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1. Introduction

This paper announces certain results concerning closed (s-1)-connected (2s+1)manifolds P, where s = 3 or 7. (Here a "manifold" is a compact oriented differential manifold). The manifolds P spilt by connected sum whenever there is a splitting by direct sum of certain invariants $(G, b, \hat{\beta})$ and in this way each manifold P can be expressed as a connected sum of "indecomposable" manifolds. Up to the addition of homotopy spheres the indecomposable manifolds P are then either classified by the invariants $(G, b, \hat{\beta})$ or there exist two distinct manifolds corresponding to a particular value of the invariants. The approach to these results is by the fact that the manifolds P occur as boundaries of handlebodies L of $\mathscr{H}(s+1)$.

Closed (s-1)-connected (2s+1)-manifolds have been classified, up to the addition of homotopy spheres, by Wall [7] excluding the cases when s = 1, 2, 3, 7. The case s = 2 has been covered by Barden [1].

I would like to express my gratitude to Professor C. T. C. Wall for suggesting this problem to me and also for his advice and encouragement during this research.

2. The invariants

Let P be a closed (s-1)-connected (2s+1)-manifold, where s = 3 or 7. The non-zero homology groups of P are $H_0(P) \cong \mathbb{Z}$, $H_s(P)$, $H_{s+1}(P)$, $H_{2s+1}(P) \cong \mathbb{Z}$ where by duality $H_{s+1}(P)$ is free abelian of the same torsion-free rank as $H_s(P)$. The cohomology groups are given by Poincaré duality and we write $G = H_s(P) \cong H^{s+1}(P)$.

 G^* denotes the torsion subgroup of G and the linking invariant is a non-singular symmetric bilinear map

$$b: G^* \times G^* \to S$$

where $S = \mathbb{Q}/\mathbb{Z}$, and which is defined in [1] and [7].

P is s-parallelisable, i.e. the tangent bundle restricted to the s-skeleton is trivial, since $\pi_{s-1}(SO) \cong 0$. $\pi_s(SO) \cong \mathbb{Z}$ and so the obstruction to triviality over the (s+1)-skeleton is a well defined element

$$\hat{\beta} \in H^{s+1}(P; \pi_s(SO)) \cong H^{s+1}(P).$$

For convenience we may regard $\hat{\beta}$ as an element of G by Poincaré duality. Since $\pi_{2s}(SO) \cong 0$ there is no further obstruction to triviality of the stable tangent bundle and so in fact $\hat{\beta}$ is the obstruction to stable parallelisability. Lemma 1.1 of Kervaire [3] expresses the Pontrjagin classes of P in terms of $\hat{\beta}$ as follows

$$2\hat{\beta} = p_1(P)$$
 when $s = 3$, $6\hat{\beta} = p_2(P)$ when $s = 7$.

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The invariants taken for P are $(G, b, \hat{\beta})$; the notation we have used here being similar to that employed in [7].

From the definition of the invariants it easily follows that the invariants of the connected sum of two manifolds are the direct sum of the two sets of invariants; i.e. if P_1 and P_2 have invariants $(G_1, b_1, \hat{\beta}_1)$ and $(G_2, b_2, \hat{\beta}_2)$ then the connected sum $P_1 \neq P_2$ has invariants $(G_1 \oplus G_2, b, (\hat{\beta}_1, \hat{\beta}_2))$ where

$$b((x_1, x_2), (y_1, y_2)) = b_1(x_1, y_1) + b_2(x_2, y_2)$$

for $x_1, y_1 \in G_1^*, x_2, y_2 \in G_2^*$.

PROPOSITION. The invariants $(G, b, \hat{\beta})$ for the manifolds P can assume the following values:

G can be any finitely generated abelian group,

 $b: G^* \times G^* \rightarrow S$ can be any non-singular symmetric bilinear map, and

 $\hat{\beta} \in G$ can be any even element.

3. Decomposition and classification of the manifolds P

A non-singular symmetric bilinear map $b: G^* \times G^* \to S$ is *irreducible* if it cannot be expressed as the proper sum of two maps and we say that a manifold P with invariants $(G, b, \hat{\beta})$ is *indecomposable* if G is finite and b is irreducible or if $G \cong \mathbb{Z}$ (when of course b = 0).

THEOREM 1. Any manifold P is a connected sum of indecomposable manifolds.

In Theorem 4 of [8] Wall determines the irreducible maps b for which the possible finite groups G^* turn out to be $\mathbb{Z}_p k$, for a prime p, and $\mathbb{Z}_2 k \oplus \mathbb{Z}_2 k$. Thus for the indecomposable manifolds $G \cong \mathbb{Z}$, $\mathbb{Z}_p k$, or $\mathbb{Z}_2 k \oplus \mathbb{Z}_2 k$.

It is clear that our invariants $(G, b, \hat{\beta})$ cannot distinguish between homotopy spheres, when we have $G \cong 0$, and so any classification result will be modulo the addition of homotopy spheres. For the statements of Theorems 2 and 3 below we make the following convention. We say that a collection of manifolds P are classified by the invariants $(G, b, \hat{\beta})$ if for two manifolds P_1 , P_2 with invariants $(G_1, b_1, \hat{\beta}_1)$, $(G_2, b_2, \hat{\beta}_2)$ whenever there exists an isomorphism $h: G_1 \to G_2$ preserving the linking invariants and sending $\hat{\beta}_1$ onto $\hat{\beta}_2$ then there exists an orientation preserving diffeomorphism $f: P_1 \to P_2 \notin \Sigma$ for some homotopy sphere $\Sigma \in \Theta_{2s+1}$, such that the induced isomorphism $f_* = h$. We say that there are two distinct manifolds corresponding to a particular value $(G, b, \hat{\beta})$ of the invariants if there exist manifolds P_1 , P_2 each with invariants $(G, b, \hat{\beta})$ such that P_1 and $P_2 \notin \Sigma$ are not diffeomorphic, by an orientation preserving diffeomorphism, for any $\Sigma \in \Theta_{2s+1}$; moreover any other P with invariants $(G, b, \hat{\beta})$ is diffeomorphic, by an orientation preserving diffeomorphism, to either $P_1 \notin \Sigma$ or $P_2 \# \Sigma$ for some $\Sigma \in \Theta_{2s+1}$.

THEOREM 2. (i) The indecomposable manifolds P with $G \cong \mathbb{Z}$ or $\mathbb{Z}_p k$, for p an odd prime, are classified by the invariants $(G, b, \hat{\beta})$.

(ii) For the indecomposable manifolds P with $G \cong \mathbb{Z}_2 k$ or $\mathbb{Z}_2 k \oplus \mathbb{Z}_2 k$ there are two cases

- (a) If $|G| \leq 4$ then the manifolds with $\hat{\beta}$ not divisible by 4 are classified by the invariants $(G, b, \hat{\beta})$; there are two distinct manifolds for each value of $(G, b, \hat{\beta})$ if $\hat{\beta}$ is divisible by 4 (when $\hat{\beta} = 0$).
- (b) If |G| > 4 then the manifolds with β divisible by 4 are classified by the invariants (G, b, β); there are two distinct manifolds for each value of (G, b, β) if β is not divisible by 4.

Any manifold P, where G contains no elements of order 2, is a connected sum of indecomposable manifolds covered in part (i) of Theorem 2 and since the invariants add together under connected sum we have

THEOREM 3. The manifolds P, where G contains no elements of order 2, are classified by the invariants $(G, b, \hat{\beta})$.

4. The manifolds P as boundaries of handlebodies L

Elements $L \in \mathcal{H}(s+1)$ are manifolds which can be formed by adding a finite number of (s+1)-handles to the (2s+2)-disc. The foundation for the results above is

THEOREM 4. For s = 3 $P = \partial L$ for some $L \in \mathcal{H}(4)$. For s = 7 $P \# \Sigma = \partial L$ for some $L \in \mathcal{H}(8)$ and $\Sigma \in \Theta_{15}$.

This is a special case of a more general result. If P is a closed s-parallelisable (s-1)-connected (2s+1)-manifold, now for any value of s, then by the method of proof of Wall's main theorem in [6] P by surgeries of type (s+1, s+1) is χ -equivalent to an s-connected manifold which of necessity is a homotopy sphere. So for some

$$\Sigma \in \Theta_{2s+1} P \# \Sigma$$

is formed by surgeries of type (s+1, s+1) from S^{2s+1} . The surgery embeddings can now be used to attach (s+1)-handles to D^{2s+2} and so obtain $L \in \mathcal{H}(s+1)$ with $\partial L = P \neq \Sigma$. As mentioned before for s = 3, 7 the manifolds P are automatically s-parallelisable and since all homotopy 7-spheres occur as boundaries of elements of $\mathcal{H}(4)$ Theorem 4 follows.

The handlebodies $\mathscr{H}(s+1)$, for s=3 or 7, are classified in [5] by invariants (H, λ, α) where $H = H_{s+1}(L)$,

$$\lambda: H \times H \to \mathbb{Z}$$

is the symmetric bilinear map given by intersection numbers and

$$\hat{\alpha} \in H^{s+1}(L; \pi_s(SO)) \cong H^{s+1}(L)$$

is the obstruction to triviality of the tangent bundle over the (s+1)-skeleton, and in fact there being no further obstruction & is the obstruction to parallelisability of L. The invariants $(H, \lambda, \&)$ add together by direct sum corresponding to taking boundary connected sums of the manifolds L.

If $P = \partial L$ then the invariants $(H, \lambda, \hat{\alpha})$ of L determine $(G, b, \hat{\beta})$ of P as described below. Letting $\hat{H} = Hom(H, \mathbb{Z}) \lambda$ defines a homomorphism

$$\pi: H \to \widehat{H}$$

by $\pi(x)(y) = \lambda(x, y)$ and then

 $G \cong \operatorname{coker} \pi$.

From this it follows that P is a homotopy sphere if and only if π is an isomorphism i.e. if and only if λ is unimodular. H' is the subgroup of \hat{H} consisting of those elements with some multiple in $Im \pi$ and a symmetric bilinear map

$$\lambda': H' \times H' \to \mathbb{Q}$$

is defined by $\lambda'(c, d) = 1/mn \lambda(a, b)$ where $\pi(a) = mc$, $\pi(b) = nd$. Elements $\{c\}, \{d\}$ of G^* in $G \cong$ coker π are represented by elements $c, d, \in H'$ and b is then given by

 $b(\{c\}, \{d\}) = \lambda'(c, d) \mod 1.$

If $i : P \subset L$ is the inclusion map then the induced homomorphism

$$i^*: H^{s+1}(L) \to H^{s+1}(P)$$

sends & onto $\hat{\beta}$.

In the situation above we say λ induces (G, b) and if λ is non-singular, when $G = G^*$, λ induces b. Examples of λ inducing the irreducible b are given in §7 of [8]. Two pairs (G, b) are equivalent if there is an isomorphism between the groups which preserves the bilinear maps and similarly for two maps λ to be equivalent. Two maps λ are stably equivalent if they are equivalent up to the addition of unimodular maps. The following result was postulated by Kneser and Puppe [4] in the case where λ is nonsingular and proved in the special case where |G| is odd.

LEMMA. Two maps λ are stably equivalent if and only if the induced pairs (G, b) are equivalent.

The corresponding result to this for quadratic forms is proved by Wall [9] and using a *p*-adic approach both these results are also proved by Durfee [2].

Since the manifolds L with unimodular λ have homotopy spheres as boundaries the Lemma together with Theorem 4 gives

THEOREM 5. If $\lambda : H \times H \to \mathbb{Z}$ induces (G, b) the homology invariants of a manifold P then $P \# \Sigma = \partial L$ for some $\Sigma \in \Theta_{2s+1}$ and some $L \in \mathscr{H}(s+1)$ with homology invariants (H, λ) .

COROLLARY. For s = 3 $P = \partial L$ for $L \in \mathcal{H}(4)$ and for s = 7 $P \# \Sigma = \partial L$ for $L \in \mathcal{H}(8)$ and $\Sigma \in \Theta_{15}$ where in each case the Stiefel–Whitney class $w_{s+1}(L) = 0$.

The Corollary follows from the Theorem since by §7 of [8] any b is induced by an even λ and $w_{s+1}(L) = 0$ exactly when α is even and by [5] and this occurs if and only if λ is even; also any homotopy 7-sphere is the boundary of a parallelisable manifold of $\mathscr{H}(4)$.

Theorem 1 follows from Theorem 5 since by a suitable choice of λ the manifold L will split by boundary connected sum giving a corresponding splitting of P by connected sum.

The Proposition giving possible values of $(G, b, \hat{\beta})$ is proved by exhibiting suitable invariants $(H, \lambda, \hat{\alpha})$ for the manifolds L. That $\hat{\beta}$ is of necessity even follows since $w_{s+1}(P) = 0$ which in turn is a consequence of Wu's formula.

For a particular value of the invariants $(G, b, \hat{\beta})$ and a fixed λ inducing (G, b) any manifold P with these invariants is, up to the addition of a homotopy sphere, the boundary of a manifold L with homology invariants (H, λ) by Theorem 5. The possible values of the tangential invariant $\hat{\alpha}$ so that the manifold with invariants $(H, \lambda, \hat{\alpha})$ has boundary with invariants $(G, b, \hat{\beta})$ gives a collection of manifolds L. Determining whether these manifolds are all diffeomorphic up to the addition by boundary connected sum of elements of $\mathscr{H}(s+1)$ whose boundaries are homotopy spheres or whether the manifolds in fact split into two classes, together with Theorem 6 below, gives the results of Theorem 2.

THEOREM 6. If L_1 , $L_2 \in \mathcal{H}(s+1)$ with $P_1 = \partial L_1$, $P_2 = \partial L_2$ and there exists an orientation preserving diffeomorphism $f: P_1 \to P_2$ then there exist manifolds

$$V_1, V_2 \in \mathscr{H}(s+1)$$

with $\partial V_1 = \partial V_2 = S^{2s+1}$ and an orientation preserving diffeomorphism

 $g: L_1 + V_1 \to L_2 + V_2$ with $g|_{P_1} = f$.

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