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WHITEHEAD GROUPS OF GENERALIZED FREE PRODUCTS

Friedhelm Waldhausen

The purpose of these notes is to describe a splitting theorem for the Whitehead group. Its application is in vanishing theorems of the sort that Wh(G) = 0 if G is a classical knot or link group.

An example of such a link group is the group with generators a, b, c, and relators

$$[a,[b,c^{-1}]] , \quad [b,[c,a^{-1}]] , \quad [c,[a,b^{-1}]]$$

where $[x,y]$ denotes the commutator $xyx^{-1}y^{-1}$. This group may look complicated, but it happens to be the group of one of the simplest links (the 'Borromean rings').

It is not their presentations that make knot groups tractable. What makes them tractable is the fact that they can be built up out of nothing by iterating a construction that I call 'generalized free product'. As this construction (or at least the motivation to look at it) is of topological origin, I will start by giving the topology flavored description.

Let X be a 'nice' topological space, e.g., a CW complex (or, if the reader prefers, a simplicial complex, or even a smooth manifold; all that matters for our purpose, is the global picture), and let Y be a closed 'nice' subspace, e.g., a subcomplex. We assume Y is bicollared in X, this means there exists an open embedding i: $Y \times R \to X$ (where R is the euclidean line) so that $i(Y \times 0) = Y$. We do not ask that Y be connected, in fact, Y may have infinitely many components.

A recipe says that in this situation, the fundamental groupoid of X can be calculated as the colimit of certain other groupoids.

Now <u>assume</u> that for every component Y_j of Y, the inclusion induced homomorphism of fundamental groups, $\pi_1 Y_j \to \pi_1 X$, is a monomorphism. Then the diagram obtained is called a <u>generalized</u> <u>free</u> <u>product</u> (g.f.p.) <u>structure</u> on $\pi_1 X$.

Let us denote X_i, $i \in I$, the components of X - Y, and Y_j, $j \in J$, the components of Y. The groups $\pi_1 X_i$ are called the <u>building</u> <u>blocks</u> of the g.f.p. structure, and the groups $\pi_1 Y_j$ are called the <u>amalgamations</u>. For the sake of uniform notation, we write

$$G = \pi_1 X \ , \quad B = \bigcup_{i \in I} \pi_1 X_i \ , \quad A = \bigcup_{j \in J} \pi_1 Y_j \ ,$$

where '$\cup$' denotes the sum ('disjoint union') in the category of groupoids.

As Y_j locally dissects X, we may pick one of its sides (arbitrarily, but forever) and denote it 'left', and the other one 'right'. There are injections of groups (well-determined up to inner automorphisms)

$$l_j \colon \pi_1 Y_j \to \pi_1 X_{l(j)} \quad \text{and} \quad r_j \colon \pi_1 Y_j \to \pi_1 X_{r(j)} \ .$$

Let F be a functor from groups to abelian groups which sends inner automorphisms to identities. Letting

$$F(B) = \bigoplus_{i \in I} F(\pi_1 X_i)$$

and similarly with F(A), we have well defined maps $F(l)\colon F(A) \to F(B)$, $F(r)\colon F(A) \to F(B)$, and $F(\iota)\colon F(B) \to F(G)$, satisfying $F(\iota) \circ F(l) = F(\iota) \circ F(r)$.

Examples of such functors F are

(1) $H_0(G)$, the integral homology in dimension 0

(2) $K_0(RG)$, the projective class group of the group algebra of G over R, and in particular, $K_0(G) := K_0(ZG)$

(3) $\tilde{K}_0(G) = \operatorname{coker}(H_0(G) \to K_0(G))$

(4) $Z_2 \oplus H_1(G)$

(5) $K_1(RG)$

(6) $Wh(G) = \operatorname{coker}(Z_2 \oplus H_1(G) \to K_1(G))$, this map being induced from $GL(Z,1) \times G \to GL(ZG,1)$

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We can now formulate the splitting theorem.

Proposition. There is an abelian group $\mathfrak{N}$ and a map δ so that the following sequence is exact

$$Wh(A) \xrightarrow{1_*-r_*} Wh(B) \xrightarrow{\iota_*} Wh(G) \xrightarrow{\delta} \mathfrak{N} \oplus \tilde{K}_0(A) \xrightarrow{(0,1_*-r_*)} \tilde{K}_0(B)$$

There is a similar sequence for the unreduced functors; the one with integral coefficients maps onto the one given, and the kernel is the Mayer Vietoris sequence of homology (as indicated in (3) and (6)). One can continue the sequence to the right (by Bass' 'contracted functor' argument).

The splitting theorem contains as special cases both the splitting theorem for a free product of groups, and the Künneth formula for extensions of the integers.

In order to deduce vanishing results from the splitting theorem, one uses the five lemma and some a priori information about the vanishing of the exotic term $\mathfrak{N}$. The trick here is not to work with an individual group G, but with the totality of groups $G \times F$, where F is a free abelian group. One can thus exploit the fact that $\tilde{K}_0(G \times F)$ is a direct summand of $Wh(G \times F \times Z) = Wh(G \times F')$. The trick works well since a g.f.p. structure on G (with building blocks B and amalgamation A, say) induces a g.f.p. structure on $G \times F$ (with building blocks $B \times F$ and amalgamation $A \times F$, and the obvious maps).

The next proposition describes such a vanishing result for the exotic term.

Proposition. In order that $\mathfrak{N} = 0$, it is sufficient that for any component A_j of A, the group algebra ZA_j be regular coherent.

Note that no condition is asked of the building blocks or the structure maps. In the case of the more general splitting theorem with R coefficients, one would correspondingly ask that RA_j be regular coherent.

(A ring is called coherent if its finitely presented modules form an abelian category; it is called regular coherent if, in addition, each finitely presented module has a finite dimensional projective resolution).

The sort of arguments used in deriving the splitting theorem , also gives information on this type of structure of rings:

Proposition. Let G have a g.f.p. structure with building blocks B and amalgamations A. For RG to be regular coherent, it is sufficient that the group algebras RB_i be regular coherent and that the group algebras RA_j be regular noetherian.

The proposition says, for example, if G is a free group, or a 2-manifold group, then ZG is regular coherent.

I will now indicate how g.f.p. structures occur in nature. This necessitates the notion of iterated g.f.p. structure. The main point in the definition is an appropriate transfinite recursion.

Notationally, it is convenient to introduce classes of groups, $C_{m,n}$, indexed by pairs of non-negative integers in lexicographical ordering. Each class contains the preceding ones. We abbreviate

$$C_m = \bigcup_n C_{m,n} , \quad C = \bigcup_m C_m .$$

Definition. (1) $C_{0,0}$ contains only the trivial group

(2) $G \in C$ if and only if G has a g.f.p. structure with all building blocks, B, and all amalgamations, A, in C_m, for some fixed m

(3) if $G \in C$, then $G \in C_m$ if and only if

all $B_i \in C_{m,n}$, for some fixed n, and

all $A_j \in C_{m-1}$

(4) if $G \in C_m$, then $G \in C_{m,n}$ if and only if all $B_i \in C_{m,n-1}$ (here $C_{m,-1}$ is to be interpreted as C_{m-1}) .

Examples. (1) $C_{m,n}$ is closed under taking subgroups.

(2) C is closed under extensions. (Proof: Let $1 \to \ker(p) \to F \xrightarrow{p} G \to 1$ be exact, with ker(p), G $\in C$. Let $G \in C_{m,n}$. The proof is by induction on (m,n). Let G have a g.f.p. structure with building blocks B_i, and amalgamations A_j. Then F has a g.f.p. structure with building blocks $p^{-1}(B_i)$ and amalgamations $p^{-1}(A_j)$.)

(The assertions under (1) and (2) will be obvious from the definition of g.f.p. structure to be given in the next section).

(3) $C_1 = C_{1,0}$ is the class of free groups.

(4) If M is a closed 2-manifold other than the projective plane, then $\pi_1 M \in C_{2,0}$.

(5) There is a large class of 3-dimensional manifolds (e.g., all compact submanifolds of the 3-sphere) whose fundamental groups are in C_3 (and even in C_2 if the manifold has non-empty boundary), however, the 'n' may be quite large.

(6) A one-relator-group is in C_2 if (and only if) the relator is not a proper power. This can be checked from Magnus' analysis of these groups (note that the groups encountered on the way as building blocks, need not be one-relator-groups). Consequently, if G is a one-relator-group, and its relator is not a proper power, then $Wh(G) = \tilde{K}_0(G) = 0$.

To conclude this section, we exploit the geometric picture to see that the general type of g.f.p. structure can be reduced, in a sense, to two rather special types. For, let X and Y be as in the beginning. We can break X at Y, and can then reconstruct X, by glueing, one by one, at the components of Y, and eventually taking a direct limit.

Each of the steps in the above procedure corresponds to a g.f.p. structure in which (by abuse of the old notation) the subspace Y is connected. There are two cases left, according to whether X - Y is connected or not.

Denote by G, A, B (resp. B_1, B_2) the fundamental groups of X, Y, and X-Y (or its components), respectively.

In the case where X - Y has two components , G is the pushout in the diagram

$$\begin{array}{ccc} A & \longrightarrow & B_1 \\ \downarrow & & \downarrow \\ B_2 & \longrightarrow & G \end{array}$$

In a classical terminology, G is the 'free product of B_1 and B_2, amalgamated at A ' , $G = B_1 *_A B_2$ in customary notation.

There is yet another description available, namely G is also the pushout in the category of groupoids in the diagram

$$\begin{array}{ccc} A \cup A & \longrightarrow & B_1 \cup B_2 \\ \downarrow & & \downarrow \\ A \times I & \longrightarrow & G \end{array}$$

Here '∪' is the sum in the category of groupoids, and I is the connected groupoid with two vertices and trivial vertex groups.

In the case where X - Y is connected, let α , β : $A \to B$ denote the two inclusion maps. Then G is the pushout in the category of groupoids in the diagram

$$\begin{array}{ccc} A \cup A & \xrightarrow{\alpha \cup \beta} & B \\ \downarrow & & \downarrow \\ A \times I & \longrightarrow & G \end{array}$$

A classical terminology is not available for this construction. Logicians have used it to construct groups with weird properties (unsolvable word problem, etc.). They sometimes refer to it (and also to a more general construction) as the 'Higman-Neumann-Neumann-Britton-extension', cf. Miller's book. It can be checked, incidentally, that for quite a few of the weird groups in this book, our method shows their Whitehead group is trivial.

An explicit description of G is this. Let T be a free cyclic group, with generator t. Then G is isomorphic to the quotient of the free product $B * T$ by the normal subgroup generated by

$$t\, \alpha(a)\, t^{-1}\, (\beta(a))^{-1} , \quad a \in A .$$

In the next section, I will give the definition of g.f.p. structures which is the most useful one to actually work with. The subsequent section is mostly devoted to a discussion of the exotic term in the splitting theorem. In the final section, some indication of proof is given for the

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Up to reformulation of some parts, essentially all of the present material has been taken from a preliminary report which was issued in fall '69 in mimeographed form. I have not included here the full proof of the splitting theorem, as I doubt if those details have any relevance to the conjecture described in the appendix.

2. Generalized free product structures, revisited.

Let the spaces X and Y be as in the preceding section. Denote $\tilde{X}$ the universal covering space of X, and $\tilde{Y}$ the induced covering space over Y. Identify G ($\approx \pi_1 X$) to the covering translation group of $\tilde{X}$, acting from the right.

The subspace $\tilde{Y}$ induces on $\tilde{X}$ a certain decomposition whose nerve is a graph, Γ, on which G acts. By a 'graph' we mean here a certain combinatorial device, consisting of its set of vertices, Γ^0, set of segments, Γ^1, and incidence relations ('initial vertex' and 'terminal vertex' of a segment, denoted $v_i(s)$ and $v_t(s)$, respectively). The elements of Γ^0 correspond to the components of $\tilde{X} - \tilde{Y}$, and the orbits Γ^0/G correspond to the components of $X - Y$. Similarly, the elements of Γ^1 correspond to the components of $\tilde{Y}$, and the orbits Γ^1/G correspond to the components of Y.

As the realization $|\Gamma|$ of Γ can be embedded as a retract in $\tilde{X}$, Γ must be a tree (i.e., the 1-complex $|\Gamma|$ is connected and simply connected).

Another property is obtained from the 'two-sidedness' of Y in X, namely the action of G on Γ preserves local orientations. By this we mean if $g \in G$ and $s \in \Gamma^1$, then $(s)g = s$ implies that g preserves the initial vertex of s. Consequently we can assume the segments of Γ are oriented in such a way that G preserves all orientations. We now define

Definition. A generalized free product structure on a group G consists of a tree Γ and an action (from the right) of G on Γ, preserving local orientations.

Remarks. (1) This is of course equivalent to our original definition. To recover that one, we need only construct Eilenberg-MacLane spaces $K(G_s,1)$ and $K(G_v,1)$ (corresponding to the stability groups of segments and vertices, one for each orbit), construct mapping cylinders and glue as prescribed by the quotient graph Γ/G. Since for the component Y_0 of Y, the map $\pi_1 Y_0 \to \pi_1 X$ is a monomorphism, $\pi_1 Y_0$ is indeed detected as the stability group of a certain segment.

(2) By our definition of g.f.p. structure, the 'set of g.f.p. structures on a group' is a certain contravariant functor, indeed a sum of representable ones. There is no corresponding assertion if we restrict attention to the two special types of g.f.p. structure considered at the end of the previous section.

We will now analyse g.f.p. structures a bit. By a basic tree in Γ we shall mean a subtree with the property that its set of vertices contains one and only one representative of every orbit Γ^0/G. A basic tree exists, e.g., one can lift a maximal tree from Γ/G. We choose a basic tree and keep it fixed henceforth, it will be denoted $\Gamma_\$$.

A segment in Γ is called non-recurrent if it is equivalent, under the action of G, to a segment in $\Gamma_\$$ (this notion depends on the choice of the basic tree, in general). Otherwise, it will be called recurrent. There exists a basic set of recurrent segments, denoted Γ_r^1. This means, Γ_r^1 contains one and only one representative of any orbit of recurrent segments, and if $s \in \Gamma_r^1$, then the initial vertex of s is in $\Gamma_\$$ (the terminal vertex of s is then necessarily not in $\Gamma_\$$). We fix a group element, denoted t_s, with the property that t_s^{-1} carries the terminal vertex of s into $\Gamma_\$$.

The element t_s just described, acts necessarily without fixed points on Γ. This can easily be seen from the existence of the distance function

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on Γ which associates to any pair of vertices the number of segments in a shortest path joining them.

If $x \in \Gamma^0$ or $x \in \Gamma^1$, we let G_x denote the stability group of x,

$$G_x = \{ g \in G \mid (x)g = x \} .$$

The condition involved in the definition of a g.f.p. structure, is equivalent to: For any segment s, and its end points $v_i(s)$ and $v_t(s)$, we have the relation of stability groups

$$G_{v_i(s)} \cap G_{v_t(s)} = G_s .$$

We let $\Gamma_{\pounds}$ denote the tree whose set of segments is

$$\Gamma^1_{\pounds} = \Gamma^1_{\$} \cup \Gamma^1_r \cup \{ (s)t_s^{-1} \mid s \in \Gamma^1_r \} .$$

For any subtree Δ of Γ, and any vertex v of Δ, we let $\Delta^1(v)$ denote the set of those segments in Δ which are incident to v. Then clearly, for any $v \in \Gamma^0_{\$}$, the set $\Gamma^1(v)$ is in one-one correspondence to the union of cosets

$$\bigcup_s G_s \backslash G_v , \quad s \in \Gamma^1_{\pounds} .$$

From this follows by an inductive argument involving distance, that G is generated by

$$G_v , \quad v \in \Gamma^0_{\$} , \quad \text{and} \quad t_s , \quad s \in \Gamma^1_r .$$

3. Modules over generalized free product structures.

The central notion is that of a certain diagram which I call a Γ-object, and which I will now describe, after some preliminaries.

Following the notation set up before, we denote building blocks of the g.f.p. structure the groupoid

$$B = \bigcup_v G_v , \quad v \in \Gamma^0_{\$} ,$$

and amalgamation the groupoid

$$A = \bigcup_s G_s \ , \quad s \in \Gamma^1_\$ \cup \Gamma^1_r \ .$$

Let Mod_{RG_v} be the category of modules over the group algebra RG_v, where R is some fixed ring with unit. We define Mod_B to be the restricted product

$$Mod_B = \chi_v \ Mod_{RG_v} \ , \quad v \in \Gamma^0_\$ \ ,$$

and similarly

$$Mod_A = \chi_s \ Mod_{RG_s} \ , \quad s \in \Gamma^1_\$ \cup \Gamma^1_r \ .$$

If $M \in Mod_B$, then $M \otimes_B G$ is defined: If, say, $M = \chi_v M_v$, $M_v \in Mod_{RG_v}$, $v \in \Gamma^0_\$$, then

$$M \otimes_B G = \bigoplus_v M_v \otimes_{RG_v} RG \ , \quad v \in \Gamma^0_\$ \ .$$

It is clear from the definition that, as an abelian group, $M \otimes_B G$ is a direct sum, indexed by <u>all</u> of Γ^0,

$$M \otimes_B G = \bigoplus_v M_v \ , \quad v \in \Gamma^0 \ .$$

If $g \in G$ is such that $(v_0)g = v$, where $v_0 \in \Gamma^0_\$$, we can write

$$M_v = M_{v_0} \otimes_{RG_{v_0}} RG_{v_0} \cdot g \ .$$

We can also consider M_v as a module over RG_v.

Similarly, if $N \in Mod_A$, then $N \otimes_A G$ is defined, and there is a direct sum decomposition of abelian groups,

$$N \otimes_A G = \bigoplus_s N_s \ , \quad s \in \Gamma^1 \ .$$

<u>Definition</u>. A Γ-<u>object</u> consists of modules $N \in Mod_A$ and $M \in Mod_B$, and a map over G,

$$\iota : M \otimes_B G \rightarrow N \otimes_A G$$

satisfying: if (for any v and s) the restriction of ι to M_v has a non-zero projection to N_s, then the segment s is incident to the vertex v.

A <u>map</u> of Γ-objects is a pair of maps, one in Mod_B and one in Mod_A, so that the obvious diagram commutes. The resulting category is abelian since the functors $\otimes_B G$ and $\otimes_A G$ are exact.

Dually, a Γ^*-object consists of modules, and a map

$$M \otimes_B G \leftarrow N \otimes_A G$$

satisfying the same sort of condition. The duality functor $\mathrm{Hom}_{RG}(\ ,RG)$ maps Γ-objects to Γ^*-objects, and vice-versa (however, in order to stay with right modules, we may have to replace the coefficient ring by its opposite).

We can be somewhat more explicit about the structure map

$$\iota : M \otimes_B G \rightarrow N \otimes_A G$$

in a Γ-object. Let us write

$$\iota_{v,s}$$

for the composition

$$M_v \rightarrow \bigoplus_{v'} M_{v'} \rightarrow \bigoplus_{s'} N_{s'} \rightarrow N_s \, .$$

Then ι is of course determined by its components $\iota_{v,s}$, $v \in \Gamma^0_{\$}$, $s \in \Gamma^1_{\pounds}$; and for fixed v, those components assemble to an (arbitrary) RG_v-map

$$M_v \rightarrow \bigoplus_s N_s \, , \quad s \in \Gamma^1(v) \, .$$

<u>Definition</u>. A Γ-<u>module</u> is a Γ-object $\iota : M \otimes_B G \rightarrow N \otimes_A G$ satisfying that ι is an isomorphism. The resulting category is denoted Mod_Γ; it is abelian.

A Γ-module is called <u>elementary</u> if N is finitely generated projective and, in addition, at most one of the component maps $\iota_{v,s}$, $v \in \Gamma^0_{\$}$, $s \in \Gamma^1_{\pounds}$, is not the zero map; this $\iota_{v,s}$ must then itself be an isomorphism.

A Γ-module is called <u>triangular</u> if it has a finite filtration with elementary subquotients.

We denote $K_0(\mathrm{Mod}_\Gamma, R)$ the class group of those objects in Mod_Γ which are made up of finitely generated projective modules, the relations coming from all exact sequences (not just split ones). Using elementary Γ-modules, we obtain a map

$$j : K_0(RA) \oplus K_0(RA) \rightarrow K_0(\mathrm{Mod}_\Gamma, R)$$

which is a split injection by an argument below (the construction of the modules denoted $P(s,v)$). The cokernel of j is denoted $\mathfrak{N}$. This is the $\mathfrak{N}$ that appears in the splitting theorem. The definition of $\mathfrak{N}$ is related to maps which are 'nilpotent' if this term is taken in a suitable sense. The vanishing theorem for $\mathfrak{N}$ will come in in somewhat disguised form: under the hypothesis that RA is regular coherent, the proposition below implies that the above map j is an isomorphism.

We now proceed to the analysis of Γ-modules. Let s be a segment of Γ, and v a vertex incident to s. Define $\Gamma_{s,v}$ to be the maximal subtree of Γ which contains v but not s. Given s, there are two such trees, $\Gamma_{s,v_i(s)}$ and $\Gamma_{s,v_t(s)}$.

Given $M \in \mathrm{Mod}_B$, then $M \otimes_B G$, considered as a module over RG_s, splits naturally as a direct sum

$$\overline{M}(s,v_i(s)) \oplus \overline{M}(s,v_t(s))$$

where, as an abelian group,

$$\overline{M}(s,v_i(s)) = \bigoplus_v M_v \,, \quad v \in \Gamma^0_{s,v_i(s)} \,.$$

Similarly, if $N \in \mathrm{Mod}_A$, then $N \otimes_A G$, considered as a module over RG_s, splits as

$$\overline{N}(s,v_i(s)) \oplus N_s \oplus \overline{N}(s,v_t(s))$$

where, as an abelian group,

$$\overline{N}(s,v_i(s)) = \bigoplus_{s'} N_{s'} \,, \quad s' \in \Gamma^1_{s,v_i(s)} \,.$$

If now $\iota\colon M \otimes_B G \to N \otimes_A G$ is a Γ-module, then

$$\iota(\overline{M}(s,v_i(s))) \subset \overline{N}(s,v_i(s)) \oplus N_s$$

and

$$\iota^{-1}(\overline{N}(s,v_i(s))) \subset \overline{M}(s,v_i(s)) \,.$$

Whence the canonical splitting

$$N_s = P(s,v_i(s)) \oplus P(s,v_t(s))$$

where

$$P(s,v_i(s)) = \mathrm{Im}(\overline{M}(s,v_i(s)) \to \overline{N}(s,v_i(s)) \oplus N_s \to N_s)$$
$$\approx \ker(\overline{M}(s,v_i(s)) \to \overline{N}(s,v_i(s))) \,,$$

and analogously with $P(s,v_t(s))$.

On the other hand, if v is a fixed vertex, and s a segment incident to v, let us denote $\Gamma_{v,s}$ the maximal subtree of Γ which is incident to s, but does not contain v. We have $\Gamma_{v,s} = \Gamma_{s,\tilde{v}}$ where $\tilde{v}$ is the other end point of s. As before, let us denote $\Gamma^1(v)$ the set of segments of Γ which are incident to v. Let $\Gamma^1_{rep}(v)$ denote a set of representatives for the quotient set $\Gamma^1(v)/G_v$; e.g., if $v \in \Gamma^0_{\$}$, then $\Gamma^1_{£}(v)$ is such a set of representatives.

Given $M \in \mathrm{Mod}_B$, then $M \otimes_B G$, considered as a module over RG_v, splits naturally as a direct sum

$$M_v \oplus \bigoplus_s \widetilde{M}(v,s) \ , \quad s \in \Gamma^1_{rep}(v)$$

where, as RG_v-module,

$$\widetilde{M}(v,s) = \overline{M}(s,\tilde{v}) \otimes_{RG_s} RG_v \ ,$$

$\overline{M}(s,\tilde{v})$ is defined as above, and $\tilde{v}$ is the other end point of s.

Similarly, if $N \in \mathrm{Mod}_A$, then $N \otimes_A G$, considered as a module over RG_v, splits as

$$\bigoplus_s N_s \otimes_{RG_s} RG_v \oplus \bigoplus_s \overline{N}(s,\tilde{v}) \otimes_{RG_s} RG_v \ , \quad s \in \Gamma^1_{rep}(v) \ .$$

If again $\iota: M \otimes_B G \to N \otimes_A G$ is a Γ-module, we can write ι as a map of RG_v-modules in the form

$$M_v \oplus \bigoplus_s \overline{M}(s,\tilde{v}) \otimes_{RG_s} RG_v \to \bigoplus_s N_s \otimes_{RG_s} RG_v \oplus \bigoplus_s \overline{N}(s,\tilde{v}) \otimes_{RG_s} RG_v \ ,$$

$$s \in \Gamma^1_{rep}(v) \ .$$

Now the restriction to the second summand is of a type considered before. Hence we obtain a map

$$M_v \oplus \bigoplus_s P(s,\tilde{v}) \otimes_{RG_s} RG_v \to \bigoplus_s N_s \otimes_{RG_s} RG_v =$$

$$\bigoplus_s P(s,v) \otimes_{RG_s} RG_v \oplus \bigoplus_s P(s,\tilde{v}) \otimes_{RG_s} RG_v$$

whose restriction to the second summand is the obvious identity. Therefore the restriction to the first summand is the sum of an isomorphism

$$\varkappa_v: M_v \to \bigoplus_s P(s,v) \otimes_{RG_s} RG_v$$

and some map

$$\lambda_v : M_v \rightarrow \bigoplus_s P(s,\tilde{v}) \otimes_{RG_s} RG_v .$$

For fixed $s \in \Gamma^1_{rep}(v)$, the composition $\lambda_v \circ \varkappa_v^{-1}$ induces an RG_v-map

$$P(s,v) \otimes_{RG_s} RG_v \rightarrow \bigoplus_{s'} P(s',\tilde{v}) \otimes_{RG_s} RG_v , \quad s' \in \Gamma^1_{rep}(v)$$

which in turn is determined by the induced RG_s-map

$$\mu_{s,v} : P(s,v) \rightarrow \bigoplus_{s'} P(s',\tilde{v}) \otimes_{RG_s} RG_v , \quad s' \in \Gamma^1_{rep}(v).$$

The target of this latter map is in fact slightly smaller since the composition of $\mu_{s,v}$ with the projection to $P(s,\tilde{v})$ is zero (inspection of the definitions shows that this composition can be factored through $\overline{M}(s,v)$).

The map now reads

$$\nu_{s,v} : P(s,v) \rightarrow P(s,\tilde{v}) \otimes_{RG_s} \widehat{RG}_v \oplus \bigoplus_{s'} P(s',\tilde{v}) \otimes_{RG_s} RG_v ,$$

$$s' \in \Gamma^1_{rep}(v) , \quad s' \neq s ,$$

where $\widehat{RG}_v(s)$ is the summand in the canonical splitting of RG_s-bi-modules

$$RG_v = RG_s \oplus \widehat{RG}_v(s) .$$

It is clear now that there is an (exact) functor

$$F : Mod_A \times Mod_A \rightarrow Mod_A \times Mod_A$$

which depends only on the g.f.p. structure (in particular it does not depend on the choice of the sets $\Gamma^1_{rep}(v)$) so that the collection of maps

$$\nu_{s,v} , \quad s \in \Gamma^1_{\$} \cup \Gamma^1_r ,$$

assembles to a map

$$\nu : P \rightarrow F(P)$$

where the first component of $P \in Mod_A \times Mod_A$ is given by the collection $P(s,v_i(s))$, $s \in \Gamma^1_{\$} \cup \Gamma^1_r$.

The original Γ-module is determined by the pair (P,ν). Conversely, a necessary and sufficient condition for (P,ν) to arise from a Γ-module, is that the map ν be nilpotent in the following sense.

Define a filtration $0 = P_0 \subset P_1 \subset \dots \subset P_j \subset \dots \subset P$ by the rule

$$P_{j+1} = \nu^{-1}(F(P_j)) .$$

Then we call ν <u>nilpotent</u> if $\bigcup P_j = P$.

<u>Remark</u>. If the g.f.p. structure comes from a product with the integers (so that we are in the situation of the classical Künneth formula) then a nilpotent ν in our sense is just a pair of nilpotent maps in the usual sense.

We will not prove here that ν is nilpotent as this follows directly from the lemma below. We note the following interpretation of ν. If $x \in P(s,v)$ then $x \in P_1$ (the first term of the filtration) if and only if there exists $y \in M_v$ so that $\iota(y) = x$.

Given $\nu\colon Q \to F(Q)$, it is convenient to consider a more general type of filtration, $0 \subset Q_1 \subset \dots \subset Q_j \subset \dots \subset Q$, which we call a <u>nil-filtration</u> if

$$\nu(Q_{j+1}) \subset F(Q_j) , \quad \text{and} \quad \bigcup Q_j = Q .$$

We say it is of <u>finite length</u>, q, if $Q_q = Q$, and we say it is <u>finitely generated</u>, if all the Q_j are.

The filtration originally derived from a Γ-module, denoted $.. \subset P_j \subset ..$ above, will certainly be of finite length if N is finitely generated, but it need not itself be finitely generated. It is clear nevertheless that there exists some finitely generated nil-filtration which is a subfiltration of the original one, and is of the same length.

We will now describe our resolution argument. Let $.. \subset Q_j \subset ..$ be a finitely generated nilfiltration of length q, associated to a Γ-module. Pick finitely generated projectives U_j in $\mathrm{Mod}_A \times \mathrm{Mod}_A$, and surjections

$$U_j \to Q_j , \quad j \geq 1 .$$

Then we can find maps $u_j\colon U_j \to F(U_{j-1})$ so that the diagrams

$$\begin{array}{ccc} U_j & \rightarrow & F(U_{j-1}) \\ \downarrow & & \downarrow \\ Q_j & \rightarrow & F(Q_{j-1}) \end{array}$$

commute. Define a filtration $0 \subset V_1 \subset \ldots \subset V_q = V$, by

$$V_i = U_1 \oplus \ldots \oplus U_i \ .$$

It is a nil-filtration for the map

$$v\colon V \rightarrow F(V) \ , \quad v = \Sigma_j \, u_j \ .$$

This map is associated to a certain triangular Γ-module in which the A-module is V, considered as an A-module via $\oplus$: $Mod_A \times Mod_A$. Furthermore there is a surjection of Γ-modules, compatible with the surjection of nil-filtrations, $V_j \rightarrow Q_j$. Define $.. \subset W_j \subset ..$ to be the kernel filtration, it is a nil-filtration for the map $w = v|W$, where $W = W_q$. If Q_1 was projective to begin with, we could have chosen $V_1 = Q_1$, and the new filtration would be of shorter length.

Now assume the amalgamation A is coherent, and Q is finitely presented. Then, as ${}_{\text{f.p.}}Mod_A$ is an abelian category, it follows that Q_j and W_j are finitely presented. Therefore we can repeat our construction <u>using the filtration</u> W_j .

On iterating the procedure we are building up, in particular, a projective resolution of Q_1. Therefore, if A is regular coherent, we can eventually reduce the length of the filtration, and so, by induction on this length, we have proved:

<u>Proposition</u>. If A is regular coherent, then any finitely presented Γ-module has a resolution by triangular Γ-modules.

(By abuse of language, we have called a Γ-module 'finitely presented' if the A-module involved is. Note that the main interest of the proposition is in the case where this A-module is actually projective).

Above we referred to the following lemma. The above application of the

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The above application of the

lemma just exploits the obvious fact that a nil-filtration does exist for a triangular Γ-module. The lemma says that there are as many maps from triangular Γ-modules as we can expect at all.

Lemma. Let $\iota: M \otimes_B G \to N \otimes_A G$ be any Γ-object.

(1) Let $y \in N_s$, $s \in \Gamma^1$, and $y \in \mathrm{Im}(\iota)$. Then y is in the image of some map from a triangular Γ-module.

(2) Let $x \in M_v$, $v \in \Gamma^0$. Then x is in the image of some map from a triangular Γ-module.

Proof. Ad (1). Let $y = \Sigma_v \iota(z_v)$, $z_v \in M_v$, $v \in \Delta^0$, where Δ is some finite subtree of Γ. The sought for triangular Γ-module is made up of rank-one free modules over the appropriate rings. There is one basis element for each vertex and segment in Δ, and there is an additional basis element for the segment s. Each of the components of the structure map is an 'identity' (i.e., it sends the basis element to the basis element), and there is one such for each incidence relation in Δ, and one additional one into the extra component. The definition of the map is automatic.

Ad (2). This follows from (1) by the same sort of splicing argument.

4. Mayer Vietoris presentations of G-modules.

Let L be a G-module (more precisely, an RG-module). A left Mayer Vietoris presentation of L is a short exact sequence

$$0 \to L \to M \otimes_B G \to N \otimes_A G \to 0$$

the right part of which is a Γ-object, as defined in the previous section.

Dually, a right Mayer Vietoris presentation is a short exact sequence

$$0 \to N \otimes_A G \to M \otimes_B G \to L \to 0$$

involving a Γ*-object.

A left or right Mayer Vietoris presentation is called f.g.p. if all the modules involved are finitely generated projective. F.g.p. left and right Mayer Vietoris presentations are interchanged by the duality map $\mathrm{Hom}_{RG}(\ ,RG)$ (with the usual proviso on the coefficient ring R). Hence it is sufficient to concentrate on either one. For us this will be the left Mayer Vietoris presentations, abbreviated MV presentations henceforth.

Remark. The concept of MV presentation is an axiomatization of a Mayer Vietoris type situation that occurs if one looks at chain complexes in the universal cover of a pair X,Y as considered in the introductory section.

Namely, if L is a chain complex over $G \approx \pi_1 X$, then 'subdividing at Y' produces an MV presentation of chain complexes

$$0 \to L \to M \otimes_B G \underset{\iota}{\to} N \otimes_A G \to 0 .$$

After the subdivision, L will have been replaced (up to a dimension shift) by the mapping cone $C(\iota)$. And the Mayer Vietoris sequence of chain complexes that one is accustomed to read off, now appears as the right Mayer Vietoris presentation which is the sequence of cones

$$0 \to C(\iota_1) \to C(\iota_2) \to C(\iota) \to 0$$

where ι_1 is the trivial inclusion $0 \to N \otimes_A G$, and

$$\iota_2 : M \otimes_B G \to N \otimes_A G \oplus N \otimes_A G$$

is the map whose components are ι_i and ι_t in the canonical sum decomposition of ι. The B-structures on the two copies of $N \otimes_A G$ come, respectively, from the two natural maps $A \to B$. The proposition below is the 'subdivision lemma' that one would naturally expect.

We will now verify that there exist quite a few MV presentations, and maps thereof. Our main tool will be certain 'standard' MV presentations, defined for a free G-module; part of the data will be a basis of the G-module; in the description we will assume that it has cardinality one. (Inspection shows that the construction below can actually be carried through for any

G-module equipped with a reduction to Mod_A). In describing free modules of the type $M \otimes_B G$, it is sometimes convenient to use a basis which does not come from Mod_B.

<u>Definition</u>. Let F be a free G-module, with basis element f. Let Δ be a finite subtree of Γ. Then the <u>standard</u> MV <u>presentation</u> <u>of</u> F,f, <u>associated</u> <u>to</u> Δ, is the following

(1) $M \otimes_B G$ is the free G-module on basis elements $\bar{m}_v$, $v \in \Delta^0$

(2) $N \otimes_A G$ is the free G-module on basis elements $\bar{n}_s$, $s \in \Delta^1$

(3) the G-structure on $M \otimes_B G$ is such that $\bar{m}_v$ generates a free RG_v-module; similarly with $N \otimes_A G$

(4) the structure map $\varkappa: F \to M \otimes_B G$ is given by $\varkappa(f) = \Sigma_v \bar{m}_v$, $v \in \Delta^0$

(5) the structure map $\iota: M \otimes_B G \to N \otimes_A G$ is given in terms of its components $\iota_{v,s}: M_v \to N_s$ by

$$\iota_{v,s}(\bar{m}_v) = \bar{n}_s \text{ , if } v = v_i(s) \text{, the initial vertex}$$

$$\iota_{v,s}(\bar{m}_v) = -\bar{n}_s \text{, if } v = v_t(s) \text{, the terminal vertex}$$

$$\iota_{v,s}(\bar{m}_v) = 0 \text{ , if } v \text{ is not incident to } s$$

(6) in order to describe the reduction of $M \otimes_B G$ to Mod_B, i.e., to define M, we must pick representatives of cosets for the various inclusions involved in the g.f.p. structure, so we assume this has been done once and forever. It is crucial here that we need only choose representatives of cosets for the inclusions of amalgamation groups in building block groups, and the elements denoted t_s in section 2, and that this choice determines representatives of all the cosets in G (this statement is the general version of the existence of the usual normal form for an element of a free product with amalgamation, it is easily proved by the use of the distance function on Γ). In particular then, we have picked for every $v \in \Delta^0$ an $x_v \in G$ so that $(v)x_v^{-1} \in \Gamma^0_\$$, the basic tree. By definition now, M is the B-module whose component at $v' \in \Gamma^0_\$$ is the direct sum $\bigoplus_v M_v \cdot x_v^{-1}$, taken over those $v \in \Delta^0$ for which $(v)x_v^{-1} = v'$.

In terms of the basis elements $m_v = \bar{m}_v \cdot x_v^{-1}$ (which live in M), we could now redefine $\varkappa(f) = \Sigma_v\, m_v \cdot x_v$

(7) the reduction of $N \otimes_A G$ to Mod_A is described similarly.

Before proceding, let us note that for any MV presentation (or even Γ-object), there is a canonical decomposition

$$\iota = \iota_i - \iota_t$$

where ι_i is defined so that its non-zero components are those $\iota_{v,s}$ for which $v = v_i(s)$, the initial vertex (this decomposition was used in the remark above). For the standard MV presentation just described, we have the important property

$$\iota_i(\varkappa(f)) = \Sigma_s\, \bar{n}_s \ , \quad s \in \Delta^1 \ .$$

<u>Proposition</u>. Let $0 \to L \to M' \otimes_B G \to N' \otimes_A G \to 0$ be any MV presentation. Let F be the free G-module on the basis element f, and let $g\colon F \to L$ be any G-map. Then for suitable Δ, the standard MV presentation of F,f, associated to Δ, admits a map of MV presentations, inducing g. Moreover, this map is uniquely determined by g.

<u>Proof</u>. By definition, $M' \otimes_B G$ is a direct sum

$$\bigoplus_v M'_v \otimes_{RG_v} RG \ , \quad v \in \Gamma^0_\$ \ .$$

Let $\tilde{g}_v$ denote the projection of $\varkappa' \circ g$ to $M'_v \otimes_{RG_v} RG$. Then we can write

$$\tilde{g}_v(f) = \Sigma_w\, a_w \cdot x_w$$

where $a_w \in M'_w$, $x_w \in G$ is a representative of a coset $G_v \backslash G$ as chosen before, and $w \in \Gamma^0$ runs through the vertices with $(w)x_w^{-1} = v$. From this formula and the fact that

$$\varkappa(f) = \Sigma_w\, m_w \cdot x_w \ , \quad w \in \Delta^0 \ ,$$

it is clear that the required B-map can be defined as soon as the finite tree Δ has been chosen so large that it contains all the vertices w for which $a_w \neq 0$.

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Next we define the required A-map, g_A, directly, by decomposing similarly the map

$$\iota^!_i \circ \varkappa' \circ g : F \to N' \otimes_A G$$

using

$$\iota_i(\varkappa(f)) = \Sigma_s \overline{n}_s = \Sigma_s n_s \cdot x_s \ , \quad s \in \Delta^1 \ .$$

The sum decompositions involved in our construction were canonical, and it is now easily seen that the maps g, g_B, g_A are compatible as required. We record the uniqueness part in a separate lemma.

Lemma. If in the above proposition, g is the zero map, then g_B and g_A must be zero maps, too.

Proof. It is enough to treat g_A. Since the source MV presentation is standard, we have

$$\iota_i(\varkappa(f)) = \Sigma_s n_s \cdot x_s \ ,$$

and on application to this element of the map $g_A \otimes G$, no cancellation is possible between the individual summands.

I will now indicate how the splitting theorem can be obtained. Following Whitehead's original treatment, a torsion element can be represented by a based free acyclic chain complex. The relations come from certain short exact sequences, called elementary expansions.

Using our machinery of MV presentations, we can now say that any chain complex over G comes, via the forgetful map, from a chain complex of MV presentations (with bases suitably). And we can also say what, in the framework of MV presentations, corresponds to elementary expansions.

Technically, the analysis boils down to situations which are blown up versions of the following simple prototype. If we have a chain complex which on the G-level (i.e., apply the forgetful map to Mod_G) is acyclic, there is still no reason that it be acyclic on the A-level (a Γ-module is an example for this). So we can try to make it acyclic on the A-level as well, using simple operations. The details are standard and there are no surprises: one just goes on killing homology groups, working up in dimension. It turns out that there is a global obstruction, and this gives the connecting map.

To illustrate the technique, we prove

Proposition. Let G have a g.f.p. structure with building blocks B and amalgamation A.

(1) If $\text{gl.dim.Mod}_A \leq n-1$, and $\text{gl.dim.Mod}_B \leq n$, then $\text{gl.dim.Mod}_G \leq n$.

(2) If the building blocks are coherent, and the amalgamations noetherian, then G is coherent.

Proof. Ad (1). Let L. be a free (n-1)-dimensional resolution of $\text{coker}(L_1 \to L_0)$. By the subdivision lemma, there is a complex of standard MV presentations over L.,

$$0 \to L. \to M. \otimes_B G \to N. \otimes_A G \to 0 .$$

Since no conditions had to be met in dimension 0, we can assume $N_0 = 0$. Now the last lemma of the previous section tells us that we can add a triangular Γ-module (or maybe a big sum of such) to the 2-chains to kill

$$\text{Im}(H_1(M. \otimes_B G) \to H_1(N. \otimes_A G))$$

and hence $H_1(M. \otimes_B G)$. Again it tells us that we can kill $H_2(N. \otimes_A G)$, and so on. But once we killed $H_{n-2}(N. \otimes_A G)$, we know that (using $H_*(N. \otimes_A G) \approx H_*(N.) \otimes_A G$, etc.) $\ker(N_{n-1} \to N_{n-2})$ must be projective since we resolved $H_1(N.)$. Similarly, $\ker(M_{n-1} \to M_{n-2})$ is projective, and we are done.

Ad (2). By a bit of diagram chasing, the assertion is reduced to proving that $\ker(L_1 \to L_0)$ is finitely generated once L_1 and L_0 are finitely generated free RG-modules. Again the subdivision lemma gives us a map of standard MV presentations over $L_1 \to L_0$. We regard it as a complex in dimensions 1 and 0, and can assume as before that $N_0 = 0$. Arguing as before, we can introduce a big sum of triangular Γ-modules into the 2-chains in order to kill

$$\text{Im}(H_1(M. \otimes_B G) \to H_1(N. \otimes_A G)) .$$

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5. Appendix.

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This time we would like to have N_2 finitely generated. But $\mathrm{Im}(N_2 \to N_1)$ is finitely generated by the noetherian hypothesis. Therefore some finite part of the big sum is already sufficient for our purpose. We have achieved now that the sequence

$$H_2(N. \otimes_A G) \to H_1(L.) \to H_1(M. \otimes_B G)$$

is short exact. But the base changes are exact. So the extreme terms can be rewritten $H_2(N.) \otimes_A G$ and $H_1(M.) \otimes_B G$, respectively. So they are finitely generated by the coherence hypothesis, and we are done.

5. Appendix.

Let $\underline{K}(C)$ denote Quillen's K-theory associated to the category-with-exact-sequences C. Here C is assumed to be equivalent to a small category, and, by definition, $\underline{K}(C) \cong$ (homotopy equivalent to) $\Omega\, Q'(C)$, the loop space of the nerve of the category $Q'(C)$, where $Q'(C)$ is small and equivalent to $Q(C)$, and $Q(C)$ is constructed from certain diagrams in C, involving the notions of 'admissible monomorphism' and 'admissible epimorphism'.

If $\underline{MV}$ denotes the category of MV presentations over a g.f.p. structure (of a group G, with building blocks B, and amalgamations A), we define $Q(\underline{MV})$ by the rule

(1) an identity map is admissible if all the modules involved in the object are finitely generated projective

(2) an epimorphism is admissible if its source and target are

(3) a monomorphism is admissible if its source, target, and cokernel are.

Similarly, we define $Q(\mathrm{Mod}_\Gamma)$.

There is a natural embedding

$$\underline{K}(\mathrm{Mod}_\Gamma) \to \underline{K}(\underline{MV})$$

whose composition with the natural projection, induced from the forgetful map,

$$\underline{K}(\underline{MV}) \to \underline{K}(\text{Mod}_G)$$

is trivial.

There is evidence that the following should be true

<u>Conjecture</u> 1. The sequence

$$\underline{K}(\text{Mod}_\Gamma) \to \underline{K}(\underline{MV}) \to \underline{K}(\text{Mod}_G)$$

has the homotopy type of a fibration, or equivalently, the long sequence of homotopy groups is exact.

(It is <u>not</u> conjectured that the map $\underline{K}(\underline{MV}) \to \underline{K}(\text{Mod}_G)$ is locally fiber homotopy trivial: indeed this is almost certainly not the case. Similarly below).

For the amalgamation A, define

$$\underline{K}(\text{Mod}_A) = \bigtimes_j \underline{K}(\text{Mod}_{A_j}) \ ,$$

the restricted product (the direct limit over the finite products) over the component groups. Similarly with $\underline{K}(\text{Mod}_B)$.

There is a natural embedding

$$\underline{K}(\text{Mod}_B) \to \underline{K}(\underline{MV})$$

so that the composition with the natural projection

$$\underline{K}(\underline{MV}) \to \underline{K}(\text{Mod}_A)$$

is trivial. The latter map has a section (in fact, there are three obvious such).

<u>Conjecture</u> 2. The sequence

$$\underline{K}(\text{Mod}_B) \to \underline{K}(\underline{MV}) \to \underline{K}(\text{Mod}_A)$$

is a homotopy fibration. Consequently

$$\underline{K}(\underline{MV}) \cong \underline{K}(\text{Mod}_A) \times \underline{K}(\text{Mod}_B) \ .$$

From the retraction $\text{Mod}_\Gamma \to \text{Mod}_A \times \text{Mod}_A$, we can conclude that

$$\underline{K}(\text{Mod}_\Gamma) \cong \underline{K}(\text{Mod}_A) \times \underline{K}(\text{Mod}_A) \times \underline{N} \ ,$$

defining $\underline{N}$. (And $\pi_0\underline{N} = \mathfrak{N}$, our old exotic term). Combining conjectures 1 and 2, and noting that two terms cancel, we obtain

Conjecture 3. There is a homotopy fibration

$$\underline{K}(Mod_A) \times \underline{N} \to \underline{K}(Mod_B) \to \underline{K}(Mod_G) .$$

Concerning the exotic space $\underline{N}$, there is the vanishing

Conjecture 4. If A is regular coherent, then $\underline{N}$ is contractible.

Conjecture 4 happens to be true, for under the regular coherence hypothesis, we can replace in the definitions of both $\underline{K}(Mod_A \times Mod_A)$ and $\underline{K}(Mod_\Gamma)$, respectively, finitely generated projectives by finitely presented modules, and can then conclude that the two spaces are equivalent. This uses the resolution of Γ-modules by triangular ones, and Quillen's theorems on reduction by resolution and devissage, respectively.

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Waldhausen, Friedhelm

Whitehead groups of generalized free products.

Algebraic K-theory, II: ''Classical'' algebraic K-theory and connections with arithmetic (*Proc. Conf., Battelle Memorial Inst., Seattle, Wash.*, 1972), *pp.* 155–179. *Lecture Notes in Math., Vol.* 342, *Springer, Berlin,* (1973).

Suppose that X is (say) a CW-complex, Y a bi-collared sub-complex, Y_j the components of Y, and X_i the components of $X-Y$. Suppose that the natural maps $Y_j \to X$ induce monomorphisms on $\pi_1(Y_j)$. Then $G=\pi_1(X)$ may be obtained as the generalized free product of the groups in the union $B=\bigcup \pi_1(X_i)$ with amalgamated subgroups in the union $A=\bigcup \pi_1(Y_j)$. If F is any functor from groups to abelian groups sending inner automorphisms to identities, define $F(A)=\oplus F(\pi_1(Y_j))$, $F(B)=\oplus F(\pi_1(X_i))$. The author proves that $\mathrm{Wh}(A)\to \mathrm{Wh}(B)\to \mathrm{Wh}(G)\to \mathfrak{N}\oplus \tilde{K}_0(A)\to \tilde{K}_0(B)$ is exact for some abelian group $\mathfrak{N}$. Among numerous consequences is the vanishing of $\mathrm{Wh}(G)$ for G the group of a knot or link in S^3.

{For the entire collection see MR0325308 (48 #3656b).}

Reviewed by *L. Neuwirth*