 8.2 and 8.3: there is a commutative diagram

$$\begin{tikzcd} L_n(\mathbb{Z}[\{1\}] \arrow[r] & L^n(C(X''), c(\gamma'')) \\ L_n(\mathbb{Z}[\pi]) \arrow[r] \arrow[u, "\text{transfer}"] & L^n(C(\tilde{X}), c(\gamma)) \arrow[u, "\text{transfer}"] \end{tikzcd}$$

If $n = 2 \bmod 4$, the top horizontal arrow is isomorphic and the left vertical arrow is zero (by [**13**, Chapter 13A]). If, moreover, $n \neq 2(2^p - 1)$, then 8.2 and 8.3 imply that the transfer on the right is also zero.

11.6. Corollary. *If $n = 2k$ is as in* 11.5, *then the Kervaire invariant $\pi^s_{2k} \to Z_2$ is zero.*

Proof. There is a commutative diagram

$$\begin{tikzcd} \pi^s_{2k} = \pi_{2k}(M(X'', \gamma'')) \arrow[r] & L^{2k}(C(X''), c(\gamma'')) \cong Z_2 \\ \pi_{2k}(M(X, \gamma)) \arrow[r] \arrow[u, "\text{geometric transfer}"'] & L^{2k}(C(\tilde{X}), c(\gamma)) \arrow[u, "\text{algebraic transfer}"] \end{tikzcd}$$

in which
the left vertical arrow is surjective (Kahn–Priddy),
the right vertical arrow is zero (by 11.5),
so that the horizontal arrow (which is the Kervaire invariant) is also zero.

To conclude the chapter, here is the geometric proof of 11.4. The idea and construction are based on I, §3.A; so we shall produce a sequence $\{P^n \mid n \geqslant 0\}$ of framed manifolds (each P^n having the homotopy type of the $[\frac{1}{2}n]$-skeleton of $\mathbb{R}P^m$, etc.) such that the sesquilinear sliding forms λ_r (cf. I, 3.A.2(ii)) are as specified in 11.4. We use the standard CW-structure on $\mathbb{R}P^m$.

Suppose that $P^0, P^1, \ldots, P^{2k}$ have already been constructed so that the sliding forms $\lambda_0, \lambda_1, \ldots, \lambda_k$ are the required ones. (Assume that $m \geqslant 2k > 2$, otherwise there is little to prove.)

Claim. *Let z be a generator of $\pi_k(P^{2k}) \cong \pi_k(\mathbb{R}P^k) \cong \mathbb{Z}$. Because of Hirsch's immersion theorem* [**6**, **5**], *and because P^{2k} is framed, z determines a regular homotopy class of immersions i_k: $S^k \to P^{2k}$. The self-intersection number $\mu(i_k)$ of this immersion equals $1 + T$ if $k = 2^p - 1$ for some integer $p > 1$, and 0 otherwise. (It belongs to $\mathbb{Z}[Z_2]$ if k is even, to $Z_2[Z_2]$ if k is odd.)*

Assuming the truth of the claim, we can easily construct framed manifolds P^{2k+1}, P^{2k+2}, etc. giving the correct sliding form λ_{k+1}.

To prove the claim, we can suppose that k is odd. (For even k, there is nothing to prove because $\mu(i_k)$ is algebraically determined by the sliding form λ_k: we have $2\mu(i_k) = \lambda_k(1-T, 1-T)$, since i_k represents the element $1-T$ in $C(\tilde{X})_k = \mathbb{Z}[Z_2]$.)

Since k is odd, there are just two regular homotopy classes of immersions $S^k \to P^{2k}$ homotopic to i_k; if i_k has self-intersection number $a \cdot 1 + b \cdot T$ (with $a, b \in Z_2$), then the immersions in the class not containing i_k have self-intersection number $(a+1) \cdot 1 + b \cdot T$.

Let i'_k: $S^k \to P^{2k}$ be an immersion which factors as follows:

$$S^k \xrightarrow{\text{double cover}} \mathbb{R}P^k \xrightarrow{\quad g \quad} P^{2k},$$

where g is an immersion and also a homotopy equivalence. Arguing exactly as in [3], one finds that the self-intersection number of i'_k is $0 \cdot 1 + 1 \cdot T \in Z_2[Z_2]$ if $k = 2^p - 1$, and 0 otherwise.

Let f: $P^{2k} \to \mathbb{R}^{2k}$ be the codimension-0 immersion determined, up to regular homotopy, by the framing of P^{2k}. Then the immersion $f \cdot i'_k$: $S^k \to \mathbb{R}^{2k}$ has self-intersection number $1 \in Z_2$ if $k = 2^p - 1$, and 0 otherwise (also by [3]). From the definition of i_k, it is clear that $f \cdot i_k$: $S^k \to \mathbb{R}^{2k}$ has self-intersection number 0 for all k, being regularly homotopic to the standard embedding. Therefore i_k and i'_k are regularly homotopic if and only if $k \neq 2^p - 1$.

Putting these observations together proves the claim.

References

1. W. Browder, 'The Kervaire invariant of framed manifolds and its generalisations', *Ann. of Math.*, 90 (1969), 157–186.
2. E. H. Brown, Jr., 'Generalisations of the Kervaire invariant', *Ann. of Math.*, 95 (1972), 368–383.
3. E. H. Brown, Jr., 'A remark concerning immersions of S^n in $\mathbb{R}^{2n}$', *Quart. J. Math. Oxford* (2), 24 (1973), 559–560.
4. A. Dold, *Halbexakte Homotopiefunktoren*, Lecture Notes in Mathematics 12 (Springer, Berlin, 1966).
5. A. Haefliger and V. Poenaru, 'La classification des immersions combinatoires', Publications Mathématiques 23 (Institut des Hautes Études Scientifiques, Paris, 1964), pp. 75–91.
6. M. Hirsch, 'Immersions of manifolds', *Trans. Amer. Math. Soc.*, 93 (1959), 242–276.
7. J. Jones and E. Rees, 'Kervaire's invariant for framed manifolds', Proceedings of Symposia 32 (American Mathematical Society, Providence, R.I., 1978), pp. 141–147.
8. J. Jones and E. Rees, 'A note on the Kervaire invariant', *Bull. London Math. Soc.*, 7 (1975), 279–282.
9. D. Kahn and S. Priddy, 'Applications of the transfer to stable homotopy theory', *Bull. Amer. Math. Soc.*, 78 (1972), 981–987.
10. F. Quinn, 'Surgery on Poincaré and normal spaces', *Bull. Amer. Math. Soc.*, 78 (1972), 262–267.
11. A. Ranicki, 'The algebraic theory of surgery I, II', *Proc. London Math. Soc.*, (3), 40 (1980), 87–283.
12. A. Ranicki, *Exact sequences in the algebraic theory of surgery*, Mathematics Notes (Princeton University Press, 1981).
13. C. T. C. Wall, *Surgery on compact manifolds* (Academic Press, London, 1970).
14. M. Weiss, 'Surgery and the generalized Kervaire invariant, I', *Proc. London Math. Soc.* (3), 51 (1985), 146–192.

Universität Bielefeld
Fakultät für Mathematik
4800 *Bielefeld* 1
West Germany