ON THE MAPS OF AN n-SPHERE INTO ANOTHER n-SPHERE

By Hassler Whitney

1. **Introduction.** It is well known that to each map[1] f of an n-sphere S^n into another one S_0^n ($n \geqq 1$ always) there corresponds a number d_f, the *degree* of f, and $d_f = d_g$ if f and g are homotopic (see §2). H. Hopf[2] has proved the converse theorem, that if $d_f = d_g$, then f and g are homotopic. The object of this note is to give an elementary proof of the latter theorem. The methods will be used and extended in later papers.

In an appendix we give somewhat briefly a proof of the theorem for the case that $d_f = 0$. This is the only case needed in the following paper; the general theorem then follows from that paper. The second proof is more intuitive geometrically than the first, but complete details would make it perhaps more lengthy.

2. **On deformations.** A *deformation* of one space S in another S_0 is a family $\phi_t(p)$ ($0 \leqq t \leqq 1$, p in S) of maps of S into S_0, continuous in both variables together. Given maps f and g of S into S_0, if there exists a deformation ϕ_t such that $\phi_0 \equiv f$ and $\phi_1 \equiv g$, we say f and g are *homotopic.* If f is homotopic to g, where $g(p) \equiv P_0$ (all p in S), we say f is *homotopic to zero,* and f may be *shrunk to the point* P_0.

Suppose S and S_0 are complexes, K_0 is a simplicial subdivision of S_0, and f maps S into S_0. Then, for a sufficiently fine simplicial subdivision K of S, the following is true. To each vertex V of K we may choose a vertex $g(V)$ of a cell of K_0 which contains $f(V)$, so that the vertices of any cell of K go into the vertices of a cell of K_0. This determines uniquely a "simplicial map" g of K into K_0, affine in each cell (see §5); moreover, f is homotopic to g.

3. **The degree of a map.** Let S_0^n be the unit n-sphere in $(n + 1)$-space, let K_0^n be a simplicial triangulation of S_0^n, and let σ_0^n be an n-cell of K_0^n. We choose K_0^n so that if P_1 is a point of σ_0^n and P_0 is the antipodal point of S_0^n, each great semicircle from P_1 to P_0 intersects the boundary $\partial\sigma_0^n$ of σ_0^n in exactly one point. By pushing along these semicircles, we define a deformation Ω_t of the identity $\Omega_0(p) \equiv p$ into a map Ω_1, where $\Omega_1(p) \equiv P_0$ for p in $S_0^n - \sigma_0^n$.

Let σ^k be a k-cell ($k \leqq n$), in fixed correspondence with a k-simplex, and let

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[1] All maps will be assumed continuous.

[2] See Alexandroff-Hopf, *Topologie*, I, Berlin, 1935, pp. 501–505. See also the reference to Lefschetz in the following paper.

f map σ^k into S_0^n. We say f is *standard* if $f(p) \equiv P_0$, or, $k = n$ and for some affine map ϕ of $\sigma^k = \sigma^n$ into σ_0^n, $f(p) \equiv \Omega_1(\phi(p))$. In any case, $f(p) \equiv P_0$ in $\partial\sigma^k$.[3] The map f of an n-complex K^n into S_0^n is *standard* if it is standard over each k-cell ($k \leqq n$).

We may orient S_0^n by orienting σ_0^n. Let K^n be a simplicial triangulation of the oriented n-sphere S^n, and let f be a standard map of K^n into S_0^n. Let σ^n be an (oriented) n-cell of K^n. If $f(p) \equiv P_0$ in σ^n, we set $d_f(\sigma^n) = 0$. Otherwise, there is a simplicial map ϕ of σ^n into (the whole of) σ_0^n such that $f(p) \equiv \Omega_1(\phi(p))$ in σ^n; we set $d_f(\sigma^n) = 1$ or -1 according as ϕ is positive or negative. We define the *degree* of f by

$$d_f = \sum_{\sigma^n} d_f(\sigma^n). \tag{3.1}$$

4. **The theorem.** In homology theory it is shown how to attach to each map f of S^n into S_0^n (both spheres oriented) an integer d_f, the degree of the map. Moreover, if f is homotopic to g, then $d_f = d_g$, and if S^n and S_0^n are triangulated and f is standard, then d_f is given by (3.1).

Suppose f and g map S^n into S_0^n, and $d_f = d_g$. Then for a sufficiently fine subdivision K^n of S^n, both f and g can be deformed into simplicial maps and hence into standard maps ϕ and ψ. As f and ϕ, also g and ψ, are homotopic, $d_\phi = d_\psi$. By Theorem 1 below, ϕ is homotopic to ψ; hence f is homotopic to g. Therefore this theorem furnishes the converse of the statements above.

THEOREM 1. *If ϕ and ψ are standard maps of S^n into S_0^n, using the same subdivision K^n of S^n, and $d_\phi = d_\psi$, then ϕ is homotopic to ψ.*

From the proof below, the following corollary is apparent.

COROLLARY. *If $\phi(V) = \psi(V) = P_0$ for a fixed vertex V of K^n, we can make V remain at P_0 throughout the deformation.*

In fact, if $n \geqq 2$, all vertices of K^n remain at P_0. If $n = 1$, we may choose the chains of cells in §8 so that in no chain do we pass over V; then V is never moved.

THEOREM 2. *For any integer γ there is a map f of S^n into S_0^n with $d_f = \gamma$.*

To prove this, subdivide S^n into $\alpha \geqq |\gamma|$ n-cells. Let ϕ map $|\gamma|$ of these cells simplicially into σ_0^n, positively or negatively according as $\gamma > 0$ or $\gamma < 0$ (if $\gamma \neq 0$), and set $f(p) = \Omega_1(\phi(p))$ in these cells and $f(p) = P_0$ elsewhere. Clearly $d_f = \gamma$. Note that the degree of the identity map of S_0^n into itself is 1.

The remainder of the paper is devoted to the proof of Theorem 1.

5. **Coördinates p_t in a cell.** Any simplicial complex K^n is homeomorphic to a complex $\bar{K}^n$ in euclidean space whose cells are straight. Using $\bar{K}^n$, we define straightness in K^n, the center of a cell (i.e., center of mass of its vertices), etc. Hence an "affine map" of one cell into another has meaning. Let σ be a cell of K^n, and a, the center of σ. For each point p of the boundary $\partial\sigma$ of σ let p_t be the point of the segment ap such that $ap_t/ap = t$.

[3] There are $(n + 1)!$ standard maps ϕ of σ^n into S_0^n with $\phi(p) \not\equiv P_0$.

6. Certain deformations of simplexes. We prove first a combinatorial lemma, needed in Lemma 2.

LEMMA 1. *Any even permutation of the letters $a_0a_1 \cdots a_n$ $(n \geqq 2)$ may be made by means of a succession of cyclic permutations, each on three of the letters.*

This is clear if $n = 2$; then any even permutation is cyclic. Suppose $n > 2$, and let $B = a_{\alpha_0} \cdots a_{\alpha_n}$ be any even permutation. If $\alpha_n \neq n$, bring a_{α_n} to the right end by a cyclic permutation; bring $a_{\alpha_{n-1}}$ next to a_{α_n}. Suppose $\alpha_0 \neq 0$. We then perform the two cyclic permutations

$$a_0 \cdots a_{\alpha_0} \cdots a_{\alpha_{n-1}} a_{\alpha_n} \to a_{\alpha_0} \cdots a_{\alpha_{n-1}} \cdots a_0 a_{\alpha_n} \to a_{\alpha_0} \cdots a_0 \cdots a_{\alpha_n} a_{\alpha_{n-1}} .$$

If $n \geqq 4$ and a_{α_1} is not now in the second place, we perform two cyclic permutations to bring it there, again interchanging $a_{\alpha_{n-1}}$ and a_{α_n}, etc. When $a_{\alpha_0}, \cdots, a_{\alpha_{n-3}}$ are in their correct places, $a_{\alpha_{n-2}}$ is also; as B is even and the above permutations are even, $a_{\alpha_{n-1}}$ and a_{α_n} are also in their correct positions.

LEMMA 2. *Let $\sigma^n = a_0 \cdots a_n$ be a simplex, and let $a_{\alpha_0} \cdots a_{\alpha_n}$ be an even permutation of its vertices. Then there is a deformation ϕ_t of σ^n in itself, such that $\phi_0(p) \equiv p$, $\phi_1(a_s) = a_{\alpha_s}$, ϕ_1 is affine, and ϕ_t for each t is a homeomorphism both in σ^n and in its boundary.*

If $n = 0$ or 1, the lemma is trivial. Suppose that $n = 2$; say $a_{\alpha_0}a_{\alpha_1}a_{\alpha_2} = a_1a_2a_0$. Let $\phi_t(a_i)$ be the point p of a_ia_{i+1} (setting $2 + 1 = 0$) for which $a_ip/a_ia_{i+1} = t$. Let ϕ_t map the segment a_ia_{i+1} into the broken line $\phi_t(a_i)a_{i+1}\phi_t(a_{i+1})$ so that, if the line were straightened, the map would be linear. For any point p_u (see §5) interior to σ^2, set $\phi_t(p_u) = (\phi_t(p))_u$. As $\phi_t(a) = a =$ center of mass of σ^2, ϕ_1 is easily seen to be affine.

Now suppose $n > 2$; consider first a cyclic permutation, changing say $a_0a_1a_2$ into $a_1a_2a_0$. Set $\sigma = a_0a_1a_2$, $\sigma' = a_3 \cdots a_n$, and let $[p, q, u]$ for p in σ, q in σ', $0 \leqq u \leqq 1$, be the point r of the segment pq for which $pr/pq = u$. Define ϕ_t in σ as above. For any point $[p, q, u]$ not in σ, set $\phi_t[p, q, u] = [\phi_t(p), q, u]$. We show that ϕ_t is a homeomorphism. Suppose $\phi_t[p, q, u] = \phi_t[p', q', u']$; then $[\phi_t(p), q, u] = [\phi_t(p'), q', u']$, which implies $\phi_t(p) = \phi_t(p')$, $q = q'$, $u = u'$;[4] as ϕ_t is a homeomorphism in σ, $p = p'$ also. Further, given $[p, q, u]$ and t, we may find a p^* for which $\phi_t(p^*) = p$; then $\phi_t[p^*, q, u] = [p, q, u]$. The other properties of ϕ_t are clear, and the lemma for this case is proved. Now take any permutation. We may obtain it by cyclic permutations as in Lemma 2; the corresponding deformations together give the required deformation.

7. Two types of deformations of S^n in S_0^n. Let ϕ' be a standard map of S^n into S_0^n, and let σ and σ' be oriented n-cells of K^n with the common $(n-1)$-face τ:

$$\sigma = a_0a_1 \cdots a_n , \qquad \sigma' = -a_0'a_1 \cdots a_n , \qquad \tau = a_1 \cdots a_n .$$

(a) Suppose $d_{\phi'}(\sigma) = 1$, $d_{\phi'}(\sigma') = 0$; we shall deform ϕ' into ϕ'' so that $d_{\phi''}(\sigma) = 0$, $d_{\phi''}(\sigma') = 1$, leaving $K^n - (\sigma + \sigma')$ fixed.

(b) Suppose $d_{\phi'}(\sigma) = 1$, $d_{\phi'}(\sigma') = -1$; we shall obtain $d_{\phi''}(\sigma) = d_{\phi''}(\sigma') = 0$.

[4] This is so if $0 < u < 1$, as we may assume.

In each case ϕ'' will be a standard map.

(a) Set $\sigma_1 = a_0a_2 \cdots a_n$, $\sigma_1' = a_0'a_2 \cdots a_n$, or if $n = 1$, then $\sigma_1 = a_0$, $\sigma_1' = a_0'$. Let θ_1 and θ_2 be the affine maps of σ_1 into τ and σ_1' determined by sending a_0 into a_1 and a_0' respectively. For each p in σ_1, let $\alpha(p, u)$ run linearly along the segments $p\theta_1(p)$ and $\theta_1(p)\theta_2(p)$ as u runs from 0 to 1 and from 1 to 2. Set

$$\phi_t'[\alpha(p, u)] = \begin{cases} \phi'[\alpha(p, u - t)] & (t \leqq u), \\ \phi'[\alpha(p, 0)] & (t > u), \end{cases} \tag{7.1}$$

and $\phi_t'(p) = \phi'(p)$ in $K^n - (\sigma + \sigma')$. As $\phi'(p) \equiv P_0$ in $\partial\sigma_1 + \partial\sigma_2$, this is clearly a deformation of $\phi' = \phi_0$ into a map $\phi'' = \phi_1$. The map ϕ'' in σ' is obtained from the map ϕ' in σ by replacing $a_0, a_1, \cdots, a_n$ (which form $+\sigma$) by $a_1, a_0', \cdots, a_n$ (which form $+\sigma'$); hence $d_{\phi''}(\sigma') = d_{\phi'}(\sigma)$. Also $d_{\phi''}(\sigma) = 0$ as $\phi''(p) \equiv P_0$ in σ, and (a) is proved.

(b) Let λ and λ' be the affine maps of σ and σ' into σ_0^n such that $\phi'(p) = \Omega_1(\lambda(p))$ in σ and $= \Omega_1(\lambda'(p))$ in σ'. Say $\sigma_0^n = b_0 \cdots b_n$,

$$\lambda(a_i) = b_{k_i}, \qquad \text{and } \lambda'(a_0') = b_{l_0}, \qquad \lambda'(a_i) = b_{l_i} \qquad (i > 0).$$

As $d_{\phi'}(\sigma') = -d_{\phi'}(-\sigma')$, and hence

$$d_{\phi'}(\sigma) = d_{\phi'}(a_0 a_1 \cdots a_n) = -d_{\phi'}(\sigma') = d_{\phi'}(a_0' a_1 \cdots a_n),$$

$b_{l_0} \cdots b_{l_n}$ is an even permutation of $b_{k_0} \cdots b_{k_n}$. Applying Lemma 2, we find a deformation λ_t' of σ' in σ_0^n such that $\lambda_0' \equiv \lambda'$, λ_1' is affine, and

$$\lambda_1'(a_0') = \lambda(a_0), \qquad \lambda_1'(a_i) = \lambda(a_i) \qquad (i > 0). \tag{7.2}$$

Set

$$\phi_t'(p) = \begin{cases} \Omega_1(\lambda_t'(p)) & p \text{ in } \sigma', \\ \phi'(p), & p \text{ in } K^n - \sigma'. \end{cases} \tag{7.3}$$

Then as $\Omega_1(\lambda_t'(p)) \equiv P_0$ in $\partial\sigma'$, ϕ_t' is a deformation of ϕ' into a map $\phi^* \equiv \phi_1'$.

For each p in τ, let $\beta(p, u)$ be the point q of the segment a_0p of σ such that $a_0q/a_0p = u$, and let $\beta'(p, u)$ be the corresponding point of the segment $a_0'p$ in σ'. As λ and λ_1' are affine, (7.2) and (7.3) give

$$\phi^*[\beta(p, u)] = \phi^*[\beta'(p, u)] \qquad (p \text{ in } \tau, 0 \leqq u \leqq 1). \tag{7.4}$$

We deform ϕ^* into ϕ'' by setting

$$\phi_t^*[\beta(p, u)] = \phi_t^*[\beta'(p, u)] = \phi^*[\beta(p, (1 - t)u)], \tag{7.5}$$

and $\phi_t^*(p) = \phi^*(p)$ in $K^n - (\sigma + \sigma')$. This is clearly a deformation; (7.4) shows that $\phi_0^* \equiv \phi^*$. As $\phi''(p) \equiv \phi_1^*(p) \equiv P_0$ in $\sigma + \sigma'$, $d_{\phi''}(\sigma) = d_{\phi''}(\sigma') = 0$.

8. **Proof of Theorem** 1. Suppose there are cells of K^n mapped positively over S_0^n by ϕ, and also cells mapped negatively. Then we can find a chain $\sigma_0, \cdots, \sigma_\nu$ of adjacent n-cells of K^n such that

$$d_\phi(\sigma_0) = 1, \quad d_\phi(\sigma_1) = 0, \quad \cdots, \quad d_\phi(\sigma_{\nu-1}) = 0, \quad d_\phi(\sigma_\nu) = -1.$$

Using (a), §7, we deform ϕ in $\sigma_0 + \sigma_1$, then in $\sigma_1 + \sigma_2$, etc.; then, using (b), §7, we deform the map in $\sigma_{\nu-1} + \sigma_\nu$. The new map ϕ' has $d_{\phi'}(\sigma_i) = 0$ $(i = 0, \cdots, \nu)$. Continue in this manner till no cells are mapped positively or none are mapped negatively over S_0^n; for definiteness, say the latter holds. Do the same for ψ. The new maps ϕ^* and ψ^* each have exactly $d_\phi = d_\psi$ cells mapped positively over S_0^n.

Suppose $d_{\phi^*}(\sigma) \neq d_{\psi^*}(\sigma)$ for some σ. Then let $\sigma_0, \sigma_1, \cdots, \sigma_\nu$ be a chain of adjacent n-cells such that

$$d_{\phi^*}(\sigma_0) = d_{\psi^*}(\sigma_\nu) = 1, \qquad d_{\phi^*}(\sigma_\nu) = d_{\psi^*}(\sigma_0) = 0,$$

$$d_{\phi^*}(\sigma_i) = d_{\psi^*}(\sigma_i) \qquad (0 < i < \nu).$$

Let $\sigma_0, \sigma_{k_1}, \cdots, \sigma_{k_s}$ be the cells of the chain for which $d_{\phi^*} = 1$. Using (a), §7, we deform ϕ^* over $\sigma_{k_s} + \sigma_{k_s+1}$ etc. until we have $d_{\phi_1^*}(\sigma_{k_s}) = 0$, $d_{\phi_1^*}(\sigma_\nu) = 1$; another succession of deformations makes $d_{\phi_2^*}(\sigma_{k_{s-1}}) = 0$, $d_{\phi_2^*}(\sigma_{k_s}) = 1$, etc. Finally $d_{\phi^{**}}(\sigma_0) = 0$, $d_{\phi^{**}}(\sigma_{k_i}) = 1$ (all i), and $d_{\phi^{**}}(\sigma_\nu) = 1$. $d_{\phi^{**}}(\sigma)$ differs from $d_{\psi^*}(\sigma)$ over fewer cells than $d_{\phi^*}(\sigma)$. Continuing in this manner, we deform ϕ^* into a map ϕ' with $d_{\phi'}(\sigma) = d_{\psi^*}(\sigma)$, all σ. ϕ' and ψ^* are standard. Applying Lemma 2, we deform ϕ' over each n-cell where necessary, to obtain ψ^*. (Compare the first half of the proof of (b), §7.) This completes the proof.

Appendix[5]

Let f be a map of S^n into S_0^n with the degree 0. We first deform it into a simplicial map and then into a standard map ϕ (see §§ 2, 3). To shrink ϕ to a point is equivalent to extending ϕ through the interior R of S^n (see the following paper, § 4). Let $\sigma_1, \cdots, \sigma_s$ and $\sigma_1', \cdots, \sigma_s'$ be the simplexes of S^n mapped positively and negatively over S_0^n respectively. Let T_i be a tube joining σ_i to σ_i' inside R. We may choose these so no two intersect, and also (to prove the corollary) so no one cuts the radius of R to the vertex V. Let $a_0 \cdots a_n$ and $a_0' \cdots a_n'$ be positive and negative orientations of σ_i and σ_i' respectively, such that $\lambda(a_i) = b_i$ and $\lambda'(a_i') = b_i$ determine simplicial maps of σ_i and σ_i' into σ_0^n, which in turn determine ϕ in σ_i and σ_i'. Now carry σ_i through T_i to σ_i', turning it so that a_i goes into a_i'; let $g_t(\sigma_i)$ be the position of σ_i after the time t. We do this so that $g_t(\sigma_i)$ does not intersect $g_{t'}(\sigma_i)$ if $t \neq t'$. (We are using a deformation theorem on simplexes in euclidean space, similar to but simpler than Lemma 2.) The definition of ϕ in R is as follows. For p not in any $g_t(\sigma_i)$, set $\phi(p) = P_0$. For p in $g_t(\sigma_i)$, choose q in σ_i so that $p = g_t(q)$, and set $\phi(p) = \phi(q)$.

HARVARD UNIVERSITY.

[5] Added in proof.