EXCISION AND RESTRICTION IN CONTROLLED K–THEORY

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Abstract. The main result is a description in terms of controlled algebra/topology of the locally finite homology theory associated with the algebraic K–theory spectrum of a ring or the algebraic K–theory spectrum of a topological space. While such a description is known from [ClPV], the one here is better suited as a receptacle for characteristic invariants.

0. Introduction

Controlled algebra, a hybrid of algebra and topology, has been used to give a description of the generalized homology theory associated with $\mathbf{K}(R)$, the (in general non–connective) algebraic K–theory spectrum of a ring R. See [PW1], [PW2], [ACFP], [V1], [V2], [CdP]. Here the ring R may be discrete, or it may be one of the brave new group rings which arise in Waldhausen's algebraic K–theory of spaces. Descriptions of this type have proved to be very useful in topology. In particular, *assembly maps* [WWa] in K–theory such as

$$\alpha\colon X_+ \wedge \mathbf{K}(\mathbb{Z}) \longrightarrow \mathbf{K}(\mathbb{Z}[\pi_1 X])$$

can often be interpreted as *forget control* maps. Hence elements in $\Omega^\infty\mathbf{K}(\mathbb{Z}[\pi_1 X])$ can often be shown to lift across the assembly to $\Omega^\infty(X_+ \wedge \mathbf{K}(\mathbb{Z}))$, provided they arise from geometric situations with sufficient control.

Examples. (i) Let X be a connected space which admits a homotopy domination by a finite CW–space. Then X determines a characteristic element $\chi_h(X)$ in $\Omega^\infty\mathbf{K}(\mathbb{Z}[\pi_1 X])$ which represents the projective class of the singular chain complex of the universal cover of X. The image of $\chi_h(X)$ in the reduced projective class group

$$\begin{aligned}\tilde{K}_0(\mathbb{Z}[\pi_1 X]) &= \operatorname{coker}\left[H_0(X;\mathbf{K}(\mathbb{Z})) \to K_0(\mathbb{Z}[\pi_1 X])\right] \\ &= \operatorname{coker}\left[\pi_0(\Omega^\infty(X_+\wedge\mathbf{K}(\mathbb{Z}))) \xrightarrow{\alpha_*} \pi_0(\Omega^\infty\mathbf{K}(\mathbb{Z}[\pi_1 X]))\right]\end{aligned}$$

is the Wall finiteness obstruction [Wa] of X. If X happens to be a compact ENR, then $\chi_h(X)$ lifts to an element $\chi(X) \in \Omega^\infty(X_+ \wedge \mathbf{K}(\mathbb{Z}))$, and so the finiteness obstruction of X vanishes. (The vanishing was first proved in greater generality by West [We] using Hilbert cube manifold methods.)

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(ii) Let $f: X \to Y$ be a homotopy equivalence between compact CW–spaces. This determines a path $\chi_h(f)$ from $f_*\chi_h(X)$ to $\chi_h(Y)$ in $\Omega^\infty\mathbf{K}(\mathbb{Z}[\pi_1 Y])$. Together with the lifts $f_*\chi(X)$, $\chi(Y)$ of $f_*\chi_h(X)$ and $\chi_h(Y)$, respectively, $\chi_h(f)$ determines an element in the Whitehead group

$$\begin{aligned} Wh(\pi_1 X) &= \operatorname{coker}\left[H_1(X;\mathbf{K}(\mathbb{Z})) \to K_1(\mathbb{Z}[\pi_1 X])\right] \\ &= \operatorname{coker}\left[\pi_1(\Omega^\infty(X_+ \wedge \mathbf{K}(\mathbb{Z}))) \xrightarrow{\alpha_*} \pi_1(\Omega^\infty\mathbf{K}(\mathbb{Z}[\pi_1 X]))\right]. \end{aligned}$$

This is the Whitehead torsion of f. If f is a homeomorphism, then $\chi_h(f)$ lifts across the assembly to a path $\chi(f)$ from $f_*\chi(X)$ to $\chi(Y)$. Hence the Whitehead torsion vanishes. (Here the vanishing was first proved by Chapman [Cha] using Hilbert cube manifold methods.)

Ranicki and Yamasaki [RY] prove the two vanishing theorems mentioned above using the assembly map in controlled K_0- and K_1-groups. Explicit constructions of the characteristic elements $\chi_h(X)$, $\chi(X)$ and characteristic paths $\chi_h(f)$ and $\chi(f)$ can be found in [DWW].

> In more detail: see 6.1–6.3 of [DWW] for $\chi_h(X)$ and $\chi_h(f)$; then 7.8 and 7.9 for $\chi(X)$ and $\chi(f)$; then 8.7–8.9 for compatibility. This gives the nonlinear versions, i. e.,
>
> $$\begin{aligned} &\chi_h(X) \in \Omega^\infty(\mathbf{A}(X)), \\ &\chi(X) \in \Omega^\infty(X_+ \wedge \mathbf{A}(*)) \end{aligned}$$
>
> where $\mathbf{A}(X)$ is Waldhausen's algebraic K–theory spectrum of X. To get the linear versions apply the natural linearization map $\mathbf{A}(X) \to \mathbf{K}(\mathbb{Z}[\pi_1(X)])$. See also 6.4 of [DWW].

But the belief that these characteristic elements and characteristic paths have been known for some time, *with the stated assumptions on X and f*, is essential to this author's understanding of the development of controlled topology and the modest contribution of this paper.

With a view to generalizations, it is appropriate to describe the situation abstractly as follows. We start with a homotopy invariant functor $\mathbf{J}$ from spaces to spectra. Such a functor comes (always) with an essentially unique *assembly map* [WWa], a natural transformation

$$X_+ \wedge \mathbf{J}(*) \to \mathbf{J}(X).$$

In addition, we assume given a rule associating to certain spaces X a characteristic element $\chi_h(X) \in \Omega^\infty\mathbf{J}(X)$. The rule also associates, with a homotopy equivalence $f: X_1 \to X_2$ between such spaces, a path $\chi_h(f)$ in $\Omega^\infty\mathbf{J}(X_2)$ from $f_*\chi_h(X_1)$ to $\chi_h(X_2)$. Interpretation of the assembly map as a forget control map gives, under suitable conditions on X which include compactness, a lift of $\chi_h(X)$ across the assembly map to an element $\chi(X) \in \Omega^\infty(X_+ \wedge \mathbf{J}(*))$. And under suitable conditions on a homotopy equivalence $f: X_1 \to X_2$ between compact spaces, the path $\chi_h(f)$ lifts to a path $\chi(f)$ from $f_*\chi(X_1)$ to $\chi(X_2)$.

The main construction of this paper extends the (now more or less standard) controlled algebra description [ACFP] of the homology theory on compact metrizable spaces associated with $\mathbf{K}(R)$ to a controlled algebra description of the corresponding locally finite homology theory on locally compact spaces with countable base.

More precisely: Let $\mathcal{L}^\bullet$ be the category of locally compact spaces with countable base, where a morphism from X_1 to X_2 is a pointed map $X_1^\bullet \to X_2^\bullet$ (the bullets indicate one–point compactifications).

> Important example (a): every proper map $X_1 \to X_2$ determines a morphism in $\mathcal{L}^\bullet$ from X_1 to X_2. Important example (b): every open embedding $j\colon X_2 \to X_1$ determines a collapse alias localization map $X_1^\bullet \to X_2^\bullet$, wrong–way; this is inverse to j on $j(X_2) \subset X_1 \subset X_1^\bullet$ and maps the complement of $j(X_2)$ in $X_1^\bullet$ to the point at infinity in $X_2^\bullet$.

Controlled algebraic K–theory methods are used here to construct a functor $\mathbf{F}$ from $\mathcal{L}^\bullet$ to spectra such that $\mathbf{F}(X) \simeq X^\bullet \wedge \mathbf{K}(R)$ under mild conditions on X.

Motivation. The controlled algebra model of $\Omega^\infty(X^\bullet \wedge \mathbf{K}(R))$ constructed here will be used in [DWW] as a receptacle for a characteristic element

$$\chi(X) \in \Omega^\infty(X^\bullet \wedge \mathbf{K}(R))$$

defined when X is an ENR, compact or not, and $R = \mathbb{Z}$ (linear case) or $R = Q\mathbb{S}^0$ (nonlinear case). This extends the folklore χ described in examples (i) and (ii) above, from compact ENR's to arbitrary ENR's (and, in the nonlinear setting, refines it). If X is a compact ENR, therefore, $\chi(X)$ is still a refinement of $\chi_h(X) \in \Omega^\infty(\mathbf{K}(\mathbb{Z}[\pi_1 X])$. The extended χ, however, is also natural with respect to inclusions of open subsets $W \hookrightarrow X$ (where X and of course W may be noncompact); that is, the map $\Omega^\infty(X^\bullet \wedge \mathbf{K}(R)) \to \Omega^\infty(W^\bullet \wedge \mathbf{K}(R))$ induced by the collapse map from $X^\bullet$ to $W^\bullet$ takes $\chi(X)$ to $\chi(W)$, up to a specified path. The naturality property turns out to be a crucial ingredient in the proof of an index theorem which states that $\chi(X)$ for a topological n–manifold X is Poincaré dual to a certain characteristic class (better, generalized cocycle) evaluated on the tangent bundle [Ki] of X. Indeed, naturality makes it possible to relate $\chi(X)$ to the 'local indices' $\chi(W)$, where W runs through the open balls in X, alias tangent spaces of X. These local indices $\chi(W)$ essentially constitute the characteristic class.

The index theorem, in turn, can be used to show that $\chi(X)$ and hence $\chi_h(X)$ for a *smooth* compact n–manifold are subject to rather severe restrictions. (The homotopy theoretic implications can first be seen when χ or χ_h are applied fiberwise, to a bundle of smooth compact manifolds; they amount to a refinement and generalization of the Bismut–Lott Riemann–Roch theorem [BiLo].) The point is that the above characteristic generalized cocycle, which in principle can be evaluated on any fiber bundle with fibers homeomorphic to $\mathbb{R}^n$, behaves in a rather trivial way when evaluated on n–dimensional *vector bundles.*

Admittedly, the construction here of controlled algebra models of $\Omega^\infty(X^\bullet \wedge \mathbf{K}(R))$ is routine, and the overlap with [ClPV], the earlier [ClP] and [ACFP] is considerable. Both [ClPV] and [ACFP] construct controlled algebra models for the homology theory on compact metrizable spaces associated with $\mathbf{K}(R)$. The constructions in [ACFP] are better suited as receptacles for characteristic elements. However, [ClPV] have a hard excision theorem in the tradition of Steenrod and Milnor [Mi] which, for a compact metrizable pair (X, Y), puts the homology groups of X, Y and X/Y in a long exact sequence. By contrast [ACFP] only have the softer version where the mapping cone of $Y \hookrightarrow X$ replaces the quotient X/Y. This paper represents a compromise between the two approaches: essentially the models of [ACFP], and the hard excision theorems of [ClVP].

Organisation. Part I of this paper is about the case where the ring R is discrete, and descriptions of the locally finite homology theory associated with $\mathbf{K}(R)$ along the lines of [PW2], [ACFP] and [ClP]. Part II is about the case where R is one of Waldhausen's brave new group rings, and descriptions of the locally finite homology theory associated with $\mathbf{K}(R)$ along the lines of [V2] and [ClPV]. Great care has been taken to model Part II on Part I, in order to avoid pitfalls.

Part I: K–Theory of categories of controlled chain complexes

1. Additive Categories and K–Theory

Let $\mathcal{A}$ be a small additive category [ML]. The category $\mathcal{A}^\sharp$ of additive contravariant functors from $\mathcal{A}$ to abelian groups is an abelian category. The Freyd embedding $\mathcal{A} \to \mathcal{A}^\sharp$ takes an object A in $\mathcal{A}$ to the contravariant functor $\mathrm{mor}_{\mathcal{A}}(?, A)$. It embeds $\mathcal{A}$ as a full subcategory, by the Yoneda lemma [ML].

1.1. Observation. *All objects of $\mathcal{A}$ become projective in $\mathcal{A}^\sharp$.*

Proof. Given a diagram $A \xrightarrow{f} B \xleftarrow{p} B'$ in $\mathcal{A}^\sharp$, where p is epic and A is in $\mathcal{A}$, we must produce $f' : A \to B'$ such that $pf' = f$. By the Yoneda correspondence, f may be regarded as an element in the abelian group $B(A)$, and this notation is justified because B is a contravariant functor on $\mathcal{A}$. Since $p_A : B'(A) \to B(A)$ is onto by assumption, we can find $f' \in B'(A)$ such that $p_A(f') = f$. By the Yoneda correspondence again, this translates into $f' : A \to B'$ such that $pf' = f$. $\square$

An $\mathcal{A}$–complex is by definition a chain complex C, graded over $\mathbb{Z}$ and bounded below, where each C_n and each $d_n : C_n \to C_{n-1}$ belong to $\mathcal{A}$. The $\mathcal{A}$–complexes form a category

$$\mathcal{C}\mathcal{A} \ ,$$

where the morphisms are the chain maps. An $\mathcal{A}$–complex C is *finite* if it is also bounded above, and *homotopy finitely dominated* if $\mathrm{id} : C \to C$ is homotopic to a chain map $f : C \to C$ such that $f_n = 0$ for all but finitely many n. The homotopy finitely dominated objects in $\mathcal{C}\mathcal{A}$ determine a full subcategory

$$\mathcal{D}\mathcal{A} \subset \mathcal{C}\mathcal{A} \, .$$

A chain map $f : C \to D$ of $\mathcal{A}$–complexes is a *cofibration* if $f_n : C_n \to D_n$ is split monic, for all $n \in \mathbb{Z}$; i.e., for each n there exists $e_n : C'_n \to D_n$ in $\mathcal{A}$ such that $f_n \oplus e_n : C_n \oplus C'_n \to D_n$ is an isomorphism. Chain homotopies between chain maps of $\mathcal{A}$–complexes are defined in the usual way, but we refrain from introducing a homotopy category. For C in $\mathcal{C}\mathcal{A}$ let

$$H_i(C) := \ker d_i / \operatorname{im} d_{i+1}$$

which has meaning in $\mathcal{A}^\sharp \supset \mathcal{A}$. A chain map $C \to D$ in $\mathcal{C}\mathcal{A}$ induces a map of graded homology objects, $H_*(C) \to H_*(D)$.

1.2. Corollary. *A chain map $e: C \to D$ of $\mathcal{A}$–complexes is a homotopy equivalence if and only if $e_*: H_*(C) \to H_*(D)$ is an isomorphism.*

Proof. It is enough to show that the mapping cone of e is contractible if e induces an isomorphism in homology [D, 3.7]. Since the mapping cone has zero homology, it is also enough to show that every $\mathcal{A}$–complex E with vanishing homology is contractible. We can regard such an E as an $\mathcal{A}^\sharp$–complex with vanishing homology, with the property that each E_n is projective, by 1.1. It is then easy to construct a chain contraction by induction over the skeletons. Since $\mathcal{A}$ is full in $\mathcal{A}^\sharp$, this contraction may also be regarded as a contraction of the $\mathcal{A}$–complex E. □

Warning. The category $\mathcal{A}$ might be an abelian category in its own right. In such a case it is important not to misread H_* in 1.2 as homology of $\mathcal{A}$–complexes; it is really the homology functor on $\mathcal{A}^\sharp$–complexes.

There are several accepted ways to define the algebraic K–theory of $\mathcal{A}$, and we can use 1.2 to show that they give essentially the same result.

(1) Call a sequence $A_1 \to A_2 \to A_3$ in $\mathcal{A}$ *short exact* if it is short exact in $\mathcal{A}^\sharp$ (equivalently, if it is split short exact). This makes $\mathcal{A}$ into an *exact* category in the sense of Quillen [Q]. Hence we may define the K–theory of $\mathcal{A}$ as $\Omega|Q\mathcal{A}|$, where Q is Quillen's construction in [Q].

(2) Call a morphism $A_1 \to A_2$ in $\mathcal{A}$ a *cofibration* if it can be extended to a short exact sequence $A_1 \to A_2 \to A_3$, and call it a *weak equivalence* if it is an isomorphism. This makes $\mathcal{A}$ into a category with cofibrations and weak equivalences in the sense of [Wd]. Hence we may define the K–theory of $\mathcal{A}$ as $\Omega|\mathcal{S}_\bullet\mathcal{A}|$, where $\mathcal{S}_\bullet$ is Waldhausen's construction in [Wd, §1.3]. The spaces $\Omega|Q\mathcal{A}|$ and $\Omega|\mathcal{S}_\bullet\mathcal{A}|$ are homotopy equivalent by [Wd, §1.9].

(3) Assume for simplicity that $\mathcal{A}$ is *idempotent complete.* (That is, for every idempotent endomorphism $p: A \to A$ in $\mathcal{A}$, there exist $q_i : A_i \to A$ for $i = 1, 2$ in $\mathcal{M}$ such that $q_1 \oplus q_2 : A_1 \oplus A_2 \to A$ is an isomorphism, and $pq_1 = q_1$, $pq_2 = 0$.) We want to re–define the K–theory of $\mathcal{A}$ in terms of $\mathcal{D}\mathcal{A}$. For this purpose we need to make $\mathcal{D}\mathcal{A}$ into a category with cofibrations and weak equivalences; we do so by using the notion of cofibration defined earlier, and homotopy equivalences as weak equivalences. Now we may define the K–theory of $\mathcal{D}\mathcal{A}$ as $\Omega|\mathcal{S}_\bullet\mathcal{D}\mathcal{A}|$. We may also embed $\mathcal{A}$ in $\mathcal{D}\mathcal{A}$ by identifying it with the full subcategory consisting of the chain complexes concentrated in dimension zero. Then [TT, Thm. 1.7.11] the inclusion $\mathcal{A} \to \mathcal{D}\mathcal{A}$, which is an *exact* functor, induces a homotopy equivalence

$$\Omega|\mathcal{S}_\bullet\mathcal{A}| \simeq \Omega|\mathcal{S}_\bullet\mathcal{D}\mathcal{A}|\,.$$

Proof. We want to apply [Wd, Thm. 1.7.1]. For this we need a *homology theory* on $\mathcal{A}$, which we have as in corollary 1.2. above, with values in $\mathcal{A}^\sharp$. We also need an exact subcategory of $\mathcal{A}^\sharp$, for which we take $\mathcal{A}$ itself. Let $\mathcal{D}^0\mathcal{A} \subset \mathcal{D}\mathcal{A}$ be the full subcategory consisting of all objects C for which $H_0(C)$ belongs to $\mathcal{A}$ and $H_i(C) = 0$ for $i \neq 0$. By Waldhausen's 1.7.1, the inclusion

$$\mathcal{D}^0\mathcal{A} \to \mathcal{D}\mathcal{A}$$

induces a homotopy equivalence of the K–theories. Now observe that every C in $\mathcal{D}^0\mathcal{A}$ has a canonical filtration by cofibrations

$$C' \hookrightarrow C'' \hookrightarrow C$$

where $C'_i = C''_i = 0$ for $i < 0$, and $C'_i = C''_i = C_i$ for $i > 0$, and C'_0 , C''_0 are image of d_1 and kernel of d_0, respectively. This definition is meaningful in $\mathcal{A}^\sharp$, and using idempotent completeness one verifies easily that C'_0 and C''_0 are indeed in the subcategory $\mathcal{A}$, at least up to isomorphism. By the additivity theorem [Wd], [MC], the identity map $\mathcal{D}^0\mathcal{A} \to \mathcal{D}^0\mathcal{A}$ is homotopic to the sum of the three maps induced by the exact endofunctors

$$C \mapsto C/C'' , \qquad C \mapsto C''/C' , \qquad C \mapsto C' .$$

But C' and C/C'' have vanishing homology and are therefore contractible, so that two of the three maps are nullhomotopic. It follows that the inclusion $\mathcal{A} \to \mathcal{D}^0\mathcal{A}$ induces a homotopy equivalence of K–theories. □

What is the homotopy type of $K(\mathcal{D}\mathcal{A})$ when $\mathcal{A}$ is *not* idempotent complete ? This is answered by the following lemma, for which we introduce the *idempotent completion* $\mathcal{A}^\wedge$ of $\mathcal{A}$. Objects of $\mathcal{A}^\wedge$ are pairs (A, p) where A is in $\mathcal{A}$ and $p: A \to A$ is idempotent. A morphism in $\mathcal{A}^\wedge$ from (A, p) to (A', p') is a morphism f from A to A' in $\mathcal{A}$ such that $p'fp = f$. There is a full embedding $\mathcal{A} \to \mathcal{A}^\wedge$ taking A to (A, id). Note that $\mathcal{A}^\wedge$ is idempotent complete, and every object of $\mathcal{A}^\wedge$ is a direct summand of some object in the subcategory $\mathcal{A}$.

1.3. Lemma. *The inclusion $K(\mathcal{D}\mathcal{A}) \to K(\mathcal{D}\mathcal{A}^\wedge))$ is a homotopy equivalence.*

Proof. This will come out of the approximation theorem [Wd, 1.6.7], provided we can establish the following:

> Given C in $\mathcal{D}\mathcal{A}$ and $f: C \to D$ in $\mathcal{D}\mathcal{A}^\wedge$, there exist a cofibration $C \to C'$ in $\mathcal{D}\mathcal{A}$ and a weak equivalence $C' \to D$ in $\mathcal{D}\mathcal{A}^\wedge$ such that the composition $C \to C' \to D$ equals f.

Argument for this: By successively adding contractible chain complexes of the form

$$\cdots \leftarrow 0 \leftarrow E_n \xleftarrow{\cong} E_{n+1} \leftarrow 0 \leftarrow 0 \leftarrow \cdots$$

to D, we can obtain an object C'' in $\mathcal{D}\mathcal{A}$. The projection $p: C'' \to D$ is a homotopy equivalence by construction. Choose $g: C \to C''$ and a homotopy h from pg to f. Together, p and h define a map from the mapping cylinder C' of g to D which extends $f: C \to D$ and which is a homotopy equivalence. □

Note that 1.3 and the preceding info tell us that $K(\mathcal{A}^\wedge) \simeq K(\mathcal{D}\mathcal{A})$. This description of $K(\mathcal{A}^\wedge)$ is useful for the following reason. The category $\mathcal{D}\mathcal{A}$, with cofibrations and weak equivalences, has a *cylinder functor* [Wd, 1.6]. The cylinder functor appears in the *fibration theorem* [Wd, 1.6.4] and in Carlsson's product lemma [Cl1]. These are some of the best tools available to establish excision properties.

The following lemma asserts that the homotopy category associated with $\mathcal{C}\mathcal{A}$ is idempotent complete; it will be needed in the proof of theorem 3.1.

1.4. Lemma. *Let C be an $\mathcal{A}$–complex, and suppose that $p: C \to C$ is idempotent up to homotopy ($pp \simeq p$). Then there exist an $\mathcal{A}$–complex D and chain maps $r: C \to D$, $s: D \to C$ such that $sr \simeq p$, and $rs \simeq \mathrm{id}_D$.*

Proof. Let $\lambda\mathcal{A}$ be the following enlargement of $\mathcal{A}$. An object of $\lambda\mathcal{A}$ is a diagram $A_0 \to A_1 \to A_2 \to \cdots$ in which all arrows are cofibrations. A morphism

$$(A_0 \to A_1 \to A_2 \to \cdots) \quad \longrightarrow \quad (B_0 \to B_1 \to B_2 \to \cdots)$$

is an element in $\lim_i \operatorname{colim}_j \operatorname{mor}_{\mathcal{A}}(A_i, B_j)$. We call such a morphism *small* if it comes from $\lim_i \operatorname{mor}_{\mathcal{A}}(A_i, B_j)$ for some $j \geq 0$. We call an object of $\lambda\mathcal{A}$ *small* if its identity morphism is small. The full subcategory of $\lambda\mathcal{A}$ determined by the small objects is equivalent to $\mathcal{A}$.

In lemma 1.5 below, we will see that a $\lambda\mathcal{A}$–complex E for which id_E is homotopic to a (dimensionwise) small chain map is homotopy equivalent to an $\mathcal{A}$–complex (alias dimensionwise small $\lambda\mathcal{A}$–complex). Hence it is enough to construct the chain complex D required in 1.4 as a $\lambda\mathcal{A}$–complex. This is easy: Let D be the mapping telescope alias iterated mapping cylinder of

$$C \xrightarrow{p} C \xrightarrow{p} C \xrightarrow{p} C \to \cdots .$$

Let $r\colon C \to D$ be the inclusion of the front of the telescope. To make $s\colon D \to C$ we need chain maps $s^{(i)}\colon C \to D$ and homotopies $s^{(i+1)}p \simeq s^{(i)}$, for $i = 0, 1, 2, \ldots$. We get them by taking $s^{(i)} := p$ for all i, and using the hypothesis $pp \simeq p$. □

1.5. Lemma. *Let E be a $\lambda\mathcal{A}$–complex with the property that* $\mathrm{id}\colon E \to E$ *is homotopic to a dimensionwise small chain map. Then there exists a homotopy equivalence $D \to E$, where D is an $\mathcal{A}$–complex.*

Proof. Suppose for induction purposes that we have already managed to construct a homotopy equivalence $D \to E$ where D is a $\lambda\mathcal{A}$–complex for which all D_i with $i < k$ are small. The identity of D is still homotopic to a dimensionwise small chain map $D \to D$. Using the homotopy extension property for the inclusion of the $(k-1)$–skeleton, we can arrange the homotopy $h = \{h_i\colon D_i \to D_{i+1}\}$ to be trivial on the $(k-1)$–skeleton. Then $dh_k + \mathrm{id}\colon D_k \to D_k$ is small, where d denotes the differential in D. Let $D_k \cong D'_k \oplus D''_k$ be a splitting such that D'_k is small and $dh_k + \mathrm{id}$ factors through $D'_k \hookrightarrow D_k$. Let $d'\colon D_{k+1} \to D'_k$ and $d''\colon D_{k+1} \to D''_k$ be the components of $d\colon D_{k+1} \to D_k$. Let B be the contractible chain complex

$$\cdots \leftarrow 0 \leftarrow D''_k \xleftarrow{\mathrm{id}} D''_k \leftarrow 0 \leftarrow \cdots$$

with the nontrivial terms in dimensions k and $k+1$. There is a chain map $B \to D$ given by

$$\begin{aligned} (-d'h_k, \mathrm{id})\colon D''_k &\longrightarrow D'_k \oplus D''_k \ , \\ -h_k\colon D''_k &\longrightarrow D_{k+1} \end{aligned}$$

in dimensions k and $k+1$, respectively. If $\mathcal{A}$ happens to be idempotent complete, then $B \to D$ is a cofibration of $\lambda\mathcal{A}$–complexes, and the cofiber D/B is the improvement of D we have been looking for. (The projection $D \to D/B$ is a homotopy equivalence and D/B has a k–skeleton which is dimensionwise small.) If $\mathcal{A}$ is not idempotent complete, we can still say that $D/B \oplus \Sigma B$ is a meaningful construction, and a $\lambda\mathcal{A}$–complex with a k–skeleton which is dimensionwise small. □

1.6. Remark. We can apply 1.4. to $q = \mathrm{id} - p$ and obtain an $\mathcal{A}$–complex E and a chain map $g\colon C \to E$ which is epic in homology, with kernel equal to that of q_*. Then $(f, g) : C \to D \oplus E$ induces an isomorphism in homology. Consequently (f, g) is a homotopy equivalence. Hence D and E are complementary direct summands of C, up to homotopy equivalence.

2. Controlled Algebra

In this article, a *control space* is a pair of spaces $(\bar{Z}, Z)$ where $\bar{Z}$ is locally compact Hausdorff, and Z is open dense in $\bar{Z}$. Informally, the set $\bar{Z} \smallsetminus Z$ is the *singular set*, whereas Z is the *nonsingular set*. A *morphism* of control spaces is a continuous *proper* map of pairs $f\colon (\bar{Z}_1, Z_1) \longrightarrow (\bar{Z}_2, Z_2)$ such that $f^{-1}(Z_2) = Z_1$. Note that we are less restrictive here than in [WWa, §4] because we allow pairs $(\bar{Z}, Z)$ with noncompact $\bar{Z}$. In any case these ideas come from [ACFP]. — A specific type of control space which will be used frequently is

$$\mathbb{J}X := (X \times [0,1], X \times [0,1[\,)$$

where X is locally compact Hausdorff.

Fix an additive category $\mathcal{A}$ and a control space $(\bar{Z}, Z)$. Following [ACFP] except for pedantic changes and notation, we make a new category $\mathcal{A}(\bar{Z}, Z)$ whose objects are certain covariant functors A from the poset of compact subsets of Z to $\mathcal{A}$. To state the conditions on these functors, we abbreviate $A(\{z\}) =: A_z$. The conditions are:

- The set of all $z \in Z$ such that $A_z \neq 0$ is closed and discrete in Z.
- For each compact $L \subset Z$, the map $\bigoplus_{z \in L} A_z \to A(L)$ induced by the inclusions $\{z\} \to L$ is an isomorphism.

Given two objects A and B in $\mathcal{A}(\bar{Z}, Z)$, a morphism $f\colon A \to B$ is a collection of morphisms $f^x_y : A_x \to B_y$ in $\mathcal{A}$, subject to the following. Firstly, for each x, there is only a finite number of y such that $f^x_y \neq 0$ or $f^y_x \neq 0$. Secondly, given any $z \in \bar{Z} \smallsetminus Z$, and a neighbourhood V of z in $\bar{Z}$, there exists another neighbourhood W of z in $\bar{Z}$ such that $f^x_y = 0$ and $f^y_x = 0$ whenever $x \in W$ and $y \notin V$. Composition of morphisms is defined by

$$(gf)^w_y := \sum_x g^x_y f^w_x \,.$$

Remark. The reader familiar with [ACFP] may have noticed that the control condition here looks more restrictive than the one in [ACFP], which reads: … *such that* $f^x_y = 0$ *whenever* $x \in W$ *and* $y \notin V$. However, it is not hard to show that the two conditions are equivalent when $\bar{Z}$ is compact. In the general case, the more symmetric condition is better, partly because we shall need it in §4 and partly because it allows for dualization. (This is important in controlled L–theory.)

Remark. A morphism $g\colon (\bar{Z}_1, Z_1) \to (\bar{Z}_2, Z_2)$ of control spaces induces a functor $g_*\colon \mathcal{A}(\bar{Z}_1, Z_1) \to \mathcal{A}(\bar{Z}_2, Z_2)$ given by $(g_*A)(L) := A(g^{-1}(L))$. Note that $(gh)_* = g_*h_*$ for morphisms g, h between control spaces, assuming that gh is defined.

For A in $\mathcal{A}(\bar{Z}, Z)$ and a closed neighbourhood U of $\bar{Z} \smallsetminus Z$ in $\bar{Z}$, let A^U be the object in $\mathcal{A}(\bar{Z}, Z)$ defined by $A^U(L) = A(L \cap U)$; this comes with a canonical monomorphism to A, and we think of it as a subobject of A. Given A and A' in $\mathcal{A}(\bar{Z}, Z)$, a *germ* of morphisms from A to A' is an equivalence class of pairs $(U, f\colon A^U \to A')$. Here $(U, f\colon A^U \to A')$ and $(W, g\colon A^W \to A')$ are *equivalent* if f^W and g^U (with domain $A^{U \cap W} = (A^U)^W = (A^W)^U$) agree.

The morphism germs in $\mathcal{A}(\bar{Z}, Z)$ are the morphisms in a new category $\mathcal{A}(\bar{Z}, Z)_\infty$ with the same objects as $\mathcal{A}(\bar{Z}, Z)$. We will be interested in the functor

$$X \mapsto F(X) := K(\mathcal{A}(\mathbb{J}X)^\wedge_\infty)\,,$$

where $\mathcal{A}(\mathbb{J}X)^\wedge_\infty$ is the idempotent completion of $\mathcal{A}(\mathbb{J}X)_\infty$, and X denotes a locally compact Hausdorff space with countable base. In what sense is it a functor ? It is clear that F is a covariant functor on the category $\mathcal{L}$ of all locally compact Hausdorff spaces with countable base, with proper maps as morphisms. However, F extends to an enlarged category $\mathcal{L}^\bullet$ with the same objects as $\mathcal{L}$, where a morphism from X to Y is a continuous pointed map $g\colon X^\bullet \to Y^\bullet$ between the one–point compactifications. To understand this extension, note that g determines a diagram

$$X \supset V \xrightarrow{g|V} Y$$

where $V = g^{-1}(Y) = X \smallsetminus g^{-1}(\infty)$. Here V is open in X and $g|V$, as a map from V to Y, is proper. We would like to define $g_* : F(X) \to F(Y)$ as the composition of $(g|V)_* : F(V) \to F(Y)$ (which is already defined) with a suitable *restriction map*

$$F(X) \to F(V)\,.$$

Unraveling the definition of F, we see that it is enough to define a restriction functor $\mathcal{A}(\mathbb{J}X)_\infty \to \mathcal{A}(\mathbb{J}V)_\infty$. Now an object A in $\mathcal{A}(\mathbb{J}X)_\infty$ is a certain functor defined on the poset of the compact subsets of $X \times [0,1[$, and can indeed be restricted to the poset of compact subsets of $V \times [0,1[$. A morphism $f\colon A \to A'$ in $\mathcal{A}(\mathbb{J}X)$ can also be restricted: discard all

$$f^x_y\colon A_x \to A'_y$$

where $x \notin V$ or $y \notin V$. While restriction does not respect composition of morphisms in $\mathcal{A}(\mathbb{J}X)$, it does respect composition of germs of morphisms, so that we have restriction functors $\mathcal{A}(\mathbb{J}X)_\infty \longrightarrow \mathcal{A}(\mathbb{J}V)_\infty$, $\mathcal{A}(\mathbb{J}X)^\wedge_\infty \longrightarrow \mathcal{A}(\mathbb{J}V)^\wedge_\infty$ and an induced map $F(X) \to F(V)$, as required.

3. Excision

3.1. Main theorem (linear version). *Suppose that X is locally compact with countable base. Let $V \subset X$ be open. Then the commutative square*

$$\begin{array}{ccc} F(X \smallsetminus V) & \xrightarrow{\text{restriction}} & F(\emptyset) = * \\ \big\downarrow{\scriptstyle\text{inclusion}} & & \big\downarrow{\scriptstyle\text{inclusion}} \\ F(X) & \xrightarrow{\text{restriction}} & F(V) \end{array}$$

is a homotopy pullback square.

First part of proof. The proof is an application of the *fibration theorem* [Wd, 1.6.4] and the *approximation theorem* [Wd, 1.6.7]. Notation: We have $\mathcal{A}(\mathbb{J}X)$ as in §2, and we indicate corresponding chain complex categories and germ categories by attaching a prefix $\mathcal{C}$ or $\mathcal{D}$ or a suffix $_\infty$ or both (when both, pass to germs *before* making chain complexes). The chain complex categories have a preferred structure of category with cofibrations and weak equivalences, as explained in §1. This will remain nameless. In addition, however, there is a *coarse* notion ω of weak equivalence in $\mathcal{DA}(\mathbb{J}X)_\infty$. Namely, a chain map $f\colon C \to D$ in $\mathcal{DA}(\mathbb{J}X)_\infty$ is a weak equivalence in the coarse sense if it becomes a weak equivalence in $\mathcal{DA}(\mathbb{J}V)_\infty$ under restriction. This gives rise to *three* new categories with cofibrations and weak equivalences:

1. $\mathcal{DA}_\omega(\mathbb{J}X)_\infty$, which is just $\mathcal{DA}(\mathbb{J}X)_\infty$ with the coarse notion of weak equivalence.

(2) $\mathcal{DA}^\omega(\mathbb{J}X)_\infty$, which is the full subcategory of $\mathcal{DA}(\mathbb{J}X)_\infty$ consisting of those objects C which become equivalent to zero in $\mathcal{DA}_\omega(\mathbb{J}X)_\infty$. Here the preferred notion of weak equivalence is the one inherited from $\mathcal{DA}(\mathbb{J}X)_\infty$.
(3) $\mathcal{DA}^\omega_\omega(\mathbb{J}X)_\infty$, which is $\mathcal{DA}^\omega(\mathbb{J}X)_\infty$ with the coarse notion of weak equivalence. This has contractible K–theory since all objects are equivalent to the zero object.

The fibration theorem implies that the square of inclusion maps

$$\begin{array}{ccc} K(\mathcal{DA}^\omega(\mathbb{J}X)_\infty) & \longrightarrow & K(\mathcal{DA}^\omega_\omega(\mathbb{J}X)_\infty) \;\simeq\; * \\ \downarrow & & \downarrow \\ K(\mathcal{DA}(\mathbb{J}X)_\infty) & \longrightarrow & K(\mathcal{DA}_\omega(\mathbb{J}X)_\infty) \end{array}$$

is a homotopy pullback square. Hence, and by 1.3, we only need to verify that, of the maps

$$K(\mathcal{DA}(\mathbb{J}(X\smallsetminus V))_\infty) \longrightarrow K(\mathcal{DA}^\omega(\mathbb{J}X)_\infty) \qquad \text{(inclusion)},$$
$$K(\mathcal{DA}_\omega(\mathbb{J}X)_\infty) \longrightarrow K(\mathcal{DA}(\mathbb{J}V)_\infty) \qquad \text{(restriction)},$$

the first is a homotopy equivalence, and the second is a map each of whose homotopy fibers is empty or contractible. Both statements follow from the approximation theorem, as we shall see. We interrupt for some definitions and remarks of a technical nature, keeping V and X as in 3.1.

3.2. Definitions. An object A in $\mathcal{A}(\mathbb{J}X)$ is *good* if, for each $t < 1$, there are only finitely many $z = (x, s)$ in $X \times [0, 1[$ such that $s < t$ and $A_z \neq 0$. Note that 'goodness' is not invariant under isomorphism in $\mathcal{A}(\mathbb{J}X)$. An $\mathcal{A}(\mathbb{J}X)$–complex C is *good* if each C_n is good.

Homework. Verification of the following facts is left to the reader.

- If A is a good object in $\mathcal{A}(\mathbb{J}V)$, then A may also be regarded as a good object in $\mathcal{A}(\mathbb{J}X)$.
- Let $f: A \to B$ be a morphism between good objects in $\mathcal{A}(\mathbb{J}V)$. Then there exists a closed neighbourhood U of $V\times\{1\}$ in $V\times[0,1]$ such that $f^U: A^U \to B$ is a morphism in $\mathcal{A}(\mathbb{J}X)$. (This is the difficult one; use the assumption that X has a countable base.)
- Let $f: C \to D$ be a chain map between $\mathcal{A}(\mathbb{J}V)$–complexes, where D is also a good $\mathcal{A}(\mathbb{J}X)$–complex. There exist closed neighbourhoods $U(n)$ of $V \times \{1\}$ in $V \times [0, 1]$, for $n \in \mathbb{Z}$, such that $C' \subset C$ defined by $C'_n := C^{U(n)}$ is a good subcomplex of C, and moreover the restriction of f, as a chain map from C' to D , is in fact a chain map of $\mathcal{A}(\mathbb{J}X)$–complexes.
- Let $h: f \simeq g$ be a chain homotopy, where $f, g: C \to D$ are chain maps of $\mathcal{A}(\mathbb{J}V)$–complexes, and D is also a good $\mathcal{A}(\mathbb{J}X)$–complex. There exist closed neighbourhoods $U(n)$ of $V \times \{1\}$ in $V \times [0, 1]$, for $n \in \mathbb{Z}$, such that $C' \subset C$ defined by $C'_n := C^{U(n)}$ is a good subcomplex of C, the restrictions of f and g are in fact chain maps of $\mathcal{A}(\mathbb{J}X)$–complexes, and the restriction of h is a homotopy between these.

Warning. Two objects A and B in $\mathcal{A}(\mathbb{J}V)$ which are good, and isomorphic in $\mathcal{A}(\mathbb{J}V)_\infty$, need not be isomorphic when regarded as objects in $\mathcal{A}(\mathbb{J}X)_\infty$. A choice of representative $f^U: A^U \to B$ for an isomorphism $A \cong B$ in $\mathcal{A}(\mathbb{J}V)_\infty$ does determine a morphism $A^U \to B$ in $\mathcal{A}(\mathbb{J}X)_\infty$, but there is no reason to think that its domain A^U is isomorphic to A in $\mathcal{A}(\mathbb{J}X)_\infty$.

Second part of proof of 3.1. Here we check that the hypotheses of the approximation theorem [Wd, 1.6.7] hold for the inclusion

$$\mathcal{DA}(\mathbb{J}(X \smallsetminus V))_\infty \longrightarrow \mathcal{DA}^\omega(\mathbb{J}X)_\infty \,.$$

We saw in the proof of 1.3 that one of the two hypotheses can sometimes be simplified when homotopies and mapping cylinders are in good supply, and the functor in question is the inclusion of a full subcategory. (The other hypothesis did not appear explicitly in the proof of 1.3 because it was obviously satisfied.) This simplification applies here, too, and what remains to be established is the following.

(1) An arrow in $\mathcal{DA}(\mathbb{J}(X \smallsetminus V))_\infty$ is a weak equivalence if and only if its image in $\mathcal{DA}^\omega(\mathbb{J}X))_\infty$ is a weak equivalence.

(2) Given D in $\mathcal{DA}^\omega(\mathbb{J}X)_\infty$, there exists D' in $\mathcal{DA}(\mathbb{J}(X \smallsetminus V))_\infty$ and a weak equivalence $D' \to D$ in $\mathcal{DA}^\omega(\mathbb{J}X)_\infty$.

We can use 1.4 to deduce (2) from a rather simpler statement:

(3) Let D be an $\mathcal{A}(\mathbb{J}X)_\infty$– complex such that the restriction $\mathrm{res}(D)$ is a contractible $\mathcal{A}(\mathbb{J}V)_\infty$–complex. Then there exist an $\mathcal{A}(\mathbb{J}(X \smallsetminus V))_\infty$–complex E and a chain map $g: E \to D$ of $\mathcal{A}(\mathbb{J}X)_\infty$–complexes which is a domination up to homotopy (has a homotopy right inverse).

To see that (3) implies (2), start with D as in (2), and find $g: E \to D$ as in (3), and a homotopy right inverse $s: D \to E$, so that $gs \simeq \mathrm{id}_D$. Then $p = sg$ is a self–map of E which we can unambiguously view as a self–map of an $\mathcal{A}(\mathbb{J}(X \smallsetminus V))_\infty$–complex. As such it satisfies $pp \simeq p$. We now use 1.4 to obtain a splitting up to homotopy, $E \simeq D' \oplus ?$. The composition $D' \to E \to D$ is a homotopy equivalence of $\mathcal{A}(\mathbb{J}X)_\infty$–complexes. Since D is homotopy finitely dominated, $p: E \to E$ is homotopic to a chain map vanishing in dimensions $\gg 0$; it follows easily that D' belongs to $\mathcal{DA}(\mathbb{J}(X \smallsetminus V))_\infty$.

We now prove (1) and (3). Actually (1) is a straightforward consequence of the fact that $\mathcal{A}(\mathbb{J}(X \smallsetminus V))_\infty$ is a *full* subcategory of $\mathcal{A}(\mathbb{J}X)_\infty$. For (3), fix D as specified there. The assumption that $\mathrm{res}(D)$ is a contractible $\mathcal{A}(\mathbb{J}V)_\infty$–complex translates into the following. There exists a cofibration $C \to D$ of $\mathcal{A}(\mathbb{J}X)_\infty$–complexes with the properties

(i) $C \hookrightarrow D$ is nullhomotopic;
(ii) the induced chain map $\mathrm{res}(C) \to \mathrm{res}(D)$ is an isomorphism;
(iii) C vanishes outside $V \times [0, 1[$.

To be more precise, we can assume that D is a good $\mathcal{A}(\mathbb{J}X)$–complex. We construct C as a subcomplex of D with the property that, for each n and each $z \in X \times [0, 1]$, the object $(C_n)_z$ in $\mathcal{A}$ is equal to $(D_n)_z$, or equal to zero.

(To ensure that (iii) holds, we must choose $(C_n)_z = 0$ if z is not in $V \times [0, 1[$; on the other hand, to ensure that (ii) holds we must, for fixed n, choose $(C_n)_z = (D_n)_z$ for 'many' $z \in V \times [0, 1[$. This is easily done by induction on n. If (i) is not satisfied, use the homework and the hypothesis that $\mathrm{res}(C) \cong \mathrm{res}(D)$ is contractible to construct a good $\mathcal{A}(\mathbb{J}X)$–subcomplex C' of C such that $\mathrm{res}(C') \hookrightarrow \mathrm{res}(C)$ is still an isomorphism, and the inclusion $C' \to C$ is nullhomotopic as a chain map of $\mathcal{A}(\mathbb{J}X)$–complexes. Then replace C by C'.)

Now D/C is a well defined and good $\mathcal{A}(\mathbb{J}X)$–complex. Since $C \hookrightarrow D$ is nullhomotopic, we can say that D is a direct summand up to homotopy of D/C. Hence we have

$$D/C \xrightarrow{g} D \xrightarrow{s} D/C$$

such that gs is homotopic to the identity. It only remains to show that D/C is isomorphic, as an $\mathcal{A}(\mathbb{J}X)_\infty$–complex, to an $\mathcal{A}(\mathbb{J}(X \smallsetminus V))_\infty$–complex E.

What we need at this point is a set map $r\colon X^\bullet \to X^\bullet \smallsetminus V$ which extends the identity on $X^\bullet \smallsetminus V$ and is continuous at every point of $X^\bullet \smallsetminus V$. To construct such an r, we observe that $X^\bullet$ is *metrizable*. Choose a metric, and for each $x \in X^\bullet$ let $r(x)$ be some point in $X^\bullet \smallsetminus V$ whose distance from x is less than twice the distance from x to $X^\bullet \smallsetminus V$. We use r to make an $\mathcal{A}(\mathbb{J}(X \smallsetminus V))$–complex $E = r_*(D/C)$, defined up to unique isomorphism by

$$(E_n)_{(x,t)} := \bigoplus_{\substack{y\in X\\ r(y)=x}} (D_n/C_n)_{(y,t)}$$

for $(x,t) \in X \times [0,1[$. The direct sum on the right is finite for each (x,t) because D/C is good; the differentials in E are defined in such a way that the evident isomorphisms $E_n \to (D/C)_n$ in $\mathcal{A}(\mathbb{J}X)$, for $n \in \mathbb{Z}$, constitute an isomorphism from E to D/C. This completes the verification of condition (3).

Third part of proof of 3.1. It remains to check that the restriction functor induces a map $K(\mathcal{D}\mathcal{A}_\omega(\mathbb{J}X)_\infty) \to K(\mathcal{D}\mathcal{A}(\mathbb{J}V)_\infty)$ all of whose homotopy fibers are contractible or empty. This will come out of a commutative diagram of categories and functors

$$\begin{array}{ccc}
\mathcal{B} & \longrightarrow & \mathcal{D}''\mathcal{A}(\mathbb{J}V)_\infty \\
\big\downarrow & & \big\downarrow \\
\mathrm{res}^{-1}\mathcal{D}'\mathcal{A}(\mathbb{J}V)_\infty & \longrightarrow & \mathcal{D}'\mathcal{A}(\mathbb{J}V)_\infty \\
\big\downarrow & & \big\downarrow \\
\mathcal{D}\mathcal{A}_\omega(\mathbb{J}X)_\infty & \xrightarrow{\mathrm{res}} & \mathcal{D}\mathcal{A}(\mathbb{J}V)_\infty\,.
\end{array}$$

All *vertical* arrows in the diagram are inclusions of full subcategories. Specifically, $\mathcal{D}'\ldots$ and $\mathcal{D}''\ldots$ are the full subcategories of $\mathcal{D}\mathcal{A}(\mathbb{J}V)_\infty$ consisting of the chain complexes which are homotopy equivalent to finite ones, and those which are actually finite, respectively; $\mathcal{B}$ is the full subcategory of $\mathcal{D}\mathcal{A}_\omega(\mathbb{J}X)_\infty$ consisting of the finite chain complexes which vanish outside $V\times[0,1[$. All categories in the diagram inherit notions of cofibration and weak equivalence from the categories in the lower row.

It is enough to show that the middle horizontal arrow is a homotopy equivalence. A very easy application of the approximation theorem shows that the upper right–hand vertical arrow is a homotopy equivalence. So it is enough to show that the two arrows in the diagram with domain $\mathcal{B}$ satisfy the conditions of the approximation theorem. The first of these conditions is trivially satisfied in both cases: morphisms which *induce* weak equivalences *are* weak equivalences. It remains to check the second condition in both cases.

The inclusion $\mathcal{B} \to \operatorname{res}^{-1}\mathcal{D}'\mathcal{A}(\mathbb{J}V)_\infty$ is full by definition, and homotopies as well as mapping cylinders are available. Hence it is enough to verify the second approximation condition, for this inclusion, in the weaker absolute form: *given D in $\operatorname{res}^{-1}\mathcal{D}'\mathcal{A}(\mathbb{J}V)_\infty$, there exist C in $\mathcal{B}$ and a weak equivalence $C \to D$.* Using the old homework, and the hypothesis that $\operatorname{res}(D)$ is finite up to homotopy equivalence, construct C in $\mathcal{B}$ and $C \to D$ such that $\operatorname{res}(C) \to \operatorname{res}(D)$ is a homotopy equivalence in $\mathcal{D}'\mathcal{A}(\mathbb{J}V)_\infty$. Then $C \to D$ is also a weak equivalence, by the very definitions (since we are using the coarse notion ω of weak equivalence). This completes the verification of the second approximation condition in this case.

Finally, to check the second approximation condition for the restriction functor $\mathcal{B} \to \mathcal{D}''\mathcal{A}(\mathbb{J}V)_\infty$, we suppose that $g\colon \operatorname{res}(C) \to D$ is a morphism in $\mathcal{D}''\mathcal{A}(\mathbb{J}V)_\infty$, with C in $\mathcal{B}$. Without loss of generality, C is an $\mathcal{A}(\mathbb{J}X)$–complex vanishing outside $V \times [0,1[$, and D is an $\mathcal{A}(\mathbb{J}V)$–complex. By the old homework, we can find a good $\mathcal{A}(\mathbb{J}V)$–subcomplex $D^\natural$ of D such that the inclusion $D^\natural \to D$ is an isomorphism in $\mathcal{D}''\mathcal{A}(\mathbb{J}V)_\infty$, and $D^\natural$ is also an $\mathcal{A}(\mathbb{J}X)$–complex. By the old homework again, we can then find a good $\mathcal{A}(\mathbb{J}X)$–subcomplex $C^\natural$ of C such that the inclusion of $\operatorname{res}(C^\natural)$ in $\operatorname{res}(C)$ is an isomorphism in $\mathcal{D}''\mathcal{A}(\mathbb{J}V)_\infty$, and g restricts to a chain map of $\mathcal{A}(\mathbb{J}X)$–complexes from $C^\natural$ to $D^\natural$. Let C' be the homotopy pushout alias double mapping cylinder of the diagram of $\mathcal{A}(\mathbb{J}X)$–complexes

$$C \xleftarrow{\supset} C^\natural \xrightarrow{g|C^\natural} D^\natural \,.$$

Then we have a canonical cofibration $C \to C'$, and a canonical chain map of $\mathcal{A}(\mathbb{J}V)$–complexes from $\operatorname{res}(C')$ to D which is a weak equivalence in $\mathcal{D}''\mathcal{A}(\mathbb{J}V)_\infty$ and extends $g\colon \operatorname{res}(C) \to D$. This completes the verification of the second approximation property in this case. $\square$

4. Homotopy invariance

In the control business, it is customary to deduce homotopy invariance from excision. However, it is also known that the deduction works better with a more stringent notion of control. I shall roughly follow [ACFP] both in making precise what the more stringent control means, and in comparing the more stringent with the less stringent one.

4.1. Theorem. *$F(X\times\,]0,1]\,)$ is contractible for any X in $\mathcal{L}$.*

4.2. Corollary. *The inclusion–induced map $F(X \times \{0\}) \to F(X \times [0,1])$ is a homotopy equivalence.*

Proof of 4.2 *modulo* 4.1. By 3.1 there is a fibration sequence up to homotopy

$$F(X \times \{0\}\,) \longrightarrow F(X \times [0,1]\,) \longrightarrow F(X\times\,]0,1]\,)\,. \quad \square$$

The proof of 4.1 will take up the entire section. Let $\rho_i = 1 - 2^{-i}$ for $i = 0,1,2,\dots$. We use the sequence $\rho = (\rho_i)$ to define a subcategory $\mathcal{A}^\rho(\mathbb{J}X) \subset \mathcal{A}(\mathbb{J}X)$, for X in $\mathcal{L}$. This subcategory contains all the objects. A morphism $f\colon A \to B$ in $\mathcal{A}(\mathbb{J}X)$ belongs to $\mathcal{A}^\rho(\mathbb{J}X)$ iff it has the following property. There exists an integer k such that $f^x_y = 0$ whenever the closed subinterval of $[0,1[$ bounded by the second coordinates of x and y contains more than k members of the sequence ρ. Let $F^\rho(X) := K(\mathcal{A}^\rho(\mathbb{J}X)^\wedge_\infty)$.

4.3. Proposition. *$F^\rho(X\times\,]0,1]\,)$ is contractible.*

Proof. Let $Z = X\times\,]0,1]$. We use a familiar Eilenberg swindle. The control space under investigation is $(Z\times[0,1], Z\times[0,1[\,)$. It has an endomorphism given by $((x,s),t)\mapsto((x,s(1+t)/2),t)$. This induces

$$\sigma\colon \mathcal{A}^\rho(\mathbb{J}Z)^\wedge_\infty \longrightarrow \mathcal{A}^\rho(\mathbb{J}Z)^\wedge_\infty\,.$$

There is an obvious natural isomorphism $\sigma(A)\to A$, for arbitrary A in $\mathcal{A}^\rho(\mathbb{J}Z)^\wedge_\infty$. Therefore

$$\sigma_*\simeq \mathrm{id}\colon F^\rho(Z)\longrightarrow F^\rho(Z)\,.$$

Further, the expression

$$A\oplus\sigma(A)\oplus\sigma^2(A)\oplus\sigma^3(A)\dots$$

is clearly meaningful in $\mathcal{A}^\rho(\mathbb{J}Z)^\wedge_\infty$, and may be regarded as a *functor* τ in the variable A. (This requires some checking which is left to the reader. The subscript ρ is essential at this point.) Note that $\tau(A)\cong A\oplus\sigma\tau(A)$ for any A in $\mathcal{A}^\rho(\mathbb{J}Z)^\wedge_\infty$. For the induced map τ_* from $F^\rho(Z)$ to itself we therefore get

$$\tau_* = \mathrm{id}_*\,\tau_*\simeq\sigma_*\tau_*\,,$$
$$\tau_*\simeq \mathrm{id}_* + \sigma_*\tau_*\,,$$

using addition in the infinite loop space $F^\rho(Z)$. This shows that the identity map of $F^\rho(Z)$ is nullhomotopic. □

The next lemma will help us reduce 4.1 to 4.3.

4.4. Lemma. *Let S be a finite collection of morphisms in $\mathcal{A}(\mathbb{J}Y)$, where Y is in $\mathfrak{L}$. There exists an automorphism $\psi\colon\mathbb{J}Y\to\mathbb{J}Y$, restricting to the identity on the singular set, such that $\psi_*(f)$ is a morphism in $\mathcal{A}^\rho(\mathbb{J}Y)_\infty$ for every $f\in S$.*

Proof. We start by constructing a sequence of continuous functions $\lambda_i\colon Y\to[0,1[$, for $i=0,1,2,\dots$, with the following properties.

(1) $\lambda_0(y)=0$, $\lambda_i(y)\geq\rho_i$ and $\lambda_i(y)>\lambda_{i-1}(y)$ for all $y\in Y$ and $i>0$.
(2) If $f^v_w\neq 0$ or $f^w_v\neq 0$ for some $f\in S$ and $v,w\in Y\times[0,1[$, then v and w are both contained in the closed region of $Y\times[0,1[$ bounded by the graphs of λ_{i-1} and λ_{i+1} , for some $i>0$.

Suppose that $\lambda_0,\dots,\lambda_n$ have already been constructed in such a way that conditions (1) and (2) are not violated. From the very definition of morphism in $\mathcal{A}(\mathbb{J}Y)$, we deduce the existence of a neighbourhood U of $Y\times\{1\}$ in $Y\times[0,1]$ such that $f^v_w=0$ and $f^w_v=0$ whenever $f\in S$, $v\in U$, and w belongs to the closed region below the graph of λ_n. We can also assume that U has empty intersection with the closed region below the graph of λ_n. Now construct λ_{n+1} in such a way that U contains the closed region of $Y\times[0,1]$ above the graph of λ_{n+1}. For example, choose a metric on Y inducing the topology, equip $[0,1]$ with the standard metric and $Y\times[0,1]$ with the product (box) metric; let

$$\lambda_{n+1}(y)=\max\{\rho_{n+1}\,1-\ell_y/2\}$$

where ℓ_y is the distance from $(y,1)$ to the complement of U. This completes the inductive step, hence the construction of the sequence (λ_i).

Now it suffices to construct ψ in such a way that ψ takes the graph of λ_i to the graph of the constant function with value ρ_i, for all $i \geq 0$. This is easily arranged as follows. Let ψ take each interval $\{y\} \times [0,1]$ to itself, by the map which takes $(y, \lambda_i(y))$ to (y, ρ_i) and is linear on the subinterval bounded by the points $(y, \lambda_i(y))$ and $(y, \lambda_{i+1}(y))$. $\square$

Proof of 4.1. Let $Y = X \times]0,1]$. Let $T \subset F(Y)$ be a finitely generated simplicial subset. The simplices of T correspond to certain finite diagrams in $\mathcal{A}(\mathbb{J}Y)^{\wedge}_{\infty}$. Each object of $\mathcal{A}(\mathbb{J}Y)^{\wedge}_{\infty}$ may in turn be regarded as an idempotent endomorphism in $\mathcal{A}(\mathbb{J}Y)_{\infty}$. Hence T involves only a finite set of morphisms in $\mathcal{A}(\mathbb{J}Y)_{\infty}$, which we can also view as a finite set S of morphisms in $\mathcal{A}(\mathbb{J}Y)$. Choose $\psi\colon \mathbb{J}Y \to \mathbb{J}Y$ as in 4.5. Then

$$\psi_*(T) \subset F^{\rho}(Y) \subset F(Y)$$

and $F^{\rho}(Y)$ is contractible by 4.3. It follows that the inclusion of $\psi_*(T)$ in $F(Y)$ is nullhomotopic. But $\psi_*\colon F(Y) \to F(Y)$ is homotopic to the identity. In fact the functor induced by ψ, from $\mathcal{A}(Y)^{\wedge}_{\infty}$ to itself, is isomorphic to the identity functor. Hence the inclusion of T in $F(Y)$ is nullhomotopic. $\square$

5. The coefficient spectrum

5.1. Proposition. *$F(X) \simeq \Omega F(X \times \mathbb{R})$ for X in $\mathfrak{L}$.*

Proof. By 3.1, the commutative square of inclusion–induced maps

$$\begin{array}{ccc}
F(X \times \{0\}) & \longrightarrow & F(X \times [0,\infty[\,) \\
\big\downarrow{\scriptstyle\subset} & & \big\downarrow \\
F(X \times]-\infty, 0]\,) & \longrightarrow & F(X \times \mathbb{R})
\end{array}$$

is a homotopy pullback square. By 4.1, the upper right hand and lower left hand terms are contractible. $\square$

We can use 5.1 to define a spectrum $\mathbf{F}(X)$, essentially with n–th term $F(X \times \mathbb{R}^n)$. The details are left to the reader.

5.2. Lemma. *Let $X = \coprod_{i \in \mathbb{N}} X_i$ where each X_i is in $\mathfrak{L}$. Restriction from X to X_i for each i induces isomorphisms*

$$\pi_* F(X) \longrightarrow \prod_i \pi_* F(X_i)\,,$$
$$\pi_* \mathbf{F}(X) \longrightarrow \prod_i \pi_* \mathbf{F}(X_i)\,.$$

Proof. We know that $F(X) \simeq K(\mathcal{D}\mathcal{A}(\mathbb{J}X)_{\infty})$ and $F(X_i) \simeq K(\mathcal{D}\mathcal{A}(\mathbb{J}X_i)_{\infty})$, and

$$\mathcal{D}(\mathcal{A}\mathbb{J}X)_{\infty} \cong \prod_i \mathcal{D}\mathcal{A}(\mathbb{J}X_i)_{\infty}\,.$$

Since each $\mathcal{D}\mathcal{A}(\mathbb{J}X_i)_{\infty}$ has a cylinder functor, we may apply the main result of [Cl1] which states that the functor K_i from categories with cofibrations and weak equivalences and cylinder functor commutes with countably infinite products. (This is nontrivial because the functor $Z \mapsto \pi_i|Z|$ on the category of (bi–)simplicial sets does not generally commute with infinite products.) $\square$

Let $\mathcal{E}^\bullet \subset \mathcal{L}^\bullet$ be the full subcategory consisting of the ENR's. Together with excision, 3.1, and homotopy invariance, 4.1, lemma 5.2 shows that $\mathbf{F}|\mathcal{E}^\bullet$ is a *pro–excisive* functor [WWp]. The main result of [WWp] now applies, showing that

$$\mathbf{F}(X) \simeq \cdots \simeq X^\bullet \wedge \mathbf{F}(*)$$

by a chain of natural weak homotopy equivalences, provided X is an ENR. The rest of this section is devoted to the study of $\mathbf{F}(*)$. Recall that $F(*) = K(\mathcal{A}(\mathbb{J}*)^\wedge_\infty)$ where $\mathcal{A}$ is the additive category which we fixed early in §2. To stress the dependence on $\mathcal{A}$, we now write $F_\mathcal{A}(*)$ and $\mathbf{F}_\mathcal{A}(*)$ instead of $F(*)$ and $\mathbf{F}(*)$.

We begin with a definition which goes back to [K]; see also [PW2, §5] and [CdP]. Let $\mathcal{A}$ be a full additive subcategory of a small additive category $\mathcal{T}$.

5.4. Definition. An $\mathcal{A}$–filtration of $\mathcal{T}$ selects for each T in $\mathcal{T}$ a family of preferred direct sum decompositions $T = T'_\alpha \oplus T''_\alpha$, with T'_α in $\mathcal{A}$, subject to the following conditions.

(i) For each T the decompositions form a filtered poset under the partial order $T'_\alpha \oplus T''_\alpha \leq T'_\beta \oplus T''_\beta$ whenever $T''_\beta \subset T''_\alpha$ and $T'_\alpha \subset T'_\beta$.
(ii) Every morphism $A \to T$ factors as $A \to T'_\alpha \to T'_\alpha \oplus T''_\alpha = T$ for some α (assuming that A is in $\mathcal{A}$ and T is in $\mathcal{T}$).
(iii) Every morphism $T \to A$ factors as $T = T'_\alpha \oplus T''_\alpha \to T'_\alpha \to A$ for some α (assuming that A is in $\mathcal{A}$ and T is in $\mathcal{T}$).
(iv) If $S = S'_\alpha \oplus S''_\alpha$ and $T' = T'_\beta \oplus T''_\beta$ are preferred decompositions, then $S \oplus T = (S'_\alpha \oplus S'_\beta) \oplus (T''_\alpha \oplus T''_\beta)$ is a preferred decomposition of $S \oplus T$, and the poset of these decompositions is dense in the poset of all preferred decompositions of $S \oplus T$.

In the situation of 5.4, define a new additive category $\mathcal{T}/\mathcal{A}$ with the same objects as $\mathcal{T}$, and

$$\hom_{\mathcal{T}/\mathcal{A}}(S,T) := \hom_{\mathcal{T}}(S,T)/k(S,T)$$

where $k(S,T)$ consists of all morphisms $S \to T$ in $\mathcal{T}$ which factor through some object of $\mathcal{A}$. Note that $\mathcal{T}/\mathcal{A}$ does not depend on the choice of a particular $\mathcal{A}$–filtration of $\mathcal{T}$.

The example to have in mind is: $\mathcal{A}$ arbitrary, $\mathcal{T} = \mathcal{A}(\mathbb{J}*)$. Up to an equivalence of categories, $\mathcal{A}$ can be identified with the full subcategory of $\mathcal{A}(\mathbb{J}*)$ consisting of all objects which become isomorphic to zero in $\mathcal{A}(\mathbb{J}*)_\infty$. In this sense, $\mathcal{A}(\mathbb{J}*)$ has an obvious $\mathcal{A}$–filtration.

5.5. Theorem [PW2]**,** [CdP]**.** *Suppose that $\mathcal{T}$ has an $\mathcal{A}$–filtration. The following square, with vertical arrows induced by the 'quotient' functors, is a homotopy pullback square:*

$$\begin{tikzcd}
K(\mathcal{A}^\wedge) \arrow[r, "\subset"] \arrow[d] & K(\mathcal{T}^\wedge) \arrow[d] \\
* \simeq K(\mathcal{A}/\mathcal{A}) \arrow[r, "\subset"] & K((\mathcal{T}/\mathcal{A})^\wedge)\,.
\end{tikzcd} \quad \square$$

Recall from [K] that a small additive category $\mathcal{M}$ is *flasque* if there exist an additive functor $\tau\colon \mathcal{M} \to \mathcal{M}$ and a natural isomorphism

$$\tau(M) \cong M \oplus \tau(M)\,.$$

Then $K(\mathcal{M})$ and $K(\mathcal{M}^\wedge)$ are contractible because τ_* is a self–map of $K(\mathcal{M})$, and of $K(\mathcal{M}^\wedge)$, such that $\tau_* + \mathrm{id} \simeq \tau_*$.

5.6. Corollary. $\Omega F_{\mathcal{A}}(*) \simeq K(\mathcal{A}^\wedge)$.

Proof. Let $\mathcal{T} = \mathcal{A}(\mathbb{J}*)$ in 5.5. Then $\mathcal{T}/\mathcal{A}$ can be identified with $\mathcal{A}(\mathbb{J}*)_\infty$. Furthermore, $K(\mathcal{T}^\wedge) \simeq *$ because $\mathcal{T}$ is flasque. $\square$

5.7. Corollary. *In the situation of* 5.4 *and* 5.5*, the following is a homotopy pullback square of spectra:*

$$\begin{array}{ccc}
\mathbf{F}_{\mathcal{A}}(*) & \xrightarrow{\subset} & \mathbf{F}_{\mathcal{T}}(*) \\
\big\downarrow & & \big\downarrow \\
* \simeq \mathbf{F}_{\mathcal{A}/\mathcal{A}}(*) & \xrightarrow{\subset} & \mathbf{F}_{\mathcal{T}/\mathcal{A}}(*)\,.
\end{array}$$

Proof. For any X in $\mathcal{L}$ the category $\mathcal{T}(\mathbb{J}X)$ is $\mathcal{A}(\mathbb{J}X)$–filtered and the quotient $\mathcal{T}(\mathbb{J}X)/\mathcal{A}(\mathbb{J}X)$ is isomorphic to $(\mathcal{T}/\mathcal{A})(\mathbb{J}X)$. This holds in particular for $X = \mathbb{R}^i$, where $i \geq 0$. $\square$

5.8. Proposition. *The functor* $\mathcal{A} \mapsto \mathbf{F}_{\mathcal{A}}(*)$ *takes equivalences of small additive categories to homotopy equivalences, and* $\mathbf{F}_{\mathcal{A}}(*)$ *is contractible if* $\mathcal{A}$ *is flasque.*

Proof. If $\mathcal{A}$ is flasque, then $\mathcal{A}(\mathbb{J}X)$ is also flasque. This holds in particular when $X = \mathbb{R}^i$ for $i \geq 0$. $\square$

Given any small additive category $\mathcal{A}$, there exist a small additive category $\mathcal{T}$ and a full additive subcategory $\mathcal{A}' \subset \mathcal{T}$ such that $\mathcal{T}$ is flasque, $\mathcal{T}$ is $\mathcal{A}'$–filtered, and $\mathcal{A}'$ is equivalent to $\mathcal{A}$. This construction is due to Karoubi [K]. We have already seen it, just after 4.2: let $\mathcal{T} := \mathcal{A}^\rho(\mathbb{J}*)$. Of course, $\mathcal{T} := \mathcal{A}(\mathbb{J}*)$ is another possibility.

Now it is easy to see that 5.6, 5.7 and 5.8 determine the functor $\mathcal{A} \mapsto \mathbf{F}_{\mathcal{A}}(*)$ up to natural homotopy equivalence. Fix $\mathcal{A} = \mathcal{A}(0)$. Choose $\mathcal{A}' \subset \mathcal{T}$ such that $\mathcal{A}'$ is equivalent to $\mathcal{A}(0)$ and $\mathcal{T}$ is flasque and $\mathcal{A}'$–filtered. Let $\mathcal{A}(1) := \mathcal{T}/\mathcal{A}'$. Repeat the process to obtain $\mathcal{A}(2)$, $\mathcal{A}(3)$, and so on. Then 5.6, 5.7 and 5.8 imply immediately

$$\Omega^{\infty+1-k}\mathbf{F}_{\mathcal{A}}(*) \simeq \Omega^{\infty+1}\mathbf{F}_{\mathcal{A}(k)}(*) \simeq K(\mathcal{A}(k)^\wedge)\,.$$

Briefly, $\Omega\mathbf{F}_{\mathcal{A}(*)}$ is the usually *non–connective* K–theory spectrum of $\mathcal{A}$.

Part II: K–Theory of categories of controlled retractive spaces

6. Retractive spaces over a control space

Throughout this chapter, we fix a control space $(\bar{Z}, Z)$ and a space $\mathfrak{S}$. The category of retractive CW–spaces over $\mathfrak{S}$, subject to appropriate finiteness conditions, will play a role similar to that of $\mathcal{A}$ in §§2–4.

6.1. Definitions. Let Y be a CW–space relative to $\mathfrak{S}$. Let $\diamond Y$ be the set of cells of Y. For $e, e' \in \diamond Y$ write $e \geq e'$ if the smallest relative CW–subspace containing e also contains e'. Let $\sec(e) := \{e' \in \diamond Y \mid e \geq e'\}$.

Suppose that Y is *dimensionwise locally finite*, that is, for every cell e in Y and every $j \geq 0$ there are only finitely many j–cells which are $\geq e$. We say that Y is *controlled over* $(\bar{Z}, Z)$ if it comes equipped with a map $\kappa\colon \diamond Y \to Z$, subject to the following conditions.

- For any compact $L \subset Z$ and $i \geq 0$, the set of i–cells in $\kappa^{-1}(L)$ is finite.
- For any $z \in \bar{Z} \smallsetminus Z$ and $i \geq 0$ and any neighbourhood V of z in Z, there exists another neighbourhood W of z in Z such that $\sec(e) \subset \kappa^{-1}(V)$ whenever e is an i–cell of Y and $\sec(e) \cap \kappa^{-1}(W) \neq \emptyset$.

Next, suppose that Y' is another CW–space relative to $\mathfrak{S}$, also dimensionwise locally finite and controlled over $(\bar{Z}, Z)$, with reference map $\kappa'\colon \diamond Y' \to Z$. Let $f\colon Y \to Y'$ be a cellular map relative to $\mathfrak{S}$. Let Y'_f be the relative mapping cylinder of f, again a CW–space relative to $\mathfrak{S}$. There is an evident map $\kappa'_f\colon \diamond Y'_f \to Z$ such that $\kappa'_f(e) = \kappa'_f(e \times]0,1[) = \kappa(e)$ for $e \in \diamond Y \subset \diamond Y'_f$ and $\kappa'_f(e) = \kappa'(e)$ for $e \in \diamond Y' \subset \diamond Y'_f$. We say that f is *dimensionwise locally finite* if Y'_f is dimensionwise locally finite. We say that f is *controlled* if Y'_f is controlled with κ'_f as reference map.

Now suppose that Y and Y' come with retractions to $\mathfrak{S}$. If f is controlled, and if it is also a map over $\mathfrak{S}$, then we call f a *controlled morphism.* The dimensionwise locally finite retractive CW–spaces over and relative to $\mathfrak{S}$, controlled over $(\bar{Z}, Z)$, and the controlled morphisms between them, constitute a category

$$\mathcal{Q}(\mathfrak{S}; \bar{Z}, Z)$$

which should be regarded as a nonlinear analogue of $\mathcal{CA}(\bar{Z}, Z)$ defined in §2 and §3. We wish to make $\mathcal{Q}(\mathfrak{S}; \bar{Z}, Z)$ into a category with cofibrations and weak equivalences. We begin with the observation that there is an obvious notion of *controlled homotopy* between two controlled maps $f, g\colon Y \to Y'$ (not necessarily maps *over* $\mathfrak{S}$) as above; namely, a controlled homotopy is a controlled map of the form $Y \rtimes [0,1] \to Y'$, where $Y \rtimes [0,1]$ is the pushout of

$$Y \times [0,1] \leftarrow \mathfrak{S} \times [0,1] \to \mathfrak{S}\,.$$

A morphism in $\mathcal{Q}(\mathfrak{S}; \bar{Z}, Z)$ is a *weak equivalence* if it is invertible up to controlled homotopy as a controlled map. A morphism in $\mathcal{Q}(\mathfrak{S}; \bar{Z}, Z)$ is a *cofibration* if, up to isomorphism in $\mathcal{Q}(\mathfrak{S}; \bar{Z}, Z)$, it is the inclusion of a relative CW–subspace.

Finally we call an object Y in $\mathcal{Q}(\mathfrak{S}; \bar{Z}, Z)$ *homotopy finitely dominated* if the identity $Y \to Y$ is controlled homotopic to a controlled map $Y \to Y$ whose image is contained in the relative i–skeleton of Y, for some i. The homotopy finitely dominated objects in $\mathcal{Q}(\mathfrak{S}; \bar{Z}, Z)$ constitute a full subcategory

$$\mathcal{R}(\mathfrak{S}; \bar{Z}, Z) \subset \mathcal{Q}(\mathfrak{S}; \bar{Z}, Z)$$

which should be regarded as a nonlinear analogue of $\mathcal{DA}(\bar{Z}, Z)$ defined in §2 and §3. It inherits from $\mathcal{Q}(\mathfrak{S}; \bar{Z}, Z)$ the structure of a category with cofibrations and weak equivalences.

6.2. Definitions. Let Y be in $\mathcal{Q}(\mathfrak{S};\bar{Z},Z)$. A (relative) CW–subspace $Y' \subset Y$ is *cofinal* in Y if the following holds: For every $i \geq 0$, there exists a neighbourhood U of $\bar{Z} \smallsetminus Z$ in $\bar{Z}$ such that the portion of the i–skeleton of Y lying over $U \cap Z$ is contained in Y'.

A *germ* of controlled maps from Y_1 to Y_2 is an equivalence class of pairs (Y_1', f) where Y_1' is cofinal in X and $f: Y_1' \to Y_2$ is a controlled map. The equivalence relation is generated by restriction, i.e. $(Y_1', f) \sim (Y_1'', f|Y_1'')$ whenever Y_1' and Y_1'' are both cofinal in Y and $Y_1'' \subset Y_1'$.

Similarly, a *germ* of morphisms from Y_1 to Y_2 is an equivalence class of pairs (Y_1', f) where Y_1' is cofinal in Y_1 and $f: Y_1' \to Y_2$ is a morphism. The equivalence relation is generated by restriction. The germs of morphisms are the morphisms in a new category

$$\mathcal{Q}(\mathfrak{S};\bar{Z},Z)_\infty$$

which has the same objects as $\mathcal{Q}(\mathfrak{S};\bar{Z},Z)$. An object Y in $\mathcal{Q}(\mathfrak{S};\bar{Z},Z)_\infty$ is *germwise homotopy finitely dominated* if the identity of Y is homotopic, as a germ of controlled maps, to a controlled map germ with image contained in the i–skeleton of Y, for some i. The germwise homotopy finitely dominated objects of $\mathcal{Q}(\mathfrak{S};\bar{Z},Z)_\infty$ determine a full subcategory

$$\mathcal{R}(\mathfrak{S};\bar{Z},Z)_\infty \subset \mathcal{Q}(\mathfrak{S};\bar{Z},Z)_\infty.$$

Note that $\mathcal{R}(\mathfrak{S};\bar{Z},Z)_\infty$ will usually have *more* objects than $\mathcal{R}(\mathfrak{S};\bar{Z},Z)$; put differently, an object of $\mathcal{Q}(\mathfrak{S};\bar{Z},Z)$ can be germwise homotopy finitely dominated without being homotopy finitely dominated.

We make $\mathcal{Q}(\mathfrak{S};\bar{Z},Z)_\infty$ and $\mathcal{R}(\mathfrak{S};\bar{Z},Z)_\infty$ into categories with cofibrations and weak equivalences in the expected way. A morphism is a *cofibration* if, up to isomorphisms, it is the germ of an inclusion of a retractive CW–subspace. A morphism is a *weak equivalence* if it is invertible up to homotopy as a germ of controlled maps.

We need nonlinear analogues of 1.4, one for $\mathcal{Q}(\mathfrak{S};\bar{Z},Z)$ and one for $\mathcal{Q}(\mathfrak{S};\bar{Z},Z)_\infty$. A good notion of homotopy does not exist in these categories, but we can introduce 'homotopy categories' $\mathcal{HQ}(\mathfrak{S};\bar{Z},Z)$ and $\mathcal{HQ}(\mathfrak{S};\bar{Z},Z)_\infty$ by adjoining formal inverses for all weak equivalences. More details on how morphisms in $\mathcal{HQ}(\mathfrak{S};\bar{Z},Z)$ and $\mathcal{HQ}(\mathfrak{S};\bar{Z},Z)_\infty$ can be thought of will be given in the proof of 6.3.

6.3. Lemma. *Let $p: Y \to Y$ be an idempotent endomorphism in $\mathcal{HQ}(\mathfrak{S};\bar{Z},Z)$. Then there exist Y' and morphisms $q: Y \to Y'$, $j: Y' \to Y$ in $\mathcal{HQ}(\mathfrak{S};\bar{Z},Z)$ such that $p = jq$ and $qj = \mathrm{id}$.*

Preliminaries for the proof. We need some more definitions. Write $\mathcal{Q}$ for $\mathcal{Q}(\mathfrak{S};\bar{Z},Z)$. Let Y_1 and Y_2 be objects of $\mathcal{Q}$. A *lax* morphism $Y_1 \to Y_2$ consists of a cellular map $Y_1 \to Y_2$ relative to $\mathfrak{S}$, and a structure of object of $\mathcal{Q}$ on the relative mapping cylinder of g which extends the given structures on Y_1 and Y_2 (cylinder front and cylinder back). A *homotopy* between lax morphisms from Y_1 to Y_2 is a lax morphism from $Y_1 \leftthreetimes [0,1]$ to Y_2.

Informally, a lax morphism from Y_1 to Y_2 consists of a cellular map $g: Y_1 \to Y_2$ relative to $\mathfrak{S}$, and a homotopy $h: r_1 \simeq r_2 g$ subject to appropriate control and local finiteness conditions. This reformulation allows us to define the composition of a lax morphisms $(g,h): Y_1 \to Y_2$ and a lax morphism $(f,j): Y_2 \to Y_3$; namely, as $(fg, jg * h)$ where $jg * h$ is a concatenated homotopy (first h, then jg). A choice of orientation–preserving homeomorphism $[0,2] \to [0,1]$ is involved, making the result a little ambiguous, but this will hardly matter in the following.

It is an exercise to show that the morphisms $Y_1 \to Y_2$ in $\mathcal{H}\mathcal{Q}$ are (in canonical bijection with) the homotopy classes of lax morphisms from Y_1 to Y_2.

More preliminaries. Let $\lambda\mathcal{Q}$ be the following enlargement of $\mathcal{Q}$. An object of $\lambda\mathcal{Q}$ is a sequence of cofibrations $Y_0 \to Y_1 \to Y_2 \to \cdots$ in $\mathcal{Q}$. A morphism

$$(Y_0 \to Y_1 \to Y_2 \to \cdots) \quad \longrightarrow \quad (Y'_0 \to Y'_1 \to Y'_2 \to \cdots)$$

is an element in $\lim_i \operatorname{colim}_j \operatorname{mor}_{\mathcal{Q}}(Y_i, Y'_j)$. A lax morphism (between these objects) is an element in $\lim_i \operatorname{colim}_j \ell\mathrm{mor}_{\mathcal{Q}}(Y_i, Y'_j)$ where $\ell\mathrm{mor}_{\mathcal{Q}}(Y_i, Y'_j)$ denotes the set of lax morphisms from Y_i to Y'_j.

Proof of 6.3. We can represent p by a lax morphism, also denoted $p\colon Y \to Y$. The mapping telescope Y' of

$$X \xrightarrow{p} X \xrightarrow{p} X \xrightarrow{p} \cdots$$

is well defined as an object of $\lambda\mathcal{Q}$. It is easy to produce lax morphisms $q\colon X \to Y'$ and $j\colon Y' \to X$ such that $jq \simeq p$ and $qj \simeq \mathrm{id}$, where the symbol $\simeq$ indicates lax homotopies. It remains to show that Y' is lax homotopy equivalent to an object in $\mathcal{Q}$. Much as in lemma 1.5, this can be deduced from the fact that Y' admits a domination (up to lax homotopy) by an object in $\mathcal{Q}$, namely, the object Y. $\square$

6.4. Lemma. *Let $p\colon Y \to Y$ be an idempotent endomorphism in $\mathcal{H}\mathcal{Q}(\mathfrak{S}; \bar{Z}, Z)_\infty$. Then there exist Y' in $\mathcal{H}\mathcal{Q}(\mathfrak{S}; \bar{Z}, Z)_\infty$ and morphisms $q\colon Y \to Y'$, $j\colon Y' \to Y$ in $\mathcal{H}\mathcal{Q}(\mathfrak{S}; \bar{Z}, Z)_\infty$ such that $p = jq$ and $qj = \mathrm{id}$.*

The proof is similar to that of 6.3.

7. Nonlinear controlled K–theory

We will now be interested in the functor F taking a locally compact space X with countable base to the K–theory space

$$F(X) := K(\mathcal{R}(\mathfrak{S}; \mathbb{J}X)_\infty)\,.$$

In what sense is it a functor ? It is clear that a proper map $g\colon X_1 \to X_2$ of locally compact spaces (with countable base) induces an exact functor

$$\mathcal{R}(\mathfrak{S}; \mathbb{J}X_1)_\infty \longrightarrow \mathcal{R}(\mathfrak{S}; \mathbb{J}X_2)_\infty\,.$$

This in turn induces $g_*\colon F(X_1) \to F(X_2)$. So F is a functor on the category $\mathcal{L}$ of locally compact spaces with countable base, with proper maps as morphisms. However, F has additional functorial properties. Let X be locally compact with countable base and let $V \subset X$ be an open subset. There is an exact restriction functor

$$\mathcal{R}(\mathfrak{S}; \mathbb{J}X)_\infty \longrightarrow \mathcal{R}(\mathfrak{S}; \mathbb{J}V)_\infty$$

taking an object Y of $\mathcal{R}(\mathbb{J}X)_\infty$ to $Y|\mathbb{J}V$, the largest relative CW–subspace of Y whose cells are mapped to $V \times [0,1[$ under $\kappa\colon \diamond Y \to X \times [0,1[$. The restriction functor induces

$$F(X) \longrightarrow F(V)\,.$$

We conclude that F extends to a functor on the enlarged category $\mathcal{L}^\bullet$ with the same objects as $\mathcal{L}$, where a morphism from X_1 to X_2 is a continuous pointed map $g\colon X_1^\bullet \to X_2^\bullet$ between the one–point compactifications.

7.1. Main theorem (nonlinear version). *Suppose that X is in $\mathcal{L}$. Let $V \subset X$ be open. Then the commutative square*

$$\begin{array}{ccc} F(X \smallsetminus V) & \xrightarrow{\text{restriction}} & F(\emptyset) = * \\ \Big\downarrow{\scriptstyle\text{inclusion}} & & \Big\downarrow{\scriptstyle\text{inclusion}} \\ F(X) & \xrightarrow{\text{restriction}} & F(V) \end{array}$$

is a homotopy pullback square.

Proof of 7.1. The proof is extremely similar to that of 3.1; the purpose of the following outline is only to make that entirely clear.

First part of proof. This is mostly organization, and an invocation of the Waldhausen theorems. It concludes with the insight that it suffices to show that, of the maps

$$\begin{aligned} &K(\mathcal{R}(\mathfrak{S}; \mathbb{J}(X \smallsetminus V))_\infty) \longrightarrow K(\mathcal{R}^\omega(\mathfrak{S}; \mathbb{J}X)_\infty) && \text{(inclusion)}, \\ &K(\mathcal{R}_\omega(\mathfrak{S}; \mathbb{J}X)_\infty) \longrightarrow K(\mathcal{R}(\mathfrak{S}; \mathbb{J}V)_\infty) && \text{(restriction)}, \end{aligned}$$

the first is a homotopy equivalence, and the second is a map all of whose homotopy fibers are contractible. Here ω is a new, coarse notion of weak equivalence in $\mathcal{R}(\mathfrak{S}; \mathbb{J}X)_\infty$; a morphism in $\mathcal{R}(\mathfrak{S}; \mathbb{J}X)_\infty$ is a weak equivalence in the coarse sense if it becomes a weak equivalence in $\mathcal{R}(\mathfrak{S}; \mathbb{J}V)_\infty$. Again $\mathcal{R}_\omega(\mathfrak{S}; \mathbb{J}X)_\infty$ is $\mathcal{R}(\mathfrak{S}; \mathbb{J}X)_\infty$ with the coarse notion of weak equivalence, and $\mathcal{R}^\omega(\mathfrak{S}; \mathbb{J}X)_\infty$ is the full subcategory of $\mathcal{R}(\mathfrak{S}; \mathbb{J}X)_\infty$ determined by the objects which are weakly equivalent to the zero object in the coarse sense. In $\mathcal{R}^\omega(\mathfrak{S}; \mathbb{J}X)_\infty$ we use the standard (nameless) notion of weak equivalence.

Second part of proof. Here the goal is to verify that the hypotheses of the approximation theorem hold for the inclusion

$$\mathcal{R}(\mathfrak{S}; \mathbb{J}(X \smallsetminus V))_\infty \longrightarrow \mathcal{R}^\omega(\mathfrak{S}; \mathbb{J}X)_\infty .$$

In fact it is enough to verify the following statement, analogous to (3) in the second part of the proof of 3.1.

> Let Y be an object in $\mathcal{Q}(\mathfrak{S}; \mathbb{J}X)_\infty$ for which $\mathrm{res}(Y)$ in $\mathcal{Q}(\mathfrak{S}; \mathbb{J}V)_\infty$ is weakly equivalent to the zero object. Then there exist Y' in $\mathcal{Q}(\mathfrak{S}; \mathbb{J}(X \smallsetminus V))_\infty$ and a lax morphism from Y' to Y in $\mathcal{Q}(\mathfrak{S}; \mathbb{J}X)_\infty$ which is a domination up to homotopy (has a homotopy right inverse as a lax morphism).

The proof of this proceeds exactly like the proof of (3) in the second part of the proof of 3.1.

Third part of proof. The goal is to show that the map induced by the appropriate restriction functor,

$$K(\mathcal{R}_\omega(\mathfrak{S}; \mathbb{J}X)_\infty) \longrightarrow K(\mathcal{R}(\mathfrak{S}; \mathbb{J}V)_\infty) ,$$

is a componentwise homotopy equivalence. This comes out of a commutative diagram of categories and functors

$$\begin{array}{ccc}
\mathcal{B} & \longrightarrow & \mathcal{R}''(\mathfrak{S};\mathbb{J}V)_\infty \\
\Big\downarrow & & \Big\downarrow \\
\mathrm{res}^{-1}\mathcal{R}'(\mathfrak{S};\mathbb{J}V)_\infty & \longrightarrow & \mathcal{R}'(\mathfrak{S};\mathbb{J}V)_\infty \\
\Big\downarrow & & \Big\downarrow \\
\mathcal{R}_\omega(\mathfrak{S};\mathbb{J}X)_\infty & \xrightarrow{\mathrm{res}} & \mathcal{R}(\mathfrak{S};\mathbb{J}V)_\infty\,.
\end{array}$$

All *vertical* arrows in the diagram are inclusions of full subcategories. Specifically, $\mathcal{R}'\dots$ and $\mathcal{R}''\dots$ are the full subcategories of $\mathcal{R}(\mathfrak{S};\mathbb{J}V)_\infty$ consisting of the objects which are weakly equivalent to (relatively) finite–dimensional ones, and the objects which are actually (relatively) finite–dimensional. $\mathcal{B}$ is the full subcategory of $\mathcal{R}_\omega(\mathfrak{S};\mathbb{J}X)_\infty$ consisting of the relatively finite–dimensional objects which vanish outside $V\times[0,1[$. All categories in the diagram inherit notions of cofibration and weak equivalence from the categories in the lower row.

It is enough to show that the two arrows in the diagram with domain $\mathcal{B}$ satisfy the conditions of the approximation theorem. This is done very much as in the corresponding passage of part three of the proof of 3.1. We omit the details. $\square$

The following statements 7.2–5 can be proved exactly like 4.1, 4.2, 5.1 and 5.2, respectively.

7.2. Theorem. *$F(X\times\,]0,1]\,)$ is contractible for any X in $\mathcal{L}$.*

7.3. Corollary. *The inclusion–induced map $F(X\times\{0\})\to F(X\times[0,1])$ is a homotopy equivalence.*

7.4. Proposition. *$F(X)\simeq\Omega F(X\times\mathbb{R})$ for X in $\mathcal{L}$.*

We can use 7.4 to define a spectrum $\mathbf{F}(X)$, essentially with n–th term $F(X\times\mathbb{R}^n)$. The details are left to the reader.

7.5. Lemma. *Let $X=\coprod_{i\in\mathbb{N}}X_i$ where each X_i is in $\mathcal{L}$. Restriction from X to X_i for each i induces isomorphisms*

$$\pi_*F(X)\longrightarrow\prod_i\pi_*F(X_i)\,,$$
$$\pi_*\mathbf{F}(X)\longrightarrow\prod_i\pi_*\mathbf{F}(X_i)\,.$$

Together, 7.1, 7.3 and 7.5 imply that $\mathbf{F}|\mathcal{E}^\bullet$ is a pro–excisive functor (compare §5); hence

$$\mathbf{F}(X)\simeq\cdots\simeq X^\bullet\wedge\mathbf{F}(*)$$

by a chain of natural weak homotopy equivalences, for X in $\mathcal{E}^\bullet$. For more information on the spectrum $\mathbf{F}(*)$, see the next §.

8. The coefficient spectrum (nonlinear case)

We keep the notation of the previous §, but assume in addition that $\mathfrak{S}$ is path–connected and pointed. Let $\mathcal{A}$ be the additive category of finitely generated free left modules over $\mathbb{Z}\pi_1\mathfrak{S}$. A retractive CW–space Y over $\mathfrak{S}$ which is controlled over

$(\bar{Z}, Z)$ has a cellular chain complex $C(Y)$ in $\mathcal{CA}(\bar{Z}, Z)$. Namely, for a compact $L \subset Z$ and $n \in \mathbb{Z}$ we let

$$C(Y)_n(L) := \bigoplus_e \bigoplus_{[\omega]} \tilde{H}_n(e^\bullet)$$

where e runs through the n–cells of $Y \smallsetminus \mathfrak{S}$ with label in L, and $[\omega]$ runs through the path classes in $\mathfrak{S}$ connecting the label of e with the base point. We use this construction with $(\bar{Z}, Z) = \mathbb{J}X$, where X is locally compact with countable base. Passage from retractive CW–spaces over $\mathfrak{S}$ which are controlled over $\mathbb{J}X$ to their cellular chain complexes induces a natural transformation

$$F_1(X) \longrightarrow F_2(X) \ ,$$

where F_1 is the functor that we called F in §7, and F_2 is the functor that we called F in §2. More precisely:

$$F_1(X) = K(\mathcal{R}(\mathfrak{S}; \mathbb{J}X)_\infty) \ ,$$
$$F_2(X) = K(\mathcal{DA}(\mathbb{J}X)_\infty) \simeq K(\mathcal{A}(\mathbb{J}X)^\wedge_\infty) \, .$$

8.1. Theorem. *The induced map $\pi_0 F_1(X) \to \pi_0 F_2(X)$ is an isomorphism.*

Proof. We proceed by direct verification. Let us say that an object of $\mathcal{DA}(\mathbb{J}X)_\infty$ is *essentially concentrated in dimension k* if, in the homotopy category $\mathcal{HDA}(\mathbb{J}X)_\infty$, it can be dominated by an object which is actually zero in all dimensions except possibly k. Let $U_2(k)$ be the Grothendieck group generated by the isomorphism classes (in $\mathcal{HDA}(\mathbb{J}X)_\infty$) of such objects. Also, an object of $\mathcal{R}(\mathfrak{S}; \mathbb{J}X)_\infty$ is *essentially concentrated in dimension k* if, in the homotopy category $\mathcal{HR}(\mathfrak{S}; \mathbb{J}X)_\infty$, compare 6.3, it can be dominated by an object which has all its cells in dimension k. Let $U_1(k)$ be the Grothendieck group generated by the isomorphism classes (in $\mathcal{HR}(\mathfrak{S}; \mathbb{J}X)_\infty$) of such objects. By inspection (and use of 6.4), linearization $U_1(k) \to U_2(k)$ is an isomorphism for $k \geq 2$. From the commutative diagram

$$\begin{array}{ccc} U_1(k) & \longrightarrow & \pi_0 F_1(X) \\ \downarrow{\scriptstyle \cong} & & \downarrow \\ U_2(k) & \xrightarrow{\cong} & \pi_0 F_2(X) \end{array}$$

with $k \geq 2$, we see that $\pi_0 F_1(X) \to \pi_0 F_2(X)$ is split onto. To complete the proof, it is enough to show that every element of $\pi_0 F_1(X)$ is in the image of the tautological homomorphism $U_1(k) \to \pi_0 F_1(X)$ for some $k \geq 2$. Now every element of $\pi_0 F_1(X)$ is a difference of classes $[Y]$ with Y in $\mathcal{R}(\mathfrak{S}; \mathbb{J}X)_\infty$. Since Y is homotopy finitely dominated, there exists $k \gg 0$ such that the quotient of Y by its $(k-1)$–skeleton Y^{k-1} is essentially concentrated in dimension k. Then $[Y]$ equals

$$[Y/Y^{k-1}] + \sum_{j<k} [Y^j/Y^{j-1}] \ = \ [Y/Y^{k-1}] + \sum_{j<k} \pm [\Sigma_{\mathfrak{S}}^{k-j}(Y^j/Y^{j-1})] \, . \quad \square$$

8.2. Corollary. *Linearization* $\mathbf{F}_1(*) \to \mathbf{F}_2(*)$ *induces isomorphisms on* π_i *for all* $i \le 0$.

Proof. Note that $\pi_i \mathbf{F}_1(*) \cong \pi_0 F_1(\mathbb{R}^{-i})$ and $\pi_i \mathbf{F}_2(*) \cong \pi_0 F_2(\mathbb{R}^{-i})$. $\square$

8.3. Proposition. $\Omega F_1(*) \simeq K(\mathcal{R}(\mathfrak{S}))$ *where* $\mathcal{R}(\mathfrak{S}) := \mathcal{R}(\mathfrak{S}; *, *)$.

Strategy of the proof. Here evidently we have to deviate a little from the pattern of §5. The proof is an application of Waldhausen's fibration theorem [Wd, 1.6.4] and his approximation theorem [Wd, 1.6.7]. We work with the following commutative diagram of categories and functors (explanations follow):

$$\begin{tikzcd}
\mathcal{R}(\mathfrak{S}; *, *) \arrow[r] \arrow[d, "\psi_1"] & \mathcal{R}(\mathfrak{S}; \mathbb{J}*) \arrow[r] \arrow[d, "="] & \mathcal{R}(\mathfrak{S}; \mathbb{J}*)_\infty \\
\mathcal{R}^u(\mathfrak{S}; \mathbb{J}*) \arrow[r] & \mathcal{R}(\mathfrak{S}; \mathbb{J}*) \arrow[r] & \mathcal{R}_u(\mathfrak{S}; \mathbb{J}*) \arrow[u, "\psi_2"']
\end{tikzcd}$$

The upper row should be self–explanatory except possibly for the left hand arrow, which is induced by the inclusion of the control space $(*, *) \cong (\{0\}, \{0\})$ in $([0,1], [0,1[\,)$. To get the lower row, we consider two notions of weak equivalence in $\mathcal{R}(\mathfrak{S}; \mathbb{J}*)$: the standard one (nameless), and the one pulled back from $\mathcal{R}(\mathfrak{S}; \mathbb{J}*)_\infty$, which we call u. The right–hand term in the lower row is $\mathcal{R}(\mathfrak{S}; \mathbb{J}*)$ with the u notion of weak equivalence, and the left–hand term in the lower row is the full subcategory of $\mathcal{R}(\mathfrak{S}; \mathbb{J}*)$ spanned by those objects which are u–equivalent to zero. The functor ψ_1 is again induced by the inclusion of $(*, *) \cong (\{0\}, \{0\})$ in $([0,1], [0,1[\,)$, and ψ_2 is forgetful. — It is enough to show:

(i) The functor ψ_1 induces a homotopy equivalence of K–theory spaces.
(ii) The functor ψ_2 induces a homotopy equivalence of $K(\mathcal{R}_u(\mathfrak{S}; \mathbb{J}*))$ with the base point component of $K(\mathcal{R}(\mathfrak{S}; \mathbb{J}*)_\infty)$.
(iii) The lower row determines a homotopy fiber sequence of K–theory spaces.
(iv) The K–theory space of $\mathcal{R}(\mathfrak{S}; \mathbb{J}*)$ is contractible.

We will deduce (i) and (ii) from the approximation theorem, (iii) from the fibration theorem, and (iv) from an Eilenberg swindle.

Proof of (i). We need to check that ψ_1 has the properties App1 and App2 formulated in [Wd, §1.6]. It is clear that App1 holds. For App2, think of ψ_1 as an inclusion. Let a morphism $f\colon Y_1 \to Y_2$ in $\mathcal{R}^u(\mathfrak{S}; \mathbb{J}*)$ be given, with Y_1 in the subcategory $\mathcal{R}(\mathfrak{S}; *, *)$; we are looking for a factorization

$$Y_1 \to ? \to Y_2$$

of f in which $Y_1 \to ?$ is a cofibration and $? \to Y_2$ is a weak equivalence. — Since Y_2 is homotopy finitely dominated (6.1), the identity $Y_2 \to Y_2$ is controlled homotopic (rel $\mathfrak{S}$) to a cellular map whose image is contained in the relative i–skeleton of Y_2 , for some i. Since Y_2 is weakly equivalent to zero in $\mathcal{R}(\mathfrak{S}; \mathbb{J}*)_\infty$, the inclusion of its i–skeleton is controlled homotopic (rel $\mathfrak{S}$) to a cellular map whose image is contained in a relative CW–subspace Y_2' of Y_2 with only finitely many cells. The labels in $[0, 1\,[$ of these finitely many cells are irrelevant for all control purposes, so we may assume that they are all equal to 0. Then Y_2' is an object of $\mathcal{R}(\mathfrak{S}; *, *)$.

The inclusion $Y_2' \to Y_2$ is a finite domination up to lax homotopy, giving rise to a lax endomorphism $p\colon Y_2' \to Y_2'$ which is idempotent up to lax homotopy. (Again, control is not an issue here.) By 6.3, we can find a morphism $j\colon Y_2'' \to Y_2'$ in $\mathcal{R}(\mathfrak{S};*,*)$ such that the composition $Y_2'' \to Y_2' \hookrightarrow Y_2$ is a weak equivalence.

Remembering $f\colon Y_1 \to Y_2$ now, we note that it factors through the weak equivalence $Y_2'' \to Y_2$ up to 'concordance'. That is, there exist a structure of object in $\mathcal{R}(\mathfrak{S};*,*)$ on $Y_1 \curlywedge [0,1]$, and a morphism $\bar{f}\colon Y_1 \curlywedge [0,1] \to Y_2$ whose restriction to $Y_1 \curlywedge \{0\} \cong Y_1$ is f, and a factorization of $\bar{f}|Y_1 \curlywedge \{1\}$ through $Y_2'' \to Y_2$. We now define Y_2''' as the pushout of

$$Y_1 \curlywedge [0,1] \leftarrow Y_1 \curlywedge \{1\} \to Y_2''.$$

Using $\bar{f}$, we still have a weak equivalence $Y_2''' \to Y_2$. We also have an inclusion of $Y_1 \cong Y_1 \curlywedge \{0\}$ in Y_2'''. The composition $Y_1 \to Y_2''' \to Y_2$ is f. This completes the proof of (i).

Proof of (ii). Let ${}_{\mathrm{ez}}\mathcal{R}(\mathfrak{S};\mathbb{J}*)_\infty \subset \mathcal{R}(\mathfrak{S};\mathbb{J}*)_\infty$ and ${}_{\mathrm{ez}}\mathcal{R}_u(\mathfrak{S};\mathbb{J}*) \subset \mathcal{R}_u(\mathfrak{S};\mathbb{J}*)$ be the full subcategories spanned by the objects having no cells in dimensions < 2. It follows from [Wd, 1.6.2] that the inclusions of these subcategories induce homotopy equivalences of the K–theories. Hence it is enough the verify (ii) with ${}_{\mathrm{ez}}\mathcal{R}(\mathfrak{S};\mathbb{J}*)_\infty$ and ${}_{\mathrm{ez}}\mathcal{R}_u(\mathfrak{S};\mathbb{J}*)$ instead of $\mathcal{R}(\mathfrak{S};\mathbb{J}*)_\infty$ and $\mathcal{R}_u(\mathfrak{S};\mathbb{J}*)$. We will deduce this simplified version of (ii) from the approximation theorem. The forgetful functor

$${}_{\mathrm{ez}}\mathcal{R}_u(\mathfrak{S};\mathbb{J}*) \longrightarrow {}_{\mathrm{ez}}\mathcal{R}(\mathfrak{S};\mathbb{J}*)_\infty$$

clearly satisfies App1. It remains to verify that is satisfies App2, weakened so as to exclude the objects in ${}_{\mathrm{ez}}\mathcal{R}(\mathfrak{S};\mathbb{J}*)_\infty$ whose class in $K_0(\mathcal{R}(\mathfrak{S};\mathbb{J}*)_\infty)$ is nonzero. As in the proof of (i), we can further reduce to a special case: thus we only have to verify that, for every Y in ${}_{\mathrm{ez}}\mathcal{R}(\mathfrak{S};\mathbb{J}*)_\infty$ with $[Y] = 0 \in K_0(\mathcal{R}(\mathfrak{S};\mathbb{J}*)_\infty)$, there exist Y' in $\mathcal{R}_u(\mathfrak{S};\mathbb{J}*)$ and a weak equivalence $\psi_2(Y') \to Y$. This amounts to saying that Y is weakly equivalent to a finite (alias finite dimensional) object in ${}_{\mathrm{ez}}\mathcal{R}(\mathfrak{S};\mathbb{J}*)_\infty$. To prove this, at last, choose $k \gg 0$ such that Y/Y^{k-1} is essentially concentrated in dimension k. This is possible since Y is homotopy finitely dominated. By hypothesis the image of $[Y]$ in $K_0(\mathcal{A}(\mathbb{J}*)_\infty)$ is zero. Also, the image of $[Y^{k-1}]$ in $K_0(\mathcal{A}(\mathbb{J}*)_\infty)$ is zero, since it comes from $K_0(\mathcal{A}(\mathbb{J}*)) = 0$ (compare 5.6). Hence the image of $[Y/Y^{k-1}]$ in $K_0(\mathcal{A}(\mathbb{J}*)_\infty)$ is zero. It follows easily that there exist a $(k+1)$–dimensional object Y' with the same $(k-1)$–skeleton as Y, and a weak equivalence $Y' \to Y$ relative to Y^{k-1}. (Here we use the assumption that Y has no cells in dimensions < 2, relative to $\mathfrak{S}$.) This completes the proof of (ii).

Proof of (iii). This looks like a straightforward application of the fibration theorem, but there is a slight difficulty. Namely, the weak equivalences in $\mathcal{R}_u(\mathfrak{S};\mathbb{J}*)$ do not satisfy the extension axiom [Wd, §1.2]. One way to fix this is to replace u by a slightly coarser notion of weak equivalence, v , which is as follows. A morphism f in $\mathcal{R}(\mathfrak{S};\mathbb{J}*)$ is a v–equivalence if $\Sigma^k f$ is an u–equivalence for $k \gg 0$. Then v does satisfy the extension axiom. So the inclusion functors

$$\mathcal{R}^v(\mathfrak{S};\mathbb{J}*) \to \mathcal{R}(\mathfrak{S};\mathbb{J}*) \to \mathcal{R}_v(\mathfrak{S};\mathbb{J}*)$$

give a homotopy fiber sequence of K–theory spaces. But [Wd, 1.6.2] implies that the inclusion $\mathcal{R}^u(\mathfrak{S};\mathbb{J}*) \to \mathcal{R}^v(\mathfrak{S};\mathbb{J}*)$ and the identity functor $\mathcal{R}_u(\mathfrak{S};\mathbb{J}*) \to \mathcal{R}_v(\mathfrak{S};\mathbb{J}*)$ induce homotopy equivalences of the K–theory spaces. The proof of (iii) is complete.

Proof of (iv). Although $\mathcal{R}(\mathfrak{S};\mathbb{J}*)$ is not an additive category, it is 'flasque' in the following sense. There exists an exact functor $\tau:\mathcal{R}(\mathfrak{S};\mathbb{J}*)\to\mathcal{R}(\mathfrak{S};\mathbb{J}*)$ such that τ and $\mathrm{id}\amalg\tau$ are related by a chain of natural weak equivalences. To describe τ, we think of $\mathbb{J}*$ as $([0,\infty],[0,\infty[\,)$. Then $\mathbb{J}*$ has an endomorphism $s\mapsto s+1$ which induces $\sigma:\mathcal{R}(\mathfrak{S};\mathbb{J}*)\to\mathcal{R}(\mathfrak{S};\mathbb{J}*)$. Define τ by

$$\tau(X)=X\amalg\sigma(X)\amalg\sigma^2(X)\amalg\ldots\,. \quad \square$$

9. A reformulation without cells

The definition in §6 of a retractive CW–space over $\mathfrak{S}$ with control in $(\bar Z,Z)$ was chosen to be as close as possible to the concept of a chain complex of $\mathcal{A}(\bar Z,Z)$–objects. (Here $\mathcal{A}$ might be the category of finitely generated free $\mathbb{Z}\pi_1(\mathfrak{S})$–modules, or just any additive category.) This made it possible to present §§6–8 as just a minor variation on §§2–5. As a result, however, it will take us more than the usual amount of work, and a π_0–sacrifice, to get cell–free versions of the main results of §§6–8. (The π_0–sacrifice is more common than it might seem: there are places in the controlled literature, such as [V1, Prop. 2.4], where it sneaks in undeclared.)

The idea is to trade retractive CW–spaces Y over $\mathfrak{S}$ with control over $(\bar Z,Z)$ for retractive spaces Y over $\mathfrak{S}\times Z$. This works if $(\bar Z,Z)$ satisfies an appropriate local contractibility condition. For now we fix $\mathfrak{S}$ and and an arbitrary control space $(\bar Z,Z)$.

9.1. Definitions. Let Y_1 and Y_2 be retractive spaces over $\mathfrak{S}\times Z$, with retractions $r_i:Y_i\to\mathfrak{S}\times Z$ for $i=1,2$. Let $f:Y_1\to Y_2$ be a map relative to $\mathfrak{S}\times Z$ (not necessarily over $\mathfrak{S}\times Z$). We say that f is *controlled* if it satisfies the following condition:

> for every $z\in\bar Z\smallsetminus Z$ and neighbourhood V of z in $\bar Z$, there exists a smaller neighbourhood W of z in $\bar Z$ such that $\{r_1(y),r_2f(y)\}\cap(\mathfrak{S}\times W)\neq\emptyset$ implies $\{r_1(y),r_2f(y)\}\subset\mathfrak{S}\times V$, for any $y\in Y_1$.

Compare 6.1. In the same spirit, we can speak of controlled homotopies between controlled maps $Y_1\to Y_2$, and of controlled homotopy equivalences $Y_1\to Y_2$.

Suppose that Y is retractive over $\mathfrak{S}\times Z$, with retraction $r:Y\to\mathfrak{S}\times Z$, and with a relative CW structure. We say that the CW–structure is *dimensionwise locally finite* and/or *controlled* if the pushout of $Y\leftarrow\mathfrak{S}\times Z\to Z$ has the corresponding properties, as a CW–space relative to Z.

Remark. Suppose that Y is retractive over $\mathfrak{S}\times Z$, with retraction $r:Y\to\mathfrak{S}\times Z$ and a controlled relative CW–structure. For each cell e of Y, choose a characteristic map $\xi_e:\mathbb{D}^{|e|}\to Y$ mapping the interior of $\mathbb{D}^{|e|}$ homeomorphically to e. Let Y' be the pushout of

$$Y\xleftarrow{s}\mathfrak{S}\times Z\to\mathfrak{S}$$

where s is the structural section. Define $\kappa:\diamond Y'\to Z$ by $\kappa(e)=r(\xi_e(0))$ (we are identifying the relative cells of Y' with those of Y). It is easy to verify that κ makes Y' into a retractive space over $\mathfrak{S}$ with control over $(\bar Z,Z)$, as defined in 6.1.

9.2. Definitions. We make a category $t\mathcal{Q}(\mathfrak{S};\bar Z,Z)$ with cofibrations and weak equivalences, as follows. The objects are the retractive spaces Y over $\mathfrak{S}\times Z$ and the morphisms are the retractive maps over $\mathfrak{S}\times Z$. A morphism $f:Y_1\to Y_2$ is a *cofibration* if, as a controlled map (controlled over $(\bar Z,Z)$, and relative to $\mathfrak{S}\times Z$) it has the controlled homotopy extension property. A morphism $f: Y_1 \to Y_2$ is a *weak equivalence* if, as a controlled map, it is invertible up to controlled homotopy.

An object Y in $t\mathcal{Q}(\mathfrak{S}; \bar{Z}, Z)$ is *homotopy finite* if there exists a weak equivalence $Y' \to Y$ where Y' comes with a controlled relative CW–structure which is locally finite and finite dimensional.

We now define $t\mathcal{R}(\mathfrak{S}; \bar{Z}, Z)$ as the full subcategory of $t\mathcal{Q}(\mathfrak{S}; \bar{Z}, Z)$ consisting of the homotopy finite objects.

9.3. Definitions. Let Y_1 and Y_2 be objects of $t\mathcal{Q}(\mathfrak{S}; \bar{Z}, Z)$. We say that a subspace Y_1' of Y_1, relative to $\mathfrak{S} \times Z$, is *cofinal* if there exists a neighbourhood U of $\bar{Z} \smallsetminus Z$ in $\bar{Z}$ such that the portion of Y_1' lying over $U \cap Z$ equals the portion of Y_1 lying over $U \cap Z$. A *germ* of controlled maps from Y_1 to Y_2 is an equivalence class of pairs (Y_1', f) where Y_1' is cofinal in Y and $f: Y_1' \to Y'$ is a controlled map. The equivalence relation is generated by restriction.

Similarly, a *germ* of morphisms from Y_1 to Y_2 is an equivalence class of pairs (Y_1', f) where Y_1' is cofinal in Y and $f: Y_1' \to Y_2$ is a morphism. The equivalence relation is generated by restriction. The germs of morphisms are the morphisms in a new category

$$t\mathcal{Q}(\mathfrak{S}; \bar{Z}, Z)_\infty$$

which has the same objects as $t\mathcal{Q}(\mathfrak{S}; \bar{Z}, Z)$. A morphism $f: Y_1 \to Y_2$ in $t\mathcal{Q}(\mathfrak{S}; \bar{Z}, Z)_\infty$ is a *cofibration* if, as a germ of controlled maps (relative to $\mathfrak{S} \times Z$), it has the controlled homotopy extension property. A morphism $f: Y_1 \to Y_2$ in $t\mathcal{Q}(\mathfrak{S}; \bar{Z}, Z)_\infty$ is a *weak equivalence* if, as a germ of controlled maps, it is invertible up to controlled homotopy.

An object Y in $t\mathcal{Q}(\mathfrak{S}; \bar{Z}, Z)_\infty$ is *germwise homotopy finite* if there exists a weak equivalence $Y' \to Y$ in $t\mathcal{Q}(\mathfrak{S}; \bar{Z}, Z)_\infty$ where Y' comes with a relative CW–structure which is locally finite and finite dimensional. The germwise homotopy finite objects of $t\mathcal{Q}(\mathfrak{S}; \bar{Z}, Z)_\infty$ determine a full subcategory

$$t\mathcal{R}(\mathfrak{S}; \bar{Z}, Z)_\infty \subset t\mathcal{Q}(\mathfrak{S}; \bar{Z}, Z)_\infty.$$

We are almost ready for a comparison theorem; what is still missing is the local contractibility condition on $(\bar{Z}, Z)$.

9.4. Definition. The control space $(\bar{Z}, Z)$ is *contractible at infinity* if the following holds. For every $z \in \bar{Z} \smallsetminus Z$ and every neighbourhood V of z in $\bar{Z}$, there exists a smaller neighbourhood W of z in $\bar{Z}$ such that the inclusion $W \cap Z \to V \cap Z$ is homotopic to a constant map.

9.5. Theorem. *Suppose that $(\bar{Z}, Z)$ is contractible at infinity and $\bar{Z}^\bullet$ has a countable base. Then there is a chain of natural homotopy equivalences relating $K(t\mathcal{R}(\mathfrak{S}; \bar{Z}, Z)_\infty)$ to the base point component of $K(\mathcal{R}(\mathfrak{S}; \bar{Z}, Z)_\infty)$.*

Proof, part 1. Let $\mathcal{B}$ be the auxiliary category defined as follows. An object of $\mathcal{B}$ is an object Y of $t\mathcal{R}(\mathfrak{S}; \bar{Z}, Z)_\infty$ with a controlled relative CW–structure which is locally finite and finite dimensional. We also insist that a choice of points y_e in each cell $e \subset Y$ has been made. A morphism $Y_1 \to Y_2$ in $\mathcal{B}$ is a morphism $Y_1 \to Y_2$ in $t\mathcal{R}(\mathfrak{S}; \bar{Z}, Z)_\infty$ which is cellular.

A straightforward application of the approximation theorem shows that the forgetful functor $\mathcal{B} \to t\mathcal{R}(\mathfrak{S}; \bar{Z}, Z)_\infty$ induces a homotopy equivalence of the K–theory spaces.

Part 2. Let $\mathcal{C} \subset \mathcal{R}(\mathfrak{S}; \bar{Z}, Z)_\infty$ be the full subcategory consisting of the finite dimensional objects. Our next goal is to show: $(*)$ the corresponding inclusion of K–theory spaces is a homotopy equivalence from $K(\mathcal{C})$ to the base point component of $K(\mathcal{R}(\mathfrak{S}; \bar{Z}, Z)_\infty)$.

An Eilenberg swindle shows that $\pi_0 K(\mathcal{C}) = 0$. (The point is that every object Y of $\mathcal{C}$ has a sequence of cofinal subobjects Y_i with $i = 1, 2, 3, \dots$ such that

$$\infty \cdot Y := \coprod_{i \geq 1} Y_i$$

is still a well defined object of $\mathcal{C}$. Then $[\infty \cdot Y] = [Y] + [\infty \cdot Y]$ in $K_0(\mathcal{C})$, hence $[Y] = 0$, hence $K_0(\mathcal{C}) = 0$ since $[Y]$ was arbitrary.) With the approximation theorem and the usual arguments involving suspension, the proof of $(*)$ now boils down to the following verification:

$(**)$ *For any object Y in $\mathcal{R}(\mathfrak{S}; \bar{Z}, Z)_\infty$ with $[Y] = 0 \in K_0(\mathcal{R}(\mathfrak{S}; \bar{Z}, Z)_\infty)$ and no cells in dimensions < 2, there is a weak equivalence $Y' \to Y$ with Y' in $\mathcal{C}$.*

Indeed we can choose $k \gg 0$ such that Y/Y^{k-1} is essentially concentrated in dimension k (compare proof of 8.1); then

$$0 = [Y] = [Y/Y^{k-1}] + \sum_{j<k} [Y^j/Y^{j-1}]$$

in $K_0(\mathcal{R}(\mathfrak{S}; \bar{Z}, Z)_\infty)$. Now the terms $[Y^j/Y^{j-1}]$ are represented by objects which come from $\mathcal{C}$; hence they are zero. Hence $[Y/Y^{k-1}] = 0$, in $K_0(\mathcal{R}(\mathfrak{S}; \bar{Z}, Z)_\infty)$ and a fortiori in $K_0(\mathcal{A}(\bar{Z}, Z)_\infty)$ where $\mathcal{A}$ is the category of finitely generated projective left $\mathbb{Z}\pi_1(\mathfrak{S})$–modules. It follows (compare proof of 8.1) that there exist a $(k+1)$–dimensional object Y' with the same $(k-1)$–skeleton as Y, and a weak equivalence $Y' \to Y$ relative to Y^{k-1}.

Part 3. It remains to show that the forgetful functor $\psi\colon \mathcal{B} \to \mathcal{C}$ defined by the remark after 9.1 induces a homotopy equivalence of K–theory spaces. We check that it satisfies the hypotheses of the approximation theorem. The first approximation property is satisfied by definition (a morphism in $\mathcal{B}$ is a weak equivalence if and only if its image morphism in $\mathcal{C}$ is a weak equivalence). The second approximation property simplifies to:

$(\natural\natural)$ *Given Y in $\mathcal{B}$ and a cofibration $f\colon \psi(Y) \to Y'$ in $\mathcal{C}$ such that the cofiber $Y'/\psi(Y)$ is concentrated in dimension k, there exist a cofibration $Y \to Y''$ in $\mathcal{B}$ and an extension of f to a weak equivalence $\psi(Y'') \to Y'$.*

Given any (relative) cell e of $Y'/\psi(Y)$, we first look for a partial solution of $(\natural\natural)$, namely: $(\natural\natural\natural)$ a cofibration $Y \to Y_e$ in $\mathcal{B}$ such that Y_e/Y has exactly one cell (that of dimension k) and an extension of f to a weak equivalence f_e from $\psi(Y_e)$ to $\psi(Y) \cup e$. Here is a systematic way to obtain such partial solutions.

Let $N(e)$ be the smallest relative CW–subspace of Y' containing e. The inverse image of $N(e)$ in Y is a relative CW–subspace $M(e)$ of Y with finitely many relative cells, all of dimension $< k$. Let $L(e)$ be the closure of the union of the images of the cells of $M(e)$ under

$$Y \xrightarrow{r} \mathfrak{S} \times Z \to Z\,.$$

If $L(e)$ is contractible in a larger subset $L'(e)$ of Z, then it is easy to find a partial solution as in $(\natural\natural\natural)$ above — with a homotopy equivalence $\psi(Y_e) \to \psi(Y) \cup e$ which is relative to $\psi(Y)$.

From a control point of view, the usefulness of such a partial solution depends on how we can control the size of $L'(e)$. Therefore choose a metric on $\bar{Z}^{\bullet}$ and let $\varepsilon(e)$ be the infimum of the diameters of those $L'(e)$. If no such $L'(e)$ exist, set $\varepsilon(e) = \infty$ and call e *inadmissible*. We can ignore the inadmissible e because the admissible ones make up a cofinal subobject of Y' , due to the local contractibility at infinity of $(\bar{Z}, Z)$.

For each admissible e, make a choice of $L'(e)$ with diameter less than $2\varepsilon(e)$. Using that particular $L'(e)$, find a partial solution (Y_e, f_e) as in ($\natural\natural\natural$). Let Y'' be the union of the Y_e along their common subspace Y. The union of the f_e is then a morphism $\psi(Y'') \to Y$ which is a weak equivalence. $\square$

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