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ON THE HOMOTOPY TYPE OF MANIFOLDS

By J. H. C. Whitehead

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1. The object of this note is to call attention to certain theorems, which follow very easily from some results due to E. Stiefel,[1] H. Seifert,[2] Hassler Whitney,[3] and myself.[4] They refer to a class of manifolds which we call the class Π, and are intended to throw light on the question, raised by W. Hurewicz,[5] whether two closed manifolds of the same homotopy type are necessarily homeomorphic. The theorems depend both on M. H. A. Newman's[6] theory of combinatorial equivalence, as re-developed by J. W. Alexander[7] and carried further in S. S., and on theorems concerning differentiable manifolds. Therefore it is necessary to give a precise meaning to the term 'manifold'.

By an n-dimensional manifold, M^n, we shall mean a class of combinatorially equivalent, simplicial complexes covering the same space, each complex being a *formal manifold*, meaning that the complement[7] of each vertex is combinatorially equivalent to $\dot{A}^n$ or to A^{n-1}, according as the vertex in question is inside M^n or on $\dot{M}^n$, where A^k stands for a closed k-simplex and $\dot{M}^n$ is the boundary of M^n. These covering complexes will be called *proper triangulations* of M^n (of course any simplicial complex covering M^n is a proper triangulation if the '*Hauptvermutung*' is true). The proper triangulations of an unbounded manifold of class C^1, or *smooth manifold*, are to be C^1-triangulations.[7a] By a smooth, bounded, n-dimensional manifold we shall mean the manifold of which a sub-complex $K_0^n \subset K^n$ is a proper triangulation, where K^n is a C^1 triangulation of a smooth, unbounded n-dimensional manifold and K_0^n is a formal manifold. By the topological product $M^n \times A^k$ we shall mean the manifold having a normal subdivision of the cell-complex $K^n \times A^k$ as a proper triangulation, where K^n is a proper triangulation of M^n. We shall use $\equiv$ to indicate combinatorial equivalence, and $M_1^n \equiv M_2^n$ will mean that $K_1^n \equiv K_2^n$, where K_i^n is a proper triangulation of M_i^n.

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[1] E. Stiefel, Comm. Math. Helvetici, 8 (1935), 305–53.

[2] H. Seifert, Math. Zeit., 41 (1936), 1–17.

[3] Hassler Whitney, Proc. N. A. S., 21 (1935), 464–8; Bull. American Math. Soc., 43 (1937), 785–805. Page references will refer to the second of these papers.

[4] J. H. C. Whitehead, Proc. London Math. Soc., 45 (1939), 243. This paper will be referred to as S. S.

[5] W. Hurewicz, Akad. Wet. Amsterdam, 29 (1936), 125.

[6] M. H. A. Newman, Akad. Wet. Amsterdam, 29 (1926), 611–41, 30 (1927), 670–3.

[7] J. W. Alexander, Annals of Math., 31 (1930), 292–320.

[7a] J. H. C. Whitehead, Annals of Math. this number, 809–824. This paper will be referred to as C. C. Relevant to the present paper are theorems 4, 5, 7 and 8 of C. C.

We now state some of our theorems, postponing the proof of theorem 1 and the definition of the class Π till §2. It is to be understood that the manifolds referred to in these theorems are connected and covered by finite complexes.

THEOREM 1. *If $M_i^n \in \Pi$ $(i = 1, 2)$ and M_1^n and M_2^n have the* same nucleus,[4] *then*

$$M_1^n \times A^k \equiv M_2^n \times A^k$$

for sufficiently large values of k.

It is shown in S. S. that, provided their fundamental group satisfies a certain condition,[8] two (finite) complexes have the same nucleus if they are of the same homotopy type. For manifolds with such a group, theorem 1 can therefore be restated with 'have the same nucleus' replaced by 'are of the same homotopy type'.

A bounded manifold M^n, which is an absolute retract (i.e. is of the same homotopy type as a single point) belongs to the class Π if it is combinatorially equivalent to a smooth manifold. If M^n is smooth we may assume that[9] $M^n \subset M_1^n \subset R^{2n+k}$ for any $k > 0$, where M_1^n is an unbounded analytic manifold and R^m is Euclidean m-space. Since M^n has the same homology and cohomology groups as a cell its normal sphere-space[9a] in R^{2n+k} is simple. Taking $k = 5$ we have, from theorem 5, below, and S. S., theorem 25, corollary 3:

THEOREM 2. *If[10] $\pi_1(M^n) = 1$, $\beta(M^n) = 0$ $(r = 1, \cdots, n)$ and M^n is smooth, then*

$$M^n \times A^{n+5} \equiv A^{2n+5}.$$

It will be seen that any (bounded) polyhedral $M^n \subset R^n$ belongs to Π. Therefore $M^n \times A^{n+5} \equiv A^{2n+5}$ if M^n is the finite region bounded by a polyhedral $(n - 1)$-sphere in R^n, or even if $\dot{M}^n$ is of the same homotopy type as $\dot{A}^n$.

The Poincaré hypothesis, in its combinatorial form and as generalized by Hurewicz[5] from $n = 3$ to any n, is equivalent to the hypothesis.

If $\dot{M}^n$ is an $(n - 1)$-sphere and if M^n is an absolute retract, then $M^n \equiv A^n$.

Discarding the condition that $\dot{M}^n$ is an $(n - 1)$-sphere, we have what may be called the extended Poincaré hypothesis, namely:

A bounded, n-dimensional manifold, which is an absolute retract is an n-element.

From theorem 2, since a k-element is the topological product of k linear segments, we have:

THEOREM 3. *The extended Poincaré hypothesis, for smooth manifolds at least, is equivalent to the hypothesis:*

If $M^p \times A^1 \equiv A^{p+1}$, then $M^p \equiv A^p$.

This theorem raises various questions, one of which can be answered very

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[8] See S. S. p. 287. See also a paper by G. Higman to be published shortly by the London Math. Soc.

[9] Hassler Whitney, Annals of Math., 37 (1936), 645–80.

[9a] Appendix, Theorem 2, corollary.

[10] $\pi_1(M^n)$ denotes the (multiplicative) fundamental group and $\beta_r(M^n)$ the (additive) r^{th} homology group of M^n.

simply, namely: 'are there manifolds $M_1^n \not\equiv M_2^n$ such that $M_1^n \times A^1 \equiv M_2^n \times A^1$?' The answer is in the affirmative. For let $M_i^3 = M_i^2 \times A^1$, where M_1^2 is a torus with one hole and M_2^2 is a 2-sphere with three holes. Then $M_1^2 \not\equiv M_2^2$. On the other hand, taking $M_i^2 \subset R^3$, it is easily verified that $M_1^3 \equiv M_2^3$, since $M_i^3 \subset R^3$ is obviously a regular neighborhood (S. S., p. 293) of two simple circuits with a single point in common. As another, and perhaps more interesting example, let M_i^3 $(i = 1, 2)$ be a lens space of type[11] (p, q_i), from which the interior of a 3-simplex A_i^3 has been removed, where $q_1q_2 \not\equiv \pm 1 \bmod p$. Then M_i^3 contracts (S. S., pp. 248 and 258) into the 2-cell, bounded by a circuit taken p times, which, taken twice, bounds a lens model of M_i^3. Therefore M_1^3 and M_2^3 have the same nucleus. It will be seen that $M_i^3 \in \Pi$, whence, by theorem 1, $M_1^3 \times A^k \equiv M_2^3 \times A^k$ for large values of k (actually for $k \geqq 6$). But M_1^3 and M_2^3 are not combinatorially equivalent. For if they were, the lens spaces $M_1^3 + A_1^3$ and $M_2^3 + A_2^3$ would be combinatorially equivalent, which they are not since[12] $q_1q_2 \not\equiv \pm 1 \bmod p$.

2. Let a proper triangulation, K^n, of a given manifold, M^n, be represented as a recti-linear complex in R^{n+k}, and let $U(K^n, R^{n+k})$ be a regular neighborhood[12a] of K^n. Then our definition of Π is: $M^n \in \Pi$ *if, and only if,*

$$U(K^n, R^{n+k}) \equiv K^n \times A^k \tag{2.1}$$

for large values of k. Provided $k \geqq n + 3$ it follows from S. S., theorems 23 and 24, that this definition is independent of the choice of the proper triangulation K^n, of the choice of the regular neighborhood $U(K^n, R^{n+k})$ and of the way in which K^n is imbedded in R^{n+k}. If $K^n \subset R^{n+k} \subset R^{n+k+l} = R^{n+k} \times R^l$ $(l > 0)$ we may take

$$U(K^n, R^{n+k+l}) = U(K^n, R^{n+k}) \times A_1^1 \times \cdots \times A_l^1 .$$

For the latter is a manifold and, by an obvious induction on l, it contracts into $U(K^n, R^{n+k})$, and hence into K^n. Therefore, if the condition (2.1) is satisfied by some $K^n \subset R^{n+k}$, it is satisfied for every $k_1 > k$ and a suitable $K^n \subset R^{n+k_1}$. Theorem 1, above, is now seen to be an immediate consequence of S. S., theorem 25.

It follows from an argument in S. S. (p. 298) that an n-sphere belongs to Π for each value of n. Moreover, if $M^n \in \Pi$ and $M_0^n \subset M^n$, then $M_0^n \in \Pi$. For let t be a semi-linear homeomorphism of $K^n \times A^k$ on $U(K^n, R^{n+k})$, where K^n is a proper triangulation of M^n which contains a sub-complex, K_0^n, covering M_0^n. Then $t(K_0^n \times A^k)$ is a manifold and contracts geometrically into

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[11] H. Seifert and W. Threlfall, *Lehrbuch der Topologie,* Leipzig (1934), 210.

[12] K. Reidemeister, Abh. Math. Sem. Hamb., 11 (1935), 102–9; Journal f. d. r. u. a. Math., 173 (1935), 164–73.

[12a] S. S. p. 293. Observe that regular neighbourhoods are not necessarily neighbourhoods in the sense of topology.

$t(K_0^n \times p)$, for any point $p \in A^k$, and (2.1) is satisfied by $t(K_0^n \times p) \subset R^{n+k}$. Therefore $t(K_0^n \times A^k)$ is a regular neighborhood of K_0^n. More generally, let $M_0^r \subset M^n$ and let a proper triangulation K_0^r, of M_0^r, be a sub-complex of K^n. If a regular neighborhood $U_0^n = U(K_0^r, K^n) \equiv K_0^r \times A^{n-r}$ we shall say that M_0^r is in *regular position*[13] in M^n. This is always the case if $r = n$, for then we may take U_0^n to be K_0^r itself.

THEOREM 4. *If $M_0^r \subset M^n$ is in regular position in M^n and $M^n \in \Pi$, then $M_0^r \in \Pi$.*

For, with the above notation, U_0^n is an n-dimensional manifold in M^n and we have shown that if $M^n \in \Pi$, then $U_0^n \in \Pi$. That is to say

$$U^{n+k} = U(U_0^n, R^{n+k}) \equiv U_0^n \times A^k$$

for some value of k and some recti-linear $U_0^n \subset K^n \subset R^{n+k}$. But U^{n+k} con tracts into U_0^n and the latter contracts into K_0^r. Therefore U^{n+k} is also a regular neighborhood of K_0^r, and if $U_0^n \equiv K_0^r \times A^{n-r}$ we have

$$U^{n+k} \equiv U_0^n \times A^k \equiv K_0^r \times A^{n-r} \times A^k \equiv K_0^r \times A^{n-r+k},$$

and the theorem is established.

With the help of theorem 4 we can dispose of the case $n = 2$. No non-orientable manifold can belong to Π. For its regular neighborhood in R^{n+k}, being an $(n + k)$-dimensional manifold in R^{n+k}, is orientable, while its topological product with a cell is not. On the other hand any orientable surface may be represented as a polyhedron in R^3 and is necessarily in regular position. Therefore it belongs to Π. Also any orientable, polyhedral surface in R^m is in regular position if[14] $m \geqq 7$. Of course theorem 1 is trivial for any closed surface, whether orientable or not. Also it follows from special arguments, as in the remarks following theorem 3, that theorem 1, with $k = 1$, is true of bounded, orientable surfaces.

Now let $M^n \subset R^{n+k}$ be a smooth manifold which, without loss of generality, we may assume to be analytic.[9]

THEOREM 5. *$M^n \in \Pi$ if its normal sphere-space[3] in R^{n+k} is simple.*

Since M^n is compact there is a positive δ such that the flat k-spaces normal to M^n at two different points do not meet at a distance less than 2δ from M^n. Therefore no two of the k-cells $E^k(p)$ meet each other, where $E^k(p)$ is the interior and boundary of a $(k - 1)$-sphere with centre p and radius δ in the normal flat k-space at p. To say that the normal sphere-space is simple is to say that k mutually orthogonal, unit vectors $e_1(p), \cdots, e_k(p)$ are defined in the normal flat k-space at each point $p \in M^n$, and that $e_\lambda(p)$ varies continuously with p.

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[13] Hassler Whitney, Annals of Math., 37 (1936), 865–78.

[14] Though this lower limit for m can probably be reduced from 7 to 5 it cannot be discarded. For if K is a knotted circuit in a 3-sphere, S^3, it may be verified that the 2-sphere $(a + b)K$ is not in regular position in the 4-sphere $(a + b)S^3$, where a and b are vertices not in S^3. (Cf. E. Artin, Abh. Math. Sem. Hamb., 4 (1925), 174–7.)

After a process of approximation, projection in $E^k(p)$, and a final normalization, we may assume that $e_\lambda(p)$ varies analytically with p. The bounded manifold M^{n+k}, which is swept out by $E^k(p)$ as p describes M^n, is then seen to be the image of $M^n \times E^k(p_0) \equiv M^n \times A^k$ in an analytic transformation which maps M^n on itself. Therefore a suitable triangulation of $M^n \times A^k$ determines a C^1-triangulation, P^{n+k} of M^{n+k}, which contains a proper triangulation of M^n as a sub-complex. Let K^{n+k} be a rectilinear model of P^{n+k} and let $K^n \subset K^{n+k}$ be the subcomplex representing M^n. By C. C., theorem 4, there is a semi-linear, topological map $F(K^{n+k}) \subset R^{n+k}$. Then $F(K^{n+k})$ ($\equiv M^n \times A^k$) is a regular neighborhood of $F(K^n)$ ($\equiv M^n$), and the theorem is established.

It follows from this theorem, and the results referred to at the beginning of §1, that $M^n \in \Pi$, where M^n is a smooth, orientable manifold, if any one of the following conditions is satisfied:

1. *M^n is closed and admits an internal parallelism, as is always the case if*[1] *$n = 3$, or for example, if M^n is a Lie group.*
2. *M^n is closed and can be represented as a manifold of class C^2 in R^{n+1} or in* R^{n+2} (Seifert[2]).
3. M^n is bounded and all its cohomology groups vanish with integral, and hence with all coefficients. It can be shown that this follows from the general theory of sphere-spaces.[3]

The sufficiency of the first condition follows from a theorem similar to theorem 23 on pp. 43 and 44 of Stiefel's paper.[1] For let $M^n \subset R^{n+k}$, where $k \geqq n + 1$, and let K^n be a triangulation of M^n. Then we successively set up outer parallelisms (i.e. parallelisms in the normal flat k-spaces) over $K^0, K^1, \cdots, K^n$, where K^r is the r-dimensional skeleton of K^n. An outer parallelism over K^r ($0 \leqq r < n$) determines an $(r + 1)$-dimensional cocycle in K^{r+1}, whose coefficients are elements of $\pi_r(G_k)$, where G_k is the group of rotations in R^k. The parallelism over K^r may be extended throughout K^{r+1} if this cocycle is zero. If it is not zero, but cohomologous to zero, then the parallelism over K^r may be replaced by one for which the corresponding cocycle is zero.[14a] Thus K^{r+1} admits an outer parallelism if the cocycle determined by the outer parallelism over K^r is cohomologous to zero. Since $r + 1 < k$ it follows from the analysis of G_k ($= V_{k,k-1}$) in §1 of Stiefel's paper, that a map $f(S^r) \subset G_k \subset G_{n+k}$, which is homotopic to a point in G_{n+k}, is homotopic to a point in G_k; also that any $f(S^r) \subset G_{n+k}$ can be deformed into a map in G_k. Therefore a lemma, analogous to the one in Stiefel's theorem 23, follows from arguments similar to those in his §3. Therefore the $(r + 1)$-dimensional cocycle in K^{r+1}, which is determined by an outer parallelism over K^r is cohomologous to zero. Finally, Stiefel's assumption that some triangulation of M^n is a sub-complex of a triangulation of R^{n+k} need not, in this case, be taken as an additional axiom. For we may assume M^n to be analytic and sub-divide it and a recti-linear triangulation of

[14a] Cf. S. Eilenberg, Annals of Math., 41 (1940), 231–51.

R^{n+k} by the van der Waerden-Lefschetz method.[15] The result will not, in general, be a C^1-triangulation, but it will suffice in setting up the outer parallelism. Alternatively we may replace M^n by a homeomorphic polyhedral $F(K^n)$, as in the proof of theorem 5, and attach a flat n-space and a flat k-space to each point of $F(K^n)$, which are respectively parallel to the tangent and normal flat spaces at the corresponding point of M^n. Then an inner parallelism in M^n determines a parallelism in the n-spaces attached to the points of $F(K^n)$, and a parallelism in the k-spaces at points of $F(K^n)$ will determine an outer parallelism for M^n.

APPENDIX

(*Extract from a letter of the author to Hassler Whitney under date of Jan. 26, 1940.*— The Editors.)

* * * I omitted to prove that G_m, the group of rotations in Euclidean metric space R^{m+1}, is r-simple for each $r \geqq 1$, as the term is used by S. Eilenberg.[16] This condition may be expressed as follows. Let X be any arcwise connected topological space, let $\tilde{X}$ be its universal covering space and let Γ_1 be the group of covering transformations of $\tilde{X}$ (i.e. the group of homeomorphisms $\gamma_1(\tilde{X}) = \tilde{X}$, such that $u\gamma_1 = u$, where $u(\tilde{X}) = X$ is a locally (1-1) map of $\tilde{X}$ on X). Then X is said to be 1-simple if $\pi_1(X)$ is Abelian, and r-simple ($r > 1$) if, and only if, any spherical map $f(S^r) \subset \tilde{X}$ is homotopic in $\tilde{X}$ to the map $\gamma_1 f(S^r)$, for each $\gamma_1 \epsilon \Gamma_1$. Let us assume that Γ_1 is a sub-group of some arcwise connected, topological group Γ, of homeomorphisms $\gamma(\tilde{X}) = \tilde{X}$, whose topology agrees with that of $\tilde{X}$, meaning that $\gamma(x)$ varies continuously with $x \epsilon \tilde{X}$ and $\gamma \epsilon \Gamma$. Then the identity in Γ, say γ_0 is joined to a given $\gamma_1 \epsilon \Gamma_1$ by a segment $\gamma_t \epsilon \Gamma$ ($0 \leqq t < 1$). Therefore $\gamma_1 f_0(S^r) = f_1(S^r)$, say, is the image of a given map, $f_0(S^r) \subset \tilde{X}$, in the deformation $f_t = \gamma_t f_0$, whence X is r-simple for any $r > 1$. Therefore, and since Γ_1 is isomorphic to $\pi_1(X)$, we have the theorem:

THEOREM 1. *If Γ_1 satisfies the above condition and is also Abelian, then X is r-simple for each $r \geqq 1$.*

Let X be an arcwise connected topological group and let $\tilde{X}$ be its universal covering group. Then Γ_1 is Abelian, and is also a sub-group of the 'left translations' $\xi \to \gamma\xi$ (also of the 'right translations' $\xi \to \xi\gamma$, since Γ_1 is not only Abelian but, if the translation $\xi \to \gamma\xi$ is identified with the element $\gamma \epsilon \tilde{X}$, then Γ_1 belongs to the centre of $\tilde{X}$). Since X is arcwise connected, so is $\tilde{X} = \Gamma$, and we have the corollary:

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[15] B. L. van der Waerden, Math. Ann., 102 (1929), 337–62. S. Lefschetz, *Topology*, New York (1930), 364. See also B. O. Koopman and A. B. Brown, Trans. American Math. Soc., 34 (1932), 231–52 and S. Lefschetz and J. H. C. Whitehead, *ibid.*, 35 (1933), 510–17.

[16] S. Eilenberg. Fund. Math. 32 (1939), 167–75.

COROLLARY. *Any arcwise connected topological group is r-simple for each $r \geqq 1$.*

The consequence of this condition which interests us here is that, if X is r-simple, then a unique element of $\Pi_r(X)$ is determined by a 'free' map $f(S^r) \subset X$, meaning a map which is independent of the base point for $\Pi_r(X)$.

Now let an orientable sphere-space $S(K^n)$ be given, where K^n is a simplicial complex and the associated spheres are ν-dimensional, and let $S(K^n)$ be simple in the r-dimensional skeleton, K^r, of K^n $(0 < r < n)$. We shall assume that $S(K^n)$ is not only orientable but oriented, meaning that the associated spheres $S^\nu(p)$ $(p \in K^n)$ and the base sphere S_0^ν are oriented, and that the defining maps $\xi\{p, S^\nu(p)\} = S_0^\nu$ are all direct. Thus the (orthogonal) transformations of S_0^ν into itself by which 'transformations of coördinates' are determined will be rotations. Let A_i^{r+1} $(i = 1, 2, \cdots)$ be the (oriented) $(r + 1)$-simplexes in K^{r+1} and, using the rotation, let $q \to \xi_i(p, q) \in S_0^\nu$ $(p \in A_i^{r+1}, q \in S^\nu(p))$ be a local coördinate system for A_i^{r+1}. Since $S(K^r)$ is simple there is a map $q \to \eta(p, q) \in S_0^\nu$ defined for each $p \in K^r$, $q \in S^\nu(p)$, such that the rotation

$$q_0 \to \phi_p(q_0) = \xi_i\{p, \eta^{-1}(p, q_0)\} \qquad (p \in \dot{A}_i^{r+1}, q_0 \in S_0^\nu)$$

varies continuously with p. In other words, $p \to \phi_p$ is a continuous map of $\dot{A}_i^{r+1}$ in $G_{\nu+1}$, and since $G_{\nu+1}$ is r-simple $p \to \phi_p$ defines a unique element $\alpha_i \in \pi_r(G_{\nu+1})$. The element α_i is independent of the coördinate system ξ_i. For if $\xi_i'(p, q)$ is a second coördinate system for A_i^{r+1}, then $p \to \xi_i'\xi_i^{-1} = \psi_0(p)$, say, is a map of A_i^{r+1} in $G_{\nu+1}$. Since A_i^{r+1} can be shrunk into a point there is a deformation ψ_t $(0 \leqq t \leqq 1)$, of $\psi_0(p)$ into the map given by $\psi_1(p) = 1$, the identity in $G_{\nu+1}$. Therefore the coördinate system ξ_i' $(= \xi_{i0}')$ may be deformed into ξ $(= \xi_{i1}')$: Thus

$$\xi_{it}'(p, q) = \psi_t(p)\{\xi_i(p, q)\},$$

remembering that $\psi_t(p)$ is a rotation of S_0^ν into itself. Therefore the map of $\dot{A}_i^{r+1}$ in $G_{\nu+1}$, which is defined by ξ_i' and η, is homotopic to the above map $p \to \phi_p$, and hence determines the same element $\alpha_i \in \pi_r(G_{\nu+1})$. Let B_λ^r be an oriented r-simplex, which is common to A_i^{r+1} and to A_j^{r+1} and let $\eta_\lambda'(p, q)$ be any coördinate system for B_λ^r, which coincides with $\eta(p, q)$ in $\dot{B}_\lambda^r$. Then the map $\xi_i\eta^{-1}$, of B_λ^r in $G_{\nu+1}$, together with the map $\xi_i\eta_\lambda'^{-1}$, in which the orientation of B_λ^r is reversed, determine an element $\beta_\lambda \in \pi_r(G_{\nu+1})$. It follows from a similar argument to the one just given that the same element, β_λ, is determined by η, η_λ' and a coördinate system ξ_j for A_j^{r+1}. Thus, if η_λ' is constructed in such a way that η, η_λ' and ξ_i determine a given element β_λ, then η, η_λ' and ξ_j lead back to the same element β_λ.

After these preliminaries it follows from arguments which are similar to some of those used by E. Stiefel[1] and by Eilenberg[14a] that

1. $$C^{r+1} = \sum_i \alpha_i A_i^{r+1}$$

   is a co-cycle, with coefficients in $\pi_r(G_{\nu+1})$,
2. If $C^{r+1} \sim 0$, then $S(K^n)$ is simple in K^{r+1}. For this is obviously so if

$C^{r+1} = 0$. If $C^{r+1} \neq$ but ~ 0, then the coördinate system n may be replaced by one for which the corresponding co-cycle vanishes.

Since $S(K^n)$ is orientable by hypothesis, it follows that it is simple in K^1, and we have the theorem:

THEOREM 2. *If the* $(r + 1)$*-dimensional co-homology group of* K^n *vanishes for each* $r = 1, \cdots, n - 1$, *with coefficients in* $\pi_r(G_{\nu+1})$, *then any orientable sphere-space* $S(K^n)$, *in which the associated spheres are* ν*-dimensional* $(\nu > 0)$, *is simple.*

If the 1-dimensional co-homology group of K^n vanishes with integral coefficients, reduced mod 2, then any sphere-space $S(K^n)$ is orientable. Also the co-homology groups vanish with all coefficients if they all vanish with integral coefficients. Hence we have the corollary:

COROLLARY. *If all the co-homology groups of* K^n *vanish, with integral coefficients, then any sphere space* $S(K^n)$ *is simple.*

Notice, on the other hand, that no condition is imposed on the $(r + 1)$-dimensional cohomology groups for those values of r such that $\pi_r(G_{\nu+1}) = 0$. Do you know if there are any, beyond $r = 2$, for any $\nu > 1$?

We also have, for the reasons indicated in my paper on homotopy types:

THEOREM 3. *If a differentiable n-dimensional manifold admits an absolute parallelism, then its normal sphere-space in* R^{2n+k} $(k > 0)$ *is simple.*

In the paper on homotopy types I was interested only in finite (i.e. closed or bounded) manifolds. But this theorem is obviously true in general, provided one requires the manifold to be a closed, but not necessarily compact, sub-set of R^{2n+k}.

BALLIOL COLLEGE, OXFORD.