THE GENERAL TYPE OF SINGULARITY OF A SET OF $2n - 1$ SMOOTH FUNCTIONS OF n VARIABLES

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1. **Introduction.** Let a region R of n-space E^n, or more generally, of a differentiable n-manifold, be mapped differentiably into m-space E^m. If $m \geq 2n$, it is always possible [1; 818], [3], by a slight alteration of the mapping function f (letting also any finite number of derivatives change arbitrarily slightly), to obtain a mapping f^* which is everywhere regular. That is, for any p in R, and any set of independent vectors $u_1, \cdots, u_n$ in R at p, f^* carries these vectors into independent vectors. Here, vector equals the vector in "tangent space" equals the differential. As a consequence, some neighborhood U of p is mapped by f in a one-one way. The object of this paper is to determine what can be obtained by slight alterations of f in case $m = 2n - 1$. It turns out that any singularities may be made into a fixed kind. (It will be shown in other papers that any smooth n-manifold may be imbedded in $(2n)$-space, and may be immersed (self-intersections allowed) in $(2n - 1)$-space.)

There are two main theorems in the paper, roughly:

(a) We may alter f arbitrarily slightly, forming f^*, for which the singular points (points where f^* is not regular) are isolated, and such that a certain condition (C) below holds at each singular point. (The self-intersection may also be made simple; cf. [3; 655, (D)].)

(b) Let f^* satisfy the condition mentioned. Then for any singular point p, we may choose coördinate systems $x_1, \cdots, x_n$ in a neighborhood of p and $y_1, \cdots, y_{2n-1}$ in a neighborhood of $f(p)$ such that f^* is given exactly by the equations (4.2). Here, f^* must have many derivatives.

Remark. As a consequence, there is a slight deformation of E^{2n-1} which carries $f(U)$ (U a neighborhood of p) into the set of points given by (4.2).

The transformations in (b) may lower the class of f^* considerably; but if f^* is of class C^∞, or analytic, the transformations will be also. The condition mentioned in (a) is the following:

(C) There is a direction through p with the following properties: (C_1) f^* maps any vector in this direction into the null vector in E^{2n-1}, but maps any other vector at p into a non-null vector. (C_2) If $g(p')$ is the derivative of $f^*(p')$ in the direction given above, for p' near p, then there is no vector in E^{2n-1} which is the image both of a vector under f^* and a vector $\neq 0$ under g, both at p.

We may phrase the second condition as follows:

(C_2') Suppose a coördinate system is chosen in which the given vector is in the x_1-direction. Then

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$$(1.1) \qquad \frac{\partial f^*}{\partial x_2}, \quad \cdots, \quad \frac{\partial f^*}{\partial x_n}, \quad \frac{\partial^2 f^*}{\partial x_1^2}, \quad \frac{\partial^2 f^*}{\partial x_1 \partial x_2}, \quad \cdots, \quad \frac{\partial^2 f^*}{\partial x_1 \partial x_n},$$

taken at p, are independent vectors.

We shall show first that the two conditions (C_2) and (C_2') are equivalent, and independent of the coördinate system chosen (with the same or opposite direction of the x_1-axis). Next we show that (C) implies regularity near p, so that p is an isolated singularity. (This fact follows also from (b); a proof is given here to help in understanding condition (C).) We then study the typical singularity mentioned in (b). Next we prove (b), and finally (a). The proof of (a) uses methods found in [3]; the other proofs are straightforward analysis.

2. **Equivalence of** (C_2) **and** (C_2'). We use f in place of f^*. Having chosen a coördinate system as in (C_2'), we note that $g(p') = \partial f(p')/\partial x_1$. If e_i is the unit vector in R at p in the direction of x_i, then by definition, f carries it into the vector $\partial f(p)/\partial x_i$. Hence, for any vectors $u = (u_1, \cdots, u_n)$ and $v = (v_1, \cdots, v_n)$ at p, f and g carry these into

$$f(u) = \sum_{i=1}^{n} u_i \frac{\partial f(p)}{\partial x_i} = \sum_{i=2}^{n} u_i \frac{\partial f(p)}{\partial x_i},$$

$$g(v) = \sum_{i=1}^{n} v_i \frac{\partial^2 f(p)}{\partial x_1 \partial x_i}.$$

Suppose (C_2') holds. Then if $f(u) = g(v)$, since the vectors (1.1) are independent, the coefficients $u_2, \cdots, u_n, v_1, \cdots, v_n$ are all 0, and $v = 0$, proving (C_2). Suppose conversely that (C_2) holds. If there is a linear relation between the vectors (1.1) with coefficients, say $u_2, \cdots, u_n, -v_1, \cdots, -v_n$, not all 0, then defining u and v as above, with $u_1 = 0$, we have $f(u) = g(v)$, and $u \neq 0$ or $v \neq 0$. By (C_2), $v = 0$, and hence $f(u) = 0$. By (C_1), u is in the x_1-direction, that is, $u_2 = \cdots = u_n = 0$, which is a contradiction.

We shall show that condition (C_2'), and hence (C), is independent of the coördinate system employed. Take two systems, each with the first axis (at p) in the given direction; then $\partial x_k/\partial x_1' \,|_p = a\delta_{1k}$, $a \neq 0$. Let T^{n-1} be the plane (in E^{2n-1}) of all directional derivatives of f at p, i.e., all vectors $f(u)$. Let $u \sim v$ denote $u - v \in T^{n-1}$. If we suppose that (C_2') holds in the first but not in the second system, we have, for some $v' = (v_1', \cdots, v_n') \neq 0$, if $g' = \partial f/\partial x_1'$,

$$\begin{aligned} g'(v') = \sum_i v_i' \frac{\partial^2 f}{\partial x_1' \partial x_i'}\bigg|_p &= \sum_i v_i' \frac{\partial}{\partial x_1'} \sum_j \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial x_i'}\bigg|_p \\ &\sim \sum_i v_i' \sum_j \sum_k \frac{\partial^2 f}{\partial x_k \partial x_j} \frac{\partial x_k}{\partial x_1'} \frac{\partial x_j}{\partial x_i'}\bigg|_p \\ &= a \sum_j \left[\sum_i v_i' \frac{\partial x_j}{\partial x_i'}\right] \frac{\partial^2 f}{\partial x_1 \partial x_j}\bigg|_p \sim 0. \end{aligned}$$

Hence, all $v_j = \sum_i v'_i \partial x_j/\partial x'_i$ are 0. But the Jacobian $| \partial x_j/\partial x'_i | \neq 0$ at p, so that the columns of $\| \partial x_j/\partial x'_i \|$ are independent, which contradicts the assumption that not all the v'_i are 0.

3. **Regularity near a singular point.** Let (C) hold for f at p; we shall show that f is regular in the rest of a neighborhood of p. Using (C'_2), let $T^{n-1}(p')$ and $T^n(p')$ be the planes determined by the $\partial f/\partial x_i$ ($i > 1$) and the $\partial^2 f/\partial x_1 \partial x_i$ respectively, taken at p'. Then in a neighborhood U_0 of p, they are of the dimensions shown, and if $p' \varepsilon U_0$ and T'^n is any n-plane sufficiently near $T^n(p)$, T'^n and $T^{n-1}(p')$ have only one point in common. Since the vectors $\partial g(p)/\partial x_1, \cdots, \partial g(p)/\partial x_n$ determine $T^n(p)$, we can take $U_1 \subset U_0$ so that for any $p'' \varepsilon U_1$, there is a plane T'^n as near $T^n(p)$ as required above which contains $g(p'') - g(p)$; it follows that if $p'' \neq p$, $g(p'') - g(p)$ is in no $T^{n-1}(p')$ ($p' \varepsilon U_0$). Now, since $g(p) = \partial f(p)/\partial x_1 = 0$, we have in particular: for any $p' \varepsilon U_1$, $p' \neq p$, $g(p')$ is not in $T^{n-1}(p')$; that is, $\partial f(p')/\partial x_1$ is independent of $\partial f(p')/\partial x_2, \cdots, \partial f(p')/\partial x_n$. Thus f is regular in U_1 except at p, as required.

4. **A typical singularity.** As is well known, a mapping of a projective plane into 3-space may be obtained by replacing a piece of the surface of a sphere by a "cross cap". A cross cap (or rather the top of one) may be described as follows: Let $T(y)$ be the plane in E^3 perpendicular to the y-axis at a given y. Take a parabola in $T(y_0)$ for $y_0 < 0$; say $z = \pm y_0 x^{\frac{1}{2}}$. As we let y increase, pull in the two sides of the parabola until they coincide (and thus form a half ray) at $y = 0$ and become a parabola reversed in direction for $y > 0$. The locus of these parabolas forms the top of the cross cap. A sphere about the origin cuts the cross cap in a curve in the form of a bent figure 8.

Using coördinates y_1, y_2, y_3 in E^3, the cross cap may be represented parametrically by the equations

$$y_1 = x_1^2, \qquad y_2 = x_2, \qquad y_3 = x_1 x_2. \tag{4.1}$$

Generalizing this, let us map E^n into E^{2n-1} by:

$$\begin{aligned} y_1 &= x_1^2, \\ y_i &= x_i \qquad (i = 2, \cdots, n), \\ y_{n+i-1} &= x_1 x_i \qquad (i = 2, \cdots, n). \end{aligned} \tag{4.2}$$

The matrix $\| \partial y_i/\partial x_j \|$, transposed, is

$$\begin{Vmatrix} 2x_1 & 0 & 0 & \cdots & 0 & x_2 & x_3 & \cdots & x_n \\ 0 & 1 & 0 & \cdots & 0 & x_1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & x_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & x_1 \end{Vmatrix}.$$

If we call the mapping $y = f(x)$, f is clearly regular except at the origin; only $\partial f/\partial x_1$ is 0 there. Also, the matrix $\| \partial^2 y_i/\partial x_1 \partial x_j \|$, transposed, has a 2 in the upper left corner, a diagonal of $n - 1$ ones in the lower right, and zero elsewhere. Combining the two matrices, with the first row $\partial y_1/\partial x_j$ omitted, we obtain a diagonal matrix, except for an interchange of rows, at the origin. Hence, the vectors (1.1) are independent, and (C) holds.

To determine the self-intersections, suppose $f(x'_1, \cdots, x'_n) = f(x_1, \cdots, x_n)$, where the points are distinct. Then $x'_i = x_i$ for $i > 1$, since $y'_i = y_i$; hence, $x'_1 \neq x_1$, and, since $y'_1 = y_1$, $x'_1 = -x_1 \neq 0$. Since $y'_{n+i-1} = y_{n+i-1}$ and $x_1 \neq 0$, $x_i = 0$ $(i > 1)$. Thus,

$$f(\alpha, 0, \cdots, 0) = f(-\alpha, 0, \cdots, 0),$$

and there are no other self-intersections.

Examining the matrix $\| \partial y_i/\partial x_j \|$, we see that at any such self-intersection, the two tangent planes have only the y_1-axis in common.

5. **Proof of** (b). We shall prove:

THEOREM 1. *Let f be a mapping of class C^s, $s = 4r + 8$, $r \geq 1$, of the region R of E^n into E^{2n-1}, and let* (C) *hold at the origin* (*assumed in R*). *Then there are curvilinear coördinate systems about* 0 *and $f(0)$, of class C^r, in terms of which f has the form* (4.2). *If f is analytic, or of class C^∞, so are the coördinate systems.*

Probably s need not be taken so large in terms of r. We shall consider the case where f is of class C^s; in the other cases, the transformations employed are clearly analytic or of class C^∞, respectively.

As a first step, choose coördinates $x_1, \cdots, x_n$ in E^n as in (C'_2); since the vectors (1.1) are independent, we may choose oblique axes $y_1, \cdots, y_{2n-1}$ in E^{2n-1} so that these vectors are unit vectors (except for $\partial^2 f/\partial x_1^2$) on the axes of

$$y_2, \quad \cdots, \quad y_n, \quad y_1, \quad y_{n+1}, \quad \cdots, \quad y_{2n-1},$$

respectively. Now take each y_k, expand it in terms of x_1 to the third order, expand the coefficient of x_1^i in terms of x_2 to the order $3 - i$, etc. This gives

$$y_k = A^k + \sum_i B_i^k x_i + \sum_{i \leq j} C_{ij}^k x_i x_j + \sum_{h \leq i \leq j} D_{hij}^k(x_1, \cdots, x_n) x_h x_i x_j .$$

(Many of the D_{hij}^k are actually functions of fewer variables.) The A^k, B_i^k and C_{ij}^k are the y_k and first and second derivatives (except for a factor) at the origin. By the choice of the y-coördinate system, therefore, many of these may be determined, and we find

$$\begin{aligned} y_1 &= x_1^2 + R_1(x_1, \cdots, x_n), \\ y_i &= x_i + R_i(x_1, \cdots, x_n) \qquad (i = 2, \cdots, n), \\ y_{n+i-1} &= x_1 x_i + R_{n+i-1}(x_1, \cdots, x_n) \qquad (i = 2, \cdots, n), \end{aligned} \tag{5.1}$$

where

$$R_k = \sum_{2 \leq i \leq j} C^k_{ij} x_i x_j + \sum_{h \leq i \leq j} D^k_{hij}(x_1, \cdots, x_n) x_h x_i x_j, \tag{5.2}$$

the R_k being of class C^s.

Set $x'_1 = x_1$, and

$$x'_i = x_i + R_i(x_1, \cdots, x_n) \qquad (i = 2, \cdots, n). \tag{5.3}$$

Then $\partial x'_i / \partial x_j |_0 = \delta_{ij}$, so that this is a transformation of coördinates of class C^s near the origin. Furthermore, $\partial x_i / \partial x'_j |_0 = \delta_{ij}$, so that

$$x_i = x'_i + \sum_{j \leq k} a^i_{jk}(x'_1, \cdots, x'_n) x'_j x'_k .$$

Substituting in (5.1) and dropping primes gives

$$\begin{aligned} y_1 &= x_1^2 + R_1(x_1, \cdots, x_n), \\ y_i &= x_i \qquad (i = 2, \cdots, n), \\ y_{n+i-1} &= x_1 x_i + R_{n+i-1}(x_1, \cdots, x_n) \qquad (i = 2, \cdots, n), \end{aligned} \tag{5.4}$$

where the new R_k have the same form (5.2).

Next we simplify the form of y_1. Since

$$\left. \frac{\partial y_1}{\partial x_1} \right|_0 = 0, \qquad \left. \frac{\partial^2 y_1}{\partial x_1^2} \right|_0 = 2,$$

we may solve $\partial y_1 / \partial x_1 = 0$ in a neighborhood of the origin, obtaining a function $x_1 = \phi(x_2, \cdots, x_n)$, of class C^{s-1}. Then, by definition,

$$2\phi(x_2, \cdots, x_n) + \frac{\partial}{\partial \phi} R_1(\phi(x_2, \cdots, x_n), x_2, \cdots, x_n) = 0. \tag{5.5}$$

Set

$$x'_1 = x_1 - \phi(x_2, \cdots, x_n), \qquad x'_i = x_i \qquad (i > 1). \tag{5.6}$$

For each $(x_2, \cdots, x_n)$ near $(0, \cdots, 0)$, expand y_1 in terms of x'_1. Since $\partial y_1 / \partial x'_1 = 0$ for $x'_1 = 0$, this gives

$$y_1 = \psi_0(x_2, \cdots, x_n) + x_1'^2 \psi_2(x'_1, x_2, \cdots, x_n); \tag{5.7}$$

ψ_0 and ψ_2 are of class C^{s-1} and C^{s-3}, respectively, by [2; Theorem 3]. Differentiating (5.5) gives

$$\left[2 + \left. \frac{\partial^2 R_1}{\partial x_1^2} \right|_0 \right] \left. \frac{\partial \phi}{\partial x_i} \right|_0 + \left. \frac{\partial^2 R_1}{\partial x_1 \partial x_i} \right|_0 = 0 \qquad (i > 1).$$

Since $\partial^2 R_1 / \partial x_1 \partial x_j |_0 = 0$ for all j, this gives $\partial \phi / \partial x_i |_0 = 0$; hence $\partial x'_i / \partial x_j |_0 = \delta_{ij}$. It follows that the Jacobian of the transformation is 1 at the origin; also $\phi(x'_2, \cdots) = \sum \phi_{ij}(x'_2, \cdots) x'_i x'_j$, so that the y_i are given by the same kinds of expressions in terms of $x'_1, x_2, \cdots$ as in terms of $x_1, x_2, \cdots$.

Next, set

$$(5.8)\qquad \begin{aligned} x_1'' &= x_1'[\psi_2(x_1', \cdots, x_n')]^{\frac{1}{2}}, \qquad x_i'' = x_i \qquad (i > 1),\\ y_1' &= y_1 - \psi_0(y_2, \cdots, y_n), \qquad y_i' = y_i \qquad (i > 1).\end{aligned}$$

Now

$$2 = \frac{\partial^2 y_1}{\partial x_1^2}\Big|_0 = \frac{\partial^2 y_1}{\partial x_1'^2}\Big|_0 = 2\psi_2\Big|_0;$$

hence $\psi_2 |_0 = 1$, and the transformations to the x_i'' and y_i' are allowable and are of class C^{s-3} in a neighborhood of the origin. Dropping primes again, we have the same equations as (5.4), with R_1 missing; the R_k are of class C^{s-3}, and have again the form (5.2). To prove this for y_{n+i-1}, we note that if $x_1' = \theta(x_1'', \cdots, x_n)$, then $\partial\theta/\partial x_1'' |_0 = 1$, and hence

$$\frac{\partial^2}{\partial x_1''\partial x_j''}[\theta x_i + R'_{n+i-1}(\theta, \cdots, x_n)]\Big|_0 = \delta_{ij} \qquad (i = 1, j \geq 1).$$

Next, set $y_i' = y_i$ $(i = 1, \cdots, n)$, and

$$(5.9)\qquad y_k' = y_k - R_k(0, y_2, \cdots, y_n) \qquad (k = n + 1, \cdots, 2n - 1).$$

Setting $k = n + i - 1$ and using (5.2), we find

$$\begin{aligned} y_k' = x_1x_i &+ \sum_{i\leq l} D_{1il}^k(x_1, \cdots, x_n)x_1x_ix_l\\ &+ \sum_{2\leq h\leq i\leq l}[D_{hil}^k(x_1, x_2, \cdots) - D_{hil}^k(0, x_2, \cdots)]x_hx_ix_l;\end{aligned}$$

if we expand the first bracketed term D in terms of x_1, and drop primes, we have

$$(5.10)\qquad \begin{aligned} y_1 &= x_1^2, \qquad y_i = x_i \qquad (i = 2, \cdots, n),\\ y_{n+i-1} &= x_1[x_i + R_{n+i-1}(x_1, \cdots, x_n)] \qquad (i = 2, \cdots, n),\end{aligned}$$

where (for $k = n + i - 1$)

$$(5.11)\qquad R_k(x_1, \cdots, x_n) = \sum_{i\leq l} E_{il}^k(x_1, \cdots, x_n)x_ix_l.$$

The new R_k are of class C^{s-4}; the E_{il}^k are of class C^{s-6}.

Our next job is to move the curve of self-intersection over onto the y_1-axis; this will result in the term in x_1^2 in (5.11) dropping out.

Examples. Take $n = 2$. If $y_3 = x_1(x_2 + x_1^2)$, then the curve $x_2 = -x_1^2$ maps into the curve of self-intersection, which is $(x_1^2, -x_1^2, 0)$ in E^3. If $y_3 = x_1(x_2 + x_1^2 + x_1x_2)$, the curve is $x_2 = -x_1^2$ again, mapping onto $(x_1^2, -x_1^2, -x_1^4)$.

As in §4, if the distinct points $(x_1', x_2', \cdots)$ and $(x_1, x_2, \cdots)$ go into the same point, then $x_i' = x_i$ for $i > 1$, and $x_1' = -x_1 \neq 0$. Using $y_k' = y_k$ for $n + i - 1 = k > n$ gives

$$(5.12)\qquad \begin{aligned} x_1[x_i + R_k(x_1, x_2, \cdots)] &= -x_1[x_i + R_k(-x_1, x_2, \cdots)],\\ x_i &= -\tfrac{1}{2}[R_k(x_1, x_2, \cdots) + R_k(-x_1, x_2, \cdots)].\end{aligned}$$

If we write these in the form $x_i + \frac{1}{2}[\quad] = 0$, then since $\partial R_k/\partial x_j\,|_0 = 0$, the functional matrix is $\|\,\delta_{ij}\,\|$ at the origin; hence, we may solve these for $x_2, \cdots, x_n$ in terms of x_1. The resulting functions are obviously even and of class $C^{s-4} = C^{4r+4}$; hence, by the theorem of the preceding paper, the solutions may be written as $\phi_i(x_1^2)$, where the ϕ_i are of class C^{2r+2}. Then

$$\phi_i(x_1^2) = -\tfrac{1}{2}[R_k(x_1, \phi_2(x_1^2), \cdots) + R_k(-x_1, \phi_2(x_1^2), \cdots)]. \tag{5.13}$$

Also, for $k > n$, we may define

$$\phi_k(x_1^2) = \tfrac{1}{2}x_1[R_k(x_1, \phi_2(x_1^2), \cdots) - R_k(-x_1, \phi_2(x_1^2), \cdots)]. \tag{5.14}$$

(Then $y_k = \phi_k(x_1^2)$ for all k on the curve of intersection.)

Make the transformation

$$\begin{aligned} x_1' &= x_1, \qquad & x_i' &= x_i - \phi_i(x_1^2) \qquad & (i &= 2, \cdots, n), \\ y_1' &= y_1, \qquad & y_i' &= y_i - \phi_i(y_1) \qquad & (i &= 2, \cdots, 2n-1). \end{aligned} \tag{5.15}$$

These have non-vanishing Jacobians near the origins, and are of class C^{2r+2}. Now $y_1' = x_1'^2$, $y_i' = x_i'$ $(i = 2, \cdots, n)$, and using (5.13) and (5.14), we find for $n + i - 1 = k > n$,

$$\begin{aligned} y_k' &= x_1'[x_i' + \phi_i(x_1'^2) + R_k(x_1', x_2, \cdots)] - \phi_k(x_1'^2) \\ &= x_1'[x_i' - \tfrac{1}{2}\{R_k(x_1', \phi_2(x_1'^2), \cdots) + R_k(-x_1', \phi_2(x_1'^2), \cdots)\} \\ &\qquad + R_k(x_1', x_2' + \phi_2(x_1'^2), \cdots) \\ &\qquad\qquad - \tfrac{1}{2}\{R_k(x_1', \phi_2(x_1'^2), \cdots) - R_k(-x_1', \phi_2(x_1'^2), \cdots)\}], \end{aligned}$$

or

$$y_k' = x_1'[x_i' + R_k'(x_1', x_2', \cdots)], \tag{5.16}$$

where the R_k', of class C^{2r+2}, are given by

$$R_k'(x_1', x_2', \cdots) = R_k(x_1', x_2' + \phi_2(x_1'^2), \cdots) - R_k(x_1', \phi_2(x_1'^2), \cdots).$$

Since $R_k'(x_1', 0, \cdots) = 0$, expanding, for each x_1', in terms of $x_2', \cdots, x_n'$ to the first order and dropping primes from the x_i' gives

$$R_k'(x_1, x_2, \cdots) = \sum_{1<j} x_j R_{kj}(x_1, x_2, \cdots). \tag{5.17}$$

Using the definition of R_k' and (5.11), we find

$$R_{kj}(0, 0, \cdots) = \frac{\partial R_k'}{\partial x_j}\bigg|_0 = \frac{\partial R_k}{\partial x_j}\bigg|_0 = 0;$$

therefore,

$$R_{kj}(x_1, x_2, \cdots) = \sum_l x_l R_{kjl}(x_1, x_2, \cdots). \tag{5.18}$$

Putting in (5.16) (with primes dropped) gives, if $k = n + i - 1$,

$$(5.19)\qquad \begin{aligned} y_1 &= x_1^2, \qquad y_i = x_i && (i = 2, \cdots, n), \\ y_k &= x_1[x_i + \sum_{j=2}^{n} x_j R_{kj}(x_1, \cdots, x_n)] && (i = 2, \cdots, n). \end{aligned}$$

The R_{kj} are of class C^{2r+1}.

Next, by the theorems of the preceding paper, we may write

$$(5.20)\qquad \begin{aligned} P_{kj}(x_1^2, x_2, \cdots) &= \tfrac{1}{2}[R_{kj}(x_1, x_2, \cdots) + R_{kj}(-x_1, x_2, \cdots)], \\ x_1 Q_{kj}(x_1^2, x_2, \cdots) &= \tfrac{1}{2}[R_{kj}(x_1, x_2, \cdots) - R_{kj}(-x_1, x_2, \cdots)], \end{aligned}$$

the P_{kj} and Q_{kj} being of class C^r. This gives

$$y_k = x_1[x_i + \sum_{j=2}^{n} x_j P_{kj}(x_1^2, x_2, \cdots)] + x_1^2 \sum_{j=2}^{n} x_j Q_{kj}(x_1^2, x_2, \cdots).$$

The next to the last transformation is: $y_i' = y_i$ $(i \leq n)$,

$$(5.21)\qquad y_k' = y_k - y_1 \sum_{j=2}^{n} y_j Q_{kj}(y_1, y_2, \cdots) \qquad (k > n).$$

The Jacobian is 1 at the origin. Now, dropping primes again, we have

$$(5.22)\qquad \begin{aligned} y_1 &= x_1^2, \qquad y_i = x_i && (i = 2, \cdots, n), \\ y_{n+i-1} &= x_1[x_i + \sum_{j=2}^{n} x_j P_{n+i-1,j}(x_1^2, x_2, \cdots)] && (i = 2, \cdots, n). \end{aligned}$$

We are now ready for the final transformation. Define functions

$$F_{kl}(u_1, \cdots, u_n) \qquad (k, l = n + 1, \cdots, 2n - 1),$$

for each fixed k, as the solutions of the linear equations

$$(5.23)\qquad \sum_{h=n+1}^{2n-1} [\delta_{n+i-1,h} + P_{hi}(u_1, \cdots)] F_{kh}(u_1, \cdots) = P_{ki}(u_1, \cdots),$$

for $i = 2, \cdots, n$. By (5.20) and (5.18), the determinant $| \delta_{n+i-1,h} + P_{hi} |$ is near 1 in a neighborhood of the origin; hence, the F_{kh} are defined and of class C^r there, and $F_{kh}(0, \cdots, 0) = 0$. Set

$$(5.24)\qquad \begin{aligned} y_i' &= y_i && (i = 1, \cdots, n), \\ y_k' &= y_k - \sum_{h=n+1}^{2n-1} F_{kh}(y_1, \cdots, y_n) y_h && (k = n + 1, \cdots, 2n - 1). \end{aligned}$$

Since, for $k \geq n + 1$,

$$\frac{\partial y_k'}{\partial y_h} = \delta_{kh} - F_{kh} \qquad (h > n),$$

and the $F_{kh} |_0 = 0$, the Jacobian $\neq 0$ at the origin, and so the transformation is allowable, and of class C^r.

We find the y'_k in terms of the x_i : Writing P_{kj} for $P_{kj}(x_1^2, x_2, \cdots)$, etc., and taking $k = n + i - 1$, $i > 1$, we have

$$\begin{aligned}\frac{1}{x_1} y'_k &= x_i + \sum_{j=2}^{n} P_{kj}x_j - \sum_{h=n+1}^{2n-1} F_{kh}[x_{h-n+1} + \sum_{j=2}^{n} P_{hj}x_j] \\ &= x_i + \sum_{j=2}^{n} x_j[P_{kj} - F_{k,n+j-1} - \sum_{h=n+1}^{2n-1} F_{kh}P_{hj}] \\ &= x_i .\end{aligned}$$

This proves the theorem.

6. **Proof of** (a). The other main theorem is:

THEOREM 2. *Let f be a mapping of class C^2 of a region R of E^n into E^{2n-1}. Then arbitrarily close to f (together with first and second derivatives) there is a mapping f^* satisfying* (C) *at each singular point. f^* may be made analytic.*

Remark. It will be clear from the proof that the theorem holds equally well with R and E^{2n-1} replaced by smooth manifolds M^n and M^{2n-1}.

Let $R_1, R_2, \cdots$ be rectangular regions in R, and choose R'_i with $\bar{R}'_i \subset R_i$, so that the R'_i cover R. Suppose we have shown how to deform f slightly so that (C) holds at any singular point in a given $\bar{R}'_\alpha$. Then carrying out deformations successively, we make it hold in $\bar{R}'_1, \bar{R}'_2, \cdots$, each time making the deformation so slight that the property is not disturbed in any preceding $\bar{R}'_i$, and so that the limit will be a mapping with the property in every $\bar{R}'_i$ and hence in R. (Since for any f_0 and $\eta(p)$, the property of f satisfying (C) at any singular point is an $(f_0, 2, \eta)$-property, as in [3; §7], the statement follows at once from [3; Lemma 12].) We consider only functions f of class C^r, $r \geq 2$; the final function may be made analytic, by [3; Lemma 9].

If f is not of class C^3, approximate to it by a function f^{**} of class C^3; thus we may suppose f is of class C^3 in the first place.

Let $\lambda(p)$ be a function of class C^∞ in R, $\equiv 1$ in $\bar{R}'_\alpha$, $\equiv 0$ in $R - R_\alpha$. (Use [3; Lemma 11], or construct it directly.) We shall define certain kinds of mappings G_{ij}, H_{ij}, with which f may be combined to give the required function satisfying (C) at all singular points in $\bar{R}'_\alpha$.

First, by the proof of [3; Lemma 18], we may suppose also that $\partial f/\partial x_2, \cdots, \partial f/\partial x_n$ are independent in $\bar{R}'_\alpha$.

Set $m = 2n - 1$. Let $v_1, \cdots, v_m$ be the unit vectors in E^m. Let $\bar{E} = E^{2mn}$ be a Euclidean space, with unit vectors (in a definite order)

$$v_{ij}, \qquad w_{ij} \qquad\qquad (i = 1, \cdots, n; j = 1, \cdots, m).$$

Corresponding to any mapping $g(p)$ of class C^3 of $\bar{R}_\alpha$ into E^m such that $\partial g(p)/\partial x_2, \cdots, \partial g(p)/\partial x_n$ are independent for each $p = (x_1, \cdots, x_n)$, define $\bar{g}^2(p), \cdots,$

$\bar{g}^n(p)$ as the numbers such that

$$g^{\odot}(p) = \frac{\partial g(p)}{\partial x_1} - \sum_{1<i} \bar{g}^i(p) \frac{\partial g(p)}{\partial x_i}$$

is orthogonal to $\partial g(p)/\partial x_2, \cdots, \partial g(p)/\partial x_n$. If $g_1(p), \cdots, g_m(p)$ are the coördinates of $g(p)$, define the corresponding mapping $g^*(p)$ of $\bar{R}_\alpha$ into $\bar{E}$ by

$$g^* = \sum_j g_j^{\odot} v_{1j} + \sum_{i,j;1<i} \frac{\partial g_j}{\partial x_i} v_{ij}$$
$$+ \sum_j \left[\frac{\partial^2 g_j}{\partial x_1^2} - 2 \sum_{1<i} \bar{g}^i \frac{\partial^2 g_j}{\partial x_1 \partial x_i} + \sum_{1<i,h} \bar{g}^h \bar{g}^i \frac{\partial^2 g_j}{\partial x_h \partial x_i} \right] w_{1j}$$
$$+ \sum_{i,j;1<i} \left[\frac{\partial^2 g_j}{\partial x_1 \partial x_i} - \sum_{1<h} \bar{g}^h \frac{\partial^2 g_j}{\partial x_h \partial x_i} \right] w_{ij} .$$

Then the $\bar{g}^i$ and $g^{\odot}$ are of class C^2 and g^* is of class C^1.

Given g as above, if we take a fixed $p_0 = (\bar{x}_1, \cdots, \bar{x}_n)$ and make the change of variables

$$x_1 = x_1' + \bar{x}_1 , \qquad x_i = x_i' - \bar{g}^i(p_0) x_1' + \bar{x}_i \qquad (i > 1),$$

we find

$$g^*(p_0) = \sum_{i,j} \frac{\partial g_j(p_0)}{\partial x_i'} v_{ij} + \sum_{i,j} \frac{\partial^2 g_j(p_0)}{\partial x_1' \partial x_i'} w_{ij} .$$

Remark. One could carry out the proof, using all $\partial g/\partial x_i$ and $\partial^2 g/\partial x_h \partial x_i$ independently; then $\bar{E}$ will have $m[n + \frac{1}{2}n(n+1)]$ dimensions.

For any sets of numbers β_{ij}, γ_{ij} ($i = 1, \cdots, n; j = 1, \cdots, m$), set

$$f_{\beta\gamma}(p) = f(p) + \lambda(p)\left[\sum_{i,j} \beta_{ij} x_i v_j + \sum_{i,j} \gamma_{ij} x_1 x_i v_j\right].$$

If the β_{ij} and γ_{ij} are small enough, $\partial f_{\beta\gamma}/\partial x_2, \cdots, \partial f_{\beta\gamma}/\partial x_n$ are independent in $\bar{R}_\alpha$, so that the $\bar{f}^i_{\beta\gamma}(p)$ etc. are defined. To find $f^*_{\beta\gamma}(p_0)$, make the change of variables described above, differentiate $f_{\beta\gamma}$ with respect to these variables, and put in the second relation above for $f^*_{\beta\gamma}$; we find (at $p_0 \varepsilon \bar{R}'_\alpha$)

$$f^*_{\beta\gamma} = \sum_j \left[\frac{\partial f_j}{\partial x_1'} + \beta_{1j} - \sum_{1<i} \bar{f}^i_{\beta\gamma} \beta_{ij} + 2\gamma_{1j} x_1 + \sum_{1<i} \gamma_{ij} x_i - \sum_{1<i} \bar{f}^i_{\beta\gamma} \gamma_{ij} x_1 \right] v_{1j}$$
$$+ \sum_{i,j;1<i} \left[\frac{\partial f_j}{\partial x_i'} + \beta_{ij} + \gamma_{ij} x_1 \right] v_{ij}$$
$$+ \sum_j \left[\frac{\partial^2 f_j}{\partial x_1'^2} + 2\gamma_{1j} - 2 \sum_{1<i} \bar{f}^i_{\beta\gamma} \gamma_{ij} \right] w_{1j} + \sum_{i,j;1<i} \left[\frac{\partial^2 f_j}{\partial x_1' \partial x_i'} + \gamma_{ij} \right] w_{ij} .$$

It follows that (all at p_0)

$$\frac{\partial f^*_{\beta\gamma}}{\partial \beta_{1j}} = v_{1j}\,, \qquad \frac{\partial f^*_{\beta\gamma}}{\partial \beta_{ij}} = -\bar{f}^i_{\beta\gamma} v_{1j} + v_{ij} \qquad (i > 1),$$

$$\frac{\partial f^*_{\beta\gamma}}{\partial \gamma_{1j}} = 2x_1 v_{1j} + 2w_{1j}\,,$$

$$\frac{\partial f^*_{\beta\gamma}}{\partial \gamma_{ij}} = x_i v_{1j} - \bar{f}^i_{\beta\gamma} x_1 v_{1j} + x_1 v_{ij} - 2\bar{f}^i_{\beta\gamma} w_{1j} + w_{ij} \qquad (i > 1).$$

Clearly the $f^*_{\beta\gamma}$ form a $(2mn)$-parameter family of mappings of $\bar{R}'_\alpha$ into $\bar{E}$ (see [3; §18]).

We wish to find, for arbitrarily small β_{ij} and γ_{ij}, a mapping $f_{\beta\gamma}$ with the property (C) at each singular point in $\bar{R}'_\alpha$. We shall construct sets $S_k \subset \bar{E}$, $k \geq 1$, and show that if a mapping g of $\bar{R}'_\alpha$ into E^m does not satisfy (C) at all singular points, then $g^*(p) \in \sum S_k$ for some $p \in \bar{R}'_\alpha$. Take $(t_{ij}, u_{ij}) \in \bar{E}$, and let t_i and u_i be the vectors in E^m with components t_{ij} and u_{ij}, respectively. Then (t, u) is in S_k if the following hold:

(a) $t_2, \cdots, t_n$ are independent;

(b) if T^{n-1} is the plane in E^m determined by them, then $t_1 \in T^{n-1}$;

(c) the vectors $t_1, \cdots, t_n, u_1, \cdots, u_n$ determine exactly a plane T^{m-k}.

Now take any mapping g, such that, at some singular point p, (C) does not hold, but $\partial g(p)/\partial x_1, \cdots, \partial g(p)/\partial x_n$ are independent. Let t_{ij} and u_{ij} be the coördinates of the corresponding $g^*(p)$:

$$g^*(p) = \sum_{i,j} t_{ij} v_{ij} + \sum_{i,j} u_{ij} w_{ij}\,,$$

i.e., $g^*(p) = (t, u)$. Since p is a singular point of g, and the $\partial g(p)/\partial x_i$ are independent for $i = 2, \cdots, n$, $\partial g(p)/\partial x_1$ lies in the plane T^{n-1} determined by them. Hence, if we change coördinates to the x'_i as above,

$$\frac{\partial g(p)}{\partial x'_1} = g^{\odot}(p) = 0.$$

Since (C) does not hold, the vectors $\partial g(p)/\partial x'_i$ $(i > 1)$ and $\partial^2 g(p)/\partial x'_1 \partial x'_i$ are dependent, and therefore determine a plane T^{m-k} with $k > 0$. But the expressions for $g^*(p)$ show that these vectors are exactly the t_i $(i > 1)$ and the u_i; hence $g^*(p) \in S_k$.

Next we show that S_k is an analytic manifold, and determine its dimension. Let (t^*, u^*) be any point of S_k. We shall show that the nearby points (t, u) of S_k are obtained by varying the t_{ij} and u_{ij} under certain restrictions; if we introduce d independent parameters describing them in an analytic fashion, we will have an analytic regular mapping of a d-manifold, and the statement will follow. First, $t_2, \cdots, t_n$ may be varied (of course slightly enough to keep them independent) in any fashion; there are thus $(n - 1)m$ parameters so far. Next, keeping these fixed, vary t_1, keeping it in the plane T^{n-1}; $n - 1$ parameters are added. Next, we consider the positions of the plane T^{m-k}. Given T^{n-1}, we determined them by naming a plane $T^{n-k} \subset T^{m-k}$ orthogonal to T^{n-1}, since $T^{n-1} \subset T^{m-k}$ and $(n-1)+(n-k) = m-k$. Now T^{n-k} is any plane in the maximal plane T^n orthogonal to T^{n-1}; it may be varied by varying each of a set of $n-k$ vectors ξ_i determining it, so that $\Delta\xi_i$ is in the plane $T^k \subset T^n$ orthogonal to T^{n-k}; thus it is determined by $k(n-k)$ parameters; see also [3; §24]. Finally, vary $u_1, \cdots, u_n$ in any manner such that, with T^{n-1}, they still determine T^{m-k}. Since they can vary (slightly) in any manner, so long as they remain in T^{m-k}, $n(m-k)$ more parameters are introduced. Therefore,

$$\begin{aligned}\dim(S_k) = d &= (n-1)m + (n-1) + k(n-k) + n(m-k) \\ &= 4n^2 - 3n - k^2.\end{aligned}$$

Consequently [3; §17], $\sum S_k$ is a denumerable sum of compact sets, each of zero $(4n^2-3n)$-extent. Therefore, using [3; Lemma 16] (see also [3; Lemma 12]), since $\bar{R}'_\alpha$ is of finite n-extent, and the $f_{\beta\gamma}$ form a $(4n^2-2n)$-parameter family, there is an arbitrarily small (β, γ), for which $f_{\beta\gamma}(p)$ is not in any S_k for any $p \in \bar{R}'_\alpha$; then $f_{\beta\gamma}$ is the required mapping. This completes the proof of the theorem. (The proof should be compared with that in [3; 677–679].)

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