SELF-LINKING AND THE GAUSS INTEGRAL IN HIGHER DIMENSIONS.

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Introduction. Let M be an oriented compact differnetiable manifold of dimension n (with or without boundary) and let $\phi\colon M \to S^n$ be a differentiable map of M into the unit sphere of dimension n. Let dO_n denote the pull-back of the volume element of S^n under ϕ and O_n the volume of the S^n. Then

$$\frac{1}{O_n}\int_M dO_n$$

is called the *Gauss integral* for ϕ. In this paper we shall prove a number of differential topological and integral geometric formulas for submanifolds of Euclidean spaces which arise from the application of this integral to certain geometric constructions.

The Gauss integral has numerous applications in geometry. In case M has no boundary, it gives the degree of ϕ. It was used by Kronecker to give a formula for the intersection number of two submanifolds of Euclidean space. In case M is an immersed hypersurface and ϕ is the Gauss map, it gives the total curvature of M. In connection with this, Chern [3], in his work on the Gauss-Bonnet theorem, expressed the Gauss integrand as the exterior derivative of a differential form. Gauss himself used the integral in his investigation of electromagnetic theory to give a formula for the linking number of two spaces curves. In fact, if C_1 and C_2 are two closed disjoint space curves and if $\phi\colon C_1 \times C_2 \to S^2$ is the map which assigns to each (x, y) the unit vector from x to y, then the Gauss integral gives the linking number of C_1 and C_2. (If C_1 and C_2 are not disjoint, however, the linking number is indeterminate.) In [2] Călugăreanu raised the question of the significance of the Gauss integral for this last map $\phi\colon C_1 \times C_2 \to S^2$ if $C_1 = C_2 \equiv C$.

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He found that if C is differentiable of class C^3 and has non-vanishing curvature then

$$\frac{1}{4\pi}\int_{C\times C} dO_2 + \frac{1}{2\pi}\int_C \tau ds = SL$$

is an integer, where τ is the torsion. He also worked out a number of geometric interpretations for this integer. In [12] Pohl gave a much simplified proof of Călugăreanu's results using the space $S(C)$ of secants of C, and he called the integer SL, appropriately, the self-linking number of the curve C. This work was the starting point of the present paper.

Pohl's approach can be generalized by using the space of secants $S(M)$ of an n-dimensional differentiable manifold M to give a higher dimensional version of Călugăreanu's formula. (This approach is summarized in Appendix B.) However, we have found a new approach to these formulas which gives a much greater variety of results. Instead of using the space of secants of a differentiable manifold M, we introduce the space $S(M,N)$ of secants of a differentiable manifold N relative to a submanifold M and study differential forms on it. A striking feature of our results is that in spite of their diversity they all follow from a single main equation.

Section 1 deals with the definition of $S(M,N)$ and the proof of the main equation. In Sections 2 through 5 we consider the case of an n-dimensional submanifold M of Euclidean $(2n+1)$-space. If n is odd, we define the torsion of M, generalizing the torsion of a space curve, and prove the generalized Călugăreanu formula. If n is even, we show that the terms in Călugăreanu's formula are zero, but our construction leads to differential topological results. We show that if v is a normal vector field on M (for example, the mean curvature vector field), then one-half the Euler characteristic of the subbundle of the normal bundle complementary to the subbundle of lines spanned by the vector v is equal to the negative of the linking number of the submanifold M with the submanifold moved a small distance along the vector field v.

The method of proof in these sections relies heavily on the use of differential forms, in particular, the forms introduced by Chern in his intrinsic proof of the Gauss-Bonnet formula. We show, in fact, that these forms actually arise in a natural and geometric fashion.

Section 6, which contains differential topological results, deals with a generalization of the even-dimensional case mentioned above. Suppose we have an imbedding f of an n-dimensional differentiable manifold, n even or odd, into Euclidean $(n+s)$-space, and suppose there exists an oriented

k-plane subbundle N of the normal bundle such that $0 < s - k \leqq n$ and $s - k$ is even. We define the *normal intersection locus* to be $f(M) \cap (N$-the zero section of $N)$. Then we show that the Poincaré dual of this locus is the Euler class of hhe $(s-k)$-plane subbundle of the normal bundle complementary to the k-plane subbundle. This result is closely related to the work of Lashof-Smale [9].

Section 7 is concerned with the case in which M is the boundary of N, the most striking result being one for *curves and surfaces in ordinary space.* Let C be a simple closed curve which bounds a compact surface immersed in E^3. Let α be the angle between the surface normal and the binormal vector of the curve at the points of C. Let SL be the self-linking number of the curve and I the sum of the indices of the non-trivial intersections of the curve with the surface. Then we show that

$$\frac{1}{2\pi}\int_C d\alpha = SL + I.$$

Section 8 introduces the concept of the Gauss integral for submanifolds and deduces further formulas. Section 9 proves a higher-dimensional version of the Fenchel-Jacobi theorem to the effect that the total torsion of a closed spherical space curve is zero. Finally, Section 10 deals briefly with deformation theory, showing the invariance of the self-linking number under nondegenerate isotopy.

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1. The secant manifold $S(M,N)$ and the main equation. We begin by recalling the definition of the abstract space $S(M)$ of secant directions of a differentiable manifold M with or without boundary. If M has no boundary, $S(M) = M \times M - D \cup T(M)$ where $D = \{(m,m) \in M \times M\}$ and $T(M)$ is the space of oriented tangent directions of M. $S(M)$ has been constructed in a canonical fashion by Pohl in [11]. If M has a boundary, he extends M a bit beyond its boundary to obtain an open manifold M^+ and defines $S(M)$ to be the closure of $M \times M - D$ in $S(M^+)$. In this case $S(M)$ is essentially the same as above, i. e., $S(M) = M \times M - D \cup T(M)$; however, we understand that, at points m of the boundary, $T(M)$ at m is to be taken as $T(M^+)$ at m.

Let $N = N^q$ be a differentiable manifold of dimension q possibly with with boundary and let $M = M^n$ be a closed submanifold of dimension $n (n \leqq q)$ with no boundary. We define $S(M,N)$ to be the closure of $M \times N - D_M$

in $S(N)$, where $D_M = \{(m,n) \in M \times N \mid m = n\}$. It follows from the construction of $S(N)$ that

$$S(M,N) = M \times N - D_M \cup M \times \partial N \cup T(N)_M,$$

where $T(N)_M$ is the restriction of $T(N)$ to M, and where we understand that, if M is a part of the boundary of N, $T(N)_M$ consists of $T(M)$ and tangent directions of N pointing to the interior of N from M. $S(M,N)$ is a manifold with boundary $M \times \partial N$ and $T(N)_M$.

Let g be a smooth C^3 map of N into oriented Euclidean $(n+q)$-space, E^{n+q}, such that g is a C^3 imbedding (or possibly immersion in the case $N = M$) in a neighborhood of M. In case N has boundary, we extend N a bit beyond its boundary to form an open manifold N^+ and then extend the map g to N^+. From now on we assume M and N are compact.

We denote the restriction of g to M by f. It is quite possible that $g(N)$ and $f(M)$ may intersect in points other than the obvious intersection. Such non-trivial intersection points will be denoted by I, i.e.,

$$I = \{(m,n) \in M \times N \mid m \neq n, g(n) = f(m)\}.$$

By means of the Thom Transversality Theorem it can be shown that, under a small deformation of g, these intersections may be made transverse. Under such a deformation the geometric entities we shall be discussing vary continuously. Hence we shall assume that the intersections of I are transverse. Because of the compactness of $f(M)$ and $g(M)$, they will be finite in number, say r. We denote them by $(m^{(a)}, n^{(a)})$, $a = 1, \cdots, r$. We surround these points by disjoint 'boxes' of small width ϵ

$$B_a = \{(m,n) = (m_i, n_j) \in M \times N - D_M \mid |m_i - m_i^{(a)}| \leqq \epsilon, |n_j - n_j^{(a)}| \leqq \epsilon, i = 1, \cdots, n; j = 1, \cdots, q\}.$$

We define a map

$$e_1 \colon S(M,N) - I \to S^{n+q-1},$$

where S^{n+q-1} is the unit $(n+q-1)$-sphere in E^{n+q}. For each

$$(m,n) \in M \times N - D_M - I,$$

we set

$$e_1(m,n) = \frac{g(n) - f(m)}{|g(n) - f(m)|}.$$

For each $t \in T(N)_M$, we set $e_1(t) = t$ regarded as a unit vector in E^{n+q}. That e_1 is a differentiable map is a clear consequence of the argument of [11].

We now assume that M and N are both oriented and we orient $M \times N$ in the canonical fashion. This induces an orientation on $S(M,N)$ and on the boxes B_a and hence on $T(N)_M$, $M \times \partial N$ and ∂B_a. We will take the orientation on ∂B_a to be that given from the "inside." Let dO_{n+q-1} be the pull-back of the volume element of S^{n+q-1} under the map e_1. Then $d(dO_{n+q-1}) = 0$, since dO_{n+q-1} is the pull-back of an $(n+q-1)$-form defined on an $(n+q-1)$-dimensional manifold. We apply Stokes' theorem to get

$$0 = \int_{M \times \partial N} dO_{n+q-1} + \int_{T(N)_M} dO_{n+q-1} - \sum_{a=1}^{r} \int_{\partial B_a} dO_{n+q-1}.$$

The minus sign preceding the last integral sum is due to the orientation given ∂B_a above.

As is shown in [1], each of the integrals in the sum is an index of an intersection times O_{n+q-1}, where O_{n+q-1} is the volume of the $(n+q-1)$-sphere. Hence the sum gives $O_{n+q-1} I(g,f)$, where $I(g,f)$ denotes the algebraic number of (non-trivial) intersections of $g(N)$ with $f(M)$, or simply the sum of the indices of the intersections. We recall that the volume of the k-sphere is

$$O_k = \frac{2\pi^{\frac{1}{2}(k+1)}}{\Gamma(\frac{1}{2}(k+1))},$$

where

$$\Gamma(\tfrac{1}{2}(k+1)) = \tfrac{1}{2}(k-1)\Gamma(\tfrac{1}{2}(k-1)); \Gamma(\tfrac{1}{2}) = \pi^{\frac{1}{2}}; \Gamma(1) = 1.$$

We now find an expression for dO_{n+q-1} in the second integral in order to integrate over the fibre. First, we choose local fields of orthonormal frames $fe_1, \cdots, e_{n+q}$ such that $e_1, \cdots, e_q$ are tangent to N at $f = f(m)$, e_1 being the map defined above, considered now as a vector-valued map, such that $e_{q+1}, \cdots, e_{n+q}$ span the normal space to N at $f = f(m)$, and such that the frames agreed with the orientation of $T(N)_M$ and E^{n+q}. We set $de_i \cdot e_j = \omega_{ij}$. Clearly, we have $dO_{n+q-1} = \omega_{12} \wedge \cdots \wedge \omega_{1n+q}$.

Next, we choose local fixed fields of orthonormal frames on M and write the fields defined above in terms of our new fields. Accordingly, let $fa_1, \cdots, a_{n+q}$ be local fixed fields of orthonormal frames such that $a_1, \cdots, a_q$ are tangent to N at $f = f(m)$, $m \in M$, and such that $a_{q+1}, \cdots, a_{n+q}$ are normal to N and hence to M at $f = f(m)$. We also require that the orientation of the frames agree with the orientation of the manifolds on which they are defined. We set $da_i \cdot a_j = \pi_{ij}$ and note that the π_{ij}'s are defined on the base manifold M.

We write

$$\begin{aligned} e_1 &= u_{11}a_1 + \cdots + u_{1q}a_q \\ &\;\;\vdots \qquad\qquad\qquad\qquad , \quad (u_{ij}) \text{ orthogonal;} \\ e_q &= u_{q1}a_1 + \cdots + u_{qq}a_q \\ e_s &= a_s, \quad s = q+1, \cdots, n+q. \end{aligned}$$

Then

$$\omega_{12} \wedge \cdots \wedge \omega_{1n+q} = \omega_{12} \wedge \cdots \wedge \omega_{1q} \wedge \omega_{11+q} \wedge \cdots \wedge \omega_{1n+q} =$$
$$(*) \quad = (dO_{q-1} + \text{terms in } \pi_{ij}\text{'s}) \wedge (u_{1i_1}\pi_{i_1q+1} \wedge \cdots \wedge u_{1i_n}\pi_{i_nn+q}),$$

where we use the Einstein's summation convention for repeated indices, the range of indices of the i_k being from 1 to q. Since the π_{ij}'s are defined on base manifold M, we have that any form of degree $> n$ in the π_{ij}'s is identically zero. Hence (*) becomes

$$dO_{q-1} \wedge u_{1i_1} \cdots u_{1i_n}\pi_{i_11+q} \wedge \cdots \wedge \pi_{i_nn+q}.$$

Thus, we arrive at our main equation, one which we shall use again and again.

$$\text{(E)} \; \frac{1}{O_{n+q-1}} \int_{M \times \partial N} dO_{n+q-1} + \frac{1}{O_{n+q-1}} \int_{T(N)_M} dO_{q-1} \wedge u_{1i_1} \cdots u_{1i_n}\pi_{i_11+q} \wedge \cdots \wedge \pi_{i_nn+q}$$
$$= I(g, \mathrm{f}).$$

We observe here that in the second term on the left-hand side we could integrate over the fibre now and obtain an integral over M. However, it proves to be simpler to do each case separately in what follows.

The major portion of this paper is devoted to the application of equation (E) to various situations depending in particular on the choice of M, N and the map g. The obvious first choice for N is simply M. In the cases where this is of interest, namely those in which M is even dimensional, equation (E) yields a result of Whitney, Lashof and Smale [13], [14], and [8]. This case is presented in Appendix A.

2. The case $N = M \times L$, $L = [0, \varepsilon]$. In this section we present results generalizing the formula of Călugăreanu [2] and the notion of self-linking of space curves [12]. We digress briefly from our main argument to recall some well-known facts about the linking of manifolds.

Let M^n and K^l be two closed smooth oriented manifolds of dimensions n and l, and let f and g be continuous maps of them into oriented Euclidean space E^{n+l+1} of dimension $n+l+1$ such that $f(M^n)$ and $g(K^l)$ do not intersect. Let S^{n+l} be the unit $n+l$-sphere centered at the origin of E^{n+l+1}.

Consider the cartesian product $M^n \times K^l$ given the canonical orientation. We define a map

$$e: M^n \times K^l \to S^{n+l}$$

by associating to each $(m,k) \in M^n \times K^l$ the unit vector in E^{n+l+1}

$$e(m,k) = \frac{g(k) - f(m)}{|g(k) - f(m)|} .$$

The degree of this map is the linking number $L(f(M^n), g(K^l))$, or simply $L(f,g)$. Let dO_{n+l} be the pull-back of the volume element of the $(n+l)$-sphere under the map e (we assume now that the maps f and g are C^1). Then, clearly

$$(1) \qquad L(f,g) = \frac{1}{O_{n+l}} \int_{M^n \times K^l} dO_{n+l},$$

where O_{n+l} is the volume of the $n+l$-sphere.

It is clear that if the maps f and g vary continuously, i.e., $f = f_t$, $g = g_t$ in such a way that the sets $f_t(M^n)$ and $g_t(K^l)$ intersect for no value of t, then the integral in (1) varies continuously, and hence, since it is integer-valued, remains constant.

Furthermore, let $e': K^l \times M^n \to S^{n+l}$ be the map $e'(k,m) = -e(m,k)$. Let α be the map of $K^l \times M^n$ onto $M^n \times K^l$ which transforms (k,m) into (m,k), and let β be the antipodal map of S^{n+l} onto itself. The degree of α is $(-1)^{nl}$, and the degree of β is $(-1)^{n+l+1}$. Since $e' = \beta e \alpha$, we obtain

$$(2) \qquad L(g,f) = (-1)^{(n+1)(l+1)} L(f,g).$$

We return now to our main line of argument. We consider the case $N = M \times L$ where L is the closed interval of real numbers $[0, \epsilon]$ and where we make the obvious identification $M \cong M \times \{0\}$. We orient L in the positive sense and then orient $M \times L$ in the canonical fashion.

We will use equation (E) where, as we have stated, $N = M \times L$ and hence is of dimension $n+1$, i.e., $q = n+1$. Let $f: M \to E^{2n+1}$ be a C^3 imbedding of M into Euclidean $(2n+1)$-space. Let v be a non-vanishing unit differentiable normal vector field on M (which always exists). We let g in the main equation (E) be defined as follows:

$$g(m,l) = f(m) + l v_{f(m)},$$

for $(m,l) \in N$, where $v_{f(m)}$ is the normal vector at the point $f(m)$. We choose ϵ small enough so that $f(M)$ and $f(M) + tv \equiv f_{t_v}(M)$ do not intersect for any value of t, $0 < t \leqq \epsilon$. This will insure that in the main equation (E) $I(g,f) = 0$, for there can be no non-trivial intersections of $g(N)$ and

$f(M)$. Finally, we denote the linking number of $f(M)$ and $f_{\epsilon_v}(M)$ by $L(f, f_{\epsilon_v})$ and $M \times \{\epsilon\}$ by M_ϵ.

We choose the frames $fa_1, \cdots, a_{2n+1}$ as follows. $a_1, \cdots, a_n$ are to be tangent to M at $f = f(m)$, a_{n+1} is to be along $v_{f(m)}$, and $a_{n+2}, \cdots, a_{2n+1}$ are to span the rest of the normal space at $f = f(m)$.

We apply equation (E) to get

$$(3) \quad -\frac{1}{O_{2n}}\int_{M\times M} dO_{2n} + \frac{1}{O_{2n}}\int_{M\times M_\epsilon} dO_{2n} + \frac{1}{O_{2n}}\int_{T(N)_M} dO_n \wedge u_{1i_1} \cdots u_{1i_n}\pi_{i_1 n+2} \wedge \cdots \wedge \pi_{i_n 2n+1} = 0.$$

Using formula (1), we see that

$$(4) \quad \frac{1}{O_{2n}}\int_{M\times M} dO_{2n} - \frac{1}{O_{2n}}\int_{T(N)_M} dO_n \wedge u_{1i_1} \cdots u_{1i_n}\pi_{i_1 n+2} \wedge \cdots \wedge \pi_{i_n 2n+1} = L(f, f_{\epsilon_v}).$$

The first integral on the left-hand side we call *the Gauss integral for the map f and the manifold M.*

Before stating the general theorems of this section, we proceed with a few special cases.

a. $n = 1$.

For $n = 1$, we have

$$\int_{T(N)_M} dO_1 \wedge u_{1i}\pi_{i3} = \int_{T(N)_M} dO_1 \wedge (u_{11}\pi_{13} + u_{12}\pi_{23}).$$

Let F stand for the fibre. We choose for our positive coordinate system giving us the proper orientation for $T(N)_M$ that which places the base coordinates first and the fibre coordinates last. Thus,

$$\int_{T(N)_M} dO_1 \wedge u_{1i}\pi_{i3} = -\int_M \int_F (u_{11}\pi_{13} + u_{12}\pi_{23})\, dO_1.$$

Using polar coordinates, $u_{11} = \cos\theta$, $u_{12} = \sin\theta$, and

$$-\int_M \int_0^\pi (\cos\theta\, \pi_{13} + \sin\theta\, \pi_{23})\, d\theta = -2\int_M \pi_{23}.$$

Equation (4) becomes

$$\frac{1}{4\pi}\int_{M\times M} dO_2 + \frac{1}{2\pi}\int_F \pi_{23} = L(f, f_{\epsilon_v}).$$

We notice immediately that if v is the principal normal vector field, we obtain a new proof for the formula of Călugăreanu [2]

$$\frac{1}{4\pi}\int_{M\times M} dO_2 + \frac{1}{2\pi}\int_M \tau ds = L(f, f_{\epsilon_v}) = SL.$$

If v is the binormal vector field, we obtain a new proof of Pohl's result [12] that SL is indeed the linking number of a curve and the same moved a small distance along the binormal. In fact, we have also shown that SL is the linking number of a curve and the same curve moved a small distance along the principal normal.

b. $n = 2$.

For $n = 2$, we have

$$\int_{T(N)_M} dO_2 \wedge (u_{1i_1} u_{1i_2} \pi_{i_1 4} \wedge \pi_{i_2 5}) = -\int_M \int_F (u_{1i_1} u_{1i_2} \pi_{4i_1} \wedge \pi_{i_2 5})\, dO_2.$$

We use polar coordinates to integrate and observe that, unless the power of the u_{1r} is even, the integral of the term involved is zero. Hence, our equation becomes

$$-\int_M \int_F u_{1i}^2 \pi_{4i} \wedge \pi_{i5} dO_2 = -\int_M d\pi_{45} \int_0^{2\pi} \int_0^{\pi/2} \cos^2 a \sin a \, da db$$

$$= -\frac{2\pi}{3}\int_M d\pi_{45}.$$

Equation (4) becomes

$$\frac{1}{O_4}\int_{M\times M} dO_4 + \frac{2\pi}{3O_4}\int_M d\pi_{45} = L(f, f_{\epsilon_v}).$$

From [4] we know that

$$-\frac{1}{2\pi}\int_M d\pi_{45}$$

is the Euler characteristic of the oriented subbundle of the normal bundle complementary to the bundle fibred by the line spanned by the vector v. We shall denote this Euler characteristic by $\chi(v^c)$. (More precisely, this integral is the Euler class of the subbundle evaluated on the fundamental class of M.) We combine our results and obtain

$$\frac{1}{O_4}\int_{M\times M} dO_4 - \tfrac{1}{2}\chi(v^c) = L(f, f_{\epsilon_v}).$$

We shall show in the next section that the Gauss integral for even dimensional manifolds is zero. Hence, we have

$$-\tfrac{1}{2}\chi(v^c) = L(f, f_{\epsilon_v}).$$

If we choose for v a unit vector along the mean curvature vector h of M, we might define the self-linking number of $f(M)$ to be

$$SL \equiv L(f, f_{\epsilon_h}) = -\tfrac{1}{2}\chi(h^c).$$

We remark here that an imbedding with non-vanishing mean curvature vector is the generic situation, that is, close to every C^3 imbedding of an n-dimensional closed smoothed manifold into Euclidean $(2n+1)$-space, there is an imbedding with non-vanishing mean curvature vector [10].

One final remark. Since $L(f, f_{\epsilon_v})$ is an integer, the Euler characteristic must always be even.

c. $n = 3$.

For $n = 3$, we have

$$\int_{T(N)_M} dO_3 \wedge (u_{1i_1}u_{1i_2}u_{1i_3}\pi_{i_15} \wedge \pi_{i_26} \wedge \pi_{i_37})$$
$$= -\int_M \int_F u_{1i_1}u_{1i_2}u_{1i_3}\pi_{i_15} \wedge \pi_{i_26} \wedge \pi_{i_37} dO_3.$$

We use polar coordinates to integrate and observe that, for all summation indices i except 4, if the power of u_{1i} is not even, an integral containing such a term is zero. Hence, we have two types of contributing terms, terms containing $u_{14}{}^3$ and terms containing a u_{14} and a $u_{1r}{}^2$, $r = 1, 2,$ or 3. Hence, our equation becomes

$$-\int_M \int_F u_{14}{}^3\pi_{45} \wedge \pi_{46} \wedge \pi_{47} dO_3$$
$$-\int_M \int_F u_{1r}{}^2 u_{14}[\pi_{r5} \wedge \pi_{r6} \wedge \pi_{47} + \pi_{r5} \wedge \pi_{46} \wedge \pi_{r7} + \pi_{45} \wedge \pi_{r6} \wedge \pi_{r7}] dO_3$$
$$= -\frac{8\pi}{15}\int_M \pi_{45} \wedge \pi_{46} \wedge \pi_{47} + \frac{4\pi}{15}\int_M \{\pi_{47} \wedge \Lambda_{56} - \pi_{46} \wedge \Lambda_{57} + \pi_{45} \wedge \Lambda_{67}\},$$

where

$$\Lambda_{\alpha\beta} = \sum_{k=1}^{3} \pi_{\alpha k} \wedge \pi_{k\beta}.$$

Equation (4) becomes

$$\frac{1}{O_6}\int_{M\times M} dO_6 + \frac{1}{O_3}\int_M \{+\pi_{45} \wedge \pi_{46} \wedge \pi_{47} - \tfrac{1}{2}(\pi_{47} \wedge \Lambda_{56} - \pi_{46} \wedge \Lambda_{57} + \pi_{45} \wedge \Lambda_{67})$$
$$= L(f, f_{\epsilon_v}).$$

If v is a unit vector along the mean curvature h of M, we have a definition of the self-linking of a three dimensional manifold in seven-space, i.e.

$$SL = L(f, f_{\epsilon_h});$$

hence the integrand in the second term on the left-hand side may be considered as a generalization of τds of a space curve. We therefore write

$$\frac{1}{O_6}\int_{M\times M} dO_6 + \frac{1}{O_3}\int_M \tau dV = L(f, f_{\epsilon_h}) = SL,$$

where dV is the volume element of the manifold M. We call τdV the torsion form of the imbedding with respect to the mean curvature vector h.

We proceed now briefly with the cases $n=4, 5$ and then shall state the general equations for n even and odd.

d. $n=4$.

For $n=4$, we have, using polar coordinates,

$$\begin{aligned}\int_M \int_F & u_{1i_1}u_{1i_2}u_{1i_3}u_{1i_4}\pi_{6i_1}\wedge\pi_{i_27}\wedge\pi_{8i_3}\wedge\pi_{i_49}dO_4\\ &= \frac{4\pi^2}{3\cdot 35}\int_M \{3\pi_{6i}\wedge\pi_{i7}\wedge\pi_{8i}\wedge\pi_{i9}+\pi_{6i}\wedge\pi_{i7}\wedge\pi_{8j}\wedge\pi_{j9}\\ &\quad +\pi_{6i}\wedge\pi_{j7}\wedge\pi_{8i}\wedge\pi_{j9}+\pi_{6i}\wedge\pi_{j7}\wedge_{8j}\wedge\pi_{j9}\} \quad i\neq j.\end{aligned}$$

Equation (4) becomes

$$\frac{1}{O_8}\int_{M\times M} dO_8 - \frac{1}{8\pi^2}\int_M \{\Omega_{67}\wedge\Omega_{89} - \Omega_{68}\wedge\Omega_{79}+\Omega_{69}\wedge\Omega_{78}\} = L(f, \epsilon_v),$$

where $\Omega_{\alpha\beta}=\sum_{k=1}^{5}\alpha_k\wedge\pi_{k\beta}$. Since the Gauss integral for even dimensional manifolds is zero, we obtain, similarly to 2,

$$-\tfrac{1}{2}\chi(v^c) = L(f, f_{\epsilon_v}),$$

where $\chi(v^c)$ is the Euler characteristic of the complementary (to v) oriented subbundle of the normal bundle.

e. $n=5$.

For $n=5$, we have, using polar coordinates,

$$\begin{aligned}-\int_M\int_F & u_{1i_1}u_{1i_2}\cdots u_{1i_5}\pi_{i_17}\wedge\pi_{i_28}\wedge\cdots\wedge\pi_{i_511}dO_5\\ &= -\frac{1}{5!}\,\frac{64\pi^2}{9\cdot 7\cdot 5\cdot 3}\int_M \epsilon_{\alpha_1\cdots\alpha_5}\pi_{6\alpha_1}\wedge\cdots\wedge\pi_{6\alpha_5}\\ &\quad +\frac{1}{3!2!}\,\frac{16\pi^2}{9\cdot 7\cdot 5\cdot 3}\int_M \epsilon_{\alpha_1\cdots\alpha_5}\pi_{6\alpha_1}\wedge\pi_{6\alpha_2}\wedge_{6\alpha_3}\wedge\Lambda_{\alpha_4\alpha_5}\\ &\quad -\frac{1}{4!}\,\frac{8\pi^2}{9\cdot 7\cdot 5}\int_M \epsilon_{\alpha_1\cdots\alpha_5}\pi_{6\alpha_1}\wedge\Lambda_{\alpha_2\alpha_3}\wedge\Lambda_{\alpha_4\alpha_5},\end{aligned}$$

where

$$\epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = \begin{cases} +1 \text{ if } \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 \text{ is an even permutation of } 7\,8\,9\,10\,11 \\ -1 \text{ if } \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 \text{ is an odd permutation of } 7\,8\,9\,10\,11 \\ \quad 0 \text{ otherwise,} \end{cases}$$

and

$$\Lambda_{\alpha\beta} = \sum_{k=1}^{5} \pi_{\alpha k} \wedge \pi_{k\beta}.$$

Equation (4) becomes

$$\frac{1}{O_{10}}\int_{M\times M} dO_{10} + \frac{1}{O_5}\epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}\Big[\frac{1}{5!}\int_M \pi_{6\alpha_1}\wedge\pi_{6\alpha_2}\wedge\pi_{6\alpha_3}\wedge\pi_{6\alpha_4}\wedge\pi_{6\alpha_5}$$
$$-\frac{1}{4}\,\frac{1}{3!2!}\int_M \pi_{6\alpha_1}\wedge\pi_{6\alpha_2}\wedge\pi_{6\alpha_3}\wedge\Lambda_{\alpha_4\alpha_5} + \frac{3}{8}\cdot\frac{1}{4!}\int_M \pi_{6\alpha_1}\wedge\Lambda_{\alpha_2\alpha_3}\wedge\Lambda_{\alpha_4\alpha_5}\Big]$$
$$= L(f, f_{\epsilon_v}).$$

Once again, if v is a unit vector along the mean curvature vector h, we define the self-linking of the imbedded five-manifold to be

$$\frac{1}{O_{10}}\int_{M\times M} dO_{10} + \frac{1}{O_5}\int_M \tau dV = L(f, f_{\epsilon_h}) \equiv SL,$$

where τdV is the torsion form of the imbedded manifold with respect to the mean curvature vector field, dV being the volume element of the five-manifold.

f. The general theorems. In order to obtain the general theorems, we integrate as in the cases $n = 1, \cdots, 5$ the second term on the left-hand side of equation (4). For n odd, we obtain

$$-\frac{1}{O_{2n}}\int_{T(N)_M} dO_n \wedge u_{1i_1}\cdots u_{1i_n}\pi_{i_1 n+2}\wedge\cdots\wedge\pi_{i_n 2n+1} = \frac{1}{O_n}\int_M \tau^* dV,$$

where $\tau^* dV$ is defined as follows. Let

$$\Phi_k = \epsilon_{\alpha_1\cdots\alpha_n}\pi_{n+1\alpha_1}\wedge\cdots\wedge\pi_{n+1\alpha_{n-2k}}\wedge\Lambda_{\alpha_{n-2k+1}\alpha_{n-2k+2}}\wedge\cdots\wedge\Lambda_{\alpha_{n-1}\alpha_n},$$

where

$$\Lambda_{\alpha\beta} = \sum_{i=1}^{n} \pi_{\alpha i}\wedge\pi_{i\beta},$$

and where

$$\epsilon_{\alpha_1\cdots\alpha_n} \begin{cases} +1 \text{ if } \alpha_1\cdots\alpha_n \text{ is an even permutation of } n+2\cdots 2n+1 \\ -1 \text{ if } \alpha_1\cdots\alpha_n \text{ is an odd permutation of } n+2\cdots 2n+1 \\ \quad 0 \text{ otherwise.} \end{cases}$$

Define

$$\Lambda = \frac{1}{\pi^{\frac{1}{2}(n+1)}} \sum_{k=0}^{\frac{1}{2}(n-1)} (-1)^k \frac{1}{1\cdot 3\cdots(n-2k)2^{\frac{1}{2}(n+1)+k}k!}\,\Phi_k.$$

Then

$$\tau^* dV = O_n \Lambda.$$

We call $\tau^* dV$ *the torsion form of the imbedded manifold with respect to the vector field* v.

We remark that such forms were first introduced by Chern [3] in his proof of the Gauss-Bonet formula. We, in a sense, have dualized his forms and have interpreted them as "torsion" forms.

THEOREM 1. *Let* $f\colon M^n \to E^{2n+1}$ *be a* C^3 *imbedding of a closed oriented differentiable manifold into Euclidean* $(2n+1)$*-space. Let* v *be a non-vanishing unit normal differentiable vector field on* M^n. *If* n *is odd,*

$$\frac{1}{O_{2n}}\int_{M\times M} dO_{2n} + \frac{1}{O_n}\int_M \tau^* dV = L(f, f_{\epsilon_v}),$$

where $L(f, f_{\epsilon_v})$ *is the linking number of the imbedded manifold with the same manifold deformed a small distance* ϵ *along the vector field* v, *and where* $\tau^* dV$ *is the torsion form of the imbedded manifold with respect to the vector field* v.

If we choose for v a unit vector along the mean curvature vector h of M^n, we have a definition of the self-linking of an odd n-dimensional manifold in $(2n+1)$-space, i. e.

$$SL = L(f, f_{\epsilon_h}),$$

and we write

$$\frac{1}{O_{2n}}\int_{M\times M} dO_{2n} + \frac{1}{O_n}\int_M \tau dV = SL,$$

where τdV is the torsion form of the imbedding with respect to the mean curvature vector h.

We proceed now to the general theorem for n even. We have

$$-\frac{1}{O_{2n}}\int_{T(N)_M} dO_n \wedge u_{1i_1}\cdots u_{1i_n}\pi_{i_1 n+2}\wedge\cdots\wedge\pi_{i_n 2n+1} =$$

$$-\frac{(-1)^{n/2}}{2^{n+1}\pi^{n/2}(\frac{1}{2}n)!}\int_M \Delta_0,$$

where

$$\Delta_0 = \epsilon_{\alpha_1\cdots\alpha_n}\Omega_{\alpha_1\alpha_2}\wedge\cdots\wedge\Omega_{\alpha_{n-1}\alpha_n};$$

$\epsilon_{\alpha_1 \cdots \alpha_n}$ is as above and

$$\Omega_{\alpha\beta} = \sum_{k=1}^{n+1} \pi_{\alpha k} \wedge \pi_{k\beta}.$$

But the integral on the right-hand side is simply $-\frac{1}{2}\chi(v^c)$ where $\chi(v^c)$ is the Euler characteristic of the complementary (to v) subbundle of the normal bundle.

THEOREM 2. *Let $f\colon M^n \to E^{2n+1}$ be a C^3 imbedding of a closed oriented differentiable manifold into Euclidean $(2n+1)$-space. Let v be a non-vanishing unit normal differentiable vector field on M^n. If n is even,*

$$-\tfrac{1}{2}\chi(v^c) = L(f, f_{\epsilon_v}),$$

where $\chi(v^c)$ is the Euler characteristic of the complementary (to v) oriented subbundle of the normal bundle and $L(f, f_{\epsilon_v})$ is as in Theorem 1.

We remark that the proof of Theorem 2 is not yet complete since we still must show that the Gauss integral for even dimensional manifolds is zero. This will be done in the next section.

If we choose for v a unit vector along the mean curvature vector h of M^n, we have a definition of the self-linking of an even dimensional manifold in $(2n+1)$-space, i. e.

$$SL \equiv L(f, f_{\epsilon_h}) = -\tfrac{1}{2}\chi(h^c),$$

3. The case $N = M \times L$, $L = [-\varepsilon, \varepsilon]$. In this section we prove that for n even the Gauss integral is indeed zero. We consider the case $N = M \times L$, where $L = [-\epsilon, \epsilon]$. Everything in equation (E) will be the same as in Section 2 with the exception of N, which in Section 2 was $M \times L$, $L = [0, \epsilon]$. g again is the map

$$g(m, l) = f(m) + l v_{f(m)}.$$

We choose ϵ small enough so that $f(M) + tv \equiv f_{t_v}(M)$ and $f(M) + sv \equiv f_{s_v}(M)$ do not intersect for any value of t and s, $-\epsilon \leqq s \neq t \leqq \epsilon$. This will insure that in the main equation (E) $I(g, f) = 0$, for there can be no non-trivial intersection of $g(N)$ and $f(M)$. The linking numbers of $f(M)$ and $f_{\epsilon_v}(M)$, $f_{-\epsilon_v}(M)$ will be denoted respectively by $L(f, f_{\epsilon_v})$, $L(f, f_{-\epsilon_v})$. Finally, we choose our frames $fa_1, \cdots, a_{2n+1}$ as in Section 2.

We apply equation (E) to get

$$(5) \quad -\frac{1}{O_{2n}} \int_{M \times M_{-\epsilon}} dO_{2n} + \frac{1}{O_{2n}} \int_{M \times M_{\epsilon}} dO_{2n} + \frac{1}{O_{2n}} \int_{T(N)_M} dO_n \wedge u_{1 i_1} \cdots u_{1 i_n} \pi_{i_1 n+2} \wedge \cdots \wedge \pi_{i_n 2n+1} = 0.$$

Using formula (1), we see that

$$L(f,f_{-\epsilon_v})-L(f,f_{\epsilon_v})=\frac{1}{O_{2n}}\int_{T(N)_M} dO_n\wedge u_{1i_1}\cdots u_{1i_n}\pi_{i_1n+2}\wedge\cdots\wedge\pi_{i_n2n+1}.$$

We integrate the right-hand similarly to the manner in Section 2. The only difference is the range of integration of the fibre. In fact, letting N^+ be the N of Section 2 and $N^\pm$ the N of this section, one can easily verify (writing dO_{2n} for $dO_n\wedge u_{1i_1}\cdots u_{1i_n}\pi_{i_1n+2}\wedge\cdots\wedge\pi_{i_n2n+1}$) that for n odd

$$\frac{1}{O_{2n}}\int_{T(N^\pm)_M} dO_{2n}=0,$$

and for n even

$$\frac{1}{O_{2n}}\int_{T(N^\pm)_M} dO_{2n}=\frac{2}{O_{2n}}\int_{T(N^+)_M} dO_{2n}.$$

Consider next $f(M)$, $f_{\epsilon_v}(M)$, $f_{-\epsilon_v}(M)$ and their respective linking numbers. By deforming $f(M)$ and $f_{\epsilon_v}(M)$ along v continuously a distance $-\epsilon$, we find that

$$L(f,f_{\epsilon_v})=L(f_{-\epsilon_v},f).$$

Using equation (2), we have

$$L(f_{-\epsilon_v},f)=(-1)^{(n+1)^2}L(f,f_{-\epsilon_v}).$$

Therefore, for n even

$$L(f_{-\epsilon_v},f)=L(f,f_{\epsilon_v})=-L(f,f_{-\epsilon_v}).$$

Thus, for n even

$$L(f,f_{\epsilon_v})=-\frac{1}{O_{2n}}\int_{T(N^+)_M} dO_{2n}.$$

Hence, using equation (4), we have

$$\frac{1}{O_{2n}}\int_{M\times M} dO_{2n}=0.$$

THEOREM 3. *Let $f\colon M^n\to E^{2n+1}$ be a C^3 imbedding of an oriented closed differentiable manifold M^n of dimension n into Euclidean $2n+1$-space. Then, if n is even,*

$$\frac{1}{O_{2n}}\int_{M\times M} dO_{2n}=0.$$

We observe now that this also completes the proof of Theorem 2.

4. The case $N=M\times L$, $L=[0,\infty]$. In this section we interpret $L(f,f_{\epsilon_v})$ as an intersection number. We consider the case $N=M\times L$, where

L is the full half-line $[0,\infty]$ including the point at infinity. Everything in equation (E) will be the same as in Section 2 with the exception of N. The map g is again

$$g(m,l)=f(m)+lv_{f(m)},$$

$(m,l)\in N$. We choose the frames $fa_1,\cdots,a_{2n+1}$ as before and apply equation (E) to get

$$-\frac{1}{O_{2n}}\int_{M\times M} dO_{2n}+\frac{1}{O_{2n}}\int_{M\times M_\infty} dO_{2n}+\frac{1}{O_{2n}}\int_{T(N)_M} dO_n\wedge u_{1i_1}\cdots u_{1i_n}\pi_{i_1n+2}\wedge\cdots\wedge\pi_{i_n2n+1}$$
$$=I(g,f),$$

where $M_\infty\cong M\times\{\infty\}$ and where, we recall, $I(g,f)$ is the algebraic number of (non-trivial) intersections of $g(N)$ with $f(M)$. We also recall that $I(g,f)$ is the sum of the indices of the intersections. Because of the connection of the intersections with the normal field v we will also call $I(g,f)$ *the sum of the indices of the normal intersections.* We will further call these intersections *forward* normal intersections because one finds the points of intersection by proceeding in a positive or "forward" sense along the vectors $v_{f(m)}$. The second integral on the left-hand side is zero because the image of $M\times M_\infty$ under the map e_1 is at most n-dimensional, since clearly $e_1(m_1,m,\infty)=e_1(m_2,m,\infty), m, m_1, m_2\in M$. Hence the degree is zero and thus the integral

$$\frac{1}{O_{2n}}\int_{M\times M_\infty} dO_{2n}$$

vanishes. We remark that these notions may be made precise by considering E^{2n+1} as the open "upper hemisphere" of S^{2n+1} so that the map of g is well defined on $M\times\{\infty\}$ and so that the map e_1 is guaranteed differentiable at points $(m,m,\infty)\in M\times M\times\{\infty\}=M\times M_\infty$.

The other two integals on the left-hand side are clearly the same as the respective integrals in Section 2, equation (3). Therefore, using equation (4), we have

$$L(f,f_{\epsilon_v})=-I(g,f).$$

We make one final observation in this section. For the case of space curves with v equal to the principal normal, $L(f,f_{\epsilon_v})=SL$. In this case, Pohl [12] has called $-I(g,f)$ the sum of the indices of forward cross-normals. There is a difference in sign because of the manner of definition of $I(g,f)$.

5. The case $N=M\times L$, $L=[-\infty,\infty]$. We consider the case where

$N = M \times L$, L being the full line including both points at infinity. We use equation (E), all things being the same as in Section 4 except, of course, for N, to get

$$-\frac{1}{O_{2n}}\int_{M\times M_{-\infty}} dO_{2n} + \frac{1}{O_{2n}}\int_{M\times M_{\infty}} dO_{2n} + \frac{1}{O_{2n}}\int_{T(N)_M} dO_n \wedge u_{1i_1}\cdots u_{1i_n}\pi_{i_1 n+2}\wedge\cdots\wedge\pi_{i_n 2n+1}$$
$$= I(g,f).$$

The first two integrals vanish for reasons analogous to those of Section 4. The third integral is clearly the same as the respective integral in Section 3, equation (5). Hence, we obtain, for n odd

$$I(g,f) = L(f,f_{-\epsilon_v}) - L(f,f_{\epsilon_v}) = 0$$

and for n even,

$$I(g,f) = L(f,f_{-\epsilon_v}) - L(f,f_{\epsilon_v}) = -2L(f,f_{\epsilon_v}).$$

If we define the sum of the indices of "backward" normal intersections analogously to the manner in which we define forward cross normals in Section 4, these two equations simply say that for n odd the sum of the indices of the forward normal intersections is equal to the sum of the indices of backward normal intersections and for n even the former is equal to minus the latter.

6. The case where N is a subbundle of the normal bundle. In this section we prove some purely differential topological results which generalize Theorem 2. Throughout this section, as always, all bundles discussed will be oriented.

Let $F: M^n \rightarrow E^{n+s}$ be a C^∞ imbedding of an oriented connected closed n-dimensional differentiable manifold into E^{n+s}. Suppose there exists a k-plane subbundle N of the normal bundle such that $0 < s - k \leqq n$ and $s - k$ is even. The normal bundle $\bar{N}$ of M consists of pairs (m, e) where $m \in M^n \equiv M$ and e is a point in the normal s-plane through $f(m)$. For the discussion that follows, to each oriented k-plane of N we add the oriented $(k-1)$-sphere of points at infinity thereby "compactifying" N; thus, in the pairs (m, e), e may take on points of the $(k-1)$-sphere at infinity. We make the obvious identification of the zero section of the normal bundle with M.

We define a map $H: \bar{N} \rightarrow E^{n+s}$ by setting $H(m,e) = e$ and a map g to be the restriction of H to N. Finally, we define a *normal intersection* to be a point $m \in M$ such that $F(m) = g(n)$ where $n = (m_0, e)$, any $m_0 \neq m$.

THEOREM 4. *Let $F: M^n \rightarrow E^{n+s}$ be a C^∞ imbedding of an oriented closed* *n-dimensional differentiable manifold into E^{n+s}. Suppose there exists an oriented k-plane subbundle N of the normal bundle such that $0 < s - k \leqq n$ and $s - k$ is even. Then the Poincaré dual of the locus of normal intersections is the Euler class of the complementary (to N) $(s-k)$-plane subbundle of the normal bundle.*

Remark. All classes are considered with real coefficients.

COROLLARY 5. *Let $F\colon M^n \to E^{2n+k}$ be a C^∞ imbedding of an even dimensional closed oriented differentiable manifold into E^{2n+k}. Let N be a k-plane subbundle of the normal bundle (which always exists). Then the sum of the indices of the normal intersections is equal to the Euler characteristic of the complementary (to N) n-plane subbundle of the normal bundle.*

The proofs use the secant manifold $S(A,N)$ where A is an arbitrary closed compact oriented differentiable singular $(s-k)$-chain in M and N is the (compactified) k-plane subbundle of the normal bundle. Our discussion in Section 1 considered $S(M,N)$ only where M was a closed submanifold of N. However, in our present case all details of that section go over since A is contained in a closed submanifold M.

We use equation (E) where for the map g we have the map defined above and for the map f we have $F|_A = g|_A$. We choose frames on A, $fa_1, \cdots, a_{n+s}$, such that $a_1, \cdots, a_n$ are tangent to M at $f = f(a)$, $a \in A$, $a_{n+1}, \cdots, a_{n+k}$ span the fibre of N and agree with its orientation at $f = f(a)$, and $a_{n+k}, \cdots, a_{n+s}$ span the rest of the normal space to M at $f = f(a)$.. Finally, we define a *normal A-intersection* to be a point $(a,n) \in A \times N$ such that $f(a) = g(n)$, where $n = (m_0, e)$, $m_0 \neq a$. Equivalently, a normal A-intersection is a point $a \in A \subset M$ of the normal intersection locus. We apply equation (E) to get

$$(6) \qquad \frac{1}{O_{n+s-1}} \int_{T(N)_A} dO_{n+k-1} \wedge u_{1i_1} \cdots u_{1i_{s-k}} \pi_{i_1 n+k+1} \wedge \cdots \wedge \pi_{i_{s-k} n+s} = I(g,f),$$

where $I(g,f)$ is clearly the sum of the indices of the normal A-intersections. The terms involving the boundaries at infinity vanish for reasons similar to those presented in Section 4.

We integrate in a maner similar to Section 2 and obtain

$$(6) = \frac{(-1)^w}{2^{2w}\pi^w w!} \int_A \Delta_0, \qquad w = \tfrac{1}{2}(s-k),$$

where

$$\Delta_0 = \epsilon_{\alpha_1 \cdots \alpha_{s-k}} \Omega_{\alpha_1\alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{s-k-1}\alpha_{s-k}},$$

$\alpha_1 \cdots \alpha_{s-k}$ being permutations of $n+k+1 \cdots n+s$, and where

$$\Omega_{\alpha\beta} = \sum_{i=1}^{n+k} \pi_{\alpha i} \wedge \pi_{i\beta}.$$

But the integral is simply

$$\chi(N^c)[A],$$

that is, the Euler class of the complementary (to N) $(s-k)$-plane sub-bundle of the normal bundle evaluated on the fundamental class of A. Combining our results, we have

$$\chi(N^c)[A] = I(g,f). \tag{7}$$

If we denote by $\mathfrak{N}$ the locus of normal intersections, then the intersection number $I(\mathfrak{N}, A)$ of $\mathfrak{N}$ with A equals $I(g,f)$. Hence we obtain

$$\chi(N^c)[A] = I(\mathfrak{N}, A).$$

But A was an arbitrary closed differentiable oriented singular $(s-k)$-chain. Hence we use the Poincaré Duality Theorem to obtain that the Euler class $\chi(N^c)$ is nothing other than the Poincaré dual of the locus of normal intersections, i. e.

$$\chi(N^c) = \mathfrak{D}\mathfrak{N}$$

where $\mathfrak{D}$ denotes the Poincaré dual. This completes the proof of Theorem 4. Corollary 5 is just the statement of equation (7) when $s-k=n$ and $A^{s-k} = M^n$.

7. The case $\partial N = M$. In this section we consider the case where the boundary of N is M. We start by proving a theorem about curves bounding surfaces in three-space.

a) Let C be a simple closed curve with non-vanishing curvature which bounds a compact surface A, C^3 immersed in Euclidean 3-space. We use the secant manifold $S(C,A)$ and the main equation (E), where for N we have A, for the map g we have the immersion of the surface A, and for f the restriction of g to C, $g\,|_C$, which as we have stated is an imbedding. We recall that the non-trivial intersection locus is $I = \{(c,a) \in C \times A \mid c \neq a, f(c) = g(a)\}$. Finally, we choose the frames $fa_1a_2a_3$ as follows. a_1 is to be tangent to C at $f = f(c)$, $c \in C$, a_3 is to be the surface normal at $f = f(c)$, and $a_2 = a_3 \times a_1$, so that a_2 is tangent to A and normal to C at $f = f(c)$.

We apply equation (E) to obtain

$$-\frac{1}{O_2}\int_{C\times C} dO_2 + \frac{1}{O_2}\int_{T(A)_C} dO_1 \wedge u_{1i}\pi_{i3} = I(g,f),$$

where $I(g,f)$ is clearly the sum of the indices of the non-trivial intersections of C with A. We integrate the second term on the left-hand side as in Section 2 to get

$$\frac{1}{O_2}\int_{T(A)_C} dO_1 \wedge u_{1i}\pi_{i3} = -\int_C \int_F u_{1i}\pi_{i3} dO_1 = -2\int_C \pi_{23},$$

where F is the fibre. $\pi_{23} = \tau_r ds$ where τ_r is called the relative torsion [7] and ds is the arc element of the curve. Hence,

$$\frac{1}{4\pi}\int_{C\times C} dO_2 + \frac{1}{2\pi}\int_C \tau_r\, ds = -I(g,f). \tag{8}$$

We next recall again the Formula of Călugăreanu

$$\frac{1}{O_2}\int_{C\times C} dO_2 + \frac{1}{O_1}\int_C \tau\, ds = SL$$

and subtract equation (8) from it to obtain

$$\frac{1}{O_1}\left[\int_C \tau\, ds - \int_C \tau_r\, ds\right] = SL + I(g,f).$$

But $\tau\, ds - \tau_r\, ds = d\alpha$, where α is the angle between the surface normal and the binormal of the curve. See the definition of $\tau_r\, ds$ in [7].

We state our theorems.

THEOREM 6. *Let C be a simple closed curve with non-vanishing curvature which bounds a compact surface A, C^3 immersed in Euclidean 3-space. Then*

$$\frac{1}{4\pi}\int_{C\times C} dO_2 + \frac{1}{2\pi}\int_C \tau_r\, ds = -I(g,f),$$

where τ_r is the relative torsion of the curve C and $I(g,f)$ is the sum of the indices of the non-trivial intersections of the curve with the surface.

THEOREM 7. *Let C be as in Theorem 6. Then, the total turning of the binormal vector of the curve with respect to the normal of the surface is equal to the sum of the indices of the non-trivial intersections of the curve with the surface plus the self-linking number of the curve, i.e.*

$$\frac{1}{2\pi}\int_C (\tau - \tau_r)\, ds = \frac{1}{2\pi}\int_C d\alpha = SL + I(g,f),$$

where α is the angle between the surface normal and the binormal of the curve.

We make a final remark about equation (8). In light of Theorem 1, this states that

$$L(f, f_{\epsilon_v}) = -I(g, f),$$

where v is either the surface normal vector field or the normal vector field to the curve which is tangent to the immersed surface A. In particular, if the surface is imbedded

$$L(f, f_{\epsilon_v}) = -I(g, f) = 0.$$

b) We generalize Theorem 6.

THEOREM 8. *Let $f\colon M^n \to E^{2n+1}$ be a C^3 imbedding of a closed oriented n-dimensional differentiable manifold which bounds an oriented $(n+1)$-dimensional differentiable compact manifold N, C^3 immersed in Euclidean $(2n+1)$-space. Then,* 1) *if n is even*

$$\tfrac{1}{2}\chi(v^c) = I(g, f),$$

where $\chi(v^c)$ is the Euler characteristic of the subbundle of the normal bundle complementary to the bundle fibred by the line spanned by the vector v which is normal to M and tangent to N, and where $I(g,f)$ is the sum of the indices of the non-trivial intersections of $g(N)$ with $f(M)$.

2) *If n is odd,*

$$\frac{1}{O_{2n}} \int_{M \times M} dO_{2n} + \frac{1}{O_n} \int_M \tau^* \, dV = -I(g, f),$$

where $\tau^* \, dV$ is the torsion form of the imbedding of M with respect to the vector field v.

The proof uses the secant manifold $S(M, N)$ and the main equation (E), where for the map g we have the immersion of the manifold N, and for f the restriction of g to M which is the imbedding. We choose the frames $fa_1, \cdots, a_{2n+1}$ as follows. $a_1, \cdots, a_n$ are to be tangent to M at $f = f(m)$, a_{n+1} is to be tangent to N along v at $f = f(m)$, and $a_{n+2}, \cdots, a_{2n+1}$ are to span the rest of the normal space to M at $f = f(m)$.

We apply equation (E) to obtain

$$-\frac{1}{O_{2n}} \int_{M \times M} dO_{2n} + \frac{1}{O_{2n}} \int_{T(N)_M} dO_n \wedge u_{1i_1} \cdots u_{1i_n} \pi_{i_1 n+2} \wedge \cdots \wedge \pi_{i_n 2n+1} = I(g, f),$$

where $I(g,f)$ is clearly the sum of the indices of the non-trivial intersections of $g(N)$ with $f(M)$. We integrate the second term on the left-hand side and proceed precisely as in Section 2 to get for n even

$$\tfrac{1}{2}\chi(v^c) = I(g,f),$$

and for n odd

$$\frac{1}{O_{2n}}\int_{M\times M} dO_{2n} + \frac{1}{O_n}\int_M \tau^* \, dV = -I(g,f),$$

where $\tau^* \, dV$ is the torsion form of the imbedding with respect to the vector field v, dV being the volume element of the manifold M.

Finally, we remark similarly to part (a) that

$$L(f, f_{\epsilon_v}) = -I(g,f),$$

and, in particular, if the manifold N is imbedded

$$L(f, f_{\epsilon_v}) = -I(g,f) = 0.$$

8. The Gauss integral for submanifolds. Until now, we have been concerned with the Gauss integral for a map f and a manifold M^n of the form

$$\frac{1}{O_{2n}}\int_{M^n\times M^n} dO_{2n}.$$

In this section we present a discussion of the Gauss integral for a map f and a restriction map $f\,|_{A^s}$, and a manifold M^n and a closed submanifold A^s of the form

$$\frac{1}{O_{n+s}}\int_{A^s\times M^n} dO_{n+s}.$$

We now make these notions precise.

Let $F\colon M^n \to E^{n+s+1}$ be a C^3 imbedding of a closed oriented differentiable manifold of dimension n into Euclidean $(n+s+1)$-space. Let $A^s = A$ be a closed oriented differentiable submanifold on $M^n = M$ and suppose there exists a non-vanishing unit normal differentiable vector field v on M. We consider the secant manifold $S(A,N)$ where $N = M \times L$, $L = [0,\epsilon]$. As in Section 2 we denote $M \times \{0\}$ and $M \times \{\epsilon\}$ by M and M_ϵ respectively. We assume that ϵ is small so that $F(M)$ and $F(M) + tv = F_{t_v}$ do not intersect for any value of t, $0 < t \leqq \epsilon$. This will insure in the use of equation (E) which follows that $I(g,f) = 0$. As always, we orient L in the positive sense and $M \times L$ in the canonical sense.

We employ again equation (E), where for the map f we have $F\,|_A$ and for map g:

$$g(m,l) = F(m) + l v_{F(m)},$$

$(m,l) \in N$. We choose the frames $f a_1, \cdots, a_{n+s+1}$ such that $a_1, \cdots, a_n$ are to be tangent to M at $f = f(a)$, $a \in A$, a_{n+1} is along $v_{f(a)}$, and $a_{n+2}, \cdots, a_{n+s+1}$

span the rest of the normal space to M at $f=f(a)$. The orientation of the frames is to be consistent with the orientations of N and E^{n+s+1}.

We apply equation (E) to obtain

$$\frac{(-1)^{n+s+1}}{O_{n+s}}\int_{A\times M} dO_{n+s}-\frac{(-1)^{n+s+1}}{O_{n+s}}\int_{A\times M_\epsilon} dO_{n+s}$$
$$+\frac{1}{O_{n+s}}\int_{T(N)_A} dO_n\wedge u_{1i_1}\cdots u_{1i_s}\pi_{i_1 n+2}\wedge\cdots\wedge\pi_{i_s n+s+1}=0,$$

the signs, as always, being determined by the induced orientation and in this case being $(-1)^{n+s+1}$ or $-(-1)^{n+s+1}$ because of the dimension and orientation of $A\times N$. Hence,

$$\frac{1}{O_{n+s}}\int_{A\times M} dO_{n+s}$$
$$+\frac{(-1)^{n+s+1}}{O_{n+s}}\int_{T(N)_A} dO_n\wedge u_{1i_1}\cdots u_{i1_s}\pi_{i_1 n+2}\wedge\cdots\wedge\pi_{i_s n+s+1}=L(f,F_{\epsilon_v}),$$

where $L(f,F_{\epsilon_v})$ is the linking number of the imbedded submanifold $f(A)$ with the imbedded manifold $F(M)$ deformed a small distance ϵ along the vector field v.

We integrate the second term on the left-hand side similarly to the manner of Section 2. We omit the details. For s odd we obtain

$$\frac{(-1)^{n+s+1}}{O_{n+s}}\int_{T(N)_A} dO_n\wedge u_{1i_1}\cdots u_{1i_s}\pi_{i_1 n+2}\wedge\cdots\wedge\pi_{i_s n+s+1}=\frac{1}{O_s}\int_A \tau_A{}^*\,dV,$$

where $\tau_A{}^*dV$ is defined as follows, dV being the volume element of A. Let

$$\Phi_k=\epsilon_{\alpha_1\cdots\alpha_s}\pi_{n+1\alpha_1}\wedge\cdots\wedge\pi_{n+1\alpha_{s-2k}}\wedge\Lambda_{\alpha_{s-2k+1}\alpha_{s-2k+2}}\wedge\cdots\wedge\Lambda_{\alpha_{s-1}\alpha_s},$$

where

$$\Lambda_{\alpha\beta}=\sum_{i=1}^{n}\pi_{\alpha i}\wedge\pi_{i\beta}$$

and

$$\epsilon_{\alpha_1\cdots\alpha_s}=\begin{cases}+1 \text{ if } \alpha_1\cdots\alpha_s \text{ is an even permutation of } n+2\cdots n+s+1\\ -1 \text{ if } \alpha_1\cdots\alpha_s \text{ is an odd permutation of } n+2\cdots n+s+1\\ \quad 0 \text{ otherwise.}\end{cases}$$

Define

$$\Lambda=\frac{1}{\pi^{\frac{1}{2}(s+1)}}\sum_{k=0}^{\frac{1}{2}(s-1)}(-1)^k\frac{1}{1\cdot 3\cdots(s-2k)\,2^{\frac{1}{2}(s+1)+k}k!}\Phi_k.$$

Then

$$\tau_A{}^*\,dV=O_s\Lambda.$$

We call $\tau_A{}^*\,dV$ the torsion form of the imbedded manifold $F(M)$ with respect to the vector field v relative to the imbedded submanifold $f(A)$.

We now proceed to the case s even. For s even we obtain

$$\frac{(-1)^{n+s+1}}{O_{n+s}}\int_{T(N)_A} dO_n \wedge u_{1i_1}\cdot\cdot\cdot u_{1i_s}\pi_{i_1 n+2}\wedge\cdot\cdot\cdot\wedge\pi_{i_s n+s+1} = \frac{(-1)^{s/2}}{2^{s+1}\pi^{s/2}(\frac{1}{2}s)\,!}\int_A \Delta_0,$$

where

$$\Delta_0 = \epsilon_{\alpha_1\cdots\alpha_s}\Omega_{\alpha_1\alpha_2}\wedge\cdot\cdot\cdot\wedge\Omega_{\alpha_{s-1}\alpha_s};$$

$\epsilon_{\alpha_1\cdots\alpha_s}$ is as above and

$$\Omega_{\alpha\beta} = \sum_{k=1}^{n+1}\pi_{\alpha k}\wedge\pi_{k\beta}.$$

But the integral on the right-hand side is simply

$$-\tfrac{1}{2}\chi(v^c)[A],$$

where $\chi(v^c)[A]$ is the Euler class of the complementary (to v) subbundle of the normal bundle of M evaluated on the fundamental class of A.

We state our theorem.

THEOREM 9. *Let A^s be a closed oriented differentiable submanifold of dimension s on an oriented closed differentiable manifold M^n of dimension n. Let $F\colon M^n \to E^{n+s+1}$ be a C^3 imbedding of M^n into Euclidean $(n+s+1)$-space and suppose there exists a non-vanishing unit normal vector field v on M^n. Then,*

1) *if s is odd*

$$\frac{1}{O_{n+s}}\int_{A^s\times M^n} dO_{n+s} + \frac{1}{O_s}\int_{A^s}\tau_A{}^*\,dV = L(f, F_{\epsilon_v}),$$

where f is the restriction of F to A^s, $\tau_A{}^*\,dV$ is the torsion form of the imbedded manifold $F(M)$ with respect to the vector field v relative to the imbedded submanifold $f(A)$, and where $L(f, F_{\epsilon_v})$ is the linking number of the imbedded submanifold $f(A)$ with the imbedded manifold $F(M)$ deformed a small distance ϵ along the vector field v.

2) *If s is even,*

$$\frac{1}{O_{n+s}}\int_{A^s\times M^n} dO_{n+s} - \tfrac{1}{2}\chi(v^c)[A^s] = L(f, F_{\epsilon_v}),$$

where $\chi(v^c)[A^s]$ is the Euler class of the complementary (to v) subbundle of the normal bundle of M^n evaluated on the fundamental class of A^s.

We notice immediately that for s even Theorem 9 implies that the Gauss integral

$$\frac{1}{O_{n+s}} \int_{A^s \times M^n} dO_{n+s}$$

is an integer. From Section 2 we know that, when $s = n$ and $A^s = M^n$, the Gauss integral is zero. It would be interesting to have more information in the general case.

9. Discussion of torsion. We now explore briefly why we have chosen to name our forms $\tau^* dV$ in Section 2 torsion forms. In this section we concern ourselves only with torsion forms with respect to the mean curvature vector field $\tau\, dV$. Of course there is the obvious fact that such forms are generalizations of the form $\tau\, ds$ for space curves.

The rest of this section deals with a generalization of the Fenchel-Jacobi Theorem [6], namely that the total torsion

$$\frac{1}{2\pi} \int_C \tau\, ds$$

of a space curve lying on a sphere is zero.

THEOREM 10. *Let $f\colon M^n \to S^{2n}$ be a C^3 imbedding of an n-dimensional oriented closed differentiable manifold into the 2n-dimensional sphere considered as a submanifold in E^{2n+1}. Then, if n is odd*

$$\frac{1}{O_n} \int_{M^n} \tau\, dV = 0,$$

where $\tau\, dV$ denotes the torsion form with respect to the mean curvature vector field.

Remark. It can easily be shown that the mean curvature vector field of such an imbedding never vanishes and, in fact, has a non-zero constant component along the outward normal of S^{2n}.

The proof uses the secant manifold $S(M^n, N)$, where $N = M^n \times L$, $L = [0, \infty]$ and the main equation (E), where for f we have the imbedding and for g the map

$$g(m, l) = f(m) + l v_{f(m)},$$

where v is the outward normal unit vector field of the sphere S^{2n} on which M^n is imbedded.

We choose the frames $f a_1, \cdots, a_{2n+1}$ as follows. $a_1, \cdots, a_n$ are to be tangent to M^n at $f = f(m)$, a_{n+1} is to be along v at $f = f(m)$ and $a_{n+2}, \cdots, a_{2n+1}$ are to be normal to M^n but tangent to S^{2n} at $f = f(m)$.

We apply equation (E) and obtain

$$+\frac{1}{O_{2n}}\int_{M^n\times M^n} dO_{2n}+\frac{1}{O_{2n}}\int_{M^n}\tau^*\,dV=0, \tag{9}$$

where $\tau^*\,dV$ is the torsion form with respect to the vector field v. The term at infinity is zero for similar reasons to those of Section 4. $I(g,f)=0$, since there clearly can be no normal intersections, v being along the outward normal of S^{2n}.

In what follows we assume for the sake of simplicity that S^{2n} is a unit sphere.

The form $\tau^*\,dV$ contains terms of the form

$$\Phi_k=\epsilon_{\alpha_1\alpha_2\cdots\alpha_n}\pi_{n+1\alpha_1}\wedge\cdots\wedge\pi_{n+1\alpha_{n-2k}}\wedge\Lambda_{\alpha_{n-2k+1}\alpha_{n-2k+2}}\wedge\cdots\wedge\Lambda_{\alpha_{n-1}\alpha_n},$$

where

$$\Lambda_{\alpha\beta}=\sum_{i=1}^{n}\pi_{\alpha i}\wedge\pi_{i\beta},$$

and where $\epsilon_{\alpha_1\alpha_2\cdots\alpha_n}$ is as in Section 2. Hence, every term of $\tau^*\,dV$ contains a term involving a term $\pi_{n+1\alpha_r}$, where α_r is $n+2, n+3,\cdots$, or $2n+1$. Since S^{2n} is a unit sphere

$$\pi_{n+1\alpha_r}=\pi_{\alpha_r},$$

where $\pi_{\alpha_r}=df\cdot a_r$. But π_{α_r} vanishes identically on M^n, since a_r is normal to the imbedding of M^n. Hence

$$\tau^*\,dV=0.$$

This implies

$$\frac{1}{O_{2n}}\int_{M^n\times M^n} dO_{2n}=0.$$

But this in turn implies by definition of SL

$$\frac{1}{O_n}\int\tau\,dV=SL,$$

where $\tau\,dV$ is the torsion form of the imbedding with respect to the mean curvature vector field.

There is left only to show that SL is zero. But this is an immediate consquence of the definition of SL and the fact that the mean curvature vector at each point of the imbedding has a component along the outward normal to S^{2n}. We simply use the same method as was used to obtain equation (9), the only difference being that v is now along the mean curvature vector. We have

$$SL=\frac{1}{O_{2n}}\int_{M\times M}dO_{2n}+\frac{1}{O_n}\int_M\tau\,dV=0,$$

since $I(g,f)$ in this case is zero for essentially the same reason it was zero in equation (9). This completes the proof of Theorem 10.

10. Deformation theory. We next investigate briefly the behaviour of the self-linking of imbedded manifolds of higher dimensions by exhibiting the invariance of SL under non-degenerate isotopy. The case of the self-linking of space curves has been investigated in [12].

By a *smooth regular deformation of closed manifolds* we mean a C^3 map $F\colon M^n\times I\to E^{2n+1}$, where M^n is the manifold and $I=[0,1]$ is the closed unit interval of real numbers, such that for fixed t, $f_t(M)=F(m,t)$ is a closed immersed manifold. F is called a non-degenerate deformation if each f_t is non-degenerate i.e. has a non-vanishing mean curvature vector at each point. We call a smooth regular deformation an isotopy if each f_t is imbedded.

Under a non-degenerate isotopy the mean-curvature vector varies continuously, which implies that the integral of the torsion form

$$\frac{1}{O_n}\int_{M^n}\tau\,dV$$

varies continuously. Now, under an isotopy the Gauss integral varies continuously. Since SL is an integer and since

$$\frac{1}{O_{2n}}\int_{M^n\times M^n}dO_{2n}+\frac{1}{O_n}\int_{M^n}\tau\,dV=SL$$

varies contiuously, SL remains an integer and is thus invariant under non-degenerate isotopy.

For the case where n is even, the invariance is obvious since $\chi(h^c)$ remains constant, where h is the mean curvature vector field, and

$$-\tfrac{1}{2}\chi(h^c)=SL$$

THEOREM 11. *The self-linking number of an n-dimensional closed oriented differentiable manifold M^n C^3 imbedded in Euclidean $(2n+1)$-space remains invariant under non-degenerate isotopy.*

Appendix A.

This portion of the appendix is devoted to the proof of the following theorem due to Whitney, Lashof and Smale [13] and [8].

THEOREM. *Let $f\colon M^n \to E^{2n}$ be a C^3 immersion of a closed orientable differentiable manifold M^n of even dimension n into Euclidean $2n$-space. Then, twice the algebraic number of self-intersections of f is equal to the Euler characteristic of the normal bundle, i.e. the normal characteristic of f.*

The proof uses equation (E) of Section 1 in the case where $N = M$ and where $f = g$ is the immersion. $I(g,f)$ is clearly twice the algebraic number of self-intersections. We choose the frames $fa_1, \cdots, a_{2n}$ as follows. $a_1, \cdots, a_n$ are to be tangent vectors to M^n at $f = f(m)$, $m \in M^n$, and $a_{n+1}, \cdots, a_{2n}$ are to be normal vectors at $f = f(m)$. The orientation of the frames are to be consistent with the orientation of M and E^{2n}.

Equation (E) gives, since ∂N is empty,

$$\text{(A1)} \quad \frac{1}{O_{2n-1}} \int_{T(M)} dO_{n-1} \wedge u_{1i_1} u_{1i_2} \cdots u_{1i_n} \pi_{i_1 n+1} \wedge \cdots \wedge \pi_{i_n 2n} = I(g,f).$$

One may proceed in a manner similar to Section 2 to integrate the left-hand side of equation (A1) and obtain

$$\frac{1}{O_{2n-1}} \int_{T(M)} dO_{n-1} \wedge u_{1i_1} \cdots u_{1i_n} \pi_{i_1 n+1} \wedge \cdots \wedge \pi_{i_n 2n} = \chi(\mathcal{N}),$$

where $\chi(\mathcal{N})$ is the Euler characteristic of the normal bundle.

For the sake of clarity we present the details for the case $n = 2$. We have

$$\text{(A2)} \quad \frac{1}{O_3} \int_{T(M)} dO_1 \wedge u_{1i_1} u_{1i_2} \pi_{i_1 3} \wedge \pi_{i_2 4} = \frac{1}{O_3} \int_M \int_F u_{1i_1} u_{1i_2} \pi_{i_1 3} \wedge \pi_{i_2 4} dO_1$$

Using polar coordinates, we have

$$\text{(A2)} \quad = \frac{1}{O_3} \int_M \int_0^{2\pi} [\cos^2\theta \pi_{13} \wedge \pi_{14} + \cos\theta \sin\theta (\pi_{13} \wedge \pi_{24} + \pi_{23} \wedge \pi_{14}) + \sin^2\theta \pi_{23} \wedge \pi_{24}] \, d\theta$$

$$\frac{1}{O_3} \cdot \pi \int_M \pi_{13} \wedge \pi_{14} + \pi_{23} \wedge \pi_{24} = -\frac{\pi}{O_3} \int_M d\pi_{34} = \frac{\pi}{O_3} \int_M N dA$$

where d denotes the exterior derivative, and were NdA is the curvature form of the connection in the normal bundle. That $d\pi_{34} = -NdA$ may be found in Chern [5]. But

$$+\frac{\pi}{O_3} \int_M NdA = +\frac{1}{2\pi} \int_M NdA = \chi(\mathcal{N}).$$

We make the final observation that the integral in (A1) is also the tangential degree of the map f.

Appendix B.

In this portion of the appendix we present a different approach to some of the ideas in Section 2. The approach is a direct generalization of Pohl's proof of Călugăreanu's formula for the self-linking of a space curve [14],

$$\frac{1}{4\pi}\int_{C\times C} dO_2 + \frac{1}{2\pi}\int_C \tau\, ds = SL.$$

In his proof, he uses the secant manifold $S(C)$, as opposed to the secant manifold $S(C, N)$ which we use in Section 2. We now give a brief indication of some of the ideas in Section 2 using only the secant manifold $S(M)$.

Accordingly, let $f\colon M^n \to E^{2n+1}$ be a C^3 imbedding of a closed orientable n-dimensional differentiable manifold $M^n = M$ into oriented Euclidean space of dimension $2n+1$. Let $S(M)$ be the space of abstract secant directions of M. We recall $S(M) = M \times M - D \cup T(M)$, where $D = \{(m, m) \in M \times M\}$ and $T(M)$ is the space of oriented tangent directions of M.

Let v be a non-vanishing unit normal vector field on M. We will call the line spanned by v at a point $f(m)$ *the normal line* at $f(m)$. The manifold of all normal lines to f we shall call the *normal manifold*.

To each $(m_1, m_2) \in M \times M - D$ we associate the unit vector $e_1(m_1, m_2)$ in E^{2n+1} directed from $f(m_1)$ to $f(m_2)$ i.e.

$$e_1(m_1, m_2) = \frac{f(m_2) - f(m_1)}{|f(m_2) - f(m_1)|}.$$

To each $t \in T(M)$ we associate $e_1(t) = t$ regarded as a unit vector in E^{2n+1}. The map $e_1\colon S(M) \to S^{2n}$ thus defined on all of $S(M)$ is shown to be differentiable in [11].

To each $(m_1, m_2) \in M \times M - D$ such that the normal line to f at $f(m_1)$ does not pass through $f(m_2)$, we associate the vector e_{2n+1}, the unit vector in the plane spanned by the normal line to f at $f(m_1)$ and the secant line $\overline{f(m_1)f(m_2)}$, orthogonal to e_1, and so oriented that $e_1 e_{2n+1}$ agrees with the orientation $e_1 v_{f(m_1)}$ where $v_{f(m_1)}$ is the normal vector to the imbedding f at $f(m_1)$. The vector function e_{2n+1} clearly extends smoothly to the boundary of $S(M)$, where it simply becomes $v_{f(m)}$.

However, e_{2n+1} is not defined at points where the imbedding intersects the normal manifold. We call these points the cross-normals of f. We will assume that f(M) crosses the normal manifold tranversally at the cross-normals. Because of the compactness of M, the cross-normals will be finite in number, say r. We shall denote them $(m_1^{(a)}, m_2^{(a)})$, $a = 1, \cdots, r$. We surround each of the cross-normals by a box $B_{a\epsilon}$ of small width ϵ in $M \times M - D \subset S(M)$,

$$B_{a\epsilon} = \{(m_1, m_2) = (m_{1i}, m_{2i}) \in M \times M - D \mid |m_{1i} - m_{1i}^{(a)}| \leqq \epsilon, |m_{2i} - m_{2i}^{(a)}| \leqq \epsilon, \quad i = 1, \cdots, n\}$$

We orient $M \times M$ in the canonical fashion. This induces an orientation on $S(M)$, hence on $T(M)$ and the boundaries of the boxes $B_{a\epsilon}$, $\partial B_{a\epsilon}$, $a = 1, \cdots, r$. We will speak of the canonical orientation on $T(M)$ to be the usual one and on $\partial B_{a\epsilon}$ to be that given from the 'inside' of the boxes.

We denote the pull-back of the volume element of the $2n$-sphere, S^{2n}, under the map e_1 by dO_{2n} and apply Stokes' Theorem to get

$$\text{(B1)} \qquad \int_{S(M)} dO_{2n} = \int_{T(M)} \Delta - \lim_{\epsilon \to 0} \sum_{a=1}^{r} \int_{\partial B_{a\epsilon}} \Delta,$$

where Δ is a differential form of degree $2n - 1$ such that its exterior derivative $d\Delta$ is equal to dO_{2n}. We will show such a form exists. The second term on the right is preceded by a minus sign since the orientation induced is from the 'outside' of the boxes and hence is opposite to the canonical orientation.

We now find an expression for Δ and for $d\Delta = dO_{2n}$, using the formalism introduced in [3]. We define local fields of orthonormal frames on $f(M)$, $fe_1, \cdots, e_{2n+1}$ such that e_1 and e_{2n+1} are the maps above now considered as vectors and such that $e_2, \cdots, e_{2n}$ complete the orthonormal frame. We set $\omega_{ij} = de_i \cdot e_j$, $i, j = 1, \cdots, 2n+1$. Clearly, $dO_{2n} = \omega_{12} \wedge \cdots \wedge \omega_{12n+1}$. In order to apply the formalism of [3], we need a well-chosen vector orthogonal to e_1; we have such a vector e_{2n+1}.

We set

$$\Omega_{\alpha\beta} = \omega_{\alpha 1} \wedge \omega_{1\beta}$$

for any α, β, and

$$\Phi_\lambda = \epsilon_{\alpha_2 \cdots \alpha_{2n}} \Omega_{\alpha_2\alpha_3} \wedge \cdots \wedge \Omega_{\alpha_{2\lambda}\alpha_{2\lambda+1}} \wedge \omega_{\alpha_{2\lambda+2}2n+1} \wedge \cdots \wedge \omega_{\alpha_{2n}2n+1},$$

where

$$\epsilon_{\alpha_2 \cdots \alpha_{2n}} = \begin{cases} +1 \text{ if } \alpha_2 \cdots \alpha_{2n} \text{ is an even permutation of } 2 \cdots 2n \\ -1 \text{ if } \alpha_2 \cdots \alpha_{2n} \text{ is an odd permutation of } 2 \cdots 2n \\ 0 \text{ otherwise.} \end{cases}$$

We set

$$\Delta^* = \frac{1}{\pi^n} \sum_{\lambda=0}^{n-1} (-1)^\lambda \frac{1}{1 \cdot 3 \cdots (2n - 2\lambda - 1) 2^{n+\lambda} \lambda!} \Phi_\lambda.$$

Finally, we set

$$\Delta = -\frac{2^{2n}\pi^n n!}{(-1)^n(2n)!}\Delta^*.$$

Then Δ is the form we are looking for, that is, it is the form whose exterior derivative $d\Delta$ is equal to dO_{2n}.

We will now present the cases $n=2,3$ as examples of the cases n even and n odd. We first notice that the first integral on the left-hand side of (B1) is simply the Gauss integral times O_{2n}

$$\int_{S(M)} dO_{2n} = \int_{M\times M} dO_{2n},$$

since the removed points of $M\times M$ to form $S(M)$ have measure zero.

a) The case $n=2$.

We have $n=2$

$$\Delta^* = \frac{1}{\pi^2}\{\frac{1}{12}\Phi_0 - \tfrac{1}{8}\Phi_1\}.$$

Substituting for Φ_0 and Φ_1, we get

$$\Delta = -\tfrac{1}{3}\{2\omega_{25}\wedge\omega_{35}\wedge\omega_{45} - \{\omega_{21}\wedge\omega_{13}\wedge\omega_{45} - \omega_{21}\wedge\omega_{14}\wedge\omega_{35} + \omega_{31}\wedge\omega_{14}\wedge\omega_{25}\}\}.$$

Let us first integrate Δ over $T(M)$. To do this, we apply Fubini's Theorem by choosing local fields of orthonormal frames $fe_1,\cdots,e_5$ such that e_1e_2 are tangent to $f(M)$ at $f=f(m)$, e_1 being the map defined above, and such that $e_3e_4e_5$ are normal at $f=f(m)$, e_5 being along $v_{f(m)}$. Next, we choose local fixed fields of orthonormal frames $fa_1,\cdots,a_5$, such that a_1, a_2 are tangent and a_3, a_4, a_5 are normal. To integrate Δ, we write

$$\begin{aligned} e_1 &= \cos\theta a_1 + \sin\theta a_2\\ e_2 &= -\sin\theta a_1 + \cos\theta a_2;\\ e_s &= a_s, \quad s=3,4,5. \end{aligned}$$

We set $da_i\cdot a_j = \pi_{ij}$ and note that the π_{ij}'s are defined on the base manifold M; hence, any form of degree greater than 2 in the π_{ij}'s is identically zero.

The first term in Δ:

$$\omega_{25}\wedge\omega_{35}\wedge\omega_{45} = (-\sin\theta\pi_{15} + \cos\theta\pi_{25})\wedge\pi_{35}\wedge\pi_{45} = 0.$$

The second:

$$\begin{aligned} \omega_{21}\wedge\omega_{13}\wedge\omega_{45} &= (\pi_{21} - d\theta)\wedge(\cos\theta\pi_{13} + \sin\theta\pi_{23})\wedge\pi_{45}\\ &= -d\theta\wedge(\cos\theta\pi_{13} + \sin\theta\pi_{23})\wedge\pi_{45}. \end{aligned}$$

Integrating over $T(M)$, we get

$$\int_{T(M)} \omega_{21} \wedge \omega_{13} \wedge \omega_{45} = -\int_M \int_0^{2\pi} (\cos\theta\pi_{13} + \sin\theta\pi_{23}) \wedge \pi_{45} d\theta = 0.$$

The third term gives a similar analysis to the second. Hence,

$$\int_{T(M)} \Delta = 0.$$

We remark here that clearly any term not containing an ω_{12} could not contribute to the integral.

So we have now

$$\int_{M\times M} dO_{2n} = -\lim_{\epsilon\to 0} \sum_{a=1}^{r} \int_{\partial B_{a\epsilon}} \Delta.$$

To integrate the right-hand side, we replace, i.e. blow up, each cross-normal $(m_1^{(a)}, m_2^{(a)})$ by the sphere of oriented tangent directions R_a to $M \times M$ at $(m_1^{(a)}, m_2^{(a)})$. We obtain thus a manifold with boundary consisting of $R_1, \cdots, R_r$ and $T(M)$. It can be shown that the frames defined on the interior of $S(M)$ above can be extended smoothly to the boundary components $R_1, \cdots, R_r$, which is all we need to integrate. Now e_1 is constant on the spheres; hence, any term in Δ containing an ω_{1j} is identically zero. Thus, we have

$$\int_{M\times M} dO_{2n} = +\sum_{a=1}^{r} \tfrac{2}{3} \int_{R_a} \omega_{25} \wedge \omega_{35} \wedge \omega_{45}.$$

The right-hand side is recognized as a sum of indices times the volume of the three-sphere times 2/3. Thus, we have established:

THEOREM. *Let $f: M^2 \to E^5$ be a C^3 imbedding of a two-dimensional oriented closed manifold into Euclidean five-space and let v be a normal vector field defined on M^2. Then*

$$\frac{1}{O_4} \int_{M\times M} dO_4 = I(v),$$

where $I(v)$ is the sum of the indices of the cross-normals.

We remark that we need the machinery of Section 3 to prove

$$\frac{1}{O_4} \int_{M\times M} dO_4 = 0.$$

This gives us the fact that $I(v) = 0$.

b) The case $n = 3$.

We have for $n=3$

$$\Delta^* = \frac{1}{\pi^3}\left\{\frac{1}{120}\Phi_0 - \frac{1}{48}\Phi_1 + \frac{1}{64}\Phi_2\right\},$$

where

$$\Phi_0 = \epsilon_{\alpha_2\cdots\alpha_6}\omega_{\alpha_2 7}\wedge\cdots\wedge\omega_{\alpha_6 7} = 120\omega_{27}\wedge\cdots\wedge\omega_{67},$$

$$\Phi_1 = \epsilon_{\alpha_2\cdots\alpha_6}\Omega_{\alpha_2\alpha_3}\wedge\omega_{\alpha_4 7}\wedge\omega_{\alpha_5 7}\wedge\omega_{\alpha_6 7}$$
$$= 12\{\Omega_{23}\wedge\omega_{47}\wedge\omega_{57}\wedge\omega_{67} - \Omega_{24}\wedge\omega_{37}\wedge\omega_{57}\wedge\omega_{67} + \cdots + \Omega_{34}\wedge\omega_{57}\wedge\omega_{67}\wedge\omega_{27}$$
$$+ \cdots + \Omega_{45}\wedge\omega_{67}\wedge\omega_{27}\wedge\omega_{37} + \cdots + \Omega_{56}\wedge\omega_{27}\wedge\omega_{37}\wedge\omega_{47}\},$$

$$\Phi_2 = \epsilon_{\alpha_2\cdots\alpha_6}\Omega_{\alpha_2\alpha_3}\wedge\Omega_{\alpha_4\alpha_5}\wedge\omega_{\alpha_6 7}$$
$$= 24\{\Omega_{23}\wedge\Omega_{45}\wedge\omega_{67} - \Omega_{23}\wedge\Omega_{65}\wedge\omega_{47} - \Omega_{23}\wedge\Omega_{46}\wedge\omega_{57}$$
$$+ \Omega_{26}\wedge\Omega_{45}\wedge\omega_{37} - \Omega_{63}\wedge\Omega_{45}\wedge\omega_{27}\}.$$

We further have

$$\Delta = \frac{8\pi^3}{15}\Delta^*,$$

where $d\Delta = dO_6$.

We proceed as in the case $n=2$ to integrate Δ over $T(M)$ by means of Fubini's Theorem. We choose local fields of orthonormal frames $fe_1,\cdots,e_7$ such that e_1, e_2, e_3 are tangent to M at $f=f(m)$, e_1 being the map defined above, and such that $e_4,\cdots,e_7$ are normal at $f=f(m)$, e_7 being along $v_{f(m)}$. Next we choose local fixed fields of othonormal frames $fa_1,\cdots,a_7$ such that a_1, a_2, a_3 are tangent and $a_4,\cdots,a_7$ are normal. To integrate Δ, we write

$$\begin{aligned} e_1 &= u_{11}a_1 + u_{12}a_2 + u_{13}a_3 \\ e_2 &= u_{21}a_1 + u_{22}a_2 + u_{23}a_3, \quad (u_{ij}) \text{ orthogonal;} \\ e_3 &= u_{31}a_1 + u_{32}a_2 + u_{33}a_3 \\ e_s &= a_s, \quad s = 4, 5, 6, 7. \end{aligned}$$

We make two important observations. Since the base manifold M is three-dimensional and since the π_{ij}'s $= da_1\cdot a_j$'s are defined on M, any form of degree greater than three in the π_{ij}'s is identically zero. Secondly, corresponding to the remark in the case $n=2$ for ω_{12}, any form not containing $\omega_{12}\wedge\omega_{13}$ will not contribute to the integral.

Hence, no form from Φ_0 contributes. Only the form

$$12\omega_{21}\wedge\omega_{13}\wedge\omega_{47}\wedge\omega_{57}\wedge\omega_{67}$$

from Φ_1 contributes. Finally, the forms

$$24\omega_{21}\wedge\omega_{13}\wedge\omega_{41}\wedge\omega_{15}\wedge\omega_{67}, \qquad -24\omega_{21}\wedge\omega_{13}\wedge\omega_{61}\wedge\omega_{15}\wedge\omega_{47},$$

and $-24\omega_{21}\wedge\omega_{13}\wedge\omega_{41}\wedge\omega_{16}\wedge\omega_{57}$ contributes from Φ_2. Hence,

$$\int_{T(M)}\Delta=\int_{T(M)}\frac{8\pi^3}{15}\Delta^*=\int_{T(M)}\{\frac{2}{15}\omega_{12}\wedge\omega_{13}\wedge\omega_{47}\wedge\omega_{57}\wedge\omega_{67}$$

$$+\tfrac{1}{5}\{\omega_{12}\wedge\omega_{13}\wedge\omega_{14}\wedge\omega_{15}\wedge\omega_{67}+\omega_{12}\wedge\omega_{13}\wedge\omega_{15}\wedge\omega_{16}\wedge\omega_{47}$$

$$-\omega_{12}\wedge\omega_{13}\wedge\omega_{14}\wedge\omega_{16}\wedge\omega_{57}\}\}.$$

We now investigate these forms in terms of the π_{ij}'s. First

$$\omega_{12}\wedge\omega_{13}=dO_2+\text{terms in the }\pi_{ij}\text{'s},$$

where dO_2 is the area element of the tangent 2-sphere. Thus,

$$\int_{T(M)}\frac{2}{15}\omega_{12}\wedge\omega_{13}\wedge\omega_{47}\wedge\omega_{57}\wedge\omega_{67}=\frac{2}{15}\int_M\int_{S^2}\pi_{47}\wedge\pi_{57}\wedge\pi_{67}dO_2$$

$$=\frac{8\pi}{15}\int_M\pi_{47}\wedge\pi_{57}\wedge\pi_{67}.$$

Next,

$$\int_{T(M)}\tfrac{1}{5}\omega_{12}\wedge\omega_{13}\wedge\omega_{14}\wedge\omega_{15}\wedge\omega_{67}=\tfrac{1}{5}\int_M\int_{S^2}(u_{11}\pi_{14}+u_{12}\pi_{24}+u_{13}\pi_{34})$$

$$\wedge(u_{11}\pi_{15}+u_{12}\pi_{25}+u_{13}\pi_{35})\wedge\pi_{67}dO_2$$

$$=\tfrac{1}{5}\int_M\int_{S^2}(u_{11}{}^2\pi_{14}\wedge\pi_{15}+u_{12}{}^2\pi_{24}\wedge\pi_{25}+u_{13}{}^2\pi_{34}\wedge\pi_{35})\wedge\pi_{67}dO_2.$$

The cross-terms vanish and only the terms involving the squares of the u_{ij}'s contribute, since the integral is over the entire two-sphere. Using polar coordinates to integrate, the above expression becomes

$$\frac{4\pi}{15}\int_M(\pi_{14}\wedge\pi_{15}+\pi_{24}\wedge\pi_{25}+\pi_{34}\wedge\pi_{35})\wedge\pi_{67}.$$

Similarly, we analyze the other forms and we obtain

$$\int_{T(M)}\Delta=\frac{4\pi}{15}\int_M\{2\pi_{47}\wedge\pi_{57}\wedge\pi_{67}-\Lambda_{45}\wedge\pi_{67}-\Lambda_{56}\wedge\pi_{47}+\Lambda_{46}\wedge\pi_{57}\},$$

where

$$\Lambda_{ab}=\sum_{i=1}^{3}\pi_{ai}\wedge\pi_{ib}.$$

The similarity between the expression here under the integral sign and that in Section 2c is immediate. We point out that here a_7 is along $v_{f(m)}$, whereas in Section 2c a_4 is along v. This accounts for the differences in subscripts.

We have thus far

$$\int_{M\times M} dO_6 - \frac{4\pi}{15}\int_M \{2\pi_{47}\wedge\pi_{57}\wedge\pi_{67} - \Lambda_{45}\wedge\pi_{67} - \Lambda_{56}\wedge\pi_{47} + \Lambda_{46}\wedge\pi_{57}\}$$

$$= -\lim_{\epsilon\to 0}\sum_{a=1}^{r}\int_{\partial B_{a\epsilon}}\Delta.$$

To integrate the right-hand side, we proceed as in the case $n=2$ and replace each cross-normal $(m_1^{(a)}, m_2^{(a)})$ by the sphere of oriented tangent directions R_a to $M\times M$ at $(m_1^{(a)}, m_2^{(a)})$. Arguing as before, we find that any term in Δ containing an ω_{1j} is identically zero. Thus we get

$$-\lim_{\epsilon\to 0}\sum_{a=1}^{r}\int_{\partial B_{a\epsilon}}\Delta = -\sum_{a=1}^{r}\frac{8}{15}\int_{R_a}\omega_{27}\wedge\omega_{37}\wedge\omega_{47}\wedge\omega_{57}\wedge\omega_{67}.$$

The right-hand side gives the sum of the indices of the cross-normals times the volume of the five-sphere times 8/15. Thus, we have established:

THEOREM. *Let $f\colon M^3\to E^7$ be a C^3 imbedding of a three-dimensional oriented closed manifold M^3 into Euclidean seven-space and let v be a normal vector field defined on M. Then*

$$\frac{1}{O_3}\int_{M\times M} dO_6 + \frac{1}{O_6}\int_M\{-\pi_{47}\wedge\pi_{57}\wedge\pi_{67} + \tfrac{1}{2}(\Lambda_{45}\wedge\pi_{67} + \Lambda_{56}\wedge\pi_{47} - \Lambda_{46}\wedge\pi_{57})\}$$

$$= -I(v),$$

where $I(v)$ is the sum of the indices of th ecross-normals.

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