In Theorem 12, we mean the (uniquely determined) products of 11 (a). In place of the following paragraphs, read: As knowing all $v_i^p \cdot u^p$ for all i uniquely determines u^p, and the correct $\frown$ satisfies (11.8), we have found the correct $\frown$. (NOTE: Theorem 11 is not used.) The end of the first paragraph in 12 should read: Find $u^p \smile u^q$ over I_0, then over R_1, ($p \leqq 2$), then for all p, by (5.12); then find $\frown$ over I_0, by Theorem 12. (We must know (11.15).) In Theorem 13, (a), add: $\phi\tau^p = 0$ if $O(\tau^p)$ is acyclic. Relation (14.9) follows directly from (14.4). After (25.7), add: $I^p \smile I^q = I^{p+q}$.

[2] *Augmentable* in Tucker, *Ann. Math.*, **34,** 191–243 (1933).

[3] See Tucker, these PROCEEDINGS, **25,** 371–374 (July, 1939). The theory was developed independently by S. Lefschetz and myself.

[4] This case of the theory is due to de Rham; see *Comm. Math. Helv.*, **4,** 151–157 (1933).

ON THE THEORY OF SPHERE-BUNDLES

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1. *Introduction.*—We give here a brief sketch of some new results in the theory of sphere-bundles;[1] in particular, further properties of the characteristic classes, a duality theorem, theorems on tangent and normal bundles to a manifold, and some examples. The results will be published later in book form.

2. *Fibre-Bundles.*—Let S_0 be a space, and G, a group of homeomorphisms of S_0 into itself. Then over any space K, the *base space*, with neighborhoods U_i, we may define a *fibre-bundle* $\mathfrak{B}(K)$ as follows. For $p \in U_i$, let $\xi_i(p)$ be a homeomorphism of S_0 into a set of points $S(p)$. Let $S(p) \cdot S(q) = 0$ if $p \neq q$. Let $\mathfrak{S}(K)$ be the space of all points on all $S(p)$, the *total space* (gefaserte Raum). Let $\xi_i(p, q)$ be the image of q in S_0 under $\xi_i(p)$. For $p \in U_i \cdot U_j$, set $\xi_{ij}(p, q) = \xi_i^{-1}(p, \xi_j(p, q))$; assume $\xi_{ij}(p) \in G$ for each p, and that it varies continuously with p. Then a topology is easily defined in $\mathfrak{S}$. The part $\mathfrak{S}(U_i)$ of $\mathfrak{S}$ over U_i is a product $U_i \times S_0$.

If S_0 is a set of μ points, and G is the group of permutations, we obtain the covering spaces of K with μ sheets. If S_0 is a subgroup of a continuous group R_0, and $G = S_0$, then the left (or right) cosets of S_0 form a space K (factor group if S_0 is normal); the total space is R_0. If $S_0 = S_0^\nu$ is a ν-sphere, and $G = G^{\nu+1}$, the orthogonal group, we have a *sphere-bundle.* If S_0 is a vector space, and G, the affine or orthogonal group, an equivalent theory is obtained.

3. *Particular Coördinate Systems.*—We use this section in the proof of the duality theorem. Let K be a complex with ordered vertices. We may use ξ_σ, defined over closed cells σ. (See TP.) We may choose them so $\xi_\sigma = \xi_{\sigma'}$ if σ and σ' have the same first vertex. Let P^r be a small closed

region in $\underline{K}$ surrounding all $D_a(\sigma^r, \sigma')$, σ' of any dimension (see I, §2); set $Q^r = \overline{K - P^r}$. Then if K^r is the r-dimensional part of K, and $e_1, \ldots, e_{\nu+1}$ are the unit points of S_0^{ν},

$$\xi_{\sigma,\sigma'}(p,e_i) = e_i \qquad (p \in Q^k\cdot\sigma\cdot\sigma', i = 1, \ldots, \nu - k).$$

Note that $\phi_i(p) = \xi_{\sigma^{r-1}}(p, e_i)$ ($p \in$ any σ^{r-1}, $i = 1, \ldots, \nu - r + 2$) defines orthogonal projections of K^{r-1} into $\mathfrak{S}$.

4. *Characteristic Classes.*—Choose $\phi_1, \ldots, \phi_{\nu - r + 2}$ as above over K^{r-1}; then for each σ^r, studying these on $\partial\sigma^r$ gives $W^r\cdot\sigma^r$, which is an integer mod 2 if $r = 1$ or $r \leqq \nu$ is even, and an integer otherwise. W^r is a cocycle whose class $\mathbf{W}^r$ is an invariant of $\mathfrak{B}$; the $\mathbf{W}^r$ characterize $\mathfrak{B}$ if $\nu \leqq 1$ or dim $(K) \leqq 3$ (see TP). We may use a general type of subdivision of the polyhedron K in defining the $\mathbf{W}^r$.

If $\mathfrak{B}$ is not orientable (TP, §4), and K' is obtained from K by replacing $[\sigma^r: \sigma^{r+1}]$ by $-[\sigma^r:\sigma^{r+1}]$ when ξ_{σ^r} and $\xi_{\sigma^{r+1}}$ give opposite orientations to the $S(p)$ ($p \in \sigma^r$) (see TP, p. 793, footnote), then K' is locally isomorphic with K, and $\mathbf{W}' = \mathbf{W}^1$. We call K' the *complex associated with* $\mathfrak{B}$. The characteristic classes are taken in K'; the theorems above hold still.

If f maps K_1 into K_2, and $\mathfrak{B}_2(K_2)$ is defined, then a bundle $\mathfrak{B}_1(K_1)$ is defined (TP, §8), and $\mathbf{W}_1^r = f'\mathbf{W}_2^r$ (f' = dual of f).

If $\nu = 2$, $\dim(K) = 4$, and $\mathbf{W}^1 = 0$, $\mathbf{W}^2 = 0$, then an invariant characterizing $\mathfrak{B}$ is obtained as follows. A triple $\phi = (\phi_1, \phi_2, \phi_3)$ of orthogonal projections of K^3 into $\mathfrak{S}(K^3)$ exists. Let $\Phi(p)$ ($p \in K^3$) map S_0 into $S(p)$ so that $\Phi(p, e_i) = e_i$ ($i = 1, 2, 3$). For each σ^4, set

$$\Psi_{\sigma^4}(p) = \xi_{\sigma^4}^{-1}(p)\Phi(p) \ (p \in \partial\sigma^4).$$

This maps $\partial\sigma^4$ into the orthogonal group G^3; as G^3 is homeomorphic with projective 3-space P^3, this defines an integer $D^4\cdot\sigma^4$, the degree of Ψ_{σ^4}. D^4 is a cocycle. If we identify two cocycles if they are cohomologous, or differ by a cocycle of the form $X^1 \smile X^1 \smile X^1 \smile X^1$ (X^1 a 1-cocycle), the class determined is the invariant.

The classes W^{2r+1} are determined from the others as follows (see I, §11):

$$W^{2r+1} = \tfrac{1}{2}\partial\omega W^{2r} \qquad (\text{if } \nu \geqq 2r).$$

5. *On Mappings into* $G^{\nu+1}$.—In the theorem just stated, and in the duality theorem, we need the following (and other more complicated) theorems. (*a*) Let f map σ^ν into $G^{\nu+1}$ so that if $\phi(p) = f(p,e_1)$, then $\phi(p) = e_1$ in $\partial\sigma^\nu$; let ϕ be of degree α. Let ψ map σ^ν into S_0^ν with the degree β, and let $\psi(p) = e_1 (p \in \partial\sigma^\nu)$. Then $\theta(p) = f(p, \psi(p))$ is of degree $\alpha + \beta$. (*b*) Take f as before; then $\phi'(p) = f^{-1}(p, e_1)$ (the point of S_0 mapped into e_1 by $f(p)$) is of degree $-\alpha$. (Use (*a*).) (*c*) Let f map σ^ν into $G^{\nu+1}$, let ϕ map $\partial\sigma^\nu$ into $S_0^{\nu-1}$ with the degree α, and suppose $f(p, \phi(p)) = e_1$ ($p \in$

$\partial\sigma^\nu$). Then $\psi(p) = f(p, e^{\nu+1})$ maps $\partial\sigma^\nu$ into the $S_1^{\nu-1}$ orthogonal to e_1 with a degree $\equiv \alpha$ (mod 2).

6. *The Duality Theorem.*—Given bundles $\mathfrak{B}_1^\lambda(K)$ and $\mathfrak{B}_2^\mu(K)$, there is a uniquely determined bundle $\mathfrak{B}_3^\nu(K)$, their *product* ($\nu = \lambda + \mu + 1$; see TP, p. 796); thus if $M^m \subset M^n$, the tangent times the normal bundle gives the part of the tangent bundle of M^n over M^m. The formula for the characteristic classes of $\mathfrak{B}_3^\nu$ is

$$\mathbf{W}_3^r = \sum_i \mathbf{W}_1^i \smile \mathbf{W}_2^{r-i}, \text{ reducing mod 2 if necessary.}$$

(See §4 and I, §12; we use W^0 = sum of vertices.) The proof is very difficult if $r \geqq 4$. We use the special $\xi_{1,\sigma}$ of §3 in $\mathfrak{B}_1$, and $\xi_{2,\sigma}$ in $\mathfrak{B}_2$, with a replaced by $a' < a$, so the P_1 and P_2 will be in "general position" (see I, §5). The projections into $\mathfrak{S}_3(K^{r-1})$ are defined successively over $Q_1^0, Q_1^1, \ldots$. For each σ^r, they are now deformed in $\partial\sigma^r$ into a simpler position, except in each $A^i = P_1^i \cdot P_2^{r-1-i} \cdot \sigma^{r-1}$ (σ^{r-1} = face of σ^r opposite first vertex of σ^r). The terms shown come from the A^i, two coming from A^{r-1}. The results of §5 and the products of I, §6, are needed.

REMARK. We do not know whether or not the individual terms $\mathbf{W}_1^i \smile \mathbf{W}_2^{r-i}$ have topologicial significance.

Reducing everything mod 2, write, for any $\mathfrak{B}$, the formal power series

$$\mathbf{W} = \sum_i \mathbf{W}^i t^i, \quad \overline{\mathbf{W}} = 1/\mathbf{W} = \sum_i \overline{\mathbf{W}}^i t^i;$$

then

$$\overline{\mathbf{W}}^0 = I|_2, \overline{\mathbf{W}}^1 = \mathbf{W}^1, \overline{\mathbf{W}}^2 = \mathbf{W}^2 + \mathbf{W}^1 \smile \mathbf{W}^1,$$

etc. The duality theorem gives then, as $\mathbf{W}_N = \mathbf{W}/\mathbf{W}_T$, etc.,

$$\mathbf{W}_N^r = \sum \mathbf{W}^i \smile \overline{\mathbf{W}}_T^{r-i}, \quad \overline{\mathbf{W}}^r = \sum \overline{\mathbf{W}}_T^i \smile \overline{\mathbf{W}}_N^{r-i}, \text{ etc.} \quad \text{(mod 2)}.$$

7. *Tangent Bundles.*—Let K be a simplicial subdivision of the manifold M^n, with ordered vertices. Each p in K may be written uniquely as $p = \sum \eta_{\lambda_i}(p) x_{\lambda_i}$, if $p \in x_{\lambda_0} \ldots x_{\lambda_r}$. Define

$$v_k(p) = \sum_{\lambda_0 < \ldots < \lambda_k} \eta_{\lambda_0}(p) \ldots \eta_{\lambda_k}(p)(x_{\lambda_k} - x_{\lambda_{k-1}}) \quad (k = 1, 2, \ldots).$$

These are continuous in K, and the first r are independent except in K^{r-1} (any r). If K^* is the usual complex dual to K, these may be used to define W^r, a cocycle in K^*. Its dual is a *characteristic cycle* C^{n-r} in K', the complex associated with $\mathfrak{B}$ (which was studied in I, §13). Note that $C^{n-(2r+1)} = \frac{1}{2}\partial\omega C^{n-2r}$. The value of $C^s \cdot \sigma^s$ ($s = n - r$, $\sigma^s = x_{\lambda_0} \ldots x_{\lambda_s}$) is as follows. Let K_1 be the subcomplex of the closed star of σ^s containing all vertices x_i with

$$\lambda_s > i > \lambda_{s-1} \text{ or } \lambda_{s-2} > i > \lambda_{s-3} \text{ or } \ldots.$$

(This includes all vertices below λ_0 if s is even.) Then $C^s \cdot \sigma^s = 1 - \chi(K_1)$ (χ = Euler-Poincaré characteristic), or this mod 2.

From this we prove: If K is the first derived of a simplicial subdivision of M, then C^s is the sum of all s-simplexes of K (properly oriented if integer coefficients are used).[2]

In the proofs of the following theorems, we study the classes over submanifolds of the given manifold, and use the duality theorem and results from §8. For $M^m \subset M^n$, let $\mathbf{W}^r$ mean the part of $\mathbf{W}_T{}^r(M^n)$ in M^m.

If a closed M^m can be imbedded in E^n (with or without singularities), then

$$0 = \mathbf{W}_N{}^{n-m}|_2 = \sum \mathbf{W}^i \smile \overline{\mathbf{W}}_T{}^{n-m-i} = \overline{W}_T{}^{n-m}.$$

Hence $\overline{W}_T{}^m = 0$ always. This gives, if $(X)^2 = X \smile X$, etc.,

closed M^2: $\mathbf{W}^2|_2 = (\mathbf{W}^1)^2$; closed M^3: $(\mathbf{W}^1)^3 = 0$;

closed M^4: $\mathbf{W}^4|_2 + (\mathbf{W}^2)^2 + \mathbf{W}^2 \smile (\mathbf{W}^1)^2 + (\mathbf{W}^1)^4 = 0$; etc.

In any M^2, for any 1-I_2-cocycle X^1, $X^1 \smile X^1 \backsim X^1 \smile W^1$. In any M^3, $\mathbf{W}^2 = \mathbf{W}^1 \smile \mathbf{W}^1$; hence (Stiefel) for orientable M^3 (closed or not), the tangent bundle is simple. In any orientable M^4, $\mathbf{W}^3 = 0$. (The proof uses facts from §10.) For any 2-I_2-cocycle X^2 in any M^4, $X^2 \smile X^2 \backsim X^2 \smile \overline{W}^2$. For any orientable M^4 in an orientable M^r, $\overline{W}_T{}^4 = W_N{}^4$.

8. *Normal Bundles.*—For any M^m in any M^n, $\mathbf{W}_N{}^{n-m}$ is the intersection of M^m with itself in M^n, which is a cohomology class of the complex associated with the normal bundle; if M^m is closed and $M^n = E^n$, then $\mathbf{W}_N{}^{n-m}|_2 = 0$, and if also M^m is orientable, then $\mathbf{W}_N{}^{n-m} = 0$. Compare PC, §20, and TP, p. 795.

If M^m is mapped regularly into M^n, but with singularities, we may deform M^m slightly into M'^m, and consider the intersections of the σ' with a neighborhood of σ' in M; then W^{n-m} is the *local intersection* thus defined. For a closed orientable $M^m \subset E^n$, the *intersection* vanishes, so that the *distant intersection* equals the local. This holds mod 2 in the non-orientable case. If $n = 2m$, and the singularities are isolated points, the distant intersection is of course 0 (mod 2); if m is odd, it vanishes, because $\{\sigma^m, \sigma'^m\} = -\{\sigma'^m, \sigma^m\}$.

Take an orientable $M^m \subset E^n$. Then the normal bundle is simple if $m =$ 1 or 2, or $n = m + 1$ or $m + 2$, or $m = 3$ and M is closed, or M is a cell. This holds if M^m is merely mapped regularly, provided that if $n = m + 2$, then m is odd, and we omit $m = 2$ if M is closed.

9. *Examples.*—Consider a cylinder, the product $T^3 = T^1 \times T^2$ of a segment and a disc. Let P^2 be one end, let $S_2{}^1$ be a segment crossing P^2,

let Q^2 be a rectangle cutting through T^3 and ending on S_2^1, and let S_1^1 be the center of Q^2, ending at p_0, the center of S_2; rather, let these be the sets after the identifications below. (1) Join the ends of T^3, and shrink ∂T^2 to a point; this forms $M_1^3 = S^1 \times S^2$, with a simple tangent bundle. (2) Join the ends, and identify opposite points of ∂T^2; then characteristic cycles are (mod 2) $C^2 = Q^2$, $C^1 = S_1^1$; $\therefore \mathbf{W}^2 \neq 0$. (3) Join the ends, reflecting one so that P^2 is joined to itself with orientation reversed, and shrink ∂T^2 to a point. We obtain M_3^3, with $\mathbf{W}^2 = 0$. (4) Join the ends as in (3), and identify opposite points of ∂T^2, forming M_4^3. Now P^2 is a projective plane, and Q^2 is a Klein bottle. Intersections are (mod 2)

$$\{P^2, P^2\} \sim 0,\ \{Q^2, P^2\} \sim S_2^1,\ \{Q^2, Q^2\} \sim S_1^1 + S_2^1;$$
$$\{P^2, S_2^1\} \sim 0,\ \{P^2, S_1^1\} \sim p_0,\ \{Q^2, S_2^1\} \sim p_0,\ \{Q^2, S_1^1\} \sim 0;$$

characteristic classes are

$$C^2 \sim (P^2 + Q^2)_2,\ C^1 \sim (S_1^1 + S_2^1)_2.$$

Define $\mathfrak{B}_1^1(M_4^3)$, with $C_1^2 \sim P^2$, $C_1^1 \sim 0$, and $\mathfrak{B}_2^1(M_4^3)$, with $C_2^2 \sim Q^2$, $C_2^1 \sim 0$; let the total spaces be M_1^4, M_2^4. We may pretend M_4^3 is in either (because $C_i^1 = 0$). Then

$$C_1^3 \sim \mathfrak{S}(Q^2),\ C_1^2 \sim \mathfrak{S}(S_1^1),\ C_1^1 \sim \mathfrak{S}(p_0);\ \ \therefore \overline{C}_1^1 \sim 0;$$
$$C_2^3 \sim \mathfrak{S}(P^2),\ C_2^2 \sim \mathfrak{S}(S_2^1),\ C_2^1 \sim \mathfrak{S}(p_0);\ \ \therefore \overline{C}_2^1 \sim \mathfrak{S}(p_0);$$

hence in M_2^4, $\mathbf{W}^3 \neq 0$, $\overline{\mathbf{W}}^3 \neq 0$. Hence (see §7) M_2^4 cannot be imbedded in E^7.

Define $\mathfrak{B}^2(M_3^3)$, with $C^2 \sim P^2$, and $C^1 \sim S_1^1$. Then we may consider $M_3^3 \subset M^5 = \mathfrak{S}(M_3^3)$, and prove (*a*) M^5 is closed and orientable, (*b*) $C^2(M^5) \sim \mathfrak{S}(p)$; hence $\overline{\mathbf{W}}^3 = \mathbf{W}^3 \neq 0$, and M^5 cannot be imbedded in E^8.

We may define $M^8 = \mathfrak{S}(S_0^4)$, with $\overline{\mathbf{W}}^4 = \mathbf{W}^4 \neq 0$; hence M^8 cannot be imbedded in E^{12}.

The complex projective plane P^{*4} cannot be imbedded in E^6, as $\mathbf{W}^2 \neq 0$. ($\mathbf{W}^2 \smile \mathbf{W}^2 = \mathbf{W}^4|_2$; $W^4 \cdot P^{*4} = \chi(P^{*4}) = 3$.) But it can be in E^7.

For any closed orientable M^4, and any cocycles X^2, X^4 (mod 2) in M^4, we may imbed M^4 in an $M^8 \subset E^{17}$, so that the part over M^4 of the characteristic classes of the normal bundle of $M^8 \subset E^{17}$ are X^2 and X^4. Hence we may make $\mathbf{W}_N^3 \neq 0$ and $\mathbf{W}_N^4 \neq 0$ also.

If we put a Klein bottle Q^2 in E^3, then the distant and local intersections are equal (mod 2); these are $\mathrm{W}_N^1 = \mathrm{W}_T^1$; $C_T^1 =$ a closed curve in Q^2. Hence the distant intersection, as a cycle, is a certain curve in Q^2, as is clear in the usual immersion of Q^2 in E^3. For P^2 in E^3, we get the "projective line" similarly.

A direct study shows: If $M^2 \subset E^4$ is closed (with or without singularities), and C^2 is the fundamental cycle of the associated complex (integer coefficients), then $W^2 \cdot C^2 = 2[\chi(M^2) + 2k]$ for some k; any k may be obtained. Hence if $\chi \not\equiv 0$ (mod 2), then a field of normal vectors never exists. Also one imbedding cannot be deformed into another with a different $W^2 \cdot C^2$.

10. *Homology Groups of Total Spaces, Etc.*—Let K be connected. Given $\mathfrak{B}(K)$, the homology groups satisfy $\mathbf{H}^r(\mathfrak{S}) \approx \mathbf{H}^r(K)$, $r < \nu$. $kS(p) \sim 0$ if and only if for some $A^{\nu+1}$, $A^{\nu+1} \cdot W^{\nu+1} = k$. (We may use $k_\lambda = k \bmod \lambda$, and $A^{\nu+1}|_\lambda$.) For $\mathfrak{B}$ oriented, a $(\nu + 1)$-I_μ-cycle A in K is the projection of such a cycle in $\mathfrak{S}$ if and only if $A \cdot W^{\nu+1} = 0_\lambda$. Now $\mathbf{H}^\nu(\mathfrak{S})$ may be described in terms of $\mathbf{W}^{\nu+1}$ and properties of $kS(p)$. If μ is the smallest integer such that $\mu S(p) \sim 0$, then $\mathbf{H}^\nu(\mathfrak{S}) \approx I_\mu \oplus \mathbf{H}^\nu(K)$ if and only if for each λ and each $(\nu + 1)$-I_λ-cycle A, $A \cdot W^{\nu+1} \equiv 0 \bmod (\lambda, \mu)$. A mapping f of a complex K' of dimension $\leqq \nu + 1$ into K is the projection of a mapping into $\mathfrak{S}$ if and only if $f'\mathbf{W}^{\nu+1} = 0$.

[1] We refer the reader to papers in these PROCEEDINGS, **21,** 464–468 (1935), and in *Bull. Am. Math. Soc.*, **43,** 785–805. We denote the latter by TP, and the preceding note, by I. Sphere-bundles were formerly called "sphere-spaces."

[2] This was a conjecture of Stiefel, *Comm. Math. Helv.*, **8,** 40 (1936).