SMOOTH MAPS TO THE PLANE AND PONTRYAGIN CLASSES PART I: LOCAL ASPECTS

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Abstract. We classify the most common local forms of smooth maps from a smooth manifold L to the plane. The word *local* can refer to locations in the source L, but also to locations in the target. The first point of view leads us to a classification of certain germs of maps, which we review here although it is very well known. The second point of view leads us to a classification of certain *multigerms* of maps.

1. Introduction

Our goal is to investigate locally uncomplicated smooth maps from a smooth manifold L of dimension $n+2$ to the plane $\mathbb{R}^2$. Where we use the word *local*, as in *locally uncomplicated*, we sometimes refer to locations in the source L, sometimes to locations in the target $\mathbb{R}^2$. The emphasis is on families of smooth maps; this is in contrast to Morse theory, where the study of individual (locally uncomplicated) smooth maps from a manifold to $\mathbb{R}$ is a central topic. We are guided by two observations.

(i) Let X be an open subspace of the space of all smooth maps $L \to \mathbb{R}^2$ defined by prohibiting certain singularities. It is a special case of a theorem due to Vassiliev [6],[7] that X has an accessible homotopy type or homology type if, loosely speaking, every smooth map $L \to \mathbb{R}^2$ can be approximated by a map which belongs to X, and moreover every smooth one-parameter family of smooth maps $L \to \mathbb{R}^2$ can be approximated by a path in X. Therefore we are inclined to define notions of locally uncomplicated map $L \to \mathbb{R}^2$ by prohibiting certain singularities or singularity types corresponding to a subset of an appropriate jet space whose codimension in the jet space is at least $n+4$.

(ii) More restrictive notions of locally uncomplicated map $L \to \mathbb{R}^2$ can be obtained by prohibiting, for every $r \geq 1$, certain configurations of r singularities (*multigerms*) in the source L, with the same image point in $\mathbb{R}^2$. The Vassiliev theorem mentioned above can be adapted to this setup [5], although it is considerably harder to say which multigerms can be prohibited without making the resulting space of locally uncomplicated smooth maps $L \to \mathbb{R}^2$ homologically or homotopically inaccessible.

These two observations raise two elementary classification problems, one for uncomplicated germs and one for uncomplicated multigerms, which we solve. The

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solution of the first problem is well known. Our account of it leads on naturally to a solution of the second problem, which we believe is new.

2. Germs of maps from the plane to the plane

The classification of the most common map germs from plane to plane up to left-right equivalence is well known. See for example [1]. (We are talking about smooth map germs f from $(\mathbb{R}^2,0)$ to $(\mathbb{R}^2,0)$. Two such germs f_0, f_1 are *left-right equivalent* if there exist diffeomorphism germs $\psi\colon(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ and $\sigma\colon(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ such that $f_1=\sigma f_0\psi^{-1}$.) We will repeat it here nevertheless and see some normal forms and tell the story of each singularity type.

2.1. **Classification.** There are six types that we consider worthy of attention: *regular*, *fold*, *cusp*, *swallowtail*, *lips* and *beak-to-beak*. The regular (alias nonsingular) type is well understood. The remaining five types are of rank 1, that is, the derivative at the origin has rank 1. (The cases where the derivative has rank 0 are uninteresting to us because their codimension is at least 4.) Among these, it is natural to distinguish between those for which the 1-jet prolongation is transverse to the rank 1 stratum (fold, cusp and swallowtail) and those for which it is not (lips and beak-to-beak). In the transverse case, the singularity set in the source is a smooth curve in the plane, passing through the origin; in the non-transverse case, it is in some way or other a singular curve, as we will see.

Fold: The normal form is $f(x,y)=(x,y^2)$. The singularity set in the source is a line (in the normal form, the x-axis) and the singularity set in the target is also a line (in the normal form, again the x-axis). The intrinsic second derivative [3] at the origin is a nondegenerate quadratic form (defined on the kernel of the first derivative, and with values in the cokernel of the first derivative).

Cusp: Normal form $f(x,y)=(x,y^3+xy)$. The derivative matrix for the normal form is

$$df(x,y)=\begin{bmatrix}1 & 0\\ y & 3y^2+x\end{bmatrix}$$

with determinant $(x,y)\mapsto 3y^2+x$. Hence the singularity set Σ in the source is the trajectory of $t\mapsto(-3t^2,t)$, a parabola. The singularity set in the target is the trajectory of $t\mapsto(-3t^2,-2t^3)$.

Swallowtail: Normal form $f(x,y)=(x,y^4+xy)$. The singularity set Σ in the source is the trajectory of $t\mapsto(-4t^3,t)$. The singularity set in the target is the trajectory of $t\mapsto(-4t^3,-3t^4)$.

Lips: Normal form $f(x,y)=(x,y^3+x^2y)$. The singularity set in the source is the set of zeros of the quadratic form $(x,y)\mapsto x^2+3y^2$, that is, a single point. It is a manifold but it does not have dimension 1.

Beak-to-beak: Normal form $f(x,y)=(x,y^3-x^2y)$. The singularity set in the source is the set of zeros of the quadratic form $(x,y)\mapsto x^2-3y^2$, that is, the union of the lines described by $x=cy$ and $x=-cy$, where $c=3^{1/2}$. It has dimension 1 but it is not a manifold. The singularity set in the target is the union of the trajectories of

$$t\mapsto(ct,-2t^3),\qquad t\mapsto(-ct,-2t^3).$$

Remark 2.1. In all these formulae, the first coordinate f_1 of f is $(x,y) \mapsto x$. The best way to understand the classification is to regard the second coordinate f_2 of f as an *unfolding* of a germ $g : (\mathbb{R},0) \to (\mathbb{R},0)$, with unfolding parameter x. The formula for g can be seen by setting $x = 0$. This gives $g(y) = y$ for the regular case, $g(y) = y^2$ for the fold, $g(y) = y^3$ for cusp, lips and beak-to-beak, and $g(y) = y^4$ for the swallowtail.[1] Each of the unfoldings can be pulled back from a miniversal unfolding with parameter space V. The miniversal unfoldings are as follows:

$$g(y) = y^2 : \quad y^2 \tag{2.1}$$
$$g(y) = y^3 : \quad y^3 + uy \tag{2.2}$$
$$g(y) = y^4 : \quad y^4 - uy^2 + vy\ . \tag{2.3}$$

This is essentially in the notation of [2, ch.15], although we use y where [2] has x. The decisive features of the germs f are therefore as follows:

(i) the corresponding germ $g : (\mathbb{R},0) \to (\mathbb{R},0)$ obtained by setting $x = 0$ in the formula for f_2 ;
(ii) the smooth map $p : (\mathbb{R},0) \to (V,0)$ (where V parametrizes the miniversal unfolding of the appropriate g) such that f_2 as an unfolding is isomorphic to p^* of the miniversal unfolding. This p is in most cases far from unique.

For us, $V = \mathbb{R}$ or $V = \mathbb{R}^2$. In the notation of [2, ch.15], the maps p are as follows: $p(x) = x \in \mathbb{R}$ for the cusp, $p(x) = (0,x) \in \mathbb{R}^2$ for the swallowtail, $p(x) = x^2 \in \mathbb{R}$ for the lips and $p(x) \mapsto -x^2 \in \mathbb{R}$ for beak-to-beak.

It is not completely trivial to justify this classification. What the above arguments prove beyond doubt is that we have a *surjective* map from isomorphism classes of 1-parameter unfoldings of germs g (such as $g(y) = y^n$, with $n = 1,2,3,4$) to the set of left-right equivalence classes of germs $f : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ whose derivative at 0 has rank 1. What remains to be done is roughly the following:

(i) to produce a "sufficiently big" list of some of the 1-parameter unfoldings of the germs g, and to determine the corresponding left-right equivalence classes of map germs $f : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$;
(ii) to show that each of these left-right equivalence classes has codimension ≤ 3 and that all remaining germs $(\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ taken together make up a subset of codimension ≥ 4.

2.2. **Unfoldings.** We start with the list of unfoldings. Every 1-parameter unfolding of a smooth function germ $g : (\mathbb{R},0) \to (\mathbb{R},0)$ with nonzero Taylor series is isomorphic to e^* of the miniversal unfolding, where

$$e : (\mathbb{R},0) \to (V,0)$$

is smooth and $V = V_g$ is the parameter space for the miniversal unfolding of g. The fact that e is usually not unique makes the classification difficult. However, some special cases are easy.

If $g(y) = y^2$, then V_g is zero-dimensional.

[1] Catastrophe theory has names for these germs g which sometimes clash with our names for the corresponding maps f. The catastrophe theory names tend to describe the projection from the fiberwise singularity set of the miniversal unfolding of g to the parameter space of the unfolding.

If $g(y) = y^3$, then V_g is 1-dimensional. The proposed normal forms for e are $e(x) = x$, $e(x) = x^2$ and $e(x) = -x^2$. If $q : (\mathbb{R}, 0) \to \mathbb{R} = V_g$ has nonzero first derivative, then we can find an invertible $h : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $q = eh$ where $e(x) = x$, and that can be used to produce the required isomorphism. Similarly, if q has zero first derivative but strictly positive second derivative, then we can find an invertible $h : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $q = eh$ where $e(x) = x^2$. Similarly, if q has zero first derivative but strictly negative second derivative, then we can find an invertible germ $h : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $q = eh$ where $e(x) = -x^2$.

So in fact the only difficult case is the case where $g(y) = y^4$. We use the miniversal unfolding given by (2.3). Hence $V = V_g$ is 2-dimensional. We want to focus on map germs $e : (\mathbb{R}, 0) \to (V, 0)$ with nonzero first derivative, not parallel to the u-axis. (The u-axis is a distinguished direction in V because it is parallel to the cusp in V obtained by projecting the fiberwise singularity set of the unfolding to V.) The corresponding 1-parameter unfolding of $g(y) = y^4$ then has the form $y^4 + e_1(x)y^2 + e_2(x)y$ with $e_2'(0) \neq 0$. Using e_2 to transform the source of e, we can reduce to a situation where $e_2(x) = x$. So we have

$$(x, y) \mapsto y^4 + p_x y^2 + xy$$

where $p_x = e_1(x)$. From example 5.11 we know that this is left-right equivalent to $(x, y) \mapsto (y^4 + xy)$, which is the swallowtail normal form.

The rest of our classification task is easier. The five singularity types, represented by the five normal forms above, are easy to distinguish by geometric properties which are invariant under left-right equivalence.

For the fold type, the singularity set Σ in the source is a smooth submanifold of dimension 1, and $f|\Sigma$ is an immersion (near 0).

For the cusp and swallowtail, the singularity set Σ in the source is still a smooth submanifold of dimension 1, but $f|\Sigma$ is not an immersion near 0. To distinguish cusp and swallowtail, it is enough to show that the curves

$$t \mapsto (-3t^2, -2t^3), \ t \mapsto (-4t^3, -3t^4)$$

are not left-right equivalent. This is obvious by looking at the second (intrinsic) derivative [2, 3] at the origin, which is nonzero in the cusp case, zero in the swallowtail case.

For the lips and beak-to-beak, the singularity set in the source is not a smooth submanifold of dimension 1; it is a point in the lips case and a "node" (two crossing lines) in the beaks-to-beaks case.

2.3. **Codimension and stratification.** We turn to the codimension and stratification analysis. Among other things we want to determine the codimension of each of the six types described above, and we want to show that all remaining singularity types taken together constitute a set of codimension > 3. We start by summarizing the analytic characterizations of the six types. We can always assume that $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ has the form

$$(x, y) \mapsto (x, f_2(x, y))$$

and $\partial f_2/\partial x$ vanishes at 0. In the singular case, we also assume that $\partial f_2/\partial y$ vanishes at 0. The following table describes the six types by means of conditions on the 4th Taylor polynomial of f_2. The conditions typically state that some term in the Taylor polynomial has to be zero (z) or nonzero (n). For example, the table states that in

the case of a cusp, the coefficients of y and y^2 must be zero while the coefficients of xy and y^3 must be nonzero (and there are no further conditions).
In the "other conditions" column of the table, b_3, d_1 and d_2 are the coefficients of y^3, xy^2 and x^2y respectively. The expression $3b_3d_2 - d_1^2$ arises when we trade xy^2 terms for x^2y terms, composing with a diffeomorphism germ (in the source) of the form $(x,y) \mapsto (x, y - kx)$ for some constant k.

y	y^2	y^3	y^4	xy	other conditions	Name
n						regular
z	n					fold
z	z	n		n		cusp
z	z	n		z	$3b_3d_2 - d_1^2 > 0$	lips
z	z	n		z	$3b_3d_2 - d_1^2 < 0$	beak-to-beak
z	z	z	n	n		swallowtail

Definition 2.2. Let P_* be the real vector space of polynomial maps $\mathbb{R}^2 \to \mathbb{R}^2$ (viewed as jets), of degree ≤ 4, with vanishing constant term. We write

$$P_* = P_*^2 \cup P_*^1 \cup P_*^0$$

where P_*^i consists of all those elements of P_* whose linear term has rank i. Let $W^{P_*} \subset P_*$ consist of the polynomials whose germ at the origin belongs to one of the types regular, fold, cusp, swallowtail, lips or beak-to-beak. Thus

$$P_*^2 \subset W^{P_*} \subset P_*^1 \cup P_*^2.$$

Let's also introduce $N \subset P_*^1$, the submanifold of those f which have the form $f(x,y) = (x, f_2(x,y))$ where f_2 has vanishing first derivative.
For P_*^2 we also write G, because it is a Lie group. The group G acts on the left and right of W^{P_*} by composition of polynomial mappings (followed by truncation to degree ≤ 4). In other words, $G \times G^{\mathrm{op}}$ acts on W^{P_*} by $(\varphi, \psi) \cdot f = \varphi f \psi$, for $\varphi, \psi \in G$ and $f \in W^{P_*}$.

Our classification attempts so far describe some orbits of this action of $G \times G^{\mathrm{op}}$ on W^{P_*}. (In particular our classification of some germs $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ up to left-right equivalence can be formulated in terms of Taylor expansions at the origin, up to degree 4 at most.) We now wish to show that W^{P_*} is open, to determine the codimensions in W^{P_*} of the six orbits, and show that the complement of W^{P_*} has codimension ≥ 4 in P_*. We have already convinced ourselves that every $g \in P_*^1$ is left-right equivalent to some $f \in N$. In other words, the restricted action map $G \times N \times G \to P_*^1$ is onto. The following lemma makes this more precise:

Lemma 2.3. *The restricted action map $G \times N \times G \to P_*^1$ is a fiber bundle.*

Proof. Let $E \subset G \times P_*^1$ be the smooth submanifold consisting of all pairs (φ, g), with $\varphi \in G$ and $g \in P_*^1$, such that the first derivative of $\varphi^{-1} g$ at the origin has image equal to the x-axis. We write our map as a composition

$$G \times N \times G \longrightarrow E \longrightarrow P_*^1$$

where the first map is given by $(\varphi, f, \psi) \mapsto (\varphi, \varphi f \psi)$ and the second map is given by $(\varphi, g) \mapsto g$. Clearly the second of these maps is a fiber bundle. To understand the first map, we fix some $(\varphi, g) \in E$. The portion of $G \times N \times G$ mapping to that is identified with the set of all $\psi \in G$ such that $\varphi^{-1} g \psi^{-1} \in N$. This condition on ψ can also be described as saying that the following commutes up to terms of order ≥ 5:

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\psi^{-1}} & \mathbb{R}^2 \\ \Big\downarrow p & & \Big\downarrow p\varphi^{-1}g \\ \mathbb{R} & \xrightarrow{=} & \mathbb{R} \end{array}$$

where $p(x, y) = x$. If we select one such ψ, and we can, then all others can be obtained from the selected one by multiplying on the left with an element of

$$H = \{\gamma \in G \mid p\gamma = p\},$$

a subgroup of G. Hence our map $G \times N \times G \longrightarrow E$ is a principal bundle with structure group H. $\square$

Lemma 2.4. *Suppose that a Lie group L acts smoothly on a smooth connected manifold M. Let $N \subset M$ be a smooth submanifold, closed as a subset of M. Suppose that the restricted action map $L \times N \to M$ is a smooth surjective submersion. Then the partition of M into L-orbits is locally diffeomorphic to the induced partition of N, multiplied with $\mathbb{R}^k$ where $k = \dim(M) - \dim(N)$.*

Proof. Given $z \in M$, choose $(g, x) \in L \times N$ such that $gx = z$. By assumption the differential of the action map $\alpha\colon L \times N \to M$ at (g, x) is a (linear) surjection $d\alpha_{(g,x)}\colon T_gL \times T_xN \to T_zM$. Its restriction to T_xN is injective since it is the differential of an embedding $N \to M$. Hence there exists a k-dimensional subspace $V \subset T_gL$ such that $d\alpha_{(g,x)}$ restricts to a linear isomorphism $V \times T_xN \to T_zM$. Now choose a smooth embedding germ $s : (V, 0) \to (L, g)$ such that the differential of s at 0 is the inclusion $V \to T_gL$. Then the germ of

$$h\colon V \times N \to M \ ; \ h(v, z) = s(v)z$$

at $(0, x)$ is a diffeomorphism germ. Clearly, for (v_1, z_1) and (v_2, z_2) in $V \times N$, the elements $h(v_1, z_1)$ and $h(v_2, z_2)$ are in the same L-orbit if and only if z_1 and z_2 are in the same L-orbit. $\square$

Putting the two lemmas together, we see that in order to understand (some of) the decomposition of P^1_* into $G \times G^{\mathrm{op}}$-orbits, it is sufficient to understand (some of) the induced decomposition of $N \subset P^1_*$. But this is already obvious from the list above. The decomposition of the affine space N can be described in terms of several linear forms on N (and a quadratic form). The linear forms are given by the coefficients $b_2, b_3, b_4, c, d_1, d_2$ of $y^2, y^3, y^4, xy, xy^2, x^2y$, respectively. The quadratic form is $q = 3b_3d_2 - d_1^2$. Let $B_2, B_3, B_4, C, Q \subset N$ be the zero sets of b_2, b_3, b_4, c, q respectively. Now we can describe the "relevant" strata (intersected with N) as follows:

- *fold*: $N \smallsetminus B_2$
- *cusp*: $B_2 \smallsetminus (B_3 \cup C)$
- *swallowtail*: $(B_2 \cap B_3) \smallsetminus (B_4 \cup C)$
- *lips*: $(B_2 \cap C) \smallsetminus Q$ (and $q > 0$)
- *beak-to-beak*: $(B_2 \cap C) \smallsetminus Q$ (and $q < 0$).

The points of N which are not in any of these strata form a closed codimension 3 algebraic subset:

$$N \smallsetminus W^{P_*} = (B_2 \cap B_3 \cap B_4) \cup (B_2 \cap B_3 \cap C) \cup (B_2 \cap C \cap Q)\ .$$

Proposition 2.5. *The complement of W^{P_*} in P_* is closed, algebraic and of codimension ≥ 4. The stratification of W^{P_*} by the six strata (alias $G \times G^{\mathrm{op}}$-orbits) is in fact a filtration by smooth submanifolds (of codimensions 0,1,2,3) as indicated in the following diagram:*

regular ∪ fold ∪ cusp ∪ swallowtail ∪ lips ∪ beak-to-beak

|

fold ∪ cusp ∪ swallowtail ∪ lips ∪ beak-to-beak

|

cusp ∪ swallowtail ∪ lips ∪ beak-to-beak

|

swallowtail ∐ lips ∐ beak-to-beak

Proof. The set $P^1_* \cap W^{P_*} = G(N \cap W^{P_*})G$ is open in P^1_* because $N \cap W^{P_*}$ is open in N. Hence $W^{P_*} = P^2_* \cup (P^1_* \cap W^{P_*})$ is open in $P^2_* \cup P^1_*$ which in turn is open in P_*. The same argument shows that W^{P_*} is algebraic in P_*, given that the two actions of G on P_* are algebraic. The codimension of $G(N \smallsetminus W^{P_*})G = P^1_* \smallsetminus W^{P_*}$ in P^1_* is ≥ 3 by lemma 2.4. Hence the codimension of $P^1_* \smallsetminus W^{P_*}$ in P_* is ≥ 4. The codimension of P^0_* in P_* is also 4.
The second statement follows from our analysis of the stratification of N, together with lemma 2.4. □

Remark 2.6. All elements of W^{P_*} are represented by proper maps $\mathbb{R}^2 \to \mathbb{R}^2$ taking the origin to itself, and have a well-defined degree. The degree is 0 in the case of a fold or swallowtail, but ± 1 in the case of a regular germ, cusp, lips or beak-to-beak. This shows that at least four of the six strata in our stratification of W^{P_*} are not connected.

3. Germs of maps from higher dimensional space to the plane

We generalize the results above by investigating (certain) smooth map germs

$$f\colon (\mathbb{R}^{2+n}, 0) \to (\mathbb{R}^2, 0)$$

for fixed $n \geq 0$. (By writing $\mathbb{R}^{2+n}$ instead of $\mathbb{R}^{n+2}$, we want to suggest a decomposition $\mathbb{R}^{2+n} \cong \mathbb{R}^2 \times \mathbb{R}^n$.) It turns out that there is an easy reduction to the case $n = 0$.

3.1. **Classification.** We begin with the classification up to left-right equivalence. Again we exclude the cases where $df(0)$ has rank 0 and note that the rank 2 case is easy. This leaves the rank 1 case. Using appropriate linear transformations of source and target, we may assume that

$$df(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & \\ 0 & 0 & 0 & 0 & \cdots & \end{bmatrix}$$

so that the image of $df(0)$ is the x-axis. Writing $p\colon \mathbb{R}^2 \to \mathbb{R}$ for the linear projection $(x,y) \mapsto x$, we can use pf as one of $2+n$ coordinates on the source and so obtain

$$f(x,y,z_1,\dots,z_n) = (x, f_2(x,y,z_1,\dots,z_n))$$

where $f_2\colon (\mathbb{R}^{2+n},0) \to (\mathbb{R},0)$ has vanishing derivative at 0. Then we require that the Hessian of f_2, restricted to $\ker(df(0))$, be not too singular: its nullspace must have dimension ≤ 1. (The cases where the nullspace has dimension ≥ 2 are considered too rare to be of interest here.) There are two cases to distinguish.

Case 1: The nullspace of that restricted Hessian has dimension 0. By the Morse lemma we can assume, after a coordinate transformation in the source (involving only the coordinates $y, z_1, \dots, z_n$), that f_2 restricted to $\ker(df(0))$ is a quadratic form alias homogeneous polynomial of degree 2. Then f_2 can be viewed as a 1-parameter deformation of the restriction of f_2 to $\ker(df(0))$. By the classification of such deformations, we may assume that the deformation is merely given by translations in the target (after another coordinate transformation in the source, of type

$$(x,y,z_1,\dots,z_n) \mapsto (x, \varphi_x(y,z_1,\dots,z_n))$$

with $\varphi_0 = \mathrm{id}$). Then we have the form

$$f(x,y,z_1,\dots,z_n) = (x, q(y,z_1,\dots,z_n) + g(x))$$

where q is a nondegenerate quadratic form in $n+1$ variables. Finally we may remove the $g(x)$ term using a coordinate transformation in the target. This gives the form

$$f(x,y,z_1,\dots,z_n) = (x, u(y,z_1,\dots,z_n))$$

where u is a nondegenerate quadratic form in $n+1$ variables. Using another linear transformation of the source coordinates $y, z_1, \dots, z_n$ and where necessary a reflection $(x,y) \mapsto (x,-y)$ in the target, we reduce further to the case where $u(y,z_1,\dots,z_n) = y^2 + q(z_1,\dots,z_n)$ for a quadratic form q in the variables $z_1,\dots,z_n$. Then we have the canonical form

$$f(x,y,z_1,\dots,z_n) = (x, y^2 + q(z_1,\dots,z_n))$$

where q is a nondegenerate quadratic form in the variables $z_1,\dots,z_n$.

Case 2: The nullspace of that restricted Hessian has dimension 1. We may assume that the nullspace is the y-axis. Let $K = \{(0,0,z_1,\dots,z_n)\} \subset \mathbb{R}^{2+n}$. By the Morse lemma applied to $f_2|_K$, we may assume that $f_2|_K$ is a nondegenerate quadratic form (after a suitable coordinate transformation in the source involving only $z_1,\dots,z_n$). Now we can view f as a 2-parameter deformation (parameters x and y) of $f_2|_K$. By the classification of such deformations, we may assume that the deformation is merely given by translations in the target. Then

$$f(x,y,z_1,\dots,z_n) = (x, f_2^r(x,y) + q(z_1,\dots,z_n))$$

where we write f_2^r to indicate a "reduced" form of f_2. In words, f_2 has the form of a function germ f_2^r which only depends on the variables x and y, and has vanishing first derivative at 0, plus a nondegenerate quadratic form q which depends only on the *other* variables $z_1,\dots,z_n$. The second derivative at 0 of f_2^r restricted to the y-axis is zero, because we are not in "case 1".

The analysis in case 1 above is fairly complete. We call this type a *fold*. In case 2, it is natural to proceed by imposing a condition: namely, that the germ

$$(x,y) \mapsto (x, f_2^r(x,y))$$

have one of the types *cusp*, *swallowtail*, *lips* or *beak-to-beak* described earlier in this section. Then we get the list of normal forms

$$\begin{aligned}
&(3.1) & \textit{Fold}: f(x,y,z_1,\ldots,z_n) &= (x, y^2 + q(z_1,\ldots,z_n)) \\
&(3.2) & \textit{Cusp}: f(x,y,z_1,\ldots,z_n) &= (x, y^3 + xy + q(z_1,\ldots,z_n)) \\
&(3.3) & \textit{Swallowtail}: f(x,y,\ldots,z_1,z_n) &= (x, y^4 + xy + q(z_1,\ldots,z_n)) \\
&(3.4) & \textit{Lips}: f(x,y,z_1,\ldots,z_n) &= (x, y^3 + x^2y + q(z_1,\ldots,z_n)) \\
&(3.5) & \textit{Beaktobeak}: f(x,y,z_1,\ldots,z_n) &= (x, y^3 - x^2y + q(z_1,\ldots,z_n)).
\end{aligned}$$

In these formulae, q is a nondegenerate quadratic form. It is easy to see that the five types are distinguishable in coordinate free terms. For example, in the cusp and swallowtail cases, the singularity set in the source is a smooth submanifold of dimension 1, but in the lips and beak-to-beak cases, it is not. The cusp case can be distinguished from the swallowtail case because the singularity sets in the target are not equivalent.

The above reduction procedure extends easily to 1-parameter families. Indeed, suppose that we have a smooth function germ $(\mathbb{R}\times\mathbb{R}^{2+n},0)\to(\mathbb{R}^2,0)$ which we want to regard as a 1-parameter family of germs

$$f_t : (\mathbb{R}^{2+n},0)\to(\mathbb{R}^2,0)$$

with $t\in\mathbb{R}$ in a neighborhood of 0. Suppose that the first derivative of each f_t at 0 has rank 1, and also that f_0 has the "reduced" form

$$f_0(x,y,z_1,\ldots,z_n) = (x, f_{0,2}^r(x,y) + q_0(z_1,\ldots,z_n))$$

where q_0 is a nondegenerate quadratic form in n variables. Then there exist diffeomorphism germs

$$\psi_t : (\mathbb{R}^{2+n},0)\to(\mathbb{R}^{2+n},0)\ , \qquad \varphi_t : (\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$$

depending smoothly on t, with $\psi_0 = \mathrm{id}$ and $\varphi_0 = \mathrm{id}$, such that $\bar{f}_t = \varphi_t f_t \psi_t$ is in reduced form,

$$\bar{f}_t(x,y,z_1,\ldots,z_n) \mapsto (x, \bar{f}_{t,2}^r(x,y) + q_t(z_1,\ldots,z_n)).$$

Here q_t is a nondegenerate quadratic form in n variables. Therefore we have proved the following lemma.

3.2. **Codimension and stratification.** Let P_* be the finite dimensional real vector space of polynomial maps $\mathbb{R}^{2+n}\to\mathbb{R}^2$ of degree ≤ 4, with vanishing constant term. We write

$$P_* = P_*^2 \cup P_*^1 \cup P_*^0$$

where P_*^i consists of the polynomials whose linear term has rank i. Let G be the set of polynomial maps of degree ≤ 4 from $\mathbb{R}^{2+n}$ to $\mathbb{R}^{2+n}$, with vanishing constant term and invertible linear term. Under composition and truncation, G becomes a group, and this group acts on the right of P_* by composition. Let $W^{P_*}\subset P_*$ be the union of the six strata *regular, fold, cusp, swallowtail, lips* and *beak-to-beak*. Let $D\subset P_*^1$ be the closed subset consisting of the elements whose second "Porteous" derivative has a nullspace of dimension > 1. Let F be the space of nondegenerate quadratic forms in n real variables $z_1, \dots, z_n$. We write G_{ol} for the old G of lemma 2.3, and N_{ol} for the old N of lemma 2.3.

Lemma 3.1. *The restricted action map*

$$\begin{array}{rcl} G_{ol} \times N_{ol} \times F \times G & \longrightarrow & P^1_* \smallsetminus D \\ (\varphi, f, q, \psi) & \mapsto & \varphi(f+q)\psi \ , \end{array}$$

where $f+q$ is shorthand for the map

$$(x, y, z_1, \dots, z_n) \mapsto (x, f_2(x,y) + q(z_1, \dots, z_n)) \ ,$$

is a surjective submersion. □

This puts us in a position to use lemma 2.4. It follows that the partition of $P^1_* \smallsetminus D$ into $G_{ol} \times G^{\mathrm{op}}$ orbits is locally diffeomorphic to the induced partition of $N_{ol} \times F$. But the latter is essentially the partition of N_{ol} into $G_{ol} \times G^{\mathrm{op}}_{ol}$ orbits multiplied with a certain partition of F where each part is a union of path components.

Corollary 3.2. *The complement of W^{P_*} in P_* is closed, algebraic and of codimension $\geq n+4$. The stratification of W^{P_*} by the six strata is given by a nested sequence of smooth algebraic subvarieties of W^{P_*} of codimensions 0, $n+1$, $n+2$, $n+3$, respectively, as indicated in the following diagram:*

$$\begin{array}{c} \text{regular} \cup \text{fold} \cup \text{cusp} \cup \text{swallowtail} \cup \text{lips} \cup \text{beak-to-beak} \\ \Big| \\ \text{fold} \cup \text{cusp} \cup \text{swallowtail} \cup \text{lips} \cup \text{beak-to-beak} \\ \Big| \\ \text{cusp} \cup \text{swallowtail} \cup \text{lips} \cup \text{beak-to-beak} \\ \Big| \\ \text{swallowtail} \coprod \text{lips} \coprod \text{beak-to-beak} \end{array}$$

It is invariant under the action of $G_{ol} \times G^{\mathrm{op}}$. The "regular" stratum is a single orbit of that action. The "fold" stratum falls into $\lfloor n/2 + 3/2 \rfloor$ orbits, and the "cusp", "swallowtail", "lips" and "beak-to-beak" strata fall into $\lfloor n/2 + 1 \rfloor$ orbits each.

Proof. Most of this has already been established. The left-right equivalence class counts are obtained by counting components of suitable spaces of nondegenerate quadratic forms, modulo sign change. In the fold case, we have to look at nondegenerate quadratic forms in $n+1$ variables. The components are classified by the signature, which can be $n+1$, $n-1$, ..., $-n-1$. If we allow sign change, as we must, only the absolute value of the signature remains, so there are $\lfloor n/2 + 3/2 \rfloor$ types. In the remaining cases, we are looking at nondegenerate quadratic forms in n variables. There are $\lfloor n/2 + 1 \rfloor$ types. □

4. Multigerms of maps

Let L be a smooth manifold and $S \subset L$ a finite nonempty subset. We are interested in *multigerms* of smooth maps $f\colon (L, S) \to (\mathbb{R}^m, 0)$. Such a multigerm is, strictly speaking, an equivalence class of pairs (U, f) where U is a neighborhood of S in L and $f: U \to \mathbb{R}^m$ is a smooth map taking all of S to 0. Two such pairs (U_0, f_0) and (U_1, f_1) are equivalent if f_0 and f_1 agree on some neighborhood of S contained in $U_0 \cap U_1$.

Definition 4.1. Two multigerms $f:(L,S) \to (\mathbb{R}^m, 0)$ and $g:(L',S') \to (\mathbb{R}^m, 0)$ are *left-right equivalent* if there exist a diffeomorphism germ $\psi:(L,S) \to (L',S')$, extending some bijection $S \to S'$, and a diffeomorphism germ $\sigma:(\mathbb{R}^m,0) \to (\mathbb{R}^m,0)$ such that $g = \sigma f \psi^{-1}$.

There is a similar notion of left-right equivalence for *multijets*. We have in mind the finite set $S_r = \{1, 2, \ldots, r\}$ for $r \geq 1$, and two elements f, g of

$$\prod_{x \in S_r} P_*$$

where P_* is the vector space of polynomial mappings of degree $\leq z$ from $\mathbb{R}^\ell$ to $\mathbb{R}^m$, with vanishing constant term, for some $z > 0$. (Soon we will take $z = 4$ and $\ell = n+2$, $m = 2$ as in previous sections.)

Definition 4.2. The multijets f and g are *left-right equivalent* if there exist z-jets of diffeomorphisms ψ from $(S_r \times \mathbb{R}^\ell, S_r)$ to $(S_r \times \mathbb{R}^\ell, S_r)$, extending some permutation of S_r , and σ from $(\mathbb{R}^m, 0)$ to $(\mathbb{R}^m, 0)$, such that $g = \sigma f \psi^{-1}$. (We have identified S_r with $S_r \times \{0\} \subset S_r \times \mathbb{R}^{n+2}$.)

Remark 4.3. For the rest of the section we take $\mathbb{R}^m = \mathbb{R}^2$ as the target manifold, and focus on source manifolds L of dimension $n+2$. Our goal is to select for each $r \geq 1$ a finite collection Γ_r of left-right equivalence classes of multigerms

$$(L, S) \longrightarrow (\mathbb{R}^2, 0)$$

where $|S| = r$, in such a way that a number of obviously desirable and less obviously desirable conditions are satisfied. The multigerms which belong to one of the types in the collection Γ_r , for some r, are called *admissible*. Among the obviously desirable conditions is

(a) Naturality: for an admissible multigerm from (L,S) to $(\mathbb{R}^2, 0)$, and any nonempty subset T of S, the induced multigerm from (L,T) to $(\mathbb{R}^2,0)$ is admissible. More generally, for any admissible multigerm f from (L,S) to $(\mathbb{R}^2,0)$, there exists a neighborhood U of S in L with the following property. For any finite nonempty subset T of U such that $f|T$ is constant, the multigerm of f at T, minus that constant, is admissible.

Suppose that (a) holds and let $f: L \to \mathbb{R}^2$ be a smooth map, where $\dim(L) = n+2$. We say that f is admissible if, for every finite nonempty subset $S \subset L$ such that $f|S$ is constant, the multigerm of f at S, minus that constant, is admissible. Conditions (b) and (c) below ensure, loosely speaking, that for L as above the cohomology of the space of admissible smooth maps $L \to \mathbb{R}^2$ admits a description in terms of the cohomology of the spaces of admissible smooth multigerms $(L,S) \to (\mathbb{R}^2, 0)$, where S runs through the finite nonempty subsets of L. (We will not explain here *how* conditions (b) and (c) ensure that; see [5] instead.) For finite nonempty $S \subset L$ and a non-admissible germ

$$g:(L,S) \longrightarrow (\mathbb{R}^2, 0),$$

a nonempty subset T of S is *a minimal bad event* if the induced multigerm $(L,T) \to (\mathbb{R}^2, 0)$ is non-admissible and T has no proper nonempty subset with the same property. A nonempty subset T of S is a *bad event* for g if it is a union of minimal bad events for g. The *size* of g is the maximum cardinality of a bad event for g. The *complexity* of g is the maximum of the integers k such that there exists a chain of bad events $T_0 \subset T_1 \subset \cdots \subset T_{k-1} \subset T_k$ where $T_i \neq T_{i+1}$ for $i = 0, \ldots, k-1$.

(b) The codimension $c_*(s)$ of the set of non-admissible multigerms of size s is at least $sn + 4$.

(c) For $k \leq s$, the codimension $c_*(s,k)$ of the set of non-admissible multigerms of size s and complexity k satisfies

$$\lim_{s\to\infty} (c_*(s,k) - sn - k) \;=\; \infty \;.$$

These conditions, in particular the meaning of codimension, will be made more precise just below. See also remark 4.4. Here is one more condition.

(d) The collections Γ_r for $r \geq 1$, taken together, are as small as they can be under the constraints expressed in conditions (a), (b) and (c).

As we shall see, conditions (a),(b),(c) and (d) are enough to determine the collections Γ_r for all $r \geq 1$.

By way of self-fulfilling prophecy, we assume that the admissible multigerms are finitely determined. In more detail, we assume that there exists a positive integer z, independent of r, such that two admissible multigerms

$$f\colon (L,S) \to (\mathbb{R}^2, 0) \;, \qquad g\colon (L',S') \to (\mathbb{R}^2,0) \;,$$

where $|S| = |S'| = r$, are left-right equivalent if and only if their z-multijets are left-right equivalent. Then we can speak of left-right equivalence classes of admissible multijets, and these can be described as subsets of the multijet space

$$\prod_{x\in S_r} P_* \tag{4.1}$$

as in definition 4.2. They are semi-algebraic subsets, being orbits of an algebraic group action. In the multijet space (4.1), the subset of non-admissible multijets of size s and complexity k (where $k \leq s \leq r$) is a semi-algebraic subset, with a minimum codimension which we denote by $c_*(s,k,r)$. Let

$$c_*(s,k) = \min_r \{c_*(s,k,r)\}.$$

It is easy to see that $c_*(s,k) = c_*(s,k,s)$. Let

$$c_*(s) = \min_k \{c_*(s,k)\}.$$

These definitions of codimension should be used in conditions (b) and (c) of remark 4.3.

Remark 4.4. Let X be the vector space of all smooth maps to $\mathbb{R}^2$ from a smooth $(n+2)$-manifold L, closed for simplicity. In $X \times L \times \cdots \times L$, form the subset of all $(f, x_1, \ldots, x_s)$ such that $x_1, \ldots, x_s$ are distinct and $S = \{x_1, \ldots, x_s\}$ is a bad event of complexity k (and size s) for the germ of f at S. Multijet transversality theorems, together with our assumptions on finite determinacy of admissible multigerms, imply that this subset has a well defined minimum codimension which turns out to be

$$c(s,k) := c_*(s,k) + 2(s-1) \;.$$

It is therefore tempting to think, but not obviously meaningful, that the subset of X consisting of all non-admissible f which have some bad event of size s and complexity k has codimension at least

$$C(s,k) := c(s,k) - s(n+2) = c_*(s,k) - sn - 2 \ .$$

We justify this idea in [5], following Vassiliev. Now condition (c) of remark 4.3 implies

$$\lim_{s\to\infty}(C(s,k)-k) \ = \ \infty$$

and the inequality in condition (b) implies $C(s) \geq 2$. These are the properties that we are after.

We now propose what Γ_r should be; later we show that the proposed Γ_r taken together satisfy the conditions in remark 4.3 and that they are characterized by those conditions.

Definition 4.5. A multijet

$$(f_x)_{x\in S_r} \in \prod_{x\in S_r} P_*$$

is admissible, hence represents one of the classes in Γ_r , if and only if

- each f_x which is singular (not regular) belongs to one of the types *fold, cusp, swallowtail, lips, beak-to-beak*;
- if f_x is of type *cusp, swallowtail, lips* or *beak-to-beak* for some $x \in S_r$, then f_y is regular or of type *fold* for all $y \in S_r \smallsetminus \{x\}$;
- *either* for all singular f_x, the images of their linear parts are distinct elements of $\mathbb{R}P^1$;
- *or* all singular f_x are of fold type, and for precisely two of them the images of their linear parts agree; in that case the two fold curves in the target make an ordinary (first order) tangency at the origin.

Consequently an element γ of Γ_r can be described by the following data:

(i) a subset T of S_r ;
(ii) a map $\zeta\colon T \to \{$*fold, cusp, swallowtail, lips, beak-to-beak*$\}$;
(iii) a cyclic ordering on T ;
(iv) a map $\iota\colon T \to \{0,1,\dots,n+1\}$;
(v) an element $j \in \{0,1\}$ and a choice of j two-element subsets $\{u,v\} \subset T$, where u and v are adjacent in the cyclic ordering of (iii).

There are some conditions on the map ζ in (ii):

(ii$_a$) if $\zeta(x) \neq$ *fold* for some $x \in T$, then $\zeta(y) =$ *fold* for all $y \in T \smallsetminus \{x\}$;
(ii$_b$) if $j = 1$ in (v), then $\zeta(x) =$ *fold* for all $x \in T$.

There is a condition on the map ι in (iv):

(iv$_a$) $\iota(x) \leq n$ for all $x \in T$ such that $\zeta(x) \neq$ *fold.*

This is to be understood as follows. If $\gamma \in \Gamma_r$ is represented by $(f_x)_{x\in S_r}$, then the subset T in (i) consists of the $x \in S_r$ for which f_x is singular, the map ζ in (ii) gives the type of each singular jet f_x , and the cyclic ordering in (iii) tells us something about the images of the linear parts of the singular f_x. In the case where f_u , f_v are of fold type and their fold curves in the target make a (first order) tangent at the origin, that is encoded in (v). The map ι in (iv) adds quadratic form information for each singular f_x , as in (3.1)-(3.5), to pin down the left-right equivalence class of f_x. More precisely, $\iota(x)$ is the dimension of the negative definite subspace for the appropriate quadratic form. We do claim that the data (i)-(v) determine a left-right equivalence class $\gamma \in \Gamma_r$, but we do not claim that they are uniquely determined by γ.

Example 4.6. Let $f = (f_x)_{x\in S_r}$ be a multijet and let $T \subset S_r$ be a minimal bad event for f. If $T = \{x\}$ has cardinality 1 then

(i) f_x is a jet which is not of type *fold, cusp, swallowtail, lips* or *beak-to-beak*.

If $T = \{x, y\}$ is of cardinality 2, then f_x and f_y are both of type *fold, cusp, swallowtail, lips* or *beak-to-beak*, and one of the following applies:

(ii) neither f_x nor f_y are of *fold* type;
(iii) exactly one of the two is of *fold* type and the image of the linear part is the same for both;
(iv) both are of fold type and their fold lines make a higher tangency (double, triple etc.) in the target.

If $T = \{x, y, z\}$ has cardinality 3, then f_x, f_y and f_z are all of type *fold, cusp, swallowtail, lips* or *beak-to-beak*, and one of the following applies:

(v) exactly one of f_x, f_y, f_z is not of fold type, with image of differential ℓ, while the other two are folds and share the image ℓ' of their linear part, making an ordinary tangency in the target, $\ell' \neq \ell$;
(vi) f_x, f_y and f_z are all of *fold* type, the image of the linear part is the same for all, and any two make an ordinary tangency in the target.

This covers all cases. So a minimal bad event has cardinality at most 3. Each of the above six cases defines a semi-algebraic subset of multijet space (4.1). Another point of view is that it defines a semi-algebraic subset in

$$\prod_{x\in T} P_*$$

(which leads to the same codimension estimates). With the second point of view it is easy to show that the codimension is bounded below by $n + 4$ in case (i), by $2n + 4$ in cases (ii),(iii) and (iv), and by $3n + 5$ in cases (v) and (vi).

Definition 4.7. Let $S_\bullet := S_0 \subset S_1 \subset \cdots \subset S_k$ be a chain of finite nonempty sets (and proper inclusions). We define

$$Y(S_\bullet) \subset \prod_{s\in S_k} P_*$$

to consist of all elements h such that S_j is a bad event for h, for $0 \le j \le k$, and there is no bad event for h strictly between S_j and S_{j+1}, for $0 \le j \le k - 1$.

Lemma 4.8. *The codimension of the semialgebraic set $Y(S_\bullet)$ in $\prod_{s\in S_k} P_*$ is at least $|S_k|n + 2k + 4$ everywhere.*

Proof. We proceed by induction on k. The case where $k = 0$ has been dealt with in example 4.6. In the case $k > 0$, let $S'_\bullet$ be the truncated chain

$$S_0 \subset S_1 \subset \cdots \subset S_{k-1} .$$

Let $T = S_k \smallsetminus S_{k-1}$. There is a projection

$$\prod_{s\in S_k} P_* \longrightarrow \prod_{s\in S_{k-1}} P_* \tag{4.2}$$

which induces a projection

$$Y(S_\bullet) \to Y(S'_\bullet). \tag{4.3}$$

Fix some $h \in Y(S'_\bullet)$. The fiber F_h of (4.3) over h is a semialgebraic subset of the fiber E_h of (4.2) over h, where E_h is a vector space,

$$E_h \cong \prod_{s \in T} P_* \ .$$

Now it is enough to show that the codimension of F_h in E_h is at least $|T|n+2$. In the case where $|T| > 1$ this is instantly clear. Indeed the codimension of F_h in E_h is at least $|T|(n+1)$, because the germs g_x for $g \in F_h$ and $x \in T$ are all singular, and the singular subset of P_* has codimension $n+1$. Suppose then that T is a singleton, $T = \{x\}$. Write F_h as a union of three semialgebraic subsets, one containing the elements g for which T is a minimal bad event, the second one containing the elements g for which T participates in a minimal bad event of cardinality 2, and the last one containing the elements g for which T participates in a minimal bad event of cardinality 3. Looking at the three cases separately, we see that the germ g_x for $g \in F_h$ is either singular and not of *fold* type, or it is of *fold* type but the direction of the fold line in the target is prescribed by h up to finite choice. Hence the codimension of F_h in $E_h \cong P_*$ is at least $n+2 = |T|n+2$. □

Theorem 4.9. *The collections Γ_r of definition* 4.5 *together satisfy conditions* (a), (b), (c) *and* (d) *of remark* 4.3.

Proof. By inspection, condition (a) is satisfied. For condition (b), let

$$Z_r \subset \prod_{x \in S_r} P_*$$

consist of all the multijets $f = (f_x)$ such that all of S_r is a bad event for f. We need to show that the codimension of Z_r in $\prod_x P_*$ is at least $rn+4$. Let

$$Q_r \subset \prod_{x \in S_r} P_*$$

consist of all the $f = (f_x)$ such that f_x is singular for every $x \in S_r$. Then $Z_r \subset Q_r$ and the codimension of Q_r in $\prod_x P_*$ is $r(n+1)$. Therefore it is enough to show that the codimension of Z_r in Q_r is at least 1 when $r = 3$, at least 2 when $r = 2$ and at least 3 when $r = 1$. That is easily done by inspection.

Next we verify condition (c). We look for lower bounds for $c_*(s,k,r)$ since $c_*(s,k) = \min_r\{c_*(s,k,r)\}$. It is understood that $k \le s \le r$. If there are no non-admissible multijets of size s and complexity k in $\prod_{x \in S_r} P_*$, then

$$c_*(s,k,r) = r \cdot \dim(P_*) > s(n+2) = sn + 2s \ .$$

If there are such multijets, then $k \ge s/3$ because minimal bad events have cardinality ≤ 3. By lemma 4.8, we have

$$c_*(s,k,r) \ge sn + 2k + 4$$

so that $c_*(s,k,r) - sn - k \ge k + 4 > s/3$. Therefore

$$c_*(s,k) - sn - k > s/3$$

which establishes condition (c).

It remains to show that $(\Gamma_r)_{r \ge 1}$ has the minimality property of condition (d).

Suppose for a contradiction that there exist alternative selections Γ'_r for $r \geq 1$ which together fulfill conditions (a), (b) and (c) of remark 4.3, and so that Γ'_q fails to contain Γ_q for some q (which we take minimal). Then there is some left-right equivalence class $\gamma \in \Gamma_q$ such that $\gamma \notin \Gamma'_q$. Let

$$g = (g_x)_{x \in S_q} \in \prod_{x \in S_q} P_*$$

be a multijet representing γ. We are going to show that

- (κ) all of S_q is a minimal bad event, for g as a non-admissible multijet in the Γ' setting;
- (λ) there exists $x_0 \in S_q$ such that the jet g_{x_0} is either regular, or singular of *fold* type, and such that the image of its linear part is not shared with that of any singular g_x where $x \in S_q \smallsetminus \{x_0\}$.

Property (κ) follows from the minimality of q, since deletion of any coordinate of g results in a multijet which belongs to Γ_{q-1} and therefore also to Γ'_{q-1}. Property (λ) holds because elements of Γ_r which fail to satisfy (λ) exist only when $r \leq 3$, and these elements belong to Γ'_r also, due to condition (b) of remark 4.3. With the information from (κ) and (λ), it is now easy to show that the selections Γ'_r together fail to satisfy condition (c) of remark 4.3.
To see this, suppose first that g_{x_0} is regular. For $t > q$, form the set K_t of all multijets

$$f = (f_x)_{x \in S_t} \in \prod_{x \in S_t} P_*$$

such that $f_x = g_x$ for $x \leq q$ while f_x is regular for $x > q$. Its codimension is a constant independent of t. For $f \in K_t$, all of S_t is a bad event because it is the union of the minimal bad events S_q and $(S_q \smallsetminus \{x_0\}) \cup \{y\}$ for $y \in S_t \smallsetminus S_q$. The complexity is the same number k_t for all $f \in K_t$. Therefore, for $t > q$, the number $c_*(t, k_t)$ is bounded above by a constant independent of t and k. It follows that condition (c) is violated.
Suppose next that g_{x_0} is singular of *fold* type. Let $U \subset \mathbb{R}P^1$ be an open interval containing the element determined by the linear part of the jet g_{x_0} but not containing any of the elements determined by the images of the linear parts of g_x for singular g_x, where $x \in S_q \smallsetminus \{x_0\}$. For $t > q$, form the set K_t of all multijets

$$f = (f_x)_{x \in S_t} \in \prod_{x \in S_t} P_*$$

such that $f_x = g_x$ for $x \leq q$ while f_x for $x > q$ is singular of *fold* type, and such that the element of $\mathbb{R}P^1$ determined by its linear part belongs to $U \subset \mathbb{R}P^1$. The codimension of K_t is

$$\text{const.} + (t - q)(n + 1) \ .$$

For $f \in K_t$, all of S_t is a bad event because it is the union of the minimal bad events S_q and $(S_q \smallsetminus \{x_0\}) \cup \{y\}$ for $y \in S_t \smallsetminus S_q$. The complexity is the same number k_t for all $f \in K_t$. Therefore, for every $t > r$ the number $c_*(t, k_t)$ is bounded above by const. $+ (t - q)(n + 1)$, and so condition (c) is still violated. □

5. Appendix: Basic results from singularity theory

We rely mostly on the excellent book by Martinet [4] for definitions and theorems. Another very readable text is [2], but that is exclusively concerned with singularities of functions (target $\mathbb{R}$).

We take the definition of an *unfolding* of a smooth map germ $(\mathbb{R}^s,0)\to(\mathbb{R}^t,0)$ from [4, ch.XIII]. For the definition of an *isomorphism* between two unfoldings (of the same map germ, and with the same parameter space) we also rely on the same source. Note that [2] has a definition (in the case $t=1$) which is slightly more restrictive in some respects, but less restrictive in other respects because it allows for a change of the parameter space.

Following [4], we call an unfolding F (with parameter space $\mathbb{R}^p$) of a smooth map germ f *universal* if every other unfolding (with parameter space $\mathbb{R}^q$, say) of f is isomorphic to h^*F for some germ $h\colon(\mathbb{R}^q,0)\to(\mathbb{R}^p,0)$. For a universal F with minimal parameter space dimension p , Martinet uses the expression *minimal universal*, which we shorten to *miniversal*. (Bröcker uses instead *versal* for Martinet's universal, and *universal* for Martinet's minimal universal.)

Definition 5.1. Let $\mathcal{E}_{s,t}$ be the real vector space of all smooth map germs from $(\mathbb{R}^s,0)$ to $\mathbb{R}^t$. In the case $t=1$, we write $\mathcal{E}_s$ instead of $\mathcal{E}_{s,t}$. In the general case, $\mathcal{E}_{s,t}$ is a module over the ring $\mathcal{E}_s$ by $(u\cdot g)(x)=u(x)\cdot g(x)$ for $u\in\mathcal{E}_s$ and $g\in\mathcal{E}_{s,t}$.

Definition 5.2. The *tangent space* Tf of a germ $f\colon(\mathbb{R}^s,0)\to(\mathbb{R}^t,0)$ is the vector subspace

$$\{df\cdot X+Y\circ f\}\ \subset\ \mathcal{E}_{s,t} \tag{5.1}$$

where X and Y run through all the vector field germs defined near the origin on $\mathbb{R}^s$ and $\mathbb{R}^t$, respectively, and df is the total derivative of f. The tangent space is typically not an $\mathcal{E}_s$ submodule. But it is an $\mathcal{E}_t$ submodule of $\mathcal{E}_{s,t}$ for the action of $\mathcal{E}_t$ on $\mathcal{E}_{s,t}$ defined in terms of f by

$$(u\cdot g)(x)=u(f(x))\cdot g(x) \tag{5.2}$$

for $u\in\mathcal{E}_t$ and $g\in\mathcal{E}_{s,t}$.

Theorem 5.3. (Main theorem on unfoldings.) *An unfolding*

$$F\colon(\mathbb{R}^p\times\mathbb{R}^s,0)\to(\mathbb{R}^p\times\mathbb{R}^t,0)$$

of a germ $f\colon(\mathbb{R}^s,0)\to(\mathbb{R}^t,0)$ *is universal if and only if the differential at* 0 *of the adjoint* $F^{\mathrm{ad}}\colon(\mathbb{R}^p,0)\to\mathcal{E}_{s,t}$ *is transverse to* Tf. □

Remark. We have not specified a norm on $\mathcal{E}_{s,t}$. Nevertheless, F^{ad} has a well defined differential at 0, the linear map $dF^{\mathrm{ad}}(0)\colon\mathbb{R}^p\to\mathcal{E}_{s,t}$ defined by

$$v\ \mapsto\ \left(x\mapsto\lim_{t\to0}\frac{F(tv,x)-F(0,x)}{t}\right) \tag{5.3}$$

for $v\in\mathbb{R}^p$ and $x\in\mathbb{R}^s$, with x sufficiently close to 0. The transversality condition means that $\mathrm{im}(dF^{\mathrm{ad}}(0))+Tf=\mathcal{E}_{s,t}$.

Corollary 5.4. *Let* $f\colon(\mathbb{R}^s,0)\to(\mathbb{R}^t,0)$ *be a germ such that* Tf *has finite codimension in* $\mathcal{E}_{s,t}$. *Suppose that*

$$g^{(1)},\dots,g^{(p)}\in\mathcal{E}_{s,t}$$

generate $\mathcal{E}_{s,t}/Tf$ as a vector space. Then $F\colon (\mathbb{R}^p \times \mathbb{R}^s, 0) \to (\mathbb{R}^p \times \mathbb{R}^t, 0)$ defined by

$$(z, x) \mapsto f(x) + \sum_i z_i g^{(i)}(x) \tag{5.4}$$

is a universal unfolding of f. □

Lemma 5.5. *Let $F, G\colon (\mathbb{R}^p \times \mathbb{R}^s, 0) \to (\mathbb{R}^p \times \mathbb{R}^t, 0)$ be unfoldings of a germ $f\colon (\mathbb{R}^s, 0) \to (\mathbb{R}^t, 0)$. If F and G are isomorphic as unfoldings of f, then the linear map*

$$dF^{\mathrm{ad}}(0) - dG^{\mathrm{ad}}(0)\colon \mathbb{R}^p \longrightarrow \mathcal{E}_{s,t}$$

factors through $Tf \subset \mathcal{E}_{s,t}$. □

Remark. This means that the composition

$$\mathbb{R}^p \xrightarrow{dF^{\mathrm{ad}}(0)} \mathcal{E}_{s,t} \xrightarrow{\text{proj.}} \mathcal{E}_{s,t}/Tf \tag{5.5}$$

is an *isomorphism invariant* of the unfolding F (of a fixed germ f, and with fixed parameter space $\mathbb{R}^p$).

We conclude this section with a few calculations of tangent spaces of germs, in increasing order of difficulty. These are used in section 2.

Example 5.6. Let $f\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be the germ given by

$$f(x, y) = (x, y^2).$$

This is one of the germs shown to be stable by Whitney in his investigation of singularities of maps from the plane to the plane. Stable germs have trivial miniversal unfoldings; equivalently, $Tf = \mathcal{E}_{2,2}$. It is also easy to verify by direct calculation that $Tf = \mathcal{E}_{2,2}$.

Example 5.7. Let $f\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be the germ given by

$$f(x, y) = (x, y^3 - xy).$$

This is again one of Whitney's stable germs. Therefore $Tf = \mathcal{E}_{2,2}$ and the miniversal unfolding of f is trivial.
As an alternative, here is a direct proof of $Tf = \mathcal{E}_{2,2}$ using the Mather-Malgrange preparation theorem. We view $\mathcal{E}_{2,2} = \mathcal{E}_{s,t}$ as a module over $\mathcal{E}_t = \mathcal{E}_2$ as in definition 5.2. We have $\mathcal{M}_t\mathcal{E}_{s,t} = \{f_1 \cdot g + f_2 \cdot h \mid g, h \in \mathcal{E}_{s,t}\}$, where the multiplication dot means ordinary multiplication of vector-valued functions by scalar functions. Therefore $\mathcal{E}_{s,t}/\mathcal{M}_t\mathcal{E}_{s,t}$ has vector space dimension 6, and is spanned by the (cosets of) the six maps

$$(x, y) \mapsto \begin{cases} (1, 0) \\ (0, 1) \\ (y, 0) \\ (0, y) \\ (y^2, 0) \\ (0, y^2). \end{cases}$$

By the preparation theorem, these six maps generate $\mathcal{E}_{s,t}$ as an $\mathcal{E}_t$ module. A slightly tedious verification shows that they are all in the $\mathcal{E}_t$-submodule Tf of $\mathcal{E}_{s,t}$. Therefore $Tf = \mathcal{E}_{s,t}$.

Example 5.8. Let $f:(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ be the germ given by

$$f(x,y)=(x,y^3+x^2y).$$

Let $W\subset\mathcal{E}_{s,t}=\mathcal{E}_{2,2}$ be the linear subspace consisting of all $k=(k_1,k_2)$ such that the first derivative of $y\mapsto k_2(0,y)$ at $y=0$ vanishes. This is clearly an $\mathcal{E}_t$-submodule of $\mathcal{E}_{s,t}$, and it contains Tf. We want to show that $Tf=W$.
We have the standard description

$$Tf=Jf+\tau f=\mathcal{E}_s\{\,(1,2xy),(0,3y^2+x^2)\,\}+\mathcal{E}_t\{\,(1,0),(0,1)\,\},$$

where $\mathcal{E}_s\{...\}$ and $\mathcal{E}_t\{...\}$ denote the $\mathcal{E}_s$ and $\mathcal{E}_t$ submodules, respectively, generated by the elements listed between the brackets. A two-fold application of [4, XV.2.1] proves that

$$Tf+\mathcal{E}_t\{\,(0,y)\,\}=\mathcal{E}_{2,2} \tag{5.6}$$

where $(0,y)$ is short for the map $(x,y)\mapsto(0,y)$. In more detail, we know from theorem 5.3 that $F(x,y,u)=(x,y^3+x^2y+uy)$ defines a universal (not miniversal) unfolding, with two unfolding parameters x and u, of the germ $y\mapsto y^3$. By [4, XV.2.1] it follows that F is a stable germ. But F is also a one-parameter unfolding of the germ f. Then [4, XV.2.1] can be applied in the opposite direction, which leads to equation (5.6).
Hence it is enough to check that $\mathcal{M}_t\cdot(0,y)\subset Tf$. As Tf is an $\mathcal{E}_t$-module, that reduces to showing that

$$\begin{array}{rcl}(0,xy)&\in&Tf\\(0,y^4+x^2y^2)&\in&Tf\;.\end{array}$$

For the first of these, write $2(0,xy)=(1,2xy)-(1,0)$ where $(1,2xy)\in Jf$ and $(1,0)\in\tau f$. For the second, write

$$9(0,y^4+x^2y^2)=3y^2(0,3y^2+x^2)+2x^2(0,3y^2+x^2)-2x^4(0,1)$$

where $3y^2(0,3y^2+x^2)\in Jf$ and $2x^2(0,3y^2+x^2)\in Jf$ and $2x^4(0,1)\in\tau f$. □

Example 5.9. Let $f:(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ be the germ given by

$$f(x,y)=(x,y^3-x^2y).$$

Again we have $Tf=W$, where $W\subset\mathcal{E}_{s,s}=\mathcal{E}_{2,2}$ is the linear subspace consisting of all $k=(k_1,k_2)$ such that the second derivative of $y\mapsto k_2(0,y)$ at $y=0$ vanishes. The proof follows the lines of example 5.8.

Example 5.10. Let $f:(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ be the germ given by

$$f(x,y)=(x,y^4+xy).$$

We want to show that Tf has codimension 1 in $\mathcal{E}_{s,t}=\mathcal{E}_{2,2}$. We have the standard description

$$Tf=Jf+\tau f=\mathcal{E}_s\{\,(1,y),(0,4y^3+x)\,\}+\mathcal{E}_t\{\,(1,0),(0,1)\,\}.$$

A two-fold application of [4, XV.2.1] proves that $Tf+\mathcal{E}_t\{\,(0,y^2)\,\}=\mathcal{E}_{s,t}$. (Follow the reasoning of example 5.8.) Hence it is enough to check that

$$\mathcal{M}_t\cdot(0,y^2)\subset Tf\;.$$

As Tf is an $\mathcal{E}_t$-module, that reduces to checking that

$$\begin{array}{rcl} (0, xy^2) & \in & Tf \\ (0, y^6 + xy^3) & \in & Tf \ . \end{array}$$

For the first of these we write

$$3(0, xy^2) = 4xy(1, y) - 4(y^4 + xy, 0) + 4y^4(1, y) - y^2(0, 4y^3 + x).$$

For the second we write

$$16(0, y^6 + xy^3) = 3x(0, 4y^3 + x) + 4y^3(0, 4y^3 + x) - 3x^2(0, 1).$$

□

Example 5.11. Let $g\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be the germ given by

$$g(x, y) = (x, y^4 + p_x y^2 + xy)$$

where $x \mapsto p_x$ is a smooth function (germ) of x, with $p_0 = 0$. We shall see that the tangent space Tg has codimension 1 in $\mathcal{E}_{2,2} = \mathcal{E}_{s,t}$. More precisely, we are going to show that g is left-right equivalent to the germ f defined by $f(x, y) = (x, y^4 + xy)$, which we investigated in example 5.10. Since Tf has codimension 1 in $\mathcal{E}_{2,2}$, it follows that Tg has codimension 1 in $\mathcal{E}_{2,2}$.
For nonzero $a \in \mathbb{R}$ define $g^a\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ by

$$g^a(x, y) = (x, y^4 + a^{-2} p_{a^3 x} y^2 + xy).$$

Then $g^1 = g$. It is easy to see that g^a is left-right equivalent to g. Indeed, $g^a = \varphi g \psi$ where $\psi(x, y) = (a^3 x, ay)$ and $\varphi(x, y) = (a^{-3} x, a^{-4} y)$.
We also define $g^0\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ by $g^0(x, y) = (x, y^4 + xy) = f(x, y)$. With these abbreviations, the germ $G\colon (\mathbb{R} \times \mathbb{R}^2, 0) \longrightarrow (\mathbb{R} \times \mathbb{R}^2, 0)$ defined by

$$G(a, x, y) = (a, g^a(x, y))$$

is smooth. (To see this, write $p_x = x \cdot u_x$ where $x \mapsto u_x$ is a smooth function. This is possible by [4, I.5.1]. Then $g^a(x, y) = (x, y^4 + a \cdot u_{a^3 x} y^2 + xy)$, which is clearly smooth as a function of a, x and y.) We think of it as a 1-parameter unfolding with parameter $a \in \mathbb{R}$ of the germ $g^0 = f$. As g^0 is finitely determined, with Tg^0 of codimension 1 etc., we know that a miniversal unfolding of g^0 is given by $F\colon (\mathbb{R} \times \mathbb{R}^2, 0) \longrightarrow (\mathbb{R} \times \mathbb{R}^2, 0)$ where

$$F(b, x, y) = (b, x, y^4 + by^2 + xy) \ .$$

By the universal property, the unfolding G is isomorphic (as an unfolding of g^0) to the pullback of F under some map germ $\beta\colon (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ relating the parameter spaces. But β must be the zero germ. (Indeed, g^a for arbitrary fixed a has a serious singularity at 0 whereas $(x, y) \mapsto (x, y^4 + by^2 + xy)$ for nonzero b, and near the origin, has only Whitney's folds and cusps.) Hence all g^a for sufficiently small $a > 0$ are left-right equivalent to $g^0 = f$. But we already saw that g^a for $a \neq 0$ is left-right equivalent to $g^1 = g$. It follows that g is left-right equivalent to f. □

Example 5.12. Let $f\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a germ of the form

$$f(x, y) = (x, f_2(x, y)) \ .$$

Let $q\colon \mathbb{R}^n \to \mathbb{R}$ be a nondegenerate quadratic form (a polynomial function, homogeneous of degree 2). Define a new germ by

$$\begin{array}{rcl} f^\sharp\colon (\mathbb{R}^{2+n},0) & \to & (\mathbb{R}^2,0) \\ (x,y,z_1,\dots,z_n) & \mapsto & (x, f_2(x,y)+q(z_1,\dots,z_n)). \end{array}$$

Let $r\colon \mathcal{E}_{2+n,2} \to \mathcal{E}_{2,2}$ be the restriction map (restriction to the xy-plane). This is clearly onto. *We have*

$$Tf^\sharp = r^{-1}(Tf)\ . \tag{5.7}$$

To prove this, we note first that $r(Tf^\sharp) \subset Tf$ and also $Tf \subset r(Tf^\sharp)$, from the definitions. Then it only remains to show

$$\ker(r) \subset Tf^\sharp\ .$$

Indeed we shall see that $\ker(r)$ is contained in $Jf^\sharp$, the subspace of $Tf^\sharp$ consisting of all $df^\sharp \cdot X$ where X is a vector field germ on $(\mathbb{R}^{2+n},0)$. Suppose then that $k=(k_1,k_2)$ is in the kernel of r. Let $\ell = df^\sharp \cdot k_1 X$ where X is the vector field with constant value $(0,\dots,0,1,0)$. Then ℓ is in $Jf^\sharp \cap \ker(r)$ and $\ell_1 = k_1$. Therefore $k-\ell = (0, k_2-\ell_2)$ is in $\ker(r)$ and we only need to prove that it is in $Jf^\sharp$. The function $k_2-\ell_2$ vanishes on the xy-plane. Therefore, by [4, I.5.1], it can be written in the form

$$(z_1,\dots,z_n,x,y) \quad\mapsto\quad \sum_{i=1}^n z_i \cdot g_i(z_1,\dots,z_n,x,y).$$

This means that $k-\ell$ can be written in the form

$$(z_1,\dots,z_n,x,y) \quad\mapsto\quad \sum_{i=1}^n g_i(z_1,\dots,z_n,x,y)\cdot(0,z_i)\ .$$

The map $(z_1,\dots,z_n,x,y) \mapsto (0,z_i)$ is in $Jf^\sharp$, due to the fact that q is nondegenerate. Since $Jf^\sharp$ is an $\mathcal{E}_{2+n}$-submodule of $\mathcal{E}_{2+n,2}$, it follows that $k-\ell \in Jf^\sharp$. □

References

[1] F. Aicardi and T. Ohmoto, *First order local invariants of apparent contours*, Topology **45** (2006), 27–45.

[2] T. Bröcker, *Differentiable germs and catastrophes*, London Math. Soc. Lecture Note Ser. 17, Cambridge University Press, 1975.

[3] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Grad. Texts in Math., Springer, 1973.

[4] J. Martinet, *Singularities of smooth functions and maps*, London Math. Soc. Lecture Note Ser. 58, Cambridge University Press, 1982.

[5] R. Reis and M. Weiss, *Manifold calculus and the discriminant method (Smooth maps to the plane and Pontryagin classes, Part III)*, in preparation.

[6] V. Vassiliev, *Complements of discriminants of smooth maps: topology and applications*, Transl. of Math. Monographs, 98, Amer. Math. Soc., Providence 1992, revised ed. 1994

[7] ———, *Topology of spaces of functions without complex singularities*, Funktsional.Anal. i Prilozhen. **23** (1989), 24–36, 96; translation in Funct. Anal. Appl. **23** (1989), 277–286.

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