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Mapping Hilbert cube manifolds to ANR's: A solution of a conjecture of Borsuk

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After Borsuk introduced the notion of absolute neighborhood retracts (ANR's) and J.H.C. Whitehead demonstrated that all ANR's have the homotopy types of cell-complexes [34], the question naturally arose as to whether compact (metric) ANR's must necessarily be homotopy-equivalent to finite cell-complexes. Borsuk expressly posed this conjecture in his address to the Amsterdam Congress in 1954 [5], and over the ensuing years considerable progress was made (see [26]), including (a) ANR's admitting "brick decompositions," by Borsuk [5], (b) the simply connected case, by de Lyra [21], (c) products with the circle, by M. Mather [22], (d) applications of Wall's obstruction to finiteness [29], (e) compact n-manifolds, by Kirby and Siebenmann [16], and (f) compact Hilbert cube manifolds and locally triangulable spaces, by Chapman [9]. The full problem remained open, however.

In this paper the conjecture is settled positively by application of Hilbert cube manifold theory. Specifically, it is shown that each compact ANR is the image of some Hilbert cube manifold (Q-manifold) by a cell-like (CE) mapping. Such mappings, between ANR's, are always homotopy-equivalences [14], [17], [20], [28] (although for more general metric compacta they need not be shape-equivalences [27]), so that by appealing to (f) above, the conjecture is settled. In the process, a considerably stronger result is established, namely, that there exists a cell-like map f from a Q-manifold onto the ANR X whose mapping cylinder $M(f)$ is itself a Q-manifold. The mapping cylinder collapse of $M(f)$ to X provides a particularly nice CE-map of a Q-manifold to the ANR.

The general outline of the proof is as follows: First, it is shown that the mapping cylinder of a CE-map from a Q-manifold to an ANR is always a Q-manifold. Second, this result is used to show that the ANR A is the image of a Q-manifold by a CE-mapping if and only if whenever A is em-

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bedded as a Z-set in each of two Q-manifolds M_1 and M_2, then the union of M_1 and M_2 along A is itself a Q-manifold. This involves a regular neighborhood theorem for ANR's in Q-manifolds.

Finally, this is combined with R.T. Miller's theorem [23] that the cone of a compact ANR is always the image of Q by a CE-mapping to obtain the general existence of CE-mappings from Q-manifolds onto ANR's. All the work generalizes to the locally compact case.

Two immediate consequences of the work, communicated to the author by E. Dyer and T.A. Chapman [12], respectively, are (a) Wall's dimension estimate [29] for the homotopy type of a finite-dimensional CW complex works for compact ANR's: if $\dim A = n$ then A is homotopy-equivalent to a polyhedron of dimension less than or equal to $\max\{3, n\}$. (b) The simple homotopy type of the Q-manifold M^Q mapped to the ANR by a CE-map is unique, so that simple homotopy theory extends full-blown to locally compact ANR's with CE-maps playing the role of collapses (see (5.5) for a more detailed statement).

Two appendices are included, the first gives a list of the basic theorems about Q-manifolds and ANR's cited in the text, while the second gives an explicit proof of Miller's Cone Theorem.

1. Notational conventions

In this paper, the only spaces considered are locally compact and metric, so AR and ANR refer to locally compact metric absolute retracts and absolute neighborhood retracts, respectively. The Hilbert cube, Q, is the countably infinite Cartesian product, $\prod_{i=1}^{\infty} I_i$, of closed intervals, $I = [0, 1]$, and a Hilbert cube manifold (Q-manifold) is a space locally homeomorphic to an open subset of Q. An important property of position in ANR's and in manifolds, generally, is *Property Z* (see [1]). A closed subset A of an ANR X has *Property Z* (is a Z-set) if the inclusion $U \backslash A \to U$ is a homotopy-equivalence for each open set U of X.

An embedding is a *Z-embedding* if its image is a Z-set. (In Q-manifolds, Z-sets coincide with the closed subsets of collared submanifolds ((A1), (A3), and (A4)), every proper mapping into a Q-manifold is approximable by Z-embeddings (A2), and, philosophically, Z-sets are analogous to PL subsets of PL manifolds in the trivial range of dimension because of a strong isotopy extension theorem (A3).)

A mapping $f\colon X \to Y$ between local compacta is *proper* if the pre-image of each compactum is compact, and two maps are *properly homotopic* (by a *proper homotopy*) if there is a homotopy between them which is a proper mapping. A mapping $f: X \to Y$ is *cell-like* (CE-) if it is a proper surjection and every point-inverse has the shape of a point (is cell-like, see [19]).

The *mapping cylinder*, $M(f)$, of a map $f: X \to Y$ is $X \times [0, 1] \cup_\alpha Y$, where $\alpha: X \times \{0\} \to X \xrightarrow{f} Y$ is as indicated. Let $\pi = \pi(f): X \times [0, 1] \to M(f)$ be the restriction of the quotient mapping and $c(f): M(f) \to Y$, the collapse $((x, t) \mapsto f(x),\ y \mapsto y)$. The two inclusions $X \to X \times \{1\} \to M(f)$ and $Y \to M(f)$ are generally unnamed, and X is identified with $X \times \{1\}$, called the *top* of $M(f)$, while Y is identified with its image in $M(f)$ called the *base* of $M(f)$. If $A \subset X$ is any closed set, then the *relative mapping cylinder*, $M(f, A)$, of *f reduced modulo* A is the result of identifying each arc $\pi(\{a\} \times I)$ for $a \in A$ in $M(f)$ to its "base" $f(a) \in Y$. The notation $\pi(f, A)$ is used for this "partial" collapse $M(f) \to M(f, A)$, and $c(f, A)$ is used for the collapse $c(f)\pi(f, A)^{-1}$ of $M(f, A)$ to its base, Y.

If $\mathcal{U}$ is an open cover of Y, then

$$\mathrm{st}\mathcal{U} = \{\mathrm{st}(U; \mathcal{U}) = \bigcup\{V \in \mathcal{U} \mid V \cap U \neq \varnothing\} \mid U \in \mathcal{U}\}\,.$$

A collection of subsets S of Y is *limited by* $\mathcal{U}$ if for each $S \in \mathcal{S}$, there is a $U \in \mathcal{U}$ containing S. Two mappings $f, g: X \to Y$ are $\mathcal{U}$-*close* if $\mathcal{S} = \{\{f(x), g(x)\} \mid x \in X\}$ is limited by $\mathcal{U}$, and a homotopy $F: X \times I \to Y$ is *limited by* $\mathcal{U}$ (is a $\mathcal{U}$-*homotopy*) if $\mathcal{S} = \{F(\{x\} \times I) \mid x \in X\}$ is limited by $\mathcal{U}$. $\mathcal{V}$ is a st^n-*refinement of* $\mathcal{U}$ if $\mathrm{st}^n(\mathcal{V})$ refines $\mathcal{U}$, where

$$\mathrm{st}^n(\mathcal{V}) = \{\mathrm{st}(W; \mathcal{V}) \mid W \in \mathrm{st}^{n-1}(\mathcal{V})\}\,.$$

Often "im(f)" is used for "the image of the map f", "bd" means "boundary of", and occasionally "1" is used for "the identity mapping"; if $X = Y \cup Z$ and $W = Y \cap Z$ then for any maps $f: Y \to T$, $g: Z \to T$ which agree on W, "$f \cup g$" is used to denote the map $h: X \to T$ defined by $h \mid Y = f$ and $h \mid Z = g$.

If X is a subset of A and of B, then $A \cup_X B$, the union of A and B along X, is the quotient of the disjoint union of A and B by the equivalence relation identifying the two copies of each point x of X.

A mapping $f: X \to Y$ is a *near homeomorphism* if it is the (uniform) limit of homeomorphisms of X onto Y.

2. The Manifold Mapping Cylinder Theorem

This section contains the crucial technical result, the Manifold Mapping Cylinder Theorem (Theorem (2.4)), which asserts that the mapping cylinder of a CE-mapping from a Q-manifold to an ANR is always a Q-manifold. It immediately establishes a regular neighborhood theory for the images of such mappings (see Section 3), allows the application of earlier mapping cylinder machinery, and for that reason is essential to the proof of the

Existence Theorem for CE-mappings.

The theorem is preceded by three lemmas. For now, let $f\colon M^Q \to X$ be a CE-mapping from a Q-manifold to an ANR.

(**2.1**) LEMMA. *If $h\colon M^Q\times I \to M^Q\times I$ is a closed embedding which is the identity on $M^Q \times \{1\}$ and is a Z-embedding on $M^Q\times\{0\}$, then for any open cover $\mathcal{U}$ of $M^Q\times I$ limiting $\{\{x\}\times I \cup h(\{x\}\times I)\mid x\in M^Q\}$, there is a $\mathcal{U}$-homeomorphism $g\colon h(M^Q\times I)\to M^Q\times I$ which is the identity on $M^Q\times\{1\}\cup h(M^Q\times\{0\})$.*

Proof. Applying the Isotopy Theorem (A3), one need only assert the existence of an open st^3-refinement of $\mathcal{U}$ which is refined by $\{\{x\}\times I\cup h(\{x\}\times I)\mid x\in M^Q\}$.

If $\varphi\colon M^Q \to (0,1)$ is a map, then the φ-level, $\varphi\cdot M^Q = \{[x,\varphi(x)]\mid x\in M^Q\}\subset M(f)$, separates $M(f)$ into an upper shank, $Z^+(\varphi) = \{[x,t]\mid x\in M^Q,\ \varphi(x)\leqq t\leqq 1\}$, homeomorphic to $M^Q\times I$, and a lower portion, $Z^-(\varphi) = \{[x,t]\mid x\in M^Q,\ 0\leqq t\leqq \varphi(x)\}$, homeomorphic to $M(f)$.

(**2.2**) LEMMA. *For any open cover $\mathcal{U}$ of $M(f)$ and mapping $\varphi\colon M\to(0,1)$, there is a closed $\mathcal{U}$-embedding $g\colon M(f)\to M(f)$, with image missing X, which is supported on $Z^-(\varphi)$.*

Proof. The base X is a Z-set in the ANR $M(f)$ (A9). Without loss of generality, we assume $Z^-(\varphi)\subset \mathrm{st}(X,\mathcal{U})$. Let $\mathcal{V}$ be an open cover of int $Z^-(\varphi)$ which is a star-refinement of $\mathcal{U}\mid \text{int } Z^-(\varphi)$ so fine that (1) any $\mathcal{V}$-mapping of int $Z^-(\varphi)$ into itself is proper and (2) st($\mathcal{V}$) is a normal cover of int $Z^-(\varphi)$ in $M(f)$, i.e., any st($\mathcal{V}$)-mapping of int $Z^-(\varphi)$ into itself extends continuously by the identity to a mapping of $M(f)$ into itself. By (A10) there is a mapping $\mathcal{V}$-close to the identity $h_1\colon \text{int } Z^-(\varphi)\to Z^-(\varphi)\backslash X$, and by (A2) it may be $\mathcal{V}$-approximated by a closed embedding, h_2. Let g be the extension of h_2 to $M(f)$.

The next lemma is the inductive step in the proof of (Theorem (2.4)), where the desired homeomorphism is obtained as an infinite composition. An additional list of terminology seems desirable here: If N^Q is a Z-set Q-submanifold of $M^Q\times I$, then it is collared (A4). If $\mathcal{V}$ is an open cover of $M^Q\times I$ and $c\colon N^Q\times[0,1)\to M^Q\times I$ is an open collar (i.e., an open embedding with $c(x,0)=x$ whenever $x\in N^Q$) which is a $\mathcal{V}$-collar (that is, for each $x\in N^Q$, $c(\{x\}\times[0,1))\subset V_x$ for some $V_x\in\mathcal{V}$) and in addition has the property that $c(N^Q\times[0,t])$ is closed in $M^Q\times I$ for each $t<1$, then the restriction of c to $N^Q\times[0,1/2]$ will be termed a *$\mathcal{V}$-trench about N^Q*. The same term will also be applied to the image $c(N^Q\times[0,1/2])$. Now let $h\colon M(f)\to M^Q\times I$ be a closed embedding which is the "identity" on $M^Q = M^Q\times\{1\}$, and let

$\mathcal{A} = \{\{x\} \times I \cup h\pi(\{x\} \times I) \mid x \in M^Q\}$.

The idea of Lemma (2.3) below is that by an arbitrarily small adjustment, one may change h to an embedding which carries, for some $\varphi: M^Q \to (0, 1/2)$, the φ-level, $\varphi \cdot M(f)$, of $M(f)$ onto the boundary of a "shallow" trench and the lower portion, $Z^-(\varphi)$, into the trench; moreover, one may also adjust this h so that the upper shank $Z^+(\varphi)$ is carried homeomorphically onto that part of $M^Q \times I$ outside the trench without moving points of $M^Q \times \{1\}$. The latter motion may not be made arbitrarily small but may be limited by any open cover $\mathcal{U}$ refined by $\mathcal{A}$. This sets up an inductive construction because the trench is a copy of $M^Q \times [0, 1]$ and $Z^-(\varphi)$ is a copy of $M(f)$. Moreover, the "depth" of the trench selected at one stage gives control on the size of the cover analogous to $\mathcal{A}$ in the next step. (See the proof of Theorem (2.4).)

(2.3) LEMMA. *For any two open covers, $\mathcal{U}$ and $\mathcal{V}$, of $M^Q \times I$ such that $\mathcal{A}$ refines $\mathcal{U}$, there are a Z-set Q-submanifold, N^Q, of $M^Q \times I$ homeomorphic to M^Q and lying in* $\mathrm{st}^-(N^Q, \mathcal{V})$, *a $\mathcal{V}$-trench, $c(N^Q \times [0, 1/2])$, about N^Q, a mapping $\varphi: M^Q \to (0, 1/2)$, and a closed $\mathcal{U}$-embedding $g: h(M(f)) \to M^Q \times I$ such that*

(1) *$g \mid M^Q \times \{1\}$ is the identity,*
(2) *$gh(\varphi \cdot M(f)) = c(N^Q \times \{1/2\})$,*
(3) *$g|h(Z^-(\varphi))$ is a $\mathcal{V}$-embedding (of $h(Z^-(\varphi))$ in $c(N^Q \times [0, 1/2])$,*
(4) *for each $x \in M^Q$, $gh\pi(\{x\} \times [0, \varphi(x)])$ lies in some member of $\mathcal{V}$, and*
(5) *$g(h(Z^+(\varphi))) = M^Q \times I \backslash c(N^Q \times [0, 1/2))$.*

Proof. Suppose that N^Q, $c: N^Q \times [0, 1) \to M^Q \times I$, φ, and g' are chosen to satisfy (1)-(4) above. Let $\alpha: M^Q \times I \to M^Q \times I \backslash c(N^Q \times [0, 1/2))$ be the natural homeomorphism which pushes "in" along c-lines, i.e.,

$$\alpha(x, t) = \begin{cases} c\left(y, \frac{1}{2}(1 + s)\right), & \text{if } (x, t) = c(y, s) \\ (x, t), & \text{if } (x, t) \notin \mathrm{im}(c) \end{cases}.$$

Suppose, further, that g' is limited by an open cover $\mathcal{W}$ of $M^Q \times I$ with the properties that (i) $\mathrm{st}(\mathcal{W})$ refines $\mathcal{U}$ and (ii) $\mathcal{W}$ is refined by the cover

$$\mathcal{A}' = \{\alpha(\{x\} \times I) \cup g'h\pi(\{x\} \times [\varphi(x), 1]) \mid x \in M^Q\}.$$

Then, by applying Lemma (2.1) to the embedding $\beta = \alpha^{-1}g'h\pi\gamma: M^Q \times I \to M^Q \times I$ and the open cover $\alpha^{-1}(\mathcal{W})$, where $\gamma: M^Q \times I \to Z^+(\varphi)$ is given by $\gamma(x, t) = (x, t + (1 - t)\varphi(x))$, one obtains a homeomorphism g'' of $\alpha^{-1}g'h(Z^+(\varphi))$ onto $M^Q \times I$ which is the identity on $M^Q \times \{1\}$ and $N^Q = \alpha^{-1}g'h(\varphi \cdot M(f))$ and which is limited by $\alpha^{-1}(\mathcal{W})$. Now the embedding $g: h(M(f)) \to M^Q \times I$ given by

$$g(x) = \begin{cases} g'(x), & \text{if } x \in h(Z^-(\varphi)) \\ \alpha g''\alpha^{-1}g'(x), & \text{if } x \in h(Z^+(\varphi)) \end{cases}$$

satisfies (5) as well as (1)-(4). (It is assumed here that $c(N^Q \times [0, 1)) \cap M^Q \times \{1\} = \emptyset$, so that there is no movement of $M^Q \times \{1\}$ by α; this follows if $\alpha(N^Q \times [0, 1))$ is sufficiently close to $h(Z)$.) The condition on $\mathcal{W}$ is ensured if g' is limited by a sufficiently fine open cover of $M^Q \times I$ and if $\mathcal{V}$ is sufficiently small with respect to $\mathcal{U}$, or, what is here equivalent, if c is a $\mathcal{V}'$-trench for a sufficiently fine cover $\mathcal{V}'$. Therefore, it suffices to show that for any open cover $\mathcal{V}'$ of $M^Q \times I$, there are a Z-set φ-submanifold N^Q, a $\mathcal{V}'$-trench $c: N^Q \times [0, 1) \to M^Q \times I$ about N^Q, a mapping $\varphi: M^Q \to (0, 1/2)$, and a $\mathcal{V}$-embedding g' of $h(M(f))$ in $M^Q \times I$ satisfying (1)-(4) above.

To obtain N^Q, c, φ and g', let $\mathcal{V}''$ be any open cover of $M^Q \times I$ such that $\mathrm{st}^7(\mathcal{V}'')$ refines $\mathcal{V}'$ and $\mathrm{st}(h(X), \mathrm{st}^3(\mathcal{V}'')) \cap M^Q \times \{1\} = \emptyset$. Let $h': h(M(f)) \to M^Q \times I$ be a closed $\mathcal{V}''$-embedding which is the identity on $M^Q \times \{1\}$ and whose image is a Z-set (A2). Let $\varphi: M^Q \to (0, 1/2)$ be a mapping such that for each $x \in M^Q$, $h\pi(\{x\} \times [0, \varphi(x)])$ lies in some element, V_x, of $\mathcal{V}''$. Let $N^Q = h'h(\varphi \cdot M^Q)$. Because N^Q is closed, there is a $\mathcal{V}''$-trench $c: N^Q \times [0, 1/2] \to M^Q \times I$ about N^Q. (Not every open collar of N^Q will serve to define a $\mathcal{V}''$-trench, for in the case that N^Q is not compact one must take care to assure that c be a closed embedding; this may be done simply by making c a $\mathcal{V}'''$-trench for an open cover $\mathcal{V}'''$ of $M^Q \times I$ which refines $\mathcal{V}''$ and has the property that for some complete metric on $M^Q \times I$, there are, for each $\varepsilon > 0$, at most finitely many elements of $\mathcal{V}'''$ of diameter as large as ε in any one given component of $M^Q \times I$.)

Now, let h'' be a closed $\mathcal{V}''$-embedding of $h'h\, M(f)$ in itself missing $h'h(X)$ which is supported on $Z^-(\varphi)$ (A9, A10, and A2), and note that $\mathrm{im}(h'') \subset h'hZ^+(\psi)$ for some mapping $\psi: M^Q \to (0, 1/2)$ with $\psi(x) < \varphi(x)$ for each $x \in M$. Let $\alpha: M^Q \times I \to M^Q \times I \backslash c(N^Q \times [0, 1/2))$ be defined as before and define $h''': h''h'(M(f)) \to M^Q \times I$ by

$$h'''(x) = \begin{cases} \alpha(x), & \text{if } x \in h(Z^+(\varphi)) \\ c\left(y, \dfrac{1}{2}\,\dfrac{t - \psi(z)}{\varphi(z) - \psi(z)}\right), & \text{if } x = h'h\pi(z, t) \text{ and } \psi(z) \leqq t \leqq \varphi(z) \end{cases}.$$

(That is, for each $y \in N^Q$ there are two arcs emanating from y; one is $c(\{y\} \times [0, 1/2])$ and the other is $h'h\pi(\{z\} \times [\psi(z), \varphi(z)])$ with $z \in M^Q$ such that $y = h'h\pi(z, \varphi(z))$. The mapping h''' moves the latter onto the former while pushing the former into $M^Q \times I \backslash c(N^Q \times [0, 1/2))$ via α.) Now let $g' = h'''h''h'$.

(**2.4**) THEOREM. *The mapping cylinder $M(f)$ of a cell-like mapping $f: M^Q \to X$ of a Q-manifold onto an* ANR *is a Q-manifold; moreover, the* *inclusion* $M^Q \hookrightarrow M(f)$ *is properly homotopic to a homeomorphism.*

Proof. Because X is a Z-set in $M(f)$ (A9), there is (A10) a map $g_0\colon M(f) \to M^Q \times (0, 1] = M(f)\backslash X$ which is the identity on $M^Q \times \{1\}$ and is limited by an open cover, such as one fine enough to guarantee that it is proper. This, in turn, may be approximated by a closed embedding $\bar{g}_0$ in $M(f)$, also the identity on $M^Q \times \{1\}$, missing X. Then $\pi^{-1}\bar{g}_0$ is a closed embedding of $M(f)$ in $M^Q \times I$ which is the "identity" on $M^Q \times \{1\}$.

If, using the notation of Lemma (2.3), we substitute $c(N^Q \times [0, 1/2])$ for $M^Q \times I$, $c(N^Q \times \{1/2\})$ for $M^Q \times \{1\}$, $Z^-(\varphi)$ for $M(f)$ and gh for h, then the analogue of $\mathfrak{A}$ is

$$\mathfrak{A}^g = \left\{c\left(\{y\} \times \left[0, \frac{1}{2}\right]\right) \cup gh\pi(\{x\} \times [0, \varphi(x)]) \,|\, x \in M^Q \text{ and } y = h'h\pi(x, \varphi(x))\right\}.$$

Now, $\mathfrak{A}^g$ does not necessarily refine $\mathcal{V}$, but it does refine $\mathrm{st}(\mathcal{V})$. Therefore, by an inductive application of Lemma (2.3), one may find

(a) a sequence $\{\mathcal{V}_i\}_{i=1}^{\infty}$ of open covers of $M^Q \times I$,

(b) a sequence $\{c_i\colon M^Q \times [0, 1/2] \to M^Q \times I\}_{i=1}^{\infty}$ of $\mathcal{V}_i$-trenches about Z-set Q-submanifolds,

(c) a sequence $\{g_i\}_{i=1}^{\infty}$ of closed $\mathcal{V}_{i-1}$-embeddings (here let

$$\mathcal{V}_0 = \{M^Q \times I\}) \quad g_i\colon g_{i-1}g_{i-2} \cdots g_1\pi^{-1}\bar{g}_0(M(f)) \longrightarrow M^Q \times I\,,$$

and

(d) a sequence $\{\varphi_i\colon M^Q \to (0, 1/2^i)\}_{i=1}^{\infty}$ of mappings together satisfying the following conditions: (Let

$$C_i = c_i\left(M^Q \times \left[0, \frac{1}{2}\right]\right) \text{ if } i \geqq 1,\ C_0 = M^Q \times I,\ h_i = g_i g_{i-1} \cdots g_1\pi^{-1}\bar{g}_0$$
$$\text{if } i \geqq 1, \text{ and } h_0 = \pi^{-1}\bar{g}_0\,.)$$

(i) $\varphi_{i+1}(x) < \varphi_i(x)$ for each $x \in M^Q$,

(ii) $h_i(Z^+(\varphi_i)) = M^Q \times I\backslash\mathrm{int}(C_i)$, for $i \geqq 1$,

(iii) g_{i+1} is supported on C_i, which lies in C_{i-1},

(iv) $g_i \,|\, h_{i-1}Z^-(\varphi_i)$ is a $\mathcal{V}_i$-embedding,

(v) for each $x \in M^Q$, $g_i g_{i-1} \cdots g_1\pi^{-1}g_0\pi(\{x\} \times [0, \varphi_i(x)])$ lies in some member of $\mathcal{V}_i$ and

(vi) given the choice of g_{i-1}, $\mathcal{V}_i$ may be chosen arbitrarily small.

If the successive covers $\mathcal{V}_i$ are inductively chosen sufficiently fine (i.e., depending upon the previous choices of $\mathcal{V}_j$'s, g_j's, etc.), then the sequence $\{h\}_{i=0}^{\infty}$ will converge to the desired homeomorphism, h, of $M(f)$ onto $M^Q \times I$. This is seen as follows: First, if ρ is a complete metric for $M^Q \times I$ and the

ρ-mesh of $\mathcal{V}_i$ is less than $1/2^i$, then $\{h_i\}_{i=0}^{\infty}$ is uniformly Cauchy and converges to a mapping $h: M(f) \to M^Q \times I$. Second, because h_0 is a closed embedding and both $M(f)$ and $M^Q \times I$ are locally compact, the $\mathcal{V}_i$'s may be chosen so fine that h is a proper map. Third, since C_i must lie within $\mathrm{st}^3(h_{i-1}(X); \mathcal{V}_i)$ by (b), (iv), and (v), and $\mathrm{im}(h_i)$ contains the complement of C_i by (i), a choice of $\mathcal{V}_i$'s such as "ρ-mesh $(\mathcal{V}_i) < 1/2^i$" suggested above will guarantee that $h(M(f)\backslash X) = M^Q \times I \backslash h(X)$, hence that h is onto. Fourth, $h \mid M(f)\backslash X$ is an open embedding by (ii) and (iii), so that if h is one-to-one on X it is a homeomorphism. However, $g_i \mid h_{i-1}(X)$ is a $\mathcal{V}_i$-embedding by (iv) and this, too, is just a matter of inductively choosing the $\mathcal{V}_i$'s sufficiently fine.

We see that as $h: M(f) \to M^Q \times I$ is the "identity" on $M^Q = M^Q \times \{1\}$ and as (A1) yields a homeomorphism $g: M^Q \to M^Q \times I$ properly homotopic to the inclusion $M^Q \to M^Q \times \{1\} \hookrightarrow M^Q \times I$, the composition $h^{-1} \circ g: M^Q \to M(f)$ is a homeomorphism properly homotopic to the inclusion $M^Q \hookrightarrow M(f)$.

(**2.5**) Corollary (to proof). *The quotient mapping* $\pi: M^Q \times I \to M(f)$ *is a near-homeomorphism*; moreover, the approximating homeomorphisms may be required to agree with* π *off any neighborhood of* $M^Q \times \{0\}$ *and to carry* $M^Q \times \{0\}$ *to a set containing* X.

Proof. Letting $\mathcal{U}$ be any open cover of $M(f)$, consider requiring, in the proof of (2.4), $\bar{g}_0$ to be limited by $\mathcal{U}$, so that $\pi^{-1}\bar{g}_0\pi$ is within $\pi^{-1}(\mathcal{U})$ of the identity. Suitable control of the g_i's will ensure that $g: M(f) \to M^Q \times I$ is the identity off $\pi^{-1}(\mathrm{st}(X, \mathcal{U}))$ and that $g\pi$ is within $\pi^{-1}(\mathcal{U})$ of the identity. Further restrictions will guarantee that the projection of $g(X)$ to $M^Q \times \{0\}$ is $\pi^{-1}(\mathcal{U})$-homotopic to the identity, so that application of (A3) yields a $\mathcal{U}$-adjustment of g to a homeomorphism h carrying X into $M^Q \times \{0\}$, still agreeing with π^{-1} off $\mathrm{st}^2(M^Q \times \{0\}, \pi^{-1}(\mathcal{U}))$. Then h^{-1} is the desired homeomorphism for suitable $\mathcal{U}$.

3. A Regular Neighborhood Theorem

The Manifold Mapping Cylinder Theorem (2.4) quickly leads to a regular neighborhood theory for ANR's which are the images under CE-mappings of Q-manifolds whenever they are embedded as Z-sets of Q-manifolds†, (3.1, 3.3, 3.4), and this in turn yields a characterization of such ANR's in terms of separating Q-manifold factors (3.5, 3.6).

(**3.1**) Theorem (Existence of Mapping Cylinder Neighborhoods). *If* $f: M^Q \to X$ *is a* CE*-mapping of a* Q*-manifold to the* ANR X *and* L^Q *is any*

* Remarked by several including R. D. Edwards.

† Certain intersection with R. T. Miller's Regular Neighborhoods [23] is inevitable here.

Q-manifold containing X as a Z-set, then X has a mapping cylinder neighborhood in L^Q, i.e., *there is an embedding* $h\colon (M(f), M^Q, X) \to (L^Q, \operatorname{bd}(\operatorname{im}(h)), X)$ *of $M(f)$ in L^Q as a neighborhood of X with $h(M^Q)$ bicollared in L^Q and h the identity on X.*

Proof. Let $i\colon X \to L^Q$ be inclusion. Replace $M(f) \xrightarrow{c(f)} X \xrightarrow{i} L^Q$ by a Z-embedding g which restricts to i on X (A2). Now $g(M(f))$ is a Z-set submanifold of L^Q and so is collared (A4). Let $\alpha\colon M(f) \times I \to L^Q$ be a closed collar of $g(M(f))$ with $\alpha(M(f) \times \{1\})$ bicollared in L^Q and equal to the boundary of $\operatorname{im}(\alpha)$.

Let $\beta\colon M(f) \to M(f) \times I$ be a homeomorphism properly homotopic to the inclusion $M(f) \to M(f) \times \{0\}$ (A1), and note that $\beta \mid X \cup M^Q$ is by (2.4) properly homotopic in $M(f) \times I$ to a closed embedding β' which is the inclusion on X and restricts to a homeomorphism of M^Q onto $M(f) \times \{1\}$. Thus, there is, by (A9) and (A3), a homeomorphism $\bar{\beta}\colon (M(f), X, M^Q) \to (M(f) \times I, X \times \{0\}, M(f) \times \{1\})$ which is the natural inclusion on X. Now $h = \alpha\bar{\beta}$ is a mapping cylinder neighborhood of X in L^Q.

Before proving the uniqueness of these regular neighborhoods, it is convenient to note the following lemma.

(**3.2**) Proposition (Uniqueness of domain for CE-mappings). *If $f\colon M^Q \to X$ and $g\colon N^Q \to X$ are* CE-*mappings of* Q-*manifolds to the* ANR X, *then for any open cover $\mathcal{U}$ of X there is a homeomorphism $h\colon M^Q \to N^Q$ such that gh is $\mathcal{U}$-close to f.*

Proof. This is immediate from the fact that in the diagram below, the maps across the top and on both sides are near-homeomorphisms.

$$
\begin{tikzcd}
M(f) \times Q \arrow[d] & (M(f) \cup_X M(g)) \times Q \arrow[l] \arrow[r] \arrow[dd] & M(g) \times Q \arrow[d] \\
M(f) & & M(g) \\
 & M(f) \cup_X M(g) \arrow[dd] & \\
M^Q \times I \arrow[uu, "\pi(f)"] \arrow[d] & & N^Q \times I \arrow[uu, "\pi(g)"'] \arrow[d] \\
M^Q \arrow[r, "f"] & X & N^Q \arrow[l, "g"']
\end{tikzcd}
.
$$

(The projections $M^Q \times I \to M^Q$, $N^Q \times I \to N^Q$, $M(f) \times Q \to M(f)$, and $M(g) \times Q \to M(g)$ are near-homeomorphisms by (A1), $\pi(f)\colon M^Q \times I \to M(f)$ and $\pi(g)\colon N^Q \times I \to M(g)$ are near-homeomorphisms by (2.5), and

$$(1 \cup c(g)) \times 1\colon (M(f) \cup_X M(g)) \times Q \longrightarrow M(f) \times Q \text{ and}$$
$$(c(f) \cup 1) \times 1\colon (M(f) \cup_X M(g)) \times Q \longrightarrow M(g) \times Q$$

are near-homeomorphisms by (2.5), (A7) or (A7a), and (A1a).)

(**3.3**) THEOREM (Uniqueness of regular neighborhoods for images of CE-mappings). *Suppose that* $f\colon M^Q \to X$ *and* $g\colon N^Q \to X$ *are* CE-*mappings of* Q-*manifolds onto the* ANR X *and that* L^Q *is a* Q-*manifold containing* X *as a* Z-*set. If* $\zeta\colon (M(f), M^Q, X) \to (L^Q, \operatorname{bd}\operatorname{im}(\zeta), X)$ *and* $\eta\colon (M(g), N^Q, X) \to (L^Q, \operatorname{bd}\operatorname{im}(\eta), X)$ *define mapping cylinder neighborhoods of* X *in* L^Q (as in (3.1)) *then for any open cover* $\mathcal{U}$ *of* L^Q *refined by* $\{\zeta c(f)^{-1}(x) \cup \eta c(g)^{-1}(x) \mid x \in X\}$, *there is an ambient* $\mathcal{U}$-*isotopy* $\Lambda\colon L^Q \times I \to L^Q$ *which is stationary on* X *and off* $\operatorname{st}(X, \mathcal{U})$ *between the identity and a homeomorphism* λ_1 *of* L^Q *such that* $\lambda_1(\operatorname{im}(\zeta)) = \operatorname{im}(\eta)$.

Proof. One first gives each neighborhood the structure of a collar of a Z-set submanifold; next, one adjusts matters so that they are given as collars of the same submanifold; lastly, one adds an "external" collar to the submanifold and deforms one neighborhood along its collar lines into the external collar and then out along the other neighborhood's collar lines onto it. Details are as follows:

Let $\mathcal{V}$ be an open st^4-refinement of $\mathcal{U}$ refined by $\{\zeta c(f)^{-1}(x) \cup \eta c(g)^{-1}(x) | x \in X\}$, let $\mathcal{W}$ be an open st^3-refinement of $\mathcal{V}$ such that $\operatorname{st}^2(X, \mathcal{W}) \subset \operatorname{image}(\zeta) \cap \operatorname{image}(\eta)$, and let $\mathcal{O}$ be an open refinement of $\mathcal{W}$ such that proper maps into L^Q are properly $\mathcal{W}$-homotopic if they are (st-$\mathcal{O}$)-close to each other. Using (2.5), replace $\pi(f)\colon M^Q \times I \to M(f)$ and $\pi(g)\colon M^Q \times I \to M(g)$ by homeomorphisms so close to them that their compositions with ζ and η are $\mathcal{O}$-close to $\zeta\pi(f)$ and $\eta\pi(g)$, and require that these replacements agree with $\pi(f)$ and $\pi(g)$ on $M^Q \times \{1\}$ and $N^Q \times \{1\}$ and that the replacement for $\pi(g)$ carry $N^Q \times \{0\}$ to a set containing X. Denote the compositions by

$$\zeta'\colon M^Q \times I \longrightarrow \operatorname{image}(\zeta) \text{ and } \eta'\colon N^Q \times I \longrightarrow \operatorname{image}(\eta)\ .$$

(This gives the collar structures.)

Note that $\zeta\pi(f)\zeta'^{-1} | \zeta'(M^Q \times \{0\})$ and $\eta\pi(g)\eta'^{-1} | \eta'(N^Q \times \{0\})$ are $\mathcal{O}$-close to the identity and that (3.2) may be invoked to give a homeomorphism $\alpha\colon \zeta'(M^Q \times \{0\}) \to \eta'(N^Q \times \{0\})$ such that $\eta\pi(g)\eta'^{-1}\alpha$ is $\mathcal{O}$-close to $\zeta\pi(f)\zeta'^{-1} | \zeta'(M^Q \times \{0\})$. Now α is (st-$\mathcal{O}$)-close to the identity, hence properly $\mathcal{W}$-homotopic to it. As such a homotopy must lie in $\operatorname{st}^2(X, \mathcal{W})$, there is by (A3) an ambient homeomorphism $\bar{\alpha}$ of L^Q extending α, supported on $\operatorname{image}(\zeta) \cap \operatorname{image}(\eta)$, and $\mathcal{V}$-close to the identity. (This gives the two neighborhoods the structure of a collar of $\eta'(N^Q \times \{0\})$.)

Let $L_1^Q = L^Q \times \{0\} \cup \eta'(N^Q \times \{0\}) \times I$ and let $\beta\colon L^Q \to L_1^Q$ be a homeomorphism with support on $\operatorname{image}(\zeta) \cap \operatorname{image}(\eta)$ which extends the inclusion $\eta'(N^Q \times \{0\}) \to \eta'(N^Q \times \{0\}) \times \{1\}$ and is such that β^{-1} is $\mathcal{V}$-close to the projection $L_1^Q \to L^Q$.

Note that now if $m \in M^Q$ then $\bar{\alpha}\zeta'(\{m\} \times I) \cup \eta'(\{n\} \times I)$ lies in an element of $\mathrm{st}^3\text{-}\mathcal{V}$ if $n \in N^Q$ is such that $\eta'(n, 0) = \bar{\alpha}\zeta'(m, 0)$ and that hence $\bar{\alpha}\zeta'(\{m\} \times I) \cup \eta'(\{n\} \times I) \cup \beta^{-1}(\eta'(n, 0) \times I)$ lies in an element of $\mathrm{st}^4\text{-}\mathcal{V}$. Using the collar structures, let $H\colon L_1^Q \times I \to L_1^Q$ be an ambient isotopy which is supported on $\mathrm{st}(X, \mathrm{st}^4\text{-}\mathcal{V}) \times \{0\} \cup \eta'(N^Q \times \{0\}) \times I$, is stationary on $\eta'(N^Q \times \{0\}) \times \{1\}$, is limited by $\beta(\mathrm{st}^4\text{-}\mathcal{V})$, and deforms $\zeta(M(f)) \times \{0\} \cup \eta'(N^Q \times \{0\}) \times I$ into $\eta'(N^Q \times \{0\}) \times I$ along the $\bar{\alpha}\zeta'$ collar lines and then back out along the η' collar lines onto $\eta(M(g)) \times \{0\} \cup \eta'(N^Q \times \{0\}) \times I$. Then $\Lambda = \{\lambda_t = \beta^{-1}h_t\beta\}_{t \in I}$ is the sought-after isotopy of L^Q.

(**3.4**) *Remark on regular neighborhoods.* If $f\colon M^Q \to X$ is not CE, but *is* proper and surjective, then X is not a Z-set in $M(f)$, which may nevertheless be a Q-manifold.

As a consequence of the existence of regular neighborhoods for ANR images of CE-mappings (3.1) and the Manifold Mapping Cylinder Theorem (2.4) one may characterize those ANR's which are the images of Q-manifolds under CE-mappings as follows:

(**3.5**) CHARACTERIZATION THEOREM. *The following are equivalent for a locally compact metric* ANR X:

(i) *X is the image of a* Q-*manifold by a* CE-*mapping*;

(ii) *whenever X is embedded as a Z-set in each of two* Q-*manifolds M^Q and N^Q, their union $M^Q \cup_X N^Q$ is a* Q-*manifold factor, and*

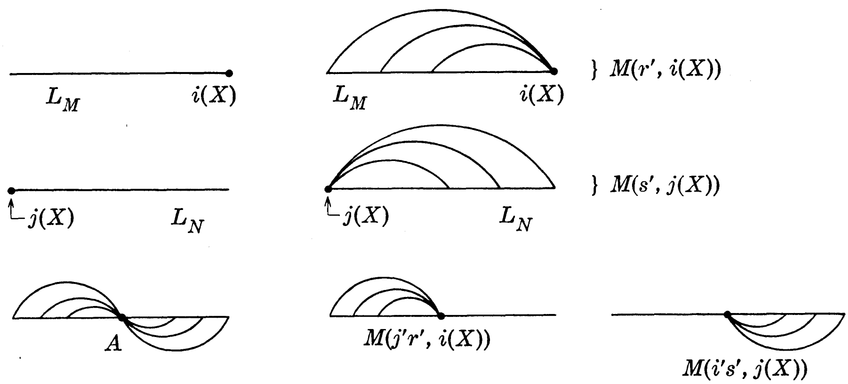
(iii) *for some pair M^Q and N^Q of* Q-*manifolds and Z-embeddings of X in them, the union $M^Q \cup_X N^Q$ along X is a* Q-*manifold factor.*

Proof. (i) $\Rightarrow$ (ii). This follows immediately from the Mapping Cylinder Theorem (A7) (or A7a) (absolute form), applied to the decomposition $M^Q \cup_X N^Q = \overline{M^Q \backslash h(M(f))} \cup M(if)$, where $i\colon X \hookrightarrow N^Q$ is the given Z-embedding, $f\colon L^Q \to X$ is a CE-mapping, and $h\colon M(f) \to M^Q$ parameterizes a regular neighborhood of X in M^Q.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i). Let $i\colon X \to M^Q$ and $j\colon X \to N^Q$ be the given embeddings, and let $r\colon U \to X$ and $s\colon V \to X$ be neighborhood retractions in M^Q and N^Q, respectively. Choose closed Q-manifold neighborhoods of X, $X \subset L_M \subset U$ and $X \subset L_N \subset V$, in U and V, respectively, such that $r' = r \mid L_M$ and $s' = s \mid L_N$ are proper. (This may be done using (A6) and (A5).)

Now consider the space $A = M(r', i(X)) \cup_X M(s', j(X))$. If $i'\colon i(X) \to L_M$ and $j'\colon j(X) \to L_N$ are the inclusions, then A may also be regarded as the union of $M(j'r', i(X))$ and $M(i's', j(X))$ along $L_M \cup_X L_N$, the last being a Q-manifold factor by hypothesis, while each of the first two is a Q-manifold factor by the Relative Mapping Cylinder Theorem (A7), (A8), (A7a). (See diagram.) Thus, A is a Q-manifold factor by (A8) (A8a), and the mapping $A \times Q \to A \to X$ is the desired CE-mapping of a Q-manifold to X.



(3.6) *Remark.* One could add to the above list:

(iv) *Whenever X is embedded as a Z-set in each of two* Q-*manifold factors, A and B, their union, $A \cup_X B$, along X is a* Q-*manifold factor.*

4. Existence of CE-mappings onto ANR's

Here the Manifold Mapping Cylinder Theorem (2.4) and the Characterization Theorem (3.5) are applied to Miller's Cone Theorem (A11) to obtain the Existence Theorem for CE-mappings.

(4.1) THEOREM. *Every locally compact* ANR, *X, is the image of a* Q-*manifold by a* CE-*mapping.*

Proof. First, assume that X is compact. Let $C(X) = X \times I/X \times \{0\}$ be its cone. By Miller's Theorem (A11), there is a CE-mapping $f\colon Q \to C(X)$. By the Manifold Mapping Cylinder Theorem (2.4), $M(f)$ is a Hilbert cube, and $C(X)$ is a Z-set of $M(f)$ by (A9). Let Y be the double of $M(f)$ about $C(X)$, $Y = M(f) \cup_{C(X)} M(f)$. Now by the Characterization Theorem (3.5), $Y \times Q$ is a Q-manifold (and a Hilbert cube, at that, as it is contractible [7], [8]).

Let Z be the result of slitting Y along $C(X)$ from the vertex half of the way to $X \times \{1\}$. ($Z = M(f) \cup_A M(f)$, where $A = X \times [1/2, 1]$.) It is immediate from the above discussion that $Z \backslash (X \times \{1/2\})$ is a Q-manifold factor. But since $C(X)$ is a Z-set of $M(f)$, so is A, and there is by (A3) a homeomorphism h of $M(f)$ such that $h|A$ is reflection about $X \times \{3/4\}$, i.e., $h(x, t) = (x, 3/2 - t)$. Then $h \cup h\colon Z \to Z$ is a homeomorphism carrying $X \times \{1/2\}$ to $X \times \{1\}$. Thus, $X \times \{1/2\}$ has a neighborhood in Z which is a Q-manifold factor and Z is a Q-manifold factor. By the Characterization Theorem (3.5), A is the image of a Q-manifold by a CE-mapping, and so is X.

In the case that X is not compact, let $C(X^*, *)$ be the cone of the one-point compactification, $X^* = X \cup \{*\}$, reduced at the compactification point $\big(C(X^*, *) = X^* \times [0, 1]/(X^* \times \{0\} \cup \{*\} \times I)\big)$. By (A8a) this is an AR and thus the image of Q by a CE-mapping $f: Q \to C(X^*, *)$. Let $M^Q = f^{-1}\big(C(X^*, *)\backslash\{v\}\big)$, where v is the vertex. Then, as $C(X^*, *)\backslash\{v\} = X \times (0, 1]$, one has a CE-mapping $f' = f \mid M^Q: M^Q \to X \times (0, 1]$. Now the proof given in the compact case goes through with

$$Y = M(f') \cup_{X \times (0,1]} M(f') \text{ and } Z = M(f') \cup_{X \times [1/2,1]} M(f') \ .$$

5. Consequences

Here are collected the consequences of the previous section.

(**5.1**) COROLLARY (Regular Neighborhood Theorem). *If the locally compact* ANR X *is embedded in a* Q-*manifold* M^Q *as a Z-set,* $f: N^Q \to X$ *is a* CE-*mapping to* X *of a* Q-*manifold, then there is a neighborhood* A *of* X *in* M^Q *and a homeomorphism* $\zeta: \big(M(f), N^Q, X\big) \to (A, \mathrm{bd}\, A, X)$ *which is the identity on* X; *if* $g: L^Q \to X$ *is another* CE-*mapping of a* Q-*manifold to* X *and* $\eta: \big(M(g), L^Q, X\big) \to (B, \mathrm{bd}\, B, X)$ *is another such neighborhood of* X *in* M^Q, *then for any open cover* $\mathfrak{U}$ *of* M^Q *such that for each* $x \in X$, *there is an element* U *of* $\mathfrak{U}$ *containing* $\zeta c(f)^{-1}(x) \cup \eta c(g)^{-1}(x)$, *there is an ambient* $\mathfrak{U}$-*isotopy* Λ *of* M^Q *which is stationary on* X *and supported on* $\mathrm{st}(X, \mathfrak{U})$ *such that* $\lambda_0 = \mathrm{id}$ *and* $\lambda_1(A) = B$.

Proof. This is (3.1), (3.3), and (4.1).

(**5.2**) COROLLARY (Sum Theorem). *The union of two* Q-*manifold factors along a common* ANR *Z-set is a* Q-*manifold factor.*

Proof. This is (3.5) and (3.6) along with (4.1).

(**5.3**) COROLLARY (Homotopy types of ANR's). *Compact* ANR's *are homotopy-equivalent to compact polyhedra; locally compact* ANR's *are proper-homotopy-equivalent to locally compact polyhedra.*

Proof. Let X be a locally compact ANR. By (4.1) there is a CE-map $f: M^Q \to X$ from a Q-manifold to X. By (A9, A9a) f is a proper homotopy equivalence. M^Q is compact if X is. Finally, by (A6), M^Q is homeomorphic to $K \times Q$ for some locally finite simplicial complex K, so $K \to K \times Q \to X$ is the proper homotopy equivalence.

(**5.4**) COROLLARY (Homotopy Dimension Estimate) (Remarked by E. Dyer). *If* X *is an n-dimensional, compact* ANR, *then* X *has the homotopy type of a compact m-dimensional simplicial complex* K, *where* $m = \max\{3, n\}$.

Proof. By (5.3) there is a finite simplicial complex, L, homotopy-equivalent to X. Since $\dim X = n$, Wall's homotopy dimension estimate [29] holds.

(5.5) THEOREM (Extension of simple homotopy theory) (Chapman [12]). *For each connected, compact, metric* ANR X, *assign the simple homotopy type of a finite cell-complex* K *for which there is a* CE-*mapping* $f\colon K \times Q \to X$, *and if* $h\colon X \to Y$ *is a homotopy equivalence between* X *and another compact metric* ANR *for which* $g\colon L \times Q \to Y$ *is a* CE-*mapping with* L *a finite cell-complex, assign to* h *the torsion* $\tau(h) = g_* i_* \tau(h') \in \mathrm{Wh}(\pi_1(Y))$ *where* $i\colon L \to L \times Q$ *is the inclusion* $x \mapsto (x, p)$ *for some* $p \in Q$ *and*

$$h' = K \hookrightarrow K \times Q \xrightarrow{f} X \xrightarrow{h} Y \hookrightarrow M(g) = L \times Q \to L$$

with "=" being a homeomorphism and other maps either inclusions or projections. This assignment is well-defined, and $\tau(h) = 0$ *if and only if* h *is homotopic to a mapping* h_1 *which is a finite composition* $X \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} X_2 \longrightarrow \cdots \xrightarrow{\alpha_n} X_n = Y$ *in which* α_i *is either a* CE-*mapping or the inclusion of the image of a* CE-*retraction into its domain and all* X_i *are compact, connected metric* ANR's.

(For a proof, see [12]. One also can assert the analogous extension for non-compact, locally compact ANR's, proper-homotopy equivalences, and infinite simple homotopy theory: See [12] again.)

Appendix A: Basic citations

For the benefit of the reader who is not familiar with the literature in Q-manifold theory, the following overview is offered as an aid.

(A1) STABILITY THEOREM ([8], implicit in [3]). *If* M^Q *is a* Q-*manifold, then the projection* $M^Q \times Q \to M^Q$ *is a near-homeomorphism.*

(A1a) *Remark.* Chapman [11] has generalized the Armentrout-Siebenmann result on CE-mappings [4], [25] to Q-manifolds, obtaining the result that any CE-mapping between Q-manifolds is a near-homeomorphism. Simple proofs of this from Theorem (2.4) of this paper have been independently discovered by A. Fathi [13] and H. Torúncyzk.

(A2) MAPPING APPROXIMATION LEMMA ([2], [8]). *Every proper mapping* $f\colon X \to M^Q$ *of a locally compact space into a* Q-*manifold is* $\mathcal{U}$-*close to a* Z-*embedding,* g, *for any open cover* $\mathcal{U}$ *of* M^Q; *moreover, if* f *is a* Z-*embedding on the closed subset* A *of* X, *then* g *may be taken to extend* $f \mid A$.

(A3) ISOTOPY THEOREM (see [2], [8]). *Let* $\mathcal{U}$ *be an open cover of the* Q-*manifold* M^Q, *and suppose* $F\colon X \times I \to M^Q$ *is a proper* $\mathcal{U}$-*homotopy between*

Z-embeddings F_0 and F_1. There is an ambient st^3-$\mathfrak{U}$ *isotopy G: $M^Q \times I \to M^Q$ such that $G_0 =$ id and $G_1 F_0 = F_1$; moreover, if F is stationary on the closed subset $A \subset X$,* i.e., *$F(x, t) = F(x, 0)$ for all t and each $x \in A$, then G may be taken to be stationary on A.*

(A4) COLLARING THEOREM ([2], [8]). *A Z-set sub-*Q*-manifold in a* Q*-manifold is collared.*

(A5) PRODUCT THEOREM ([30], [31], [33], [6], [18], [8]). *The product of a locally finite cell-complex and a Hilbert cube is always a* Q*-manifold.*

(A6) TRIANGULATION THEOREM ([9], [10]). *Each* Q*-manifold is homeomorphic to the product with Q of a locally finite simplicial complex.*

(A7) RELATIVE MAPPING CYLINDER THEOREM ([33], see also [31] and [8]). *If f: $X \to Y$ is any mapping between compact* Q*-manifold factors, then $M(f, A)$ is a* Q*-manifold factor for each closed $A \subset X$.*

(A7a) *Remark*. By induction using (A6), one may readily extend (A7) to locally compact X, A, and Y, provided that f is proper.

(A8) SUM THEOREM FOR Q-MANIFOLD FACTORS ([31]). *If X, Y, and $X \cap Y$ are* Q*-manifold factors, then so is $X \cup Y$.*

(A8a) This has been extended to the locally compact case by L. S. Newman, Jr. [24] using the observation that if A is a locally compact ANR, then the cone $C(A^*, *)$ reduced at infinity of the one-point compactification of A is a compact AR.

The next results connect Z-sets, CE-mappings and mapping cylinders.

(A9) THEOREM ([14], see also [17], [20], [28]). *If f: $X \to Y$ is a* CE*-mapping between* ANR'*s, then the base, Y, of $M(f)$ has Property Z in $M(f)$.*

(A9a) *Remark*. [14], [17], and [28] explicitly treat the infinite-dimensional case and show that a CE-mapping between ANR's is a proper homotopy equivalence.

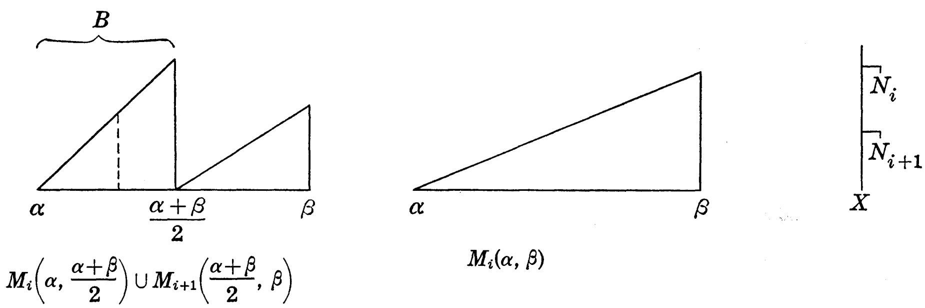
(A10) THEOREM ([15]). *If A is a Z-set in the* ANR *X, then for every open cover $\mathfrak{U}$ of X there is a homotopy inverse f: $X \to X \backslash A$ of the inclusion $X \backslash A \to X$ with the property that both homotopies to the identity may be limited by $\mathfrak{U}$.*

(A11) Miller's Cone Theorem ([23], see also Appendix B). *The cone over each compact* ANR *is the image of Q under a* CE*-mapping.*

Appendix B: A proof of Miller's Cone Theorem

Let X be any compact metric ANR and $C(X)$ its cone. Assume $X \subset Q$.

Let $r: U \to X$ be a retraction to X of a neighborhood in Q, and let $\{N_i\}_{i=1}^{\infty}$ be a sequence of closed neighborhoods of X in U such that (a) $X = \bigcap_{i=1}^{\infty} N_i$, (b) the topological boundary of each N_i is a bicollared Q-submanifold of N_{i-1}, with that of N_1 bicollared in Q, and (c) $r \mid N_i$ is $1/2^i$-homotopic to the identity in N_{i-1}, if $i > 1$. For each i and each closed interval $[\alpha, \beta]$ let $M_i(\alpha, \beta)$ be the copy of $M(r \mid N_i)$ with $[\alpha, \beta]$ replacing the interval $[0, 1]$.



LEMMA. *There is a* CE-*mapping of* $M_i(\alpha, \beta) \times Q$ *onto* $[M_{i+1}(\alpha, (\alpha+\beta)/2) \cup M_{i+1}((\alpha+\beta)/2, \beta)] \times Q$ *which changes neither* Q-*coordinate of any point by more than* $1/2^{i-1}$ *and is the identity on* $N_{i+1} \times \{\beta\} \times Q$ *and* $X \times \{\alpha\} \times Q$.

Proof. Let

$$B = N_i \times \left[\frac{2\alpha + \beta}{4}, \frac{\alpha + \beta}{2}\right] \subset M_i\left(\alpha, \frac{\alpha + \beta}{2}\right).$$

Then $B \times Q$ and $(B \cup M_{i+1}((\alpha + \beta)/2, \beta)) \times Q$ are Q-manifolds (A7), so if $c: B \cup M_{i+1}((\alpha+\beta)/2, \beta) \to B$ is the collapse (induced from that of $M_{i+1}((\alpha+\beta)/2, \beta)$), then $c \times \mathrm{id}_Q$ may be approximated arbitrarily closely by a homeomorphism, f (A1a); moreover, because $c \mid N_{i+1} \times \{\beta\}$ is $1/2^{i+1}$-homotopic to the natural map $j: N_{i+1} \times \{\beta\} \to N_{i+1} \times \{(\alpha + \beta)/2\}$ in B, one may use (A3) to adjust f to agree with $j \times \mathrm{id}_Q$ on $(N_{i+1} \times \{\beta\}) \times Q$, to be the identity on $(N_i \times \{(2\alpha+\beta)/4\}) \times Q$, and to remain within $1/2^i$ of $c \times \mathrm{id}_Q$. Now, if $g: M_i(\alpha, \beta) \times Q \to M_i(\alpha, (\alpha+\beta)/2) \times Q$ merely reparameterizes the interval $[\alpha, \beta]$, then $f^{-1}g$ is a homeomorphism of $M_i(\alpha, \beta) \times Q$ onto $M_i(\alpha, (\alpha + \beta)/2) \times Q \cup M_{i+1}((\alpha + \beta)/2, \beta) \times Q$ which is the identity on $N_{i-1} \times \{\beta\} \times Q \cup X \times \{\alpha\} \times Q$ and changes neither Q-coordinate of any point by as much as $1/2^i$. Using the collar of $\mathrm{bd}(N_{i+1})$ in $\overline{N_i \backslash N_{i+1}}$ and the fact that $r \mid N_i$ is within $1/2^i$ of the identity, it is easy to construct a CE-retraction $h: M_i(\alpha, (\alpha + \beta)/2) \to M_{i+1}(\alpha, (\alpha + \beta)/2)$ changing no point's Q-coordinate by as much as $1/2^i$. Now $hf^{-1}g$ is the desired CE-mapping.

THEOREM (Miller [23]). *The cone of any compact, metric* ANR *is the image of the Hilbert cube by a cell-like mapping.*

Proof. For each i, let $M_i = \bigcup_{n=1}^{2^i-1} M_i(n/2^i, (n+1)/2^i) \cup C_i$ where $C_i = N_i \times [0, 1/2^i] \cup Q \times \{0\}$. Because of the bicollar of the boundary of N_2, $(Q \times [0,1]) \times Q$ maps to $(Q \times \{0\} \cup N_2 \times I) \times Q$ by a CE-mapping f_1, and using the lemma and the bicollared boundaries of N_i for each i, $((Q \times \{0\}) \cup (N_2 \times I)) \times Q$ maps onto $M_2 \times Q$ by a CE-mapping, f_2, and $M_i \times Q$ maps onto $M_{i+1} \times Q$ by a CE-mapping, f_{i+1}, which is within $1/2^{i-2}$ of the identity. (For simplicity's sake, explicit choices of metric representations of the M_i have been omitted, raising the possibility of some confusion on the part of the reader concerning where this convergence is to be carried out. One way to make this explicit is to embed each $M_i(\alpha, \beta)$ in $Q \times [\alpha, \beta] \times Q$ by approximating closely the map

$$[x, t] \longrightarrow \frac{t-\alpha}{\beta-\alpha}(x, \beta, 0) + \frac{\beta-t}{\beta-\alpha}(r(x), \alpha, 0)$$

where $Q = \prod_{i=1}^{\infty} [0,1]_i$ and the algebraic operations use this convex structure. This provides embeddings of M_i in $Q \times I \times Q$, which then may serve as an ambient space for convergence procedures.) Thus, the sequence $\{g_i = f_i \cdots f\}_{i=1}^{\infty}$ is uniformly Cauchy and defines a limit map $g: Q \times [0, 1] \times Q \longrightarrow (X \times [0, 1] \cup Q \times \{0\}) \times Q$.

To verify that g is cell-like, note that if $p: M_i \longrightarrow [0, 1]$ is projection onto the mapping cylinder coordinate, then for each i,

$$pf_i\left(M_{i-1}\left(\frac{m}{2^{i-1}}, \frac{m+1}{2^{i-1}}\right)\right) \subset \left[\frac{m}{2^{i-1}}, \frac{m+1}{2^{i-1}}\right].$$

If $x \in (X \times [0, 1] \cup Q \times \{0\}) \times Q$ and U is any open neighborhood of $g^{-1}(x)$ in $Q \times [0, 1] \times Q$, one may find an open neighborhood V of x in $(X \times [0, 1] \cup Q \times \{0\}) \times Q$ whose closure contracts in

$$W = (X \times [0, 1] \cup Q \times \{0\}) \times Q \backslash g(Q \times [0, 1] \times Q \backslash U).$$

One may also find an n so large and a homeomorphism $h: Q \times [0, 1] \times Q \longrightarrow M_n \times Q$ so close to g_n that V contracts in $h(U) \cap W$ and $hg^{-1}(x)$ may be deformed along the mapping cylinder lines of $M_n \times Q$ to a subset of $(N_n \times \{t\} \times Q) \cap V$ for some t, while staying within $h(U)$. So $hg^{-1}(x)$ contracts in $h(U)$ and $g^{-1}(x)$ contracts in U, which suffices to show that g is UV^{∞}, hence CE. If $\pi: (X \times [0, 1] \cup Q \times \{0\}) \times Q \longrightarrow C(X)$ is projection followed by identification of $Q \times \{0\}$ to a point, then $\pi g: Q \times [0, 1] \times Q \longrightarrow C(X)$ is a cell-like mapping of a Hilbert cube to $C(X)$.

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