SMOOTH MAPS TO THE PLANE AND PONTRYAGIN CLASSES PART II: HOMOTOPY THEORY

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Abstract. It is known that in the integral cohomology of $BSO(2m)$, the square of the Euler class is the same as the Pontryagin class in degree $4m$. Given that the Pontryagin classes extend *rationally* to the cohomology of $BS\mathrm{TOP}(2m)$, it is reasonable to ask whether the same relation between the Euler class and the Pontryagin class in degree $4m$ is still valid in the rational cohomology of $BS\mathrm{TOP}(2m)$. In this paper we use smoothing theory and tools from homotopy theory to reformulate the hypothesis, and variants, in a differential topology setting and in a functor calculus setting.

1. Introduction

We are going to discuss several related types of hypotheses. Among them are the A type which is a group of statements about characteristic classes of bundles with structure group $\mathrm{TOP}(n)$ for some n, and the B type which is a group of statements about spaces of smooth regular (i.e., nonsingular) maps to $\mathbb{R}^2$. Our purpose here in Part II is to use smoothing theory and functor calculus to explore logical dependencies between these hypotheses.

Our hypothesis' labelling system has letters A,B and C for flavors, superscripts s, m and w (strong, medium and weak) as strength indicators, and occasionally superscripts e and o (even and odd) as parity indicators. Our aim in later parts, such as [19], will be to prove some of the B hypotheses, which appear to be the most accessible. We do not use results from Part I [18] here.

The A hypotheses relate $\mathrm{TOP}(n)$ to $\mathrm{O}(n)$ and to $\mathrm{G}(n)$, the grouplike topological monoid of homotopy self-equivalences of the sphere S^{n-1}. There is a diagram of inclusions

$$\mathrm{O}(n) \longrightarrow \mathrm{TOP}(n) \longrightarrow G(n) \tag{1.1}$$

leading to a similar diagram of classifying spaces.

Hypothesis $\mathbf{A}^w$. *If $4i > 2n$, then $p_i = 0$ in $H^{4i}(B\mathrm{TOP}(n);\mathbb{Q})$, where p_i is the rational Pontryagin class of Thom and Novikov* [23, 17].

Hypothesis $\mathbf{A}^m$. *In $H^{4m}(BS\mathrm{TOP}(2m);\mathbb{Q})$ the equation $e_{2m}^2 = p_m$ holds, where e_{2m} is the Euler class.*

Hypothesis $\mathbf{A}^s$. *The unique class in $H^{4m}(B\mathrm{G}(2m+1);\mathbb{Q})$ which extends the squared Euler class in $H^{4m}(BS\mathrm{G}(2m);\mathbb{Q})$ also extends the Pontryagin class in $H^{4m}(B\mathrm{TOP}(2m+1);\mathbb{Q})$.*

Date: December 29, 2010.

2000 *Mathematics Subject Classification.* Primary 57D20; Secondary 55F40, 57D10.

This project was generously supported by the Engineering and Physical Sciences Research Council (UK), Grant EP/E057128/1.

The analogues of these equations are well known to hold for $B\mathrm{O}(n)$, respectively $B\mathrm{SO}(2m)$, respectively $B\mathrm{O}(2m+1)$. They are nontrivial as stated. Indeed, although $B\mathrm{TOP}$ is rationally homotopy equivalent to $B\mathrm{O}$ by [23, 17, 12], it is not true that $B\mathrm{TOP}(n)$ is rationally equivalent to $B\mathrm{O}(n)$ for all or most finite n. The reasons for the known discrepancies are quite deep. They were given in a short paper by Farrell and Hsiang [5], building on smoothing theory, Waldhausen's work on h-cobordisms and algebraic K-theory, and on concordance stability theorems which were only supplied some years later by Igusa [10].

For a formulation of the B hypotheses, fix a positive integer n. We will think of the product $D^n \times D^2$ as a smooth manifold with corners. Let $\mathscr{R} = \mathscr{R}(n,2)$ be the space of regular (alias nonsingular) smooth maps from $D^n \times D^2$ to $\mathbb{R}^2$ which agree near the boundary with the projection to the second factor $D^2 \subset \mathbb{R}^2$. Each $f \in \mathscr{R}$ has a derivative ∇f which is a map from $D^n \times D^2$ to the space $V = V(n,2)$ of surjective linear maps $\mathbb{R}^{n+2} \to \mathbb{R}^2$. Moreover ∇f is constant near the boundary, with value the standard projection from $\mathbb{R}^{n+2}$ to $\mathbb{R}^2$ which we regard as the basepoint of V. So ∇f can be viewed as an element of the $(n+2)$-fold loop space $\Omega^{n+2}V$, and it turns out (remark 2.12) to belong to the base point component $\mathscr{V} = \Omega^{n+2}_0 V$. Therefore we have

$$\begin{array}{rccc} \nabla\colon & \mathscr{R} & \longrightarrow & \mathscr{V} \\ & f & \mapsto & \nabla f\ , \end{array} \tag{1.2}$$

a map of based spaces, which is an embedding. It is equivariant for the conjugation actions of $S^1 = \mathrm{SO}(2)$ on source and target. In somewhat more detail, S^1 acts on the source of (1.2) by $(a,f) \mapsto af(I_n \oplus a^{-1})$ for f in $\mathscr{R}$ and $a \in S^1$. There is a matching action of S^1 on the target of (1.2). We leave it to the reader to make that explicit; beware that S^1 also acts nontrivially on the factor D^2 of $D^n \times D^2$ which is concealed in the symbol Ω^{n+2} within the definition of $\mathscr{V}$.
It is not very hard to produce an *integral* nullhomotopy for (1.2), both for odd and even n, but we will not do so here. It is much harder to produce a nullhomotopy which takes the S^1-actions into account. This is what the B hypotheses are about. As a matter of language, we shall say that an S^1-map $f\colon X \to Y$ between based spaces with S^1-action is *weakly S^1-nullhomotopic* if the composite map

$$X_{\mathrm{cw}} \xrightarrow{\ g\ } X \xrightarrow{\ f\ } Y$$

is based S^1-nullhomotopic. Here X_{cw} is a based S^1-CW-space with free S^1-cells (away from the basepoint) and g is a based S^1-map which is also a weak homotopy equivalence.

Hypothesis $\mathbf{B}^{w|o}$. *For odd n, the S^1-map $\nabla\colon \mathscr{R} \longrightarrow \mathscr{V}$ admits a rational weak S^1-nullhomotopy.*

Hypothesis $\mathbf{B}^{m|e}$. *For even n, the S^1-map $\nabla\colon \mathscr{R} \longrightarrow \mathscr{V}$ admits a rational weak S^1-nullhomotopy.*

Remark 1.1. The space $\mathscr{V}$ is rationally an Eilenberg-MacLane space $K(\mathbb{Q}, n-1)$ when n is odd, and a $K(\mathbb{Q}, n-3)$ when n is even, $n > 2$. Consequently, the action of S^1 on the target of ∇ is trivial in a weak (rational) sense so that ∇ determines a cohomology class $z \in H^{n-1}(\mathscr{R}_{\mathrm{h}S^1}, \star_{\mathrm{h}S^1}; \mathbb{Q})$, respectively $z \in H^{n-1}(\mathscr{R}_{\mathrm{h}S^1}, \star_{\mathrm{h}S^1}; \mathbb{Q})$, where $\mathrm{h}S^1$ denotes a homotopy orbit construction (alias Borel construction). In this language the two hypotheses above state that $z = 0$.

Remark 1.2. It will be shown in remark 2.13 that the S^1-map $\nabla\colon \mathscr{R} \longrightarrow \mathscr{V}$ does not always admit an *integral* weak S^1-nullhomotopy.

There are more hypotheses in the body of the paper. They are hypotheses $\mathrm{B}^{s|o}$, $\mathrm{B}^{s|e}$, $\mathrm{B}^{m|o}$ in section 2 and hypotheses C^s and C^m in section 3. The C hypotheses sit between the A hypotheses and the B hypotheses, and are formulated in the language of functor calculus. Our main business is to show that all the s hypotheses are equivalent, and all the m hypotheses are equivalent.

2. Smoothing theory and further hypotheses

The main input from smoothing theory is the following general theorem due to Morlet [16]. See also the thorough exposition of Morlet's result in [12] and the earlier work [7], which was later to appear in print.

Theorem 2.1. *The space of smooth structures on a closed topological manifold M of dimension $m \neq 4$ is homotopy equivalent to the space of vector bundle structures on the topological tangent (micro-)bundle of M.*
For a compact topological m-manifold M with smooth boundary, $m \neq 4$, the space of smooth structures on M extending the given structure on ∂M is homotopy equivalent to the space of vector bundle structures on the topological tangent bundle of M extending the prescribed vector bundle structure over ∂M.

Remark 2.2. 1 There is a homotopy lifting principle for vector bundle structures on fibre bundles with fiber $\mathbb{R}^m$. Namely, if $E \to X \times [0,1]$ is such a fiber bundle and a vector bundle structure has been chosen on $E|_{X\times 0}$, then this vector bundle structure admits an extension to all of E. This has the following homotopy theoretic consequence. Given a map $c : X \to B\mathrm{TOP}(m)$, the space of vector bundle structures on the associated fiber bundle on X with fiber $\mathbb{R}^m$ is homotopy equivalent to the space of maps $\tilde{c}\colon X \times [0,1] \to B\mathrm{TOP}(m)$ which satisfy $\tilde{c}(x,0) = c(x)$ and map $X \times 1$ to $B\mathrm{O}(m) \subset B\mathrm{TOP}(m)$.

Example 2.3. Let $\mathscr{Y}_n$ be the space of smooth structures on D^n extending the standard smooth structure on S^{n-1}. Then

$$\mathscr{Y}_n \;\simeq\; \Omega^n(\mathrm{TOP}(n)/\mathrm{O}(n)) \;.$$

Furthermore, there is a homotopy fiber sequence

$$\mathscr{R} \longrightarrow \Omega^2\mathscr{Y}_n \longrightarrow \mathscr{Y}_{n+2}$$

where $\mathscr{R} = \mathscr{R}(n,2)$ as previously defined. It is obtained as follows: Any smooth regular map $f\colon D^n \times D^2 \to D^2$ satisfying our boundary conditions is a smooth fiber bundle by Ehresmann's theorem. The underlying bundle of topological manifolds is canonically trivial relative to the given trivialization over the boundary ∂D^2. (Its structure group, the group of topological automorphisms of D^n extending the identity on the boundary, is contractible by the Alexander trick.) Hence f determines a family, parametrized by D^2, of smooth structures on D^n extending the standard smooth structure on S^{n-1}. This family is of course trivialized over the boundary ∂D^2. The resulting "integrated" smooth structure on the total space of the bundle is equal to the standard structure on $D^n \times D^2$ by assumption. These observations lead to the stated homotopy fiber sequence, and we conclude

$$\mathscr{R} \;\simeq\; \Omega^{n+2}\mathrm{hofiber}[\mathrm{TOP}(n)/\mathrm{O}(n) \to \mathrm{TOP}(n+2)/\mathrm{O}(n+2)] \;. \tag{2.1}$$

Example 2.4. Let $\mathscr{H}_n$ be the space of smooth structures on $D^{n-1}\times[0,1]$ extending the standard structure on $(D^{n-1}\times 0)\cup(\partial D^{n-1}\times[0,1])$. Reasoning as in the previous example we have a homotoppy equivalence

$$\mathscr{H}_n \;\simeq\; \Omega^{n-1}\big(\mathrm{hofiber}\big[\mathrm{TOP}(n-1)/\mathrm{O}(n-1)\to \mathrm{TOP}(n)/\mathrm{O}(n)\big]\big)\;,$$

if $n\geq 6$. The space $\mathscr{H}_n$ can also be viewed as the space of smooth h-cobordisms on D^{n-1}, since the space of topological h-cobordisms on D^{n-1} is contractible by the Alexander trick.

Example 2.5. With the notation of the previous examples, there is a homotopy fiber sequence

$$\mathrm{Aut}_{\mathrm{diff}}(D^n)\longrightarrow \mathrm{Aut}_{\mathrm{top}}(D^n)\longrightarrow \mathscr{Y}_n$$

where $\mathrm{Aut}_{\mathrm{diff}}(D^n)$ is the space of diffeomorphisms $D^n\to D^n$ which extend the identity $S^{n-1}\to S^{n-1}$, and $\mathrm{Aut}_{\mathrm{top}}(D^n)$ is the topological analogue. This homotopy fiber sequence is obtained by considering the action of $\mathrm{Aut}_{\mathrm{top}}(D^n)$ on $\mathscr{Y}_n$, and the stabilizer subgroup of the base point in $\mathscr{Y}_n$. By the Alexander trick, $\mathrm{Aut}_{\mathrm{top}}(D^n)$ is contractible. Hence $\mathrm{Aut}_{\mathrm{diff}}(D^n)\;\simeq\;\Omega^{n+1}(\mathrm{TOP}(n)/\mathrm{O}(n))$.

We now state a strong version $\mathrm{B}^{s|o}$ of hypothesis $\mathrm{B}^{w|o}$, motivated in part by smoothing theory. For this we introduce the space

$$V_{\mathrm{G}}=\mathrm{hofiber}[B\mathrm{G}(n)\to B\mathrm{G}(n+2)],\quad V_{\mathrm{TOP}}=\mathrm{hofiber}[B\mathrm{TOP}(n)\to B\mathrm{TOP}(n+2)]$$

also known informally as $\mathrm{G}(n+2)/\mathrm{G}(n)$ and $\mathrm{TOP}(n+2)/\mathrm{TOP}(n)$, respectively. Also, we define the spaces $\mathscr{V}_{\mathrm{G}}=\Omega^{n+2}_0V_{\mathrm{G}}$ and $\mathscr{V}_{\mathrm{TOP}}=\Omega^{n+2}_0V_{\mathrm{TOP}}$. The composition

$$\mathscr{R}\xrightarrow{\;\nabla\;}\mathscr{V}\xrightarrow{\;\mathrm{inc.}\;}\mathscr{V}_{\mathrm{G}}$$

admits a canonical integral weak S^1-nullhomotopy. This is obvious from the smoothing theory model for $\mathscr{R}$ which we saw in equation (2.1). The model amounts to a homotopy fiber sequence of spaces with S^1-action

$$\mathscr{R}\xrightarrow{\;\nabla\;}\mathscr{V}\xrightarrow{\;\mathrm{inc.}\;}\mathscr{V}_{\mathrm{TOP}}\;.$$

Hypothesis $\mathbf{B}^{s|o}$. *For odd n, the S^1-map $\nabla\colon\mathscr{R}\longrightarrow\mathscr{V}$ admits a rational weak S^1-nullhomotopy which refines the canonical S^1-nullhomotopy of the composition*

$$\mathscr{R}\xrightarrow{\;\nabla\;}\mathscr{V}\longrightarrow\mathscr{V}_{\mathrm{G}}\;.$$

The space $S\mathrm{G}(\mathbb{R}^n)=S\mathrm{G}(n)$ can be investigated using the homotopy fiber sequence

$$\Omega^{n-1}_1S^{n-1}\longrightarrow S\mathrm{G}(n)\longrightarrow S^{n-1}\tag{2.2}$$

where the right-hand map is evaluation at the base point of S^{n-1}. For even $n>0$ we can deduce immediately

$$S\mathrm{G}(n)\simeq_{\mathbb{Q}}S^{n-1}\quad,\quad BS\mathrm{G}(n)\simeq_{\mathbb{Q}}K(\mathbb{Q},n).$$

For odd $n>1$ the connecting homomorphisms $\pi_{n-1}S^{n-1}\to\pi_{n-2}\Omega^{n-1}_1S^{n-1}$ in the long exact homotopy group sequence of (2.2) are rational isomorphisms, as can be seen by comparing (2.2) with the homotopy fiber sequence

$$S\mathrm{O}(n-1)\to S\mathrm{O}(n)\to S^{n-1}\;.$$

Therefore

$$S\mathrm{G}(n) \simeq_{\mathbb{Q}} K(\mathbb{Q}, 2n-3) \quad , \quad BS\mathrm{G}(n) \simeq_{\mathbb{Q}} K(\mathbb{Q}, 2n-2)\ .$$

For $B\mathrm{G}(n)$ with arbitrary $n \geq 2$ we obtain therefore

$$B\mathrm{G}(n) \simeq_{\mathbb{Q}} \begin{cases} K(\mathbb{Q}, n)_{\mathrm{h}\mathbb{Z}/2} & n \text{ even},\ n > 0 \\ K(\mathbb{Q}, 2n-2) & n \text{ odd},\ n > 1 \end{cases} \tag{2.3}$$

where $\mathbb{Z}/2$ acts by sign change on the $\mathbb{Q}$ in $K(\mathbb{Q}, n)$. (Here we use an extended notion of rational homotopy equivalence $\simeq_{\mathbb{Q}}$, allowing for maps between path-connected based spaces which induce an isomorphism of fundamental groups and rational isomorphisms of the higher homotopy groups.) Furthermore, it follows from (2.2) and (2.3) that for odd $n > 1$, the diagram

$$S\mathrm{G}(n-1) \xrightarrow{\text{inc.}} S\mathrm{G}(n) \xrightarrow{\text{eval. at } \star} S^{n-1} \tag{2.4}$$

is a rational homotopy fiber sequence. These calculations can be summarized as follows. In the case of even n, the twisted Euler class $H^n(B\mathrm{G}(n); \mathbb{Z}^t)$ (with local coefficients $\mathbb{Z}^t$ determined by the nontrivial action of $\pi_1 B\mathrm{G}(n) \cong \mathbb{Z}/2$ on $\mathbb{Z}$) detects the entire rational homotopy of $B\mathrm{G}(n)$. In the case of odd $n > 1$, there is a class in $H^{2n-2}(B\mathrm{G}(n); \mathbb{Q})$ which detects the entire rational homotopy of $B\mathrm{G}(n)$, and extends the squared Euler class $e^2 \in H^{2n-2}(BS\mathrm{G}(n-1); \mathbb{Q})$. (This class made an earlier appearance in hypothesis A^s.)

Remark 2.6. Let n be an odd integer. We have seen that $\mathscr{V}$ is rationally an Eilenberg-Mac Lane $K(\mathbb{Q}, n-1)$. We saw above that $S\mathrm{G}(n)$ and $S\mathrm{G}(n+2)$ are also rationally Eilenberg-Mac Lane spaces $K(\mathbb{Q}, 2n-3)$ and $K(\mathbb{Q}, 2n+1)$, respectively. It follows from this that

$$\pi_i(\mathscr{V}_G) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{if } i = n-1, n-4 \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

The inclusion $\mathscr{V} \to \mathscr{V}_G$ is an injection in rational homotopy and it follows from the long exact homotopy sequence that $\mathrm{hofiber}[\mathscr{V} \to \mathscr{V}_G]$ is rationally an Eilenberg-Mac Lane $K(\mathbb{Q}, n-5)$. We can therefore think of hypothesis $\mathrm{B}^{s|o}$ as stating the vanishing of a cohomology class $z \in H^{n-5}(\mathscr{R}_{\mathrm{h}S^1}, \star_{\mathrm{h}S^1}; \mathbb{Q})$.

To state a medium variant of hypothesis $\mathrm{B}^{w|o}$ and a strong variant of $\mathrm{B}^{s|e}$ we need to introduce $\mathscr{R}^!$, the subspace of $\mathscr{R}$ consisting of those $f \in \mathscr{R}$ which satisfy

$$f(x_1, \ldots, x_n, z_1, z_2) = (f_1(x_1, \ldots, x_n, z_1, z_2), z_2)$$

for all $(x_1, \ldots, x_n, z_1, z_2) \in D^n \times D^2$. There is an analogue $V^! \subset V$ consisting of the linear surjections $\mathbb{R}^{n+2} \to \mathbb{R}^2$ which satisfy the same condition, and a corresponding subspace $\mathscr{V}^! \subset \mathscr{V}$. There is a commutative diagram

$$\begin{array}{ccc} \mathscr{R}^! & \xrightarrow{\nabla^!} & \mathscr{V}^! \\ \downarrow & & \downarrow \\ \mathscr{R} & \xrightarrow{\nabla} & \mathscr{V} \end{array}$$

where the vertical arrows are inclusion maps. If n is odd, the term $\mathscr{V}^!$ is rationally contractible as $V^! \simeq S^n$. Therefore ∇ has a preferred rational nullhomotopy on $\mathscr{R}^!$. Suppose that n is even. Then by smoothing theory the composition

$$\mathscr{R}^! \longrightarrow \mathscr{V}^! \longrightarrow \mathscr{V}^!_G$$

has a preferred nullhomotopy, where $\mathscr{V}^!_G = \Omega^{n+2}_0 \mathrm{hofiber}[B\mathrm{G}(n) \to B\mathrm{G}(n+1)]$. The map from $\mathscr{V}^!$ to $\mathscr{V}^!_G$ is a rational homotopy equivalence. Therefore again ∇ has a preferred rational nullhomotopy when restricted to $\mathscr{R}^!$.

Hypothesis $\mathrm{B}^{m|o}$. *For odd n, the S^1-map $\nabla : \mathscr{R} \longrightarrow \mathscr{V}$ admits a rational weak S^1-nullhomotopy which (as an ordinary nullhomotopy) extends ??? the preferred nullhomotopy of ∇ on $\mathscr{R}^!$.*

Hypothesis $\mathrm{B}^{s|e}$. *For even n, the S^1-map $\nabla : \mathscr{R} \longrightarrow \mathscr{V}$ admits a rational weak S^1-nullhomotopy which (as an ordinary nullhomotopy) extends the preferred nullhomotopy of ∇ on $\mathscr{R}^!$.*

In the remainder of this section we explain how the various A hypotheses are related with each other and with some of the B hypotheses.

Proposition 2.7. *Hypothesis A^s implies hypothesis A^m.*

Proof. We reformulate hypothesis A^m as follows: $p_m \in H^{4m}(BS\mathrm{TOP}(2m); \mathbb{Q})$ is the *restriction* of the squared Euler class in $H^{4m}(BS\mathrm{G}(2m); \mathbb{Q})$. From this reformulation, it is clear that hypothesis A^s implies hypothesis A^m. □

Proposition 2.8. *Hypothesis A^m implies hypothesis A^w.*

Proof. Assuming hypothesis A^m holds, we have $p_i = e^2_{2i} \in H^{4i}(BS\mathrm{TOP}(2i); \mathbb{Q})$. Hence p_i is zero in $H^{4i}(BS\mathrm{TOP}(n); \mathbb{Q})$ when $n < 2i$. The restriction homomorphism

$$H^{4i}(B\mathrm{TOP}(n); \mathbb{Q}) \longrightarrow H^{4i}(BS\mathrm{TOP}(n); \mathbb{Q})$$

is injective, as $BS\mathrm{TOP}(n)$ is homotopy equivalent to a double cover of $B\mathrm{TOP}(n)$. Therefore p_i is zero in $H^{4i}(B\mathrm{TOP}(n); \mathbb{Q})$ for $n < 2i$. □

Proposition 2.9. *Hypothesis A^w implies hypothesis $\mathrm{B}^{w|o}$.*

Proof. Hypothesis A^w implies that for odd n, the inclusion of pairs

$$(B\mathrm{O}, B\mathrm{O}(n)) \longrightarrow (B\mathrm{TOP}, B\mathrm{TOP}(n))$$

has a rational left homotopy inverse. We justify that using obstruction theory: to make such a left homotopy inverse, we only need to find a factorization up to homotopy as in the diagram

$$\begin{array}{ccc} & & B\mathrm{O}(n) \\ & \nearrow & \downarrow \\ B\mathrm{TOP}(n) & \longrightarrow & B\mathrm{TOP} \simeq_{\mathbb{Q}} B\mathrm{O} \,. \end{array}$$

To find the factorization we use the Postnikov tower of $B\mathrm{O}(n) \to B\mathrm{O}$. The homotopy groups of the homotopy fiber $\mathrm{O}/\mathrm{O}(n)$ are rationally nontrivial (of rank 1) only in dimensions $4i-1$ when $4i > 2n$, and the corresponding Postnikov invariants are (images of) the Pontryagin classes $p_i \in H^{4i}(B\mathrm{O})$ for $4i > 2n$. As these p_i map to zero in $H^{4i}(B\mathrm{TOP}(n))$ by hypothesis A^w, the factorization exists.
It follows that for odd n, the inclusion

$$\mathrm{O}/\mathrm{O}(n) \to \mathrm{TOP}/\mathrm{TOP}(n)$$

has a rational homotopy left inverse. From that we want to deduce that the inclusion

$$\mathrm{O}(n+2)/\mathrm{O}(n) \to \mathrm{TOP}(n+2)/\mathrm{TOP}(n)$$

has a rational homotopy left inverse respecting S^1-actions in the weak sense. (Then smoothing theory does the rest.) To begin with we observe that the commutative diagram

$$\begin{array}{ccc} \mathrm{O}(n+2)/\mathrm{O}(n) & \xrightarrow{a} & \mathrm{O}/\mathrm{O}(n) \\ \downarrow & & \downarrow b \\ \mathrm{TOP}(n+2)/\mathrm{TOP}(n) & \xrightarrow{c} & \mathrm{TOP}/\mathrm{TOP}(n) \end{array}$$

can be viewed as a diagram of S^1-spaces and S^1-maps. Moreover the S^1 actions in the right-hand column are trivial in the weak sense. For example we approximate O by $\mathrm{O}(j)$ for some $j \gg 0$ and write $\mathbb{R}^j = \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{j-n-2}$ and let S^1 act by conjugation via its standard action on the $\mathbb{R}^2$ factor. It is clear that this action of S^1 is trivial on $\mathrm{O}(j-2)/\mathrm{O}(n)$ where we think of $\mathrm{O}(j-2)$ as the orthogonal group of $\mathbb{R}^n \times 0 \times \mathbb{R}^{j-n-2}$. Letting j tend to infinity we see that the action of S^1 on $\mathrm{O}/\mathrm{O}(n)$ which we have used is trivial in the weak sense. At the same time it was constructed to ensure that the upper horizontal map in the diagram is an S^1-map. Similar reasoning takes care of $\mathrm{TOP}/\mathrm{TOP}(n)$ and the lower horizontal map.
Since the inclusion $\mathrm{O}(n+2)/\mathrm{O}(n) \longrightarrow \mathrm{O}/\mathrm{O}(n)$ admits a rational left homotopy inverse and the S^1-action on $\mathrm{O}(n+2)/\mathrm{O}(n)$ is weakly (rationally) trivial, we now have left homotopy inverses (respecting S^1-actions in the weak sense) for arrows a and b in the diagram. Composing these left inverses with c, we have the required map $\mathrm{TOP}(n+2)/\mathrm{TOP}(n) \to \mathrm{O}(n+2)/\mathrm{O}(n)$. □

Proposition 2.10. *Hypothesis* A^m *implies hypothesis* $\mathrm{B}^{m|e}$ *.*

Proof. Hypothesis A^m implies that for even n, the inclusion of pairs

$$(BS\mathrm{O}, BS\mathrm{O}(n)) \longrightarrow (BS\mathrm{TOP}, BS\mathrm{TOP}(n))$$

has a rational left homotopy inverse. Again we justify this by obstruction theory. To make such a left homotopy inverse, we only need to find a factorization up to homotopy as in the diagram

(2.6)
$$\begin{array}{ccc} & & BS\mathrm{O}(n) \\ & \nearrow & \downarrow \\ BS\mathrm{TOP}(n) & \longrightarrow & BS\mathrm{TOP} \simeq_{\mathbb{Q}} BS\mathrm{O} \ . \end{array}$$

To find the factorization we use the Postnikov tower of $BS\mathrm{O}(n) \to BS\mathrm{O}$. The homotopy groups of the homotopy fiber $\mathrm{O}/\mathrm{O}(n)$ are rationally nontrivial (of rank 1) only in dimensions $4i-1$ when $4i > 2n$, and in dimension n. The Postnikov invariant corresponding to $\pi_{2n}(\mathrm{O}/\mathrm{O}(n))$ lives in $H^{2n+1}(BS\mathrm{O}) = 0$, and so the corresponding stage of the Postnikov tower is rationally $BS\mathrm{O} \times K(\mathbb{Q}, n)$. The next Postnikov invariant therefore lives in $H^{2n}(BS\mathrm{O} \times K(\mathbb{Q}, n))$ and is equal to the difference between $p_{n/2} \in H^{2n}(BS\mathrm{O})$ and the square of the fundamental class of $K(\mathbb{Q}, n)$, both pulled back to the product. We know this because the relation $p_{n/2} = e^2$ must hold in the next stage of the Postnikov tower. The higher Postnikov invariants are simply pullbacks of the higher Pontryagin classes p_i where $4i > 2n$. From the analysis of the Postnikov tower of $BS\mathrm{O}(n) \to BS\mathrm{O}$ and hypothesis A^m it is clear that the required factorization in diagram 2.6 exists. From here we can proceed as in the proof of lemma 2.9. □

Proposition 2.11. *Hypothesis A^s implies hypothesis $B^{s|o}$.*

Proof. In this proof, we write $\Phi\mathrm{TOP}(n)$ and $\Phi\mathrm{O}(n)$ for the homotopy fibers of $\mathrm{TOP}(n) \to \mathrm{G}(n)$ and $\mathrm{O}(n) \to \mathrm{G}(n)$, respectively. Hypothesis A^s implies that for odd n, the inclusion of pairs

$$(B\Phi\mathrm{O}, B\Phi\mathrm{O}(n)) \longrightarrow (B\Phi\mathrm{TOP}, B\Phi\mathrm{TOP}(n))$$

has a rational left homotopy inverse. We justify that using obstruction theory: to make such a left homotopy inverse, we only need to find a factorization up to homotopy as in the diagram

$$\begin{array}{ccc} & & B\Phi\mathrm{O}(n) \\ & \nearrow & \downarrow \\ B\Phi\mathrm{TOP}(n) & \longrightarrow & B\Phi\mathrm{TOP} \simeq_{\mathbb{Q}} B\Phi\mathrm{O} \ . \end{array}$$

To find the factorization we use the Postnikov tower of $B\Phi\mathrm{O}(n) \to B\Phi\mathrm{O}$. The homotopy groups of

$$\mathrm{hofiber}[B\Phi\mathrm{O}(n) \to B\Phi\mathrm{O}] = \Phi\mathrm{O}/\Phi\mathrm{O}(n)$$

are rationally nontrivial (of rank 1) only in dimensions $4i-1$ when $4i > 2n-4$, and the corresponding Postnikov invariants are (images of) the Pontryagin classes $p_i \in H^{4i}(B\Phi\mathrm{O})$ for $4i > 2n-4$. As these p_i map to zero in $H^{4i}(B\Phi\mathrm{TOP}(n))$ by hypothesis A^s , the factorization exists.
It follows that for odd n, the inclusion of $\Phi\mathrm{O}/\Phi\mathrm{O}(n)$ in $\Phi\mathrm{TOP}/\Phi\mathrm{TOP}(n)$ has a rational homotopy left inverse. From here we can proceed as in the proof of lemma 2.9 to deduce that the inclusion

$$\Phi\mathrm{O}(n+2)/\Phi\mathrm{O}(n) \to \Phi\mathrm{TOP}(n+2)/\Phi\mathrm{TOP}(n)$$

has a rational homotopy left inverse respecting S^1-actions in the weak sense. The homotopy fiber of this inclusion is of course again an $(n+2)$-fold delooping of $\mathscr{R}$, while $\Omega^{n+2}(\Phi\mathrm{O}(n+2)/\Phi\mathrm{O}(n))$ is $\mathrm{hofiber}[\mathscr{V} \to \mathscr{V}_{\mathrm{G}}]$. Therefore we may conclude that the forgetful map $\mathscr{R} \longrightarrow \mathrm{hofiber}[\mathscr{V} \to \mathscr{V}_{\mathrm{G}}]$ admits a rational nullhomotopy with S^1-equivariance in the weak sense. □

Remark 2.12. To show that $\nabla\colon \mathscr{R} \longrightarrow \Omega^{n+2}V$ lands in the base point component of $\Omega^{n+2}V$, we introduce certain approximations X and X' to $\mathscr{R}$ and $\Omega^{n+2}V$, respectively. Let

$$E \to \mathbb{R}P^1 \ , \qquad E' \to \mathbb{R}P^1$$

be the fiber bundles such that E_ℓ for $\ell \in \mathbb{R}P^1$ is the space of regular maps from $D^n \times D^2$ to $\mathbb{R}^2/\ell$ which extend the canonical projection on the boundary, and $E'_\ell = \Omega^{n+2}(\mathrm{O}(n+2)/\mathrm{O}(\mathbb{R}^n \oplus \ell))$. Let X and X' be the corresponding section spaces. Then there is a commutative diagram

$$\begin{array}{ccc} \mathscr{R} & \longrightarrow & \Omega^{n+2}V \\ \downarrow & & \downarrow \\ X & \longrightarrow & X' \end{array} \tag{2.7}$$

obtained by taking $f \in \mathscr{R}$ to the compositions $p_\ell f$ where $p_\ell\colon \mathbb{R}^2 \to \mathbb{R}^2/\ell$ is the projection, and $g \in V$ to $p_\ell g$. The right-hand vertical arrow induces a bijection in

π_0 when $n \geq 2$. (This is based on the commutative diagram

$$\begin{array}{ccccc} \Omega^{n+2}S^n & \longrightarrow & \Omega^{n+2}V & \longrightarrow & \Omega^{n+2}S^{n+1} \\ \downarrow{\scriptstyle \text{stab.}} & & \downarrow & & \downarrow{\scriptstyle =} \\ \Omega^{n+3}S^{n+1} & \longrightarrow & X' & \longrightarrow & \Omega^{n+2}S^{n+1} \end{array}$$

where the rows are fibrations. It is known that the left-hand map induces an isomorphism in π_0 for $n \geq 2$; see [22, ch 9, ex D6].) Therefore, in the cases $n \geq 2$, it is more than enough to show that the lower horizontal map in diagram 2.7 is nullhomotopic. In fact the map $E \to E'$ is fiberwise nullhomotopic over $\mathbb{R}P^1$. This follows easily from the smoothing theory interpretation of the fibers E_ℓ. Namely,

$$E_\ell \;\simeq\; \Omega^{n+2}\text{hofiber}\left[\frac{\mathrm{O}(n+2)}{\mathrm{O}(\mathbb{R}^n \oplus \ell)} \longrightarrow \frac{\mathrm{TOP}(n+2)}{\mathrm{TOP}(\mathbb{R}^n \oplus \ell)}\right]$$

and the map to $E'_\ell = \Omega^{n+2}(\mathrm{O}(n+2)/\mathrm{O}(\mathbb{R}^n \oplus \ell))$ is the obvious forgetful map.
On the other hand, if $n \leq 1$, the space $\mathscr{R}$ is homotopy equivalent to the space of smooth automorphisms of $D^n \times D^2$ extending the identity on the boundary. The latter space is connected (well known in case $n = 0$, and a consequence of the affirmed Smale conjecture in case $n = 1$).

Remark 2.13. We finish this section by showing, as promised in remark 1.2, that the S^1-map $\nabla\colon \mathscr{R} \longrightarrow \mathscr{V}$ does not admit an *integral* weak S^1-nullhomotopy. We take $n = 19$ and, for a contradiction, make the assumption (i) that ∇ admits a weak S^1-nullhomotopy localized at the prime 691. In the remainder of this remark, we always localize away from 2. There is a homotopy fiber sequence (see example 2.3)

$$\Omega^{22}\big(\mathrm{O}(21)/\mathrm{O}(19)\big) \xrightarrow{\;g\;} \Omega^{22}\big(\mathrm{TOP}(21)/\mathrm{TOP}(19)\big) \longrightarrow \mathscr{R}$$

of based S^1-spaces, classified by $\nabla\colon \mathscr{R} \to \mathscr{V}$. So if (i) holds, then (ii) the map g admits a weak homotopy left inverse as a based S^1-map, localized at 691.
We now recall the following construction of the looped Pontryagin classes on O(21). Let B be the Grassmannian of oriented codimension 2 linear subspaces of $\mathbb{R}^{21}$. This is the same as the orbit space of the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^{21}$ by a free action of S^1. Since the Stiefel manifold is homotopy equivalent (away from the prime 2) to S^{39}, it follows that B has the same cohomology as $\mathbb{C}P^{19}$. There is a fiber bundle $E \longrightarrow B$ whose fiber E_H over $H \in B$ (an oriented codimension 2 linear subspace of $\mathbb{R}^{21}$) is the coset space $\mathrm{O}(21)/\mathrm{O}(H)$, which we consider as a based space. An element $x \in \mathrm{O}(21)$ determines a section σ_x of $E \to B$ such that $\sigma_x(H)$ is the coset $x \cdot \mathrm{O}(H) \in \mathrm{O}(21)/\mathrm{O}(H)$. By adjunction this gives a based map

$$\mathrm{O}(21) \longrightarrow \Gamma(E \to B) \tag{2.8}$$

where $\Gamma(E \to B)$ is the section space of the fiber bundle. Now let $E^? \to B$ be the fibration obtained from $E \to B$ by fiberwise killing of homotopy groups in dimensions > 39. The fibers of $E^? \to B$ are Eilenberg-MacLane spaces $K(\mathbb{Z}[\frac{1}{2}], 39)$, each with a preferred base point, so that the section space can be identified with the space of maps from B to $K(\mathbb{Z}[\frac{1}{2}], 39)$. In making this identification we have a choice, and we make it so that the fundamental class of the manifold $\mathrm{O}(21)/\mathrm{O}(H)$,

for $H \in B$, corresponds to the number $2 \in \mathbb{Z}[\frac{1}{2}] = H_{39}(K(\mathbb{Z}[\frac{1}{2}], 39))$. Then, composing (2.8) with the inclusion

$$\Gamma(E \to B) \to \Gamma(E^? \to B) \ ,$$

we have constructed a map

$$\mathrm{O}(21) \longrightarrow \mathrm{map}(B, K(\mathbb{Z}, 39)) \ . \tag{2.9}$$

Since $H^*(B) \cong H^*(\mathbb{C}P^{19})$, the target has a canonical splitting (*explain*)

$$\mathrm{map}(B, K(\mathbb{Z}, 39)) \simeq \prod_{i=0}^{19} K(\mathbb{Z}, 39-2i)$$

so that we can write (2.9) in the form

$$\mathrm{O}(21) \longrightarrow \prod_{i=0}^{19} K(\mathbb{Z}[\tfrac{1}{2}], 39-2i) \quad .$$

We take it as known[1] that this map is the (looped, total, integral) Chern alias Pontryagin class away from the prime 2. We loop 22 times to obtain

$$\Omega^{22}\mathrm{O}(21) \longrightarrow \prod_{i=0}^{19} K(\mathbb{Z}[\tfrac{1}{2}], 17-2i) \tag{2.10}$$

which is still the 23 times looped total Pontryagin class. This map could also have been obtained more directly from a map

$$\Omega^{22}\mathrm{O}(21) \longrightarrow \Gamma(\Omega^{22}_B E \to B)$$

analogous to (2.8). Note that the fiber bundle $\Omega^{22}_B E \to B$ is associated to the canonical principal S^1-bundle on B via the (base-point preserving) conjugation action of S^1 on $\Omega^{22}(\mathrm{O}(21)/\mathrm{O}(19)) = \Omega\mathscr{V}$. Therefore, using consequence (ii) of assumption (i), we find that the map (2.10) admits a factorization through $\Omega^{22}\mathrm{TOP}(21)$ at the prime 691. It follows that the 23 times looped Pontryagin classes on $\Omega^{22}\mathrm{O}(21)$ extend to $\Omega^{22}\mathrm{TOP}(21)$, not just rationally, but at the prime 691. But it is well known that this is false. Specifically [15, §20] the generator of

$$\pi_1(\Omega^{22}\mathrm{O}(21)) \cong \pi_{24}B\mathrm{O}(21) \cong_{691} \mathbb{Z}$$

has scalar product 11! times a small power of 2 with the Pontryagin class p_6, but its image in $\pi_1(\Omega^{22}\mathrm{TOP}(21)) \cong \pi_{24}B\mathrm{TOP}(21)$ is known to be divisible by 691. (This is due to the existence of elements of order 691 in $\pi_{23}(\mathrm{TOP}/\mathrm{O})$, better known as the group of oriented smooth homotopy spheres of dimension 23 modulo oriented diffeomorphism.)

3. Orthogonal calculus

Let $\mathscr{J}$ be the category of finite dimensional real vector spaces with inner product. For objects V and W in $\mathscr{J}$, the morphism space $\mathrm{mor}(V, W)$ is the Stiefel manifold of linear maps $V \to W$ respecting the inner product. Let $\mathfrak{bo}$, $\mathfrak{bt}$ and $\mathfrak{bg}$ be the continuous functors on $\mathscr{J}$ given for V in $\mathscr{J}$ by $V \mapsto B\mathrm{O}(V)$, $V \mapsto B\mathrm{TOP}(V)$ and $V \mapsto B\mathrm{G}(V)$, respectively, where $\mathrm{G}(V)$ is the topological group-like monoid of homotopy equivalences $S(V) \to S(V)$. By orthogonal calculus [27], the functors $\mathfrak{bo}$, $\mathfrak{bt}$ and $\mathfrak{bg}$ determine spectra $\Theta\mathfrak{bo}^{(i)}$, $\Theta\mathfrak{bt}^{(i)}$ and $\Theta\mathfrak{bg}^{(i)}$ with an action of $\mathrm{O}(i)$,

[1] ... but we do not know of a good reference.

for any integer $i > 0$. These are the i-th derivatives at infinity of $\mathfrak{bo}$, $\mathfrak{bt}$ and $\mathfrak{bg}$, respectively. In the spirit of orthogonal calculus, we may assume that they are $\mathrm{O}(i)$-CW spectra and that only free $\mathrm{O}(i)$-cells are involved (in particular the actions are free away from base points). The inclusions

$$\begin{array}{ccc} \mathfrak{bo} & \longrightarrow & \mathfrak{bt} \\ & \searrow & \downarrow \\ & & \mathfrak{bg} \end{array}$$

determine a similar comutative triangle of maps of spectra

$$\begin{array}{ccc} \Theta\mathfrak{bo}^{(i)} & \longrightarrow & \Theta\mathfrak{bt}^{(i)} \\ & \searrow & \downarrow \\ & & \Theta\mathfrak{bg}^{(i)} \end{array} \tag{3.1}$$

which respect the actions of $\mathrm{O}(i)$. The Taylor tower of $\mathfrak{bo}$ consists of approximations $\mathfrak{bo} \to T_i\mathfrak{bo}$ for every $i \geq 0$, and homotopy fiber sequences

$$L_i\mathfrak{bo} \to T_i\mathfrak{bo} \to T_{i-1}\mathfrak{bo}$$

for every $i > 0$, where $L_i\mathfrak{bo}$ can be described as

$$L_i\mathfrak{bo}(V) \;\simeq\; \Omega^\infty\left(\left((V \otimes \mathbb{R}^i)^c \wedge \Theta\mathfrak{bo}^{(i)}\right)_{\mathrm{hO}(i)}\right)$$

(by a chain of natural homotopy equivalences). The functor $T_0\mathfrak{bo}$ is essentially constant, $T_0\mathfrak{bo}(V) \simeq B\mathrm{O}$ by a chain of natural homotopy equivalences. The natural transformation $\mathfrak{bo} \to T_i\mathfrak{bo}$ has a universal property, in the *initial* sense: it is the best approximation of $\mathfrak{bo}$ from the right by a polynomial functor of degree $\leq i$.
Similarly, $\mathfrak{bt}$ and $\mathfrak{bg}$ have a Taylor tower whose layers $L_i\mathfrak{bt}$ and $L_i\mathfrak{bg}$ for $i > 0$ are determined by the spectra $\Theta\mathfrak{bt}^{(i)}$ and $\Theta\mathfrak{bg}^{(i)}$, respectively. Also, $T_0\mathfrak{bt}$ is essentially constant with value $B\mathrm{TOP}$ and $T_0\mathfrak{bg}$ is essentially constant with value $B\mathrm{G}$. There is one aspect in which $\mathfrak{bt}$ differs substantially from $\mathfrak{bo}$ and $\mathfrak{bg}$: the Taylor towers of $\mathfrak{bo}$ and $\mathfrak{bg}$ are known [2] to converge to $\mathfrak{bo}$ and $\mathfrak{bg}$ respectively, that is,

$$\mathfrak{bo}(V) \simeq \operatorname*{holim}_i T_i\mathfrak{bo}(V)\;, \qquad \mathfrak{bg} \simeq \operatorname*{holim}_i T_i\mathfrak{bg}(V)\;.$$

It is not known whether this holds for $\mathfrak{bt}$, and Igusa's work on concordance stability [10] indicates that it will not be easy to decide.
This chapter analyses the Taylor towers of the three functors $\mathfrak{bo}$, $\mathfrak{bt}$ and $\mathfrak{bg}$ up to stage 2 at most, concentrating on rational aspects. This is done in part for illustration of methods and techniques, and we are quite aware that Arone [2] has already given an exhaustive, integral and very pretty description of the Taylor tower of $\mathfrak{bo}$.

We begin with the orthogonal calculus analysis of the functor $\mathfrak{c}\colon \mathscr{J} \to \mathscr{T}_*$ given by $V \mapsto V^c$, where $\mathscr{T}_*$ is the category of based spaces and V^c is the one-point compactification of the vector space V. Below, we will denote the symmetric group in n letters by Σ_n.

Proposition 3.1. *The functor $\mathfrak{c}$ is rationally polynomial of degree 2, except for a deviation at $V = 0$. The first and second derivative spectra are*

$$\Theta\mathfrak{c}^{(j)} \simeq (\mathrm{O}(j)/\Sigma_j)_+ \wedge \Omega^{j-1}\underline{S}^0\;,$$

for $j = 1, 2$, where the $\mathrm{O}(j)$-action is trivial on the sphere spectrum and is the usual action on $\mathrm{O}(j)/\Sigma_j$.

Proof. We mainly have to show that the natural map from $\mathfrak{c}(V)$ to $T_2\mathfrak{c}(V)$ is a rational homotopy equivalence when $V \neq 0$. The EHP tradition gives us, for every based connected CW-space X, a natural map

$$X \longrightarrow \mathrm{hofiber}[\Omega^\infty\Sigma^\infty X \xrightarrow{\mathrm{sq}} \Omega^\infty\Sigma^\infty((X \wedge X)_{\mathrm{rh}\mathbb{Z}/2})] \ . \tag{3.2}$$

The subscript $rh\mathbb{Z}/2$ is for a reduced homotopy orbit construction, that is

$$(X \wedge X) \wedge_{\mathbb{Z}/2} E\mathbb{Z}/2_+ \ .$$

The map sq is defined as follows (sketch). We use configuration space models for source and target of sq. In particular $\Omega^\infty\Sigma^\infty X$ is understood to be the classifying space of a topological category $\mathscr{C}_1$ whose objects are pairs (S, f) where S is a finite set and $f\colon S \to X$ is any map. (The f in (S, f) can vary continuously, which makes the object set into a space.) A morphism from (R, g) to (S, f) is an injective map $h\colon R \to S$ such that $fh = g$ and $f(k) = * \in X$ whenever $k \notin \mathrm{im}(h)$. We describe $\Omega^\infty\Sigma^\infty((X \wedge X)_{\mathrm{rh}\mathbb{Z}/2})$ as the classifying space of a topological category $\mathscr{C}_2$ whose objects are pairs (S, f) where S is a finite set with free action of $\mathbb{Z}/2$ and $f\colon S \to X \wedge X$ is a $\mathbb{Z}/2$-map. A morphism from (R, g) to (S, f) is an injective $\mathbb{Z}/2$-map $h\colon R \to S$ such that $fh = g$ and $f(k) = * \in X$ whenever $k \notin \mathrm{im}(h)$. Now sq is the map induced by the functor $\mathscr{C}_1 \to \mathscr{C}_2$ which takes an object (S, f), with $f\colon S \to X$, to the object $(S \times S \smallsetminus d(S), f^2)$ where $d(S) \subset S \times S$ is the diagonal, with

$$f^2(s, t) = (f(s), f(t)) \in X \wedge X \ .$$

With these models, we can identify the subspace X of $\Omega^\infty\Sigma^\infty X$ as the classifying space of the subcategory $\mathscr{C}_0 \subset \mathscr{C}_1$ spanned by the objects (S, f) where S is a singleton. Then it is clear that the functor $\mathscr{C}_1 \to \mathscr{C}_2$ just constructed is trivial on $\mathscr{C}_0$. Therefore we have constructed a map from $B\mathscr{C}_0$ to the homotopy fiber of $B\mathscr{C}_1 \to B\mathscr{C}_2$, as promised. (The method is due to Segal [20].) It is well known that the map (3.2) is a rational homotopy equivalence for simply connected X, and also when $X = S^1$.

Now we may specialize by taking $X = V^c = \mathfrak{c}(V)$ for nonzero V in $\mathscr{J}$. Then sq becomes a natural transformation

$$\Omega^\infty\Sigma^\infty V^c \longrightarrow \Omega^\infty\Sigma^\infty((V^c \wedge V^c)_{\mathrm{rh}\mathbb{Z}/2}) \tag{3.3}$$

defined for $V \neq 0$. It is clear that the source functor $\Omega^\infty\Sigma^\infty V^c$ is homogeneous of degree 1 and corresponds to the spectrum

$$\mathrm{O}(1)_+ \wedge \underline{S}^0 \ .$$

We will now show that the target functor is homogeneous of degree 2. This amounts to writing $\Sigma^\infty((V^c \wedge V^c)_{\mathrm{rh}\mathbb{Z}/2})$ in the form $((V \otimes \mathbb{R}^2)^c \wedge \Psi)_{\mathrm{hO}(2)}$ where Ψ is a spectrum with an action of $\mathrm{O}(2)$. This is easily done by taking

$$\Psi = \Big(\frac{\mathrm{O}(2)}{\mathbb{Z}/2}\Big)_+ \wedge \underline{S}^0 \ .$$

Finally it remains to extend sq so that it is defined also for $V = 0$. For that we have a formal argument which, unfortunately, does not tell us what sq looks like when $V = 0$. Namely, any polynomial functor $\mathfrak{p}$ of degree $\leq n$ from $\mathscr{J}$ to based spaces is determined (up to natural equivalences) by its restriction to the full subcategory

of $\mathscr{J}$ spanned by all objects of dimension $\geq k$, for a fixed k. This is shown by the formula

$$\mathfrak{p}(V) \xrightarrow{\simeq} \operatorname*{holim}_{0\neq U\leq\mathbb{R}^{n+1}} \mathfrak{p}(V\oplus U)\ . \tag{3.4}$$

The same principle applies to natural transformations between polynomial functors. In particular we have shown that (3.3) is defined for all V in $\mathscr{J}$, including $V=0$. Now we can say that the functor

$$V\mapsto \mathfrak{q}(V) = \text{hofiber}[\ \Omega^\infty\Sigma^\infty V^c \xrightarrow{\text{sq}} \Omega^\infty\Sigma^\infty((V^c\wedge V^c)_{\text{rh}\mathbb{Z}/2})\ ]$$

is polynomial of degree 2. The EHP transformation

$$\mathfrak{c}(V)\to \text{hofiber}[\ \Omega^\infty\Sigma^\infty V^c \xrightarrow{\text{sq}} \Omega^\infty\Sigma^\infty((V^c\wedge V^c)_{\text{rh}\mathbb{Z}/2})\ ] \tag{3.5}$$

is therefore defined for all V and factors through $T_2\mathfrak{c}$. As (3.5) is $(3d-c)$-connected where $d=\dim(V)$ and c is a fixed constant, it follows (using the explicit formula for $T_2\mathfrak{c}$) that the induced map from $T_2\mathfrak{c}(V)$ to $\mathfrak{q}(V)$ is a homotopy equivalence for all V. This gives us the formulae for $\Theta\mathfrak{c}^{(1)}$ and $\Theta\mathfrak{c}^{(2)}$. As noted before, (3.5) is a rational homotopy equivalence when $V\neq 0$.
Let us now look at the case $V=0$. We can use the above configuration space model to make a commutative square

$$\begin{array}{ccc} \coprod_{n\geq 0} B\Sigma_n & \longrightarrow & \coprod_{n\geq 0} B(\mathbb{Z}/2\wr\Sigma_n) \\ \downarrow & & \downarrow \\ \Omega^\infty\Sigma^\infty S^0 & \xrightarrow{\text{sq}} & \Omega^\infty\Sigma^\infty((S^0\wedge S^0)_{\text{rh}\mathbb{Z}/2})\ . \end{array}$$

Here the upper lefthand term is (homotopy equivalent to) the classifying space of the category of finite sets with their isomorphisms and the upper righthand term is the classifiying space of the category of finite sets with free $\mathbb{Z}/2$-action and their isomorphisms. The horizontal map on top is given as before by squaring and deleting the diagonal. The lower horizontal map is defined as the canonical extension of (3.3), via equation (3.4). Consequently, the map sq in the lower row has the following effect on elements $x\in\pi_0(\Omega^\infty\Sigma^\infty S^0)\cong\mathbb{Z}$ of nonnegative degree:

$$x\mapsto x^2-x\ .$$

At first sight this suggests that the homotopy fiber of sq in the lower row might have exactly two path components, like S^0. But that is not the case because π_1 of $\Omega^\infty\Sigma^\infty((S^0\wedge S^0)_{\text{rh}\mathbb{Z}/2})$ is $\mathbb{Z}/2\times\mathbb{Z}/2$, larger than π_1 of $\Omega^\infty\Sigma^\infty S^0$. Therefore the homotopy fiber of sq in the lower row has at least four components, and so is not rationally homotopy equivalent to S^0. □

We now generalize the functor $\mathfrak{c}$ of the previous proposition to include wedge sums of shifts of $\mathfrak{c}$. Let $J=(k_j)_{j=1,2,\dots}$ be a (finite or infinite) sequence of nonnegative integers. Then we define $\mathfrak{c}_J$ to be the functor

$$V\mapsto \bigvee_j \mathfrak{c}(V\oplus\mathbb{R}^{k_j})\ . \tag{3.6}$$

Proposition 3.2. *The functor $\mathfrak{c}_J$ is rationally polynomial of degree 2, except possibly for a deviation at $V = 0$. The first and second derivative spectra are as follows:*

$$\Theta\mathfrak{c}_J^{(1)} = \bigvee_j \mathrm{O}(1)_+ \wedge \underline{S}^{k_j} \ ,$$

$$\Theta\mathfrak{c}_J^{(2)} = \left(\bigvee_j \mathrm{O}(2)_+ \wedge_{\Sigma_2} \Omega\underline{S}^{2k_j} \right) \ \vee \ \left(\bigvee_{j<\ell} \mathrm{O}(2)_+ \wedge \Omega\underline{S}^{k_j+k_\ell} \right) \ ,$$

where $\mathrm{O}(i)$ acts by translation on the first smash factor (and Σ_2 acts on $\Omega\underline{S}^{2k_j} = \Omega(S^{k_j} \wedge S^{k_j} \wedge \underline{S}^0)$ by permuting the two copies of S^{k_j}).

Proof. The case where the sequence J is finite is very similar to proposition 3.1 and we leave this to the reader. The general case can be deduced from the case where the sequence J is finite by a direct limit argument. It is crucial to observe that the operator T_2 commutes with direct limits up to equivalence. □

Now we shall sketch the orthogonal calculus analysis of $\mathfrak{bo}$, concentrating on rational aspects where that saves energy. Much more detailed results can be found in [2].

Proposition 3.3. *The functor $\mathfrak{bo}$ is rationally polynomial of degree 2, in the sense that the canonical map $\mathfrak{bo}(V) \to T_2\mathfrak{bo}(V)$ is a rational homotopy equivalence, for every $V \neq 0$. The derivative spectra of $\mathfrak{bo}$ are as follows*

(i) $\Theta\mathfrak{bo}^{(1)} \simeq \underline{S}^0$ *with trivial action of* $\mathrm{O}(1)$*;*
(ii) $\Theta\mathfrak{bo}^{(2)} \simeq \Omega\underline{S}^0$ *with rationally trivial action of* $\mathrm{O}(2)$*.*

Proof. By definition the spectrum $\Theta\mathfrak{bo}^{(1)}$ is made up of the based spaces

$$\Theta\mathfrak{bo}^{(1)}(n) = \mathrm{hofiber}[\mathfrak{bo}(\mathbb{R}^n) \to \mathfrak{bo}(\mathbb{R}^{n+1})] \ \simeq \ S^n$$

and so turns out to be a sphere spectrum $\underline{S}^0$. The generator of $\mathrm{O}(1)$ acts on $\Theta\mathfrak{bo}^{(1)}(n)$ alias $S^n = \mathbb{R}^n \cup \{\infty\}$ via $-\mathrm{id}\colon \mathbb{R}^n \to \mathbb{R}^n$. The structure maps

$$S^1 \wedge \Theta\mathfrak{bo}^{(1)}(n) \to \Theta\mathfrak{bo}^{(1)}(n+1)$$

are $\mathrm{O}(1)$-maps, where we use the standard conjugation action on S^1 and the resulting diagonal action on $S^1 \wedge \Theta\mathfrak{bo}^{(1)}(n)$. Therefore, strictly speaking, the structure maps are in a twisted relationship to the actions of $\mathrm{O}(1)$ on the various $\Theta\mathfrak{bo}^{(1)}(n)$, but there are mechanical ways to untwist this (see also ??? below) and the result is a sphere spectrum with trivial action of $\mathrm{O}(1)$.
For the description of the second derivative spectrum we reduce this to proposition 3.1. We have a natural homotopy fiber sequence

$$\mathfrak{c}(V) \longrightarrow \mathfrak{bo}(V) \longrightarrow \mathfrak{bo}(V \oplus \mathbb{R})$$

inducing a corresponding homotopy fiber sequence of spectra

$$\Theta\mathfrak{c}^{(2)} \longrightarrow \Theta\mathfrak{bo}^{(2)} \wedge S^0 \longrightarrow \Theta\mathfrak{bo}^{(2)} \wedge S^2 \ ,$$

where we think of S^2 as $(\mathbb{R}^2)^c$ and the maps of this homotopy fiber sequence preserve the $\mathrm{O}(2)$ actions (in particular, the $\mathrm{O}(2)$-action on $\Theta\mathfrak{bo}^{(2)} \wedge S^2$ is the diagonal one). Consequently, we have that

$$\Theta\mathfrak{c}^{(2)} \simeq \Omega(\Theta\mathfrak{bo}^{(2)} \wedge (S^2/S^0)) \simeq \Theta\mathfrak{bo}^{(2)} \wedge S^1_+. \tag{3.7}$$

Taking homotopy orbit spectra for the action of $SO(2)$ and using our previous formula for $\Theta\mathfrak{c}^{(2)}$ we obtain that

$$\Omega\underline{S}^0 \simeq \Theta\mathfrak{bo}^{(2)} .$$

This equivalence does not fully keep track of $O(2)$ actions, but it does allow us to say that orientation reversing elements of $O(2)$ act by self-maps homotopic to the identity. Consequently the action of the orthogonal group on $\Theta\mathfrak{bo}^{(2)}$ is rationally trivial in a weak sense. To see that $\mathfrak{bo}$ is rationally polynomial of degree 2 we consider the commutative diagram

$$\begin{array}{ccccc} \mathfrak{c}(V) & \longrightarrow & \mathfrak{bo}(V) & \longrightarrow & \mathfrak{bo}(V\oplus\mathbb{R}) \\ \downarrow & & \downarrow & & \downarrow \\ T_2\mathfrak{c}(V) & \longrightarrow & T_2\mathfrak{bo}(V) & \longrightarrow & T_2\mathfrak{bo}(V\oplus\mathbb{R}) \end{array} ,$$

where the rows are homotopy fiber sequences. If $V \neq 0$ the left-hand vertical arrow is a rational equivalence (in particular, it induces an isomorphism of fundamental groups) and so the right-hand square is rationally a homotopy pullback square. Therefore, by iteration,

$$\begin{array}{ccc} \mathfrak{bo}(V) & \longrightarrow & \mathfrak{bo}(V\oplus\mathbb{R}^\infty) \\ \downarrow & & \downarrow \\ T_2\mathfrak{bo}(V) & \longrightarrow & T_2\mathfrak{bo}(V\oplus\mathbb{R}^\infty) \end{array} ,$$

is rationally a homotopy pullback square. Here the right-hand column is a homotopy equivalence. So the left-hand column is a rational homotopy equivalence. □

Proposition 3.4. *The functor $\mathfrak{bg}$ is rationally polynomial of degree 2, for all $V \neq 0$. The derivative spectra of $\mathfrak{bg}$ satisfy the following:*

(i) *the natural transformation $\Theta\mathfrak{bo}^{(1)} \to \Theta\mathfrak{bg}^{(1)}$ is a homotopy equivalence;*

(ii) *the inclusion induced map $\Theta\mathfrak{bo}^{(2)} \to \Theta\mathfrak{bg}^{(2)}$ fits into a commutative triangle of spectra with $O(2)$ action*

$$\begin{array}{ccc} \Theta\mathfrak{bo}^{(2)} & & \\ \downarrow & \searrow & \\ \mathrm{map}(S^1, \Theta\mathfrak{bo}^{(2)}) & \xrightarrow[\simeq_{\mathbb{Q}}]{} & \Theta\mathfrak{bg}^{(2)} \end{array}$$

where $O(2)$ acts in the standard manner on S^1 and the vertical arrow is given by inclusion of the constant maps.

Proof. We begin by showing (i) and (ii). It is known [6, 26] that the map from $O(n+1)/O(n)$ to $G(n+1)/G(n)$, induced by inclusion, is $(2n-c)$-connected (for a small constant c independent of n). It follows immediately that the natural map $\Theta\mathfrak{bo}^{(1)} \to \Theta\mathfrak{bg}^{(1)}$ is a homotopy equivalence. Therefore $\Theta\mathfrak{bg}^{(1)}$ is a sphere spectrum with a trivial action of $O(1)$. For the second derivative spectrum we use the homotopy fiber sequence

$$\Omega^V_{\pm1}\mathfrak{c}(V) \longrightarrow \Omega\mathfrak{bg}(V\oplus\mathbb{R}) \longrightarrow \mathfrak{c}(V) , \tag{3.8}$$

where $\Omega^V(X)$ is the space of pointed maps from V^c to X (X a based space) and the ±1 singles out the degree ±1 components. In the lemma 3.6 below we show

that the functor $V \mapsto \Omega^V_{\pm 1}\mathfrak{c}(V)$ is rationally polynomial of degree 1, for all nonzero V. Therefore the homotopy fiber sequence (3.8) induces a rational equivalence of second derivative spectra,

$$\Omega(\Theta\mathfrak{bg}^{(2)} \wedge S^2) \simeq \Theta\mathfrak{c}^{(2)} \ .$$

More precisely, we have a commutative diagram of second derivative spectra

$$\begin{array}{ccc} \Omega(\Theta\mathfrak{bo}^{(2)} \wedge S^2) & & \\ \downarrow & \searrow^{\text{eval.}} & \\ \Omega(\Theta\mathfrak{bg}^{(2)} \wedge S^2) & \xrightarrow{\text{eval.}} & \Theta\mathfrak{c}^{(2)} \end{array} \ , \tag{3.9}$$

where the horizontal map is a homotopy equivalence. We saw in the proof of proposition 3.3 that the diagonal arrow in the diagram is in fact the map

$$\Omega(\Theta\mathfrak{bo}^{(2)} \wedge S^2) \ \longrightarrow \ \Omega(\Theta\mathfrak{bo}^{(2)} \wedge (S^2/S^0)) \simeq \Theta\mathfrak{bo}^{(2)} \wedge S^1_+ \ , \tag{3.10}$$

where we use (3.7). Therefore we can change triangle (3.9) into a commutative triangle

$$\begin{array}{ccc} \Omega(\Theta\mathfrak{bo}^{(2)} \wedge S^2) & & \\ \downarrow & \searrow^{(3.10)} & \\ \Omega(\Theta\mathfrak{bg}^{(2)} \wedge S^2) & \xrightarrow[\simeq_{\mathbb{Q}}]{} & \Theta\mathfrak{bo}^{(2)} \wedge S^1_+ \end{array} \ . \tag{3.11}$$

By undoing the looping and the double suspension, we obtain the commutative triangle

$$\begin{array}{ccc} \Theta\mathfrak{bo}^{(2)} & & \\ \downarrow & \searrow & \\ \Theta\mathfrak{bg}^{(2)} & \xrightarrow[\simeq_{\mathbb{Q}}]{} & \Omega^2(\Theta\mathfrak{bo}^{(2)} \wedge (S^2/S^0)) \end{array} \ . \tag{3.12}$$

There is a homotopy equivalence from $\Omega^2(\Theta\mathfrak{bo}^{(2)} \wedge (S^2/S^0))$ to $\mathrm{map}(S^1, \Theta\mathfrak{bo}^{(2)})$, preserving $\mathrm{O}(2)$-actions, which we describe in adjoint form by

$$S^1_+ \wedge \Omega^2(\Theta\mathfrak{bo}^{(2)} \wedge (S^2/S^0)) \longrightarrow \Theta\mathfrak{bo}^{(2)}.$$

Namely, every choice of point z in S^1 determines a nullhomotopy for the inclusion $S^0 \to S^2$ and thereby a map $S^2/S^0 \to S^2 \vee S^1 \to S^2$. So we have

$$z_+ \wedge \Omega^2(\Theta\mathfrak{bo}^{(2)} \wedge (S^2/S^0)) \longrightarrow \Omega^2(\Theta\mathfrak{bo}^{(2)} \wedge S^2) \simeq \Theta\mathfrak{bo}^{(2)}$$

for every $z \in S^1$, and using these for all z gives the required map. It is equivariant for the diagonal $\mathrm{O}(2)$-action on the source.

The homotopy fiber sequence (3.8) implies that the approximation $\mathfrak{bg}(V) \to T_2\mathfrak{bg}(V)$ is a rational homotopy equivalence when $\dim(V) \geq 2$. We need to improvise to show that it is also a rational homotopy equivalence when $V = \mathbb{R}$. There is a commutative diagram

$$\begin{array}{ccc} \mathfrak{bg}(\mathbb{R}) & \longrightarrow & \mathfrak{bg}(\mathbb{R}^2) \\ \downarrow & & \downarrow \\ T_2\mathfrak{bg}(\mathbb{R}) & \longrightarrow & T_2\mathfrak{bg}(\mathbb{R}^2) \\ \downarrow & & \downarrow \\ T_1\mathfrak{bg}(\mathbb{R}) & \longrightarrow & T_1\mathfrak{bg}(\mathbb{R}^2) \end{array}$$

where the outer rectangle and the lower square are rationally homotopy pullbacks by inspection. See also remark 3.5 below. It follows that the upper square is rationally a homotopy pullback. Therefore the map from $\mathfrak{bg}(\mathbb{R})$ to $T_2\mathfrak{bg}(\mathbb{R})$ is a rational equivalence. □

Remark 3.5. Let M be a closed smooth manifold and $\underline{E}$ a CW-spectrum. A Poincare duality principle identifies $\mathrm{map}(M, \underline{E})$ with

$$\underline{F} = \int_{x \in M} \Omega^{T_x M} \underline{E}.$$

Here the spectra $\Omega^{T_x M}\underline{E}$ for $x \in M$ together make up a fibered spectrum over M, and $\underline{F}$ is the ordinary spectrum obtained by passing to total spaces and collapsing the zero sections. Such an identification can also be used when M and $\underline{E}$ come with actions of a compact Lie group G. In particular, for $G = \mathrm{O}(2)$ and $M = S^1$ with the standard action of $\mathrm{O}(2)$ and $\underline{E} = \Omega\underline{S}^0$ with the trivial action, the spectrum $\mathrm{map}(M, \underline{E})$ can be described as $\Omega^2(S^1_+ \wedge \underline{S}^0)$ with the following action of $\mathrm{O}(2)$: trivial on the $\underline{S}^0$ factor, standard on the S^1_+ factor, *adjoint* action on one of the loop coordinates (the action of $\mathrm{O}(2)$ on its Lie algebra).

Lemma 3.6. *The functor $V \mapsto \Omega^V_{\pm 1}\mathfrak{c}(V)$ is rationally polynomial of degree 1, for all nonzero V.*

Proof. More precisely, we want to show that for nonzero V the approximation $\Omega^V_{\pm 1}\mathfrak{c}(V) \to T_1(\Omega^V_{\pm 1}\mathfrak{c})(V)$ is a rational homotopy equivalence of componentwise nilpotent spaces. Let $\mathfrak{a}_u(V) = \Omega^V_{\pm 1}\mathfrak{c}(V)$ and let $\mathfrak{a}$ be the slightly simpler functor defined by $\mathfrak{a}(V) = \Omega^V\mathfrak{c}(V)$. It is enough to show that $\mathfrak{a}$ is rationally polynomial of degree ≤ 1, because there is a homotopy pullback square

$$\begin{array}{ccc} \mathfrak{a}_u(V) & \longrightarrow & \mathfrak{a}(V) \\ \downarrow & & \downarrow \\ \{\pm 1\} & \longrightarrow & \mathbb{Z} \end{array}$$

(for $V \neq 0$) where the functors in the lower row are of degree 0 and hence of degree ≤ 1. From the proof of proposition 3.1 we have a rational homotopy fiber sequence

$$\mathfrak{a}(V) \longrightarrow \Omega^V\Omega^\infty\Sigma^\infty V^c \longrightarrow \Omega^V\Omega^\infty\Sigma^\infty(V^c \wedge V^c)_{\mathrm{rh}\mathbb{Z}/2}\ .$$

We want to show this is natural in V. Given a morphism $j\colon V \to W$, a vector $w \in W$ perpendicular to its image, and a configuration (S, f) with $f\colon S \to V^c$, we obtain a new configuration $(S, jf + w)$ where $jf + w$ is a map from S to W^c. Letting w vary, we obtain a map

$$(W/V)^c \wedge \Omega^\infty\Sigma^\infty V^c \longrightarrow \Omega^\infty\Sigma^\infty W^c$$

and consequently

$$\Omega^\infty\Sigma^\infty V^c \longrightarrow \Omega^{W/V}\Omega^\infty\Sigma^\infty W^c\ .$$

After applying Ω^V, we finally obtain

$$\Omega^V\Omega^\infty\Sigma^\infty V^c \longrightarrow \Omega^W\Omega^\infty\Sigma^\infty W^c.$$

This promotes $V \mapsto \Omega^V\Omega^\infty\Sigma^\infty V^c$ to a functor. Similarly, given $j\colon V \to W$ and w as before, and a configuration (S, f) with $f\colon S \to V^c \wedge V^c$, where S comes with a free action of $\mathbb{Z}/2$ and f is a $\mathbb{Z}/2$-map, we obtain a new configuration

$(S, (j \oplus j)f + (w \oplus w))$ where $(j \oplus j)f + (w \oplus w)$ is a $\mathbb{Z}/2$-map from S to $W^c \wedge W^c$. Letting w vary, we obtain therefore

$$\Omega^V\Omega^\infty\Sigma^\infty((V^c \wedge V^c)_{\mathrm{rh}\mathbb{Z}/2}) \longrightarrow \Omega^W\Omega^\infty\Sigma^\infty((W^c \wedge W^c)_{\mathrm{rh}\mathbb{Z}/2}).$$

This promotes $V \mapsto \Omega^V\Omega^\infty\Sigma^\infty((V^c \wedge V^c)_{\mathrm{rh}\mathbb{Z}/2})$ to a functor.
It is immediately clear that the functor $V \mapsto \Omega^V\Omega^\infty\Sigma^\infty V^c$ is polynomial of degree 0. We also need to show that $V \mapsto \Omega^V\Omega^\infty\Sigma^\infty((V^c \wedge V^c)_{\mathrm{rh}\mathbb{Z}/2})$ is polynomial of degree 1. At this point it is convenient to write $V^c \wedge V^c \cong (V \oplus V)^c = (\delta V \oplus \alpha V)^c$ where $\delta V \subset V \oplus V$ and $\alpha V \subset V \oplus V$ are the diagonal and antidiagonal linear subspaces, respectively. There is a natural equivalence

$$\Omega^\infty\Sigma^\infty((\alpha V^c)_{\mathrm{rh}\mathbb{Z}/2}) \longrightarrow \Omega^V\Omega^\infty\Sigma^\infty((V^c \wedge V^c)_{\mathrm{rh}\mathbb{Z}/2})$$

obtained by adjunction from a map

$$\delta V^c \wedge \Omega^\infty\Sigma^\infty((\alpha V^c)_{\mathrm{rh}\mathbb{Z}/2}) \longrightarrow \Omega^\infty\Sigma^\infty((V^c \wedge V^c)_{\mathrm{rh}\mathbb{Z}/2})$$

made by pushing configurations around. This leaves us with the task of showing that

$$V \mapsto \Omega^\infty\Sigma^\infty((\alpha V^c)_{\mathrm{rh}\mathbb{Z}/2}) \tag{3.13}$$

is polynomial of degree 1. Here αV is V with the action of $\mathbb{Z}/2 = \mathrm{O}(1)$ which has the generator acting by $-\mathrm{id}$. But (3.13) is exactly the definition of the homogeneous functor of degree one associated with the sphere spectrum with the trivial action of $\mathrm{O}(1)$. □

Remark 3.7. Let $\mathfrak{bg}^u$ be defined by $\mathfrak{bg}^u(V) = \mathrm{hofiber}[\mathfrak{bg}(V) \to T_0\mathfrak{bg}(V)]$. Then clearly $L_i\mathfrak{bg}^u = L_i\mathfrak{bg}$ for $i > 0$. From the rational homotopy fiber sequence (2.4) we obtain, for odd $n \geq 2$, another rational homotopy fiber sequence

$$S^{n-1} \longrightarrow \mathfrak{bg}^u(\mathbb{R}^{n-1}) \longrightarrow \mathfrak{bg}^u(\mathbb{R}^n)\ .$$

It follows that the inclusion-induced map $\mathfrak{bg}^u(\mathbb{R}^{n-1}) \longrightarrow \mathfrak{bg}^u(\mathbb{R}^n)$ is not rationally nullhomotopic. As $L_2\mathfrak{bg}$ is concentrated in odd dimensions and $L_1\mathfrak{bg}$ is concentrated in even dimensions, it follows also that $\mathfrak{bg}^u(V)$ is not naturally rationally identifiable with a product $L_2\mathfrak{bg}(V) \times L_1\mathfrak{bg}(V)$. The extension $L_2\mathfrak{bg} \longrightarrow T_2\mathfrak{bg}^u \longrightarrow L_1\mathfrak{bg}$ is therefore rationally nontrivial.

We now turn our attention to the (rational) Taylor tower of the functor $\mathfrak{bt}$. The inclusion $\mathfrak{bo} \to \mathfrak{bt}$ induces a rational homotopy equivalence $T_0\mathfrak{bo} \to T_0\mathfrak{bt}$, which just restates the Thom-Novikov result $B\mathrm{O} \simeq_{\mathbb{Q}} B\mathrm{TOP}$. The spectrum $\Theta\mathfrak{bt}^{(1)}$ has a complete rational description with action of $\mathrm{O}(1)$, due to Waldhausen, Borel and Farrell-Hsiang [25, 4, 5]. Below, $\underline{A}(\star)$ is Waldhausen's A-theory spectrum, also known as the algebraic K-theory spectrum of the ring spectrum $\underline{S}^0$, and $\underline{K}(\mathbb{Q})$ is the algebraic K-theory of the ring $\mathbb{Q}$.

Proposition 3.8. *The functor $\mathfrak{bt}$ has first derivative spectrum*

$$\Theta\mathfrak{bt}^{(1)} \simeq \underline{A}(\star) \simeq_{\mathbb{Q}} \underline{K}(\mathbb{Q}).$$

The $\mathrm{O}(1)$ action is the standard duality action on K-theory. Hence,

$$\Theta\mathfrak{bt}^{(1)} \ \simeq_{\mathbb{Q}}\ \Theta\mathfrak{bo}^{(1)} \vee \bigvee_{i=1}^{\infty} \underline{S}^{4i+1}$$

where the summand $\bigvee_{i=1}^{\infty} \underline{S}^{4i+1}$ has the sign change involution.

Proof. By definition the spectrum $\Theta\mathfrak{bt}^{(1)}$ is made up of spaces

$$\Theta\mathfrak{bt}^{(1)}(n) = \mathrm{TOP}(n+1)/\mathrm{TOP}(n)$$

with structure maps analogous to those of $\Theta\mathfrak{bo}^{(1)}$. The identification of the spectrum $\{\mathrm{TOP}(n+1)/\mathrm{TOP}(n) \,|\, n \in \mathbb{N}\}$ with $\underline{A}(\star)$ comes from [25]. It relies on the smoothing theory description of spaces of smooth h-cobordisms over D^n, as in example 2.4. Modulo that it is a central part of Waldhausen's development of the algebraic K-theory tradition in h-cobordism theory, which started with the h-cobordism and s-cobordism theorems [21, 14, 3, 11]. The identification of the canonical O(1)-action on the spectrum $\{\mathrm{TOP}(n+1)/\mathrm{TOP}(n) \,|\, n \in \mathbb{N}\}$ with the $\mathbb{Z}/2$-action on $\underline{A}(\star)$ by (Spanier-Whitehead) duality is due to [24], again going through h-cobordism theory. The inclusion-induced map $\Theta^{(1)}\mathfrak{bo} \to \Theta^{(1)}\mathfrak{bt}$, alias $\underline{S}^0 \to \underline{A}(\star)$, admits an *integral* homotopy left inverse (with weak O(1)-equivariance). The rational equivalence $\underline{A}(\star) \simeq_{\mathbb{Q}} \underline{K}(\mathbb{Q})$ is a consequence of the rational equivalence between the sphere spectrum (as a ring spectrum) and the Eilenberg-MacLane spectrum $H\mathbb{Q}$. The calculation of the rational homotopy groups of $\underline{K}(\mathbb{Q})$ follows from the calculation of the rational cohomology groups of $B\mathrm{GL}(\mathbb{Q})$, due to Borel [4]. The result is

$$\pi_n(\underline{K}(\mathbb{Q})) \otimes \mathbb{Q} \;\cong\; \begin{cases} \mathbb{Q} & n = 0 \\ \mathbb{Q} & n = 5, 9, 13, 17, \dots \\ 0 & \text{otherwise.} \end{cases}$$

The action of O(1) on $\pi_n(\underline{K}(\mathbb{Q}))$ is trivial for $n = 0$ and nontrivial (sign change) for $n = 5, 9, 13 \dots$. □

4. Natural transformations between homogeneous functors

Our analysis of the functor $\mathfrak{c}_J$ in proposition 3.2 gives us a tool for studying the space of natural transformations

$$\mathrm{nat}_\star(\mathfrak{a}_1, \mathfrak{a}_2)$$

in certain situations where $\mathfrak{a}_1$ is homogeneous of degree 1 and $\mathfrak{a}_2$ is homogeneous of degree 2. (For a detailed definition of these spaces of natural transformations, see the beginning of section 5. We are assuming that all functors in sight are cofibrant. Also the functors are typically from $\mathscr{J}$ to $\mathscr{T}_\star$, the category of based spaces. Natural transformations are assumed to respect base points unless otherwise stated.) We take $\mathfrak{a}_1 = T_1\mathfrak{c}_J = L_1\mathfrak{c}_J$ while $\mathfrak{a}_2$ remains unspecified until later.

Lemma 4.1. *There is a homotopy fiber sequence of spaces of natural transformations*

$$\mathrm{nat}_\star(L_2\mathfrak{c}_J, \mathfrak{a}_2) \leftarrow \mathrm{nat}_\star(T_2\mathfrak{c}_J, \mathfrak{a}_2) \leftarrow \mathrm{nat}_\star(T_1\mathfrak{c}_J, \mathfrak{a}_2).$$

Proof. There is a more obvious homotopy fiber sequence

$$\mathrm{nat}_\star(L_2\mathfrak{c}_J, \mathfrak{a}_2) \leftarrow \mathrm{nat}_\star(T_2\mathfrak{c}_J, \mathfrak{a}_2) \leftarrow \mathrm{nat}_\star(\mathfrak{m}, \mathfrak{a}_2)$$

where $\mathfrak{m}$ is the mapping cone of the forgetful map $L_2\mathfrak{c}_J \to T_2\mathfrak{c}_J$. Furthermore we have $\mathrm{nat}_\star(\mathfrak{m}, \mathfrak{a}_2) \simeq \mathrm{nat}_\star(T_2\mathfrak{m}, \mathfrak{a}_2)$ by the universal property of $T_2\mathfrak{m}$. Therefore it is enough to show that the natural map $\mathfrak{m} \to T_1\mathfrak{c}_J$ induces an equivalence of second Taylor approximations,

$$T_2\mathfrak{m} \to T_2T_1\mathfrak{c}_J \simeq T_1\mathfrak{c}_J \ .$$

By a direct limit argument we may assume that J is a finite sequence. Also, from the explicit form of the operator T_2 , it is enough to show that $\mathfrak{m}(V) \to T_1\mathfrak{c}_J(V)$

is $(3d-c)$-connected for all V, where $d=\dim(V)$ and c is a constant independent of V. This is a special case of the following observation related to the Blakers-Massey homotopy excision theorem: If $f\colon Y\to Z$ is a based map where Z is k-connected and f is ℓ-connected, then the canonical map from the mapping cone of hofiber$(f)\to Y$ to Z is $(k+\ell-c)$ connected. We apply this with f equal to the map $T_2\mathfrak{c}_J(V)\to T_1\mathfrak{c}_J(V)$. □

Two of the spaces in the homotopy fiber sequence of lemma 4.1 are easy to understand. The space $\mathrm{nat}_\star(L_2\mathfrak{c}_J,\mathfrak{a}_2)$ can be understood, since we understand natural transformations between homogeneous functors of the same degree (see lemma 4.3). The space $\mathrm{nat}_\star(T_2\mathfrak{c}_J,\mathfrak{a}_2)$ can be understood because

$$\mathrm{nat}_\star(T_2\mathfrak{c}_J,\mathfrak{a}_2)\simeq\mathrm{nat}_\star(\mathfrak{c}_J,\mathfrak{a}_2)$$

and $\mathfrak{c}_J$ behaves in many ways like a (co)representable functor as can be seen by the following lemma.

Lemma 4.2. *Let $\mathscr{J}^{[1]}$ be the full subcategory of $\mathscr{J}$ spanned by the objects $0,\mathbb{R}$. Then the functor $\mathfrak{c}_J$ is freely generated by its restriction to $\mathscr{J}^{[1]}$. In particular, we have*

$$\mathrm{nat}_\star(\mathfrak{c}_J,\mathfrak{a}_2)\cong\mathrm{nat}_\star(\mathfrak{c}_J|_{\mathscr{J}^{[1]}},\mathfrak{a}_2|_{\mathscr{J}^{[1]}})\cong\prod_j\Omega^{k_j}\mathrm{hofiber}[\mathfrak{a}_2(0)\to\mathfrak{a}_2(\mathbb{R})]\ .$$

Proof. We show this in the case $\mathfrak{c}_J=\mathfrak{c}$ (the general case follows similarly). By the free generation statement we mean that $\mathfrak{c}$ is the left Kan extension of its restriction $\mathfrak{c}|_{\mathscr{J}^{[1]}}$. More explicitly, given V in $\mathscr{J}$ and a point $y\in\mathfrak{c}(V)$ we can find A in $\mathscr{J}^{[1]}$, an $x\in\mathfrak{c}(A)$ and $f\colon A\to V$ such that $f_*(x)=y$. The triple (A,x,f) is unique up to the obvious relations. In the cases where $A=\mathbb{R}$, we can always choose x and f such that $x\in[0,\infty]$. Then we see that $\mathrm{nat}_\star(\mathfrak{c},\mathfrak{a}_2)$ can be identified with the space of pairs of based maps (f,g) making the diagram

$$\begin{array}{ccc}\{0,\infty\} & \xrightarrow{\mathrm{incl}} & [0,\infty]\\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g}\\ \mathfrak{a}_2(0) & \xrightarrow{\mathrm{incl}_*} & \mathfrak{a}_2(\mathbb{R})\end{array}$$

commute, where $\{0,\infty\}=\mathfrak{c}(0)$ and $[0,\infty]\subset\mathfrak{c}(\mathbb{R})$. This space of pairs (f,g) is just the homotopy fiber of the inclusion-induced map $\mathfrak{a}_2(0)\to\mathfrak{a}_2(\mathbb{R})$. □

Lemma 4.3. *Let $\mathfrak{g}$ and $\mathfrak{f}$ be two homogeneous functors, from $\mathscr{J}$ to based spaces, of the same degree $n>0$. Then*

$$\mathrm{nat}_\star(\mathfrak{g},\mathfrak{f})\simeq\mathrm{map}^{\mathrm{hO}(n)}(\Theta\mathfrak{g}^{(n)},\Theta\mathfrak{f}^{(n)})\ ,$$

where the righthand side is the space of weak $\mathrm{O}(n)$-maps.

This is obvious from the classification of homogeneous functors [27].

Example 4.4. Suppose that $\mathfrak{a}_2$ is the homogeneous functor of degree 2 corresponding to the spectrum $\underline{H}\mathbb{Q}$ with trivial action of $\mathrm{O}(2)$, and $\mathfrak{a}_1$ is $T_1\mathfrak{c}_J$ where $k_j>0$

for all $j = 1, 2, \ldots$. Then by lemma (4.1) we have

$$\begin{aligned}
\mathrm{nat}_\star\,(T_2\mathfrak{c}_J, \mathfrak{a}_2) &\simeq \mathrm{nat}_\star\,(\mathfrak{c}_J, \mathfrak{a}_2)\\
&\simeq \textstyle\prod_j \Omega^{k_j}\mathrm{hofiber}[\mathfrak{a}_2(0) \to \mathfrak{a}_2(\mathbb{R})]\\
&\simeq \textstyle\prod_j \Omega^{k_j}\Omega^\infty\mathrm{hofiber}[(S^0 \wedge \underline{H}\mathbb{Q})_{h\mathrm{O}(2)} \to (S^2 \wedge \underline{H}\mathbb{Q})_{h\mathrm{O}(2)}]\\
&\simeq \textstyle\prod_j \Omega^{k_j+1}\Omega^\infty\left(((S^2/S^0) \wedge \underline{H}\mathbb{Q})_{h\mathrm{O}(2)}\right)\\
&\simeq \textstyle\prod_j \Omega^{k_j+1}\Omega^\infty(S^1 \wedge \underline{H}\mathbb{Q})\\
&\simeq \textstyle\prod_j \Omega^{k_j}\Omega^\infty\underline{H}\mathbb{Q}\\
&\simeq \textstyle\prod_j \Omega^{k_j}\mathbb{Q}
\end{aligned}$$

which is contractible. By lemma 4.3 we have the homotopy equivalence

$$\mathrm{nat}_\star\,(L_2\mathfrak{c}_J, \mathfrak{a}_2) \simeq \mathrm{map}^{\mathrm{hO}(n)}(\Theta\mathfrak{c}_J^{(2)}, \underline{H}\mathbb{Q})$$

and by lemma 3.2 the righthand side is contractible. From the homotopy fiber sequence of lemma 4.1 it follows that $\mathrm{nat}_\star\,(\mathfrak{a}_1, \mathfrak{a}_2)$ is contractible.
For another example which we will need, keep $\mathfrak{a}_1$ as above and let $\mathfrak{a}_2$ be the homogeneous functor corresponding to $\mathrm{map}(S^1, \underline{H}\mathbb{Q})$ with $\mathrm{O}(2)$ acting via its standard action on S^1. A calculation similar to the above shows that $\mathrm{nat}_\star\,(\mathfrak{a}_1, \mathfrak{a}_2)$ is again contractible.

Example 4.5. Let us look at spaces of natural transformations $\mathrm{nat}(\mathfrak{a}_1, \mathfrak{a}_2)$ where base points are ignored, although we still assume that $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are functors from $\mathscr{J}$ to $\mathscr{T}_\star$. There is a homotopy fiber sequence

$$\mathrm{nat}_\star\,(\mathfrak{a}_1, \mathfrak{a}_2) \to \mathrm{nat}(\mathfrak{a}_1, \mathfrak{a}_2) \to \mathfrak{a}_2(0)$$

where the right-hand arrow is given by evaluation at the base point in $\mathfrak{a}_1(0)$. That arrow has a right inverse given by the inclusion of constant natural transformations. If $\mathrm{nat}_\star\,(\mathfrak{a}_1, \mathfrak{a}_2)$ is contractible, and $\mathfrak{a}_2$ takes values in infinite loop spaces, as in the above example 4.4, then it follows that $\mathrm{nat}(\mathfrak{a}_1, \mathfrak{a}_2)$ is homotopy equivalent to $\mathfrak{a}_2(0)$. In other words the space of all constant natural transformations from $\mathfrak{a}_1$ to $\mathfrak{a}_2$ is homotopy equivalent (by inclusion) to the space of all natural transformations from $\mathfrak{a}_1$ to $\mathfrak{a}_2$.

Example 4.6. Let $J = (k_j)_{j=1,2,\ldots}$ be a sequence of integers such that $k_1 = 0$ and $k_j > 0$ for all $j > 1$. Let $\mathfrak{a}_1 = T_1\mathfrak{c}_J$ and $\mathfrak{b} = T_1\mathfrak{c}$, and $\mathfrak{a}_2$ as in example 4.4. The map $p\colon \mathfrak{c}_J \to \mathfrak{c}$, collapse to the first wedge summand, gives a map $\mathfrak{a}_1 \to \mathfrak{b}$. We wish to show that

$$p^*\colon \mathrm{nat}(\mathfrak{b}, \mathfrak{a}_2) \to \mathrm{nat}(\mathfrak{a}_1, \mathfrak{a}_2) \tag{4.1}$$

is a homotopy equivalence. By the same reduction that we used in example 4.5, it is enough to show that $p^*\colon \mathrm{nat}_\star\,(\mathfrak{b}, \mathfrak{a}_2) \to \mathrm{nat}_\star\,(\mathfrak{a}_1, \mathfrak{a}_2)$ is a homotopy equivalence. Going through calculations as in example 4.4, we can see that

$$\mathrm{nat}_*(T_2\mathfrak{c}, \mathfrak{a}_2) \to \mathrm{nat}_*(T_2\mathfrak{c}_J, \mathfrak{a}_2)\ , \qquad \mathrm{nat}_*(L_2\mathfrak{c}, \mathfrak{a}_2) \to \mathrm{nat}_*(L_2\mathfrak{c}_J, \mathfrak{a}_2)$$

are homotopy equivalences, hence $p^*\colon \mathrm{nat}_\star\,(\mathfrak{b}, \mathfrak{a}_2) \to \mathrm{nat}_\star\,(\mathfrak{a}_1, \mathfrak{a}_2)$ is a homotopy equivalence by lemma 4.1. Therefore the map (4.1) is a homotopy equivalence.

Example 4.7. There is a slight generalization of the previous example where we assume $\mathfrak{a}_1$ is a retract of $T_1\mathfrak{c}_J$ and $\mathfrak{b}$ is a retract of $T_1\mathfrak{c}$. More precisely we assume

there exists a commutative diagram of natural transformations

$$\begin{array}{ccccc} \mathfrak{a}_1 & \longrightarrow & T_1\mathfrak{c}_J & \longrightarrow & \mathfrak{a}'_1 \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle T_1 p} & & \downarrow \\ \mathfrak{b} & \longrightarrow & T_1\mathfrak{c} & \longrightarrow & \mathfrak{b}' \end{array}$$

such that the horizontal compositions are homotopy equivalences. Then the same conclusion holds, i.e. the map $q^*\colon \mathrm{nat}(\mathfrak{b}, \mathfrak{a}_2) \to \mathrm{nat}(\mathfrak{a}_1, \mathfrak{a}_2)$ is a homotopy equivalence.

5. Splitting hypotheses

In our formulations of the C hypotheses below, we often use the word *weak* in an informal way. As a rule this has an interpretation in the language of model categories [8], [9]. Less formally, let us assume that we are dealing with a category $\mathscr{C}$ which comes with a notion of *homotopy* between morphisms and with notions of *weak equivalence* and *cofibrant object*. The following axiom should hold: any diagram

$$\begin{array}{ccc} & & B' \\ & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & B \end{array}$$

in $\mathscr{C}$, where g is a weak equivalence, can be completed to a triangle

$$\begin{array}{ccc} & \nearrow & B' \\ & \swarrow & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & B \end{array}$$

commutative up to homotopy.
Then, for objects A and B in $\mathscr{C}$, a *weak morphism* from A to B consists of a cofibrant object A', a weak equivalence $\rho\colon A' \to A$ and a genuine morphism $A' \to B$. The weak equivalence $\rho\colon A' \to A$ is a *cofibrant replacement* (alias cofibrant resolution), and where it exists it has some uniqueness features. If the category $\mathscr{C}$ has additional structure (e.g. a model category structure), then it may be possible to define a simplicial set of morphisms from A' to B (in the examples stated below this is clear) and this can play the role of a weak mapping space $\mathrm{map}(A, B)$. Its weak homotopy type should be independent of the choice of cofibrant resolution of A. Our examples are of two types.

Example 5.1. Let $\mathscr{C}$ be the category of all functors from $\mathscr{J}$ to spaces. Such a functor is *cofibrant* if it is a retract of a functor which is built from functors of the form

$$V \mapsto D^i \times \mathrm{mor}(W, V)$$

for fixed $W \in \mathscr{J}$ and $i \geq 0$. See [27, A.2] for more details. We write $\mathrm{nat}_\star(\ ,\ )$ instead of $\mathrm{map}(\ ,\ )$ for the spaces of natural transformations.

Example 5.2. Let $\mathscr{C}$ be the category of spaces with Γ-action, or a suitable category of spectra with Γ-action, for Γ a fixed compact Lie group (often we take $\Gamma = \mathrm{O}(2)$, or $\Gamma = S^1$). Here the cofibrant objects are those which can be built from "free" pieces such as $D^k \times \Gamma$ or stable analogues.

With these instructions, the reader should be able to decode our formulations of the C hypotheses in a mechanical fashion. As the mechanical method tends to produce long-winded statements, we offer more hands-on formulations in remark 5.5 below.

Hypothesis C^s . *As an* $\mathrm{O}(2)$*-map, the inclusion* $\Theta\mathfrak{bo}^{(2)} \to \Theta\mathfrak{bt}^{(2)}$ *admits a (rational, weak) left inverse over* $\Theta\mathfrak{bg}^{(2)}$*.*

Hypothesis C^m . *As an* $\mathrm{O}(2)$*-map, the inclusion* $\Theta\mathfrak{bo}^{(2)} \to \Theta\mathfrak{bt}^{(2)}$ *admits a (rational, weak) left inverse.*

We now give alternative formulations of hypotheses C^s and C^m . The rest of this chapter will be devoted to proving that these new statements are in fact equivalent to our original hypotheses.

Proposition 5.3. *Hypothesis* C^s *is equivalent to the statement that the inclusion* $\mathfrak{bo} \to \mathfrak{bt}$ *admits a (weak, rational) left inverse over* $\mathfrak{bg}$*.*

Proposition 5.4. *Hypothesis* C^m *is equivalent to saying that the inclusion* $\mathfrak{bo} \to \mathfrak{bt}$ *admits a (weak, rational) left inverse.*

Remark 5.5. Explicit formulations of the 2 hypotheses and 2 propositions above are as follows. In hypothesis C^m , we mean that there exists a spectrum Υ with action of $\mathrm{O}(2)$ and an $\mathrm{O}(2)$-map $\Theta\mathfrak{bt}^{(2)} \to \Upsilon$ such that the composition

$$\Theta\mathfrak{bo}^{(2)} \to \Theta\mathfrak{bt}^{(2)} \to \Upsilon$$

is a rational homotopy equivalence. (There is no need to arrange that $\mathfrak{bt}^{(2)}$ is cofibrant.) In hypothesis C^s we mean the same but in addition Υ comes with an $\mathrm{O}(2)$-map to $\Theta\mathfrak{bg}^{(2)}$ and the $\mathrm{O}(2)$-map $\Theta\mathfrak{bt}^{(2)} \to \Upsilon$ is over $\Theta\mathfrak{bg}^{(2)}$. In proposition 5.4 we mean that there exist a third functor $\mathfrak{d}$ and a natural transformation $\mathfrak{bt} \to \mathfrak{d}$ such that the composition $\mathfrak{bo} \to \mathfrak{bt} \to \mathfrak{d}$ is a rational equivalence. In proposition 5.3 we mean the same but in addition, $\mathfrak{d}$ comes with a natural transformation to $\mathfrak{bg}$ and $\mathfrak{bt} \to \mathfrak{d}$ is over $\mathfrak{bg}$.

We recall how a functor $\mathfrak{d}$ from $\mathscr{J}$ to $\mathscr{T}_*$ determines a sequence of spectra $\Theta\mathfrak{d}^{(i)}$, for $i = 1, 2, 3, \ldots$ (for more details on this and what follows, see [27]). The category $\mathscr{J}$ is contained in a larger category $\mathscr{J}_i$ enriched over based spaces and the functor $\mathfrak{d}$ has a right Kan extension $\mathfrak{d}^{(i)}$, from $\mathscr{J}_i$ to $\mathscr{T}_*$. An explicit formula for $\mathfrak{d}^{(i)}$ is

$$\mathfrak{d}^{(i)}(V) = \mathrm{hofiber}\left[\mathfrak{d}(V) \longrightarrow \operatorname*{holim}_{0 \neq U \leq \mathbb{R}^{i+1}} \mathfrak{d}(V \oplus U)\right] \tag{5.1}$$

where we use a topologically enhanced homotopy limit. Instead of saying that $\mathfrak{d}^{(i)}$ is defined on $\mathscr{J}_i$ we can also pretend that $\mathfrak{d}^{(i)}$ is defined on $\mathscr{J}$ and comes with the following additional structure: a natural map

$$\sigma : (V \otimes \mathbb{R}^i)^c \wedge \mathfrak{d}^{(i)}(W) \longrightarrow \mathfrak{d}^{(i)}(V \oplus W) \tag{5.2}$$

with the expected associativity and unital properties. Moreover $\mathfrak{d}^{(i)}$ comes with an action of $\mathrm{O}(i)$, obvious from the explicit formula, such that σ is equivariant. (It is equivariant for the diagonal action of $\mathrm{O}(i)$ on the source. By specializing to $V = \mathbb{R}$ and $W = \mathbb{R}^i$ we obtain a spectrum with twisted action of $\mathrm{O}(i)$ where the structure maps involve smash product with a sphere $(\mathbb{R}^i)^c$ on which $\mathrm{O}(i)$ acts nontrivially. This can be untwisted. Besides, it is essential in the following that we don't specialize too soon.)

Remark 5.6. We need to state and explain a few facts to be used later.

(a) The inclusion $B\mathrm{O} \to B\mathrm{TOP}$ is a rational homotopy equivalence.
(b) The canonical actions of O on $\Theta\mathfrak{bo}^{(i)}$ and $\Theta\mathfrak{bt}^{(i)}$ for $i \geq 1$ are special cases of a "natural action" of O on all spectra.

Statement (a) is well known (see [12]). It can, of course, be restated as saying that the inclusion of zeroth Taylor approximations $T_0\mathfrak{bo} \to T_0\mathfrak{bt}$ is a rational homotopy equivalence.

Regarding (b), suppose that $\mathfrak{d}$ is a continuous functor from $\mathscr{J}$ to based spaces, and that $\mathfrak{d}(\mathbb{R}^\infty) = \mathrm{hocolim}_n \mathfrak{d}(\mathbb{R}^n)$ is path-connected. Then the spectra $\Theta^{(i)}\mathfrak{d}$ (for $i = 1, 2, 3, \dots$) are defined, and each $\Theta^{(i)}\mathfrak{d}$ comes with an action of $\mathrm{O}(i)$. What matters here is that they also come with an action of $\Omega\mathfrak{d}(\mathbb{R}^\infty)$, commuting (in the weak sense) with the action of $\mathrm{O}(i)$. This can be seen along the following lines. Fix $n \geq 0$ and $x \in \mathfrak{d}(\mathbb{R}^n)$. Let $s_n\mathfrak{d}_x$ be defined by $s_n\mathfrak{d}_x(V) = \mathfrak{d}(\mathbb{R}^n \oplus V)$, with base point equal to the image of x under the inclusion-induced map $\mathfrak{d}(\mathbb{R}^n) \to \mathfrak{d}(\mathbb{R}^n \oplus V)$. The functor $s_n\mathfrak{d}_x$ determines spectra

$$\Theta s_n \mathfrak{d}_x^{(i)} \tag{5.3}$$

for $i \geq 1$. As x runs through $\mathfrak{d}(\mathbb{R}^n)$ these spectra constitute a fibered spectrum over $s_n\mathfrak{d}(0) = \mathfrak{d}(\mathbb{R}^n)$, with fiberwise action of $\mathrm{O}(i)$. However it is also true by (5.2) that there is a canonical homotopy equivalence

$$\sigma\colon (\mathbb{R}^n \otimes \mathbb{R}^i)^c \wedge \Theta\mathfrak{d}^{(i)} \longrightarrow \Theta s_n \mathfrak{d}_x^{(i)} \tag{5.4}$$

if x is the standard base point in $\mathfrak{d}(\mathbb{R}^n)$. In the left-hand side we have the diagonal action of $\mathrm{O}(i)$, with $\mathrm{O}(i)$ acting on $\mathbb{R}^i$ in the standard way. In adjoint notation, this becomes a homotopy equivalence

$$\Theta\mathfrak{d}^{(i)} \longrightarrow \Omega^{\mathbb{R}^n \otimes \mathbb{R}^i} \Theta s_n \mathfrak{d}_x^{(i)}\ . \tag{5.5}$$

Letting x vary again, we conclude that $\Theta\mathfrak{d}^{(i)}$ extends to a fibered spectrum over $\mathfrak{d}(\mathbb{R}^n)$. Now it is easy to let n tend to infinity. Instead of saying that $\Theta\mathfrak{d}^{(i)}$ extends to a fibered spectrum over $\mathfrak{d}(\mathbb{R}^\infty)$, we can also say that $\Omega\mathfrak{d}(\mathbb{R}^\infty)$ acts on $\Theta\mathfrak{d}^{(i)}$. This action, for $i = 1, \dots, k$, is one of the ingredients in a "stagewise" description of the k-Taylor approximation $T_k\mathfrak{d}$ of $\mathfrak{d}$. That is also how we will use it below, with $k = 2$.

We now specialize by taking $\mathfrak{d} = \mathfrak{bo}$, while i remains unspecified. Then $\Omega\mathfrak{bo}(\mathbb{R}^\infty)$ is the group O. It is easy to see how the spectra (5.3) depend on $x \in \mathfrak{bo}(\mathbb{R}^n) = B\mathrm{O}(n)$ if we think of x as an n-dimensional vector space V_x (since $B\mathrm{O}(n)$ is a Grassmannian). Indeed, we can make an identification of based spaces

$$s_n\mathfrak{bo}_x(W) \xrightarrow{\ \simeq\ } \mathfrak{bo}(W \oplus V_x)$$

(see also remark 5.7 below). Therefore

$$s_n\mathfrak{bo}_x^{(i)}(\mathbb{R}^m) \xrightarrow{\ \simeq\ } \mathfrak{bo}^{(i)}(\mathbb{R}^m \oplus V_x)$$

so that we have an identification of spectra

$$\Theta s_n \mathfrak{bo}_x^{(i)} \xrightarrow{\ \simeq\ } \{\mathfrak{bo}^{(i)}(\mathbb{R}^m \oplus V_x) \mid m \geq 0\} \xleftarrow{\ \simeq\ } (V_x \otimes \mathbb{R}^i)^c \wedge \Theta\mathfrak{bo}^{(i)}$$

where the second map uses (5.2). Summarizing, we have a canonical homotopy equivalence

$$\Theta s_n \mathfrak{bo}_x^{(i)} \ \cong\ (V_x \otimes \mathbb{R}^i)^c \wedge \Theta\mathfrak{bo}^{(i)}$$

extending (5.4). This means that the action of $\mathrm{O}(n) \subset \mathrm{O}$ on $\Theta\mathfrak{bo}^{(i)}$ is obtained essentially by writing

$$\Theta\mathfrak{bo}^{(i)} \simeq \Omega^{\mathbb{R}^n \otimes \mathbb{R}^i}\left((\mathbb{R}^n \otimes \mathbb{R}^i)^c \wedge \Theta\mathfrak{bo}^{(i)}\right)$$

and letting $\mathrm{O}(n)$ act exclusively on the $\mathbb{R}^n$ factor in $(\mathbb{R}^n \otimes \mathbb{R}^i)^c$. We now let n tend to infinity and thereby complete our sketch proof of (b) in the case of $\mathfrak{bo}$. The case of $\mathfrak{bt}$ is similar, so long as we are not interested in the action of $\mathrm{TOP} \simeq \Omega\mathfrak{bt}(\mathbb{R}^\infty)$ on $\Theta\mathfrak{bt}^{(i)}$, but only in the action of the subgroup $\mathrm{O} \simeq \Omega\mathfrak{bo}(\mathbb{R}^\infty)$.

Remark 5.7. We have used the following fact: for any n-dimensional real vector space V, there is a canonical identification of $B\mathrm{O}(V)$ with $B\mathrm{O}(n)$. More precisely, we can specify a contractible space Y of (unbased) maps from $B\mathrm{O}(V)$ to $B\mathrm{O}(n)$ such that every element of Y is a homotopy equivalence. Let Y be the space of pairs (f, g) where $f\colon B\mathrm{O}(V) \to B\mathrm{O}(n)$ is continuous and g is an isomorphism from the universal vector bundle on $B\mathrm{O}(V)$ to f^* of the universal vector bundle on $B\mathrm{O}(n)$. This has the curious consequence that $\mathfrak{bo}(V)$ is "the same" for all V of a fixed dimension n, if we are willing to ignore base points. That was the basis for our proof of fact (b) in remark 5.6.

Remark 5.8. The Taylor tower in orthogonal calculus (of a functor $\mathfrak{d}$ from $\mathscr{J}$ to $\mathscr{T}_*$) has considerable formal similarities with the Postnikov tower of a based, connected CW-space X. The essentially constant functor $T_0\mathfrak{d}$ plays the part of $\pi_1(X)$ in Postnikov theory. The spectra $\Theta^{(i)} = \Theta\mathfrak{d}^{(i)}$ for $i > 0$ play the part of the homotopy groups $\pi_i = \pi_i(X)$, where $i > 1$. The fact that π_i is a module over $\pi_1(X)$ is analogous to the fact that $\Theta^{(i)}$ extends to a fibered spectrum $z \mapsto \Theta^{(i)}_z$ on the space $T_0\mathfrak{d}(0)$. The inductive construction of the stages X_i of the Postnikov tower of X is best described by means of homotopy pullback squares

$$\begin{array}{ccc} X_i & \xrightarrow{\text{proj.}} & X_{i-1} \\ \downarrow{\scriptstyle \text{proj.}} & & \downarrow{\scriptstyle \kappa_i} \\ B\pi_1(X) & \xrightarrow{\text{0-section}} & (B^{i+1}\pi_i)_{\mathrm{h}\pi_1(X)} \end{array} .$$

Here the lower right-hand term is the homotopy orbit construction for the action of $\pi_1(X)$ on $B^{i+1}\pi_i$, so that there is a projection from it to $B\pi_1(X)$ with Eilenberg-MacLane fiber $B^{i+1}\pi_i$. By analogy with that, there is a homotopy pullback square

$$\begin{array}{ccc} T_i\mathfrak{d} & \xrightarrow{\text{proj.}} & T_{i-1}\mathfrak{d} \\ \downarrow{\scriptstyle \text{proj.}} & & \downarrow{\scriptstyle \kappa_i} \\ T_0\mathfrak{d} & \xrightarrow{\text{0-section}} & \mathfrak{f}[\Theta^{(i)}_\bullet] \end{array} .$$

Here $\mathfrak{f}[\Theta^{(i)}_\bullet]$ is a functor which is essentially determined by the space $T_0\mathfrak{d}(0)$ and the fibered spectrum

$$z \mapsto S^1 \wedge \Theta^{(i)}_z$$

on it, with the fiberwise action of $\mathrm{O}(i)$. There is a forgetful projection from $\mathfrak{f}[\Theta^{(i)}_\bullet]$ to $T_0\mathfrak{d}$, and for every point $z \in T_0\mathfrak{d}(0)$ the homotopy fiber of that projection at z is the homogeneous functor of degree i from $\mathscr{J}$ to $\mathscr{T}_*$ determined by the spectrum

$$S^1 \wedge \Theta^{(i)}_z$$

with action of $\mathrm{O}(i)$.

Proof of prop. 5.4. Assuming hypothesis C^m, we will construct a functor $\mathfrak{d}$ and a map $\mathfrak{bt} \to \mathfrak{d}$ such that the composition $\mathfrak{bo} \to \mathfrak{bt} \to \mathfrak{d}$ is a rational equivalence. We construct $\mathfrak{d}$ inductively, starting with $T_0\mathfrak{d}$ and then building $T_1\mathfrak{d}$ and finally $T_2\mathfrak{d}$. Let $T_0\mathfrak{d} = T_0\mathfrak{bt}$ and let $T_1\mathfrak{d}$ be the homotopy pullback of $T_0\mathfrak{bt} \to T_0\mathfrak{bg} \leftarrow T_1\mathfrak{bg}$. To make $T_2\mathfrak{d} = \mathfrak{d}$ we need a fibered spectrum $\Upsilon_\bullet$ over $T_0\mathfrak{d}(0) = B\mathrm{TOP}$, with fiberwise action of $\mathrm{O}(2)$, and a map $\kappa_2 : T_1\mathfrak{d} \to \mathfrak{f}[\Upsilon_\bullet]$ which is a natural transformation over $T_0\mathfrak{d}$. Here $\mathfrak{f}[\Upsilon_\bullet]$ is the functor essentially determined as in remark 5.8 by the fibered spectrum $S^1 \wedge \Upsilon_\bullet$ over $B\mathrm{TOP}$, with the fiberwise action of $\mathrm{O}(2)$. Then we define $T_2\mathfrak{d}$ by a homotopy pullback square

$$\begin{array}{ccc} T_2\mathfrak{d} & \xrightarrow{\text{proj.}} & T_1\mathfrak{d} \\ \downarrow{\scriptstyle \text{proj.}} & & \downarrow{\scriptstyle \kappa_2} \\ T_0\mathfrak{d} & \xrightarrow{\text{0-section}} & \mathfrak{f}[\Upsilon_\bullet]\ . \end{array}$$

We already have a spectrum Υ from our hypothesis, with an $\mathrm{O}(2)$-map $\Theta\mathfrak{bt}^{(2)} \to \Upsilon$ such that $\Theta\mathfrak{bo}^{(2)} \to \Theta\mathfrak{bt}^{(2)} \to \Upsilon$ is a rational equivalence. We may as well assume that the homotopy groups of Υ are rational vector spaces. Given the shallow nature of the actions of O on $\Theta\mathfrak{bo}^{(2)}$ and $\Theta\mathfrak{bt}^{(2)}$, as explained in remark 5.6, it is easy to promote Υ to a fibered spectrum over $B\mathrm{O}$. Furthermore since $\pi_*\Upsilon$ is rational, this also promotes Υ to a fibered spectrum over $B\mathrm{TOP}$ and we now have a diagram of fibered spectra over $B\mathrm{TOP}$,

$$\Theta\mathfrak{bo}^{(2)}_\bullet \to \Theta\mathfrak{bt}^{(2)}_\bullet \to \Upsilon_\bullet\ .$$

We now wish to define $\kappa_2 : T_1\mathfrak{d} \to \mathfrak{f}[\Upsilon_\bullet]$ so as to make the following diagram commutative:

$$\begin{array}{ccc} T_1\mathfrak{bt} & \longrightarrow & T_1\mathfrak{d} \\ \downarrow{\scriptstyle \kappa_2} & & \vdots{\scriptstyle \kappa_2} \\ \mathfrak{f}[\Theta\mathfrak{bt}^{(2)}_\bullet] & \longrightarrow & \mathfrak{f}[\Upsilon_\bullet] \end{array} \tag{5.6}$$

(The horizontal maps are obvious.) The first thing to notice in the above is that all the functors in the commutative diagram 5.6 are over $T_0\mathfrak{bt}$, which is rationally the same as $T_0\mathfrak{bo}$. This, of course, we can regard as a space $T_0\mathfrak{bo}(0)$, since $T_0\mathfrak{bo}$ is essentially a constant functor. We now pick a point $x \in T_0\mathfrak{bo}(0)$ and solve the factorization problem over this point. Namely, we have a corresponding diagram of homotopy fiber functors

$$\begin{array}{ccc} (L_1\mathfrak{bt})_x & \xrightarrow{r_1} & (L_1\mathfrak{d})_x \\ \downarrow{\scriptstyle \kappa_2} & & \vdots{\scriptstyle \kappa_2} \\ \mathfrak{f}[\Theta\mathfrak{bt}^{(2)}_x] & \xrightarrow{r_2} & \mathfrak{f}[\Upsilon_x]\ . \end{array} \tag{5.7}$$

The functors in the lower row are determined by the spectra $\Theta\mathfrak{bt}^{(2)}_x$ and Υ_x with action of $\mathrm{O}(2)$, viewed as fibered spectra over a point. Taking $\mathfrak{a}_1 = (L_1\mathfrak{bt})_x$, $\mathfrak{b} = (L_1\mathfrak{d})_x$ and $\mathfrak{a}_2 = \mathfrak{f}[\Upsilon_x]$ we are in the situation of example 4.7. Therefore the

map $\kappa_2 : (L_1\mathfrak{d})_x \to \mathfrak{f}[\Upsilon_x]$ exists and, together with a homotopy from $r_2\kappa_2$ to $\kappa_2 r_1$ in (5.7), is unique up to contractible choice. This establishes the existence and uniqueness of κ_2 in diagram (5.6). □

Proof. (of prop 5.3) This proceeds mostly like the proof of proposition 5.4. The construction of $T_1\mathfrak{d}$ as the homotopy pullback of $T_0\mathfrak{bt} \to T_0\mathfrak{bg} \leftarrow T_1\mathfrak{bg}$ automatically gives us a functor over $T_1\mathfrak{bg}$. The main difference arises when we wish to define $\kappa_2 : T_1\mathfrak{d} \to \mathfrak{f}[\Upsilon_\bullet]$ so as to make the following diagram commutative with preferred homotopies:

(5.8)
$$\begin{array}{ccccc} T_1\mathfrak{bt} & \longrightarrow & T_1\mathfrak{d} & \longrightarrow & T_1\mathfrak{bg} \\ \downarrow{\scriptstyle \kappa_2} & & \downarrow{\scriptstyle \kappa_2} & & \downarrow{\scriptstyle \kappa_2} \\ \mathfrak{f}[\Theta\mathfrak{bt}^{(2)}_\bullet] & \longrightarrow & \mathfrak{f}[\Upsilon_\bullet] & \longrightarrow & \mathfrak{f}[\Theta\mathfrak{bg}^{(2)}_\bullet] \end{array}$$

Notice that the outer rectangle is already commutative by our hypothesis. Again we take $x \in T_0\mathfrak{bo}(0)$ with image $y \in T_0\mathfrak{bg}(0)$ and form the corresponding homotopy fibers:

(5.9)
$$\begin{array}{ccccc} (L_1\mathfrak{bt})_x & \xrightarrow{r_1} & (L_1\mathfrak{d})_x & \xrightarrow{s_1} & (L_1\mathfrak{bg})_y \\ \downarrow{\scriptstyle \kappa_2} & & \downarrow{\scriptstyle \kappa_2} & & \downarrow{\scriptstyle \kappa_2} \\ \mathfrak{f}[\Theta\mathfrak{bt}^{(2)}_x] & \xrightarrow{r_2} & \mathfrak{f}[\Upsilon_x] & \xrightarrow{s_2} & \mathfrak{f}[\Theta\mathfrak{bg}^{(2)}_y] \end{array}$$

We then determine the broken arrow as in the previous proof to make the left-hand square commute with preferred homotopy. Now the right-hand square is also commutative with preferred homotopy because $s_2\kappa_2$ and $\kappa_2 s_1$ are both solutions to the problem of factoring $s_2 r_2 \kappa_2$ through r_1. Such solutions are again unique by example 4.7. □

Proposition 5.9. *Hypothesis* C^m *implies hypothesis* A^m *.*

Proof. We start by reformulating hypothesis A^m as a statement about $B\mathrm{TOP}(2m)$ instead of $BS\mathrm{TOP}(2m)$. The Euler class e in $H^{2m}(BS\mathrm{TOP}(2m); \mathbb{Q})$ comes from an Euler class e_t in $H^{2m}(B\mathrm{TOP}(2m); \mathbb{Q}^t)$ where $\mathbb{Q}^t$ is a "twisted" local coefficient system, the twist being determined by the first Stiefel-Whitney class of the universal euclidean bundle on $B\mathrm{TOP}(2m)$. The (hypothetical) equation

$$e_t^2 = p_m \in H^{4m}(B\mathrm{TOP}(2m); \mathbb{Q})$$

is equivalent to $e^2 = p_m \in H^{4m}(BS\mathrm{TOP}(2m); \mathbb{Q})$.
By proposition 5.4, we have a functor splitting $\mathfrak{bo} \to \mathfrak{bt} \to \mathfrak{d}$ such that the composition $\mathfrak{bo} \to \mathfrak{d}$ is a rational homotopy equivalence. Therefore, for $2m$-dimensional V, we can speak of the Euler class e_t in $H^{2m}(\mathfrak{d}(V); \mathbb{Q}^t)$ and the Pontryagin class $p_m \in H^{4m}(\mathfrak{d}(V); \mathbb{Q})$. For these we have $e_t^2 = p_m$. It is therefore enough to show that under the map $\mathfrak{bt} \to \mathfrak{d}$, the class $p_m \in H^{4m}(\mathfrak{d}(V); \mathbb{Q})$ pulls back to the Pontryagin class in $H^{4m}(\mathfrak{bt}(V); \mathbb{Q})$ and the class $e_t \in H^{2m}(\mathfrak{d}(V); \mathbb{Q}^t)$ pulls back to the twisted Euler class in $H^{2m}(\mathfrak{bt}(V); \mathbb{Q}^t)$. For the Pontryagin classes this follows from

the commutativity of the diagram

$$\begin{array}{ccc} \mathfrak{bt}(V) & \longrightarrow & T_0\mathfrak{bt}(V) \\ \downarrow & & \downarrow \simeq_{\mathbb{Q}} \\ \mathfrak{d}(V) & \longrightarrow & T_0\mathfrak{d}(V)\ . \end{array}$$

(The Pontryagin classes come from the right-hand column.) For the Euler classes it follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathfrak{bt}(V) & \longrightarrow & T_1\mathfrak{bt}(V) \\ \downarrow & & \downarrow \simeq_{\mathbb{Q}} \\ \mathfrak{d}(V) & \longrightarrow & T_1\mathfrak{d}(V)\ . \end{array}$$

(The Euler classes come from the right-hand column.) The right-hand vertical arrow in the last diagram is a rational homotopy equivalence by proposition 3.8, since V is even dimensional. □

Proposition 5.10. *Hypothesis* C^s *implies hypothesis* A^s *.*

Proof. Suppose hypothesis C^s holds. By proposition 5.3, we have a functor splitting $\mathfrak{bo} \to \mathfrak{bt} \to \mathfrak{d}$ over $\mathfrak{bg}$ such that the composition $\mathfrak{bo} \to \mathfrak{d}$ is a rational homotopy equivalence. Therefore, for $2m$-dimensional V, we can speak of the Euler class e_t in $H^{2m}(\mathfrak{d}(V); \mathbb{Q}^t)$ and the Pontryagin class $p_m \in H^{4m}(\mathfrak{d}(V); \mathbb{Q})$. For these we have $e_t^2 = p_m$. Let $q_m \in H^{4m}(\mathfrak{bg}(V \oplus \mathbb{R}); \mathbb{Q})$ be the unique class which extends the class $e_t^2 \in H^{4m}(\mathfrak{bg}(V); \mathbb{Q})$. It is then clear that q_m also extends $p_m \in H^{4m}(\mathfrak{d}(V); \mathbb{Q})$. It is therefore enough to show that under the map $\mathfrak{bt} \to \mathfrak{d}$, the class $p_m \in H^{4m}(\mathfrak{d}(V); \mathbb{Q})$ pulls back to the Pontryagin class in $H^{4m}(\mathfrak{bt}(V); \mathbb{Q})$. This follows as in the proof of proposition 5.9. □

6. Orthogonal calculus and smoothing theory

In this section we turn our attention to the relation between the B and C hypotheses. Smoothing theory allows us to reformulate the B hypotheses so that they fit into the orthogonal calculus framework.

Remark 6.1. Let Γ be a compact Lie group. We are interested in spaces and spectra with an action of the group Γ. Traditionally there are two frameworks for this. In the "naive" framework, objects are ordinary spaces or spectra with an action of Γ and a Γ-map between two such spaces or spectra is considered an equivalence if it is an ordinary (weak) homotopy equivalence. In particular any Γ-space X is considered equivalent to $E\Gamma \times X$, which comes with a free action of Γ. In the equivariant framework, a Γ-space is still a space with an action of Γ, but a Γ-map $X \to Y$ is not considered an equivalence unless it induces (ordinary) equivalences $X^H \to Y^H$ of the fixed point spaces for closed subgroups $H \leq \Gamma$. Furthermore a Γ-spectrum Θ in the equivariant framework is a family of based spaces Θ_V indexed by (selected) finite dimensional real representations of Γ and the structure maps $V^c \wedge \Theta_W \longrightarrow \Theta_{V \oplus W}$ are Γ-maps. There is an elaborate theory of fixed point spectra corresponding to subgroups $H \leq \Gamma$. See e.g. [1, 13].

For orthogonal calculus purposes, the naive setting is the right one. Nevertheless, most examples of Γ-spectra which arise in orthogonal calculus have some features

reminiscent of the equivariant setting which we strive to suppress. The following questions arise frequently:

(i) Given a fixed representation V of Γ, a sequence of based Γ-spaces $(X_{nV})_{n\in\mathbb{N}}$ and based Γ-maps $V^c \wedge X_{nV} \to X_{(n+1)V}$ (with the diagonal action on the source) can we build a naive Γ-spectrum $\underline{X}$ from these data? Also, given two such sequences $(X_{nV})_{n\in\mathbb{N}}$ and $(Y_{nV})_{n\in\mathbb{N}}$, and compatible based Γ-maps $f_n : X_{nV} \to Y_{nV}$, can we build a Γ-map $f : \underline{X} \to \underline{Y}$?

(ii) In these circumstances, can the Γ-map f be recovered if we only know the based Γ-maps $\Omega^{nW} f_n : \Omega^{nW} X_{nV} \to \Omega^{nW} Y_{nV}$ for all n, where W is another representation of Γ ? We are willing to assume that the Y_{nV} are Eilenberg-MacLane spaces and the adjoints of the maps $V^c \wedge Y_{nV} \to Y_{(n+1)V}$ are homotopy equivalences $Y_{nV} \to \Omega^V Y_{(n+1)V}$.

The following propositions try to answer these questions.

Proposition 6.2. *Given a fixed representation V of Γ, a sequence of based Γ-spaces $(X_{nV})_{n\in\mathbb{N}}$ and based Γ-maps $V^c \wedge X_{nV} \to X_{(n+1)V}$, the spaces*

$$\operatorname*{hocolim}_{n\to\infty} \Omega^{nV}(S^m \wedge X_{nV})$$

(for $m \geq 0$) form an Ω-spectrum $\underline{X}$.

Proof. There are obvious structure maps

$$\operatorname*{hocolim}_{n\to\infty} \Omega^{nV}(S^m \wedge X_{nV}) \to \Omega\big(\operatorname*{hocolim}_{n\to\infty} \Omega^{nV}(S^{m+1} \wedge X_{nV})\big) \tag{6.1}$$

We need to show that these are weak homotopy equivalences. Suppose to begin with that there exists n_0 such that, for all $n \geq n_0$, the Γ-map $V^c \wedge X_{nV} \to X_{(n+1)V}$ is a homeomorphism. Then X_{nV} is $(\dim(nV) - k)$-connected for a constant k independent of n. It follows by Freudenthal's theorem that the maps (6.1) are weak homotopy equivalences, since they can be written in the form

$$\operatorname*{hocolim}_{n\to\infty} \Omega^{nV}(S^m \wedge X_{nV}) \to \operatorname*{hocolim}_{n\to\infty} \Omega^{nV}\left(\Omega(S^{m+1} \wedge X_{nV})\right).$$

The general case follows from this special case by a direct limit argument. □

Proposition 6.3. *Keeping the notation of proposition 6.2, let W be another representation of Γ and assume Γ is connected. If $d = \dim(V)$ is greater than $e = \dim(W)$, the canonical homomorphisms*

$$g^n : H^k(\underline{X}_{\mathrm{h}\Gamma}; \mathbb{Q}) \to H^{k+nd-en}((\Omega^{nW} X_{nV})_{\mathrm{rh}\Gamma}, \star; \mathbb{Q}) \ ,$$

induce an injection

$$g : H^k(\underline{X}_{\mathrm{h}\Gamma}; \mathbb{Q}) \to \prod_n H^{k+nd-en}((\Omega^{nW} X_{nV})_{\mathrm{rh}\Gamma}, \star; \mathbb{Q}) \ .$$

Proof. By the universal coefficient theorem in rational homology, it is enough to show that the corresponding homology homomorphisms g_n produce a surjection

$$\bigoplus_n H_{k+nd-en}((\Omega^{nW} X_{nV})_{\mathrm{rh}\Gamma}, \star; \mathbb{Q}) \longrightarrow H_k(\underline{X}_{\mathrm{h}\Gamma}; \mathbb{Q}) \ .$$

We note that $\oplus_n g_n$ is defined by the composition

$$\begin{array}{c} \bigoplus_n H_{k+nd-ne}((\Omega^{nW}X_{nV})_{\mathrm{rh}\Gamma},\star;\mathbb{Q}) \\ \downarrow \\ \bigoplus_n H_{k+nd}((S^{nW}\wedge\Omega^{nW}X_{nV})_{\mathrm{rh}\Gamma},\star;\mathbb{Q}) \\ \downarrow \\ \operatorname*{colim}_n\ H_{k+nd}((X_{nV})_{\mathrm{rh}\Gamma},\star;\mathbb{Q}) \cong H_k(\underline{X}_{\mathrm{h}\Gamma};\mathbb{Q})\ , \end{array} \tag{6.2}$$

where the top arrow is the suspension isomorphism and the bottom one is induced by the obvious maps $S^{nW}\wedge\Omega^{nW}X_{nV}\to X_{nV}$. Now we need to show the bottom arrow is onto. By a direct limit argument (as in the previous proposition) we can reduce this to the case where X_{nV} is $(nd-p)$-connected for a constant p independent of n. It then follows that the map

$$S^{nW}\wedge\Omega^{nW}X_{nV}\to X_{nV}$$

is $(2nd-ne-q)$-connected for some constant q independent of n. For large enough n we have $2nd-ne-q=n(d-e)-q+nd>k+nd$, so that the n-th summand in the middle term of diagram (6.2) maps onto $H_{k+nd}((X_{nV})_{\mathrm{rh}\Gamma},\star;\mathbb{Q})$. □

Remark 6.4. In proposition 6.3, we can interpret an element f of $H^k(\underline{X}_{\mathrm{h}\Gamma};\mathbb{Q})$ as a homotopy class of (weak) Γ-maps from $\underline{X}$ to $\underline{Y}\simeq S^k\wedge H\mathbb{Q}$. We may also assume that $\underline{Y}$ is constructed from based Γ-spaces Y_{nV} and Γ-maps

$$V^c\wedge Y_{nV}\to Y_{(n+1)V}$$

whose adjoints are homotopy equivalences $Y_{nV}\to\Omega^V Y_{(n+1)V}$. The image of f in $H^{k+nd-ne}((\Omega^{nW}X_{nV})_{\mathrm{rh}\Gamma},\star;\mathbb{Q})$ is the map $\Omega^{nW}f_n$ from $\Omega^{nW}X_{nV}$ to $\Omega^{nW}Y_{nV}$ induced by f. The content of the lemma is that f is determined by these images $\Omega^{nW}f_n$. This gives an affirmative answer to the question in (ii) of remark 6.1.

Remark 6.5. The proof of proposition 6.3 proves more than what is stated. In fact, given any infinite subset S of the natural numbers, the composition of the injection g with the projection

$$\prod_{n\in\mathbb{N}} H^{k+nd-en}((\Omega^{nW}X_{nV})_{\mathrm{rh}\Gamma},\star;\mathbb{Q})\longrightarrow\prod_{n\in S} H^{k+nd-en}((\Omega^{nW}X_{nV})_{\mathrm{rh}\Gamma},\star;\mathbb{Q})$$

is still an injection.

Proposition 6.6. *Hypotheses* $\mathrm{B}^{m|e}$ *implies hypothesis* C^m *.*

Proof. Let $\Theta_{\mathfrak{bo}}=\Theta\mathfrak{bo}^{(2)}$, $\Theta_{\mathfrak{bt}}=\Theta\mathfrak{bt}^{(2)}$ and $\Theta_{\mathfrak{bo}\to\mathfrak{bt}}=\mathrm{hofiber}[\Theta_{\mathfrak{bo}}\to\Theta_{\mathfrak{bt}}]$. We need to show that the forgetful map $\Theta_{\mathfrak{bo}\to\mathfrak{bt}}\to\Theta_{\mathfrak{bo}}$ is rationally nullhomotopic with weak O(2)-invariance. As $\Theta_{\mathfrak{bo}}$ is rationally an Eilenberg-MacLane spectrum concentrated in dimension -1 with trivial O(2)-action, this amounts to showing that a class δ in the spectrum cohomology $H^{-1}((\Theta_{\mathfrak{bo}\to\mathfrak{bt}})_{\mathrm{h}O(2)};\mathbb{Q})$ vanishes. A transfer argument shows that the restriction induced homomorphism

$$H^{-1}((\Theta_{\mathfrak{bo}\to\mathfrak{bt}})_{\mathrm{h}O(2)};\mathbb{Q})\longrightarrow H^{-1}((\Theta_{\mathfrak{bo}\to\mathfrak{bt}})_{\mathrm{h}S^1};\mathbb{Q})$$

is injective. Therefore we only have to show that δ is zero in $H^{-1}((\Theta_{\mathfrak{bo}\to\mathfrak{bt}})_{\mathrm{h}S^1};\mathbb{Q})$, or equivalently that the forgetful map $\Theta_{\mathfrak{bo}\to\mathfrak{bt}}\to\Theta_{\mathfrak{bo}}$ is rationally nullhomotopic with weak S^1-invariance. This is equivalent to showing that the forgetful map

$$\Omega^V\Theta_{\mathfrak{bo}\to\mathfrak{bt}}\to\Omega^V\Theta_{\mathfrak{bo}}$$

is rationally nullhomotopic with weak S^1-invariance, where V is the standard 2-dimensional representation of S^1. Again this is equivalent to the vanishing of a cohomology class $\eta \in H^{-3}((\Omega^V \Theta_{\mathfrak{bo}\to\mathfrak{bt}})_{\mathrm{h}S^1}; \mathbb{Q})$. By proposition 6.3 this will follow from showing that the cohomology classes $\eta_n \in H^{-3+2n-n}((\Omega^n X_{nV})_{\mathrm{rh}\Gamma}, \star; \mathbb{Q})$ determined by η are zero for all even n, where

$$X_{nV} = \Omega^V \mathrm{hofiber}[\mathfrak{bo}^{(2)}(\mathbb{R}^n) \to \mathfrak{bt}^{(2)}(\mathbb{R}^n)] \ .$$

To show this we use the commutative diagram of S^1-spaces

$$\begin{array}{ccc}
\Omega^n\Omega^V \mathrm{hofiber}[\mathfrak{bo}^{(2)}(\mathbb{R}^n) \to \mathfrak{bt}^{(2)}(\mathbb{R}^n)] & \xrightarrow{\eta_n} & \Omega^n\Omega^V \mathfrak{bo}^{(2)}(\mathbb{R}^n) \\
\downarrow & & \downarrow{\scriptstyle \tau} \\
\Omega^n\Omega^V \mathrm{hofiber}\left[\frac{\mathrm{O}(n+2)}{\mathrm{O}(n)} \to \frac{\mathrm{TOP}(n+2)}{\mathrm{TOP}(n)}\right] & \longrightarrow & \Omega^n\Omega^V \frac{\mathrm{O}(n+2)}{\mathrm{O}(n)} \\
\downarrow{\scriptstyle \simeq} & & \uparrow \\
\mathscr{R}(n,2) & \xrightarrow{\nabla} & \mathscr{V} \ .
\end{array}$$

The map τ is a rational homotopy equivalence on base point components. (Indeed, there is a homotopy fiber sequence

$$\mathfrak{bo}^{(2)}(\mathbb{R}^n) \xrightarrow{\tau} \frac{\mathrm{O}(n+2)}{\mathrm{O}(n)} \longrightarrow \Gamma(E \to \mathbb{R}P^1) \ ,$$

where $E \to \mathbb{R}P^1$ is the fiber bundle with fiber given by $E_L = \mathrm{O}(\mathbb{R}^{n+2})/\mathrm{O}(\mathbb{R}^n \oplus L)$. It is easy to check that the base point component of $\Omega^n\Omega^V\Gamma(E \to \mathbb{R}P^1)$ is rationally contractible.) Therefore by hypothesis $\mathrm{B}^{m|e}$, all these classes η_n are zero. □

Proposition 6.7. *Hypotheses* $\mathrm{B}^{m|o}$ *implies hypothesis* C^m.

Proof. As in the previous proof, this is equivalent to the vanishing of a cohomology class $\eta \in H^{-3}((\Omega^V \Theta_{\mathfrak{bo}\to\mathfrak{bt}})_{\mathrm{h}S^1}; \mathbb{Q})$. By proposition 6.3 this will follow from showing that the cohomology classes $\eta_n \in H^{-3+2n-n}((\Omega^n X_{nV})_{\mathrm{rh}\Gamma}, \star; \mathbb{Q})$ determined by η are zero for all odd n, where

$$X_{nV} = \Omega^V \mathrm{hofiber}[\mathfrak{bo}^{(2)}(\mathbb{R}^n) \to \mathfrak{bt}^{(2)}(\mathbb{R}^n)] \ .$$

We now use slightly unusual descriptions of $\mathrm{hofiber}[\mathfrak{bo}^{(2)}(\mathbb{R}^n) \to \mathfrak{bt}^{(2)}(\mathbb{R}^n)]$ and $\mathfrak{bo}^{(2)}(\mathbb{R}^n)$ given by

$$\mathfrak{bo}^{(2)}(\mathbb{R}^n) \simeq \operatorname*{holim}_{0\neq U\leq\mathbb{R}^2} \frac{\mathrm{O}(\mathbb{R}^n \oplus U)}{\mathrm{O}(\mathbb{R}^n)} \ ,$$

$$\mathrm{hofiber}[\mathfrak{bo}^{(2)}(\mathbb{R}^n) \to \mathfrak{bt}^{(2)}(\mathbb{R}^n)] \simeq \operatorname*{holim}_{0\neq U\leq\mathbb{R}^2} \mathrm{hofiber}\left[\frac{\mathrm{O}(\mathbb{R}^n \oplus U))}{\mathrm{O}(\mathbb{R}^n)} \to \frac{\mathrm{TOP}(\mathbb{R}^n \oplus U))}{\mathrm{TOP}(\mathbb{R}^n)}\right] .$$

The vanishing of the classes η_n will therefore follow if, in the commutative square

$$\begin{array}{ccc}
\Omega^n\Omega^V \mathrm{hofiber}\left[\frac{\mathrm{O}(n+1)}{\mathrm{O}(n)} \to \frac{\mathrm{TOP}(n+1)}{\mathrm{TOP}(n)}\right] & \xrightarrow{f} & \Omega^n\Omega^V \frac{\mathrm{O}(n+1)}{\mathrm{O}(n)} \\
\downarrow & & \downarrow \\
\Omega^n\Omega^V \mathrm{hofiber}\left[\frac{\mathrm{O}(n+2)}{\mathrm{O}(n)} \to \frac{\mathrm{TOP}(n+2)}{\mathrm{TOP}(n)}\right] & \xrightarrow{g} & \Omega^n\Omega^V \frac{\mathrm{O}(n+2)}{\mathrm{O}(n)} \ ,
\end{array}$$

we can find compatible rational nullhomotopies for f and g, with weak $\mathrm{O}(1)$-invariance in the case of f and weak $\mathrm{O}(2)$-invariance in the case of g. Because in the upper right-hand term the base point component is rationally contractible and in the lower right-hand term the base point component is rationally an Eilenberg-MacLane space, finding these compatible nullhomotopies amounts to showing that a certain rational cohomology class vanishes. A transfer argument then implies that we can weaken the invariance requirements so that the groups are $S\mathrm{O}(2)$ in the case of g and $S\mathrm{O}(1)$ in the case of f. But then, by smoothing theory, we have exactly the content of hypothesis $\mathrm{B}^{m|o}$. □

Proposition 6.8. *Hypothesis C^m implies hypotheses $B^{m|e}$ and $B^{m|o}$.*

Proof. By proposition 5.4 we obtain a (rational, weak) splitting for the inclusion $\mathfrak{bo} \to \mathfrak{bt}$. It follows that, for even $n \geq 2$, the inclusion

$$\Omega^{n+2}(\mathrm{O}(n+2)/\mathrm{O}(n)) \longrightarrow \Omega^{n+2}(\mathrm{TOP}(n+2)/\mathrm{TOP}(n)) \tag{6.3}$$

admits a rational splitting with weak S^1-invariance. Consequently, the map from the homotopy fiber of (6.3) to the source is rationally nullhomotopic with weak S^1-invariance. But this is precisely the smoothing theory model for $\nabla\colon \mathscr{R} \to \mathscr{V}$. The proof for odd n is similar.

□

Proposition 6.9. *Hypothesis $B^{s|o}$ implies hypothesis C^s.*

Proof. We reason as in the proof of lemma 6.6. Let $\Theta_{\mathfrak{bo}\to\mathfrak{bg}}$ be the second derivative spectrum of $U \mapsto \mathrm{hofiber}[\mathfrak{bo}(U) \to \mathfrak{bg}(U)]$. We have to show that the map of spectra

$$\delta\colon \Theta_{\mathfrak{bo}\to\mathfrak{bt}} \longrightarrow \Theta_{\mathfrak{bo}\to\mathfrak{bg}}$$

induced by $\mathfrak{bt} \to \mathfrak{bg}$ is rationally nullhomotopic with weak $\mathrm{O}(2)$-invariance. This time $\Theta_{\mathfrak{bo}\to\mathfrak{bg}}$ is again (rationally) an Eilenberg-Mac Lane spectrum $\Omega^3\underline{H}\mathbb{Q}$. As before, it is enough to show that δ is rationally nullhomotopic with weak S^1-invariance. This is equivalent to showing that the forgetful map

$$\Omega^V\Theta_{\mathfrak{bo}\to\mathfrak{bt}} \to \Omega^V\Theta_{\mathfrak{bo}\to\mathfrak{bg}}$$

is rationally nullhomotopic with weak S^1-invariance, where V is the standard 2-dimensional representation of S^1. Again this is equivalent to the vanishing of a cohomology class $\eta \in H^{-5}((\Omega^V\Theta_{\mathfrak{bo}\to\mathfrak{bt}})_{\mathrm{h}S^1};\mathbb{Q})$. By proposition 6.3 this will follow from showing that the cohomology classes $\eta_n \in H^{-5+2n-n}((\Omega^n X_{nV})_{\mathrm{rh}S^1},\star;\mathbb{Q})$ determined by η are zero for all odd n, where

$$X_{nV} = \Omega^V\mathrm{hofiber}[\mathfrak{bo}^{(2)}(\mathbb{R}^n) \to \mathfrak{bt}^{(2)}(\mathbb{R}^n)]\ .$$

By remark 2.6 and the homotopy pullback square in lemma 6.10 below, we find that η_n sits in a commutative diagram of S^1-spaces

$$\begin{array}{ccc} \Omega^n X_{nV} & \xrightarrow{\eta_n} & \mathrm{hofiber}\left[\mathfrak{bo}^{(2)}(\mathbb{R}^n) \to \mathfrak{bg}^{(2)}(\mathbb{R}^n)\right] \\ \downarrow & & \downarrow{\scriptstyle \simeq_{\mathbb{Q}}} \\ \mathscr{R}(n,2) & \xrightarrow{\nabla} & \mathrm{hofiber}\left[\mathscr{V}(n,2) \to \mathscr{V}_{\mathrm{G}}(n,2)\right] \end{array}$$

By hypothesis $\mathrm{B}^{s|o}$, all these classes η_n are zero. □

Lemma 6.10. *For odd n, the following is a rational homotopy pullback square:*

$$\begin{array}{ccccc}
\mathfrak{bo}^{(2)}(\mathbb{R}^n) & & \longrightarrow & & \mathfrak{bg}^{(2)}(\mathbb{R}^n) \\
\Big\downarrow{\scriptstyle \text{proj}} & & & & \Big\downarrow{\scriptstyle \text{proj}} \\
\text{hofiber}\,[\mathfrak{bo}(\mathbb{R}^n) & \longrightarrow & \mathfrak{bo}(\mathbb{R}^{n+2})] & \longrightarrow & \text{hofiber}\,[\mathfrak{bg}(\mathbb{R}^n) \longrightarrow \mathfrak{bg}(\mathbb{R}^{n+2})]\ .
\end{array}$$

Proof. The homotopy fiber of the left hand vertical map is

$$\text{hofiber}\begin{bmatrix} \text{hofiber}[\ \mathfrak{bo}(\mathbb{R}^n) \longrightarrow \operatorname*{holim}_{0\neq U\leq\mathbb{R}^2} \mathfrak{bo}(\mathbb{R}^n\oplus U)\ ] \\ \Big\downarrow \\ \text{hofiber}[\ \mathfrak{bo}(\mathbb{R}^n) \longrightarrow \operatorname*{holim}_{0\neq U\leq\mathbb{R}^2} \mathfrak{bo}(\mathbb{R}^n\oplus \mathbb{R}^2)\ ] \end{bmatrix}$$

which simplifies to

$$\Omega\,\text{hofiber}\begin{bmatrix} \operatorname*{holim}_{0\neq U\leq\mathbb{R}^2} \mathfrak{bo}(\mathbb{R}^n\oplus U) \\ \Big\downarrow \\ \operatorname*{holim}_{0\neq U\leq\mathbb{R}^2} \mathfrak{bo}(\mathbb{R}^n\oplus \mathbb{R}^2) \end{bmatrix}.$$

By a Fubini argument this simplifies to

$$\Omega \operatorname*{holim}_{0\neq U\neq\mathbb{R}^2} \text{hofiber}[\mathfrak{bo}(\mathbb{R}^n\oplus U)\to\mathfrak{bo}(\mathbb{R}^n\oplus\mathbb{R}^2)].$$

The homotopy limit here is just the space of sections of a fiber bundle on $\mathbb{R}P^1$ whose fiber over $U\in\mathbb{R}P^1$ is $\mathrm{O}(\mathbb{R}^n\oplus\mathbb{R}^2)/\mathrm{O}(\mathbb{R}^n\oplus U)$. An analogous calculation gives

$$\Omega \operatorname*{holim}_{0\neq U\neq\mathbb{R}^2} \text{hofiber}[\mathfrak{bg}(\mathbb{R}^n\oplus U)\to\mathfrak{bg}(\mathbb{R}^n\oplus\mathbb{R}^2)]$$

for the right-hand side vertical homotopy fiber. Now it is sufficient to show that

$$\begin{array}{ccc}
\mathfrak{bo}(\mathbb{R}^{n+1}) & \longrightarrow & \mathfrak{bo}(\mathbb{R}^{n+2}) \\
\Big\downarrow & & \Big\downarrow \\
\mathfrak{bg}(\mathbb{R}^{n+1}) & \longrightarrow & \mathfrak{bg}(\mathbb{R}^{n+2})
\end{array}$$

is a rational homotopy pullback square. This follows easily, e.g. by looking at the vertical homotopy fibers, from our rational calculations of $S\mathrm{O}(k)$ and $S\mathrm{G}(k)$. □

Proposition 6.11. *Hypothesis C^s implies hypothesis $B^{s|o}$.*

Proof. This comes from smoothing theory, similar to the proof of proposition 6.8. Alternatively, we already know that hypothesis C^s implies hypothesis A^s (proposition 5.10), which in turn implies hypothesis $\mathrm{B}^{s|o}$ (proposition 2.11). □

Proposition 6.12. *Hypothesis C^s implies hypothesis $B^{s|e}$.*

Proof. This follows from the same smoothing theory argument used in the previous proof. □y

Proposition 6.13. *Hypothesis $B^{s|e}$ implies hypothesis C^s.*

Proof. Reasoning as in the proof of proposition 6.9, we need to show that the cohomology classes $\eta_n \in H^{-5+2n-n}((\Omega^n X_{nV})_{\mathrm{rh}\Gamma}, \star; \mathbb{Q})$ determined by η are zero for all *even* n, where

$$X_{nV} = \Omega^V \mathrm{hofiber}[\mathfrak{bo}^{(2)}(\mathbb{R}^n) \to \mathfrak{bt}^{(2)}(\mathbb{R}^n)] \ .$$

As in the proof of proposition 6.7, the classes η_n will vanish if, in the commutative square

$$\begin{array}{ccc} \Omega^n\Omega^V \mathrm{hofiber}\left[\frac{\mathrm{O}(n+1)}{\mathrm{O}(n)} \to \frac{\mathrm{TOP}(n+1)}{\mathrm{TOP}(n)}\right] & \xrightarrow{\ f\ } & \Omega^n\Omega^V \mathrm{hofiber}\left[\frac{\mathrm{O}(n+1)}{\mathrm{O}(n)} \to \frac{\mathrm{G}(n+1)}{\mathrm{G}(n)}\right] \\ \downarrow & & \downarrow \\ \Omega^n\Omega^V \mathrm{hofiber}\left[\frac{\mathrm{O}(n+2)}{\mathrm{O}(n)} \to \frac{\mathrm{TOP}(n+2)}{\mathrm{TOP}(n)}\right] & \xrightarrow{\ g\ } & \Omega^n\Omega^V \mathrm{hofiber}\left[\frac{\mathrm{O}(n+2)}{\mathrm{O}(n)} \to \frac{\mathrm{G}(n+2)}{\mathrm{G}(n)}\right] \end{array} \ ,$$

we can find compatible rational nullhomotopies for f and g, with weak $\mathrm{O}(1)$-invariance in the case of f and weak $\mathrm{O}(2)$-invariance in the case of g. The upper right-hand term has a rationally contractible base point component and in the lower right-hand term the base point component is rationally an Eilenberg-MacLane space. We can therefore weaken the invariance requirements so that the groups are $S\mathrm{O}(2)$ in the case of g and $S\mathrm{O}(1)$ in the case of f. By smoothing theory, we have exactly the content of hypothesis $\mathrm{B}^{s|e}$. □

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