Exotic spheres and the Whitehead space

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I. Review: An obstruction theory. — Let Θ_n be the group of diffeomorphism classes of oriented smooth homotopy spheres of dimension n. Let $\mathrm{Diff}(\mathbb{S}^{n-1})$ be the simplicial group of orientation preserving diffeomorphisms $\mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$. For $n > 5$, the natural homomorphism from $\pi_0\mathrm{Diff}(\mathbb{S}^{n-1})$ to Θ_n is surjective by Smale's h–cobordism theorem, and injective by Cerf's pseudo–isotopy theorem. It is easily seen that $\pi_0\mathrm{Diff}(\mathbb{S}^{n-1}) \cong \pi_0\mathrm{Diff}^\partial(I^{n-1})$, where $\mathrm{Diff}^\partial(I^{n-1})$ is the simplicial group of diffeomorphisms $I^{n-1} \to I^{n-1}$ which restrict to the identity near the boundary ∂I^{n-1}. Summarizing, there is an isomorphism $\Theta_n \cong \pi_0\mathrm{Diff}^\partial(I^{n-1})$ for $n > 5$.

Question. — Given a class in $\pi_0\mathrm{Diff}^\partial(I^{n-1})$ and given an integer q with $0 \le q \le n-1$, is there a representative f of that class such that

$$(\star)\qquad \begin{array}{ccc} I^{n-1} & \xrightarrow{f} & I^{n-1} \\ \downarrow & & \downarrow \\ I^q & \xrightarrow{=} & I^q \end{array}$$

commutes ? (The vertical arrows are the standard projections.)

Let $V_q \subset \mathrm{Diff}^\partial(I^{n-1})$ be the simplicial group of all those f for which $(\star)$ commutes. Then

$$\{1\} = V_{n-1} \subset V_{n-2} \subset \dots \subset V_1 \subset V_0 = \mathrm{Diff}^\partial(I^{n-1})\,.$$

To describe the simplicial set V_j/V_{j+1} of left cosets of V_{j+1} in V_j we will use the following notation. For a smooth compact manifold M, let $\mathcal{C}(M)$ be the (simplicial) group of pseudoisotopies alias concordances of M; that is, diffeomorphisms $M \times I \to M \times I$ which are the identity near $M \times \{0\}$ and $\partial M \times I$. There are (almost obvious) homotopy equivalences

$$V_j \simeq \Omega^j \mathrm{Diff}^\partial(I^{n-1-j})\,, \qquad\qquad V_j/V_{j+1} \simeq \Omega^j \mathcal{C}(I^{n-2-j})\,.$$

They lead to exact sequences

$$\pi_{j+1}\mathcal{C}(I^{n-2-j}) \longrightarrow \pi_0 V_{j+1} \longrightarrow \pi_0 V_j \longrightarrow \pi_j \mathcal{C}(I^{n-2-j})$$

and therefore to an obstruction theory for the study of $\pi_0 V_0 \cong \Theta_n$. For more details, see [2]. Now it is well known that the spaces $\mathcal{C}(I^q)$ are closely related to the algebraic K–theory of $\mathbb{Z}$. More precisely, there are stabilisation maps

$$\cdots \to \mathcal{C}(I^{q-1}) \to \mathcal{C}(I^q) \to \mathcal{C}(I^{q+1} \to \cdots$$

and a map

$$\mathcal{C}(I^\infty) := \lim_q \mathcal{C}(I^q) \;\longrightarrow\; \Omega^2(\mathrm{BGL}(\infty,\mathbb{Z})^+);$$

see [3], [4], [5]. Hence there is a connection between homotopy spheres and algebraic K–theory.

II. Tying the obstructions together. — To elucidate this connection, we will construct:

(i) a fibration $p\colon E \to \mathbb{S}^{n-2}$ whose fibers have the homotopy type of $\mathcal{C}(I^{n-2})$;
(ii) an involution $\tau\colon E \to E$ sucht that $p\tau = \alpha p$, where α is the antipodal involution on $\mathbb{S}^{n-2}$;
(iii) for each oriented smooth homotopy sphere, a homotopy class of equivariant sections of p.

Here are the details:

(i) *The fibration.* — Let $\mathbb{D}^{n-1} \subset \mathbb{R}^{n-1}$ be the unit disk. For each $x \in \mathbb{S}^{n-2} = \partial\mathbb{D}^{n-1}$ let

$$W(x) = \{y \in \mathbb{S}^{n-2} \mid \langle x, y\rangle \geq 0\}\,,$$

where $\langle\,,\,\rangle$ is the usual inner product. Let E_x be the group of diffeomorphisms f from $\mathbb{D}^{n-1}$ to $\mathbb{D}^{n-1}$ for which $f(z) = z$ whenever $z \in W(x)$. Then clearly $E_x \simeq \mathcal{C}(I^{n-2})$.

(ii) *The involution.* — For each $x \in \mathbb{S}^{n-2}$ there is a surjection

$$\begin{aligned}\mathrm{pr}_x\colon W(x) \times I &\to \mathbb{D}^{n-1}\\ (v, t) &\mapsto v - 2t\langle v, x\rangle \cdot x.\end{aligned}$$

For each $f \in E_{\alpha(x)}$ (where $\alpha(x)$ is the antipode of x) let $\partial f\colon W(x) \to W(x)$ be the restriction of f to $W(x)$. Define $\partial f \times I\colon \mathbb{D}^{n-1} \to \mathbb{D}^{n-1}$ by

$$(\partial f \times I)(\mathrm{pr}_x(v, t)) := \mathrm{pr}_x(\partial f(v), t).$$

The map

$$E_{\alpha(x)} \longrightarrow E_x \quad ; \quad f \mapsto (\partial f \times I)^{-1} \cdot f$$

is a homeomorphism. Letting x vary in $\mathbb{S}^{n-2}$, one obtains an involution $\tau\colon E \to E$ for which $p\tau = \alpha p$.

(iii) *The equivariant section $\psi(\Sigma^n)$ associated with a smooth homotopy sphere Σ^n.* — We represent Σ^n by a diffeomorphism $f\colon \mathbb{D}^{n-1} \to \mathbb{D}^{n-1}$ which fixes $\partial\mathbb{D}^{n-1} = \mathbb{S}^{n-2}$ pointwise. For each $x \in \mathbb{S}^{n-2}$ we can view f as an element of the fiber E_x , by the very definition of E_x. The section $\psi(\Sigma_n)$ of $p\colon E \to \mathbb{S}^{n-2}$ obtained in this way is indeed equivariant. (It can be shown that the section is non–equivariantly nullhomotopic; hence it is a 2–primary invariant.)

Now let $\mathbf{Wh}(*) = \mathbf{Wh}^{\mathrm{DIFF}}(*)$ be the spectrum associated with the infinite loop space $B^2\mathcal{C}(I^\infty)$. This comes with a canonical involution tw, independent of n, which we will regard as an operation of $\mathbb{Z}/2 = \pi_1(\mathbb{R}P^\infty)$ on $\mathbf{Wh}(*)$.

Remark. — One can view $\psi(\Sigma^n)$ as an element of

$$H_n(\mathbb{R}P^\infty; \mathbf{Wh}(*)^{tw}) = \pi_n(\mathbb{S}^\infty_+ \wedge_{\mathbb{Z}/2} \mathbf{Wh}(*)).$$

Sketch proof. — Let $\hat{p}\colon \hat{E} \to \mathbb{R}P^{n-2}$ be the fibration obtained from $p\colon E \to \mathbb{S}^{n-2}$ by passage to the orbit spaces $\hat{E} = E_{\mathbb{Z}/2}$ and $\mathbb{R}P^{n-2} = (\mathbb{S}^{n-2})_{\mathbb{Z}/2}$. Then $\psi(\Sigma^n)$ is a section of $\hat{p}$, in other words a 'twisted map' from $\mathbb{R}P^{n-2}$ to $\mathcal{C}(I^{n-2})$. This leads to a twisted map from $\mathbb{R}P^{n-2}$ to $\mathcal{C}(I^\infty)$ by means of a stabilisation of the fibers of $\hat{p}$. But $\mathcal{C}(I^\infty)$ is a representing space for the representable cofunctor $H^0(\ ; \Sigma^{-2}\mathbf{Wh}(*))$. The twisted map $\mathbb{R}P^{n-2} \to \mathcal{C}(I^\infty)$ therefore represents an element of $H^0(\mathbb{R}P^{n-2}; \Sigma^{-2}\mathbf{Wh}(*)^{tw'}$ where $tw'\colon \mathbf{Wh}(*) \to \mathbf{Wh}(*)$ is an appropriate involution (which depends on n). To complete the proof, one uses the homomorphisms

$$\begin{aligned}H^0(\mathbb{R}P^{n-2}; \Sigma^{-2}\mathbf{Wh}(*)^{tw'}) &\xrightarrow{\cong} H_{n-2}(\mathbb{R}P^{n-2}; \Sigma^{-2}\mathbf{Wh}(*)^{tw})\\ &\xrightarrow{\cong} H_n(\mathbb{R}P^{n-2}; \mathbf{Wh}(*)^{tw}) \longrightarrow H_n(\mathbb{R}P^\infty; \mathbf{Wh}(*)^{tw})\,;\end{aligned}$$

the first of these is a Poincaré duality isomorphism, the second is a suspension isomorphism, and the third is induced by the inclusion $\mathbb{R}P^{n-2} \subset \mathbb{R}P^\infty$.

For any positive integer q and sufficiently large n (larger than an integer $n(q)$ depending on q), the following conditions are equivalent:

(i) the smooth homotopy sphere Σ^n can be obtained from a diffeomorphism $f \in \mathrm{Diff}^\partial(I^{n-1})$ making the square $(\star)$ above commutative;
(ii) the invariant $\psi(\Sigma^n) \in H_n(\mathbb{R}P^\infty; \mathbf{Wh}(*)^{tw})$ belongs to the image of the homomorphism induced by inclusion,

$$H_n(\mathbb{R}P^{n-q-2}; \mathbf{Wh}(*)^{tw}) \longrightarrow H_n(\mathbb{R}P^\infty; \mathbf{Wh}(*)^{tw}).$$

Moreover (i) implies (ii) for all $n > q$.

The existence of an invariant having roughly the properties of ψ was conjectured by Bruce Williams (Notre Dame) in a more general setting, along with the theorem just below. A joint article is in preparation.

III. Calculations. — Recall that for $n \geq 4$, the surgery obstruction group $L_{n+1}(\mathbb{Z})$ is isomorphic to the bordism group of smooth compact stably framed manifolds M^{n+1} whose boundary ∂M is a homotopy sphere. The isomorphism is obtained by associating to such an M its signature divided by 8 if $n+1 = 4k$, and its Kervaire invariant if $n+1 = 4k+2$. This description of $L_{n+1}(\mathbb{Z})$ gives us a homomorphism

$$\begin{array}{rcl} \psi\partial\colon L_{n+1}(\mathbb{Z}) & \longrightarrow & H_n(\mathbb{R}P^\infty; \mathbf{Wh}(*)^{tw}) \\ {[M]} & \mapsto & \psi(\partial M)\,. \end{array}$$

Waldhausen has obtained a map of spectra

$$\lambda\colon \mathbf{Wh}(*) \longrightarrow \mathbf{K}(\mathbb{Z}).$$

This map commutes with the standard involutions; note that $\mathbf{K}(\mathbb{Z})$ has a standard involution coming from the involutions $\mathrm{GL}(m,\mathbb{Z}) \to \mathrm{GL}(m,\mathbb{Z})\,;\ A \mapsto (A^t)^{-1}$. Hence there is an induced homomorphism

$$\lambda\colon H_n(\mathbb{R}P^\infty; \mathbf{Wh}(*)^{tw}) \longrightarrow H_n(\mathbb{R}P^\infty; \mathbf{K}(\mathbb{Z})^{tw})\,.$$

Our goal is to describe the composite homomorphism

$$\lambda\psi\partial\colon L_{n+1}(\mathbb{Z}) \longrightarrow H_n(\mathbb{R}P^\infty; \mathbf{K}(\mathbb{Z})^{tw}).$$

On needs to know that, for $n = 4k-1$, there is a Poincaré duality isomorphism

$$H_k(\mathbb{R}P^n; \mathbf{K}(\mathbb{Z})^{tw}) \cong H^{n-k}(\mathbb{R}P^n; \mathbf{K}(\mathbb{Z})^{tw}).$$

The element corresponding to $1 \in H^0(\mathbb{R}P^n; \mathbf{K}(\mathbb{Z})^{tw})$ under this duality will be called the fundamental class $[\mathbb{R}P^n] \in H_n(\mathbb{R}P^n; \mathbf{K}(\mathbb{Z})^{tw})$.

Theorem. — *For $x \in L_{n+1}(\mathbb{Z})$ one has*

$$\lambda\psi\partial(x) = \begin{cases} (\text{signature of } x)\cdot[\mathbb{R}P^n] & \text{if } n+1 = 4k \\ 0 & \text{otherwise} \end{cases}$$

in $H_n(\mathbb{R}P^\infty; \mathbf{K}(\mathbb{Z})^{tw})$.

Suppose now that $n = 4k-1$ (where $k > 1$) and let Σ^n be the Milnor exotic sphere. It can be regarded as the boundary of the generator of $L_{4k}(\mathbb{Z}) \cong \mathbb{Z}$. Using the above theorem and easy calculations with real topological K–theory, one can estimate the order of $\psi(\Sigma^n)$ in

$H_n(\mathbb{R}P^\infty; \mathbf{Wh}(*)^{tw})$. It is divisible by 2^{2k-3} if k is even, and by 2^{2k-2} if k is odd. Using the Atiyah–Hirzebruch spectral sequence, one can deduce:

Corollary. — *Let $\beta(k)$ be the number of indices i such that $0 \leq i \leq 4k-1$ and $\pi_i(\mathbf{Wh}(*))$ has odd order (note that 1 is odd). Then $\beta(k) \leq 2k+3$ if k is even, and $\beta(k) \leq 2k+2$ if k is odd.*

This result has also been obtained by Bökstedt and Waldhausen, see [1], with completely different methods.

A more detailed investigation of the Atiyah–Hirzebruch spectral sequence gives the following. Let $f \in \mathrm{Diff}^\partial(I^{4k-2})$ represent Milnor's exotic sphere.

Corollary. — *It is impossible to choose f in such a way that the diagram*

$$\begin{array}{ccc} I^{4k-2} & \xrightarrow{f} & I^{4k-2} \\ \downarrow & & \downarrow \\ I^2 & \xrightarrow{=} & I^2 \end{array}$$

(where the vertical arrows are the standard projections) commutes.

In fact one encounters a nonzero obstruction in $\pi_3(\mathbf{Wh}(*)) \cong \mathbb{Z}/2$. (There is no obstruction in $\pi_2(\mathbf{Wh}(*))$ because that group is zero by Cerf's pseudoisotopy theorem.) It is surprising that such a simple description of the generator of $\pi_3(\mathbf{Wh}(*)) \cong \mathbb{Z}/2$ exists; until 1983, the structure of $\pi_3(\mathbf{Wh}(*))$ was unknown. See also [1].

References

[1] M. Bökstedt and F. Waldhausen, *The map $BG \to A(\star) \to Q\mathbb{S}^0$*, in Proc 1983 Princeton conf. John Moore's birthday, Ann. Math. Studies 113, Princeton University Press, 1987, pp. 418–431.

[2] A. Hatcher, *Concordance spaces, higher simple homotopy theory, and applications*, in Algebraic and geometric topology, (Proc. Symp. Pure Math., Stanford University 1976), Proc. Symp Pure Math. vol XXXII, Amer. Math. Soc., Providence RI, 1978, pp. 3–21.

[3] J.-L. Loday, *Homotopie des espaces de concordances (d'après Waldhausen)*, in Séminaire Bourbaki 1977/1978, Springer Lect. Notes vol. 710, 1979, pp. 187–205.

[4] F. Waldhausen, *Algebraic K-theory of spaces – a manifold approach*, in Current trends in algebraic topology, 1981 London Ontario conf., Canadian Math. Soc. Conf. Proceedings 2, Part 1, Amer. Math. Soc., 1982, pp. 141–184.

[5] ——, *Algebraic K-theory of spaces*, in Proc. of 1983 Rutgers conf. on algebraic topology, Springer Lecture Notes vol. 1126, 1984, pp. 318–419.