THE MAPS OF AN n-COMPLEX INTO AN n-SPHERE

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1. **Introduction.** The classes of maps of an n-complex into an n-sphere were classified by H. Hopf[1] in 1932. Recently, W. Hurewicz[2] has extended the theorem by replacing the n-sphere by much more general spaces. Freudenthal[3] and Steenrod[4] have noted that the theorem and proof are simplified by using real numbers reduced mod 1 in place of integers as coefficients in the chains considered. We shall give here a statement of the theorem which seems the most natural; the proof is quite simple. As in the original proof by Hopf, we shall base it on a more general extension theorem.

The fundamental tool of the paper is the relation of "coboundary";[5] it has come into prominence in the last few years.

In later papers we shall classify the maps of a 3-complex into a 2-sphere and of an n-complex into projective n-space.

I. Elementary facts

2. **Boundaries and coboundaries.** Let K be a complex, with oriented cells σ_i^r (not necessarily simplicial) of dimension r, $r = 0, \cdots, n$. Let $\partial_{ij}^r = 1, -1,$ or 0 according as σ_i^{r-1} is positively, negatively, or not at all, on the boundary of σ_j^r. An r-chain C^r is a linear form $\Sigma\alpha_i\sigma_i^r$, the α_i being integers (or elements of an abelian group). The *boundary* (or *contraboundary*) and *coboundary* of C^r are defined by

$$(2.1)\qquad \partial\Big(\sum_i \alpha_i\sigma_i^r\Big) = \sum_{i,j} \alpha_i\partial_{ji}^r\sigma_j^{r-1}, \qquad \delta\Big(\sum_i \alpha_i\sigma_i^r\Big) = \sum_{i,j} \alpha_i\partial_{ij}^{r+1}\sigma_j^{r+1}.$$

As in the ordinary theory, we say C^r is a *cocycle* if its coboundary vanishes, and C^r is *cohomologous* to D^r, $C^r \backsim D^r$, if $C^r - D^r$ is a coboundary. The relation $\delta\delta C^r = 0$ (easily proved; equivalent to $\partial\partial C^r = 0$) says that every coboundary

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[1] H. Hopf, Commentarii Mathematici Helvetici, vol. 5 (1932), pp. 39–54. See also Alexandroff-Hopf, *Topologie* I, Ch. XIII. A recent proof has been given by S. Lefschetz, Fund. Math., vol. 27 (1936), pp. 94–115. In Lemma 3 he gives a new proof of the theorem of the preceding paper; the author does not understand how the final map is made simplicial.

[2] W. Hurewicz, Proc. Kön. Akad. Wet. Amsterdam, vols. 38–39 (1935–36); in particular, vol. 39, pp. 117–126. The full paper will appear in the Annals of Math.

[3] H. Freudenthal, Compositio Math., vol. 2 (1935), footnote 8.

[4] Unpublished.

[5] This is discussed briefly in §2. For further details, see our paper *On matrices of integers*, pp. 35–45 of this volume of this Journal. We refer to this paper as I. The relation of Theorems 2, 3 and 4 to the theorems as stated by Hopf are made apparent by the theorems in I. *The present paper is independent of* I.

is a cocycle. Hence we may define the difference group of the group of r-cocycles over the group of r-boundaries, forming the r-th *cohomology group.*[6]

3. **Normal maps of cells into S_0^n.** Let S_0^n be the (oriented) unit n-sphere in $(n+1)$-space. Let f map the (oriented) n-cell σ^n into S_0^n. We say f is *normal* if $f(p) \equiv P_0$, a fixed point of S_0^n, for p in the boundary $\partial\sigma^n$ of σ^n. This is equivalent to identifying the points of $\partial\sigma^n$ in σ^n, forming an n-sphere S^n, and mapping this sphere into S_0^n. Hence we may define the degree[7] $d_f(\sigma^n)$. If f and g are normal in σ^n and $d_f(\sigma^n) = d_g(\sigma^n)$, then we may deform f into g, keeping $\partial\sigma^n$ at P_0, by II, corollary.

Any map f of σ^r into S_0^n, $r < n$, may be shrunk to P_0: we deform f into a simplicial map, and apply Ω_t (see II, §3). P_0 being assumed a vertex of K_0^n, if $\partial\sigma^r$ is at P_0 it remains there during the deformation.

If K is any complex, let K^r be the subcomplex of K containing all its cells of dimension $\leqq r$. The map f of K into S_0^n is *normal* if $f(p) \equiv P_0$ for p in K^{n-1}. Suppose σ^n or S^n is subdivided into cells σ_i^n, and f is a normal map of it into S_0^n. Then the $d_f(\sigma_i^n)$ are defined, and

$$d_f(\sigma^n) \text{ or } d_f(S^n) = \sum_i d_f(\sigma_i^n). \tag{3.1}$$

To show this, subdivide σ^n or S^n further, so that we can deform f into a simplicial map, and apply Ω_1 (see II, §3). The above quantities are unchanged, and (3.1) is now a consequence of II, (3.1).

4. **On deformations.** We shall need the following elementary results. Let $K \times I$ be the product of K and the unit interval I, consisting of all pairs (p, t), p in K, $0 \leqq t \leqq 1$. The deformation $\phi_t(p)$ of K in S_0^n is equivalent to the map $\Phi(p, t) = \phi_t(p)$ of $K \times I$ into S_0^n. Hence ϕ_0 is homotopic to ϕ_1 if and only if Φ, defined over $K \times 0 + K \times 1$, may be extended over $K \times I$.

Let f map the boundary $\partial\sigma^r$ of σ^r into S_0^n. Then f is homotopic to zero (in $\partial\sigma^r$) if and only if it may be extended through σ^r. For the deformation $f_t(p)$ (p in $\partial\sigma^r$) into $f_1(p) \equiv P$ is equivalent to the map $f(p_{1-t})$ (see II, §5) $= f_t(p)$ of σ^r into S_0^n.

LEMMA 1. *If $\phi \equiv \phi_0$ maps σ^n into S, and the deformation ϕ_t of ϕ is defined over $\partial\sigma^n$, then its definition may be extended over σ^n.*

We define ϕ_t in σ^n by

$$\phi_t(p_u) = \begin{cases} \phi(p_{(1+t)u}) & \left(0 \leqq t \leqq \dfrac{1}{u} - 1\right), \\ \phi_{t+1-\frac{1}{u}}(p) & \left(\dfrac{1}{u} - 1 \leqq t \leqq 1\right). \end{cases} \tag{4.1}$$

[6] This is the character group of the homology group with numbers mod 1 as coefficient group.

[7] See pp. 46–50 of this volume of this Journal; we refer to this paper as II.

LEMMA 2. *Any map ϕ of K into S_0^n may be deformed into a normal one; all cells already at P_0 we may keep fixed.*

We deform the map successively so that $K^0, K^1, \cdots, K^{n-1}$ are at P_0. Suppose K^{r-1} is at P_0 (if $0 < r < n$). As each $\partial\sigma^r$ is at P_0, we may deform each σ^r into P_0, keeping $\partial\sigma^r$ at P_0 (see §3). This deformation, defined over K^r, is extended over all $(r + 1)$-cells, $(r + 2)$-cells, etc., by Lemma 1. It is now defined over K, and K^r is at P_0.

5. **Parts of cocycles.** Let K' be a subcomplex of K. Any r-chain C of K may be written $C' + C''$, the coefficients of cells of $K - K'$[8] [of K'] being zero in C' [in C'']. We say C' is *part* of C. Clearly the chain C' in K' is part of a cocycle if and only if $\delta C'$ cobounds in $K - K'$, i.e., if and only if for some chain C'' in $K - K'$, $\delta C' = \delta C''$. The $(r + 1)$-chains are chains of K.

6. **The product $K \times I$.** We subdivide $K \times I$ (see §4) by means of all cells $\sigma_i^r \times I$ (σ_i^r in K). Orient the cells $\sigma_i^r \times 0$ and $\sigma_i^r \times 1$ like the σ_i^r, and orient each $(r + 1)$-cell $\sigma_i^r \times I$ so that $\sigma_i^r \times 1$ is on its boundary positively. Then

$$(6.1) \qquad \delta(\sigma_i^r \times 0) = -\sigma_i^r \times I + \cdots, \qquad \delta(\sigma_i^r \times 1) = \sigma_i^r \times I + \cdots,$$

$$(6.2) \qquad \delta(\sigma_i^r \times I) = -\sum_j \partial_{ij}^{r+1}(\sigma_j^{r+1} \times I).$$

To prove (6.2), say $\delta(\sigma_i^r \times I) = A_{ij}^{r+1}(\sigma_j^{r+1} \times I) + \cdots$. Then

$$\delta\delta(\sigma_i^r \times 1) = \delta[(\sigma_i^r \times I) + \sum_j \partial_{ij}^{r+1}(\sigma_j^{r+1} \times 1)]$$
$$= (A_{ij}^{r+1} + \partial_{ij}^{r+1})(\sigma_j^{r+1} \times I) + \cdots = 0,$$

and $A_{ij}^{r+1} = -\partial_{ij}^{r+1}$. The first equation in (6.1) is clear for $r = 0$; it is proved in succession for $r = 1, 2, \cdots$ by considering the coefficient of $\sigma_i^r \times I$ in $\delta\delta(\sigma_i^{r-1} \times 0)$.

THEOREM 1. *Let C_0 and C_1 be n-chains in $K = K^n$, and let D_0 and D_1 be the corresponding chains in $K \times 0$ and $K \times 1$. Then $D_0 + D_1$ (as a chain in $K \times I$) is part of a cocycle if and only if $C_0 \backsim C_1$ in K.*

Say

$$C_0 = \sum a_i\sigma_i^n, \qquad C_1 = \sum b_i\sigma_i^n.$$

Consider any n-chain

$$(6.3) \qquad D = D_0 + D_1 + \sum h_j(\sigma_j^{n-1} \times I);$$

then, by (6.1) and (6.2),

$$\delta D = -\sum a_i(\sigma_i^n \times I) + \sum b_i(\sigma_i^n \times I) - \sum h_j\partial_{ji}^n(\sigma_i^n \times I)$$

$$(6.4) \qquad = \sum_i [b_i - a_i - \sum_j h_j\,\partial_{ji}^n](\sigma_i^n \times I).$$

[8] $K - K'$ is in general not a subcomplex of K, i.e., is not closed in K.

Suppose $D_0 + D_1$ is part of a cocycle D; then (6.4) set $= 0$ gives

$$\delta\Big(\sum_j h_j \sigma_j^{n-1}\Big) = \sum_{i,j} h_j \partial_{ji}^n \sigma_i^n = \sum_i (b_i - a_i)\sigma_i^n = C_1 - C_0,$$

and $C_0 \backsim C_1$. Conversely, suppose $C_1 - C_0 = \delta(\sum h_j \sigma_j^{n-1})$; then the last set of equations shows that the bracket in (6.4) vanishes, and hence D, defined by (6.3), is a cocycle.

II. The theorems

7. The extension theorem. We shall prove

THEOREM 2. *Let f be a normal map of the subcomplex K' of $K = K^{n+1}$ into S_0^n. Then f can be extended over K if and only if the chain*

(7.1) $$D' = \sum_{\sigma_i^n \text{ in } K'} d_f(\sigma_i^n)\sigma_i^n$$

in K' is part of a cocycle.

First suppose D' is part of a cocycle $D = \sum a_i \sigma_i^n$:

(7.2) $$a_i = d_f(\sigma_i^n) \qquad (\sigma_i^n \text{ in } K'), \qquad \sum_i a_i \partial_{ij}^{n+1} = 0 \ \ (\text{all } j).$$

f maps $(K')^{n-1}$ into P_0; set $f(p) \equiv P_0$ in K^{n-1}. Let f map each σ_i^n not in K' into S_0^n with the degree a_i (see II, Theorem 2); then (7.2) holds for all σ_i^n. Consider any $(n+1)$-cell σ_j^{n+1} of $K - K'$. Using (3.1), we find

(7.3) $$\begin{aligned} d_f(\partial\sigma_j^{n+1}) &= d_f\Big(\sum_i \partial_{ij}^{n+1}\sigma_i^n\Big) = \sum_i \partial_{ij}^{n+1} d_f(\sigma_i^n) \\ &= \sum_i \partial_{ij}^{n+1} a_i = 0. \end{aligned}$$

Hence f, considered only in $\partial\sigma_j^{n+1}$, is homotopic to zero (II, Theorem 1), and f may be extended over σ_j^{n+1} (see §4). Thus we extend f throughout K.

Now suppose f is extended throughout K. By Lemma 2, we deform f into a normal map, leaving $(K')^{n-1}$, and hence also K', fixed. Call the new map f again, and define the a_i and D by (7.2). Then D' is part of D. By §4, f, in each $\partial\sigma_j^{n+1}$, is homotopic to zero; hence (7.3) holds, and D is a cocycle.

Remark. If f is any map of K' into S_0^n, we may deform it into a normal map ϕ, by Lemma 2. From Lemma 1, it is apparent that f can be extended over K if and only if ϕ can be. Define D' by (7.1). By Theorem 2, $\delta D'$ has zero coefficients over cells of K', and is therefore a chain, which is clearly a cocycle, of $K'' = K - K'$. By Theorem 3, Remark, if f is also deformed into the normal map ψ, defining the chain C' of K', then $C' \backsim D'$ in K', and hence for some H in K',

$$C' - D' = (\delta H)' = \delta H - (\delta H)''.$$

Therefore $\delta C' - \delta D' = \delta[(\delta H)'']$, which lies in K''. Thus the cohomology class in K'' of $\delta D'$ is uniquely determined by f, and we have (using Theorem 2): *f may be extended over K if and only if its cohomology class thus defined in K'' is $\backsim 0$ in K''.*

8. **The classes of maps of K^n into S_0^n.** If we put two maps of K^n into S_0^n into the same class if they are homotopic, the maps fall into classes, the *homotopy classes.* To any normal map f of K^n into S_0^n we let correspond a chain C_f as in (7.1).

THEOREM 3. *The normal maps ϕ and ψ of $K = K^n$ into S_0^n are homotopic if and only if $C_\phi \backsim C_\psi$.*

Set $\Phi(p \times 0) = \phi(p), \Phi(p \times 1) = \psi(p)$; then ϕ is homotopic to ψ if and only if Φ may be extended through $K \times I$ (see §4). If D_0 and D_1 correspond to C_ϕ and C_ψ in $K \times 0$ and $K \times 1$, Theorem 2 shows that this is possible if and only if $D' = D_0 + D_1$ is part of a cocycle in $K \times I$. By Theorem 1, this is true if and only if $C_\phi \backsim C_\psi$.

Remark. If K is of any dimension and ϕ and ψ are homotopic, then C_ϕ and C_ψ are cocycles and $C_\phi \backsim C_\psi$. The first statement follows from Theorem 2; the second follows on considering ϕ and ψ in K^n alone.

THEOREM 4. *The classes of maps of K^n into S_0^n are in $(1 - 1)$ correspondence with the elements of the n-th cohomology group of K with integer coefficients. The correspondence is given by deforming the map f into a normal one and taking the cohomology class of the resulting cocycle. In particular, f is homotopic to zero if and only if the corresponding cohomology class is zero.*

The deformation is possible, by Lemma 2. The cohomology class is uniquely determined by f, and non-homotopic maps determine different classes, by Theorem 3. Finally, to each cohomology class corresponds a map; we take a cocycle C of the class, and let f map each σ^n normally into S_0^n with the degree equal to its coefficient in C (see II, Theorem 2).

9. **The Theorem of Hurewicz.** Let Q_0 be a fixed point of a space S. Then the classes of maps of S_0^r into S for which P_0 goes into Q_0 form an abelian group, the *r-th homotopy group* of S.[9] If f maps σ^n [or S_0^n] into S, and $f(p) \equiv Q_0$ in $\partial\sigma^n[f(P_0) = Q_0]$, we may call the corresponding homotopy element the *degree* $d_f(\sigma^n)[d_f(S_0^n)]$ of f. (If $S = S_0^n$, the n-th homotopy group is the group of integers, as was seen in II, so that this is a natural generalization of the term degree.) The fundamental formula (3.1) holds still. The theorems of the preceding paper become matters of definition. The proofs in the present paper hold without change, and we have a new version of the Theorem of Hurewicz:

THEOREM 5. *Theorems 2, 3 and 4 hold if we replace S_0^n by any locally contractible space S_0 whose r-th homotopy groups vanish for $r < n$, and replace the integers by the n-th homotopy group of S as coefficient group in the chains and cohomology classes.*

Hurewicz also shows that in the above space S_0 the n-th homotopy group is the same as the n-th homology group with integer coefficients.

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[9] See Hurewicz, loc. cit. We assume a knowledge of the fundamental properties of homotopy groups.