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GEOMETRIC TOPOLOGY: MANIFOLDS AND STRUCTURES

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The term " geometric topology " has gradually been gaining currency in the last few years: you may wonder what the subject is all about. The object of this talk is to explain just that: to introduce the concepts involved and the main problems, and to discuss some of the most important results that have been obtained up to now.

The most basic concept in geometry is that of euclidean space, and the main branches of geometry involve the study of the various structures which it carries: linear, algebraic, differentiable, topological, etc. Many types of structure are defined by pseudogroups.

A *pseudogroup* Φ on E is a category whose objects are the open subsets of E, and whose morphisms must be continuous, invertible in Φ, and locally defined. Thus if $G\Phi$ is the set of all germs (at all points) of morphisms of Φ, and $\phi : U \to V$ is a homeomorphism whose germ at each point of U belongs to $G\Phi$, then $\phi \in \Phi$.

Φ is transitive if for all $x, y \in E$ there is a germ in $G\Phi$ with source x and target y.

The most important examples of pseudogroups are:

C^r: ϕ and ϕ^{-1} must be of class C^r. As special cases we have C^0 (the largest pseudogroup), C^∞ and C^ω, where C^ω denotes real analytic. In the complex case we have the pseudogroup C^Ω of complex analytic maps.

Lip, maps satisfying a local Lipschitz condition.

Maps preserving Lebesgue measure, or just orientation.

Nash, ϕ (and ϕ^{-1}) is an algebraic map, which is also C^ω.

Affine maps, or piecewise affine (usually called piecewise linear, or *PL*) maps: here the pieces come from a locally finite partition of U into polyhedra.

Trivial, identity maps only (the smallest pseudogroup) or translations (the smallest transitive one).

For any (closed) subgroup G of $GL(E)$, consider C^r (for some $r \geqslant 1$) maps whose derivative at each point is in G; interesting cases are the symplectic and orthogonal groups, orthogonal similitudes (giving conformal structure), maps preserving a subspace (giving foliations) or—in the case E is Hilbert space—invertible maps of the form I plus a compact operator, giving Fredholm structures.

Foliations lead to a wide variety of pseudogroups. Suppose E, F are Euclidean spaces, Φ a pseudogroup on $E \times F$ and Ψ a pseudogroup on F. Then $\mathscr{F}(\Phi, \Psi)$ is the pseudogroup on $E \times F$ of maps whose germs ϕ belong to a commutative diagram

$$\begin{array}{ccccc} E \times F & \supset & U & \xrightarrow{\phi} & E \times F \\ \downarrow p & & \downarrow & & \downarrow p \\ F & \supset & p(U) & \xrightarrow{\psi} & F \end{array}$$

with $\phi \in \Phi$, $\psi \in \Psi$. One can further specify a pseudogroup X on E, and require the restriction of ϕ to each leaflet $U \cap (E \times x)$ to belong to X.

We now come to manifolds. Let M be a topological space, E a euclidean space. A *chart* on M with model E is a pair (U, ϕ) where U is open in M, $\phi: U \to E$ an embedding with $\phi(U)$ open. An *atlas* is a collection $\{(U_\alpha, \phi_\alpha)\}$ of charts with $\cup U_\alpha = M$: if M has such an atlas, it is called a manifold modelled on E. Usually one requires also that M is Hausdorff and paracompact. If two charts (U_α, ϕ_α) and (U_β, ϕ_β) overlap, we have a coordinate transformation

$$g_{\alpha\beta}: \phi_\alpha(U_\alpha \cap U_\beta) \xrightarrow{\phi_\alpha^{-1}} U_\alpha \cap U_\beta \xrightarrow{\phi_\beta} \phi_\beta(U_\alpha \cap U_\beta).$$

If Φ is a pseudogroup on E, an atlas $\{(U_\alpha, \phi_\alpha)\}$ on M is a Φ-*atlas* if each coordinate transformation $g_{\alpha\beta}$ is in Φ. Two Φ-atlases A, A' are *compatible* if $A \cup A'$ is a Φ-atlas. The union of all Φ-atlases compatible with a given one, A, is still a Φ-atlas: clearly a maximal one. A maximal Φ-atlas on M is called a Φ-*structure*: thus each Φ-atlas defines a unique Φ-structure.

For examples, we have smooth (C^∞) structure, real or complex analytic structure, orientation, flat structure (take Φ = affine maps), PL-structure, immersion in E (take Φ = identity maps), and foliations of various kinds.

Having defined structures on manifolds, we must say what we mean by structures on morphisms (i. e. maps) of manifolds. The corresponding notion (less standard) is as follows. Given pseudogroups Φ on E, Ψ on F a *morphism* $\Omega: \Phi \to \Psi$ is a locally defined family of continuous maps from open sets in E to open sets in F, which is closed under composition on the right with maps in Φ and on the left with maps in Ψ: thus $\Psi \circ \Omega \circ \Phi \subset \Omega$. This notion seems more fundamental than that of pseudogroup; note also that in nearly all examples above of pseudogroups we first chose an Ω with $\Omega \circ \Omega \subset \Omega$ and then considered the invertible morphisms of Ω.

Examples are easy to supply, for example C^r (non-invertible) maps define a morphism $C^s \to C^t$ whenever $r \leqslant s$, $t \leqslant \omega$. So do C^r-immersions (note that embeddings are not locally defined), or more generally maps whose jacobians everywhere have rank $\geqslant k$. Another good example is provided by piecewise smooth maps: $PL \to C^\infty$; here again we can restrict to immersions with jacobian of maximal rank everywhere it is defined.

If M has a Φ-structure, N a Ψ-structure, $f: M \to N$ is a continuous map and $\Omega: \Phi \to \Psi$, then we call f an Ω-*morphism* if for all charts (U, ϕ) of M, (V, ψ) of N with $f(U) \subset V$, the composite $\psi \circ f \circ \phi^{-1}: \phi(U) \to \psi(V)$ belongs to Ω. Again, it suffices to check this for each chart of a (non-maximal) Φ-atlas of M.

Not all structures are defined by atlases. For example, we may be given a Ψ-manifold F, and a morphism $\Omega: \Phi \to \Psi$; then an Ω-map $M \to F$ can be regarded as constituting a certain type of structure on the Φ-manifold M. Write $\mathscr{S}(M)$ for the set of such maps: in many cases this will be endowed with a natural topology, e. g. C^r (uniform convergence on compact sets: not the fine topology here).

More generally we may have a Φ-*bundle* B with fibre F: by definition this assigns functorially to each $\phi: U \to U'$ in Φ a map $B(\phi): U \times F \to U' \times F$ over ϕ. Using these to glue over charts defines a bundle $B(M)$ over any Φ-manifold M. Now a structure will be defined as a section of $B(M)$ satisfying local conditions which can be specified by assigning a Φ-*sheaf* $\mathcal{S}$ of sections of B. Here for each open U in E, $\mathcal{S}(U)$ is a collection of maps $U \to F$; $\mathcal{S}$ is locally defined, i. e. is a sheaf; and for $\phi \in \Phi$, $B(\phi)$ transforms the sections $U \to U \times F$ which are graphs of members of $\mathcal{S}(U)$ into graphs of members of $\mathcal{S}(U')$.

Note. — It is simpler axiomatically to define $\mathcal{S}$ and omit B, but this takes us too far away from the geometry. One should consider a Φ-bundle or Φ-sheaf as a bundle or sheaf over E, endowed with the Grothendieck topology induced by Φ.

The most obvious example of Φ-bundle is the tangent bundle. This also has analogues in the topological and PL cases which originated with Milnor's work on microbundles. We also have the associated bundles of tensors (with, perhaps, symmetry conditions) in the traditional sense of differential geometry, the tangent bundles of higher order, and the bundle of connections: note particularly the classical cases of the Riemann bundle, and the bundle of (tangent) p-forms. Another example is the bundle normal to the foliation, if Φ defines a foliation. Also for each of the vector bundles above we have the associated projective bundle and frame bundle, and more generally, Grassmann and Stiefel bundles.

The possible sheaves $\mathcal{S}$ have a wide variety. In each case we may consider all continuous, or (perhaps) all differentiable (of some class C^r) sections—holomorphic in the complex case: examples are Riemann metrics, tangent 1-forms and connections. More generally, we could restrict the local maps $U \to F$ to lie in some suitable preassigned class Ω. E. g. for vector bundles in the differentiable case, we can consider smooth sections transverse to the zero section. Indeed, some of the most fruitful illustrations come by imposing such conditions on derivatives: assuming sufficient differentiability, given a Φ-bundle B there is an extended bundle E^rB of r-jets of sections of B. Now for any sub-Φ-bundle E^r_0B of E^rB, we can consider those sections of B whose r-jets are sections of E^r_0B (equivalently, those sections of E^r_0B which come from B: the *integrable* ones in the usual terminology). As one concrete geometrical illustration, we can take B the Riemann bundle, $r = 1$ and consider metrics with everywhere positive (or everywhere negative) sectional curvatures.

I now consider the problem of existence and classification of structures of a given type on a given manifold: this is of course a global problem since Euclidean space possesses structures of all types. More generally, I am interested in when the existence of one type of structure implies the existence of another. For classification one needs a notion of equivalence: a general definition which seems to cover all cases of interest in geometric topology (though not in differential geometry) is this:

Two structures α, β of a given type on M are *concordant* if there is a structure γ of this type on $M \times I$ inducing α on $M \times 0$ and β on $M \times 1$.

Of course, this needs to be made explicit in each case, but it is usually obvious how to interpret the definition. A stronger relation is *isotopy*: here one demands a level-preserving homeomorphism F of $M \times I$ — i. e. $F(m, t) = (f_t(m), t)$ — with $f_0 =$ identity and $f_1^*\alpha = \beta$. Frequantly, F and F^{-1} are also supposed differentiable. In many cases (e. g. smooth or PL structures on topological manifolds of dimension $\geqslant 6$) concordance and isotopy are equivalent: but this is always a tricky technical question. The most interesting problem of this type, where a structure is a diffeomorphism of the smooth manifold M onto a fixed manifold M_0, has been studied by Cerf and Wagoner.

Now suppose we are comparing structures of two different types, say Φ and Φ'. We will suppose that a Φ-structure implies a Φ'-structure—this holds trivially, for example, if we have two pseudogroups $\Phi \subset \Phi'$, or we are considering appropriately restricted sections of two bundles B, B' with a morphism $B \to B'$. The simplest sort of result is that for any M (perhaps satisfying some side conditions), each Φ'-structure is induced by a Φ-structure, unique up to concordance. Some results of this kind, where $\Phi \subset \Phi'$ are pseudogroups, are

Whitney, 1936: for $1 \leqslant r \leqslant s \leqslant \omega$, comparing C^s and C^r,

Nash, 1952: the same, with C^s replaced by Nash,

Moise, 1952; Bing, 1959: comparing PL and C^0 in dimension 3;

the result is also known, due to work of many authors, comparing topological, differentiable and trivial structures in the infinite dimensional case (here, a trivial structure is an open immersion in Hilbert space). For references see, for example, the talks of Anderson and Kuiper at this congress.

When the above simple result does not apply, one looks for a theorem of the following kind, which I will describe as an *obstruction theory*: it reduces the problem to one in homotopy theory, concerning only continuous maps. Such a theorem specifies first a space X and a functor providing for each Φ-structure on M a continuous map $M \to X$, determined up to homotopy (typically, a structure of class C^1 on M^m gives, via the tangent bundle—which has structure group $GL_m \simeq O_m$—a classifying map $M \to BO_m$). Similarly for Φ' we have an X'. There should also be a map $X \to X'$, which we may suppose a fibration, such that for any Φ-structure on M and the induced Φ'-structure, the diagram

$$\begin{array}{ccc} & \nearrow & X \\ M & & \downarrow \\ & \searrow & X' \end{array}$$

commutes up to a (preferred) homotopy. The theorem will then say that (subject perhaps to some side conditions on M), given M with Φ'-structure, the equivalence classes of Φ-structures on M which induce it (or something equivalent—but usually we can hit the structure on the nose) correspond bijectively to homotopy classes of lifts $M \to X$ of the given map $M \to X'$.

Such a theorem has some applications by its very nature—for example, take M contractible. But for effective work, information on the spaces X and X' is essential, and to obtain such information is often a central problem in geometric topology. Some such theorems are as follows:

Smoothing theory (due to the work of many people) gives an obstruction theory to imposing C^r structures ($r \geqslant 1$) on PL-manifolds. This is technically difficult since $C^r \not\subset PL$: instead one needs the result of Whitehead, 1940. The corresponding spaces here are usually denoted by $BO \to BPL$; the former has been familiar for many years, some striking results on the latter were obtained by Sullivan, 1970. Next we have the results of Kirby and Siebenmann, 1969 on imposing PL structures on topological manifolds of dimension $\geqslant 5$. Here the only obstruction to existence of a PL-structure on M is a cohomology class in $H^4(M; \mathbb{Z}_2)$. See also Eells' talk at this congress for an account of Fredholm structures.

Recent results of Haefliger, 1970 (see also below) have provided obstruction theories for existence of foliations and of complex analytic structures: indeed I think that a result is obtained for any pseudogroup Φ. A side condition is needed: that M is open (i. e. has no compact unbounded component). Nothing is known about the obstruction groups except the results of Bott which he discussed yesterday.

For structures of the second type, most known results are subsumed in the following theorem of M. L. Gromov, 1969. Take $\Phi = C^\infty$, let B be a differentiable Φ-bundle, E^rB the bundle of r-jets of sections of B (as above) and E^r_0B an *open* subbundle of E^rB. Let $\mathscr{S}(M)$ be the space of sections of $B(M)$ whose r-jets map into $E^r_0B(M)$; $T(M)$ the space of sections of $E^r_0B(M)$—thus taking r-jets defines a map $j^r\colon \mathscr{S}(M) \to T(M)$. Give $T(M)$ the compact-open topology, and topologise $\mathscr{S}(M)$ as a subspace of it.

THEOREM — *If M is open, $j^r\colon \mathscr{S}(M) \to T(M)$ is a weak homotopy equivalence.*

The proof is an improvement of that of the Smale-Hirsch, 1959 classification of immersions, and is not unduly difficult.

Many examples of applications were mentioned in the talk of Gromov at the congress. The immersion case is when $B(M)$ is a trivial bundle $M \times F$; a point of $E^1B(M)$ can be identified with a linear map of a tangent space of M to one of F, so a section of $E^1B(M)$ can be identified as a map of tangent bundles $TM \to TF$, and we let $E^1_0B(M)$ be the injective linear maps. Results corresponding to this case can now also be formulated in the PL and topological cases, and proved in the same manner—the difficult step was an isotopy extension theorem. See Haefliger and Poenaru, 1966 and Lees, 1969.

Other suitable $E^1_0B(M)$, for the same B, are maps of rank $\geqslant k$ (some fixed k)—previously treated by Sidnie Feit, 1968—and maps whose projection on the normal bundle of a prescribed foliation of F is surjective—this case was discovered independently by Phillips.

It is clearly of great interest to determine in particular cases whether or not the result is valid also for closed manifolds. For immersions $M \to F$ this is well-known to be the case provided $\dim M < \dim F$. Mrs Feit's result allows M closed if $k < \dim F$. A recent result of Feldman allows M to be a circle, considering curves immersed in the Riemannian manifold F with everywhere nonzero geodesic curvature, provided $\dim F \geqslant 3$. The underlying condition seems to be that F has at least one dimension " to spare ". Note that no advantage is gained by removing a point from M, applying the result, and attempting to reinsert the point: consider submersions $M \to \mathbb{R}$.

The classification of immersions can be made the basis of a proof of many of the theorems cited above. Put rather too crudely, the idea is this: if $\dim M = \dim V$, we have an immersion $M \to V$, and V carries a Φ-structure, then one is induced on M by using the immersion to pull charts on V back to M. There are two ways to make this the basis of a proof. One is to take $V = E$ and work by induction on coordinate charts of M. This method, which needs a special argument if M is closed, was explained in Lashof's talk at the congress. The other is to use the theorem in the case $\dim M < \dim V$, which leads (rather easily) to obtaining a Φ-structure on $M \times \mathbb{R}^q$ for some q, and then use a stability theorem of the type: a Φ-structure on $M \times \mathbb{R}$ is concordant to the product of a Φ-structure on M and the natural one on $\mathbb{R}$. The product theorem for comparison of C^r and PL structures is due to Cairns, 1961 and

A subtler use of Gromov's result to obtain structure theorems was made by Haefliger, 1970. His idea is to contemplate bundles over any space X with fibre E with (roughly) a Φ-structure on each fibre and a " foliation " transverse to the fibres. By a general argument (Ed. Brown's representability theorem), he obtains a classifying space for such structures on X. If now X is a manifold modelled on E, and the bundle is equivalent to the tangent bundle of X, the theorem implies the existence of a section transverse to the foliation. The local projections of the section on the fibres now induce a Φ-structure on M.

I will conclude with an example which does not quite fit into the above framework. Instead of beginning with a topological space which is locally euclidean, start with a space which is only prescribed up to homotopy type. To substitute for the local condition, I insist that a strong form of the Poincaré duality theorem holds. The most interesting question here is whether the prescribed homotopy type contains a manifold. The simplest result concerns the relative case when we have a pair (Y, X) satisfying Lefschetz duality. Suppose also that X, Y are connected and that the inclusion map $X \to Y$ induces an isomorphism of fundamental groups. Then, in dimensions $\geqslant 6$, there is an obstruction theory for existence of a corresponding manifold.

As with Gromov's theorem one can define (semi-simplicially) spaces $\mathscr{S}(Y)$ and $T(Y)$ and generalise this theory to obtain a homotopy equivalence $\mathscr{S}(Y) \to T(Y)$. If the corresponding map is considered now in the case when Y satisfies Poincaré (not Lefschetz) duality, it turns out that the homotopy type of the mapping fibre $\mathscr{L}(Y)$ depends only on $\pi_1(Y)$ and on dim Y (mod 4)—provided this dimension $\geqslant 5$. Although explicit calculation is difficult, the spaces $\mathscr{L}(Y)$ are gradually being determined, and I have learnt several new results at this congress. For details of what is known, see my forthcoming book, Wall, 1970.

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