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# WHAT DOES THE CLASSIFYING SPACE OF A CATEGORY CLASSIFY?

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(*communicated by Graham Ellis*)

*Abstract*

The classifying space of a small category classifies sheaves whose values are contravariant functors from that category to sets and whose stalks are representable.

## Introduction

Let $\mathcal{C}$ be a small category. Contravariant functors from $\mathcal{C}$ to the category of sets, and natural transformations between them, will be called *$\mathcal{C}$-sets* and *$\mathcal{C}$-maps*, respectively. The category of $\mathcal{C}$-sets shares many good properties with the category of sets. (In short, it is a *topos*. See [**4**] or [**7**]. Here we will not make any explicit use of this fact.) The $\mathcal{C}$-sets which are of the form $b \mapsto \mathrm{mor}_{\mathcal{C}}(b, c)$ for fixed $c \in \mathcal{C}$, and any isomorphic ones, are called *representable*. By the Yoneda lemma, the representable $\mathcal{C}$-sets form a full subcategory of the category of all $\mathcal{C}$-sets which is equivalent to $\mathcal{C}$.

We will be concerned with *sheaves* of $\mathcal{C}$-sets on a topological space $X$. For such a sheaf, and $x \in X$, the *stalk* $\mathscr{F}_x$ is again a $\mathcal{C}$-set. It is the direct limit of the $\mathcal{C}$-sets $\mathscr{F}(U)$ where $U$ runs through the open neighborhoods of $x$.

**Theorem 0.1.** *The classifying space $B\mathcal{C}$ classifies sheaves of $\mathcal{C}$-sets with representable stalks.*

**Notation, terminology, clarifications.**

Let $\mathscr{F}$ be any sheaf of $\mathcal{C}$-sets on $X$. We may regard $\mathscr{F}$ as a contravariant functor $(c, U) \mapsto \mathscr{F}^{(c)}(U)$ in two variables (where $c \in \mathrm{ob}(\mathcal{C})$ and $U$ is open in $X$). Specializing one of the variables, we obtain $\mathscr{F}^{(c)}$, a sheaf of sets on $X$, and $\mathscr{F}(U)$, a $\mathcal{C}$-set.

Let $\mathscr{L}$ be any sheaf of sets on $X$. The *espace étalé* of $\mathscr{L}$, denoted $\mathrm{Spé}(\mathscr{L})$, is the (disjoint) union of the stalks $\mathscr{L}_x$, suitably topologized. See [**2**, II.1, ex.1.13] for details. The sheaf $\mathscr{L}$ can be identified with the sheaf of continuous (partial) sections of the projection $\mathrm{Spé}(\mathscr{L}) \to X$.

The projection $\mathrm{Spé}(\mathscr{L}) \to X$ is an étale map alias *local homeomorphism*. [But it happens frequently that $X$ is Hausdorff while $\mathrm{Spé}(\mathscr{L})$ is not.] The construction $\mathrm{Spé}(\mathscr{L})$ leads to a good notion of *pullback* of sheaves: for a map $v\colon Y \to X$, the

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pullback $v^*\mathscr{L}$ is defined in such a way that $\mathrm{Spé}(v^*\mathscr{L}) = v^*\mathrm{Spé}(\mathscr{L})$. More generally, for a sheaf $\mathscr{F}$ of $\mathcal{C}$-sets on $X$ and $v: Y \to X$, the pullback $v^*\mathscr{F}$ is defined in such a way that $\mathrm{Spé}((v^*\mathscr{F})^{(c)}) = v^*\mathrm{Spé}(\mathscr{F}^{(c)})$.

Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves of $\mathcal{C}$-sets on $X$, both with representable stalks. Let $e_0, e_1: X \to X \times [0,1]$ be given by $e_0(x) = (x, 0)$ and $e_1(x) = (x, 1)$. The sheaves $\mathscr{F}$ and $\mathscr{G}$ are *concordant* if there exists a sheaf of $\mathcal{C}$-sets $\mathscr{H}$ on $X \times [0,1]$, again with representable stalks, such that $e_0^*\mathscr{H} \cong \mathscr{F}$ and $e_1^*\mathscr{H} \cong \mathscr{G}$.

The precise meaning of theorem 0.1 is as follows. Suppose that $X$ has the homotopy type of a CW-space. There is a natural bijection from the homotopy set $[X, B\mathcal{C}]$ to the set of concordance classes of sheaves of $\mathcal{C}$-sets on $X$ with representable stalks.

*Remark.* Suppose that $\mathcal{C}$ is a group. To be more precise, suppose that $\mathcal{C}$ has just one object $c$ and $\mathrm{mor}(c,c)$ is a group. Let $\mathscr{F}$ be a sheaf of $\mathcal{C}$-sets on a space $X$. If the stalks of $\mathscr{F}$ are all representable, then the projection $\mathrm{Spé}(\mathscr{F}) \to X$ is a principal $\mathrm{mor}(c,c)$-bundle. Indeed any choice of an open $U$ and $s \in \mathscr{F}^{(c)}(U)$ determines a bundle chart

$$\mathrm{Spé}(\mathscr{F}|U) \cong \mathrm{mor}(c,c) \times U\,.$$

In this situation, concordant sheaves of $\mathcal{C}$-sets on $X$ (with representable stalks) are isomorphic, because “concordant” implies “isomorphic” for principal $\mathrm{mor}(c,c)$-bundles.

*Remark.* The question in the title undoubtedly has many correct answers and a few have already been given elsewhere. Moerdijk [**7**, Introd.] has a result like theorem 0.1 in which the representability condition on stalks is replaced by a weaker condition, that of being *principal*. To explain what a principal $\mathcal{C}$-set is, we start with the following standard definitions:

- The *transport category* of a $\mathcal{C}$-set $S$ has objects $(c, x)$ where $c$ is an object of $\mathcal{C}$ and $x \in S(c)$. A morphism from $(c, x)$ to $(d, y)$ is a morphism $g: c \to d$ in $\mathcal{C}$ such that the induced map $S(g): S(d) \to S(c)$ takes $y$ to $x$. (Some people would call this the opposite of the transport category of $S$.)
- A category $\mathcal{D}$ is *filtered* if
  - it has at least one object;
  - for any two objects $d_1$, $d_2$ in $\mathcal{D}$ there exists another object $d_3$ and morphisms $d_1 \to d_3$, $d_2 \to d_3$;
  - for any two morphisms in $\mathcal{D}$ with the same source and target, say $f, g: a \to b$, there exists a coequalizer (a morphism $h: b \to c$ in $\mathcal{D}$ such that $hf = hg$).

Now a $\mathcal{C}$-set is *principal* if its transport category is filtered. (This is not exactly the terminology which Moerdijk uses. He calls a sheaf of $\mathcal{C}$-sets on $X$ a *principal $\mathcal{C}^{\mathrm{op}}$-bundle* if the transport category of each stalk, as defined above, is filtered.) A representable $\mathcal{C}$-set $S$ is certainly principal, since the transport category of $S$ has a terminal object. The converse does not hold.

For example, suppose that $\mathcal{C}$ itself is a filtered category which does not have a terminal object. Define a $\mathcal{C}$-set $S$ in such a way that $S(c)$ has exactly one element, for every object $c$ in $\mathcal{C}$. Then the transport category of $S$ is equivalent to $\mathcal{C}$, so $S$ is principal. But $S$ is not representable, since a representing object would be a terminal object for $\mathcal{C}$. It follows that there exist sheaves of $\mathcal{C}$-sets on some spaces $X$ which do not satisfy the condition of theorem 0.1 but which are principal $\mathcal{C}^{\mathrm{op}}$-bundles according to Moerdijk's definition. (Take $X$ to be a point.)

Moerdijk [**7**] takes the discussion much further by considering topological categories, which I have not attempted to do.

Another precursor of theorem 0.1 is due to tom Dieck (1972, unpublished). He used a notion of $\mathcal{C}$*-bundle* defined in terms of a bundle atlas. His result was rediscovered in [**5**, thm 4.1.2].

Theorem 0.1 is anticipated and illustrated to some extent in [**6**].

## 1. The canonical sheaf on $B\mathcal{C}$

We are going to construct a sheaf $\mathscr{E}$ of $\mathcal{C}$-sets on $B\mathcal{C}$ which will eventually turn out to be "universal". Recall to begin with that $B\mathcal{C}$ is the geometric realization of the simplicial set whose $n$-simplices are the contravariant functors $[n] \to \mathcal{C}$, where $[n]$ is the linearly ordered set $\{0, 1, 2, \ldots, n\}$. (There are historical reasons for insisting on *contravariant* functors $[n] \to \mathcal{C}$; the formulae for boundary operators look more familiar in the case where $\mathcal{C}$ is a group or monoid.) Now suppose that $U$ is open in $B\mathcal{C}$ and $c$ is an object of $\mathcal{C}$.

**Definition 1.1.** An element of $\mathscr{E}^{(c)}(U)$ is a "function" which to every $\alpha\colon [n]^{\mathrm{op}} \to \mathcal{C}$ and $x \in B[n]^{\mathrm{op}} \cong \Delta^n$ with $\alpha_* x \in U$ assigns a morphism $s(\alpha, x)\colon c \to \alpha(0)$ in $\mathcal{C}$. The function is required to be

- locally constant in the second variable, so that for $y \in \Delta^n$ sufficiently close to $x$, with $\alpha_* y \in U$, we have $s(\alpha, y) = s(\alpha, x)$;
- natural in the first variable. That is, for an order-preserving $g\colon [m] \to [n]$ and $y \in \Delta^m$, we have
$$s(\alpha, g_* y) = \alpha(0, g(0)) \circ s(\alpha g, y)$$
where $\alpha(0, g(0))\colon \alpha(g(0)) \to \alpha(0)$ is the morphism in $\mathcal{C}$ induced by the unique morphism $0 \to g(0)$ in $[n]$.

The contravariant dependence of $\mathscr{E}^{(c)}(U)$ on $U$ and $c$ is obvious. The sheaf property is also obvious. Because of the naturality condition, an element $s$ of $\mathscr{E}^{(c)}(U)$ is determined by its values $s(\alpha, x)$ for *nondegenerate* $\alpha\colon [n]^{\mathrm{op}} \to \mathcal{C}$ and $x \in \Delta^n \smallsetminus \partial\Delta^n$. Then $\alpha$ and $x$ are determined by $\alpha_* x \in U$; in particular $n$ is the dimension of the cell (in the canonical CW-decomposition of $B\mathcal{C}$) to which $\alpha_* x$ belongs. (Regarding *cells*, the convention used here is that the cells of a CW-space are pairwise disjoint, and each cell is homeomorphic to some euclidean space. This is in agreement with [**1**], for example.)

**Lemma 1.2.** *Fix a nondegenerate $\beta\colon [m]^{\mathrm{op}} \to \mathcal{C}$ and $y \in \Delta^m \smallsetminus \partial\Delta^m$. The stalk of $\mathscr{E}$ at $\beta_* y$ is the contravariant functor $c \mapsto \mathrm{mor}(c, \beta(0))$.*

*Proof.* Any point of $B\mathcal{C}$ can be uniquely written as $\alpha_* z$ where $\alpha\colon [n]^{\mathrm{op}} \to \mathcal{C}$ is nondegenerate and $z \in \Delta^n \smallsetminus \partial\Delta^n$. If $\alpha_* z$ is sufficiently close to $\beta_* y$, then some degeneracy of $\beta$ will be a face of $\alpha$. That is, there are an order-preserving surjection $f\colon[k]\to[m]$ and an order-preserving injection $g\colon[k]\to[n]$ such that $\alpha g=\beta f$. And moreover, there will be $w\in\Delta^k$ such that $f_*w=y$ and $z$ is close to $g_*w$. For $s$ in the stalk of $\mathscr{E}^{(c)}$ at $\beta_*y\in B\mathcal{C}$, we then have

$$\begin{aligned} s(\alpha,z)&=s(\alpha,g_*w)=\alpha(0,g(0))\circ s(\alpha g,w)\ ,\\ s(\alpha g,w)&=s(\beta f,w)=s(\beta,f_*w)=s(\beta,y)\ ,\end{aligned}$$

so that $s(\alpha,z)=\alpha(0,g(0))\circ s(\beta,y)$. Hence $s$ is determined by $s(\beta,y)\in\mathrm{mor}(c,\beta(0))$. To establish the existence of a germ $s$ with prescribed value $s(\beta,y)$, we proceed differently. Suppose inductively that the values of $s$ at points near $\beta_*y$ and in the $(n-1)$-skeleton of $B\mathcal{C}$ have already been determined consistently, for some fixed $n>m$. For an $n$-simplex $\alpha\colon[n]^{\mathrm{op}}\to\mathcal{C}$ we have an attaching map from $\partial\Delta^n$ to the $(n-1)$-skeleton of $B\mathcal{C}$. Hence $s(\alpha,x)$ is already determined for $x$ in some open subset $V$ of $\partial\Delta^n$. As a function on $V$, denoted informally $s|V$, it satisfies the continuity and naturality conditions of definition 1.1 (mutatis mutandis). We now have to find an open $W\subset\Delta^n$ such that $V=W\cap\partial\Delta^n$ and an extension of $s|V$ from $V$ to $W$. This is easy. For example, $\Delta^n$ can be identified with a cone on $\partial\Delta^n$ and $W$ could then be defined as the cone on $V$ minus the cone point. Then $s|V$ has a unique extension from $V$ to $W$. $\square$

For an object $c$ of $\mathcal{C}$, let $(c\downarrow\mathcal{C})$ be the "under" category associated with $c$. The objects of $(c\downarrow\mathcal{C})$ are the morphisms in $\mathcal{C}$ with source $c$, and the morphisms of $(c\downarrow\mathcal{C})$ are morphisms in $\mathcal{C}$ *under* $c$. The classifying space $B(c\downarrow\mathcal{C})$ is contractible since $(c\downarrow\mathcal{C})$ has an initial object. The forgetful map $B(c\downarrow\mathcal{C})\to B\mathcal{C}$ has a canonical factorization

$$B(c\downarrow\mathcal{C})\ \xrightarrow{\ \lambda_c\ }\ \mathrm{Sp\acute{e}}(\mathscr{E}^{(c)})\ \xrightarrow{\ \text{proj.}\ }\ B\mathcal{C}\ .$$

Indeed, any point of $B\mathcal{C}$ can be uniquely written as $\alpha_*z$ where $\alpha\colon[n]^{\mathrm{op}}\to\mathcal{C}$ is nondegenerate and $z\in\Delta^n\smallsetminus\partial\Delta^n$. Lifting $\alpha_*z$ to $B(c\downarrow C)$ amounts to specifying a morphism $c\to\alpha(n)$ in $\mathcal{C}$; lifting $\alpha_*z$ to $\mathrm{Sp\acute{e}}(\mathscr{E}^{(c)})$ amounts to specifying a morphism $c\to\alpha(0)$ in $\mathcal{C}$. Clearly a morphism $c\to\alpha(n)$ determines a morphism $c\to\alpha(0)$ by composition with $\alpha(0,n)\colon\alpha(n)\to\alpha(0)$.

The map $\lambda_c$ will be useful in the proof of

**Proposition 1.3.** *The space* $\mathrm{Sp\acute{e}}(\mathscr{E}^{(c)})$ *is weakly contractible.*

*Proof.* Let $\bar{B}\mathcal{C}$ be the fat realization of the nerve of $\mathcal{C}$, obtained by ignoring the degeneracy operators. The quotient map $q\colon\bar{B}\mathcal{C}\to B\mathcal{C}$ is a quasifibration with contractible fibers. To see this, note that the fat realization of any simplicial set $Z$ can be described as the ordinary realization of another simplicial set $\bar{Z}$ whose $n$-simplices are triples $(k,f,x)$ where $x\in Z_k$ and $f\colon[n]\to[k]$ is an order-preserving surjection. The forgetful simplicial map $\bar{Z}\to Z$ is a Kan fibration with contractible fibers; hence the induced map of (lean) geometric realizations, $|\bar{Z}|\to|Z|$, is a quasifibration with contractible fibers. See [**3**].

Let $E=\mathrm{Sp\acute{e}}(\mathscr{E}^{(c)})$, let $r\colon E\to B\mathcal{C}$ be the projection, and let $\bar{E}=q^*E$. In the pullback square

$$\begin{array}{ccc} \bar{E} & \xrightarrow{\bar{r}} & \bar{B} \\ {\scriptstyle r^*q}\downarrow & & \downarrow{\scriptstyle q} \\ E & \xrightarrow{r} & B \end{array}$$

the map $q$ is a quasifibration with contractible fibers and $r$ is a local homeomorphism. It follows that $r^*q$ is a quasifibration with contractible fibers, and consequently a weak homotopy equivalence.

It remains to prove that $\bar{E}$ is contractible, or equivalently, that the canonical map $\bar{\lambda}_c : \bar{B}(c \downarrow C) \to \bar{E}$, the fat version of $\lambda_c$, is a weak homotopy equivalence. Suppose therefore that $Y$ is any finitely generated $\Delta$-set (= "simplicial set without degeneracy operators") and let

$$f : |Y| \to \bar{E}$$

be any map. We want to show that, up to a homotopy, $f$ lifts to $\bar{B}(c \downarrow C)$. The argument has two parts.

(i) *If $\bar{r}f : |Y| \to \bar{B}$ is induced by a map of the underlying $\Delta$-sets, then $f$ admits a unique factorization through $\bar{B}(c \downarrow C)$.*

(ii) *Modulo iterated barycentric subdivision of $|Y|$, and a homotopy of $f$, the composition $\bar{r}f$ is indeed induced by a map of the underlying $\Delta$-sets.*

For the proof of (i), we may assume that $Y$ is generated by a single $n$-simplex, so $|Y| = \Delta^n$. Suppose that $\bar{r}f : \Delta^n \to B\mathcal{C}$ is the characteristic map of an $n$-simplex $\alpha : [n]^{\mathrm{op}} \to \mathcal{C}$ in the nerve of $\mathcal{C}$. The extra information contained in $f$ amounts to compatible morphisms $u_i : c \to \alpha(i)$ for $i = 0, 1, \ldots, n$; clearly all $u_i$ are determined by $u_n$. Together, $u_n$ and $\alpha$ determine an $n$-simplex in the nerve of $(c \downarrow \mathcal{C})$.

For the proof of (ii), we note that the first barycentric subdivision of $\bar{B}\mathcal{C}$ can be described as $\bar{B}\mathcal{C}'$ for another category $\mathcal{C}'$. An object of $\mathcal{C}'$ is a simplex of the nerve of $\mathcal{C}$; a morphism from an $m$-simplex $\alpha$ to an $n$-simplex $\beta$ is an injective order-preserving $v : [m] \to [n]$ with $v^*\beta = \alpha$. The functor $\mathcal{C}' \to \mathcal{C}$ given by $\alpha \mapsto \alpha(0)$ induces a $\Delta$-map from the nerve of $\mathcal{C}$ to the nerve of $\mathcal{C}'$, and then a map

$$\varphi_1 : \bar{B}\mathcal{C} \to \bar{B}\mathcal{C}' .$$

This map is not a homeomorphism. There is of course another (well-known) map $\varphi_0 : \bar{B}\mathcal{C}' \to B\mathcal{C}$ which is a homeomorphism. What is important here is that $\varphi_0$ and $\varphi_1$ are homotopic in an obvious way, by a homotopy $(\varphi_t)_{t \in [0,1]}$. (Each track of the homotopy is a straight line segment, or a single point, in a simplex of $\bar{B}\mathcal{C}$.) The homotopy $(\varphi_t \varphi_0^{-1})_{t \in [0,1]}$, from the identity of $\bar{B}\mathcal{C}$ to $\varphi_1 \varphi_0^{-1}$, has a unique lift to a homotopy

$$(\psi_t : \bar{E} \times [0,1] \to \bar{E})_{t \in [0,1]}$$

with $\psi_0 = \mathrm{id}$. (To verify this claim, compare the pullbacks of $\bar{E}$ under the maps

$$\bar{B}\mathcal{C} \times [0,1] \xrightarrow{\varphi\varphi_0^{-1}} \bar{B}\mathcal{C} , \qquad \bar{B}\mathcal{C} \times [0,1] \xrightarrow{\text{proj.}} \bar{B}\mathcal{C} .$$

They are homeomorphic as spaces over $\bar{B}\mathcal{C} \times [0,1]$.) We can similarly look at iterated barycentric subdivisions of $\bar{B}\mathcal{C}$. They all have two canonical maps $\varphi_0, \varphi_1$ $\bar{B}\mathcal{C}$, one being a homeomorphism and the other being simplicial, and these two maps are homotopic by a homotopy $(\varphi_t)_{t\in[0,1]}$. Again, the homotopy $(\varphi_t\varphi_0^{-1})_{t\in[0,1]}$ has a unique lift to a homotopy $(\psi_t\colon \bar{E} \times [0,1] \to \bar{E})_{t\in[0,1]}$ with $\psi_0 = \mathrm{id}$. Coming back now to maps $|Y| \to \bar{E}$, any such map is homotopic to a map $f$ such that $\bar{r}f$ is induced by a $\Delta$-map from some iterated barycentric subdivision of $Y$ to some iterated barycentric subdivision of the nerve of $\mathcal{C}$. Compose $f$ with $\psi_1$ from the above homotopy. Then $\bar{r}\psi_1 f$ is induced by a $\Delta$-map from $Y$ to the nerve of $\mathcal{C}$. $\square$

## 2. Resolutions

The previous section gives us a method to "convert" a map $f\colon X \to B\mathcal{C}$ into a sheaf of $\mathcal{C}$-sets on $X$ with representable stalks, by $f \mapsto f^*\mathscr{E}_{\mathcal{C}}$. Going in the opposite direction is more difficult. From a sheaf $\mathscr{F}$ of $\mathcal{C}$-sets on $X$, we shall construct a "resolution" $p_{\mathscr{F}}\colon X_{\mathscr{F}} \to X$ and a map $\pi_{\mathscr{F}}\colon X_{\mathscr{F}} \to B\mathcal{C}$. It turns out that $p_{\mathscr{F}}$ is a homotopy equivalence if $\mathscr{F}$ has representable stalks and $X$ is a CW-space. Then we can choose a homotopy inverse $p_{\mathscr{F}}^{-1}$ and obtain a map $\pi_{\mathscr{F}} p_{\mathscr{F}}^{-1}\colon X \to B\mathcal{C}$, well defined up to homotopy.

Let $\mathcal{O}(X)$ be the poset of open subsets of a space $X$, ordered by inclusion. Let $\mathscr{F}$ be a sheaf of $\mathcal{C}$-sets on $X$. We can regard $\mathscr{F}$ as a contravariant functor from $\mathcal{O}(X)\times\mathcal{C}$ to sets. The functor $\mathscr{F}$ determines a *transport category* $\mathcal{T}_{\mathscr{F}}$ whose objects are the triples $(U,c,s)$ consisting of an object $U$ in $\mathcal{O}(X)$, an object $c$ in $\mathcal{C}$, and $s \in \mathscr{F}^{(c)}(U)$. A morphism from $(U,c,s)$ to $(V,d,t)$ is a morphism $U \to V$ in $\mathcal{O}(X)$ together with a morphism $f\colon c \to d$ in $\mathcal{C}$ such that $f^*(t)|U = s$. Let $\tau$ be the tautological functor (taking $U \in \mathcal{O}(X)$ to the space $U$) from $\mathcal{O}(X)$ to spaces, and let $\varphi\colon \mathcal{T}_{\mathscr{F}} \to \mathcal{O}(X)$ be the forgetful functor. Put

$$X_{\mathscr{F}} := \mathrm{hocolim}\ \tau\varphi\,.$$

This comes with a canonical projection $p_{\mathscr{F}}\colon X_{\mathscr{F}} \to X$, induced by the obvious natural inclusions $\tau\varphi(U,c,s) \to X$. There is also a projection $X_{\mathscr{F}} \to B\mathcal{T}_{\mathscr{F}}$ which we can compose with the forgetful map $B\mathcal{T}_{\mathscr{F}} \to B\mathcal{C}$. This gives

$$\pi_{\mathscr{F}}\colon X_{\mathscr{F}} \to B\mathcal{C}\,.$$

**Proposition 2.1.** *If $\mathscr{F}$ has representable stalks, then the projection $p_{\mathscr{F}}\colon X_{\mathscr{F}} \to X$ is a weak homotopy equivalence.*

The proof relies on a few lemmas which in turn rely on the notion of a microfibration. Recall that a map $p\colon E \to B$ is a *Serre fibration* if it has the homotopy lifting property for homotopies $X\times[0,1] \to B$, with prescribed "initial" lift $X \to E$, where $X$ is a CW-space. [It is enough to check this in all cases where $X$ is a disk.] A map $p\colon E \to B$ is a *Serre microfibration* if, for any homotopy $h\colon X \times [0,1] \to B$ with prescribed initial lift $\bar{h}_0\colon X \to E$, there exist a neighborhood $U$ of $X \times \{0\}$ in $X \times [0,1]$ and a map $\bar{h}\colon U \to E$ such that $p\bar{h} = h|U$ and $\bar{h}(x,0) = \bar{h}_0(x)$ for all $x \in X$. In that case the map $\bar{h}$ is a *microlift* of $h$. [Again it is enough to check the micro-lifting property in all cases where $X$ is a disk.]

**Lemma 2.2.** *Let $p: E \to B$ be a Serre microfibration. If $p$ has weakly contractible fibers, then it is a Serre fibration.*

*Notes on the proof.* This is essentially due to G. Segal [**8**, A.2]. The hypotheses here are slightly more general, though. There is a short inductive argument as follows. The induction step consists in showing that if $p: E \to B$ is a Serre microfibration with contractible fibers, then so is the projection $p^I : E^I \to B^I$. Here $I = [0,1]$, and the mapping spaces $E^I = \mathrm{map}(I, E)$ and $B^I = \mathrm{map}(I, B)$ come with the compact-open topology. The Serre microfibration property for $p^I$ is obvious, so it is enough to establish the weak contractibility of the fibers of $p^I$. Suppose therefore given a map $\gamma: I \to B$ and a map $f: \mathbb{S}^n \times I \to E$ which covers $\gamma$, so that $pf(z,t) = \gamma(t)$ for $z \in \mathbb{S}^n$ and $t \in I$. We must extend $f$ to a map $g: \mathbb{D}^{n+1} \times I \to E$ which covers $\gamma$. But that is easy: Use a sufficiently fine subdivision of $I$ into subintervals $[a_r, a_{r+1}]$ so that partial extensions

$$g_r : \mathbb{D}^{n+1} \times [a_r, a_{r+1}] \to E$$

of $f$ can be constructed, with $pg_r(z,t) = \gamma(t)$ for $z \in \mathbb{D}^{n+1}$ and $t \in [a_r, a_{r+1}]$. Then improve $g_r$ if necessary, on a small neighborhood of $\mathbb{D}^{n+1} \times \{a_r\}$ in $\mathbb{D}^{n+1} \times [a_r, a_{r+1}]$, to ensure that $g_r(z, a_r) = g_{r-1}(z, a_r)$ for $z \in \mathbb{D}^{n+1}$.

The induction beginning consists in showing that $p$ has the path lifting property. (That is, given a path $\gamma: I \to B$ and $a \in E$ with $p(a) = \gamma(0)$, there exists a path $\omega: I \to E$ with $p\omega = \gamma$ and $\omega(0) = a$.) But that is also easy. □

**Lemma 2.3.** *Let $\tau$ be the tautological functor from $\mathcal{O}(X)$ to spaces and let $K$ be a compact subset of* hocolim $\tau$. *Then there exist a finite full sub-poset $\mathcal{P} \subset \mathcal{O}(X)$ and a subfunctor $\kappa$ of $\tau|\mathcal{P}$ with compact values such that $K \subset$* hocolim $\kappa$.

*Remarks.* The fullness assumption means that $U, V \in \mathcal{P}$ and $U \subset V$ imply $U \leqslant V$ in $\mathcal{P}$. By a subfunctor $\kappa$ of $\tau|\mathcal{P}$ is meant a selection of subspaces $\kappa(U) \subset \tau(U) = U$, one for each $U \in \mathcal{P}$, such that $\kappa(V) \subset \kappa(U)$ if $V \leqslant U$ in $\mathcal{P}$.

The lemma is closely related to an observation for which I am indebted to Larry Taylor: *The mapping cylinder $C$ of the inclusion of the open unit interval in the closed unit interval is not homeomorphic to a subset of* $[0,1]^2$. This is easy to verify, although surprising. The two endpoints of the closed unit interval, viewed as elements of the mapping cylinder $C$, don't have countable neighborhood bases; hence $C$ is not even metrizable. Equally surprising, and more to the point, is the following. Let $K$ be a compact subset of $C$. Then there exists a compact subinterval $L$ of the open unit interval such that $K$ is contained in the mapping cylinder of the inclusion $L \to [0,1]$. For the proof, exhaust the open unit interval by an ascending sequence of compact subintervals $L_i$. Suppose if possible that for each $i$ there exists $x_i \in K$ which is not contained in the mapping cylinder of the inclusion $L_i \to [0,1]$. Then the $x_i$ form an infinite discrete closed subset of $K$, which contradicts the compactness of $K$.

*Proof of lemma 2.3.* The classifying space $B\mathcal{O}(X)$ is a simplicial complex. This has one $n$-simplex for each subset of $\mathcal{O}(X)$ of the form $\{U_0, U_1, \ldots, U_n\}$ where $U_{i-1}$ is a proper subset of $U_i$, for $i \in \{1, 2, \ldots, n\}$. The image of $C$ under the projection hocolim $\tau \to B\mathcal{O}(X)$ is contained in a compact simplicial subcomplex $B\mathcal{O}(X)$, and without loss of generality we can assume that the subcomplex has the form $B\mathcal{P}$ for a finite full sub-poset $\mathcal{P}$ of $\mathcal{O}(X)$. For each simplex $S$ of $B\mathcal{P}$, let $e(S) \subset B\mathcal{P}$ be the "cell" determined by $S$, so that $e(S)$ is locally closed in $B\mathcal{P}$ and $B\mathcal{P}$ is the disjoint union (but not the coproduct in general) of the $e(S)$ for the simplices $S$ of $B\mathcal{P}$. Let $U(S)$ be the smallest of the open sets corresponding to the vertices of $S$. The inverse image of $e(S)$ for the projection

$$\operatorname{hocolim} \tau \longrightarrow B\mathcal{O}(X)$$

is identified with $e(S) \times U(S)$. Its intersection with $K$ is contained in a subset of the form $e(S) \times L(S)$, where $L(S) \subset U(S)$ is compact. (This can be proved as in the remark just above.) Choose such an $L(S)$ for every simplex $S$ in $B\mathcal{P}$. For an element $U$ of $\mathcal{P}$ let

$$\kappa(U) := \bigcup_{S \text{ with } U(S) \subset U} L(S).$$

Then $\kappa(V) \subset \kappa(U)$ for $V, U \in \mathcal{P}$ with $V \subset U$, and each $\kappa(U)$ is compact. □

**Corollary 2.4.** *The projection $p_{\mathscr{F}}: X_{\mathscr{F}} \to X$ is a Serre microfibration.*

*Proof.* Write $p = p_{\mathscr{F}}$. Let $q: X_{\mathscr{F}} \to B\mathcal{T}_{\mathscr{F}}$ be the standard projection (from the homotopy colimit to the classifying space of the indexing category). As we just discovered, the formula $y \mapsto (p(y), q(y))$ need not define an embedding of $X_{\mathscr{F}}$ in $X \times B\mathcal{T}_{\mathscr{F}}$, but it certainly defines an injective map and we can use that to label elements of $X_{\mathscr{F}}$. In particular, let $h: \mathbb{D}^i \times [0,1] \longrightarrow X$ be a homotopy with an "initial lift" $H_0: \mathbb{D}^i \longrightarrow X_{\mathscr{F}}$, so that $h(z,0) = pH_0(z)$. We need to find $\varepsilon > 0$ and a map $H: \mathbb{D}^i \times [0,\varepsilon] \longrightarrow X_{\mathscr{F}}$ such that $pH = h$ on $\mathbb{D}^i \times [0,1]$ and $H(z,0) = H_0(z)$ for $z \in \mathbb{D}^i$. The plan is to define $H$ in such a way that

$$(pH(z,t), qH(z,t)) = (h(z,t), qH_0(z)),$$

for $(z,t) \in \mathbb{D}^i \times [0,\varepsilon]$, which means that $qH: \mathbb{D}^i \times [0,1] \longrightarrow B\mathcal{T}_{\mathscr{F}}$ is a constant homotopy. By lemma 2.3, the plan is sound, giving a well defined and continuous map $\mathbb{D}^i \times [0,\varepsilon] \longrightarrow X_{\mathscr{F}}$ for sufficiently small $\varepsilon$. □

**Lemma 2.5.** *The projection $p_{\mathscr{F}}: X_{\mathscr{F}} \longrightarrow X$ has contractible fibers.*

*Proof.* The fiber over $x \in X$ is identified with the homotopy colimit of the (contravariant, set-valued) functor

$$(U, c) \mapsto \mathscr{F}^{(c)}(U)$$

where $U$ runs through the open subsets of $X$ containing $x$, and $c$ runs through the objects of $\mathcal{C}$. By a well-known Fubini principle for homotopy colimits, it is homotopy equivalent to the double homotopy colimit

$$\operatorname*{hocolim}_{c} \ \operatorname*{hocolim}_{U \ni x} \mathscr{F}^{(c)}(U).$$

In this expression the inside homotopy colimit is a homotopy colimit of sets (i.e., discrete spaces) taken over a directed poset, and therefore the canonical map

$$\operatorname*{hocolim}_{U \ni x} \mathscr{F}^{(c)}(U) \longrightarrow \operatorname*{colim}_{U \ni x} \mathscr{F}^{(c)}(U)$$

is a homotopy equivalence. Therefore

$$\operatorname*{hocolim}_{c} \ \operatorname*{hocolim}_{U \ni x} \mathscr{F}^{(c)}(U) \ \simeq \ \operatorname*{hocolim}_{c} \mathscr{F}_x^{(c)}$$

where $\mathscr{F}_x^{(c)}$ is the stalk of $\mathscr{F}^{(c)}$ at $x$. But the stalk functor $\mathscr{F}_x$ is representable by assumption. The homotopy colimit of a representable functor is contractible. $\square$

*Proof of proposition 2.1.* Apply lemma 2.2 and note that a Serre fibration with weakly contractible fibers is a weak homotopy equivalence. $\square$

## 3. Classification of sheaves up to concordance

**Lemma 3.1.** *Let $\mathscr{F}_0$ and $\mathscr{F}_1$ be two sheaves of $\mathcal{C}$-sets on $X$, both with representable stalks. Let $g\colon \mathscr{F}_0 \to \mathscr{F}_1$ be a binatural transformation. Then $\mathscr{F}_0$ and $\mathscr{F}_1$ are concordant.*

*Proof.* Let $e_0\colon X \to X \times I$ be given by $e_0(x) = (x, 0)$ and let $p\colon X \times I \to X$ be the projection. For an object $c$ in $\mathcal{C}$ and an open subset $U$ of $X \times [0,1]$, let $U_0 = e_0^{-1}(U)$ and let $\mathscr{G}^{(c)}(U)$ be the set of pairs $(s,t) \in \mathscr{F}_0(U_0) \times p^*\mathscr{F}_1(U)$ such that $gs = e_0^* t$. Now $\mathscr{G}$ is a sheaf of $\mathcal{C}$-sets on $X \times I$ with representable stalks. Its restrictions to $X \times \{0\}$ and $X \times \{1\}$ are identified with $\mathscr{F}_0$ and $\mathscr{F}_1$, respectively. $\square$

**Corollary 3.2 (to proposition 2.1 and lemma 3.1).** *Let $\mathscr{F}$ be a sheaf of $\mathcal{C}$-sets on $X$ with representable stalks. Suppose that $X$ is a CW-space. Then*

$$\pi_{\mathscr{F}} p_{\mathscr{F}}^{-1}\colon X \to B\mathcal{C}$$

*is a classifying map for $\mathscr{F}$. That is, $(\pi_{\mathscr{F}} p_{\mathscr{F}}^{-1})^*\mathscr{E}$ is concordant to $\mathscr{F}$, with $\mathscr{E}$ as in definition 1.1.*

*Proof.* Abbreviate $p = p_{\mathscr{F}}$, $\pi = \pi_{\mathscr{F}}$. It is enough to show that the sheaves $\pi^*\mathscr{E}$ and $p^*\mathscr{E}$ on $X_{\mathscr{F}}$ are concordant. By lemma 3.1, it is then also enough to make a map from $\pi^*\mathscr{E}$ to $p^*\mathscr{F}$. That is what we will do, using the “étale” point of view. Therefore let $z \in X_{\mathscr{F}}$. We need to compare the stalk of $\mathscr{F}$ at $p(z) \in X$ with the stalk of $\mathscr{E}$ at $\pi(z) \in B\mathcal{C}$. The point $z$ maps to some cell in $B\mathcal{T}_{\mathscr{F}}$ which corresponds to a nondegenerate diagram

$$(U_0, c_0) \leftarrow (U_1, c_1) \leftarrow \cdots \leftarrow (U_{k-1}, c_{k-1}) \leftarrow (U_k, c_k)$$

in $\mathcal{O}(X) \times \mathcal{C}$, with $p(z) \in U_k$, and an element $s_0 \in \mathscr{F}^{(c_0)}(U_0)$. The stalk of $\mathscr{E}$ at $\pi(z)$ is then represented by the object $c_0$. The germ of $s_0$ near $p(z)$ amounts to a morphism from $c_0$ to the object which represents the stalk of $\mathscr{F}$ at $p(z)$; equivalently, by the Yoneda lemma, $s_0$ determines a $\mathcal{C}$-map from the stalk of $\mathscr{E}$ at $\pi(z)$ to the stalk of $\mathscr{F}$ at $p(z)$. Letting $z$ vary now, and selecting an object $c$ in $\mathcal{C}$, we obtain a map over $X_{\mathscr{F}}$ from Spé$(\pi^*\mathscr{E}^{(c)})$ to Spé$(p^*\mathscr{F}^{(c)})$. This is continuous (verification left to the reader) and natural in $c$, and therefore amounts to a map between sheaves of $\mathcal{C}$-sets on $X_{\mathscr{F}}$, from $\pi^*\mathscr{E}$ to $p^*\mathscr{F}$. $\square$

*Proof of theorem 0.1.* Suppose that $X$ is a CW-space. Let $g: X \to B\mathcal{C}$ be any map and put $\mathscr{F} = g^*\mathscr{E}$. We have to show that $g$ is homotopic to $\pi p^{-1}$, where $\pi = \pi_{\mathscr{F}}$ and $p = p_{\mathscr{F}}$. We note that $X_{\mathscr{F}}$ also has the homotopy type of a CW-space since it is a homotopy colimit of open subsets of $X$ (all of which have the homotopy type of CW-spaces). Hence $p$ is a homotopy equivalence. Therefore, showing $g \simeq \pi p^{-1}$ amounts to showing that $gp \simeq \pi$.

Now recall that $X_{\mathscr{F}}$ was constructed as the homotopy colimit of a functor $\tau\varphi$ from a certain category $\mathcal{T}_{\mathscr{F}}$ with objects $(U, c, s)$ to the category of spaces. The maps $gp: X_{\mathscr{F}} \to B\mathcal{C}$ and $\pi: X_{\mathscr{F}} \to B\mathcal{C}$ both have a factorization of the following kind:

$$\underset{(U,c,s)}{\mathrm{hocolim}}\ U \xrightarrow{\ v\ } \underset{c}{\mathrm{hocolim}}\ \mathrm{Sp\acute{e}}(\mathscr{E}^{(c)}) \xrightarrow{\ w\ } B\mathcal{C}\,.$$

Here $v$ is (in both cases) induced by a natural transformation from the functor $(U, c, s) \mapsto U$ to the functor $(U, c, s) \mapsto \mathrm{Sp\acute{e}}(\mathscr{E}^{(c)})$, given by the maps

$$U \longrightarrow \mathrm{Sp\acute{e}}(\mathscr{E}^{(c)}) \quad ; \quad x \mapsto (x, s)\,.$$

In the case of $gp$, the second map $w = w_0$ in the factorization is determined by the projections $\mathrm{Sp\acute{e}}(\mathscr{E}^{(c)}) \to B\mathcal{C}$. In the case of $\pi$, the map $w = w_1$ is the composition of the projection to $B\mathcal{T}_{\mathscr{F}}$ and the forgetful map $B\mathcal{T}_{\mathscr{F}} \to B\mathcal{C}$.

Consequently it is now sufficient to show that $w_0$ and $w_1$ are *weakly homotopic*, which is to say, $w_0 f \simeq w_1 f$ for any map $f$ from a CW-space to the common source of $w_0$ and $w_1$. It is enough to check this for a particular $f$ which is a weak equivalence. A good choice of such an $f$ is the map

$$\underset{c}{\mathrm{hocolim}}\ B(c \downarrow \mathcal{C}) \longrightarrow \underset{c}{\mathrm{hocolim}}\ \mathrm{Sp\acute{e}}(\mathscr{E}^{(c)})$$

induced by the natural maps $\lambda_c: B(c \downarrow \mathcal{C}) \to \mathrm{Sp\acute{e}}(\mathscr{E}^{(c)})$. By proposition 1.3, this $f$ is indeed a weak homotopy equivalence. The maps $w_0 f$ and $w_1 f$ are easily seen to be homotopic. Indeed, each $(c \downarrow \mathcal{C})$ has two obvious functors two $\mathcal{C}$, one given by $(c \to d) \mapsto d$ and the other by $(c \to d) \mapsto c$. These are related by a natural transformation, which determines a homotopy $h^{(c)}$ between the induced maps $B(c \downarrow \mathcal{C})$ to $B\mathcal{C}$. Integrating the homotopies $h^{(c)}$ one obtains a homotopy $w_0 f \simeq w_1 f$. $\square$

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