ON THE INERTIA GROUPS OF CERTAIN MANIFOLDS

DAVID L. WILKENS

1. *Introduction*

For a closed compact smooth m-manifold M, $m \geqslant 5$, the inertia group $I(M) = \{\Sigma \in \Theta_m;$ there is an orientation-preserving diffeomorphism between M and $M \# \Sigma\}$, where $\#$ denotes the connected sum and Θ_m is the group of homotopy m-spheres. Where the inertia groups of specific manifolds have been determined, the results tend to fall into two extremes. Either necessary conditions for $\Sigma \in I(M)$ are strong enough to show that $I(M)$ is trivial (for example Schultz [7] shows that $I(M) \simeq 0$ when M is a product of ordinary spheres) or sufficient conditions for $\Sigma \in I(M)$ are weak enough to show $I(M) = \Theta_m$ or $\Theta_m(\partial\pi)$ (for example Kawakubo [3] shows that $I(S^3 \times \Sigma^{10}) = \Theta_{13}$, where Σ^{10} is a generator of the 3-component of $\Theta_{10} \simeq Z_2 \oplus Z_3$). The purpose of this paper is to determine the inertia groups of a class of manifolds and so give examples where $I(M)$ falls between 0 and Θ_m.

We consider closed $(m-1)$-connected $(2m+1)$-manifolds P, where $m = 3$ or 7, and in Theorem 1 determine the inertia groups $I(P)$, the results depending on a tangential invariant $\hat{\beta} \in H^{m+1}(P)$. These results do not apply to the case where $H^{m+1}(P)$ and Θ_{2m+1} contain elements of the same order and here $I(P)$ seems to depend on the divisibility of multiples of $\hat{\beta}$. As a contribution to this case we determine the inertia groups of certain of these manifolds in §6 and at the same time show that in general the inertia group of a connected sum is not the sum of the respective inertia groups. The methods employed in this paper apply to other $(m-1)$-connected $(2m+1)$-manifolds where $m \neq 3$ or 7, especially where $m = 3$ (mod 4), but we restrict ourselves to $m = 3$ or 7 since the results here are more precise.

2. *Invariants and Theorem* 1

Let P be a closed $(m-1)$-connected $(2m+1)$-manifold where $m = 3$ or 7. The non-zero homology and cohomology groups of P occur in dimensions 0, m, $m+1$, $2m+1$ and since $\pi_{m-1}(SO) \simeq 0$ it follows that P is m-parallelisable. $\pi_m(SO) \simeq Z$ and the obstruction to triviality of the tangent bundle over the $(m+1)$-skeleton is a well defined element

$$\hat{\beta} \in H^{m+1}(P; \pi_m(SO)) \simeq H^{m+1}(P).$$

Since $\pi_{2m}(SO) \simeq 0$ there is no further obstruction to triviality of the stable tangent bundle and so $\hat{\beta}$ is the obstruction to stable parallelisability of P. For G a finitely generated Abelian group and $\hat{\beta}$ an even element of G there exists a manifold P with $H^{m+1}(P) \simeq G$ and tangential invariant $\hat{\beta}$ (see [10] or [11]). By Lemma 1.1 of Kervaire [4] the Pontrjagin classes of P in terms of $\hat{\beta}$ are given by

$$2\hat{\beta} = p_1(P) \quad \text{for} \quad m = 3, \qquad 6\hat{\beta} = p_2(P) \quad \text{for} \quad m = 7.$$

The dimensions of the manifolds we are considering are 7 and 15 and here the

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groups of homotopy spheres are $\Theta_7 = \Theta_7(\partial\pi) \simeq \mathbb{Z}_{28}$ and $\Theta_{15} \simeq \mathbb{Z}_{2^6.127} \oplus \mathbb{Z}_2$, $\Theta_{15}(\partial\pi) \simeq \mathbb{Z}_{2^6.127}$. The main result is

THEOREM 1. *Let P be a closed $(m-1)$-connected $(2m+1)$-manifold, $m = 3$ or 7.*

(i) *If $\hat{\beta}$ is of finite order then $I(P) \simeq 0$.*

(ii) *If $H^{m+1}(P) * \Theta_{2m+1} \simeq 0$ and r is the largest integer dividing $\hat{\beta}$ then $I(P)$ consists exactly of those elements of $\Theta_{2m+1}(\partial\pi)$ divisible by $r/4$.*

In (ii) $*$ denotes the torsion product; so the condition $H^{m+1}(P) * \Theta_{2m+1} \simeq 0$ means that $H^{m+1}(P)$ and Θ_{2m+1} have no elements of the same order, i.e. $H^4(P)$ does not contain elements of order 2 or 7 for $m = 3$ and $H^8(P)$ does not contain elements of order 2 or 127 for $m = 7$. Where this condition does not hold and $\hat{\beta}$ is of infinite order the situation is more complex and will be considered in §6.

It follows from part (ii) of the theorem that any subgroup of $\Theta_{2m+1}(\partial\pi)$ occurs as the inertia group $I(P)$ for some P and this is also true if we add the restriction $H^{m+1}(P) \simeq \mathbb{Z}$. By Theorem 3 of [**11**] the manifolds P with $H^{m+1}(P)$ torsion-free are classified up to the addition of homotopy spheres by invariants $(H^{m+1}(P), \hat{\beta})$. These are topological invariants, since by [**6**] rational Pontrjagin classes are topological invariants and these determine $\hat{\beta}$, and hence P admits exactly $|\Theta_{2m+1}/I(P)|$ differential structures. So for example we have

COROLLARY. *If r is any divisor of 28 for $m = 3$ or of $2^7.127$ and $r \geqslant 2$ for $m = 7$ then there exists a manifold P with $H^{m+1}(P) \simeq \mathbb{Z}$ which admits exactly r differential structures.*

3. *Proof of Theorem* 1

As part of the statement of Theorem 1 we have $I(P) \subset \Theta_{2m+1}(\partial\pi)$, which has significance only for $m = 7$ when $\Theta_{15}(\partial\pi)$ is smaller than Θ_{15}, and as a first step we establish that this is so.

$\mathscr{H}(n)$ denotes the collection of those handlebodies formed from the $2n$-disc by attaching n-handles. The boundary ∂L of an element $L \in \mathscr{H}(m+1)$ is a closed m-parallelisable $(m-1)$-connected $(2m+1)$-manifold P.

PROPOSITION 2. *If P is a closed m-parallelisable $(m-1)$-connected $(2m+1)$-manifold and $\Sigma \in I(P)$, then $\Sigma = \partial V$ for some $V \in \mathscr{H}(m+1)$.*

COROLLARY. *If P is a closed 6-connected 15-manifold then $I(P) \subset \Theta_{15}(\partial\pi)$.*

The corollary follows from the proposition since, as we have already remarked, a 6-connected 15-manifold is automatically 7-parallelisable and by Theorem 4 of [**8**] the elements of $\Theta_{15}(\partial\pi)$ are exactly those homotopy spheres which occur as boundaries of elements of $\mathscr{H}(8)$.

To prove the proposition we refer to [**9**], where it is shown that the homotopy group $\pi_m(P)$ of a closed m-parallelisable $(m-1)$-connected $(2m+1)$-manifold P can be killed by surgeries of type $(m+1, m+1)$. Thus P is χ-equivalent by surgeries of type $(m+1, m+1)$ to an m-connected manifold which of necessity must be a homotopy sphere $\Sigma \in \Theta_{2m+1}$. It follows that $P \# (-\Sigma)$ can be formed by surgeries of type $(m+1, m+1)$ from $S^{2m+1} = \Sigma \# (-\Sigma)$, where $-\Sigma$ denotes Σ with the opposite orientation.

Now since $I(P) = I(P \# \Sigma)$ for any $\Sigma \in \Theta_{2m+1}$, to prove the proposition we may assume without loss of generality that P is formed by surgeries of type $(m+1, m+1)$ from S^{2m+1}. So for any homotopy sphere Σ it will follow that $P \# \Sigma$ is formed by surgeries of type $(m+1, m+1)$ from Σ.

Suppose that $\Sigma \in I(P)$; then combining the surgeries above we have that Σ is formed by surgeries of type $(m+1, m+1)$ from $P \# \Sigma = P$, which in turn is formed by surgeries of type $(m+1, m+1)$ from S^{2m+1}. Thus Σ is formed by surgeries of type $(m+1, m+1)$ from S^{2m+1} and so by attaching $(m+1)$-handles to the disc D^{2m+2} to correspond to these surgeries we obtain $V \in \mathscr{H}(m+1)$ with $\Sigma = \partial V$.

4. *Necessary conditions for Theorem* 1

In this section the necessary conditions for $\Sigma \in I(P)$ given by Theorem 1 are established under the assumption, justified by §3, that $\Sigma \in \Theta_{2m+1}(\partial\pi)$. The methods employed here are inspired to some extent by Browder's paper [**1**].

LEMMA 3. *Let $L_1, L_2 \in \mathscr{H}(m+1)$ and $f: \partial L_1 \to \partial L_2$ be an orientation-preserving diffeomorphism. If $M = L_1 \cup_f (-L_2)$ then $\tau(M) = \tau(L_1) - \tau(L_2)$.*

Here $-L_2$ denotes L_2 with the opposite orientation and M is formed by gluing L_1 to L_2 by the diffeomorphism f. τ denotes the signature of a manifold which is the signature of the intersection matrix for m odd and is zero for m even. The lemma is easily proved by using the relationship between the various cohomology sequences.

Now take $m = 3$ or 7 and for $L \in \mathscr{H}(m+1)$ with $\partial L = P$ suppose $\Sigma \in I(P)$ when $\Sigma \in \Theta_{2m+1}(\partial\pi)$ and there exists an orientation-preserving diffeomorphism $f: P \to P \# \Sigma$. $\Sigma = \partial W$ for some parallelisable manifold $W \in \mathscr{H}(m+1)$ (Σ bounds a parallelisable manifold which by surgery can be made m-connected and so an element of $\mathscr{H}(m+1)$) and we define $M = (L+W) \cup_f (-L)$ by identifying $P \# \Sigma$ and P by f; $L+W$ denotes the boundary connected sum of L and W.

M is a closed m-connected $(2m+2)$-manifold and $\pi_m(SO) \simeq \mathbb{Z}$ so that the obstruction to a cross-section of the tangent bundle over the $(m+1)$-skeleton is a well-defined element

$$\gamma \in H^{m+1}(M; \pi_m(SO)) \simeq H^{m+1}(M).$$

PROPOSITION 4. *Let M be as above and $\gamma \in H^{m+1}(M)$; then*

$$\frac{\tau(W)}{8} = \frac{\tau(M)}{8} = \frac{\gamma^2}{8} \bmod \begin{cases} 28 & \text{for } m = 3 \\ 2^6.127 & \text{for } m = 7. \end{cases}$$

Proof. $\tau(L+W) = \tau(L) + \tau(W)$ and so, by Lemma 3, $\tau(M) = \tau(W)$. Hirzebruch's Index Theorem and the integrability of the $\hat{A}$-genus (see [**2**]) are now used to relate $\tau(M)$ and γ^2.

If $m = 3$ when M is an 8-manifold the $\hat{A}$-genus is given by

$$\hat{A}(M) = \hat{A}_2(p_1, p_2) = \frac{1}{2^7.45}(-4p_2 + 7p_1{}^2),$$

Hirzebruch's Index Theorem gives

$$\tau(M) = L_2(p_1, p_2) = (1/45)(7p_2 - p_1^2)$$

and eliminating the Pontrjagin class p_2 from these two equations gives

$$\tau(M) = p_1^2/4 - 8.28\hat{A}.$$

Now M is a spin manifold and so $\hat{A}$ is an integer, whence

$$\tau(M)/8 = (p_1^2/32) \bmod 28.$$

By Lemma 1.1 of Kervaire [4] $p_1 = 2\gamma$ and so

$$\tau(M)/8 = (\gamma^2/8) \bmod 28.$$

If $m = 7$, when M is a 16-manifold the non-zero Pontrjagin classes are p_2 and p_4 and the $\hat{A}$-genus is given by

$$\hat{A}(M) = \hat{A}_4(0, p_2, 0, p_4) = \frac{1}{2^{15}.3^4.5^2.7}(-192p_4 + 208p_2^2).$$

Hirzebruch's Index Theorem gives

$$\tau(M) = L_4(0, p_2, 0, p_4) = \frac{1}{3^4.5^2.7}(381p_4 - 19p_2^2)$$

and eliminating p_4 from the two equations gives

$$\tau(M) = (p_2^2/36) - 2^9.127\hat{A}.$$

M is a spin manifold and so $\hat{A}$ is an integer, whence

$$\frac{\tau(M)}{8} = \frac{p_2^2}{8.36} \bmod 2^6.127.$$

By [4], $p_2 = 6\gamma$ and so

$$\frac{\tau(M)}{8} = \frac{\gamma^2}{8} \bmod 2^6.127.$$

PROPOSITION 5. *Manifolds P and M are as above with tangential invariants $\hat{\beta} \in H^{m+1}(P)$ and $\gamma \in H^{m+1}(M)$.*

(i) *If, for an integer N, $N\hat{\beta}$ is divisible by Nr then r/N divides γ^2.*

(ii) *If $H^{m+1}(P)$ has no element of order 2 and $N\hat{\beta}$ is divisible by Nr then $2r/N$ divides γ^2.*

(iii) *If $\hat{\beta}$ has finite order then $\gamma^2 = 0$.*

COROLLARY. *If $N\hat{\beta}$ is divisible by Nr and $\Sigma \in I(P)$, then Σ is divisible by $r/(8N)$. Moreover, if $H^{m+1}(P)$ has no element of order 2 then Σ is divisible by $r/(4N)$.*

Before proving the proposition we use the result to establish the necessary conditions for Theorem 1 and also the corollary above.

For P a closed $(m-1)$-connected $(2m+1)$-manifold, $m = 3$ or 7, by Theorem 4 of [11] $P \# \Sigma = \partial L$ for some $L \in \mathscr{H}(m+1)$ and $\Sigma \in \Theta_{2m+1}$. Now $I(P) = I(P \# \Sigma)$ and so to determine the inertia group we may assume without loss of generality that $P = \partial L$ and so apply the results of this section. Now if $\Sigma \in I(P)$ there exists an orientation-preserving diffeomorphism $f\colon P \to P \# \Sigma$ and so the manifold M can be formed. By [5], Σ as an element of $\Theta_{2m+1}(\partial\pi)$ is determined by $\big(\tau(W)/8\big)$ mod $|\Theta_{2m+1}(\partial\pi)|$, and so by Proposition 4 and parts (i) and (ii) of Proposition 5 the corollary follows. Part (i) and the necessary conditions for part (ii) of Theorem 1 are given by Proposition 4 together with part (ii), taking $N = 1$, and part (iii) of Proposition 5.

Proof of Proposition 5

Consider the following commutative diagram of cohomology groups.

$$\begin{tikzcd}
 & & & 0 \arrow[dd] & & & \\
 & 0 \arrow[drr] & & & & 0 \arrow[dll] & \\
 & & & H^m(P) \arrow[dll, "\delta_1"'] \arrow[drr, "\delta_2"] \arrow[dd] & & & \\
 & H^{m+1}(L, P) \arrow[dddd, bend right, "j_1^*"'] & & & & H^{m+1}(L+W, P) \arrow[dddd, bend left, "j_2^*"] & \\
 & & & H^{m+1}(M, P) \arrow[ull, "\simeq"'] \arrow[urr, "\simeq"] \arrow[dd] & & & \\
 & H^{m+1}(M, L+W) \arrow[uu, "l_1^*"] \arrow[urr] \arrow[drr, "m_1^*"'] & & & & H^{m+1}(M, L) \arrow[uu, "l_2^*"'] \arrow[ull] \arrow[dll, "m_2^*"] & \\
 & & & H^{m+1}(M) \arrow[dll, "k_1^*"] \arrow[drr, "k_2^*"'] \arrow[dd] & & & \\
 & H^{m+1}(L) \arrow[dl] \arrow[drr, "i_1^*"'] & & & & H^{m+1}(L+W) \arrow[dr] \arrow[dll, "i_2^*"] & \\
0 & & & H^{m+1}(P) \arrow[dll] \arrow[drr] \arrow[dd] & & & 0 \\
 & 0 & & & & 0 & \\
 & & & 0 & & &
\end{tikzcd}$$

The diagram gives rise to the Mayer–Vietoris sequence

$$0 \to H^m(P) \xrightarrow{\Delta} H^{m+1}(M) \xrightarrow{\Phi} H^{m+1}(L) \oplus H^{m+1}(L+W) \xrightarrow{\Psi} H^{m+1}(P) \to 0,$$

where

$$\Delta = m_1{}^* l_1{}^{*-1}\delta_1 = -m_2{}^* l_2{}^{*-1}\delta_2, \quad \Phi(x) = k_1{}^*(x) \oplus k_2{}^*(x) \quad \text{for} \quad x \in H^{m+1}(M),$$

and

$$\Psi(y \oplus z) = i_1{}^*(y) - i_2{}^*(z) \quad \text{for} \quad y \in H^{m+1}(L), \quad z \in H^{m+1}(L+W).$$

$$H^{m+1}(L+W) \simeq H^{m+1}(L) \oplus H^{m+1}(W)$$

and

$$H^{m+1}(L+W, P) \simeq H^{m+1}(L, P) \oplus H^{m+1}(W, \Sigma)$$

and in this way $j_2{}^*$ is identified with $j_1{}^* \oplus t^*$, where the isomorphism

$$t^* : H^{m+1}(W, \Sigma) \to H^{m+1}(W)$$

is induced by the inclusion map $t : W \to (W, \Sigma)$.

$\partial(L+W) = P \# \Sigma$ and $H^{m+1}(P \# \Sigma) \simeq H^{m+1}(P) \oplus H^{m+1}(\Sigma) \simeq H^{m+1}(P)$ and so the induced homomorphism $H^{m+1}(L+W) \to H^{m+1}(P \# \Sigma)$ of the inclusion $P \# \Sigma \to L+W$ can be identified with the homomorphism given by $i_1{}^*$ on $H^{m+1}(L)$ with $H^{m+1}(W)$ being mapped trivially. In forming M, $P \# \Sigma$ is identified with P by the diffeomorphism $f : P \to P \# \Sigma$ and by the identification

$$H^{m+1}(P \# \Sigma) \simeq H^{m+1}(P).$$

f^* can be regarded as an automorphism $f^* : H^{m+1}(P) \to H^{m+1}(P)$, and so $i_2{}^*$ is identified with the homomorphism given by $f^* i_1{}^*$ on $H^{m+1}(L)$ with $H^{m+1}(W)$ being mapped trivially. For convenience in what follows we write $n_1 = m_1{}^* l_1{}^{*-1}$ and $n_2 = m_2{}^* l_2{}^{*-1}$.

The obstruction to a cross-section of the tangent bundle over the $(m+1)$-skeleton of L is a well-defined element

$$\hat{\alpha} \in H^{m+1}(L; \pi_m(SO)) \simeq H^{m+1}(L)$$

and since W is parallelisable the corresponding obstruction for $L+W$ is also $\hat{\alpha} \in H^{m+1}(L+W) \simeq H^{m+1}(L) \oplus H^{m+1}(W)$. Since γ, $\hat{\alpha}$ and $\hat{\beta}$ are all obstructions to cross-sections of the tangent bundle over the $(m+1)$-skeletons of the corresponding manifolds, it follows that $k_1{}^*(\gamma) = \hat{\alpha}$, $k_2{}^*(\gamma) = \hat{\alpha}$, $i_1{}^*(\hat{\alpha}) = i_2{}^*(\hat{\alpha}) = \hat{\beta}$.

(i) $N\hat{\beta}$ is divisible by Nr and so $N\hat{\beta} = Nrx$ for some $x \in H^{m+1}(P)$. Now since $i_1{}^*(\hat{\alpha}) = \hat{\beta}$ we have $N\hat{\alpha} = NrX + Y$ for $X, Y \in H^{m+1}(L)$, where $i_1{}^*(X) = x$ and $i_1{}^*(Y) = 0$. By exactness, $Y = j_1{}^*(a) = j_2{}^*(a)$ for some $a \in H^{m+1}(L, P)$.

Suppose that the isomorphism $f^* : H^{m+1}(P) \to H^{m+1}(P)$ has $f^*(x) = x+z$; then as f is a diffeomorphism $f^*(\hat{\beta}) = \hat{\beta}$ when $f^*(N\hat{\beta}) = N\hat{\beta}$ and we have $Nrz = 0$. So if $z = i_1{}^*(Z)$ for $Z \in H^{m+1}(L)$ then $i_1{}^*(NrZ) = 0$ and by exactness $NrZ = j_1{}^*(b)$ for some $b \in H^{m+1}(L, P)$.

Referring to the Mayer–Vietoris sequence, we have

$$\Psi(X+Z \oplus X) = i_1{}^*(X+Z) - i_2{}^*(X) = i_1{}^*(X+Z) - f^* i_1{}^*(X) = 0.$$

By exactness

$$X+Z \oplus X = \Phi(c) = k_1{}^*(c) \oplus k_2{}^*(c) \quad \text{for some} \quad c \in H^{m+1}(M).$$

Now

$$\begin{aligned}\Phi(N\gamma - Nrc) &= k_1{}^*(N\gamma - Nrc) \oplus k_2{}^*(N\gamma - Nrc)\\ &= \big(N\hat{\alpha} - Nr(X+Z)\big) \oplus (N\hat{\alpha} - NrX) = (Y - NrZ) \oplus Y\\ &= j_1{}^*(a-b) \oplus j_2{}^*(a) = k_1{}^* n_1(a-b) \oplus k_2{}^* n_2(a)\\ &= \Phi\big(n_1(a-b) + n_2(a)\big)\end{aligned}$$

since $k_1{}^* n_2 = k_2{}^* n_1 = 0$. Therefore, by exactness of the Mayer–Vietoris sequence,

$$N\gamma - Nrc = n_1(a-b) + n_2(a) + \Delta(d)$$

for some $d \in H^m(P)$. Squaring this equation, $\big(n_1(a-b)\big)^2 = -(a-b)^2$, since M is formed from $-L$ i.e. L with the opposite orientation, and $n_2(a)^2 = a^2$. All other products on the right-hand side are zero since they can be factored through $H^{2m+2}(M, L+W \cup L) \simeq 0$. Hence

$$N^2 \gamma^2 - 2N^2 rc.\gamma + N^2 r^2 c^2 = 2a.b - b^2,$$

$$\text{i.e. } N^2 \gamma^2 = 2N^2 rc.\gamma - N^2 r^2 c^2 + b.(2a-b)$$

$$= 2N^2 rc.\gamma - N^2 r^2 c^2 + NrZ.(2a-b)$$

since $b.(2a-b) = j_1{}^*(b).(2a-b)$ and hence

$$\gamma^2 = 2rc.\gamma - r^2 c^2 + (r/N)Z.(2a-b), \tag{*}$$

showing that γ^2 is divisible by r/N.

(ii) $\hat{\beta}$ is always an even element (see [11]) and so we can assume that r is even. Since $Nrz = 0$ and $H^{m+1}(P)$ has no element of order 2 we must have $Nrz/2 = 0$. Thus $i_1{}^*(NrZ/2) = 0$, when, since $H^{m+1}(L, P)$ is torsion-free, it follows that b is even and so from (*) γ^2 is divisible by $2r/N$.

(iii) If $\hat{\beta}$ has finite order N then $N\hat{\beta} = 0$ is divisible by any integer, and so by (i) γ^2 is divisible by any integer, whence $\gamma^2 = 0$.

5. *Sufficient conditions for Theorem* 1

Suppose $P = \partial L$ for $L \in \mathscr{H}(m+1)$, where $m = 3$ or 7. We recall from [8] relations between invariants of L and P.

$H = H_{m+1}(L)$ is torsion-free and intersection numbers give a symmetric bilinear map

$$\lambda : H \times H \to \mathbb{Z}.$$

The obstruction to triviality of the tangent bundle over the $(m+1)$-skeleton is an element $\hat{\alpha} \in H^{m+1}(L)$. By using the identification $H^{m+1}(L) \simeq \hat{H} = \mathrm{Hom}(H, \mathbb{Z})$, $\hat{\alpha}$ can be regarded as a homomorphism $\hat{\alpha} : H \to \mathbb{Z}$ and λ and $\hat{\alpha}$ satisfy the relation $\lambda(x, x) = \hat{\alpha}(x) \bmod 2$ for all $x \in H$. The manifolds of $\mathscr{H}(m+1)$ are classified by the invariants $(H, \lambda, \hat{\alpha})$.

A manifold L splits as a boundary connected sum $L_1 + L_2$ (when the boundary P splits as a connected sum $\partial L_1 \# \partial L_2$) if and only if λ splits i.e. if $H = H_1 \oplus H_2$ and $\lambda(x, y) = 0$ for all $x \in H_1$, $y \in H_2$. Then $L = L_1 + L_2$, where L_1 and L_2 have invariants $(H_1, \lambda_1, \hat{\alpha}_1)$ and $(H_2, \lambda_2, \hat{\alpha}_2)$, where $\lambda_1 = \lambda|_{H_1 \times H_2}$, $\lambda_2 = \lambda|_{H_2 \times H_2}$ and $\hat{\alpha} = \hat{\alpha}_1 \oplus \hat{\alpha}_2$.

λ induces a homomorphism $\pi : H \to \hat{H}$ by the rule $\pi(x)(y) = \lambda(x, y)$. By identifying $H^{m+1}(L, P)$ with H and $H^{m+1}(L)$ with $\hat{H}$ in the cohomology sequence

$$0 \to H^m(P) \to H^{m+1}(L, P) \xrightarrow{j^*} H^{m+1}(L) \xrightarrow{i^*} H^{m+1}(P) \to 0,$$

where $i^*(\hat{\alpha}) = \hat{\beta}$, j^* is identified with π so that $H^{m+1}(P) \simeq \operatorname{coker} \pi$ and $H^m(P) \simeq \ker \pi$, which is isomorphic to the torsion-free part of $H^{m+1}(P)$. P is then a homotopy sphere Σ if and only if π is an isomorphism i.e. if and only if λ is unimodular. In this case $\hat{\alpha}$ regarded as an element of $\hat{H}$ determines a unique element

$\chi \in H$ by the rule $\chi . x = \hat{\alpha}(x)$ for all $x \in H$, i.e. $\pi(\chi) = \hat{\alpha}$ (here $\chi . x$ denotes $\lambda(\chi, x)$) and by Theorem 4 of [8] Σ as an element of $\Theta_{2m+1}(\partial\pi)$ is given by

$$\frac{\chi^2-\tau}{8} \bmod \begin{cases} 28 & \text{for } m = 3 \\ 2^6 . 127 & \text{for } m = 7 \end{cases}$$

where τ is the signature of L i.e. the signature of λ.

To complete the proof of Theorem 1 the following must be proved.

Let $H^{m+1}(P) * \Theta_{2m+1} \simeq 0$ and r be the largest integer dividing $\hat{\beta}$. If $\Sigma \in \Theta_{2m+1}(\partial\pi)$ is divisible by $r/4$ then $\Sigma \in I(P)$.

We can as usual assume $P = \partial L$ with $L \in \mathscr{H}(m+1)$. Since $|\Theta_{2m+1}(\partial\pi)|$ has only factors 2 and p, where $p = 7$ for $m = 3$ and $p = 127$ for $m = 7$, it is only the powers of 2 and p in r that are of importance, and so we may assume that r has no other factors.

$\pi : H \to \hat{H}$ and let $H_1 = \ker \pi$ when $H = H_1 \oplus H_2$ for some H_2 and L splits as a boundary connected sum $L_1 + L_2$ where $\lambda_1 = 0$. $P = P_1 \# P_2$ splits accordingly with $H^{m+1}(P_1) \simeq$ the torsion-free part of $H^{m+1}(P)$ and $H^{m+1}(P_2) \simeq$ the torsion subgroup of $H^{m+1}(P)$. The tangential invariant $\hat{\beta} = \hat{\beta}_1 \oplus \hat{\beta}_2$ and, since $H^{m+1}(P) * \Theta_{2m+1} \simeq 0$, $H^{m+1}(P_2)$ has no elements of order 2 or p and so $\hat{\beta}_2 \in H^{m+1}(P_2)$ is divisible by any power of 2 or p. Thus r is the largest integer dividing $\hat{\beta}_1$ and since $I(P_1 \# P_2) \supset I(P_1) + I(P_2) \supset I(P_1)$ it is enough to show that $\Sigma \in I(P_1)$.

It is enough therefore to prove the result for a manifold P with $H^{m+1}(P)$ torsion-free and $P = \partial L$ where L has $\lambda = 0$. Here $H^{m+1}(L)$ is mapped isomorphically onto $H^{m+1}(P)$ and so r is the largest integer dividing $\hat{\alpha}$. Thus if H has rank k there exists a basis $e_1, e_2, \ldots, e_k$ of H with $\hat{\alpha}(e_1) = r$, $\hat{\alpha}(e_i) = 0$ for $i \neq 1$. Then

$$L = L_1 + L_2 + \ldots + L_k$$

where $H^{m+1}(L_i) \simeq \mathbb{Z}$ with generator e_i. Correspondingly $P = P_1 \# P_2 \# \ldots \# P_k$ where $P_i = \partial L_i$ and $H^{m+1}(P_i) \simeq \mathbb{Z}$ and P_1 will have tangential invariant $\hat{\beta}$. Since

$$I(P) \supset I(P_1) + I(P_2) + \ldots + I(P_k) \supset I(P_1),$$

it is enough to show that $\Sigma \in I(P_1)$.

It is enough therefore to prove the result for a manifold P where $H^{m+1}(P) \simeq \mathbb{Z}$ and $P = \partial L$ where $H^{m+1}(L) \simeq \mathbb{Z}$ with generator e. L has intersection matrix [0], i.e. $\lambda = 0$, and $\hat{\alpha}(e) = r$.

Let $V \in \mathscr{H}(m+1)$ have $H_{m+1}(V) \simeq \mathbb{Z}$ with generator f and intersection matrix [1] and $\hat{\alpha}(f) = \pm 1$. Then $\chi = \pm f$ and the signature $\tau = 1$ so that $(\chi^2 - \tau)/8 = 0$, showing that $\partial V = S^{2m+1}$. $L + V$ has intersection matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

with respect to the basis e, f of $H^{m+1}(L+V) \simeq H^{m+1}(L) \oplus H^{m+1}(V) \simeq \mathbb{Z} \oplus \mathbb{Z}$. Consider the change of basis to e, $F = e + f$ which gives the same intersection matrix. Then $L + V = L + V'$, where V' has $H_{m+1}(V') \simeq \mathbb{Z}$ with generator F and intersection matrix [1]. Thus $\partial(L+V) = P = P \# \Sigma = \partial(L+V')$, where $\Sigma = \partial V'$, whence $\Sigma \in I(P)$. For V', $\hat{\alpha}(F) = \hat{\alpha}(e+f) = r \pm 1$ and so $\chi = (r \pm 1) F$ and

$$\frac{\chi^2-\tau}{8} = \frac{(r\pm 1)^2 - 1}{8} = \frac{r(r\pm 2)}{8}.$$

Σ as an element of $\Theta_{2m+1}(\partial\pi)$ is given by

$$\frac{r(r\pm 2)}{8} \bmod \begin{cases} 28 & \text{for } m = 3 \\ 2^6.127 & \text{for } m = 7. \end{cases}$$

We consider the factors 2 and 7 or 127 of $|\Theta_{2m+1}(\partial\pi)|$ separately and show that if r is not divisible by 7 or 127 then either $r(r+2)/8$ or $r(r-2)/8$ is not divisible by 7 or 127, and if the highest power of 2 in r is 2^k then either $r(r+2)/8$ or $r(r-2)/8$ is divisible by 2^{k-2} but not by 2^{k-1} for $k \geqslant 2$ and either $r(r+2)/8$ or $r(r-2)/8$ is odd for $k = 1$ (r is always even). Once it is established that there is a homotopy sphere $\Sigma \in I(P)$ not divisible by 7 or 127 or divisible by 2^{k-2} and not by 2^{k-1} for $k \geqslant 2$ or not divisible by 2 for $k = 1$ then the required subgroup of $\Theta_{2m+1}(\partial\pi)$ can be generated to prove the result.

If 7 or 127 does not divide r then 7 or 127 does not divide $(r+2)$ or $(r-2)$ and so 7 or 127 does not divide $r(r+2)/8$ or $r(r-2)/8$.

For $k \geqslant 2$, if 2^k divides r but 2^{k+1} does not then $(r+2)$ is divisible by 2 but not by 4 and hence $r(r+2)/8$ is divisible by 2^{k-2} but not by 2^{k-1}.

Finally, if 2 divides r but 4 does not then either $(r+2)$ or $(r-2)$ is divisible by 4 but not by 8 and hence $r(r+2)/8$ or $r(r-2)/8$ is odd.

6. *Inertia groups of connected sums*

In this section inertia groups of closed $(m-1)$-connected $(2m+1)$-manifolds P, $m = 3$ or 7, are considered where $H^{m+1}(P) * \Theta_{2m+1} \not\simeq 0$. In this case the corollary to Proposition 5 can give a stronger condition for $\Sigma \in I(P)$ than that given in part (ii) of Theorem 1. For example if P has $H^{m+1}(P) \simeq \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_p$ (where $p = 7$ for $m = 3$ and $p = 127$ for $m = 7$) and $\hat{\beta} = 2^{10}p^2 \oplus 2 \oplus 1$ then $2p\hat{\beta}$ is divisible by $2^{11}p^3$, i.e. $r = 2^{10}p^2$, and so by the corollary $\Sigma \in I(P)$ is divisible by $2^{10}p^2/(8.2p) = 2^6 p$, showing that $I(P) \simeq 0$.

For any two manifolds P_1 and P_2, $I(P_1 \# P_2) \supset I(P_1)+I(P_2)$ and if P_1 and P_2 each have $H^{m+1}(P_i) * \Theta_{2m+1} \simeq 0$ it follows from part (ii) of Theorem 1 that $I(P_1 \# P_2) = I(P_1)+I(P_2)$. This is also true if both P_1 and P_2 have tangential invariants of finite order when by part (i) of Theorem 1

$$I(P_1 \# P_2) = I(P_1) = I(P_2) \simeq 0.$$

If in general it were true that

$$I(P_1 \# P_2) = I(P_1)+I(P_2)$$

then Theorem 1 would determine $I(P)$ for any of our manifolds P, since we have already shown that $P = P_1 \# P_2$, where $H^{m+1}(P_1)$ is torsion-free and $H^{m+1}(P_2)$ is finite, and in this case $I(P)$ would depend on the torsion-free part of the tangential invariant $\hat{\beta}$. This is not, however, true in general, as we show by the following examples.

Let manifolds L_1, L_2, $L_3 \in \mathscr{H}(m+1)$, whose boundaries are P_1, P_2, P_3, be given as follows, where $p = 7$ for $m = 3$ and $p = 127$ for $m = 7$, and where we take $k \geqslant 4$ for $m = 3$ and $k \geqslant 8$ for $m = 7$.

L_1 has $H_1 = H_{m+1}(L_1) \simeq \mathbb{Z}$ with generator e_1, intersection matrix [0], and tangential invariant given by $\hat{\alpha}(e_1) = 2^k p$ (i.e. $\hat{\alpha} = 2^k p\hat{e}_1 \in \hat{H}_1$). $P_1 = \partial L_1$ then has

$H^{m+1}(P_1) \simeq \mathbb{Z}$ with a generator e_1', where the tangential invariant is $\hat{\beta}_1 = 2^k p e_1'$.

L_2 has $H_2 = H_{m+1}(L_2) \simeq \mathbb{Z}$ with generator e_2, intersection matrix $[p]$, and tangential invariant given by $\hat{\alpha}(e_2) = 1$ (i.e. $\hat{\alpha} = \hat{e}_2 \in \hat{H}_2$). $P_2 = \partial L_2$ then has $H^{m+1}(P_2) \simeq \mathbb{Z}_p$ with a generator e_2' where the tangential invariant is $\hat{\beta}_2 = e_2'$.

L_3 has $H_3 = H_{m+1}(L_3) \simeq \mathbb{Z}$ with generator e_3, intersection matrix $[2^{k-1}]$, and tangential invariant given by $\hat{\alpha}(e_3) = 2$ (i.e. $\hat{\alpha} = 2\hat{e}_3 \in \hat{H}_3$). $P_3 = \partial L_3$ then has $H^{m+1}(P_3) \simeq \mathbb{Z}_{2^{k-1}}$ with a generator e_3', where the tangential invariant $\hat{\beta}_3 = 2e_3'$.

In addition let $V \in \mathscr{H}(m+1)$ have $H^{m+1}(V) \simeq \mathbb{Z} \oplus \mathbb{Z}$ with basis f_1, f_2 and corresponding intersection matrix

$$\begin{array}{cc} f_1 & f_2 \\ \multicolumn{2}{c}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \end{array}$$

and tangential invariant given by $\hat{\alpha}(f_1) = 2$, $\hat{\alpha}(f_2) = 0$. The signature τ of V is zero and $\chi = 2f_2$ so that $\chi^2 = 4{f_2}^2 = 0$, whence $\partial V = S^{2m+1}$.

$L_1 + L_2 + V$ has intersection matrix

$$\begin{array}{cccc} e_1 & e_2 & f_1 & f_2 \\ \multicolumn{4}{c}{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \end{array}$$

Consider the change of basis of $H_{m+1}(L_1 + L_2 + V) \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ to e_1, $E_2 = e_1 + e_2 - 2^{k-1} p f_1, f_1$, $F_2 = 2^{k-1} e_2 - 2^{2k-3} p f_1 + f_2$, which gives the same intersection matrix.

$$\hat{\alpha}(E_2) = \hat{\alpha}(e_1) + \hat{\alpha}(e_2) - 2^{k-1} p \hat{\alpha}(f_1) = \hat{\alpha}(e_2) + 2^k p - 2^{k-1} p 2 = \hat{\alpha}(e_2) = 1.$$

So $L_1 + L_2 + V = L_1 + L_2 + V'$, where V' has intersection matrix

$$\begin{array}{cc} f_1 & F_2 \\ \multicolumn{2}{c}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \end{array}$$

Thus $P_1 \# P_2 = P_1 \# P_2 \# \Sigma$ where $\Sigma = \partial V'$. Now

$$\hat{\alpha}(F_2) = 2^{k-1} \hat{\alpha}(e_2) - 2^{2k-3} p \hat{\alpha}(f_1) + \hat{\alpha}(f_2) = 2^{k-1}(1 - 2^{k-1} p).$$

$\hat{\alpha}(f_1) = 2$, so that $\chi = 2^{k-1}(1 - 2^{k-1} p) f_1 + 2F_2$. The signature of V' is zero and so Σ is an element of $\Theta_{2m+1}(\partial\pi)$ is given by

$$\frac{\chi^2}{8} = \frac{2.2^{k-1}(1-2^{k-1}p).2}{8} \bmod \begin{cases} 28 & \text{for } m = 3 \\ 2^6.127 & \text{for } m = 7 \end{cases}$$

$$= 2^{k-2} \bmod \begin{cases} 28 & \text{for } m = 3 \\ 2^6.127 & \text{for } m = 7. \end{cases}$$

$\Theta_7 = \Theta_7(\partial\pi) \simeq \mathbb{Z}_4 \oplus \mathbb{Z}_7$ and $\Theta_{15}(\partial\pi) \simeq \mathbb{Z}_{2^6} \oplus \mathbb{Z}_{127}$ and so Σ generates the $\mathbb{Z}_7$ component of Θ_7 for $m = 3$ and the $\mathbb{Z}_{127}$ component of $\Theta_{15}(\partial\pi)$ for $m = 7$.

Now 2^k divides the tangential invariant $\hat{\beta}_1 \oplus \hat{\beta}_2 = 2^k p e_1' + e_2'$ of $P_1 \# P_2$ and so by the corollary to Proposition 5 with $N = 1$ any $\Sigma \in I(P_1 \# P_2)$ is divisible by 2^{k-2}. We therefore have

$$I(P_1 \# P_2) \simeq \begin{cases} \mathbb{Z}_7 & \text{for} \quad m = 3 \\ \mathbb{Z}_{127} & \text{for} \quad m = 7. \end{cases}$$

$L_1 + L_3 + V$ has intersection matrix

$$\begin{array}{c} \begin{matrix} e_1 & e_3 & f_1 & f_2 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2^{k-1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Consider the change of basis of $H_{m+1}(L_1 + L_3 + V) \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ to e_1, $E_3 = e_1 + e_3 - 2^{k-1} p f_1$, f_1, $F_2' = p e_3 - 2^{k-2} p^2 f_1 + f_2$, which gives the same intersection matrix.

$$\hat{\alpha}(E_3) = \hat{\alpha}(e_1) + \hat{\alpha}(e_3) - 2^{k-1} p \hat{\alpha}(f_1) = \hat{\alpha}(e_3) + 2^k p - 2^{k-1} p 2 = \hat{\alpha}(e_3) = 2.$$

So $L_1 + L_3 + V = L_1 + L_3 + V''$, where V'' has intersection matrix

$$\begin{array}{c} \begin{matrix} f_1 & F_2' \end{matrix} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

Thus $P_1 \# P_3 = P_1 \# P_3 \# \Sigma$, where $\Sigma = \partial V''$. Now

$$\hat{\alpha}(F_2') = p\hat{\alpha}(e_3) - 2^{k-2} p^2 \hat{\alpha}(f_1) + \hat{\alpha}(f_2) = 2p(1 - 2^{k-2} p).$$

$\hat{\alpha}(f_1) = 2$, so that $\chi = 2p(1 - 2^{k-2} p) f_1 + 2F_2'$. The signature of V'' is zero and so Σ as an element of $\Theta_{2m+1}(\partial \pi)$ is given by

$$\frac{\chi^2}{8} = \frac{2.2p(1 - 2^{k-2} p)\, 2}{8} \bmod \begin{cases} 28 & \text{for} \quad m = 3 \\ 2^6.127 & \text{for} \quad m = 7 \end{cases}$$

$$= p \bmod \begin{cases} 28 & \text{for} \quad m = 3 \\ 2^6.127 & \text{for} \quad m = 7. \end{cases}$$

Σ therefore generates the $\mathbb{Z}_4$ component of Θ_7 for $m = 3$ and the $\mathbb{Z}_{26}$ component of $\Theta_{15}(\partial \pi)$ for $m = 7$. Now p divides the tangential invariant $\beta_1 \oplus \beta_3 = 2^k p e_1' + 2e_3'$ of $P_1 \# P_3$ and so, by the corollary to Proposition 5 with $N = 1$, any $\Sigma \in I(P_1 \# P_3)$ is divisible by p. We therefore have

$$I(P_1 \# P_3) \simeq \begin{cases} \mathbb{Z}_4 & \text{for} \quad m = 3 \\ \mathbb{Z}_{26} & \text{for} \quad m = 7. \end{cases}$$

$I(P_1 \# P_2 \# P_3)$ contains both $I(P_1 \# P_2)$ and $I(P_1 \# P_3)$ and so we have

$$I(P_1 \# P_2 \# P_3) \simeq \begin{cases} \mathbb{Z}_4 \oplus \mathbb{Z}_7 & \text{for} \quad m = 3 \\ \mathbb{Z}_{26} \oplus \mathbb{Z}_{127} & \text{for} \quad m = 7 \end{cases}$$

i.e. $I(P_1 \# P_2 \# P_3) = \Theta_{2m+1}(\partial \pi)$.

Theorem 1 can be applied to the manifolds P_1, P_2, P_3 and $P_2 \# P_3$ to show that in each case the inertia groups are trivial. So, to summarize, the inertia groups of P_1, P_2, P_3 and their connected sums are

$$I(P_1) = I(P_2) = I(P_3) = I(P_2 \# P_3) \simeq 0$$

$$I(P_1 \# P_2) \simeq \begin{cases} \mathbb{Z}_7 & \text{for } m = 3 \\ \mathbb{Z}_{127} & \text{for } m = 7 \end{cases}$$

$$I(P_1 \# P_3) \simeq \begin{cases} \mathbb{Z}_4 & \text{for } m = 3 \\ \mathbb{Z}_{2^6} & \text{for } m = 7 \end{cases}$$

$$I(P_1 \# P_2 \# P_3) = \Theta_{2m+1}(\partial\pi) \simeq \begin{cases} \mathbb{Z}_{28} & \text{for } m = 3 \\ \mathbb{Z}_{2^6.127} & \text{for } m = 7. \end{cases}$$

The methods used in the examples above give sufficient conditions for $\Sigma \in I(P)$ for certain manifolds P where $H^{m+1}(P) * \Theta_{2m+1} \not\simeq 0$. Together with necessary conditions as in the corollary to Proposition 5, it seems that the general result for $I(P)$ should be along the following lines.

CONJECTURE. *Let P be a closed $(m-1)$-connected $(2m+1)$-manifold, where $m = 3$ or 7. If r and s are the largest integers such that $2^k \beta$ and $p^l \hat{\beta}$ are divisible by 2^{2k+r} and p^{2l+s} respectively, for any integers k and l, then $I(P)$ consists exactly of those elements of $\Theta_{2m+1}(\partial\pi)$ divisible by $2^{r-2} p^s$ ($p = 7$ for $m = 3$ and $p = 127$ for $m = 7$).*

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Department of Pure Mathematics,
The University,
Birmingham B15 2TT.