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COBORDISM GROUPS OF IMMERSIONS

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§ I.

THE general problem of describing the category of differential manifolds and maps is extremely complicated. Thom has introduced the technique of cobordism to simplify this problem in much the same way that the techniques of homotopy simplify the problem of describing the structure of a topological space. The family of immersions of differential manifolds in Euclidean spaces is still a very complicated part of the above category. In this paper we apply Thom's technique to obtain a rough description of that family. As often happens, the result is described in homotopy terms; just as the cobordism groups of embeddings turn out to be the homotopy groups of the Thom spaces of the classifying vector bundles, the cobordism groups of immersions turn out to be the reduced stable homotopy groups of those spaces. This identification is made by relying heavily on Hirsch's theorems describing immersions.

Let R be the real numbers, $e_1, e_2, e_3, \ldots$ the standard basis vectors of R^∞ and $\pi_j : R^\infty \to R$ the jth co-ordinate projection. Also, G_{rs} is the r-planes through 0 in R^{r+s} and ξ_{rs} the canonical r-plane bundle over G_{rs}.

Leaving details aside for the moment, we will say roughly that the sum of two immersions f and g of closed k-manifolds M and N respectively is $f \sqcup g$ if $M \sqcap N = \varnothing$ and undefined otherwise. The two immersions will be said to be cobordant if there is a manifold X such that ∂X is the disjoint union of M and N, and if there is an immersion $F: X \to \pi_{n+k+1}^{-1}[0, 1]$, transverse regular along $\pi_{n+k+1}^{-1}(\{0\} \sqcup \{1\})$, such that $F \mid M = f$ and $F(x) = g(x) + e_{n+k+1}$ if $x \in N$. The relation of cobordism turns out to be an equivalence relation and the immersions of closed k-manifolds in R^{n+k} form an abelian group $\mathcal{N}_k(n)$ modulo cobordism. The bigraded group $\Sigma_{pq}\mathcal{N}_p(q)$ has a natural structure of a bigraded ring, commutative with respect to the total degree $p + q$. Then,

THEOREM 5. *$\Sigma_{pq}\mathcal{N}_p(q) \otimes Z_0$ is a bigraded polynomial algebra with generators $f_0, f_1, f_2, \ldots$ such that $f_i \in \mathcal{N}_{2+4i}(2)$.*

Thus there is a real difference between cobordism of immersions and the usual cobordism, since in the latter case all elements have order 2—we make no orientation assumptions here. A further difference is in the appearance of odd torsion,

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THEOREM 6. *For p prime and odd and* $m \geqq \kappa$, *where* $\kappa = (p-1)/2$, $\mathcal{N}_i(2m)$ *has no p-torsion for* $i < 2m + 2p^2 - 2p - 1$ *and does have p-torsion for* $i = 2m + 2p^2 - 2p - 1$.

The case of odd codimension is more like the usual case,

THEOREM 4. $\mathcal{N}_k(2n+1)$ *is 2-primary.*

For $k \leqq 4$, the groups $\mathcal{N}_k(n)$ are the following, with some domains of generating immersions indicated:

codim = n = / dim = k	1	2	3	4	5	6
1	Z_2 S^1	0	0	0	0	0
2	Z_8 P_2	$Z+Z_2$ S^2, P_2	Z_2 P_2	Z_2	Z_2	Z_2
3	Z_2 S^3	Z_2 S^3	Z_2 S^3	0	0	0
4	0	Z_2 $P_2 \times P_2$	Z_4+Z_2	$Z+Z_2+Z_2$	Z_2+Z_2	Z_2+Z_2

where S^k is the k-sphere and P_k is k-dimensional projective space.

A more general precise description is possible in a few cases,

PROPOSITION 3. $\mathcal{N}_k(k) = \mathcal{N}_k + Z$ *if k is even,* $\mathcal{N}_k(k) = \mathcal{N}_k + Z_2$ *if k is odd, where* $\mathcal{N}_k$ *is the usual (unoriented) k-th cobordism group.*

PROPOSITION 4. $\mathcal{N}_{4s}(4s-1) = \mathcal{N}_{4s} + Z_2$ *if s is not a power of* 2, *and* $\mathcal{N}_{4s}(4s - 1 =)$ $\mathcal{N}_{4s}/(P_{4s}) + Z_4$ *if s is a power of* 2, *where* (P_{4s}) *is the usual cobordism class of* P_{4s}. *A class of order* 4 *is represented by any immersion of* P_2k *with codimension* $2^k - 1$ *if* $k > 1$.

For the case of odd codimension, the following proposition does something to explain the appearance of 2-primary torsion other than 2-torsion,

PROPOSITION 1. *If* $f: M^k \to R^{k+n}$ *is an immersion with n odd and if the normal Stiefel–Whitney class* $\bar{\omega}_n(M) \neq 0$, *then f is not weakly homotopic to* $\alpha \circ f$ *where* $\alpha : R^{n+k} \to R^{n+k}$ *is affine and orientation reversing.*

"Weak homotopy" is an equivalence relation among immersions that is coarser than regular homotopy. In the computations of the above cases, it is usually an application of Proposition 1 that gives rise to 2-primary torsion other than 2-torsion.

The computations are made possible by two theorems. The first is a modification of Thom's theorem on cobordism. If ξ^n is a vector bundle of dimension n over a finite complex X, then we will define cobordism groups $\mathcal{N}_k(\xi)$, for $k + 2 < \dim \xi$, of "ξ-manifolds", where a ξ-manifold will consist of a manifold M^k together with a bundle homotopy class of bundle maps $\tau(M^n) \times \varepsilon^{n-k} \to \xi^n$. Then, if $T(\hat{\xi})$ is the Thom space of bundle $\hat{\xi} \to X$ such that $\xi \oplus \hat{\xi}$ is trivial and $\dim \hat{\xi} = m > k + 1$, the first theorem is

THEOREM 1. *There is an isomorphism* $\mathcal{N}_k(\xi) \overset{\approx}{\to} \pi_{m+k}(T(\hat{\xi}))$.

The other theorem follows from Hirsch's paper on immersions. Let G_{sr} and ξ_{sr} be as above. Then,

THEOREM 2. *There is a natural isomorphism* $\mathcal{N}_k(n) \to \mathcal{N}_k(\xi_{k+2,n})$.

From Theorems 1 and 2 it follows that there is a natural isomorphism

$$\mathcal{N}_k(n) = \pi_{k+2+k}(T(\xi_{n,k+2} \times \varepsilon^{k+2-n})) = \mathcal{H}_{k+n}(T(\xi_{n,k+2})),$$

where $\mathcal{H}_i(Y)$ is the reduced stable homotopy group of Y. Since $\mathcal{H}_{k+n}(T(\xi_{n,k+2})) \to \mathcal{H}_{k+n}(T(\xi_n))$ is an isomorphism, it follows that Theorem 3 is true,

THEOREM 3. *There is an isomorphism* $\mathcal{N}_k(n) \to \mathcal{H}_{k+n}(T(\xi_n))$.

Some of the groups $\mathcal{H}_*(T(\xi_n))$ may then be computed by means of the Adams spectral sequence, and by means of secondary Stiefel–Whitney numbers, defined as follows. If B_i is the classifying space for i-dimensional vector bundles, then by taking mapping cylinders in the sequence $\ldots \to B_i \to B_{i+1} \to \ldots$, one may assume that $\ldots \sqsubset B_i \sqsubset B_{i+1} \sqsubset \ldots$. Let $\Gamma \in H^*(B_m, B_n; Z_2)$. Then if $f: M^k \to R^{n+k}$ represents an element of $\ker(\mathcal{N}_k(n) \to \mathcal{N}_k(m))$, where the map $\mathcal{N}_k(n) \to \mathcal{N}_k(m)$ is the natural map induced by $R^{k+n} \sqsubset R^{k+m}$, and $F: X \to R^{m+k+1}$ is an immersion bounding f followed by $R^{k+n} \sqsubset R^{k+m}$, one may define uniquely and naturally the class $\Gamma(F, f) \in H^*(X, M; Z_2)$ by setting $\Gamma(F, f)$ equal to $\nu F^*\Gamma$ where νF: $(X, M) \to (B_m, B_n)$ is the classifying map for the normal bundles of the immersions F and f. Now, $H^*(B_m, B_n; Z_2) \sqsubset H^*(B_m; Z_2)$, so $\Gamma(F, f)$ may be regarded as a polynomial in the Stiefel–Whitney classes, and if $\dim \Gamma(F, f) = k+1$, the relative Stiefel–Whitney number $\Gamma(F, f).[X, M]$ is defined by evaluating on the fundamental class mod 2 of the pair (X, M). The usefulness of these numbers arises from the following proposition.

PROPOSITION 2. *If* $f: M^k \to R^{k+n}$ *is in the same cobordism class as* $g: N^k \to R^{k+n}$, *and* f *followed by* $R^{k+n} \sqsubset R^{k+m}$ *bounds the immersion* F *of* X, *and* g *followed by* $R^{n+k} \sqsubset R^{k+m}$ *bounds the immersion* G *of* Y, *and the Stiefel–Whitney number* $\Gamma[Z] = 0$ *for every closed manifold* Z *immersible with codimension* m, *then*

$$\Gamma(F, f) \cdot [X, M] = \Gamma(G, g) \cdot [Y, N].$$

Thus the secondary Stiefel–Whitney number $\Gamma[f] = \Gamma(F, f).[X, M]$ is well defined, and Γ defines a homomorphism

$$\ker(\mathcal{N}_k(n) \to \mathcal{N}_k(m)) \to Z_2.$$

In this paper we will prove the above theorems and defer the other proofs to a later paper.

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§II. COBORDISM THEORIES

1. Definition of immersion cobordism groups

We begin with the definition of $\mathcal{N}_k(n)$, the cobordism group of immersions of k-

manifolds in R^{k+n}. First, let $\mathscr{S}_k(n)$ be the set of all immersions of closed k-manifolds in R^{k+n}. If $f \in \mathscr{S}_k(n)$ and $g \in \mathscr{S}_k(n)$ and (domain f) $\sqcup$ (domain g) $= \varnothing$, let $f+g$ be $f \sqcup g$. If (domain f) $\sqcap$ (domain g) $\neq \varnothing$, let $f+g$ be undefined. Then $+$ is an associative, commutative, not always defined pairing $\mathscr{S}_k(n) \times \mathscr{S}_k(n) \to \mathscr{S}_k(n)$. For the rest of this section, the projection π_{n+k+1} will be written π; in $\mathscr{S}_k(n)$ we introduce a relation $\sim$ as follows.

Definition. Let f and g be two members of $\mathscr{S}_k(n)$. Let domain $f = M$ and domain $g = N$. Then f and g are said to be *cobordant*, or $f \sim g$, if and only if there is a $(k+1)$-manifold X and an immersion $F: X \to R^{n+k+1}$ such that

(1) $\mathscr{d}X = M \times 0 \sqcup N \times 1$

(2) $F(X) \sqsubset \pi^{-1}[0, 1]$

(3) $F(x) \in \pi^{-1}(0) \sqcup \pi^{-1}(1)$ if and only if $x \in \mathscr{d}X$.

(4) For some Riemannian metric on X, $F(\exp_{(m,0)}tu) = f(m) + te_{n+k+1}$ for $0 \leqq t < \varepsilon$ and $(m, 0) \in M \times 0$, and $F(\exp_{(m,1)}tu) = g(m) + (1-t)e_{n+k+1}$ for $0 \leqq t < \varepsilon$ and $(m, 1) \in N \times 1$, where $\varepsilon > 0$ is sufficiently small, exp is defined by the Riemannian metric, and u is the inward normal field to $\mathscr{d}X$.

It is immediate that this relation is reflexive and symmetric. It is also transitive. The next step is to see that $\mathscr{S}_k(n)/(\sim)$ is an abelian group in a natural way; this group will be $\mathscr{N}_k(n)$. But it is immediate that if $f_1 + g_1$ and $f_2 + g_2$ are defined and $f_1 \sim f_2$ and $g_1 \sim g_2$ then $f_1 + g_1 \sim f_2 + g_2$ so $\mathscr{S}_k(n)/(\sim)$ inherits an additive structure. It is also clear that representatives with disjoint domains may be found for any two classes of $\mathscr{S}_k(n)/(\sim)$, so that the addition $+$ in $\mathscr{N}_k(n)$ is defined for any two elements. Associativity follows from these properties in $\mathscr{S}_k(n)$. The empty immersion acts as a unit. It is easy to see furthermore that any immersion f followed by reflection across any hyperplane of R^{n+k} represents the negative of the cobordism class represented by f; in fact, f followed by an affine transformation σ is a representative of sign (det σ).$[f]$, where $[f]$ is the cobordism class of f.

The graded group $\Sigma_{pq}\mathscr{N}_p(q)$ has a natural structure of a bigraded ring, defined as follows. First let $\varphi_{ij}: R^i \times R^j \to R^{i+j}$ be defined by

$$\phi_{ij}(x_1e_1 + \ldots + x_ie_i, x_{i+1}e_1 + \ldots + x_{i+j}e_j) = x_1e_1 + \ldots + x_{i+j}e_{i+j}.$$

Then define the product of two immersions f and g into R^a and R^b respectively by $\varphi_{ab}(f \times g) = f \cdot g$. Since $\varphi_{a+b,c} \circ (\varphi_{ab} \times 1) = \varphi_{a,b+c} \circ (1 \times \varphi_{bc})$, the product is associative. It is clearly distributive in $\mathscr{S}_k(n)$ and behaves with respect to cobordism, so that it defines an associative, distributive product in $\Sigma_{pq}\mathscr{N}_p(q)$ such that $\mathscr{N}_k(n) \times \mathscr{N}_l(m) \to \mathscr{N}_{k+l}(m+n)$.

The immersion $f \cdot g$ is an immersion of $M \times N$. Define the immersion f^*g to be the immersion of $N \times M$ given by $f^*g(y, x) = \varphi_{ab}(f(x), g(y))$.

Claim 1. If f is an immersion in R^a and g an immersion in R^b, then f^*g is cobordant to $f \cdot g$.

Claim 2. If f and g are as above, then f^*g is cobordant to $(-1)^{ab}g \cdot f$. It follows from these two claims that the product defined above is commutative with respect to the total degree; the proofs of both claims are trivial.

Finally, we will need also the fact that in any class of $\mathcal{N}_k(n)$ there is a connected representative. A straightforward construction based on connected sums proves this fact.

2. Definition of ξ-cobordism groups, Theorem 1

Let $\xi \to B$ be an n-plane bundle. If $m \leqq n$, an m-manifold M will be said to have a ξ-structure if there is a bundle map $\tau(M) \times \varepsilon^{n-m} \to \xi$. A ξ-structure for M will be a bundle homotopy class f of bundle maps $\tau(M) \times \varepsilon^{n-m} \to \xi$. A ξ-manifold will be a pair consisting of an m-manifold M together with a ξ-structure f.

For $n > m$, define the bundle map $j_i : \tau(M) \times \varepsilon^{n-m} \to \tau(M) \times \varepsilon^{n-m}$ by $j_i(t, e_1, \ldots, e_i, \ldots, e_{n-m}) = (t, e_1, \ldots, -e_i, \ldots, e_{n-m})$ where $t \in \tau(M)$ and $e_k \in \varepsilon$. Clearly, any two bundle maps j_i and j_k are bundle homotopic. Thus $f \circ j_i = f \circ j$ is well-defined. If $n \geqq m+1$, we say that the two closed ξ-manifolds (M,f) and (M_1,f_1) are ξ-cobordant, or $(M,f) \sim (M_1,f_1)$ if and only if there is a ξ-manifold with boundary (X, h) such that

(1) $\partial X = M \times 0 \sqcup M_1 \times 1$

(2) $h \circ \gamma_{M,X} \times 1^{n-m-1} = f'$ where $f'(x, 0) = f(x)$

(3) $h \circ \gamma_{M_1,X} \times 1^{n-m-1} = f_1' \circ j$ where $f_1'(x, 1) = f_1(x)$ and where $\gamma_{M,X}$ is the bundle homotopy class of bundle maps obtained by choosing any Riemann metric on X, letting u be the inward normal field of $M \times 0 \sqsubset X$ and letting $(t, s) \to (t, su)$ be a representative of $\gamma_{M,X}$. $\gamma_{M_1,X}$ is similarly defined. A long but straightforward argument shows that this relation is reflexive and transitive. If $n \geqq m+2$, then the cobordism relation will also be symmetric, and so an equivalence relation. By defining $(M,f) + (M_1,f_1)$ to be $(M \sqcup_{dis j} M_1, f + f_1)$, one obtains a commutative and associative pairing of ξ-manifolds which respects the cobordism relation. Thus the pairing + will induce a pairing + on the equivalence classes, which, being associative, commutative, always defined and possessing inverses, turns the equivalence classes into a group, which will be denoted by $\mathcal{N}_m(\xi)$. (The negative of the equivalence class of (M,f) is the equivalence class of $(M, f \circ j)$.)

Given $\xi \to B$, where B is a finite complex, one can always find a k-plane bundle $\eta \to B$ such that $\xi \oplus \eta = B \times \varepsilon^n$. Let $T(\eta)$ be the Thom space of η. The cobordism group $\mathcal{N}_m(\eta)$ may be computed by means of Thom's theorem suitably interpreted:

THEOREM 1. *$\mathcal{N}_m(\xi) = \pi_{m+k}(T(\eta))$ provided that η has been chosen so that $k = \dim \eta \geqq m+2$.*

The proof of this theorem is nearly identical to the proof of Thom's cobordism theorems. The following is a sketch of the necessary adjustments.

If $\alpha \to A$ is an a-plane bundle, let $GL(\alpha) \to A$ be the associated bundle with fiber $GL(a)$, obtained from the principal bundle of α by letting the left action $GL(a) \times GL(a) \to GL(a)$ of the group $GL(a)$ on the fiber $GL(a)$ be given by $(g, g_1) \to g g_1 g^{-1}$. Thus, each cross-section c_1 of $GL(\alpha)$ will define a bundle map $\alpha \to \alpha$, covering the identity. Every bundle map covering the identity will be defined by a unique cross-section of $GL(\alpha)$. The bundle $V_b(\alpha)$ will be the bundle of b-frames of α; the principal bundle of α is then $V_a(\alpha)$; it is not $GL(\alpha)$.

If c_1 is a cross-section of $GL(\alpha)$, and β is another vector bundle over A, then $c_1 \oplus 1$ will be the cross-section of $GL(\alpha \oplus \beta)$ corresponding to the bundle map $\alpha \oplus \beta \to \alpha \oplus \beta$ which is defined by c_1 on α and the identity on β. If c_1 is homotopic to c_0 via cross-sections then $c_1 \oplus 1$ is homotopic to $c_0 \oplus 1$ via cross-sections, so we may as well write $c \oplus 1$ for the cross-section homotopy class of cross-sections determined by the cross-section homotopy class of cross-sections c.

LEMMA 1. *If (A, B) is a pair of regular cell complexes and* $\dim(A - B) < \dim \alpha = a$, *where α is a vector bundle over A, then for any vector bundle β of dimension b over A, if c_0 is a cross-section of $GL(\alpha \oplus \beta)$ such that $c_0 \mid B = d_0 \oplus 1$ for some d_0, then c_0 is homotopic modulo B to a cross-section $c_1 = d_1 \oplus 1$ such that $d_1 \mid B = d_0$.*

COROLLARY 1. *If* $\dim A < \dim \alpha$, *then for any β, every cross-section of $GL(\alpha \oplus \beta)$ is homotopic to one of the form $c_1 \oplus 1$ where c_1 is a cross-section of $GL(\alpha)$.*

COROLLARY 2. *If* $\dim A < 1 + \dim \alpha$, *then for any β, the mapping $c \to c \oplus 1$ establishes a 1–1 correspondence between the homotopy classes of cross-sections of $GL(\alpha)$ and the homotopy classes of cross-sections of $GL(\alpha \oplus \beta)$.*

Let α and β be vector bundles over A, of dimensions a and b respectively, having trivial Whitney sum. Let γ and δ be vector bundles over G, of dimensions a and b respectively, also having trivial Whitney sum. Pick a framing $e_1, \ldots, e_{a+b}$ of $\alpha \oplus \beta$, and a framing $l_1, \ldots, l_{a+b}$ of $\gamma \oplus \delta$. Then for any map $g_1 : A \to G$, we define a bundle map $fr_{g_1} : \alpha \oplus \beta \to \gamma \oplus \delta$ covering g_1 by setting $fr_{g_1}(z, p_1 e_1(z) + \ldots + p_{a+b} e_{a+b}(z)) = (g_1(z), p_1 l_1(g_1(z)) \ldots p_{a+b} l_{a+b}(g_1(z)))$ for any $z \in A$ and $p_i \in R$. If c_1 is a cross-section of $GL(\alpha \oplus \beta)$, let $fr_{g_1} c_1$ be the composition of fr_{g_1} and the map defined by c_1. The bundle homotopy class of this map depends only on the homotopy class of g_1 and the cross-section homotopy class of c_1, so we may speak of $fr_g c$ where g is a homotopy class of maps and c is a cross-section class of cross-sections of $GL(\alpha \oplus \beta)$.

LEMMA 2. *The mapping $(g, c) \to fr_g c$ establishes a 1–1 correspondence between the bundle homotopy classes of bundle maps $\alpha \oplus \beta \to \gamma \oplus \delta$ and the set of pairs (g, c) consisting of a homotopy class g of maps $A \to G$ and a homotopy class of cross-sections.*

LEMMA 3. *If A is a complex such that* $\dim A + 2 \leqq \dim \beta$, *then for each bundle homotopy class of bundle maps $\alpha \to \gamma$, there is a unique bundle homotopy class f' of bundle maps $\beta \to \delta$ covering the same homotopy class of maps $A \to G$ and satisfying $f \oplus f' = fr_g$.*

Proof. Pick representatives $f_1 \in f$ and $g_1 \in g$ such that f_1 covers g_1. For any $z \in A$, pick a basis $x_1, \ldots, x_a$ of α_z and let $y_1, \ldots, y_a \in (\alpha \oplus \beta)_z$ be the points such that $fr_{g_1}(y_i) = f_1(x_i)$. Define the linear injection $\varphi_{1z} : \alpha_z \to (\alpha \oplus \beta)_z$ by $\varphi_{1z}(x_i) = y_i$. Then φ_{1z} is well defined independently of the choice of the basis $x_1, \ldots, x_a$, and the collection of φ_{1z}'s defines a homomorphic injection $\varphi_1 : \alpha \to \alpha \oplus \beta$ such that $fr_{g_1} \circ \varphi_1 = f_1$. With respect to some Riemann metric on $\alpha \oplus \beta$, let $\zeta = \varphi_1(\alpha)^{\perp}$. Now, $\dim \zeta = \dim \zeta$ and $\varphi_1(\alpha) \oplus \zeta = \alpha \oplus \beta$ is trivial so there is a j such that $\zeta \times \varepsilon^j = \beta \times \varepsilon^j$. Since $\dim \beta \geqq \dim A + 2$, β is equivalent to ζ.

Let $\psi_1 : \beta \to \zeta$ be an equivalence. Then $\varphi_1 \oplus \psi_1 : \alpha \oplus \beta \to \alpha \oplus \beta$ will be a bundle map covering the identity. Let $\gamma \oplus \delta \xrightarrow{\rho} \delta$ be the natural projection. Then

$$\chi_1 = \rho \circ fr_{g_1}(\phi_1 \oplus \psi_1)$$

is a bundle map $\beta \to \delta$ covering g_1.

Now we have at least one bundle homotopy class h of bundle maps $\beta \to \delta$, covering the homotopy class of maps $g : A \to G$. The bundle homotopy class $f \oplus h$ of bundle maps $\alpha \oplus \beta \to \gamma \oplus \delta$ also covers g so, by Lemma 2, there is a unique homotopy class c of cross-sections of $GL(\alpha \oplus \beta)$ such that $f \oplus h = fr_g c$. By Lemma 1, Corollary 2, there is a unique homotopy class of cross-sections K of $GL(\beta)$ such that $c = 1 \oplus K$. Let K^{-1} be the class of cross-sections defined by the class of bundle maps inverse to the class of bundle maps defined by K. Then we may define f' to be $h \circ K^{-1}$. As for uniqueness, suppose that both f' and k satisfy $f \oplus f' = fr_g = f \oplus k$. Clearly there is a homotopy class s of cross-sections of $GL(\beta)$ such that $f' \circ s = k$. Thus $fr_g = f \oplus k = (f \oplus f') \circ (1 \oplus s) = fr_g(1 \oplus s)$. By Lemma 2, $1 \oplus s = 1$. By Lemma 1, Corollary 2, this fact implies that $s = 1$, so $f' = k$, and f' is well defined.

COROLLARY. *If* $\dim A + 2 \leqq \dim \alpha$ *too, then the mapping* $f \to f'$ *is a 1–1 correspondence between the bundle homotopy classes of maps* $\alpha \to \gamma$ *and the bundle homotopy classes of maps* $\beta \to \delta$.

Now it is possible to prove the theorem. For each closed ξ-manifold (M, f), we define a homotopy class of maps $S^{m+k} \to T(\eta)$ which depends only on the ξ-cobordism class of (M, f). Since $k \geqq n \geqq m + 2$, an embedding $r : M \subset R^{m+k}$ always exists, and any two embeddings are isotopic. Taking M as embedded, let $M \times \varepsilon^{n-m}$ be the trivial bundle over M obtained by restricting the normal bundle of R^{m+k} in $R^{m+k} \times R^{k-m}$ to M. Let $v(M)$ be the normal bundle of $r(M)$ in R^{m+k}. Pick a framing $l_1, \ldots, l_{n+k}$ in $\xi \oplus \eta$ and let $e_1, \ldots, e_{n+k}$ be the standard framing of R^{n+k}. For each $r : M \subset R^{m+k}$, the standard framing will restrict to a framing of $\tau(r(M)) \oplus v(r(M)) \times \varepsilon^{n-m}$. The bundle homotopy class f determines the bundle homotopy class $f \circ (r_* \times 1^{n-m})^{-1} : \tau(r(M)) \times \varepsilon^{n-m} \to \xi$. By the corollary to Lemma 3, there is a unique bundle homotopy class f' of bundle maps $v(r(M)) \to \eta$ covering the same homotopy class of maps $r(M) \to B$ as $f \circ (r_* \times 1^{n-m})^{-1}$ and satisfying $f \circ (r_* \times 1^{n-m})^{-1} \oplus f' = fr_{g \circ r} - 1$.

Identifying $v(r(M))$ with a suitable tubular neighborhood of $r(M)$ in $R^{m+k} \subset S^{m+k}$ and extending each $f_1' \in f'$ in the usual way to a map $S^{m+k} \to T(\eta)$, f' defines a homotopy class of maps $S^{m+k} \to T(\eta)$. Now, the argument follows the usual channel.

3. Relation between immersion cobordism and ξ-cobordism, Theorem 2 and Theorem 3

In the following, we will identify $\tau(R^a)$ with $R^a \times R^a$ by parallel translation. Also, let G_{kn} be the k-planes in n-space and let $\xi_{kn} \to G_{kn}$ be the canonical k-plane bundle over G_{kn}. Let $\pi : R^{n+k+1} \to R$ be the projection parallel to R^{n+k}. If M is a k-dimensional manifold and f_0 and f_1 will be said to be *weakly homotopic* if there is an immersion $H : M \times I \to R^{n+k+1}$ such that

(1) $H(m, t) = f_0(m) + te_{n+k+1}$ for $0 \leqq t < \varepsilon$,

(2) $H(m, t) = f_1(m) + te_{n+k+1}$ for $1 - \varepsilon < t \leqq 1$,

where $0 < \varepsilon < \frac{1}{2}$.

LEMMA 4. *The weak homotopy classes of immersions of* M *in* R^{n+k} *are in one to one* *correspondence with the bundle homotopy classes of bundle maps* $\tau(M) \times \varepsilon \to \xi_{k+1,n}$ *under the correspondence* $f \to \rho_{kn} \circ (f_* \times 1)$ *where* $\rho_{kn} : \xi_{kn} \times \varepsilon \to \xi_{k+1,n}$ *is the standard map.*

Proof. Hirsch shows that the bundle homotopy classes of vector bundle monomorphisms $\tau(M) \to \tau(R^{n+k})_0$ are in 1–1 correspondence with the regular homotopy classes of immersions $M \to R^{n+k}$. However, the vector bundle monomorphisms $\tau(M) \to \tau(R^{n+k})_0$ define bundle maps $\tau(M) \to \xi_{kn}$ and vice versa, in such a way that bundle homotopy classes go into bundle homotopy classes. Thus, the regular homotopy classes of immersions $M \to R^{n+k}$ are in 1–1 correspondence with the bundle homotopy classes of bundle maps $\tau(M) \to \xi_{kn}$. The same argument shows that the regular homotopy classes of immersions $M \times (0, 1) \to R^{n+k+1}$ are in 1–1 correspondence with the bundle homotopy classes of bundle maps $\tau(M \times (0,1)) \to \xi_{k+1,n}$. However, these are just the same as the bundle homotopy classes of bundle maps $\tau(M) \times \varepsilon \to \xi_{k+1,n}$. On the other hand, each immersion $M \times (0, 1) \to R^{n+k+1}$ determines an immersion $M \to R^{n+k+1}$ with normal field. Hirsch shows that this immersion with normal field is regularly homotopic in R^{n+k+1} to an immersion $f: M \to R^{n+k}$ with normal field $m \to (f(m), v)$. Thus every bundle homotopy class of bundle maps $\tau(M) \times \varepsilon \to \xi_{k+1,n}$ contains a bundle map of the form $\rho_{kn} \circ (g_* \times 1)$ where $g_* : \tau(M) \to \xi_{kn}$ arises from an immersion. It follows that the mapping $f \to \rho_{kn} \circ (f_* \times 1)$ sends the regular homotopy classes of immersions of M in R^{n+k} *onto* the bundle homotopy classes of bundle maps $\tau(M) \times \varepsilon \to \xi_{k+1,n}$.

Let f_0 and f_1 be weakly homotopic via H. Then we define a homotopy from $\rho_{kn} \circ (f_{0*} \times 1)$ to $\rho_{kn} \circ (f_{1*} \times 1)$ as follows: If $V \in \tau(M)_m$ and $r \in R$, then

$$H_t(V, r) H_{*(m,t)}\left(V, r\frac{d}{dt}\right), \qquad H_{*(m,t)}(\tau(M+I)_{(m,t)}).$$

Thus, the map above may be factored through weak homotopy classes of immersions to obtain the result that the map $f \to \rho_{kn} \circ (f_* \times 1)$ sends the weak homotopy classes of immersions $M \to R^{n+k}$ *onto* the bundle homotopy classes of bundle maps $\tau(M) \times \varepsilon \to \xi_{k+1,n}$.

Now suppose that f_0 and f_1 are immersions and that $\rho_{kn} \circ (f_{0*} \times 1)$ is homotopic to $\rho_{kn} \circ (f_{1*} \times 1)$ via the homotopy H_t. We may assume that H_t is constant for $t \in [0, \frac{1}{3}) \sqcup (\frac{2}{3}, 1]$. This homotopy defines a bundle map $\tau(M \times I) \xrightarrow{H} \xi_{k+1,n}$ by $H(V, r(d/dt)) = H_t(V, r)$ for $(V, r(d/dt)) \in \tau(M \times I)_{(m,t)}$. This map in turn defines an equivariant map $\tau(M \times I) \xrightarrow{G} \tau(R^{n+k+1})_0$. Pick any continuous map $J: M \times I \to \pi^{-1}[0, 1]$ such that

$$J(m, t) = f_0(m) + tv \quad \text{for} \quad 0 \leqq t < \tfrac{1}{3}$$
$$J(m, t) = f_1(m) + tv \quad \text{for} \quad \tfrac{2}{3} < t \leqq 1.$$

Then, identifying $\tau(R^{n+k+1})$ with $R^{n+k+1} \times \tau(R^{n+k+1})_0$ by parallel translation, the map

$$(J, H) : (V, r(d/dt)) \to (J(m, t), G(V, r(d/dt))),$$

where $(V, r(d/dt)) \in \tau(M \times I)_{(m,t)}$, is an equivariant map $\tau(M \times I) \to \tau(R^{n+k+1})$ which is induced by an $M \times I$-immersion of $M \times [0, \frac{1}{3})$ and $M \times (\frac{2}{3}, 1]$ on these two sets. Then by Hirsch's Theorem 5.7, [3], there is an immersion $M \times I \to \pi^{-1}[0, 1]$ extending the above two immersions. In other words, f_0 and f_1 are weakly homotopic and the lemma is proved.

Before continuing, we need the following lemma.

LEMMA 5. *Let $s \to c_s$ be a C^∞ path in S^{m-1}. Then there is a unique path $s \to \rho_s$ in $SO(m)$ such that*

(1) $\rho_s(c_0) = c_s$

(2) $\rho_{s*} : \tau(S^{m-1})_{c_0} \to \tau(S^{m-1})_{c_s}$ *is parallel translation along the given path from c_0 to c_s.*

LEMMA 6. *Let $\alpha \to A$ and $\beta \to B$ be C^∞ vector bundles of the same dimension. Suppose that any two non-zero vector fields in $\alpha \times \varepsilon$ are homotopic via non-zero fields. If $f_0, f_1 : \alpha \to \beta$ are two bundle maps such that $f_0 \times 1$ and $f_1 \times 1$ are bundle homotopic, then f_0 and f_1 are already bundle homotopic.*

Proof. The proof is a straightforward application of Lemma 5.

Now we arrive at the lemmas we were seeking, Lemmas 7 and 8.

LEMMA 7. *The weak homotopy classes of immersions $M^k \to R^{k+n}$ are in 1–1 correspondence with the bundle homotopy classes of bundle maps $\tau(M) \times \varepsilon^2 \to \xi_{k+2,n}$ under the correspondence $f \to \rho_{k+1,n} \circ (\rho_{k,n} \circ (f_* \times 1) \times 1)$.*

Proof. Since $(G_{ij})^l \sqsubset G_{l,j}$ for $l \leqq i$, every bundle homotopy class of bundle maps $\tau(M) \to \varepsilon^2 \to \xi_{k+2,n}$ contains an element of the form $\rho_{k+1,n} \circ (f \times 1)$ where $f : \tau(M) \times \varepsilon \to \xi_{k+1,n}$ is a bundle map. Thus the mapping of the lemma is onto.

To show that it is one to one, it will suffice to show that if f_0 and f_1 are bundle maps $\tau(M) \times \varepsilon \to \xi_{k+1,n}$ and $\rho_{k+1,n} \circ (f_0 \times 1)$ is bundle homotopic to $\rho_{k+1,n} \circ (f_1 \times 1)$, then f_0 is bundle homotopic to f_1. Since $(G_{ij})^l \sqsubset G_{l,j}$ for $l \leqq i$, we may suppose immediately that $f_0 \times 1$ is bundle homotopic to $f_1 \times 1$. The obstructions to finding homotopies between unit fields in $\tau(M) \times \varepsilon^2$ are zero so any two-unit fields in $\tau(M) \times \varepsilon^2$ are homotopic. Then, by Lemma 6, f_0 is bundle homotopic to f_1, which proves Lemma 7.

Say $\partial X^{k+1} = M \times 0 \sqcup M' \times 1$ and let $F : \tau(X) \times \varepsilon \to \xi_{k+2,n}$ be a bundle map. We can find a bundle map $K : \tau(X) \to \xi_{k+1,n}$ such that $\rho_{k+1,n} \circ K$ is bundle homotopic to F. Now, K defines two bundle homotopy classes of bundle maps

$$K \circ \gamma_{M,X} : \tau(M) \times \varepsilon \to \xi_{k+1,n}$$

and

$$K \circ \gamma_{M',X} : \tau(M) \times \varepsilon \to \xi_{k+1,n}.$$

Each of these two classes defines a weak homotopy class of immersions of M, respectively M'. Choose representatives $f : M \to R^{n+k}$ and $g : M' \to R^{n+k}$. Let X have a Riemann metric, let u be its inward normal unit field along X and define the immersion G of a tubular neighborhood of ∂X in R^{n+k+1} by

$$G(\exp_{(m,0)} tu) = f(m) + te_{n+k+1} \quad \text{for} \quad m \in M \quad \text{and} \quad 0 \leqq t \leqq \varepsilon$$

$$G(\exp_{(m,1)} tu) = g(m) + (1-t)e_{n+k+1} \quad \text{for} \quad m \in M' \quad \text{and} \quad 0 \leqq t \leqq \varepsilon,$$

where $\varepsilon > 0$ is sufficiently small.

Now, $\rho_{k+1,n} \circ G_*$ is homotopic to K restricted to the tubular neighborhood above, so we can find an extension H of $\rho_{k+1,n} \circ G_*$ which is homotopic to K. Let $J : X \to \pi^{-1}[0, 1] \sqsubset$

R^{n+k+1} be any continuous extension of G. Then the map $(J, \beta \circ H) : \tau(X) \to R^{n+k+1} \times \tau(R^{n+k+1})_0$ (where $\beta : \xi_{k+1,n} \to R^{n+k+1}$ is the natural equivariant map), is an equivariant map given by an X-immersion on a neighborhood of $\mathscr{d}X$. By Hirsch's Theorem 5.7, there is then an immersion h of X, arbitrarily close to J and agreeing with J in a neighborhood of $\mathscr{d}X$. By choosing J appropriately, we may insure that range $(h) \sqsubset \pi^{-1}[0, 1]$ so f and g are cobordant immersions via h. Clearly any two weakly homotopic immersions are cobordant, so any immersion in the weak homotopy class of immersions determined by $F \circ (\gamma_{M,X} \times 1)$ is cobordant to any immersion in the weak homotopy class determined by $F \circ (\gamma_{M',X} \times 1) \circ j$. That these bundle homotopy classes of bundle maps determine weak homotopy classes of immersions follows from Lemma 7. Thus we have proved

LEMMA 8. *If* domain f, $\rho_{k+2,n} \circ (\rho_{k+1,n}(f_* \times 1) \times 1))$ *and* (domain g, $\rho_{k+2,n} \circ (\rho_{k+1,n} \circ (g_* \times 1) \times 1))$ *are cobordant* $\xi_{k+2,n}$*-manifolds, then* f *and* g *are cobordant immersions.*

Now, the mapping $f \to$ (domain f, $\rho_{k+2,n} \circ (\rho_{k+1,n} \circ (f_* \times 1) \times 1))$ induces a homomorphism $\mathscr{N}_k(n) \to \mathscr{N}_k(\xi_{k+2,n})$ by an argument similar to that proving Lemma 8. By Lemmas 4 and 7, this mapping is onto, and by Lemma 8, it is one to one. Thus we have

THEOREM 2. *The correspondence* $f \to$ (domain f, $\rho_{k+2,n} \circ (\rho_{k+1,n}(f_* \times 1) \times 1))$ *induces an isomorphism* $\mathscr{N}_k(n) \to \mathscr{N}_k(\xi_{k+2,n})$.

We have that $\xi_{k+2,n} \oplus \xi_{n,k+2}$ is trivial so that theorem 1 implies that $\mathscr{N}_k(\xi_{k+2,n})$ is isomorphic to $\pi_{m+k}(T(\xi_{n,k+2} \times \varepsilon^{m-n}))$ for $m > k + 1$. However, this last group is just the reduced stable homotopy group $\mathscr{H}_{n+k}(T(\xi_{n,k+2}))$. Since $(G_{ij})^l \sqsubset G_{l,j}$ for $l \leqq i$, we have by an application of the Whitehead theorem that $\mathscr{H}_i(T(\xi_{n,k+2})) \to \mathscr{H}_i(T(\xi_n))$ is an isomorphism for $i \leqq k$.

THEOREM 3. *There is a natural isomorphism*

$$\mathscr{N}_k(n) \to \mathscr{H}_{n+k}(T(\xi_n)).$$

§III. GENERAL PROPERTIES

1. 2-torsion, Theorem 4

For any prime $p \neq 2$ and for $p = 0$, E. Thomas shows that $H_*(B_n; Z_p) = Z_p[\not{p}_1, \ldots, \not{p}_{n/2}]$ with dim $\not{p}_i = 4i$. It follows that for n odd, $H_*(B_n, B_{n-1})$ is 2-primary. Thus $\mathscr{H}_*(B_n, B_{n-1}) = \mathscr{H}_*(T(\xi_n))$ is 2-primary for n odd, and we have

THEOREM 4. *If n is odd then* $\mathscr{N}_k(n)$ *is 2-primary.*

2. Free part, Theorem 5

For any ring K, let K_0 be the orientation sheaf of ξ_n with coefficients in K. Then the sheaf exact sequence

$$0 \to B_n \times F \to (F + F) \to F_0 \to 0$$

where F is a field and $(F + F)$ is the sheaf with fiber $F + F$ on which $\pi_1(B_n)$ acts by interchanging terms, leads to the exact sequence

$$\ldots \to H^*(B_n; F) \to H^*(B_n; (F + F)) \to H^*(B_n; F_0) \to \ldots$$

which becomes

$$\ldots \to H^*(B_n; F) \xrightarrow{\pi^*} H^*(Bo_n; F) \to H^*(B_n; F_0) \ldots$$

on introducing $\pi : Bo_n \to B_n$, the double cover of B_n, and applying Leray's theorem. If F does not have characteristic 2 and $n = 2m$, then the sequence becomes

$$0 \to F[p_1, \ldots, p_m] \to F[p_1, \ldots, p_{m-1}, \chi_0(\hat{n})] \to H^*(B_n; F_0) \to 0$$

where $p_i \to p_i$ for $i < m$ and $p_m \to \chi_0(n)^2$. Thus, additively,

$$H^*(B_n; F_0) = \chi(n)F[p_1, \ldots, p_{m-1}],$$

where $\chi(n) =$ image of $\chi_0(n)$. Under the Thom isomorphism φ, we have $\varphi(\chi(n)\mathscr{M}) = p_m\mathscr{M} \in H^*(T(\xi_n); F)$ where $\mathscr{M}$ is any element of $F[p_1, \ldots, p_{m-1}]$. If $n = 2m + 1$, then $H^*(T(\xi_{2m+1}); F) = 0$.

Using a filtration by skeletons, we obtain a spectral sequence for $\mathscr{H}_*(T(\xi_n))$ that shows that the natural map

$$\mathscr{H}_r(T(\xi_n)) \to H_r(T(\xi_n))$$

is an isomorphism modulo finite group, so

$$\mathscr{H}_r(T(\xi_n)) \otimes Z_0 \to H_r(T(\xi_n); Z_0)$$

is an isomorphism.

Using the naturality of the Thom isomorphism, we obtain the equality

$$t(f^*\phi(\chi(n)\mathscr{M}) \cdot S^{l+r} = b(f)^*(\chi(n)\mathscr{M}) \cdot \begin{bmatrix}\text{fund. normal} \\ \text{class of } M\end{bmatrix}$$

for any bundle map $f: v(M) \to \xi_n \times \varepsilon^{l-n}$ where $t(f): S^{l+r} \to T(\xi_n \times \varepsilon^{l-n})$ is the map defined by f and $b(f)$ is the base map of f. Thus we may find a basis of immersions

$$\{f_{\chi\mathscr{M}} \mid \mathscr{M} \text{ an elementary monomial in } Z_0[p_1, \ldots, p_m]\}$$

for $\mathscr{N}_+(n) \otimes Z_0$, where n is even, such that $\chi\mathscr{M} \cdot [f_{\chi\mathscr{M}}] \neq 0$ and $\chi\mathscr{N} \cdot [f_{\chi\mathscr{N}}] \neq 0$ for $\mathscr{M} \neq \mathscr{N}$, and where

$$p \cdot [f] = df^*(p) \cdot [\text{fundamental normal class of domain } f]$$

the map df: domain $f \to B_n$ being the classifying map of the normal bundle of f. Now we can prove Theorem 5,

THEOREM 5. *$\Sigma_{pq}\mathscr{N}_p(q) \otimes Z_0$ is a bigraded polynomial ring with generators $f_i \in \mathscr{N}_{2+4i}(2) \otimes Z_0$, where $i = 0, 1, 2, \ldots$.*

COROLLARY. *$\Sigma_{pq}\mathscr{N}_p(q) \otimes Z_0$ is the tensor algebra of $\mathscr{N}_*(2) \otimes Z_0$.*

Proof of theorem. Consider the natural map $\xi_{2n} \times \xi_{2m} \xrightarrow{\psi} \xi_{2n+2m}$. Let Z_0o' be the orientation sheaf of $\xi_{2n} \to B_{2n}$ over Z_0 and Z_0o'' that of $\xi_{2m} \to B_{2m}$ and Z_0o that of ξ_{2n+2m}.

We have the commutative diagram

$$\begin{array}{ccc} H^*(B_{2n}; Z_0o') \otimes H^*(B_{2m}; Z_0o'') & \to & H^*(B_{2n} \times B_{2m}; Z_0o' \otimes Z_0o'') \\ \downarrow \varphi\otimes\varphi & & \downarrow \varphi \\ H^*(T(\xi_{2n}); Z_0) \otimes H^*(T(\xi_{2m}); Z_0) & \to & H^*(T(\xi_{2n+2m}); Z_0) \end{array}$$

in which the vertical maps are given by Thom isomorphisms and the lower horizontal map is an isomorphism. Thus the upper horizontal map is an isomorphism.

From the behavior of the map $\xi o_{2n} \times \xi o_{2m} \xrightarrow{\psi o} \xi o_{2n+2m}$ of oriented classifying bundles, we see that the map

$$H^*(B_{2n+2m}; Z_0o) \xrightarrow{\psi^*} H^*(B_{2n} \times B_{2m}; \psi^*(Z_0o)) = H^*(B_{2n}; Z_0o') \otimes H^*(B_{2m}; Z_0o'')$$

is described by

$$\psi^*\chi_{(2n+2m)} \smile \mathscr{M}(\ldots, P_r, \ldots) = (\chi(2n) \times \chi(2m)) \smile \mathscr{M}\Big(\ldots, \sum_{i+j=r} p_{i'} \times p_{j''}, \ldots\Big)$$

where $\sum_\gamma P_r$ is the Pontrjagin class of $\xi_{2n+2m} \to B_{2n+2m}$ and $\sum_i p_{i'}$ is the Pontrjagin class of $\xi_{2n} \to B_{2n}$ and $\sum_j p_{j''}$ is the Pontrjagin class of $\xi_{2m} \to B_{2m}$.

From now on we follow with little change the proof in Milnor's notes on characteristic classes that the oriented cobordism ring is a polynomial ring. Let $\omega = (a_1, \ldots, a_r)$ be a partition of A and let $s_\omega(\sigma)$ be the polynomial in the symmetric functions $\sigma_1, \ldots, \sigma_A$ of the indeterminates $t_1, \ldots, t_A$, defined by $s_\omega(\sigma) = \Sigma(t_1^{a_1} \ldots t_r^{a_r})$, where the sum is taken over all the proper permutations of $t_1, \ldots, t_A$. Set $s_\phi(\sigma) = 1$. For $c = 1 + c_1 + \ldots$ where $\deg c_i = 4i$, define $s_\omega(c)$ to be $s_\omega(\sigma)$ with σ_i replaced by the term of degree $4i$ in c. Let $a = 1 + a_1 + \ldots$ and $b = 1 + b_1 + \ldots$ where $\deg a_i = \deg b_i = 4i$. Then Thom has shown that

$$s_\omega(a \,.\, b) = \sum_{\omega'\omega''=\omega} s_{\omega'}(a) \,.\, s_{\omega''}(b).$$

Consequently,

$$\psi^*\chi_{(2n+2m)}s_\omega(P) = \sum_{\omega'\omega''=\omega} \chi(2n)s_{\omega'}(p') \times \chi(2m)s_{\omega''}(p''),$$

so that, if $f\colon M^{4k+2n} \to R^{4k+4n}$ and $g : N^{4l+2m} \to R^{4l+4m}$ are immersions, then

$$(*) \qquad \chi(2n + 2m)s_\omega[f \cdot g] = \sum_{\omega'\omega''=\omega} \chi(2n)s_{\omega'}[f]\chi(2m)s_{\omega''}[g],$$

where we abbreviate $b(f)^*\chi(2n)s_\omega$ (Pontrjagin class).[fundamental normal class of M] by $\chi(2n)s_\omega[f]$, etc.

Let $l(\omega)$ be the length of the partition ω. Then for $n \geqq l(\omega)$, define $s_{n\omega}$ to be $\chi(2n)s_\omega$ (Pontrj. class). Recall that $H^*(T(\xi_{2i}); Z_0)$ is dual to $H_*(T(\xi_{2i}); Z_0)$, which is naturally isomorphic to $\mathscr{H}_*(T(\xi_{2i})) \otimes Z_0$. It follows that we may pick immersions $f_0, f_1, \ldots$ such that $s_{1,0}[f_0] \neq 0$, $s_{1,1}[f_1] \neq 0$, $s_{1,2}[f_2] \neq 0, \ldots, s_{1,j}[f_j] \neq 0, \ldots$. If $\omega = (a_1, \ldots, a_r)$ is a partition of k then for $n \geqq l(\omega)$ define $f_{n\omega}$ to be the immersion $f_0^{n-l(\omega)} f_{a_1} \cdots f_{a_r}$. Then

$$s_{n\mu}[f_{n\omega}] = \sum_{(\mu_1,\ldots,\mu_r)=\omega} (s_{10}f_0)^{n-l(\omega)} s_{1\mu_1}f_{a_1} \cdots s_{1\mu_r}f_{a_r}$$

by (*). Thus, unless μ refines ω, we have $s_{n\mu}[f_{n\omega}] = 0$. If we number the partitions of k in such a way that if ω_i refines ω_j then $i<j$, then the matrix $\|s_{n\omega_i}[f_{n\omega_j}]\|$ is triangular with non-zero diagonal elements, so it is non-singular. It follows immediately that the cobordism classes of the immersions $f_{n\omega_j}$ are linearly independent in $\mathscr{H}_{4k+2n}(2n)\otimes Z_0$.

The number of such classes is number $\{\omega \mid |\omega|=k,\ l(\omega)\leqq n\}$. But this number is just number $\{\omega \mid |\omega|=k,\ \max(\omega)\leqq n\}$. Finally, number $\{\omega \mid |\omega|=k,\ \max(\omega)\leqq n\}$ is the rank of $H^{4k}(B_{2n};Z_0)\approx\mathscr{H}_{4k+2n}(2n)\otimes Z_0$, so the $f_{n\omega}$ span $\mathscr{H}_{4k+2n}(2n)\otimes Z_0$ and the theorem is proved.

3. *p*-torsion, Theorem 6

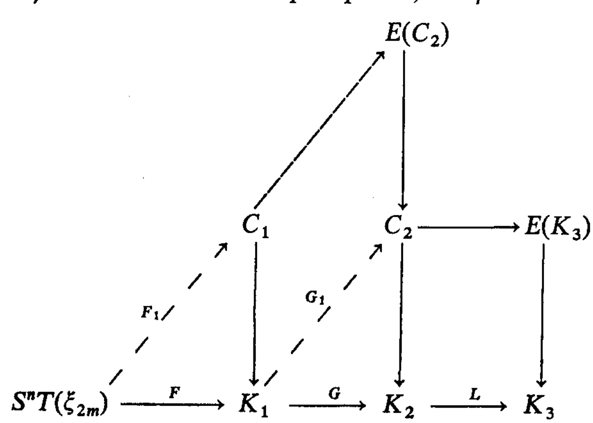
We can use the facts outlined so far and the Adams spectral sequence to find the smallest i for which $\mathscr{N}_i(2m)$ contains p-torsion. Let p be an odd prime and let $\kappa=(p-1)/2$.

THEOREM 6. *If $m\geqq\kappa$, then $\mathscr{N}_i(2m)$ has no p-torsion for $i<2m+2p^2-2p-1$, but it does have p-torsion if $i=2m+2p^2-2p-1$. If the partition ω with $|\omega|<\kappa p$ contains no entry equal to κ, then there is an immersion $f(\omega)$ with dimension $2m+4|\omega|$ and codimension $2m$ such that $\chi(2m)s_\omega[f(\omega)]$ is not divisible by p. If ω does contain an entry equal to κ, then $\chi(2m)s_\omega[f]$ is divisible by p for every immersion f.*

Proof. Let s_ω denote the same polynomial in the Pontrjagin classes as above. Let $n>8m^2+2m$. For each partition ω, let $K(\omega)$, $K(\omega,1)$, and $K(\omega,2)$ be Eilenberg–Maclane spaces of types $K(Z,n+4m+4|\omega|)$, $K(Z,n+4m+4|\omega|+5)$, and $K(Z,n+4m+4|\omega|+10)$ with fundamental classes $g(\omega)$, $h(\omega)$, and $l(\omega)$ respectively. Recall that $X\to K(Z_p,r)$ may be factored through $K(Z,r)\to K(Z_p,r)$ if $H^{r+1}(X;Z)=0$. Then consideration of the Adams spectral sequence constructed with a minimal resolution supplies us with three maps,

$$\begin{array}{rcl} S^nT(\xi_{2m}) \xrightarrow{F} & \Pi\{K(\omega)\mid \omega \text{ is } \kappa\text{-free}, |\omega|<p\kappa\} & = K_1 \\ & \big\downarrow G & \\ & \Pi\{K(\omega,1)\mid \omega \text{ is } \kappa\text{-free}, |\omega|<p\kappa\} & = K_2 \\ & \big\downarrow L & \\ & \Pi\{K(\omega,2)\mid \omega \text{ is } \kappa\text{-free}, |\omega|<p\kappa\} & = K_3, \end{array}$$

such that $F^*g(\omega)=S^np_ms_\omega$, $G^*h(\omega)=\beta\wp^1g(\omega)$, and $L^*l(\omega)=\beta\wp^1h(\omega)$, where p_m is the m-th Pontrjagin class and $\wp^1$ is the first Steenrod p-th power, and β is the Bockstein operator,



For any space X, let $E(X) \to X$ be the fiberspace of paths of X. Let $C_2 \to K_2$ be the bundle induced from $E(K_3) \to K_3$ by L. It is easy to check that $\pi_i(C_2) = \pi_i(K_2) + \pi_{i+1}(K_3)$ and that G lifts to $G_1 : K_1 \to C_2$. Let $C_1 \to K_1$ be the bundle induced from $E(C_2) \to C_2$ by G_1. It is easy to check that $\pi_i(C_1) = \pi_i(K_1) + \pi_{i+1}(K_2) + \pi_{i+2}(K_3)$ and that F lifts to $F_1 : S^nT(\xi_{2m}) \to C_1$. Then one may check that F_1 induces an isomorphism in Z_p-cohomology up to dimension $4m + n + 4 + p - 1$ and so an isomorphism in homotopy modulo non-p-torsion up to dimension $4m + n + 4 + p - 2$. Also, the differentials in the Adams spectral sequence are zero up to dimension $4m + n + 4 + p - 2$, so the duals of the $g(\omega)$'s above survive to the E^∞ term and there determine duals of the $S^n p_m s_\omega$'s. It remains only to show that $\mathscr{H}_{4m+2p^2-2p-1}(T(\xi_{2m}))$ does have p-torsion. This will follow from the fact that if $u \in H^r(X; Z_p)$ is spherical and $\beta \mathscr{P}^1 u = 0$ then $\mathscr{H}_{r+2p-3}(X)$ has p-torsion. It suffices to exhibit u; let ω_s be the partition of $2\kappa^2$ consisting of s entries 2κ and $2(\kappa - s)$ entries κ. Then $u = (2s_{\omega_0} - s_{\omega_1} + s_{\omega_2} - s_{\omega_3} + \ldots + (-1)^\kappa s_{\omega_\kappa})p_m$ will do.

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