HOMOLOGY WITHOUT SIMPLICES

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1. Introduction

Homology theory is the oldest part of algebraic topology. It was created in the last years of the 19th century by Poincaré . In those days, our modern concepts of topological space and metric space did not yet exist. Concepts like *continuous map* and *homeomorphism* existed under different names. Many of the instances of topological spaces (as we would say) which Poincaré studied were subsets of $\mathbb{R}^n$ which could be decomposed into finitely many *simplices*.[1] He saw that various integers which could be extracted from such a decomposition[2] did not depend on the decomposition: they were (suspected by him to be) *homeomorphism invariants*. He did not prove all of that by himself, but it was all proved in the next 50 years as his revolutionary ideas were developed in an orderly fashion.
It is safe to say that all algebraic topologists are in awe of Poincaré for his creation of homology theory. Perhaps it is for this reason that even in the more modern textbooks on homology theory, we still see those simplices all over the place. Many influential and less influential mathematicians with a more detached view of algebraic topology have found this aspect of algebraic topology annoying. Although I am as much in awe of Poincaré as anybody, I believe they have a point. Homology theory can take some new viewpoints on board and ditch some old ones. And if that makes it look less miraculous, then so be it.

2. Continuity is a local property

Topological spaces and continuous maps form a category.
A category $\mathcal{C}$ consists of a collection of *objects*, $\mathrm{ob}(\mathcal{C})$; for any two objects a, b , a set $\mathrm{mor}_{\mathcal{C}}(a, b)$, or $\mathrm{mor}(a, b)$ for short, whose elements are the *morphisms* from a to b; and some additional data. The additional data are as follows: for any three objects a, b, c, a map is specified called *composition*,

$$\mathrm{mor}(b, c) \times \mathrm{mor}(a, b) \longrightarrow \mathrm{mor}(a, c)\ ;\ \ (f, g) \mapsto f \circ g\ .$$

[1] *Simplices* is the plural of *simplex*. A simplex of dimension k in $\mathbb{R}^n$ is a (compact) subset of $\mathbb{R}^n$ having the form

$$\Big\{ \sum_{i=0}^{k} s_i v(i) \ \Big|\ s_i \geq 0,\ \sum_{i=0}^{k} s_i = 1 \Big\}$$

where $v(0), v(1), \ldots, v(k) \in \mathbb{R}^n$ are chosen vectors such that $v(1)-v(0), v(2)-v(0), \ldots, v(k)-v(0)$ are linearly independent. A simplex of dimension 0 is a single point, a simplex of dimension 1 is an edge, a simplex of dimension 2 is a triangle, a simplex of dimension 3 is a tetrahedron, and so on.

[2] The simplest and best known example is the Euler characteristic $r_0 - r_1 + r_2 - r_3 + r_4 - \cdots$ where r_k is the number of k-simplices in the decomposition. You probably know this from Euler's polyhedron formula, $V - E + F = 2$.

This is required to satisfy the associative law: $(f \circ g) \circ h = f \circ (g \circ h)$ in $\mathrm{mor}(a, d)$ whenever $f \in \mathrm{mor}(c, d)$, $g \in \mathrm{mor}(b, c)$ and $h \in \mathrm{mor}(a, b)$. It is also required to have two-sided units: that is, for every object a there is a distinguished element $\mathrm{id}_a \in \mathrm{mor}(a, a)$, the identity morphism of a, which acts like a two-sided unit for composition.
In our example, the category $\mathcal{T}$ of topological spaces and continuous maps, the objects are the topological spaces $X, Y, ...$ and $\mathrm{mor}_{\mathcal{T}}(X, Y)$ is the set of continuous maps from X to Y. Composition of morphisms is composition of continuous maps and the identity morphism of X is the identity map $X \to X$.

The following observation about continuous maps relates categorical notions to the concept of an open covering. Let X and Y be topological spaces and let $(U_j)_{j \in J}$ be an open covering of X. In other words each U_j is an open subset of X and the union of all U_j is X.

Lemma 2.1. *Let $(f_j : U_j \to Y)_{j \in J}$ be a family of continuous maps. Suppose that, for all $i, j \in J$ the maps f_i and f_j agree on $U_i \cap U_j$. Then there exists a unique continuous map $f : X \to Y$ such that f agrees with f_j on U_j , for every $j \in J$.*

The proof is easy. The lemma is a precise formulation of what I meant by writing that continuity is a local property. The lemma could be formulated with more category language. We could say: let a family of morphisms $(f_j)_{j \in J}$ be given, where $f_j \in \mathrm{mor}(U_j\ , Y)$. There are preferred morphisms in $\mathrm{mor}(U_i \cap U_j\ , U_i)$ and $\mathrm{mor}(U_i \cap U_j\ , U_j)$, the inclusion maps. Composition with these defines maps from $\mathrm{mor}(U_i\ , Y)$ and from $\mathrm{mor}(U_j\ , Y)$ to $\mathrm{mor}(U_i \cap U_j\ , Y)$ which we may call *restriction maps*. Suppose that f_j and f_i are taken to the same element of $\mathrm{mor}(U_i \cap U_j\ , Y)$ by these restriction maps, for all $i, j \in J$. Then there exists a unique $f \in \mathrm{mor}(X, Y)$ such that ... (*reader: finish sentence*). The meaning of mor is $\mathrm{mor}_{\mathcal{T}}$ throughout.

3. The spirit of algebra in topology

Let us think about ways to turn the category $\mathcal{T}$ of topological spaces into something else which feels more algebraic.

One brave step in that direction is to divide the morphism sets $\mathrm{mor}_{\mathcal{T}}(X, Y)$ into equivalence classes using the relation *homotopic*. We obtain a new category $\mathcal{HT}$ with the same objects as $\mathcal{T}$. The morphism set $\mathrm{mor}_{\mathcal{HT}}(X, Y)$ is $\mathrm{mor}_{\mathcal{T}}(X, Y)$ modulo homotopy, that is, the set of homotopy classes of continuous maps from X to Y. (*Reader: define composition of morphisms in $\mathcal{HT}$. Explain why it is well defined.*) The category $\mathcal{HT}$ certainly feels more algebraic than $\mathcal{T}$. For example, $\mathrm{mor}_{\mathcal{T}}(S^1, S^1)$ is uncountably infinite, but $\mathrm{mor}_{\mathcal{HT}}(S^1, S^1)$ is countably infinite and we have a preferred bijection from it to $\mathbb{Z}$, the set of integers. Also, for a simply connected space Y, the set $\mathrm{mor}_{\mathcal{HT}}(S^n, Y)$ has a preferred structure of abelian group (where $n > 1$ is an integer). Every continuous map $Y_0 \to Y_1$ between simply connected spaces induces a homomorphism of abelian groups from $\mathrm{mor}_{\mathcal{HT}}(S^n, Y_0)$ to $\mathrm{mor}_{\mathcal{HT}}(S^n, Y_1)$. This is all very good, but here we take the view that $\mathcal{HT}$ is too difficult. Passing to homotopy classes is a good idea, but we keep it for later.

Right now we aim to conjure the spirit of algebra not by thinking *homotopy*, but by thinking of ways to add and subtract elements of $\mathrm{mor}_{\mathcal{T}}(X, Y)$. One way to achieve this is to enlarge the category $\mathcal{T}$ to a category $\mathcal{PAT}$ with the same objects as $\mathcal{T}$, in such a way that

$$\mathrm{mor}_{\mathcal{PAT}}(X,Y) = \text{ free abelian group generated by } \mathrm{mor}_{\mathcal{T}}(X,Y)\ .$$

This means that a morphism in $\mathcal{PAT}$ from a topological space X to a topological space Y is a linear combination, with integer coefficients, of ordinary continuous maps from X to Y. For example, if you have some continuous maps from X to Y, say f, g and h, then you may write down $5f - 3g + 2h$ and view that as a morphism from X to Y in $\mathcal{PAT}$. Composition of morphisms in $\mathcal{PAT}$ works like this: if I have a morphism from X to Y, say $5f - 3g + 2h$, and another morphism from W to X, say $-6p + q + 7r$, then the composition is

$$-30fp + 5fq + 35fr + 18gp - 3gq - 21gr - 12hp + 2hq + 14hr\ .$$

Simplifications can sometimes be made; for example if it happens that $fp = hr$, then we may write $-16fp + 5fq + 35fr + 18gp - 3gq - 21gr - 12hp + 2hq$ instead.

The definition of $\mathcal{PAT}$ is simple-minded. It looks as if it cannot possibly do much good. We may want to scream out against the idiocy of it, but let us formulate our objections in a more civilised manner. Then we shall see that it is a good first attempt.[3] A wise objection to $\mathcal{PAT}$ is that it does not satisfy the analogue of lemma 2.1. The following examples illustrate that.

Example 3.1. Let $X = \{1, 2, \ldots, p\}$ and $Y = \{1, 2, \ldots, q\}$, finite sets with the discrete topology. For $i \in X$ let $U_i \subset X$ be the singleton $\{i\}$. The U_i form an open covering of X, and clearly $U_i \cap U_j = \emptyset$ if $i \neq j$. The abelian group $\mathrm{mor}_{\mathcal{PAT}}(\emptyset, Y)$ is free on one generator (the unique map from $\emptyset$ to Y), while $\mathrm{mor}_{\mathcal{PAT}}(U_i, Y)$ is a free abelian group on q generators. It follows that the subgroup

$$B \subset \prod_{i=1}^{p} \mathrm{mor}_{\mathcal{PAT}}(U_i, Y)$$

consisting of all $(s_1, s_2, \ldots, s_p)$ such that $s_1, \ldots, s_p$ restrict to the same element of $\mathrm{mor}_{\mathcal{PAT}}(\emptyset, Y)$ is free abelian of rank $pq - p + 1$. But $\mathrm{mor}_{\mathcal{PAT}}(X, Y)$ is a free abelian group on q^p generators. The homomorphism $\mathrm{mor}_{\mathcal{PAT}}(X, Y) \to B$ determined by the inclusions $U_i \to X$ cannot be injective if $q \geq 2$ and $p \geq 2$.

Example 3.2. Take $X = \mathbb{R}P^n$ and $Y = S^n \subset \mathbb{R}^{n+1}$. Make an open covering of X with $n + 1$ open subsets $U_0, U_1, \ldots, U_n$ where U_i consists of all the points whose i-th coordinate is $\neq 0$. (Think of points in $\mathbb{R}P^n$ as antipodal pairs in S^n ; then the i-th coordinate of such a point is well defined up to sign. We number the coordinates from 0 to n.) For every i there are two obvious continuous maps $f_i, g_i : U_i \to S^n$; one of these takes an element x in U_n to the unique $y \in S^n$ which represents x and has $y_i > 0$, while the other takes x in U_n to the unique $y \in S^n$ which represents x and has $y_i < 0$. The formal sum (alias linear combination) $f_i + g_i$ is an element in $\mathrm{mor}_{\mathcal{PAT}}(U_i, S^n)$. It is clear that $f_i + g_i$ and $f_j + g_j$ agree on $U_i \cap U_j$. But there is no element of $\mathrm{mor}_{\mathcal{PAT}}(\mathbb{R}P^n, S^n)$ whose restriction to U_i is $f_i - g_i$ for every $i \in \{0, 1, \ldots, n\}$. The reason is that, for any fixed i, neither f_i nor g_i extend continuously to $\mathbb{R}P^n$.

[3] The $\mathcal{P}$ in $\mathcal{PAT}$ does not stand for *pre*-anything, as you may have thought, but was chosen to remind you of Perceval, knight of the Arthurian legends. He achieved much in his lifetime because he had a simple mind and a pure heart.

Therefore we proceed to make a new category $\mathcal{AT}$ by making certain improvements to $\mathcal{PAT}$. Many good features of $\mathcal{PAT}$ will be kept:

- The objects of $\mathcal{AT}$ are the topological spaces.
- The morphism sets $\mathrm{mor}_{\mathcal{AT}}(X, Y)$ are abelian groups.
- Composition of morphisms
$$\mathrm{mor}_{\mathcal{AT}}(X, Y) \times \mathrm{mor}_{\mathcal{AT}}(W, X) \longrightarrow \mathrm{mor}_{\mathcal{AT}}(W, Y)$$
is bilinear.
- The abelian groups $\mathrm{mor}_{\mathcal{PAT}}(X, Y)$ and $\mathrm{mor}_{\mathcal{AT}}(X, Y)$ are related by a preferred homomorphism $\mathrm{mor}_{\mathcal{PAT}}(X, Y) \longrightarrow \mathrm{mor}_{\mathcal{AT}}(X, Y)$ which respects composition and, in the case $X = Y$, identity morphisms.

The method which we will use to make $\mathcal{AT}$ from $\mathcal{PAT}$ is old and well-established, but not as old as homology theory[4].

Definition 3.3. An element of $\mathrm{mor}_{\mathcal{AT}}(X, Y)$ can be specified by giving an open covering $(U_j)_{j\in J}$ of X, and for each $j \in J$ an element
$$s_j \in \mathrm{mor}_{\mathcal{PAT}}(U_j, Y)$$
such that the following condition is satisfied. For $i, j \in J$ and $x \in U_i \cap U_j$, there exists an open neighbourhood W of x in $U_i \cap U_j$ such that the images (under restriction) of s_i and s_j in $\mathrm{mor}_{\mathcal{PAT}}(W, Y)$ are the same.
Two such data sets $(U_j\,, s_j)_{j\in J}$ and $(V_k\,, t_k)_{k\in K}$ determine the same element of $\mathrm{mor}_{\mathcal{AT}}(X, Y)$ if and only if for every $x \in X$ there exist $j \in J$ and $k \in K$ such that $x \in U_j \cap V_k$, and an open neighbourhood W of x in $U_j \cap V_k$ such that the images (under restriction) of s_i and t_k in $\mathrm{mor}_{\mathcal{PAT}}(W, Y)$ are the same.

Example 3.4. For $X = \emptyset$ and arbitrary Y, there are many interesting open coverings of X, but the best of them is the one which has no open sets at all. It follows that $\mathrm{mor}_{\mathcal{AT}}(\emptyset, Y) = 0$. This makes an interesting contrast with $\mathrm{mor}_{\mathcal{PAT}}(\emptyset, Y) \cong \mathbb{Z}$.

Example 3.5. Suppose that $X = \star$ is a singleton. Then clearly $\mathrm{mor}_{\mathcal{AT}}(X, Y)$ is the same as $\mathrm{mor}_{\mathcal{PAT}}(X, Y)$, which is a free abelian group generated by the set underlying Y. More generally, let $X = \{1, 2, \dots, p\}$ where $p > 0$ (finite set with the discrete topology) and Y is arbitrary. Using the open covering of X with subsets $\{i\}$ for $i \in X$, we find that $\mathrm{mor}_{\mathcal{AT}}(X, Y)$ is a product of p copies of $\mathrm{mor}_{\mathcal{AT}}(\star, Y)$.

Reader: explain in detail how composition of morphisms in $\mathcal{AT}$ should be defined and why it is bilinear. Define the preferred homomorphism from $\mathrm{mor}_{\mathcal{PAT}}(X, Y)$ to $\mathrm{mor}_{\mathcal{AT}}(X, Y)$. Formulate and prove the analogue of lemma 2.1 for $\mathcal{AT}$.

Lemma 3.6. *The set $\mathrm{mor}_{\mathcal{AT}}(X, Y)$ has a preferred structure of abelian group.*

Idea of proof. Let elements a and b in $\mathrm{mor}_{\mathcal{AT}}(X, Y)$ be given by data sets $(U_j\,, s_j)_{j\in J}$ and $(V_k\,, t_k)_{k\in K}$, respectively. Then $(U_j \cap V_k)_{(j,k)\in J\times K}$ is an open covering of X. For each (j, k) in $J \times K$ we have $s_j|U_j \cap V_k \in \mathrm{mor}_{\mathcal{PAT}}(U_j \cap V_k, Y)$ and $t_k|U_j \cap V_k \in \mathrm{mor}_{\mathcal{PAT}}(U_j \cap V_k, Y)$. Form their sum r_{jk} in $\mathrm{mor}_{\mathcal{PAT}}(U_j \cap V_k, Y)$. The data set $(U_j \cap V_k\,, r_{jk})_{(j,k)\in J\times K}$ determines an element in $\mathrm{mor}_{\mathcal{AT}}(X, Y)$. That element is $a + b$. □

[4]It is called *sheafification* and belongs to sheaf theory, which goes back to the 1940s: Leray 1945, Cartan Seminar 1948 according to Wikipedia

4. Homotopy relation, homology and cohomology

It is easy to talk about homotopy in the category $\mathcal{AT}$. Say that two elements $a, b \in \mathrm{mor}_{\mathcal{AT}}(X, Y)$ are *homotopic* if there exists an element

$$h \in \mathrm{mor}_{\mathcal{AT}}(X \times [0,1], Y)$$

such that $hi_0 = a$ and $hi_1 = b$, where $i_0 : X \to X \times [0,1]$ and $i_1 : X \to X \times [0,1]$ are defined by $x \mapsto (x, 0)$ and $x \mapsto (x, 1)$, respectively. In such a case we say that h is a homotopy from a to b. Often it is convenient to use other intervals: $[1,2]$ or $[0,2]$ etc. instead of $[0,1]$.

Lemma 4.1. *The homotopy relation is an equivalence relation on*

$$\mathrm{mor}_{\mathcal{AT}}(X \times [0,1], Y) \ .$$

This may look obvious, but there is a small difficulty in proving transitivity of the relation. Let $h \in \mathrm{mor}_{\mathcal{AT}}(X \times [0,1], Y)$ be a homotopy from a to b and let $g \in \mathrm{mor}_{\mathcal{AT}}(X \times [1,2], Y)$ be a homotopy from b to c. We may hope to find an element of $\mathrm{mor}_{\mathcal{AT}}(X \times [0,2], Y)$ which agrees with h on $X \times [0,1]$ and with g on $X \times [1,2]$. But such an element of $\mathrm{mor}_{\mathcal{AT}}(X \times [0,2], Y)$ may not exist. One problem is that we cannot easily make an open covering of $X \times [0,2]$ from open coverings of $X \times [0,1]$ and $X \times [1,2]$. There is a trick to overcome this problem. Let $\psi : [0,1] \to [0,1]$ be a continuous map which is constant with value 0 on $[0, \varepsilon]$ and constant with value 1 on $[1-\varepsilon, 1]$. Replace h by h^ψ , the composition of $h \in \mathrm{mor}_{\mathcal{AT}}(X \times [0,1], Y)$ with the map $(x,t) \mapsto (x, \psi(t))$ from $X \times [0,1]$ to $X \times [0,1]$. It is easy to show that there exists an element of $\mathrm{mor}_{\mathcal{AT}}(X \times [0,2], Y)$ which agrees with h^ψ on $X \times [0,1]$ and with g on $X \times [1,2]$.

Lemma 4.2. *The homotopy relation is compatible with composition of morphisms in $\mathcal{AT}$. Consequently there is a homotopy category $\mathcal{HAT}$, with the same objects as $\mathcal{AT}$ and $\mathcal{T}$, such that*

$$\mathrm{mor}_{\mathcal{HAT}}(X, Y) = \frac{\mathrm{mor}_{\mathcal{AT}}(X, Y)}{\text{homotopy relation}} \ .$$

The proof is straightforward.

We import $\mathcal{HT}$ notation to $\mathcal{HAT}$ by writing $[f]$ for the homotopy class of a morphism $f \in \mathrm{mor}_{\mathcal{AT}}(X, Y)$ when it seems useful.

Example 4.3. Let $\star$ be a singleton (topological space with one element). For a topological space Y, the abelian group $\mathrm{mor}_{\mathcal{HAT}}(\star, Y)$ is isomorphic to the free abelian group generated by $\pi_0 Y$, the set of path components of Y. (*Reader: prove it.*) More generally, suppose that X is a discrete space (a set with the discrete topology where every subset is open). Then $\mathrm{mor}_{\mathcal{HAT}}(X, Y)$ is isomorphic to the free abelian group generated by the set of all maps from X to $\pi_0 Y$. (*Prove it.*)

Example 4.4. For a topological space X, the abelian group $\mathrm{mor}_{\mathcal{HAT}}(X, \star)$ is isomorphic to the abelian group of continuous maps from X to $\mathbb{Z}$ (where $\mathbb{Z}$ has the discrete topology). (*Prove it.*) More generally, suppose that Y is a discrete space. Then $\mathrm{mor}_{\mathcal{HAT}}(X, Y)$ is isomorphic to the abelian group of all continuous maps from X to $\mathbb{Z}Y$. Here $\mathbb{Z}Y$ is the free abelian group generated by the set Y, with the discrete topology. (*Prove it.*)

Definition 4.5. *For an integer $n \geq 0$, the n-*th homology group *of Y is the abelian group*

$$H_n(Y;\mathbb{Z}) := \frac{\mathrm{mor}_{\mathcal{HAT}}(S^n, Y)}{\mathrm{mor}_{\mathcal{HAT}}(\star, Y)} \ .$$

Comment. Let $c\colon S^n \to \star$ be the constant map and let $e\colon \star \to S^n$ be the inclusion of the standard base point, so that $ce = \mathrm{id}_\star$. Pre-composition with c defines a homomorphism $c^*\colon \mathrm{mor}_{\mathcal{HAT}}(\star, Y) \longrightarrow \mathrm{mor}_{\mathcal{HAT}}(S^n, Y)$ and pre-composition with e defines a homomorphism $e^*\colon \mathrm{mor}_{\mathcal{HAT}}(S^n, Y) \longrightarrow \mathrm{mor}_{\mathcal{HAT}}(\star, Y)$. The composition e^*c^* is the identity on $\mathrm{mor}_{\mathcal{HAT}}(\star, Y)$. This fact allows us to think of $\mathrm{mor}_{\mathcal{HAT}}(\star, Y)$ as a direct summand of $\mathrm{mor}_{\mathcal{HAT}}(S^n, Y)$.

Example 4.6. Using example 4.3, it is easy to show that $H_0(X;\mathbb{Z})$ is (isomorphic to) the free abelian group generated by $\pi_0 X$.

Definition 4.7. *For an integer $n \geq 0$, the n-*th cohomology group *of X is the abelian group*

$$H^n(X;\mathbb{Z}) := \frac{\mathrm{mor}_{\mathcal{HAT}}(X, S^n)}{\mathrm{mor}_{\mathcal{HAT}}(X, \star)} \ .$$

Example 4.8. Using example 4.3, it is easy to show that $H^0(X;\mathbb{Z})$ is (isomorphic to) the abelian group of continuous maps from X to $\mathbb{Z}$.

For topological spaces X_1, X_2, Y_1 and Y_2 and morphisms $f \in \mathrm{mor}_{\mathcal{AT}}(X_1, Y_1)$ and $g \in \mathrm{mor}_{\mathcal{AT}}(X_2, Y_2)$, there is a morphism

$$f \otimes g \in \mathrm{mor}_{\mathcal{AT}}(X_1 \times X_2, Y_1 \times Y_2)$$

defined as follows. Represent f and g by data sets $(U_j\ , s_j)_{j\in J}$ and $(V_k\ , t_j)_{k\in K}$ where $(U_j)_{j\in J}$ and $(V_k)_{k\in K}$ are open coverings of X_1 and X_2 respectively, each s_j is a $\mathbb{Z}$-linear combination of continuous maps from U_j to Y_1 and each t_k is a $\mathbb{Z}$-linear combination of continuous maps from V_k to Y_2. More precisely, suppose that

$$s_j = \sum_p \alpha_p u_p\ , \qquad t_k = \sum_q \beta_q v_q\ .$$

Let $s_j \otimes t_k = \sum_{p,q} (\alpha_p \beta_q)\ u_p \times v_q$ where $(u_p \times v_q)(x_1, x_2) := (u_p(x_1), v_q(x_2))$. Then the data set

$$(U_j \times V_k\ , s_j \otimes t_k)_{(j,k)\in J\times K}$$

represents a morphism in $\mathcal{AT}$ from $X_1 \times X_2$ to $Y_1 \times Y_2$.
This construction has some interesting consequences for cohomology. To formulate these let's note that there is an important map $\mu\colon S^m \times S^n \longrightarrow S^{m+n}$. The easiest way to understand this is to think of S^m, S^n and S^{m+n} as the one point-compactifications of $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^m \times \mathbb{R}^n$, respectively. Then we can write

$$\mu\colon (\mathbb{R}^m \cup \infty) \times (\mathbb{R}^n \cup \infty) \longrightarrow (\mathbb{R}^m \times \mathbb{R}^n \cup \infty)$$

and define μ to be the identity on $\mathbb{R}^m \times \mathbb{R}^n$. Points in the source with at least one coordinate equal to ∞ are mapped to ∞.

Definition 4.9. Let $u \in H^m(X_1;\mathbb{Z})$ and $v \in H^n(X_2;\mathbb{Z})$. The external product

$$u \times v \in H^{m+n}(X_1 \times X_2;\mathbb{Z})$$

is obtained by choosing representatives f in $\text{mor}_{\mathcal{AT}}(X_1, S^m)$ and g in $\text{mor}_{\mathcal{AT}}(X_2, S^n)$ for u and v, respectively, and composing $f \otimes g \in \text{mor}_{\mathcal{AT}}(X_1 \times X_2, S^m \times S^n)$ with $\mu\colon S^m \times S^n \to S^{m+n}$. In the case $X_1 = X_2 = X$, the cup product

$$u \cup v \in H^{m+n}(X;\mathbb{Z})$$

of u and v is obtained by choosing representatives f and g as before and composing $f \otimes g \in \text{mor}_{\mathcal{AT}}(X \times X, S^m \times S^n)$ with μ on one side and with the diagonal map $X \to X \times X$ on the other.

Lemma 4.10. *The cup product is bilinear and associative. Hence the cohomology groups $H^n(X;\mathbb{Z})$, taken together for $n \geq 0$ and equipped with the cup product, form a graded ring.*

The proof is straightforward, but it is worthwhile to expand on the meaning. A *graded ring* can be defined as a sequence of abelian groups $A_0, A_1, A_2, A_3, \dots$ together with bilinear maps $f_{mn}\colon A_m \times A_n \to A_{m+n}$, subject to associativity. If we insist on a unit, and here we do, that unit should be an element $1 \in A_0$.

The graded ring formed by the cohomology groups $H^n(X;\mathbb{Z})$ for all $n \geq 0$ has a unit. This is the element $1 \in H^0(X;\mathbb{Z})$ represented by the continuous map $X \to S^0$ which has the constant value *not the base point*. Beware that in the case where X is empty, we have $H^0(X;\mathbb{Z}) = 0$ and so the unit element agrees with the zero element.

5. Gluing and fracturing

Lemma 2.1 has a weak analogue in the homotopy category $\mathcal{HT}$, as follows. Suppose that $X = V \cup W$ where V and W are open in X.

Proposition 5.1. *Assume that the covering of X by V and W admits a subordinate partition of unity*[5]*. Let $a \in \text{mor}_{\mathcal{HT}}(V, Y)$ and $b \in \text{mor}_{\mathcal{HT}}(W, Y)$ be any morphisms such that the restrictions of a and b in $\text{mor}_{\mathcal{HT}}(V \cap W, Y)$ agree. Then there exists c in $\text{mor}_{\mathcal{HT}}(X, Y)$ which restricts to a in $\text{mor}_{\mathcal{HT}}(V, Y)$ and to b in $\text{mor}_{\mathcal{HT}}(W, Y)$.*

Proof. Choose a partition of unity $\{\psi_1, \psi_2\}$ subordinate to the covering of X by V and W. Choose continuous maps $f\colon V \to Y$ and $g\colon W \to Y$ representing a and b, respectively, and a homotopy $(h_t)_{t \in [0,1]}$ from $f|V \cap W$ to $g|V \cap W$. The homotopy exists by assumption. Define $e\colon X \to Y$ by $e(x) = f(x)$ for $x \notin W$, $e(x) = g(x)$ for $x \notin V$, and $e(x) = h_{\psi_2(x)}(x)$ for $x \in V \cap W$. Let c be the homotopy class of e. □

Comment. The partition of unity certainly exists if X is metrisable; more generally it exists if X is a normal space[6].

It is straightforward to formulate and prove an analogue of proposition 5.1 in $\mathcal{HAT}$. Again suppose that $X = V \cup W$ where V and W are open in X.

Proposition 5.2. *Assume that the covering of X by V and W admits a subordinate partition of unity. Let $a \in \text{mor}_{\mathcal{HAT}}(V, Y)$ and $b \in \text{mor}_{\mathcal{HAT}}(W, Y)$ be any morphisms such that the restrictions of a and b in $\text{mor}_{\mathcal{HAT}}(V \cap W, Y)$ agree. Then there exists c in $\text{mor}_{\mathcal{HAT}}(X, Y)$ which restricts to a in $\text{mor}_{\mathcal{HAT}}(V, Y)$ and to b in $\text{mor}_{\mathcal{HAT}}(W, Y)$.*

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[5]This means that there exist continuous functions $\psi_1\colon X \to [0,1]$ and $\psi_2\colon X \to [0,1]$ such that $\psi_1 + \psi_2 \equiv 1$ and ψ_1 has support in V while ψ_2 has support in W. The support of a continuous $u\colon X \to \mathbb{R}$ is the closure of $\{x \in X \mid u(x) \neq 0\}$.

[6]This means that two disjoint closed subsets of X admit disjoint open neighbourhoods.

Proof. Choose a partition of unity $\{\psi_1, \psi_2\}$ subordinate to the covering of X by V and W. Choose $f \in \mathrm{mor}_{\mathcal{AT}}(V,Y)$ and $g \in \mathrm{mor}_{\mathcal{AT}}(W,Y)$ representing a and b, respectively, and a homotopy $(h_t)_{t\in[0,1]}$ from $f|V\cap W$ to $g|V\cap W$. Together, the morphisms f, g and h define a single morphism u in $\mathcal{AT}$ from the double mapping cylinder[7] Z of

$$V \longleftarrow V\cap W \longrightarrow W$$

to Y. Define $j\colon X \to Z$ by $j(x) = x \in V$ for $x \notin W$, $j(x) = x \in W$ for $x \notin V$, and $j(x) = (x, \psi_2(x)) \in V\cap W \times [0,1]$ for $x \in V \cap W$. Let c be the homotopy class of the composition uj. □

Let us try to interchange the roles of X and Y in proposition 5.1. Let V and W be open subsets of Y. Suppose that $f\colon X \to V$ and $g\colon X \to W$ become homotopic when viewed as maps from X to Y. Does there exist a map $e : X \to V \cap W$ such that e composed with the inclusion $V\cap W \to V$ is homotopic to f and e composed with $V \cap W \to W$ is homotopic to g ? In general the answer is no.

Example 5.3. Let $Y = S^1 \vee D^2$ (wedge sum, obtained by identifying chosen base points of S^1 and D^2). Choose open sets V and W in Y in such a way that V contains $S^1 \vee S^1$ and the inclusion $S^1\vee S^1 \to V$ is a homotopy equivalence, while W contains D^2 and the inclusion $D^2 \to W$ is a homotopy equivalence. Then $V \cup W = Y$ and, when we pass from $\mathcal{T}$ to $\mathcal{HT}$, the diagram of inclusions

$$\begin{array}{ccc} V\cap W & \longrightarrow & V \\ \big\downarrow & & \big\downarrow \\ W & \longrightarrow & V\cup W = Y \end{array}$$

becomes (isomorphic to)

$$\begin{array}{ccc} S^1 & \xrightarrow{\ \alpha\ } & S^1\vee S^1 \\ \big\downarrow & & \big\downarrow \beta \\ \star & \longrightarrow & S^1 \end{array}$$

where α is the inclusion of the first wedge summand and β is the collapse of the first wedge summand. Now let $X = S^1$. There is only one map g from X to $\star$. With some understanding of the fundamental group of $S^1 \vee S^1$, it is easy to construct a map $f\colon X \to S^1 \vee S^1$ which is *not* homotopic to a map of the form αe, for $e\colon X \to S^1$, while βf *is* nullhomotopic.

In order to dispel the idea that this phenomenon might go away when all spaces in sight are simply connected, I want to describe a similar example where higher dimensional spheres take the place of circles. Let $Y = S^2 \vee D^3$ and choose open

[7] The double mapping cylinder of a diagram of spaces

$$A \xleftarrow{\ f\ } B \xrightarrow{\ g\ } C$$

is the quotient space of the disjoint union of A, C and $B \times [0,1]$ by the relations $(b,0) \sim f(b) \in A$ and $(b,1) \sim g(b) \in C$ for $b \in B$. In the situation above where $A = V$, $C = W$ and $B = V \cap W$, the double mapping cylinder should not be confused with the union of $V \times \{0\}$, $W \times \{1\}$ and $(V \cap W) \times [0,1]$ *as a subspace of* $X \times [0,1]$. There is an obvious bijection from one to the other but it often fails to be a homeomorphism. For example, if $X = [0,1]$ and $V = [0,1)$, $W = (0,1]$, then the double mapping cylinder of $V \leftarrow V \cap W \to W$ is not metrisable (thanks Larry Taylor), whereas any subspace of $X \times [0,1]$ is of course metrisable.

sets V and W by analogy with the above, in such a way that $V \cup W = Y$ and the diagram in $\mathcal{T}$ of inclusions

$$\begin{array}{ccc} V \cap W & \longrightarrow & V \\ \downarrow & & \downarrow \\ W & \longrightarrow & V \cup W = Y \end{array}$$

becomes, in $\mathcal{HT}$, isomorphic to

$$\begin{array}{ccc} S^2 & \xrightarrow{\alpha} & S^2 \vee S^2 \\ \downarrow & & \downarrow \beta \\ \star & \longrightarrow & S^2 \end{array}$$

where α is the inclusion of the first wedge summand and β is the collapse of the first wedge summand. Now let $X = S^3$. There is only one map g from X to $\star$. With some understanding of the third homotopy group of $S^2 \vee S^2$ (which goes a little beyond these lecture notes), it is possible to construct a map $f\colon X \to S^2 \vee S^2$ which is *not* homotopic to a map of the form αe, for $e\colon X \to S^2$, while βf *is* nullhomotopic.

The following theorem and especially its corollary should therefore come as a surprise. Indeed we have reached the high point of the homology drama. Notation for the theorem: V and W are open subsets of Y such that $V \cup W = Y$. The space X is assumed to be *paracompact*[8] and A is a closed subset of X.

Theorem 5.4 (Homotopy decomposition theorem)**.** *Let $G : X \times [0,1] \to Y$ be a morphism in $\mathcal{AT}$ which is zero on a neighborhood of $X \times \{0\}$. Then there exists a decomposition $G = G^V + G^W$, where $G^V\colon X \times [0,1], \to V$ and $G^W\colon X \times [0,1] \to W$ are morphisms in $\mathcal{AT}$, both zero on a neighborhood of $X \times \{0\}$. If G is also zero on some neighbourhood of $A \times I$, then it can be arranged that G_V and G_W are zero on a neighbourhood of $A \times I$.*

The proof of this is hard and we will postpone it for section 11.

Corollary 5.5. *Let $V, W \subset Y$ be open subsets, $V \cup W = Y$. Let $a \in \mathrm{mor}_{\mathcal{HAT}}(X, V)$ and $b \in \mathrm{mor}_{\mathcal{HAT}}(X, W)$ be such that the images of a and b in $\mathrm{mor}_{\mathcal{HAT}}(X, Y)$ agree. Then there exists $c \in \mathrm{mor}_{\mathcal{HAT}}(X, V \cap W)$ whose image in $\mathrm{mor}_{\mathcal{HAT}}(X, V)$ is a and whose image in $\mathrm{mor}_{\mathcal{HAT}}(X, W)$ is b.*

Proof. Let $f \in \mathrm{mor}_{\mathcal{AT}}(X, V)$ represent a and let $g \in \mathrm{mor}_{\mathcal{AT}}(X, W)$ represent b. Choose a homotopy $K\colon X \times [0,1] \to Y$ from 0 to $g - f$. It is easy to arrange this in such a way that K is zero on a neighbourhood of $X \times \{0\}$. Use the theorem to obtain a decomposition $K = K^V + K^W$. Let K_1^V and K_1^W be the restrictions of K^V and K^W to $X \times \{1\}$. Then f and $f + K_1^V$ are homotopic as morphisms $X \to V$, by the homotopy $fp + K^V$, where p is the projection $X \times [0,1] \to X$. Similarly $g = f + K_1^V + K_1^W$ and $f + K_1^V$ are homotopic as morphisms $X \to W$. Finally, $f + K_1^V = g - K_1^W$ lands in $V \cap W$ by construction. □

[8]Look it up in any book on general topology. *Metric* implies *paracompact* and *paracompact* implies *normal*.

6. Mayer-Vietoris sequence in homology

A sequence of abelian groups $(A_n)_{n\in\mathbb{Z}}$ together with homomorphisms

$$f_n\colon A_n \to A_{n-1}$$

for all $n \in \mathbb{Z}$ is called an *exact sequence of abelian groups* if the kernel of f_n is equal to the image of f_{n+1} , for all $n \in \mathbb{Z}$. More generally, we sometimes have to deal with diagrams of abelian groups and homomorphisms in the shape of a string

$$A_n \to A_{n-1} \to A_{n-2} \to \cdots \to A_{n-k} \ .$$

Such a diagram is *exact* if the kernel of each homomorphism in the string is equal to the image of the preceding one, if there is a preceding one.

Notation: $H_n(Y)$ is short for $H_n(Y;\mathbb{Z})$ and I is short for the interval $[0,1]$.

Definition 6.1. *Alternative definition of homology*: For a space Y, and $n \geq 0$, re-define $H_n(Y)$ as the abelian group of homotopy classes of morphisms $I^n \to Y$ in $\mathcal{AT}$ which vanish on some neighbourhood of ∂I^n.

Comment. In this definition, we regard two morphisms $I^n \to Y$ which vanish on some neighborhood of ∂I^n as *homotopic* if they are related by a homotopy $I^n \times I \to Y$ which vanishes on some neighbourhood of $\partial I^n \times I$.
To relate the old definition of $H_n(Y)$ to the new one, we make a few observations. Given $f\colon I^n \to Y$ in $\mathcal{AT}$ which vanishes on some neighbourhood of ∂I^n, we immediately obtain a morphism in $\mathcal{AT}$ from the quotient $I^n/\partial I^n$ to Y. To view this as a morphism $g\colon S^n \to Y$, we pretend $S^n = \mathbb{R}^n \cup \infty$ and specify a homeomorphism $u\colon I^n/\partial I^n \to \mathbb{R}^n \cup \infty$ taking base point to base point. We are specific enough if we say that u is smooth and orientation preserving on $I^n \smallsetminus \partial I^n$ (i.e., the Jacobian determinant is everywhere positive). Conversely, given $g\colon S^n \to Y$ in $\mathcal{AT}$ representing an element of $H_n(Y)$ according to the old definition, we may subtract a suitable constant to arrange that g is zero on the base point of S^n. We can also assume that g is zero on a neighbourhood of the base point; if not, compose with a morphism $S^n \to S^n$ in $\mathcal{T}$ which is homotopic to the identity and takes a neighbourhood of the base point to the base point. Then gu is a morphism $I^n/\partial \to Y$ in $\mathcal{AT}$ which can also be viewed as a morphism $I^n \to Y$ vanishing on a neighbourhood of ∂I^n.

Definition 6.2. Suppose that Y comes with two open subspaces V and W such that $V \cup W = Y$. The *boundary homomorphism*

$$\partial : H_n(Y) \to H_{n-1}(V \cap W)$$

is defined as follows, using the alternative definition of H_n. Let $x \in H_n(Y)$ be represented by a morphism $G : I^n \to Y$ in $\mathcal{AT}$ which vanishes near ∂I^n. Think of G as a homotopy, $G : I^{n-1} \times I \to Y$. Choose a decomposition $G = G^V + G^W$ as in theorem 5.4. Arrange that G^V and G^W vanish on a neighbourhood of $\partial I^{n-1} \times I$. Let $\partial(x)$ be the class of the morphism

$$G_1^W : I^{n-1} \to V \cap W$$

which vanishes near ∂I^{n-1}.

We must show that this is well defined. There were two choices involved: the choice of representative G, and the choice of decomposition $G = G^V + G^W$. For the moment, keep G fixed, and let us see what happens if we try another decomposition of G. Any other decomposition will have the form

$$(G^V + E) + (G^W - E)$$

where $E : I^{n-1} \times I \to V \cap W$ is a morphism in $\mathcal{A}\mathfrak{T}$ which vanishes on $\partial I^{n-1} \times I$ and on $I^{n-1} \times \{0\}$. We need to show that $G_1^W - E_1$ is homotopic (rel boundary of I^{n-1}) to G_1^W. But E_1 is homotopic to zero by the homotopy E.
Next we worry about the choice of representative G. Let F be another representative of the same class x, and let $L : I \times I^n \to Y$ be a homotopy from F to G. (Writing the factor I on the left will help us to avoid confusion.) We can think of L as a homotopy in a different way:

$$(I \times I^{n-1}) \times I \longrightarrow Y\,.$$

Then we can apply the homotopy decomposition theorem and choose a decomposition $L = L^V + L^W$ where L^V and L^W vanish on $I \times \partial I^{n-1} \times I$. We then find that L_1^W is a morphism from $X = I \times I^{n-1}$ to $V \cap W$ which we may regard as a homotopy (now with parameters written on the left). The homotopy is between G_1^W and F_1^W, provided the decompositions $G = G^V + G^W$ and $F = F^V + F^W$ are the ones obtained by restricting the decomposition $L = L^V + L^W$. □

The boundary homomorphisms ∂ can be used to make a sequence of abelian groups and homomorphisms

$$\begin{array}{ccccc}
\cdots \longrightarrow H_{n+1}(Y) & & & & \\
\downarrow \partial & & & & \\
H_n(V \cap W) & \longrightarrow & H_n(V) \oplus H_n(W) & \longrightarrow & H_n(Y) \\
& & & & \downarrow \partial \\
& & & & H_{n-1}(V \cap W) \longrightarrow \cdots
\end{array}$$

where $n \in \mathbb{Z}$. (Set $H_n(X) = 0$ for $n < 0$ and any space X. The unlabelled homomorphisms in the sequence are as follows: $H_n(V) \oplus H_n(W) \to H_n(Y)$ is $j_{V*} + j_{W*}$, the sum of the two maps given by composition with the inclusions $j_V : V \to Y$ and $j_W : W \to Y$, and $H_n(V \cap W) \to H_n(V) \oplus H_n(W)$ is $(e_{V*}, -e_{W*})$, where e_{V*} and e_{W*} are given by composition with the inclusions $e_V : V \cap W \to V$ and $e_W : V \cap W \to W$.) The sequence is called the homology *Mayer-Vietoris* sequence of Y and V, W.

Theorem 6.3. *The homology Mayer-Vietoris sequence of Y and V, W is exact.*[9]

Proof. (i) Exactness of the pieces $H_n(V \cap W) \to H_n(V) \oplus H_n(W) \to H_n(Y)$ follows from corollary 5.5, for all $n \in \mathbb{Z}$. (Take $X = S^n$ in corollary 5.5. Therefore it is more convenient to use the standard definition of H_n at this point.)
(ii) Next we look at pieces of the form

$$H_n(V) \oplus H_n(W) \longrightarrow H_n(Y) \xrightarrow{\ \partial\ } H_{n-1}(V \cap W)\,.$$

[9] View this as a sequence of abelian groups and homomorphisms indexed by the integers, by setting for example $A_{3n} = H_n(Y)$ for $n \geq 0$, $A_{3n+1} = H_n(V) \oplus H_n(W)$ for $n \geq 0$, $A_{3n+2} = H_n(V \cap W)$ for $n \geq 0$, and $A_m = 0$ for all $m \leq 0$.

The cases $n < 0$ are trivial. In the case $n = 0$, the claim is that the homomorphism $H_0(V) \oplus H_0(W) \to H_0(Y)$ is surjective. This is a pleasant exercise. Now assume $n > 0$. It is clear from the definition of ∂ that the composition of the two homomorphisms is zero. Suppose then that $[G] \in H_n(Y)$ is in the kernel of ∂, where $G : I^n \to Y$ vanishes on a neighbourhood of ∂I^n. We must show that $[G]$ is in the image of $H_n(V) \oplus H_n(W) \to H_n(Y)$. As above, we think of G as a homotopy, $I^{n-1} \times I \to Y$, which we decompose, $G = G^V + G^W$ as in theorem 5.4, where G^V and G^W vanish on a neighbourhood of $\partial I^{n-1} \times I$. We can also arrange that the homotopies G^V and G^W are stationary (do not depend on the "time" variable) on neighbourhoods of $I^{n-1} \times \{0\}$ and $I^{n-1} \times \{1\}$. The assumption $\partial[G] = 0$ then means that the zero map

$$I^{n-1} \to V \cap W$$

is homotopic to G^W_1 , say by a homotopy $L : I^{n-1} \times I \to V \cap W$ which vanishes on a neighbourhood of $\partial I^{n-1} \times I$. We can arrange that L is stationary on neighbourhoods of $I^{n-1} \times \{0\}$ and $I^{n-1} \times \{1\}$. Then $G^V + L$ and $G^W - L$ are morphisms from $I^{n-1} \times I = I^n$ to V and W, respectively. Both vanish on a neighborhood ∂I^n. Hence they represent elements in $H_n(V)$ and $H_n(W)$ whose images in $H_n(Y)$ add up to $[G]$.

(iii) We show that the composition

$$H_{n+1}(Y) \xrightarrow{\ \partial\ } H_n(V \cap W) \longrightarrow H_n(V) \oplus H_n(W)\ .$$

is zero. We can assume $n \geq 0$. Represent an element in $H_n(Y)$ by $G : I^n \times I \to Y$, vanishing on a neighbourhood of the entire boundary; decompose as usual, and obtain $\partial[G] = [G^W_1]$. Now $G^W_1 = -G^V_1$ viewed as a morphism $I^n \to V$ in $\mathcal{AT}$ is homotopic to zero by the homotopy $-G^V$ vanishing on a neighbourhood of $\partial I^{n-1} \times I$. Therefore $\partial[G]$ maps to zero in $H_n(V)$. A similar calculation shows that it maps to zero in $H_n(W)$.

(iv) Finally let $f : I^n \to V \cap W$ be a morphism in $\mathcal{AT}$ which vanishes on a neighbourhood of ∂I^n, and suppose that $[f] \in H_n(V \cap W)$ is in the kernel of the homomorphism $H_n(V \cap W) \to H_n(V) \oplus H_n(W)$. Choose a homotopy $G^V : I^n \times I \to V$ from zero to $-f$, and another homotopy $G^W : I^n \times I \to W$ from zero to f, both vanishing on a neighborhood of $\partial I^n \times I$, and both stationary near $I^n \times \{0, 1\}$. Then $G := G^V + G^W$ vanishes on the entire boundary of $I^n \times I$, hence represents a class $[G] \in H_{n+1}(Y)$. It is clear that $\partial[G] = [f]$. □

7. Homology of spheres

Proposition 7.1. *The homology groups of S^1 are $H_0(S^1) \cong \mathbb{Z}$, $H_1(S^1) \cong \mathbb{Z}$ and $H_k(S^1) = 0$ for all $k \neq 0, 1$.*

Proof. Choose two distinct points p and q in S^1. Let $V \subset S^1$ be the complement of p and let $W \subset S^1$ be the complement of q. Then $V \cup W = S^1$. Clearly V is homotopy equivalent to a point, W is homotopy equivalent to a point and $V \cap W$ is homotopy equivalent to a discrete space with two points. Therefore $H_k(V) \cong H_k(W) \cong \mathbb{Z}$ for $k = 0$ and $H_k(V) \cong H_k(W) = 0$ for all $k \neq 0$. Similarly $H_k(V \cap W) \cong \mathbb{Z} \oplus \mathbb{Z}$ for $k = 0$ and $H_k(V \cap W) = 0$ for all $k \neq 0$. The exactness of the Mayer-Vietoris sequence associated with the covering of S^1 by V and W implies immediately that $H_k(S^1) = 0$ for $k \neq 0, 1$. The part of the Mayer-Vietoris sequence which remains interesting after this observation is

$$0 \longrightarrow H_1(Y) \xrightarrow{\partial} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_0(Y) \longrightarrow 0$$

By example 4.6, the group $H_0(Y)$ is isomorphic to $\mathbb{Z}$. The homomorphism from $\mathbb{Z} \oplus \mathbb{Z}$ to $H_0(Y)$ is onto by exactness, so its kernel is isomorphic to $\mathbb{Z}$. Hence the image of the homomorphism $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is isomorphic to $\mathbb{Z}$, so its kernel is again isomorphic to $\mathbb{Z}$. Now exactness at $H_1(Y)$ leads to the conclusion that $H_1(Y) \cong \mathbb{Z}$. □

Theorem 7.2. *The homology groups of S^n (for $n > 0$) are*

$$H_k(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } k = n \\ \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We proceed by induction on n. The induction beginning is the case $n = 1$ which we have already dealt with separately in proposition 7.1. For the induction step, suppose that $n > 1$. We use the Mayer-Vietoris sequence for S^n and the open covering $\{V, W\}$ with $V = S^n \smallsetminus \{p\}$ and $W = S^n \smallsetminus \{q\}$ where $p, q \in S^n$ are the north and south pole, respectively. We will also use the homotopy invariance of homology. This gives us

$$H_k(V) \cong H_k(W) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

because V and W are homotopy equivalent to a point, and

$$H_k(V \cap W) \cong \begin{cases} \mathbb{Z} & \text{if } k = n-1 \\ \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

by the induction hypothesis, since $V \cap W$ is homotopy equivalent to S^{n-1}. Furthermore it is clear what the inclusion maps $V \cap W \to V$ and $V \cap W \to W$ induce in homology: an isomorphism in H_0 and (necessarily) the zero map in H_0 for all $k \neq 0$. Thus the homomorphism

$$H_k(V \cap W) \longrightarrow H_k(V) \oplus H_k(W)$$

from the Mayer-Vietoris sequence takes the form

$$\mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$$

when $k = 0$, and

$$\mathbb{Z} \longrightarrow 0$$

when $k = n - 1$. In all other cases, its source and target are both zero. Therefore the exactness of the Mayer-Vietoris sequence implies that $H_0(S^n)$ and $H_n(S^n)$ are isomorphic to $\mathbb{Z}$, while $H_k(S^n) = 0$ for all other $k \in \mathbb{Z}$. □

Theorem 7.3. *Let $f : S^n \to S^n$ be the antipodal map. Then $f_* : H_n(S^n) \to H_n(S^n)$ is multiplication by $(-1)^{n+1}$.*

Proof. We proceed by induction again. For the induction beginning, we take $n = 1$. The antipodal map $f : S^1 \to S^1$ is homotopic to the identity, so that $f^* : H_1(S^1) \to H_1(S^1)$ has to be the identity, too. For the induction step, we use the setup and notation from the previous proof. Exactness of the Mayer-Vietoris sequence for S^n and the open covering $\{V, W\}$ shows that

$$\partial : H_n(S^n) \longrightarrow H_{n-1}(V \cap W)$$

is an isomorphism. The diagram

$$\begin{array}{ccc} H_{n-1}(V \cap W) & \xleftarrow{\ \partial\ } & H_n(S^n) \\ {\scriptstyle f_*}\Big\uparrow & & {\scriptstyle f_*}\Big\uparrow \\ H_{n-1}(V \cap W) & \xleftarrow{\ \partial\ } & H_n(S^n) \end{array}$$

is meaningful because f takes $V \cap W$ to $V \cap W$. But the diagram is not commutative (i.e., it is not true that $f_* \circ \partial$ equals $\partial \circ f_*$). The reason is that f interchanges V and W, and it does matter in the Mayer-Vietoris sequence which of the two comes first. Therefore we have instead

$$f_* \circ \partial = -\partial \circ f_*$$

in the above square. By the inductive hypothesis, the f_* in the left-hand column of the square is multiplication by $(-1)^n$, and therefore the f^* in the right-hand column of the square must be multiplication by $(-1)^{n+1}$. □

8. The usual applications

Theorem 8.1. (Brouwer's fixed point theorem). *Let $f : D^n \to D^n$ be a continuous map, where $n \geq 1$. Then f has a fixed point, i.e., there exists $y \in D^n$ such that $f(y) = y$.*

Proof. Suppose for a contradiction that f does not have a fixed point. For $x \in D^n$, let $g(x)$ be the point where the ray (half-line) from $f(x)$ to x intersects the boundary S^{n-1} of the disk D^n. Then g is a smooth map from D^n to S^{n-1}, and we have $g|S^{n-1} = \mathrm{id}_{S^{n-1}}$. Summarising, we have

$$S^{n-1} \xrightarrow{\ j\ } D^n \xrightarrow{\ g\ } S^{n-1}$$

where j is the inclusion, $g \circ j = \mathrm{id}_{S^{n-1}}$. Therefore we get

$$H_{n-1}(S^{n-1}) \xrightarrow{\ j_*\ } H_{n-1}(D^n) \xrightarrow{\ g_*\ } H_{n-1}(S^{n-1})$$

where $g_* j_* = \mathrm{id}$. Thus the abelian group $H_{n-1}(S^{n-1})$ is isomorphic to a direct summand of $H_{n-1}(D^n)$. But from our calculations above, we know that this is not true. If $n > 1$ we have $H_{n-1}(D^n) = 0$ while $H_{n-1}(S^{n-1})$ is not trivial. If $n = 1$ we have $H_{n-1}(D^n) \cong \mathbb{Z}$ while $H_{n-1}(S^{n-1}) \cong \mathbb{Z} \oplus \mathbb{Z}$. □

Let $f : S^n \to S^n$ be any continuous map, $n > 0$. The induced homomorphism $f_* : H_n(S^n) \to H_n(S^n)$ is multiplication by some number $n_f \in \mathbb{Z}$, since $H_n(S^n)$ is isomorphic to $\mathbb{Z}$.

Definition 8.2. The number n_f is the *degree* of f.

Remark. The degree n_f of $f : S^n \to S^n$ is clearly an invariant of the homotopy class of f.

Example 8.3. According to theorem 7.3, the degree of the antipodal map $S^n \to S^n$ is $(-1)^{n+1}$.

Proposition 8.4. *Let $f\colon S^n \to S^n$ be a continuous map. If $f(x) \neq x$ for all $x \in S^n$, then f is homotopic to the antipodal map, and so has degree $(-1)^{n+1}$. If $f(x) \neq -x$ for all $x \in S^n$, then f is homotopic to the identity map, and so has degree* 1.

Proof. Let $g\colon S^n \to S^n$ be the antipodal map, $g(x) = -x$ for all x. Assuming that $f(x) \neq x$ for all x, we show that f is homotopic to g. We think of S^n as the unit sphere in $\mathbb{R}^{n+1}$, with the usual notion of distance. We can make a homotopy $(h_t : S^n \to S^n)_{t\in[0,1]}$ from f to g by "sliding" along the unique minimal geodesic arc from $f(x)$ to $g(x)$, for every $x \in S^n$. In other words, $h_t(x) \in S^n$ is situated $t \cdot 100$ percent of the way from $f(x)$ to $g(x)$ along the minimal geodesic arc from $f(x)$ to $g(x)$. (The important thing here is that $f(x)$ and $g(x)$ are not antipodes of each other, by our assumptions. Therefore that minimal geodesic arc is unique.)
Next, assume $f(x) \neq -x$ for all $x \in S^n$. Then, for every x, there is a unique minimal geodesic from x to $f(x)$, and we can use that to make a homotopy from the identity map to f. □

Corollary 8.5. (Hairy ball theorem). *Let ξ be a tangent vector field* (explanations follow) *on S^n. If $\xi(z) \neq 0$ for every $z \in S^n$, then n is odd.*

Comments. A tangent vector field on $S^n \subset \mathbb{R}^{n+1}$ can be defined as a continuous map ξ from S^n to the vector space $\mathbb{R}^{n+1}$ such that $\xi(x)$ is perpendicular to (the position vector of) x, for every $x \in S^n$. We say that vectors in $\mathbb{R}^{n+1}$ which are perpendicular to $x \in S^n$ are *tangent* to S^n at x because they are the velocity vectors of smooth curves in $S^n \subset \mathbb{R}^n$.

Proof. Define $f : S^n \to S^n$ by $f(x) = \xi(x)/\|\xi(x)\|$. Then $f(x) \neq x$ and $f(x) \neq -x$ for all $x \in S^n$, since $f(x)$ is always perpendicular to x. Therefore f is homotopic to the antipodal map, and also homotopic to the identity. It follows that the antipodal map is homotopic to the identity. Therefore n is odd by theorem 7.3. □

9. Mayer-Vietoris sequence in cohomology

Let X be a space with two open subsets V and W such that $V \cup W = X$. We aim to construct a long exact sequence relating the cohomology groups of $V \cap W$, V, W and X, analogous to the Mayer-Vietoris sequence in homology. The cohomology version needs to rely very much on the fracturing results of section 5, especially the homotopy decomposition theorem 5.4. Therefore we assume throughout this section that X, V, W and $V \cap W$ are paracompact.[10]

Choose a continuous map $\psi\colon X \to [0,1]$ such that ψ has support in W and $1-\psi$ has support in V. This amounts to saying that ψ and $1-\psi$ form a partition of unity subordinate to the covering of X by V and W. The coboundary homomorphism

$$\partial\colon H^n(V \cap W) \longrightarrow H^{n+1}(X)$$

is defined as follows. Any element of $H^n(V \cap W)$ can be represented by a morphism $f\colon V \cap W \to S^n$ in $\mathcal{AT}$. Let $q\colon S^n \times [0,1] \to S^{n+1}$ take $(x_0, \dots, x_n) \in S^n$ to $(sx_0, \dots, sx_n, 2t-1) \in S^{n+1}$ where $s = \sqrt{1-(2t-1)^2}$. Make a morphism

$$F\colon X \longrightarrow S^{n+1}$$

[10] The paracompactness assumptions can be avoided at a high price: tampering with the definition of the cohomology groups themselves. Some options will be looked at in later sections.

in $\mathcal{AT}$ as follows. On $V \cap W \subset X$ let F be the composition of q with the self-explanatory morphism $f \times \psi$ from $V \cap W$ to $S^n \times [0,1]$. On the complement of the support of $\psi + 1$, let F be constant with value $(0,0,\dots,0,-1)$ and on the complement of the support of $1-\psi$, let it be constant with value $(0,0,\dots,0,+1)$. This defines F on three open subsets of X whose union is all of X; the definition is consistent on overlaps. The homomorphism ∂ takes the class of f to the class of F. (This is easily seen to be well defined, i.e., the class of F depends only on the class of f, not on a choice of representative, nor on the choice of ψ.)

The coboundary homomorphisms ∂ can be used to make a sequence of abelian groups and homomorphisms

$$
\begin{array}{ccccc}
\cdots \longleftarrow H^{n+1}(X) & & & & \\
\uparrow{\scriptstyle\partial} & & & & \\
H^n(V\cap W) & \longleftarrow & H^n(V)\oplus H^n(W) & \longleftarrow & H^n(X) \\
 & & & & \uparrow{\scriptstyle\partial} \\
 & & & & H^{n-1}(V\cap W) \longleftarrow \cdots
\end{array}
$$

where $n \in \mathbb{Z}$. (Set $H^n(Y) = 0$ for $n < 0$ and any space Y. The unlabelled homomorphisms in the sequence are as follows: $H^n(X) \to H^n(V) \oplus H^n(W)$ is (j_V^*, j_W^*) for the inclusions $j_V : V \to X$ and $j_W : W \to X$, and the homomorphism $H^n(V) \oplus H^n(W) \to H^n(V \cap W)$ is $e_V^* \oplus -e_W^*$, where e_V^* and e_W^* are given by composition with the inclusions $e_V : V \cap W \to V$ and $e_W : V \cap W \to W$.) The sequence is called the cohomology *Mayer-Vietoris* sequence of X and V, W.

Theorem 9.1. *The cohomology Mayer-Vietoris sequence of X and V, W is exact.*

For the proof we need a much better understanding of the coboundary operator, and for that, lemma 9.2 and corollary 9.3 below. Let A be a paracompact space. Let $J^{n+1}(A)$ be the group of *relative* homotopy classes of morphisms $A \times [0,1] \to S^{n+1}$ in $\mathcal{AT}$ which are zero on $A \times \{0,1\}$. The word *relative* means that we only allow homotopies $(A \times [0,1]) \times [0,1] \to S^{n+1}$ which are zero on $(A \times \{0,1\}) \times [0,1]$.

Lemma 9.2. *There is an isomorphism $H^n(A) \to J^{n+1}(A)$.*

Proof. Represent an element of $H^n(A)$ by a morphism $f : A \to S^n$ in $\mathcal{AT}$ such that the composition of f with the unique map $S^n \to \star$ in $\mathcal{T}$ is zero (in $\mathcal{AT}$). Let

$$\Sigma f : A \times [0,1] \longrightarrow S^{n+1}$$

be the composition of $f \times [0,1] : A \times [0,1] \longrightarrow S^n \times [0,1]$ in $\mathcal{AT}$ with the map $q : S^n \times [0,1] \longrightarrow S^{n+1}$ in $\mathcal{T}$. It is easy to check that Σf vanishes on $A \times \{0,1\}$. The rule $[f] \mapsto [\Sigma f]$ is a well-defined homomorphism from $H^n(A)$ to $J^{n+1}(A)$.
For a homomorphism in the other direction, represent an element of $J^{n+1}(A)$ by a morphism $F : A \times [0,1] \to S^{n+1}$ which is zero on $A \times \{0,1\}$. Without loss of generality, F is zero on a neighbourhood of $A \times \{0\}$. Let M be the complement of $q(S^n \times \{1\})$ in S^{n+1} and let N be the complement of $q(S^n \times \{0\})$ in S^{n+1}. Choose a decomposition $F = F^M + F^N$ as in the homotopy decomposition theorem 5.4. Then $F_1^N = -F_1^M$ is a morphism from $A \cong A \times \{1\}$ to $M \cap N$, and the inclusion $S^n \to M \cap N$ is a homotopy equivalence. So F_1^N represents an element in $H^n(A)$. (To verify that this depends only on the class of F in $J^{n+1}(A)$, proceed as in

section 6, after definition 6.2.)
We have constructed two homomorphisms; denote them by $u\colon H^n(A) \to J^{n+1}(A)$ and $v\colon J^{n+1}(S) \to H^n(A)$. To show that vu is the identity on $H^n(A)$, start with $f\colon A \to S^n$ representing an element of $H^n(A)$. Choose a continuous map ψ from $[0,1]$ to $[0,1]$ which takes a neighbourhood of 0 to 0 and is the identity on $[1/2,1]$. Put $F = \Sigma f \circ g$ where $g\colon A\times[0,1] \to A\times[0,1]$ is defined by $(x,t)\mapsto(x,\psi(t))$. Now F is zero on a neighbourhood of $A\times\{0\}$. Let F^N be the composition of F with the map $A\times[0,1]\to A\times[0,1]$ taking (a,t) to (a,t) for $t\le 1/2$ and to $(a,1/2)$ otherwise. Let $F^M = F - F^N$. The decomposition $F = F^M + F^N$ has the properties that we require, and $F_1^N = f$. Therefore vu is the identity.
Finally we show that $\ker(v) = 0$. Let $F\colon A\times[0,1]\to S^{n+1}$ be a morphism in $\mathcal{AT}$ which is zero on $A\times\{1\}$ and on a neighbourhood of $A\times\{0\}$. Suppose that F has a decomposition $F = F^M + F^N$ as in the homotopy decomposition theorem, and that F_1^N represents zero in $H^n(A)$. Then it follows that F_1^N is homotopic to zero as a morphism $A\to S^n$. Using such a homotopy, it is easy to show that $[F] = [G] \in J^{n+1}(A)$ where $G\colon A\times[0,1]\to S^{n+1}$ is another morphism in $\mathcal{AT}$ which is zero on $A\times\{1\}$ and on a neighbourhood of $A\times\{1\}$, and admits a decomposition $G = G^M + G^N$ as in the homotopy decomposition theorem, with $G_1^N = 0$ (strictly, not just up to homotopy). Then we can write $[G] = [G^M] + [G^N] \in J^{n+1}(A)$. Now it is enough to show that $[G^M] = 0$ and $[G^N] = 0$ in $J^{n+1}(A)$. This is easy as M and N are contractible. ☐

Corollary 9.3. *Concatenation in $J^{n+1}(A)$ agrees with addition; reversal is the same as sign change.*

Proof and explanation. For morphisms $F, G\colon A\times[0,1]\to S^{n+1}$ in $\mathcal{AT}$, both vanishing on a neighbourhood of $A\times\{0,1\}$, there is a morphism

$$A\times[0,2]\to S^{n+1}$$

restricting to F on a neighbourhood of $A\times[0,1]$ and to G composed with the translation $(a,t)\mapsto(a,t-1)$ on a neighbourhood of $A\times[1,2]$. Reparameterising, we view this as a morphism $F*G\colon A\times[0,1]\to S^{n+1}$ and call it the concatenation of F and G. This operation is compatible with the homotopy relation and so induces a well defined map

$$J^{n+1}(A)\times J^{n+1}(A)\longrightarrow J^{n+1}(A)$$

which we still call concatenation. To show that it agrees with the addition in $J^{n+1}(A)$ we ask what happens to $[F*G]$ under the abelian group isomorphism

$$v\colon J^{n+1}(A)\to H^n(A)$$

from the proof of lemma 9.2. By inspection, $v[F*G] = v[F] + v[G]$.
In a similar spirit, for a morphism $F\colon A\times[0,1]\to S^{n+1}$ vanishing on a neighbourhood of $A\times\{0,1\}$, there is a morphism $rF\colon A\times[0,1]\to S^{n+1}$ obtained by composing F with the homeomorphism $(a,t)\mapsto(a,1-t)$ from $A\times[0,1]$ to itself. By inspection, $v[rF] = -v[F]$ so that the reversal operation $[F]\mapsto[rF]$ agrees with sign change, $[F]\mapsto -[F]$. ☐

Let X^e be the double mapping cylinder of

$$V \leftarrow V\cap W \to W\ .$$

The projection $X^e \to X$ is a homotopy equivalence.[11] Therefore $H^n(X)$ can be identified with $H^n(X^e)$ and the coboundary can be thought of as a homomorphism

$$\partial\colon H^n(V\cap W) \longrightarrow H^{n+1}(X^e)\ .$$

By lemma 9.2, we may also describe it in the form

$$\partial\colon J^{n+1}(V\cap W) \to H^{n+1}(X^e)$$

And now there is a very elementary description. Namely, an element of $J^{n+1}(V\cap W)$ can be represented by a morphism $F\colon (V\cap W)\times[0,1] \to S^{n+1}$ which is zero on a neighbourhood of $(V\cap W)\times\{0,1\}$. The morphism F extends to a morphism $X^e \to S^{n+1}$ which is zero on $V\times\{0\}$ and $W\times\{1\}$. That morphism represents $\partial[F]$. It is not difficult (left to the reader) to prove agreement with the original definition of the coboundary operator ∂.

Proof of theorem 9.1. Exactness at $H^n(V)\oplus H^n(W)$ means that given a in $H^n(V)$ and b in $H^n(W)$ such that the images of a and b in $H^n(V\cap W)$ agree, there exists $c\in H^n(X)$ restricting to $a\in H^n(V)$ and to $b\in H^n(W)$. Represent a by a morphism $f\colon V\to S^n$ in $\mathcal{A}\mathfrak{T}$ such that the composition of f with the constant map $S^n\to\star$ (in $\mathfrak{T}$) is homotopic to zero, and represent b by a morphism $g\colon W\to S^n$ in $\mathcal{A}\mathfrak{T}$ such that the composition of g with $S^n\to\star$ is homotopic to zero. Then the restrictions of a and b to $V\cap W$ are homotopic as morphisms from $V\cap W$ to S^n. Apply proposition 5.2 to find a morphism $k\colon X\to S^n$ whose restrictions to V and W are homotopic to f and g, respectively. Let c be the class of k in $H^n(X)$.
The elements of $H^{n+1}(V)$ and $H^{n+1}(W)$ that we obtain by restricting an element of the form $\partial(b)\in H^{n+1}(X)$, where $b\in H^n(V\cap W)$, are represented by morphisms $V\to S^{n+1}$ and $W\to S^{n+1}$ in $\mathcal{A}\mathfrak{T}$ which factor through a proper subset of S^{n+1}. They are therefore zero. Showing exactness at $H^{n+1}(X)$ then boils down to showing that given $c\in H^{n+1}(X)$ such that the restrictions of c in $H^{n+1}(V)$ and $H^{n+1}(W)$ are both zero, there exists $b\in H^n(V\cap W)$ such that $\partial(b)=c$. To show this, represent c by a morphism $f\colon X^e\to S^{n+1}$ in $\mathcal{A}\mathfrak{T}$ such that the composition with $S^{n+1}\to\star$ is homotopic to zero. Then the restrictions $f|V$, from V to S^{n+1}, and $f|W$, from W to S^{n+1}, are homotopic to zero. Without loss of generality therefore, f is *equal* to zero on a neighbourhood of $V\cup W$ in the double mapping cylinder X^e. (If not, it is easy to make a homotopy from f to another morphism $X^e\to S^{n+1}$ having that property.) Therefore $c=[f]$ is in the image of ∂, viewed as a homomorphism from $J^{n+1}(V\cap W)$ to $H^{n+1}(X^e)$.
It remains to show exactness at $H^n(V\cap W)$. Using lemma 9.2, we can reformulate the task as follows. Suppose that $[F]\in J^{n+1}(V\cap W)$ has

$$\partial[F]=0\in H^{n+1}(X^e)\ .$$

Then we need to show that $[F]$ is in the image of the homomorphism $e_V^*\oplus -e^*W$ from $J^{n+1}(V)\oplus J^{n+1}(W)$ to $J^{n+1}(V\cap W)$. Without loss of generality,

$$F\colon (V\cap W)\times[0,1]\to S^{n+1}$$

is zero on a neighbourhood of $(V\cap W)\times\{0,1\}$. Note that the composition of F with the unique map $S^{n+1}\to\star$ is strictly zero because it is strictly zero on $(V\cap W)\times\{0,1\}$. The assumption $\partial[F]=0\in H^{n+1}(X^e)$ means that F, viewed as a morphism from X^e to S^{n+1}, is homotopic to zero. Let $G\colon X^e\times[0,1]\to S^{n+1}$ be a

[11]A homotopy inverse is given by the map $X\to X^e$ taking $x\in X$ to $(x,0)$ when $x\notin W$, to $(x,1)$ when $x\notin V$, and to $(x,\psi(x))$ when $x\in V\cap W$.

homotopy from F to 0. We can assume that it is stationary on a neighbourhood of $X^e \times \{0,1\}$. The restrictions of G to $V \times \{0\} \times [0,1]$ and $W \times \{1\} \times [0,1]$ determine elements of $J^{n+1}(V)$ and $J^{n+1}(W)$, respectively, which we may call $[G']$ and $[G'']$. The homotopy G, restricted to $(V \cap W) \times [0,1] \times [0,1]$, can be viewed as a homotopy to zero from the concatenation of three morphisms $(V \cap W) \times [0,1] \to S^{n+1}$, all vanishing on a neighbourhood of $(V \cap W) \times \{0,1\}$: the reverse of G' restricted to $(V \cap W) \times [0,1]$, then F, and then G'' restricted to $(V \cap W) \times [0,1]$. By corollary 9.3, this means that $[F] = e_V^*[G'] - e_W^*[G'']$. □

10. Products revisited

11. Proof of the homotopy decomposition theorem

References

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