

Pure Maths.

Closed $(s-1)$ - Connected $(2s+1)$ - Manifolds, $s = 3, 7$.

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by

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SYNOPSIS

This thesis constitutes a study of certain aspects of closed $(s-1)$ - connected differential manifolds of dimension $(2s+1)$, particularly in the cases where the integer $s = 3$ or 7 . These manifolds throughout are denoted by the letter P .

Chapter 0 is an introductory chapter in which some definitions are given, some established results are quoted and an attempt is made to fix some of the notation to be used later.

Chapter 1 provides a basis for later chapters by showing that the manifolds P (for $s = 3$ or 7) occur, up to the addition of homotopy spheres, as boundaries of handlebodies $\mathcal{H}(s+1)$, whose elements are manifolds formed by adding $(s+1)$ - handles to the $(2s+2)$ - disc. Members of $\mathcal{H}(s+1)$ are denoted by the letter L . Invariants $(G, b, \hat{\beta})$ are defined for the manifolds P where $G = H_s(P)$, b is the linking invariant and $\hat{\beta} \in H^{s+1}(P)$ is a tangential invariant. For the manifolds L invariants $(H, \lambda, \hat{\alpha})$ are defined under which the manifolds are classified by Wall (43), where $H = H_{s+1}(L)$, λ is given by intersection numbers and $\hat{\alpha} \in H^{s+1}(L)$ is a tangential invariant.

(ii)

The way in which the invariants $(H, \lambda, \hat{\alpha})$ of L determine the invariants $(G, b, \hat{\beta})$ of $P = \partial L$ is described and this allows the possible values the invariants $(G, b, \hat{\beta})$ can take for manifolds P to be determined.

Chapter 2 simplifies the manifolds P by connected sum. The invariants $(G, b, \hat{\beta})$ add together by direct sum for connected sums of manifolds P and the results of chapter 2 are roughly a converse to this in that whenever the invariants $(G, b, \hat{\beta})$ split by direct sum the corresponding manifold P splits by connected sum accordingly. From this follows the definition of indecomposable manifolds P , which are manifolds whose invariants cannot be split by direct sum, and the result that any manifold P is a connected sum of indecomposable manifolds.

Closed $(s-1)$ - connected $(2s+1)$ - manifolds have been classified, up to the addition of homotopy spheres, by Wall (VI) except for the cases $s = 1, 2, 3, 7$ and the case $s = 2$ has been dealt with by Barden (1). Partial results along these lines have also been obtained by Tamura (38, 40). Chapter 3 is concerned with the classification of indecomposable manifolds P where

$s = 3$ or 7 . It is shown that the indecomposable manifolds P are classified, up to the addition of homotopy spheres, by the invariants $(G, b, \hat{\beta})$ except for certain values of $\hat{\beta}$, which are determined, where G contains elements of order 2, when there are two distinct manifolds.

Chapter 4 considers embeddings and immersions in euclidean space. An immersion theorem for closed 2 - connected 7 - manifolds is given after which embeddings of manifolds $L \in \mathcal{H}(s+1)$, for general values of s , are considered. These results restricted to the boundaries of the manifolds L give rise to embedding theorems for the manifolds P .

Chapter 5 considers the inertia groups $I(P)$ of the manifolds P , briefly for general values of s but mainly for $s = 3$ or 7 . Theorem 5.6 determines $I(P)$ for certain manifolds P covering the cases where either G is torsion - free or G is finite. For manifolds restricted to either of these cases the inertia group of a connected sum of two manifolds is the sum of the inertia groups, however it is shown that this is not true in

(iv)

general by giving examples of manifolds P and P' where

$H_s(P)$ is torsion - free and $H_s(P')$ is finite and where

$$I(P \amalg P') \neq I(P) \oplus I(P').$$

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CHAPTER 0

PRELIMINARY DEFINITIONS, RESULTS AND NOTATION

The purpose of this chapter is to provide some background material that will be used in the remainder of the thesis. This is done by giving definitions for the concepts required or by providing suitable references. Also several well known results that will be needed later are stated and we establish to some extent the general notation we will be using.

Algebraic Topology

As a general reference book for algebraic topology we use Spanier, Algebraic Topology (34); Hilton and Wylie (11) provides a further reference for Homology Theory while Hu (16) gives another reference for Homotopy Theory.

We will use singular homology and cohomology with cup and cap products defined in the usual way. For a topological pair (X,A) the q -dimensional singular homology and cohomology groups of (X,A) , when the coefficient group is the integers, are denoted by $H_q(X,A)$

and $H^q(X,A)$ respectively. If the coefficient group is the abelian group G then the corresponding homology and cohomology groups are denoted by $H_q(X,A;G)$ and $H^q(X,A;G)$.

Since we will be dealing with highly connected manifolds the Hurewicz Isomorphism Theorem, proofs of which are given in (16) and (34)m provides a result relating the homotopy and homology groups of our spaces.

The Hurewicz Isomorphism Theorem

If $n > 1$ and X is an $(n-1)$ - connected topological space then

$$\Pi_n(X) \simeq H_n(X).$$

Bundles

The books of Husemoller, Fibre Bundles (17) and Steenrod, The Topology of Fibre Bundles (36) provide all that is needed in the way of background material for the theory of fibre bundles and vector bundles. We will be concerned with obstruction theory with regard to the existence of cross-sections of bundles and this is covered in part III of Steenrod (36) and also in Chapter 8 of Spanier (34).

Manifolds

The basic definitions of differential topology can be located in the book of Munkres, Elementary Differential Topology (28) or in the printed notes of either Wall, Differential Topology (47) or Milnor, Topology from the Differentiable Viewpoint (26). We therefore assume the definitions of the following terms: topological manifold (with or without boundary), differential structure, differential manifold, submanifold, smooth map, diffeomorphism, tangent bundle, normal bundle, immersion, embedding.

For M a compact topological m -manifold with boundary ∂M orientability can be defined simply as follows. M is orientable if $H_m(M, \partial M) \cong \mathbb{Z}$, where an orientation is a given generator of the group; and M is non-orientable if $H_m(M, \partial M) \cong 0$. An orientable topological manifold with a given orientation is said to be oriented. If M is an oriented topological manifold then $-M$ denotes the manifold with the opposite orientation.

We will be concerned on the whole with compact oriented C^∞ - differential manifolds and so for convenience from now on the single word "manifold" will mean a compact oriented C^∞ - differential manifold which may or may not possess a boundary.

For a manifold M the tangent bundle of M is denoted by τ_M and the stable tangent bundle of M is then the bundle $\tau_M \oplus \theta$ where $\oplus$ denotes the Whitney sum of vector bundles and θ is the trivial line bundle.

Associated with the tangent bundle and stable tangent bundle of a manifold are three parallelisability definitions which we now distinguish.

A manifold M is parallelisable if the tangent bundle τ_M is trivial.

A manifold M is stably parallelisable if the stable tangent bundle $\tau_M \oplus \theta$ is trivial, when M is said to be a Π -manifold.

A manifold M is s-parallelisable for an integer s if the tangent bundle τ_M restricted to the s -dimensional skeleton of M (with respect to a decomposition of M as a CW - complex) is trivial. Note that whenever the expression 's-parallelisable' occurs s will be an integer and will not stand for stable.

We now define in slightly more detail the ideas of connected sum and boundary connected sum, cobordisms and handlebodies, and surgery.

These concepts which allow manifolds to be constructed from other manifolds will be used extensively in later chapters.

Connected Sum and Boundary Connected Sum

For two m -manifolds M_1 and M_2 let $i_1: D^m \rightarrow M_1$ and $i_2: D^m \rightarrow M_2$ be two embeddings such that i_1 is orientation preserving and i_2 is orientation reversing (we must ensure that the embeddings are disjoint from the boundaries of M_1 and M_2 , if they exist) then the connected sum $M_1 \# M_2$ is the manifold formed from the disjoint union of $M_1 \setminus i_1(\text{int } D^m)$ and $M_2 \setminus i_2(\text{int } D^m)$ by identifying corresponding points of $i_1(S^{m-1})$ and $i_2(S^{m-1})$, the orientation of $M_1 \# M_2$ being that which induces the given orientations of M_1 and M_2 . By the Disc Theorem and the technique of rounding of corners it can be shown that $M_1 \# M_2$ is independent up to orientation preserving diffeomorphism of the choice of the embeddings i_1 and i_2 and is a well defined m -manifold. It follows that connected sum is an associative and commutative operation on m -manifolds for which the m -sphere S^m serves as identity element.

In a similar way boundary connected sum can be defined. For two n -manifolds N_1 and N_2 with non-empty boundaries ∂N_1 and ∂N_2 let $i_1: D^{n-1} \rightarrow \partial N_1$ and $i_2: D^{n-1} \rightarrow \partial N_2$ be two embeddings, i_1 orientation

preserving and i_2 orientation reversing, then the boundary connected sum $N_1 + N_2$ is the manifold formed from the disjoint union of N_1 and N_2 by identifying corresponding points of $i_1(D^{n-1})$ and $i_2(D^{n-1})$, the orientation of $N_1 + N_2$ being that which induces the given orientations of N_1 and N_2 . Similarly as for connected sum it follows that $N_1 + N_2$ is a well defined n -manifold. It also follows that boundary connected sum is an associative and commutative operation on n -manifolds with boundary for which the n -disc D^n serves as identity element.

It also follows simply from the definitions that we have the following relation between connected sum and boundary connected sum

$$\partial(N_1 + N_2) = \partial N_1 \amalg \partial N_2$$

i.e. the boundary of a boundary connected sum is the connected sum of the boundaries.

Cobordism

To define cobordism it is convenient to be able to talk of manifolds with a corner. For an m -manifold with corner M we allow the co-ordinate neighbourhoods to map into open sets of $\{(x_1, \dots, x_n) \in \mathbb{R}^n; x_1 \geq 0, x_2 \geq 0\}$ and the corner $\bigwedge M$, an $(n-2)$ -manifold,

is then the set of points which map to $x_1 = x_2 = 0$. By the technique of straightening a corner a unique new differential structure could be put on M with the effect of rounding the corner and giving an m -manifold with boundary.

For the discussion here of cobordism and of handlebodies later we refer particularly to parts IV and V of Wall (47).

If M is an m -manifold such that $\partial_- M$ and $\partial_+ M$ are disjoint manifolds with union ∂M , the boundary of M , then we say that M is a cobordism of $\partial_- M$ to $\partial_+ M$. If M is an m -manifold with corner and the manifolds $\partial_- M$, $\partial_c M$, $\partial_+ M$ make up the boundary such that $\partial_- M$ and $\partial_+ M$ are disjoint and $\partial \partial_c M = \bigwedge M = \partial(\partial_- M \cup \partial_+ M)$ then we say that M is a cobordism with corner of $\partial_- M$ to $\partial_+ M$.

We now suppose that M is a cobordism of $\partial_- M$ to $\partial_+ M$ which may or may not have a corner.

Of interest here is the Duality Theorem, a proof of which is given in (47), which gives a relation between homology and cohomology groups associated with M and gives rise to the familiar Poincaré and Lefschetz Duality Theorems.

Duality Theorem

$$H_r(M, \partial_- M) \approx H^{m-r}(M, \partial_+ M).$$

Poincaré Duality Theorem

If M is an m -manifold without boundary

$$H_r(M) \approx H^{m-r}(M).$$

Lefschetz Duality Theorem

If M is an m -manifold with boundary ∂M

$$H_r(M) \approx H^{m-r}(M, \partial M)$$

$$H^r(M) \approx H_{m-r}(M, \partial M).$$

Stronger versions of cobordism can be introduced and one of the most useful is the idea of h -cobordism when the inclusions $\partial_- M \subset M$ and $\partial_+ M \subset M$ are required to be homotopy equivalences.

The h -cobordism Theorem, a proof of which is given by Wall (47), was proved originally by Smale and a more explicit version was produced later by Milnor in the book Lectures on the h -Cobordism Theorem (27).

The h-Cobordism Theorem

If $m \geq 6$ and M is a simply - connected h -cobordism of $\partial_- M$ to $\partial_+ M$ then M is diffeomorphic to $\partial_- M \times I$.

The theorem is proved by Smale and Wall using the theory of handlebodies, defined below, while Milnor's proof uses the related concept of Morse functions. The proofs of Smale and Milnor do not however cover the case where M may have a corner.

Handlebodies

The theory of handlebodies was introduced by Smale (33) and we preserve some of the notation used there.

Suppose M is a cobordism of $\partial_- M$ to $\partial_+ M$ and that $f: S^{r-1} \times D^{m-r} \rightarrow \partial_+ M$ is an embedding. Introduce a corner along $f(S^{r-1} \times D^{m-r})$ and glue $D^r \times D^{m-r}$ to M by identifying under the map f . The resulting manifold is M with an r -handle attached by f , written as $M \cup_f h^r$. If we have a sequence of attached handles $N = M \cup_{f_1} h_1^r \cup \dots \cup_{f_k} h_k^r$ then this is described as a handle presentation of N on M , and if the maps f_i are not specified a handle decomposition. In particular if $M = P \times I$ (with $\partial_- M = P \times 0$ and $\partial_+ M = P \times 1$) we speak of a handle decomposition of N with base P .

It can be shown that any manifold M has a handle decomposition with base $\partial_- M$. However the following theorem, which we name the Handle Decomposition Theorem, relates the number of handles of each dimension needed in such a decomposition to the number of generators of the homology groups of the pair $(M, \partial_- M)$, provided we make simple connectivity assumptions.

The Handle Decomposition Theorem

If $m \geq 6$ and $M, \partial_- M, \partial_+ M$ are all simply - connected and $H_r(M, \partial_- M) \cong F \oplus T$, where F is a free abelian group of rank β_r and T is a finite group with τ_r generators. Then M has a handle decomposition with base $\partial_- M$ with $\tau_{r-1} + \beta_r + \tau_r$ r -handles for each r .

A proof of the Handle Decomposition Theorem, which was originally proved by Smale excluding the case where M may have corners, is given by Wall (47).

Of interest are those handlebodies that can be obtained by attaching handles to an m -disc D^m i.e. we take $M = D^m$, $\partial_- M = \emptyset$, $\partial_+ M = S^{m-1}$ in the definitions above. We denote by $\mathcal{H}(m, k, r)$ the set of manifolds that can be formed by attaching k r -handles to the

m -disc D^m . We let $\mathcal{H}(n) = \bigcup_{k \geq 0} (2n, k, n)$ denote the set of manifolds that can be formed by attaching n -handles to the $2n$ -disc D^{2n} .

Surgery

The paper of Milnor (25) provides the basis definitions for an introduction to surgery.

To perform a surgery on a manifold M the idea is to remove an embedded product $S^p \times D^{q+1}$, where $m = p + q + 1$, leaving a boundary $S^p \times S^q$ and replacing this by the product $D^{p+1} \times S^q$ which has the same boundary.

More precisely let M be an m -manifold and $f: S^p \times D^{q+1} \rightarrow M$ an orientation preserving embedding, where $p + q + 1 = m$, and let $\chi(M, f)$ denote the manifold obtained from the disjoint union of $M \setminus f(S^p \times 0)$ and $D^{p+1} \times S^q$ by identifying $f(u, tv)$ with (tu, v) for each $u \in S^p$, $v \in S^q$ and $0 < t \leq 1$. If M^1 is a manifold diffeomorphic to $\chi(M, f)$ by an orientation preserving diffeomorphism then we say that M^1 can be obtained from M by surgery of type $(p + 1, q + 1)$. It is clear that the boundary of M remains unchanged in this process so that $\partial M^1 = \partial M$. Also surgery is symmetric in the sense that M can be obtained from M^1 by surgery of type $(p + 1, q + 1)$.

If there are a sequence of manifolds $M = M_1, M_2, \dots, M_{r-1}, M_r = M^1$ such that each M_{i+1} is obtained from M_i by a surgery then M and M^1 are said to be χ - equivalent.

It is clear that surgery is related to the process of adding handles in that whenever a handle is added to a manifold M the result is to perform a surgery on the boundary ∂M .

Homotopy Spheres

In (21) the group of homotopy spheres $\mathcal{H}_n$ is defined and largely determined by Kervaire and Milnor.

An n -manifold Σ is a homotopy n -sphere if Σ has no boundary and has the homotopy type of the n -sphere S^n . (We will throughout tend to use the symbol Σ to denote a homotopy sphere).

$\mathcal{H}_n$ denotes the set of k -cobordism classes of homotopy n -spheres. The connected sum operation is defined on $\mathcal{H}_n$ and it can be shown that this sum gives $\mathcal{H}_n$ the structure of an abelian group with the sphere S^n serving as the identity element.

All homotopy spheres are stably parallelisable but not every homotopy sphere occurs as the boundary of a parallelisable manifold; those homotopy sphere that do bound parallelisable manifolds form a subgroup $\underline{\mathbb{H}}_n(\partial\Pi)$ of $\mathbb{H}_n$ (originally in (27) bp_{n+1} was used to denote this subgroup).

Related to $\mathbb{H}_n$ is the Thom group Γ_n whose elements consist of differential manifolds homeomorphic to S^n and where the group operation is again given by the connected sum. By Smale's solution of the generalised Poincare conjecture (33) showing that if $n \geq 5$ a homotopy n -sphere Σ is homeomorphic to S^n and by the $\mathbb{H}$ -Cobordism Theorem which shows that $\mathbb{H}$ -cobordism is equivalent to diffeomorphism for simply-connected manifolds of dimension ≥ 5 it follows that $\mathbb{H}_n = \Gamma_n$ for $n \geq 5$.

Kervaire and Milnor show that the groups $\mathbb{H}_n$ are finite, with the possible exception of $n = 3$, and in the cases in which we are particularly interested when $n = 7, 15$ the groups are as follows:

$$\mathbb{H}_7 \simeq \underline{\mathbb{H}}_7(\partial\Pi) \simeq Z_{28}$$

$$\mathbb{H}_{15} \simeq Z_{8128} \oplus Z_2$$

$$\mathbb{H}_{15}(\partial\Pi) \simeq Z_{8128}.$$

Characteristic Classes

Certain characteristic classes of manifolds will be of interest and in particular for a manifold M the Pontrjagin classes $p_i(M) \in H^{4i}(M)$ and the Stiefel - Whitney classes $w_i(M) \in H^i(M; \mathbb{Z}_2)$. These concepts are defined and discussed in Hirzebruch's book, Topological Methods in Algebraic Geometry (13) and in the printed notes of Milnor, Lectures on Characteristic Classes (24). Wu's formula, which describes a method of calculating the Stiefel - Whitney classes of a given manifold, is described in the books of Spanier (34) and Husemoller (17).

Notation

This thesis is closely related to the collection of papers by Wall that appeared in Topology from 1963 to 1967 under the general heading of Classification Problems in Differential Topology; and in particular the papers (I), (III), (VI) and (Q). Apart from the notation already established in this chapter and other commonly accepted notation we therefore endeavour to keep as near possible to the notation used in these papers, so as to provide easy reference.

We will be investigating closed $(s-1)$ -connected $(2s+1)$ -manifolds (where closed means without boundary), particularly in the cases $s=3,7$, and we will use the letter P to denote these manifolds. Handlebodies in $\mathcal{K}(s+1)$, again particularly in the cases $s=3,7$ will be much used and we will denote elements of $\mathcal{K}(s+1)$ by the letter L .

Results will be numbered according to the chapter in which they occur and to their position in the chapter, and the end of a proof will be denoted by the symbol $\S$.

CHAPTER 1

THE MANIFOLDS P AS

BOUNDARIES OF THE MANIFOLDS L

In this chapter invariants $(G, b, \hat{\beta})$ are defined for closed $(s-1)$ -connected $(2s+1)$ -manifolds P , where $s = 3$ or 7 . For manifolds $L \in \mathcal{H}(s+1)$, again where $s = 3$ or 7 , invariants $(H, \lambda, \hat{\alpha})$ are defined by Wall, see (I), (III) and (43). It is shown that the manifolds P occur as boundaries of the manifolds L , up to the addition of a homotopy sphere for $s = 7$, and the way in which the invariants of L determine those of $P = \partial L$ is discussed. These results are collected together in Theorem 1.6. The final result of the chapter, Theorem 1.8, determines what possible values the invariants $(G, b, \hat{\beta})$ can take for the manifolds P .

The Invariants of P

Before we restrict ourselves to the case when $s = 3$ or 7 we first consider the general case where P is a closed $(s-1)$ -connected $(2s+1)$ -manifold for any integer $s \geq 1$.

Since P is $(s-1)$ -connected and is oriented it follows from the Hurewicz Isomorphism Theorem and Poincare Duality that the non-zero homology groups occur in dimensions $0, s, s+1, 2s+1$ where $H_0(P) \cong \mathbb{Z}$ and $H_{2s+1}(P) \cong \mathbb{Z}$ with preferred generators. We write G for the group $H_s(P)$ and let $\hat{G} = \text{Hom}(G, \mathbb{Z})$ the group of homomorphisms from G to $\mathbb{Z}$. $\hat{G}$ is isomorphic to the torsion-free part of G and so by duality we have $H_{s+1}(P) \cong \hat{G}$. The non-zero homology groups of P are thus:

$$H_0(P) \cong \mathbb{Z}, H_s(P) = G, H_{s+1}(P) \cong \hat{G}, H_{2s+1}(P) \cong \mathbb{Z}.$$

By Poincare Duality the non-zero cohomology groups of P are

$$H^0(P) \cong \mathbb{Z}, H^s(P) \cong \hat{G}, H^{s+1}(P) \cong G, H^{2s+1}(P) \cong \mathbb{Z}.$$

Let G^* denote the torsion subgroup of G . The linking invariant is a non-singular bilinear form

$$b: G^* \times G^* \rightarrow S \quad (\text{where } S = \mathbb{Q}/\mathbb{Z})$$

and various definitions are given in (21), (1), (45) and (VI) as well as below. The exact sequence of coefficient groups

$$0 \rightarrow \mathbb{Z} \xrightarrow{k} Q \xrightarrow{\ell} S \rightarrow 0$$

induces an exact homology sequence

$$\dots \rightarrow H_{s+1}(P; Q) \xrightarrow{\ell_*} H_{s+1}(P; S) \xrightarrow{\eta} H_s(P; \mathbb{Z}) \xrightarrow{k_*} H_s(P; Q) \rightarrow \dots$$

where η is the Bockstein homomorphism.

Suppose $y \in G^* \subset H_s(P)$ then since y is a torsion element $k_*(y) = 0$ and hence $y = \eta(z)$ for some element $z \in H_{s+1}(P; S)$. Let $z' \in H^s(P; S)$ be the unique element obtained from z by Poincare Duality i.e. z' is the element such that $\mu \cap z' = z$ where μ is the generator of $H_{2s+1}(P) \simeq \mathbb{Z}$ defining the orientation of P . For an element $x \in G^* \subset H_s(P)$ we have

$$x \cap z' \in H_0(P; S) \simeq S \quad \text{and the linking invariant}$$

is defined by

$$b(x, y) = x \cap z'.$$

Also if $y' \in H^{s+1}(P)$ is the element dual to $y \in G^*$ by Poincare Duality then $\eta(z') = y'$ in the exact cohomology sequence

$$\dots \rightarrow H^s(P; Q) \xrightarrow{\ell_*} H^s(P; S) \xrightarrow{\eta} H^{s+1}(P, Z) \xrightarrow{k_*} H^{s+1}(P; Q) \rightarrow \dots$$

b is well defined for suppose $\eta(w') = y'$ for $w' \in H^s(P; S)$ then $\eta(z' - w') = 0$ and $z' - w' = \ell_*(c)$ for some element $c \in H^s(P; Q)$.

Now $x \cap z' - x \cap w' = x \cap (z' - w') = x \cap \ell_*(c) = \ell_*(x \cap c)$. But

$\ell_*(\cap c): H_s(P; Z) \rightarrow H_0(P; Q) \simeq Q$ is a homomorphism and as x has finite order its image $\ell_*(x \cap c) = 0$ showing that $x \cap w' = x \cap z'$.

The map b can be shown to be $(-1)^{s+1}$ - symmetric, see for example (1).

We now consider tangential aspects of the manifolds P .

The obstruction to triviality of the tangent bundle over the s -skeleton of P lies in the cohomology group $H^s(P; \Pi_{s-1}(SO))$. Providing this obstruction vanishes the obstruction to triviality over the $(s+1)$ -skeleton lies in $H^{s+1}(P; \Pi_s(SO))$; and provided both these obstructions vanish the final obstruction to stable parallelisability of P can be located in $H^{2s+1}(P; \Pi_{2s}(SO))$.

For $s = 5, 6 \bmod 8$ the groups $\Pi_{s-1}(SO)$, $\Pi_s(SO)$ and $\Pi_{2s}(SO)$ are all trivial and so the manifolds P in these cases are in fact Π -manifolds.

For $s = 3, 7 \bmod 8$ $\Pi_{s-1}(SO) \approx 0$ and so in these cases the manifolds P are s -parallelisable.

Lamma 1.1

If $s \geq 1$ and P is a closed $(s-1)$ -connected $(2s+1)$ -manifold where $s \neq 2, 4, 8$ then the Stiefel - Whitney classes $w_r = 0$ for $r \neq 0$.

Proof

The Stiefel - Whitney classes are given by Wu's formula, see for example page 259 of (17). Since the non-zero cohomology groups

with coefficient group Z_2 occur in dimensions $0, s, s+1, 2s+1$ it follows that the Wu class $v = 1 + v_s$ for $1 \in H^0(P; Z_2) \approx Z_2$ and some element $v_s \in H^s(P; Z_2)$. The Stiefel - Whitney classes of P are then given by $w(P) = Sq(v)$ where $Sq = \sum_{i \geq 0} Sq^i$ is a sum of the Steenrod squares

$$Sq^i: H^k(P; Z_2) \rightarrow H^{k+i}(P; Z_2).$$

Since $Sq^0 = \text{identity}$ and $Sq^i(x) = 0$ for $i > \dim x$ it follows that the Stiefel - Whitney classes $w_0(P) = 1$, $w_s(P) = v_s$ and $w_{s+1}(P) = Sq^1(v_s) = Sq^1(w_s)$ all others being zero. It is thus sufficient to prove the result to show that $w_s = 0$.

The class $v_s = w_s$ is defined uniquely by the formula $Sq^s(x) = x v_s = x w_s$ for each $x \in H^{s+1}(P; Z_2)$.

We deal first with the case when s is an odd integer.

The Adem relation for the Steenrod squares, see (37), for $0 < a < 2b$ is

$$Sq^a Sq^b = \sum_{j=0}^{\lfloor a/2 \rfloor} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j$$

and by taking $a = 1$ and $b = s-1$ we obtain

$$Sq^1 Sq^{s-1} = (s-2) Sq^s Sq^0 = (s-2) Sq^s$$

Since $s-2$ is odd and we are working moduls 2 we have

$$Sq^s = Sq^1 Sq^{s-1}$$

For any $x \in H^{s+1}(P; Z_2)$ since

$$Sq^{s-1}(x) \in H^{2s}(P; Z_2) \approx 0 \text{ we have}$$

$$Sq^s(x) = Sq^1 Sq^{s-1}(x) = 0.$$

Hence $xw_s = 0$ for all $x \in H^{s+1}(P; Z_2)$ giving $w_s = 0$.

For the case when s is even we consider the fibration

$SO_{s-1} \rightarrow SO \xrightarrow{p} SO/SO_{s-1}$ where p is the natural projection. Let p induce a homomorphism

$$p_*: \Pi_{s-1}(SO) \rightarrow \Pi_{s-1}(SO/SO_{s-1})$$

which induces the homomorphism

$$p_{**}: H^s(P; \Pi_{s-1}(SO)) \rightarrow H^s(P; \Pi_{s-1}(SO/SO_{s-1}))$$

$w_s \in H^s(P; \Pi_{s-1}(SO/SO_{s-1})) \approx H^s(P; Z_2)$ is the obstruction to a cross-

section of the SO/SO_{s-1} - bundle associated with the tangent bundle over the s -skeleton of P , and so if $\gamma \in H^s(P; \Pi_{s-1}(SO))$ is the obstruction to a cross-section of the tangent bundle over the s -skeleton of P it follows that $p_{**}(\gamma) = w_s$. We prove the Lemma by showing that $p_* = 0$ for $s \neq 2, 4, 8$ whence $p_{**} = 0$ and hence $w_s = 0$.

p_* lies in the exact sequence of homotopy groups

$$\cdots \rightarrow \Pi_{s-1}(SO) \xrightarrow{p_*} \Pi_{s-1}(SO/SO_{s-1}) \rightarrow \Pi_{s-2}(SO_{s-1}) \rightarrow \Pi_{s-2}(SO) \rightarrow \Pi_{s-2}(SO/SO_{s-1}) \rightarrow \cdots$$

given by the fibration. Now $\Pi_{s-1}(SO/SO_{s-1}) \simeq Z_2$ and $\Pi_{s-2}(SO/SO_{s-1}) \simeq 0$ and the homotopy groups of SO below are given by Bott (2) while the homotopy groups of SO_{s-1} are given by Kervaire (20).

For $s \equiv 0, 4 \pmod{8}$ with $s \neq 4$ or 8 we have $\Pi_{s-1}(SO) \simeq Z$, $\Pi_{s-2}(SO_{s-1}) \simeq Z_2$ and $\Pi_{s-2}(SO) \simeq 0$ and so by exactness $p_* = 0$.

For $s \equiv 2 \pmod{8}$ with $s \neq 2$ we have $\Pi_{s-1}(SO) \simeq Z_2$, $\Pi_{s-2}(SO_{s-1}) \simeq Z_2 \oplus Z_2$, $\Pi_{s-2}(SO) \simeq Z_2$ and so by exactness again $p_* = 0$.

For $s \equiv 6 \pmod{8}$ we have $\Pi_{s-1}(SO) \simeq 0$ and in fact from above we know that in this case that the manifolds P are Π - manifolds and so the results follows trivially. $\S$.

We now restrict ourselves to the case where P is a closed $(s-1)$ - connected $(2s+1)$ - manifold and $s = 3$ or 7 and we construct a tangential invariant.

The tangent bundle has a cross-section over the s -skeleton of P (since $\Pi_{s-1}(SO) \simeq 0$) and as $\Pi_s(SO) \simeq \mathbb{Z}$ the obstruction to a cross-section over the $(s+1)$ - skeleton of P is a well-defined element $\hat{\beta} \in H^{s+1}(P; \Pi_s(SO)) = H^{s+1}(P)$. Since $\Pi_{2s}(SO) \simeq 0$ it follows that if $\hat{\beta}$ is zero there is no further obstruction to stable parallelisability and so $\hat{\beta}$ is the obstruction to stable parallelisability. In fact by James and Thomas, Theorem 1.7 (18), stable equivalence is the same as equivalence for 7 - plane bundles over a complex of dimension ≤ 7 so that for $s = 3$ $\hat{\beta}$ is the obstruction to parallelisability of P .

Lemma 1.1 of Kervaire (19) gives the following relations between $\hat{\beta}$ and the Pontrjagin classes of P . If $s=3$ then $2\hat{\beta} = p_1(P)$ and if $s=7$ then $6\hat{\beta} = p_2(P)$.

Lemma 1.2

$\hat{\beta} \in H^{s+1}(P)$ is even

Proof

The projection map p in the fibration

$$SO_s \xrightarrow{\ell} SO \xrightarrow{p} SO/SO_s$$

$$\text{induces } p_*: \Pi_s(SO) \rightarrow \Pi_s(SO/SO_s)$$

$$\quad \quad \quad \uparrow \quad \quad \quad \uparrow$$

$$\quad \quad \quad \mathbb{Z} \quad \quad \quad \mathbb{Z}_2$$

$$\text{and } p_{**}: H^{s+1}(P) \rightarrow H^{s+1}(P; \mathbb{Z}_2)$$

$\hat{\beta}$ is the obstruction to a cross-section of the tangent bundle and w_{s+1} is the obstruction to a cross-section of the associated SO/SO_s - bundle over the $(s+1)$ - skeleton of P and so $p_{**}(\hat{\beta}) = w_{s+1}$ which is zero by Lemma 1.1.

The exact homotopy sequence

$$\Pi_{s+1}(SO/SO_s) \rightarrow \Pi_s(SO) \xrightarrow{\ell_*} \Pi_s(SO) \xrightarrow{p_*} \Pi_s(SO/SO_s) \rightarrow \Pi_{s-1}(SO_s)$$

$$\quad \quad \quad \uparrow \quad \quad \quad \uparrow \quad \quad \quad \uparrow \quad \quad \quad \uparrow$$

$$\quad \quad \quad \mathbb{0} \quad \quad \quad \mathbb{Z} \quad \quad \quad \mathbb{Z} \quad \quad \quad \mathbb{Z}_2 \quad \quad \quad \mathbb{0}$$

gives the exact cohomology sequence

$$\dots \rightarrow H^{s+1}(P) \xrightarrow{\ell^{**}} H^{s+1}(P) \xrightarrow{p^{**}} H^{s+1}(P; \mathbb{Z}_2) \rightarrow \dots$$

ℓ_* is given by multiplication by 2 so the image of ℓ_{**} contains only even elements of $H^{s+1}(P)$. Since $p_{**}(\hat{\beta}) = 0$ $\hat{\beta}$ is in the image of ℓ_{**} and is thus even. $\S$.

The invariants we will use for the manifolds P are $(G, b, \hat{\beta})$, which we have now defined, where $G = H_s(P)$, $b: G^* \times G^* \rightarrow S$ is a non-singular symmetric bilinear form, and $\hat{\beta} \in H^{s+1}(P)$ is the obstruction to stable parallelisability.

It is clear that if we construct the connected sum of two closed $(s-1)$ -connected $(2s+1)$ -manifolds P_1 and P_2 the result $P_1 \# P_2$ will also be a closed $(s-1)$ -connected $(2s+1)$ -manifold.

Lemma 1.3

If the invariants of manifolds P_1 and P_2 are $(G_1, b_1, \hat{\beta}_1)$ and $(G_2, b_2, \hat{\beta}_2)$ respectively then the invariants of the connected sum $P_1 \# P_2$ are $(G_1 \oplus G_2, b, \hat{\beta}_1 + \hat{\beta}_2)$ where

$$b(x_1 \oplus x_2, y_1 \oplus y_2) = b_1(x_1, y_1) + b_2(x_2, y_2)$$

for $x_1, y_1 \in G_1^*$ and $x_2, y_2 \in G_2^*$.

i.e. Taking the connected sum of two manifolds corresponds to taking the direct sum of their invariants.

Proof

$H_s(P_1 \# P_2) \approx H_s(P_1) \oplus H_s(P_2)$ by considering the Mayer-Vietoris sequence of the connected sum. Homology classes in the same summand have the same intersection as before while homology classes in different summands have zero intersection giving rise to the formula for b . A suitable cell decomposition of $P_1 \# P_2$ will have cells that are contained in P_1 or P_2 and so the obstruction to a cross-section of the tangent bundle over the $(s+1)$ -skeleton will be the sum $\hat{\beta}_1 \oplus \hat{\beta}_2 \in H^{s+1}(P_1) \oplus H^{s+1}(P_2) \approx H^{s+1}(P_1 \# P_2)$ of the two obstructions for P_1 and P_2 . $\square$.

The Invariants of L

The invariants $(H, \lambda, \hat{\alpha})$ which we define for manifolds $L \in \mathcal{H}(s+1)$, where $s=3$ or 7 , provide a classification. The construction of the invariants and their properties are discussed in (I), (III) and (43) and we therefore quote the results in this section without proof.

Since $L \in \mathcal{H}(s+1)$ its non-zero homology groups occur in dimensions 0 and $s+1$. We write H for $H_{s+1}(L)$ which is a free abelian group with the number of elements in a basis being equal to the number of $(s+1)$ -handles attached to the $(2s+2)$ -disc to form L .

The non-zero homology groups of L are thus

$$H_0(L) \cong Z, \quad H_{s+1}(L) = H.$$

We write $\hat{H} = \text{Hom}(H, Z)$, the group of homomorphisms from H to Z (of course since H is free $H \cong \hat{\hat{H}}$).

For the homology and cohomology groups of a general chain complex $\{C_q\}$ and a coefficient group G we have a homomorphism

$$h: H^q(C; G) \rightarrow \text{Hom}(H_q(C), G)$$

given by
$$h([f])([\sum c_q]) = \sum f(c_q),$$

where $[f]$ and $[\sum c_q]$ are general elements of $H^q(C; G)$ and $H_q(C)$ respectively, $[]$ denoting a cohomology or homology class. The Universal Coefficient Theorem for cohomology, see for example page 243 of (34), gives a short exact sequence

$$0 \rightarrow \text{Ext}(H_{q-1}(C), G) \rightarrow H^q(C; G) \xrightarrow{h} \text{Hom}(H_q(C), G) \rightarrow 0$$

and the sequence is split. $\text{Ext}(A, B)$ for abelian groups A and B denotes the extension group of A by B and if either A is free or B is divisible $\text{Ext}(A, B) = 0$.

Thus for the cohomology groups of L we have isomorphisms

$$h: H^q(L) \longrightarrow \text{Hom}(H_q(L), \mathbb{Z})$$

since the homology of L is free. The non-zero cohomology groups of L are then

$$H^0(L) \simeq \mathbb{Z}, \quad H^{s+1}(L) \simeq \hat{H}.$$

The homology and cohomology groups of the pair $(L, \partial L)$ are given by Lefschetz Duality. The non-zero homology groups are

$$H_{s+1}(L, \partial L) \simeq \hat{H}, \quad H_{2s+2}(L, \partial L) \simeq \mathbb{Z}.$$

The non-zero cohomology groups are

$$H^{s+1}(L, \partial L) \simeq H, \quad H^{2s+2}(L, \partial L) \simeq \mathbb{Z}.$$

Intersection numbers for $H = H_{s+1}(L)$ determine a symmetric bilinear map

$$\lambda: H \times H \rightarrow \mathbb{Z}.$$

Formulated in terms of cup and cap products if $x, y \in H$ and η is the generator of $H_{2s+2}(L, \partial L) \simeq \mathbb{Z}$ defining the orientation of L and

$x', y' \in H^{s+1}(L, \partial L)$ such that $x = \eta \cap x'$ and $y = \eta \cap y'$ then λ is defined by

$$\lambda(x, y) = \eta \cap (x' \cup y').$$

The cup product is bilinear and since

$$y' \cup x' = (-1)^{(s+1)^2} x' \cup y'$$

it follows that for general s the bilinear map λ is $(-1)^{s+1}$ -symmetric and so is our case where $s = 3$ or 7 λ is symmetric.

By the Hurewicz Isomorphism Theorem $H \simeq \Pi_{s+1}(L)$ and from results of Haefliger (8) an element of H defines a unique embedding of S^{s+1} in L . The normal bundle of each embedding corresponding to an element of H is classified by an element of $\Pi_s(SO_{s+1})$ and hence we obtain a map

$$\alpha: H \rightarrow \Pi_s(SO_{s+1}).$$

Following this map by the suspension homomorphism

$$S: \Pi_s(SO_{s+1}) \rightarrow \Pi_s(SO_{s+2}) \simeq \Pi_s(SO)$$

we obtain a homomorphism

$$S\alpha: H \rightarrow \Pi_s(SO)$$

For $s = 3$ or 7 $\Pi_s(SO) \approx \mathbb{Z}$ (see (2)) and so we obtain an element

$$\hat{\alpha} = S\alpha \in \text{Hom}(H, \mathbb{Z}) = \hat{H}.$$

We denote the corresponding element in $H^{s+1}(L) \approx \hat{H}$ also by $\hat{\alpha}$. The element $\hat{\alpha} \in H^{s+1}(L)$ is then the obstruction to obtaining a

cross-section of the tangent bundle over the $(s+1)$ -skeleton of L .

If $\hat{\alpha} = 0$ there is no further obstruction to triviality of the tangent bundle of L and so $\hat{\alpha}$ is in fact the obstruction to parallelisability of P .

Lemma 1.1 of Kervaire (19) relates $\hat{\alpha}$ to the Pontrjagin classes of L . If $s = 3$ then $2\hat{\alpha} = p_1(L)$ and if $s = 7$ then $6\hat{\alpha} = p_2(L)$.

The invariants that are taken for L are $(H, \lambda, \hat{\alpha})$ where $H = H_{s+1}(L)$ is a free abelian group, $\lambda: H \times H \rightarrow \mathbb{Z}$ is a symmetric bilinear map and $\hat{\alpha}$ is the obstruction to parallelisability.

λ and $\hat{\alpha} \in \hat{H}$ are related by

$$\lambda(x, x) = \hat{\alpha}(x) \pmod{2}.$$

i.e. $\lambda(x, x)$ and $\hat{\alpha}(x)$ have the same parity for any element $x \in H$ (This is proved in §9 of (III) and also in (43)).

The following classification theorem is due to Wall.

Proposition 1.4

The manifolds $L \in \mathcal{H}(s+1)$, where $s = 3$ or 7 , are classified by the invariants $(H, \lambda, \hat{\alpha})$. i.e. For two manifolds $L_1, L_2 \in \mathcal{H}(s+1)$ with invariants $(H_1, \lambda_1, \hat{\alpha}_1), (H_2, \lambda_2, \hat{\alpha}_2)$ respectively if there exists an isomorphism $k: H_1 \rightarrow H_2$ preserving intersection numbers (i.e. $\lambda_2(k(x), k(y)) = \lambda_1(x, y)$) and with $\hat{\alpha}_2 \circ k = \hat{\alpha}_1$ then there exists an orientation preserving diffeomorphism $f: L_1 \rightarrow L_2$ with $f_* = k$.

Taking the boundary connected sum of manifolds $L_1, L_2 \in \mathcal{H}(s+1)$ gives another manifold $L_1 + L_2 \in \mathcal{H}(s+1)$ whose invariants are related to those of L_1 and L_2 by the following result (proved in (I) and (43)).

Proposition 1.5

If the invariants of manifolds L_1 and L_2 are $(H_1, \lambda_1, \hat{\alpha}_1)$ and $(H_2, \lambda_2, \hat{\alpha}_2)$ respectively then the invariants of the boundary connected sum $L_1 + L_2$ are $(H_1 \oplus H_2, \lambda, \hat{\alpha}_1 \oplus \hat{\alpha}_2)$ where for $x_1, y_1 \in H_1, x_2, y_2 \in H_2$

$$\lambda(x_1 \oplus x_2, y_1 \oplus y_2) = \lambda_1(x_1, y_1) + \lambda_2(x_2, y_2).$$

i.e. taking the boundary connected sum of two manifolds corresponds to taking the direct sum of their invariants.

Relations Between the Invariants of P and L

For $L \in \mathcal{H}(s+1)$ the boundary ∂L is a closed $(s-1)$ -connected $(2s+1)$ -manifold. Suppose $P = \partial L$ then since the invariants $(H, \lambda, \hat{\alpha})$ of L classify the manifolds L these invariants determine the invariants $(G, b, \hat{\beta})$ of P . The way in which this occurs is described below. We are of course still assuming $s = 3$ or 7 .

The symmetric bilinear map

$$\lambda: H \times H \rightarrow Z$$

formed by intersection number defines a homomorphism

$$\Pi: H \rightarrow H$$

where for $x, y \in H$ $\Pi(x): H \rightarrow Z$ is the homomorphism given by

$$\Pi(x)(y) = \lambda(x, y).$$

The homology of P can be calculated from the chain groups

$$C_{s+1} = H \text{ and } C_s = \hat{H} \text{ with the boundary homomorphism } \Pi: H \rightarrow \hat{H}.$$

Thus $G = H_s(P) \approx \text{co ker } \Pi$ and $\hat{G} \approx H_{s+1}(P) \approx \text{ker } \Pi$. This result is

Lemma 3 of (43).

It follows that $G \approx 0$ and P is therefore a homotopy sphere if and only if Π is an isomorphism, which is the case if and only if λ is unimodular.

The exact sequence

$$0 \rightarrow \hat{G} \rightarrow H \xrightarrow{\Pi} \hat{H} \rightarrow G \rightarrow 0$$

obtained from the computation of the homology of P is in essence the homology sequence of the pair (L, P)

$$0 \rightarrow H_{s+1}(P) \rightarrow H_{s+1}(L) \xrightarrow{j_*} H_{s+1}(L, P) \xrightarrow{\partial} H_s(P) \rightarrow 0$$

The homomorphism $j_*: H_{s+1}(L) \rightarrow H_{s+1}(L, P)$ induced by the inclusion $j: L \hookrightarrow C(L, P)$ is given by Π followed by the isomorphisms $\hat{H} \approx H^{s+1}(L)$ and $H^{s+1}(L) \approx H_{s+1}(L, P)$. For convenience we will regard these sequences as being the same by identifying $H_{s+1}(L, P)$ with $\hat{H}$ and j_* with Π to give the sequence

$$\begin{array}{ccccccc} 0 & \rightarrow & H_{s+1}(P) & \rightarrow & H_{s+1}(L) & \xrightarrow{\Pi} & H_{s+1}(L, P) \xrightarrow{\partial} H_s(P) \rightarrow 0 \\ & & \downarrow \scriptstyle \cong & & \downarrow \scriptstyle \cong & & \downarrow \scriptstyle \cong \\ & & \hat{G} & & H & & \hat{H} \\ & & & & & & \downarrow \scriptstyle \cong \\ & & & & & & G \end{array}$$

We now describe how $\lambda: H \times H \rightarrow Z$ determines the linking invariant $b: G^* \times G^* \rightarrow S$ and to do this we introduce a non-singular symmetric bilinear map $\hat{\lambda}: H' \times H' \rightarrow Q$. The construction of $\hat{\lambda}$ is defined

in (Q) to which we refer for details.

The subgroup H' of $\hat{H}$ consists of those elements $c \in \hat{H}$ such that mc is in the image of the homomorphism $\Pi: H \rightarrow H$ for some integer m . The non-singular symmetric bilinear map

$$\hat{\lambda}: H' \times H' \rightarrow Q$$

is defined as follows. If $c, d \in H'$ then for some integers m, n we have $mc = \Pi(a)$, $nd = \Pi(b)$ for some elements $a, b \in H$; we now set

$$\hat{\lambda}(c, d) = \frac{1}{m \cdot n} \lambda(a, b).$$

$\hat{\lambda}$ is well-defined by (Q) and since λ is a symmetric bilinear map it follows that $\hat{\lambda}$ must also be. Factoring by $\ker \Pi \lambda$ induces

$$\lambda': H/\ker \Pi \times H/\ker \Pi \rightarrow Z$$

which will be non-singular and will still give $\hat{\lambda}$ by the method described above and so it will follow that $\hat{\lambda}$ is non-singular.

Now if elements $x, y \in G^* \subset G$ are such that x has order m and y has order n then if $\partial(c) = x$, $\partial(d) = y$ for element $c, d \in \hat{H}$ we have $\partial(mc) = 0$ and $\partial(nd) = 0$ giving $mc = \Pi(a)$ and $nd = \Pi(b)$ for some elements $a, b \in H$. Hence $c, d \in H'$ and the linking invariant is given by:

$$b(x,y) = \hat{\lambda}(c,d) \bmod 1.$$

And going back to the map λ we have

$$b(x,y) = \frac{1}{m \cdot n} \lambda(a,b) \bmod 1.$$

The fact that $\hat{\lambda}$ does give a well-defined non-singular symmetric bilinear form $b: G^* \times G^* \rightarrow S$ follows from Lemma 4 of (Q). The formula above can be verified by using the algebraic definition of the linking invariant b given in this chapter and by diagram chasing techniques in commutative diagrams including the isomorphisms

$$H^s(P; S) \rightarrow \text{Hom}(H_s(P), S), \quad H^{s+1}(L, P; S) \rightarrow \text{Hom}(H_{s+1}(L, P), S),$$

$$H^{s+1}(L, P; Q) \rightarrow \text{Hom}(H_{s+1}(L, P), Q), \quad H^{s+1}(L, P) \rightarrow \text{Hom}(H_{s+1}(L, P), Z),$$

$$H^{s+1}(L) \rightarrow \text{Hom}(H_{s+1}(L), Z). \quad \text{However if the geometric definition}$$

of the linking invariant by linking numbers, as given in (VI), is used the result follows rather more simply, see proof of Theorem 8 in (VI).

We now consider the elements $\hat{\alpha} \in H^{s+1}(L)$ and $\hat{\beta} \in H^{s+1}(P)$.

Since $\hat{\alpha}$ and $\hat{\beta}$ are both obstruction to a cross-section of the tangent bundles over the $(s+1)$ - skeletons of L and P respectively it follows that if $i: P \subset L$ is the inclusion inducing $i^*: H^{s+1}(L) \rightarrow H^{s+1}(P)$ then $i^*(\hat{\alpha}) = \hat{\beta}$.

Theorem 1.6

If P is a closed $(s-1)$ -connected $(2s+1)$ -manifold, where $s = 3$ or 7 , then:

(i) For $s = 3$ $P = \partial L$ for some manifold $L \in \mathcal{H}(4)$.

For $s = 7$ $P \neq \Sigma = \partial L$ for some manifold $L \in \mathcal{H}(8)$

where $\Sigma \in \textcircled{H}_{15}$ is a homotopy sphere.

(ii) If $P = \partial L$ for $L \in \mathcal{H}(s+1)$ and P and L have invariants

$(G, b, \hat{\beta})$ and $(H, \lambda, \hat{\alpha})$ respectively then

$G \cong \text{coker } \Pi$ where $\Pi : H \rightarrow \hat{H}$ is the homomorphism given by

$$\Pi(a)(b) = \lambda(a, b) \text{ for } a, b \in H. \quad b(x, y) = \frac{1}{m \cdot n} \lambda(a, b)$$

where $x, y \in G^*$ have orders m, n respectively $\partial(c) = x$, $\partial(d) = y$

for $c, d \in \hat{H}$ and $\Pi(a) = mc$, $\Pi(b) = nd$ where $\partial : \hat{H} \rightarrow G$.

$i^*(\hat{\alpha}) = \hat{\beta}$ where $i : P \subset L$ is the inclusion map.

Proof

Part (ii) is just a display of the results we have discussed above and so it remains only to prove part (i).

The main theorem of Wall's paper (45) states:

Let W be an m -parallelisable $(2m+1)$ -manifold. If the boundary of W

has no homology in dimensions $m, m+1$ then W is χ - equivalent to an m - connected manifold.

The result is proved by showing that if W is assumed to be $(m-1)$ - connected the homotopy group $\Pi_m(W)$ can be killed by a sequence of surgeries of type $(m+1, m+1)$.

The manifold P is s -parallelisable and $(s-1)$ - connected and its boundary is vacuum so it follows by this method that P is χ - equivalent, by a sequence of surgeries of type $(s+1, s+1)$, to an s - connected manifold which of necessity must be a homotopy sphere $\Sigma \in \bigoplus_{2s+1} H$. Now a surgery can be chosen to avoid an embedded disc in P and so it follows that $P \# (-\Sigma)$ can be formed by a sequence of surgeries of type $(s+1, s+1)$ from the sphere $S^{2s+1} = \Sigma \# (-\Sigma)$.

If $f: S^s \times D^{s+1} \rightarrow S^{2s+1}$ is the embedding for the first surgery then as $S^{2s+1} = \partial D^{2s+2}$ we use f as the attaching map to add an $(s+1)$ - handle h^{s+1} to D^{2s+2} . Then $\partial(D^{2s+2} \cup h^{s+1})$ is the manifold obtained from S^{2s+1} after the surgery. We continue in the same way for the other surgeries using the embeddings as attaching maps for $(s+1)$ - handles and if $P \# (-\Sigma)$ is obtained from S^{2s+1} by a sequence of k surgeries we obtain a handlebody $L \in \mathcal{H}(2s+2, k, s+1)$ with boundary $P \# (-\Sigma)$. This proves the result for $s=7$.

Now a homotopy $(2s+1)$ - sphere that bounds a parallelisable manifold bounds a parallelisable manifold $W \in \mathcal{H}(s+1)$ since if the original manifold were not s - connected it could be made so by surgery. Every homotopy 7- sphere Σ bounds a parallelisable manifold and so $\Sigma = \partial W$ where $W \in \mathcal{H}(4)$ is parallelisable and then $P = P \# (-\Sigma) \# \Sigma = \partial(L+W)$ where $L+W \in \mathcal{H}(4)$. §.

Possible Values for the Invariants $(G, b, \hat{\beta})$

By Lemma 1.2 the invariant $\hat{\beta} \in H^{s+1}(P)$ of a manifold P is even and in this section it is shown that this is essentially the only restriction on the possible values the invariants $(G, b, \hat{\beta})$ can take. First we consider the maps λ and $\hat{\lambda}$ associated with a manifold L .

For a manifold $L \in \mathcal{H}(s+1)$, where $s = 3$ or 7 , suppose that $H = H_{s+1}(L)$ has rank r and that $e_1, \dots, e_r$ is a basis for H . Then the symmetric bilinear map $\lambda: H \times H \rightarrow \mathbb{Z}$ can be represented by the intersection matrix

$$A = \begin{matrix} & e_1, \dots, \dots, e_r \\ \left[\begin{array}{cccc} \lambda(e_1, e_1) & \dots & \dots & \lambda(e_1, e_r) \\ & \dots & & \\ & & \dots & \\ \lambda(e_r, e_1) & \dots & \dots & \lambda(e_r, e_r) \end{array} \right] \end{matrix}$$

A is a symmetric integral $r \times r$ matrix. If another basis were chosen for H and M is the matrix representing this basis in terms of the original basis, then M will be an integral matrix of determinant ± 1 and the intersection matrix in terms of the new basis will be $M^t A M$. Thus λ is represented by an integral congruence class $\{A\}$ of symmetric integral $r \times r$ matrices.

If the homomorphism $\Pi: H \rightarrow \hat{H}$ is a monomorphism, which occurs if and only if the group $G = H_s(P)$ where $P = \partial L$ is a finite group of order N say, then $\Pi(H)$ will be a subgroup of $\hat{H}$ of index N. It will then follow in this case that $\det A = \pm N$. Also we will have $H' = \hat{H}$ and hence a symmetric bilinear map $\hat{\lambda}: \hat{H} \times \hat{H} \rightarrow Q$. Corresponding to the basis $e_1, \dots, e_r$ for H if we take the dual basis $\hat{e}_1, \dots, \hat{e}_r$ for $\hat{H}$ then by definition the homomorphism Π is given by:

$$\Pi(e_i) = \lambda(e_i, e_1) \hat{e}_1 + \dots + \lambda(e_i, e_r) \hat{e}_r$$

for $i = 1, \dots, r$. $\hat{\lambda}$ can be represented by the rational symmetric $r \times r$ matrix

$$\hat{A} = \begin{bmatrix} \hat{e}_1, \dots, \hat{e}_r \\ \hat{\lambda}(\hat{e}_1, \hat{e}_1) \dots \hat{\lambda}(\hat{e}_1, \hat{e}_r) \\ \dots \\ \hat{\lambda}(\hat{e}_r, \hat{e}_1) \dots \hat{\lambda}(\hat{e}_r, \hat{e}_r) \end{bmatrix}$$

If we write the product $\hat{A}\hat{A} = (a_{ij})$ $i, j=1, \dots, r$

then

$$\begin{aligned} a_{ij} &= \lambda(e_i, e_1) \hat{\lambda}(\hat{e}_1, \hat{e}_j) + \dots + \lambda(e_i, e_r) \hat{\lambda}(\hat{e}_r, \hat{e}_j) \\ &= \hat{\lambda}(\lambda(e_i, e_1) \hat{e}_1 + \dots + \lambda(e_i, e_r) \hat{e}_r, \hat{e}_j) \\ &= \hat{\lambda}(\Pi(e_i), \hat{e}_j) \\ &= \hat{\lambda}(\Pi(e_i), \frac{\det C}{\det A}) \end{aligned}$$

where C is formed from A by replacing the j th column by

$$\begin{bmatrix} \Pi(e_1) \\ \vdots \\ \Pi(e_r) \end{bmatrix}$$

$$\text{Then } a_{ij} = \frac{\det B}{\det A}$$

where B is the matrix formed from A by replacing the j th column by

$$\begin{bmatrix} \lambda(e_1, e_i) \\ \vdots \\ \lambda(e_r, e_i) \end{bmatrix}$$

since $\hat{\lambda}(\Pi(e_i), \Pi(e_k)) = \lambda(e_i, e_k) = \lambda(e_k, e_i)$.

Thus if $i \neq j$ then B has its i th and j th columns identical and so

$\det B = 0$ and $a_{ij} = 0$.

If $i = j$ then $B = A$ and $a_{ii} = 1$. So (a_{ij}) is the identity matrix and A and $\hat{A}$ are inverses of each other. Thus $\det \hat{A} = \pm \frac{1}{N}$. The boundary of L will be a homotopy sphere then if and only if $\det A = \pm 1$ when $\det \hat{A} = \pm 1$.

Two manifolds L_1, L_2 with invariants $(H_1, \lambda_1, \hat{\alpha}_1), (H_2, \lambda_2, \hat{\alpha}_2)$ are diffeomorphic by Proposition 1.4 if and only if there exists an isomorphism $k: H_1 \rightarrow H_2$ preserving intersection numbers and sending $\hat{\alpha}_1$ to $\hat{\alpha}_2$. By preserving intersection numbers we mean that if $e_1, \dots, e_r$ is a basis for H_1 and $E_1 = k(e_1), \dots, E_r = k(e_r)$ is the corresponding basis for H_2 then the intersection matrices A_1 and A_2 with respect to the bases $e_1, \dots, e_r$ of H_1 and $E_1, \dots, E_r$ of H_2 must be identical.

Since the manifolds L are classified by invariants $(H, \lambda, \hat{\alpha})$ and by Proposition 1.5 for boundary connected sums if $H = H_1 \oplus H_2$ and λ splits as a direct sum of λ_1 on H_1 and λ_2 on H_2 (i.e. $\lambda(x_1 \oplus x_2, y_1 \oplus y_2) = \lambda_1(x_1, y_1) + \lambda_2(x_2, y_2)$ for $x_1, y_1 \in H_1, x_2, y_2 \in H_2$) then $L = L_1 + L_2$, the boundary connected sum, with invariants $(H_1, \lambda_1, \hat{\alpha}_1)$ for L_1 and $(H_2, \lambda_2, \hat{\alpha}_2)$ for L_2 where $\hat{\alpha} = \hat{\alpha}_1 \oplus \hat{\alpha}_2$. In terms of intersection matrices if there exists a basis $e_1, \dots, e_r$ of H such that the intersection matrix of L is of the form:

$$A = \begin{array}{c} e_1, \dots, e_q, e_{q+1}, \dots, e_r \\ \left[\begin{array}{cc|cc} & & & \\ & B & & 0 \\ \hline & 0 & & C \\ & & & \end{array} \right] \end{array}$$

then $L = L_1 + L_2$ with invariants $(H_1, \lambda_1, \hat{\alpha}_1)$ for L_1 and $(H_2, \lambda_2, \hat{\alpha}_2)$ for L_2 where H_1 has basis $e_1, \dots, e_q$ and H_2 has basis $e_{q+1}, \dots, e_r$.

The intersection matrices of λ_1 and λ_2 with respect to these bases are B and C respectively, and if $\hat{\alpha} = \alpha_1 \hat{e}_1 + \dots + \alpha_r \hat{e}_r$ then $\hat{\alpha}_1 = \alpha_1 \hat{e}_1 + \dots + \alpha_q \hat{e}_q$ and $\hat{\alpha}_2 = \alpha_{q+1} \hat{e}_{q+1} + \dots + \alpha_r \hat{e}_r$.

If a manifold L has invariants $(H, \lambda, \hat{\alpha})$ then the only relation between the invariants is that $\lambda(x, x)$ and $\hat{\alpha}(x)$ must have the same parity for $x \in H$. If $e_1, \dots, e_r$ is a basis for H and in terms of the dual basis $\hat{\alpha} = \alpha_1 \hat{e}_1 + \dots + \alpha_r \hat{e}_r$ then the parity of the coefficients $\alpha_1, \dots, \alpha_r$ is determined by λ . $\hat{\alpha}(e_i) = \alpha_i$ and so α_i must have the same parity as the diagonal element $\lambda(e_i, e_i)$ of the intersection matrix $A = (\lambda(e_i, e_j))$.

We now consider non-singular symmetric bilinear forms $B; G \times G \rightarrow S$ where G is a finite abelian group; these forms are discussed in §5 of (Q). b is said to be the direct sum of the non-singular symmetric bilinear forms $b_1: G_1 \times G_1 \rightarrow S$ and $b_2: G_2 \times G_2 \rightarrow S$ if $G = G_1 \oplus G_2$ and $b(x_1 \oplus x_2, y_1 \oplus y_2) = b_1(x_1, y_1) + b_2(x_2, y_2)$ for $x_1, y_1 \in G_1$ and $x_2, y_2 \in G_2$, and b_1 and b_2 are said to generate b .

b is irreducible if it cannot be expressed as a proper direct sum of two forms. The following result is essentially Theorem 4 of (Q).

Proposition 1.7

Any non-singular symmetric bilinear form $b: G \times G \rightarrow S$, where G is finite abelian group, is generated by the following irreducible non-singular symmetric bilinear forms.

1. $G = Z_p^k$, where p is a positive odd prime and $k \geq 1$.
 $b: Z_p^k \times Z_p^k \rightarrow S$ is given by $b(x, x) = p^{-k}$ for a generator x of Z_p^k .
2. $G = Z_p^k$, where p is a positive odd prime and $k \geq 1$.
 $b: Z_p^k \times Z_p^k \rightarrow S$ is given by $b(x, x) = n p^{-k}$ for a generator x of Z_p^k , where $n \neq 1$ is prime to p .
3. $G = Z_2^k$, where $k \geq 1$ and $b: Z_2^k \times Z_2^k \rightarrow S$ is given by
 $b(x, x) = 2^{-k}$ for a generator x of Z_2^k .
4. $G = Z_2^k$, where $k \geq 2$ and $b: Z_2^k \times Z_2^k \rightarrow S$ is given by
 $b(x, x) = -2^{-k}$ for a generator x of Z_2^k .

5. $G = Z_2^k$, where $k \geq 3$ and $b: Z_2^k \times Z_2^k \rightarrow S$ is given by $b(x,x) = 5 \cdot 2^{-k}$ for a generator x of Z_2^k .
6. $G = Z_2^k$, where $k \geq 3$ and $b: Z_2^k \times Z_2^k \rightarrow S$ is given by $b(x,x) = -5 \cdot 2^{-k}$ for a generator x of Z_2^k .
7. $G = Z_2^k \oplus Z_2^k$, where $k \geq 1$ and $b: (Z_2^k \oplus Z_2^k) \times (Z_2^k \oplus Z_2^k) \rightarrow S$ is given by $b(x,x) = b(y,y) = 0$ and $b(x,y) = 2^{-k}$ for generators x, y of $Z_2^k \oplus Z_2^k$.
8. $G = Z_2^k \oplus Z_2^k$, where $k \geq 2$ and $b: (Z_2^k \oplus Z_2^k) \times (Z_2^k \oplus Z_2^k) \rightarrow S$ is given by $b(x,x) = b(y,y) = 2^{1-k}$ and $b(x,y) = 2^{-k}$ for generators x, y of $Z_2^k \oplus Z_2^k$.

Theorem 1.8

The invariants $(G, b, \hat{\beta})$ for closed $(s-1)$ -connected $(2s+1)$ -manifolds P , where $s = 3$ or 7 , can assume the following values:

G can be any finitely generated abelian group. $b: G^* \times G^* \rightarrow S$ can be any non-singular symmetric bilinear form.

$\hat{\beta} \in H^{s+1}(P) \simeq G$ can be any even element.

Proof

Since we are dealing with compact manifold $G = H_s(P)$ is necessarily finitely generated, b by definition is a non-singular symmetric bilinear form and by Lemma 1.2 $\hat{\beta}$ is even, so we have to show that all these possible values can occur.

By Lemma 1.3 the invariants $(G, b, \hat{\beta})$ add together by direct sum if connected sums of the manifolds P are taken. Thus if we can prove the result of the theorem for manifolds P with G and b given by the 8 irreducible forms of Proposition 1.7 and if we can also prove the result for manifolds P with $G \approx Z$ then by taking connected sums we will have the result of the theorem in general. We give examples of manifolds P with the required invariants as boundaries of manifolds $L \in \mathcal{H}(s+1)$.

Consider a manifold $L \in \mathcal{H}(s+1)$ with $H \approx Z$ and $\lambda = 0$, then $\Pi = 0$ and if $P = \partial L$ $G \approx \text{coker } \Pi \approx Z$ and in the cohomology sequence

$$0 \rightarrow H^s(P) \rightarrow H^{s+1}(L, P) \xrightarrow{\cong} H^{s+1}(L) \xrightarrow{i^*} H^{s+1}(P) \rightarrow 0$$

$$\begin{array}{ccccccc} \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathbb{R} & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} \\ G \approx Z & & H \approx Z & & H \approx Z & & G \approx Z \end{array}$$

i^* maps $\hat{\alpha}$ isomorphically to $\hat{\beta}$. $\hat{\alpha}$ is required to be even since $\lambda = 0$ and since any even value of $\hat{\alpha}$ can occur for a manifold L it

follows that any even value of $\hat{\beta}$ can occur for a manifold P.

For the 8 irreducible forms we give rational symmetric matrices $\hat{A}$ of order $\pm \frac{1}{N}$, where N is the order of G, and which when reduced modulus 1 give the form b. Then the inverse matrix A of $\hat{A}$ will be integral and can be taken as the intersection matrix of a manifold L; $P = \partial L$ will then have invariants G and b. The matrices $\hat{A}$ here are also given in § 7 of (Q).

For case (7) we take the matrix

$$\hat{A} = \begin{matrix} & \begin{matrix} \hat{e}_1 & \hat{e}_2 \end{matrix} \\ \begin{bmatrix} 0 & 2^{-k} \\ 2^{-k} & 0 \end{bmatrix} & \end{matrix} \quad \text{For a corresponding manifold L if}$$

$\hat{\alpha} = \alpha_1 \hat{e}_1 + \alpha_2 \hat{e}_2$ then since $\lambda(e_1, e_1) = \lambda(e_2, e_2) = 0$ α_1 and α_2 must be even. Under the map

$$\begin{array}{ccc} i*: & H^{s+1}(L) & \rightarrow & H^{s+1}(P) \\ & \downarrow & & \downarrow \\ & \hat{H} \approx Z \oplus Z & & G \approx Z_2^k \oplus Z_2^k \end{array}$$

$\hat{e}_1$ and $\hat{e}_2$ will go to generators x and y of $Z_2^k \oplus Z_2^k$ and so by a suitable choice of the even integers α_1, α_2 we can obtain any even element in G for $\hat{\beta}$.

For case (8) we take the matrix

$$\hat{A} = \begin{matrix} & \begin{matrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \end{matrix} \\ \begin{matrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{matrix} & \begin{bmatrix} 2^{1-k} & 2^{-k} & 0 \\ 2^{-1} & 2^{1-k} & 1 \\ 0 & 1 & d \end{bmatrix} \end{matrix} \quad \text{where } 3d = 2^{k+1} + (-1)^k$$

For a corresponding manifold L if $\hat{\alpha} = \alpha_1 \hat{e}_1 + \alpha_2 \hat{e}_2 + \alpha_3 \hat{e}_3$ then since $\lambda(e_1, e_1)$ and $\lambda(e_2, e_2)$ are both even (they are both divisible by 2^{k-1}) α_1 and α_2 are even.

$$\begin{array}{ccc} \text{Under the map } i^*: & H^{s+1}(L) & \rightarrow H^{s+1}(P) \\ & \mathbb{R} & \mathbb{R} \\ & \hat{H} \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & G \simeq \mathbb{Z}_2^k \oplus \mathbb{Z}_2^k \end{array}$$

$\hat{e}_1$ and $\hat{e}_2$ are sent to generators x and y of $\mathbb{Z}_2^k \oplus \mathbb{Z}_2^k$ and so by a suitable choice of the even integers α_1, α_2 we can obtain any even element in G for $\hat{\beta}$.

The remaining cases (1) - (6) are all of the form

$G \simeq \mathbb{Z}_p^k$ and $b: \mathbb{Z}_p^k \times \mathbb{Z}_p^k \rightarrow S$ given by $b(x, x) = n p^{-k}$ for some integer n prime to p , where p is a prime including the case $p=2$. We can assume $0 < n < p^k$ and we take the matrix:

$$\hat{A} = \begin{matrix} & \hat{e}_1 & \hat{e}_2 & & & & \hat{e}_{r+1} \\ \left[\begin{array}{cccccccc} n p^{-k} & 1 & 0 & \dots & \dots & \dots & 0 \\ 1 & a_1 & 1 & \dots & \dots & \dots & \dots \\ 0 & 1 & a_2 & \dots & \dots & \dots & \dots \\ \cdot & \cdot & \cdot & \dots & \dots & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \dots & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \dots & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \dots & a_{r-1} & 1 \\ 0 & \dots & \dots & \dots & 0 & 1 & a_r \end{array} \right. \end{matrix}$$

where the integers $a_1, \dots, a_r$ are given by the Euclidean algorithm where we have

$$\begin{aligned} 1 &= n d_1 - p^k d_2 \quad (0 < d_2 < d_1 \text{ since } 0 < n < p^k) \\ d_1 &= a_1 d_2 - d_3 \\ &\vdots \\ d_{r-1} &= a_{r-1} d_r - 1 \\ d_r &= a_r. \end{aligned}$$

It follows by induction that d_{r+1-i} is the determinant of the last i rows and columns and hence p^{-k} is the determinant of the whole matrix.

For a corresponding manifold L under the map

$$i^*: \begin{array}{c} H^{s+1}(L) \\ \cong \\ \hat{H} \end{array} \rightarrow \begin{array}{c} H^{s+1}(P) \\ \cong \\ G \simeq \mathbb{Z}_p^k \end{array}$$

$\hat{e}_1$ maps to a generator x of $\mathbb{Z}_p^k$.

Now if p is odd every element of $\mathbb{Z}_p^k$ is even and the inverse image of an element under i^* contains both even and odd multiples of $\hat{e}_1$; hence whether α_1 is restrained to be even or odd we can choose suitable values such that $\alpha_1 \hat{e}_1$ maps to any element of $\mathbb{Z}_p^k$ showing that $\hat{\beta}$ can take any value in $\mathbb{Z}_p^k$.

For $p = 2$ then α_1 is even since $\lambda(e_1, e_1)$ is even (it is divisible by 2^k) and by a suitable choice of the even integer α_1 we can obtain any even element in $\mathbb{Z}_2^k$ for $\hat{\beta}$. § .

CHAPTER 2

DECOMPOSITIONS OF THE MANIFOLDS P

This chapter is basically concerned with the question of which invariants (H, λ) can occur for a manifold L whose boundary P has invariants (G, b) (the tangential invariants $\hat{\alpha}$ and $\hat{\beta}$ play no part in this chapter). Theorem 2.7 shows that if (H, λ) induces (G, b) algebraically then a manifold P with invariants (G, b) , up to the addition of a homotopy sphere, occurs as the boundary of a manifold L with invariants (H, λ) . As a corollary it follows that any manifold P is a connected sum of indecomposable manifolds, which are manifolds whose invariants are algebraically irreducible. The problem is considered from a purely algebraic point of view to start with and this allows us to make clear a few terms that were used in Chapter 1 without a precise definition. The main result in this section is Theorem 2.2 which shows that the notion of stable equivalence defined for maps λ corresponds to the notion of equivalence defined for pairs (G, b) .

Symmetric Bilinear Maps

We consider symmetric bilinear maps $\lambda: H \times H \rightarrow Z$ where H is a free abelian group of finite rank.

If $e_1, \dots, e_r$ is a basis for H then λ can be represented by the matrix $A = (\lambda(e_i, e_j))$ which will be a symmetric integral $r \times r$ matrix.

λ defines a homomorphism $\Pi: H \rightarrow \hat{H}$ by the rule $\Pi(a)(b) = \lambda(a, b)$ for $a, b \in H$.

λ is said to be non-singular if Π is a monomorphism; this coincides with the matrix A being non-singular i.e. $\det A \neq 0$.

λ is said to be unimodular if Π is an isomorphism; this coincides with the matrix A being unimodular i.e. $\det A = \pm 1$. (In (Q) 'non-singular' is used where we use 'unimodular').

λ is said to be even if $\lambda(a, a)$ is even for all $a \in H$ and λ is said to be odd otherwise. Thus λ is even if the diagonal elements of A are all even and λ is odd if A has at least one odd diagonal element.

The direct sum of two maps $\lambda_1: H_1 \times H_1 \rightarrow Z$ and $\lambda_2: H_2 \times H_2 \rightarrow Z$ is the map $\lambda_1 + \lambda_2: H_1 \oplus H_2 \times H_1 \oplus H_2 \rightarrow Z$ given by

$$\lambda_1 + \lambda_2(a_1 \oplus a_2, b_1 \oplus b_2) = \lambda_1(a_1, b_1) + \lambda_2(a_2, b_2) \quad \text{for } a_1, b_1 \in H_1, \\ a_2, b_2 \in H_2. \quad \text{If for a basis } e_1, \dots, e_r \text{ of } H_1 \quad \lambda_1 \text{ has matrix } A_1 \text{ and} \\ \text{for a basis } f_1, \dots, f_s \text{ of } H_2 \quad \lambda_2 \text{ has matrix } A_2 \text{ then for the basis} \\ e_1, \dots, e_r, f_1, \dots, f_s \text{ of } H_1 \oplus H_2 \quad \lambda_1 + \lambda_2 \text{ will have the matrix}$$

$$\begin{bmatrix} A_1 & & 0 \\ & & \\ 0 & & A_2 \end{bmatrix}$$

For a map $\lambda: H \times H \rightarrow Z$ we define a subgroup H' of $\hat{H}$ as follows. An element $c \in H'$ if there exists an integer m such that mc is in the image of Π . We define a non-singular symmetric bilinear map $\hat{\lambda}: H' \times H' \rightarrow Q$ as follows. For $c, d \in H'$ there exist integers m, n such that $mc = \Pi(a)$, $nd = \Pi(b)$ for some elements $a, b \in H$. We set $\hat{\lambda}(c, d) = \frac{1}{m \cdot n} \lambda(a, b)$.

If λ is non-singular then $H' = \hat{H}$ and we have $\hat{\lambda}: \hat{H} \times \hat{H} \rightarrow Q$. If $e_1, \dots, e_r$ is a basis for H and $\hat{e}_1, \dots, \hat{e}_r$ is the dual basis for $\hat{H}$ then $\hat{\lambda}$ can be represented by the matrix $A = (\hat{\lambda}(\hat{e}_i, \hat{e}_j))$ which will be a non-singular symmetric rational $r \times r$ matrix. We have shown in Chapter 1 that A and $\hat{A}$ are inverses of each other. Since λ is

non-singular coker Π is a finite group G of order N say and it follows that $\det A = \pm N$ while $\det \hat{A} = \pm \frac{1}{N}$.

Two maps $\lambda_1: H_1 \times H_1 \rightarrow Z$ and $\lambda_2: H_2 \times H_2 \rightarrow Z$ are equivalent if there exists an isomorphism $k: H_1 \rightarrow H_2$ such that $\lambda_1(a,b) = \lambda_2(k(a),k(b))$ for all $a,b \in H_1$. If H_1 has a basis $e_1, \dots, e_r$ and matrix A_1 and H_2 has a basis $f_1, \dots, f_r$ and matrix A_2 then λ_1 and λ_2 are equivalent if A_1 and A_2 are integrally congruent i.e. there exists an $r \times r$ integral matrix M with $\det M = \pm 1$ and $M^t A_1 M = A_2$.

Two maps $\lambda_1: H_1 \times H_1 \rightarrow Z$ and $\lambda_2: H_2 \times H_2 \rightarrow Z$ are defined to be stably equivalent if there exist unimodular maps λ_1^u and λ_2^u such that $\lambda_1 + \lambda_1^u$ is equivalent to $\lambda_2 + \lambda_2^u$. In terms of matrices if A_1 and A_2 are the matrices of λ_1 and λ_2 respectively then λ_1 and λ_2 are stably equivalent if there exist symmetric unimodular integral matrices B_1 and B_2 such that $\begin{bmatrix} A_1 & | & 0 \\ \hline 0 & | & B_1 \end{bmatrix}$ and $\begin{bmatrix} A_2 & | & 0 \\ \hline 0 & | & B_2 \end{bmatrix}$ are integrally congruent.

If λ_1 and λ_2 are non-singular then the conditions for equivalence and stable equivalence can be stated in terms of the matrices $\hat{A}_1$ and $\hat{A}_2$ which represent $\hat{\lambda}_1$ and $\hat{\lambda}_2$. λ_1 and λ_2 will be equivalent if the matrices $\hat{A}_1$ and $\hat{A}_2$ are integrally congruent. λ_1 and λ_2 will be stably equivalent if there exist symmetric unimodular integral matrices $\hat{B}_1$ and $\hat{B}_2$ such that $\begin{bmatrix} \hat{A}_1 & | & 0 \\ \hline 0 & | & \hat{B}_1 \end{bmatrix}$ and $\begin{bmatrix} \hat{A}_2 & | & 0 \\ \hline 0 & | & \hat{B}_2 \end{bmatrix}$ are integrally congruent.

We now consider non-singular symmetric bilinear maps

$b: G^* \times G^* \rightarrow S$ ($S = Q/Z$) where G^* is the torsion subgroup of a finitely generated abelian group G . Here non-singular means that the homomorphism $\phi: G^* \rightarrow \text{Hom}(G^*, S)$ given by $\phi(x)(y) = b(x, y)$ is a monomorphism; since G^* is finite it follows that ϕ is an isomorphism. The pair (G, b) consists of a finitely generated abelian group G and a non-singular symmetric bilinear map $b: G^* \times G^* \rightarrow S$.

The direct sum of two maps $b_1: G_1^* \times G_1^* \rightarrow S$ and $b_2: G_2^* \times G_2^* \rightarrow S$ is the map $b_1 + b_2: G_1^* \oplus G_2^* \times G_1^* \oplus G_2^* \rightarrow S$ given by $b_1 + b_2(x_1 \oplus x_2, y_1 \oplus y_2) = b_1(x_1, y_1) + b_2(x_2, y_2)$ for $x_1, y_1 \in G_1^*$ and $x_2, y_2 \in G_2^*$.

The direct sum of two pairs (G_1, b_1) and (G_2, b_2) is the pair $(G_1 \oplus G_2, b_1 + b_2)$.

By Proposition 1.7 any non-singular symmetric bilinear map $b: G^* \times G^* \rightarrow S$ is generated by the irreducible maps which can be labelled under headings (1) - (8) and where $G^* = Z_p^k$, for a prime p , or $Z_2^k \oplus Z_2^k$.

Two maps $b_1: G^*_1 \times G^*_1 \rightarrow S$ and $b_2: G^*_2 \times G^*_2 \rightarrow S$ are equivalent if there exists an isomorphism $f: G^*_1 \rightarrow G^*_2$ such that $b_1(x,y) = b_2(f(x), f(y))$ for all $x,y \in G^*_1$.

Two pairs (G_1, b_1) and (G_2, b_2) are equivalent if there exists an isomorphism $f: G_1 \rightarrow G_2$ such that $b_1(x,y) = b_2(f(x), f(y))$ for all $x,y \in G^*_1$.

We now describe how a symmetric bilinear map $\lambda: H \times H \rightarrow Z$ induces a pair (G, b) . We set $G = \text{coker } \Pi$ and let $\hat{G} = \text{Hom}(G, Z)$, which is isomorphic to the torsion-free subgroup of G , and G^* is the torsion subgroup of G . Then $\hat{G} \simeq \ker \Pi$ and we have a short exact sequence

$$0 \rightarrow \hat{G} \rightarrow H \xrightarrow{\Pi} \hat{H} \xrightarrow{\partial} G \rightarrow 0$$

For an element $x \in G^*$ suppose $\partial(c) = x$ for some $c \in \hat{H}$. Since x has finite order m say $\partial(mc) = 0$ when $mc = \Pi(a)$ for some element $a \in H$ and so $c \in H'$. If $y \in G^*$ with $\partial(d) = y$ then $d \in H'$ and we define a non-singular bilinear map $b: G^* \times G^* \rightarrow S$ by $b(x,y) = \hat{\lambda}(c,d)$ modulo 1. That b is a well defined symmetric bilinear map follows by Lemma 4 of (Q). We say that λ induces (G, b) . However if G is finite when $G = G^*$ we say simply that λ induces b , and this will occur if and only if λ is non-singular.

It is clear that if two maps λ_1 and λ_2 induce pairs (G_1, b_1) and (G_2, b_2) respectively then the sum $\lambda_1 + \lambda_2$ induces $(G_1 \oplus G_2, b_1 + b_2)$.

Examples of maps λ inducing the irreducible maps b of Proposition 1.7 were given in the proof of Theorem 1.8 by exhibiting a symmetric rational matrix $\hat{A}$ of determinant $\pm \frac{1}{N}$, where N is the order of $G = G^*$, and which when reduced modulo 1 represented b . A the inverse of $\hat{A}$, a symmetric integral matrix, was then the matrix of λ .

For case (7) we took

$$\hat{A} = \begin{bmatrix} 0 & 2^{-k} \\ 2^{-k} & 0 \end{bmatrix}$$

For case (8) we took

$$\hat{A} = \begin{bmatrix} 2^{1-k} & 2^{-k} & 0 \\ 2^{-k} & 2^{1-k} & 1 \\ 0 & 1 & d \end{bmatrix} \quad \text{where } 3d = 2^{k+1} + (-1)^k$$

For cases (1) - (6) we took

$$\hat{A} = \begin{bmatrix} np^{-k} & 1 & 0 & \dots & 0 \\ 1 & a_1 & 1 & \dots & \\ 0 & 1 & a_2 & \dots & \\ \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & a_{r-1} & 1 \\ 0 & \dots & 0 & 1 & a_r \end{bmatrix}$$

whose determinant is p^{-k} and the integers $a_1, \dots, a_r$ are determined by the Euclidean algorithm.

By a more careful choice of the matrices $\hat{A}$ above we obtain the result.

Lemma 2.1

Any pair (G, b) is induced by an even λ .

Proof

Suppose $\hat{G}$ has rank r and let H be a free abelian group of rank r and $\lambda_F: H \times H \rightarrow \mathbb{Z}$ zero. Then λ_F induces the pair $(\hat{G}, 0)$ and if a map λ_T induces (G^*, b) , i.e. λ_T induces b , then $\lambda_F + \lambda_T$ induces (G, b) .

Since $\lambda_F = 0$ is even it remains only to show that an even λ_T exists.

Since b is generated by the irreducible maps it follows that we need only prove the result for these irreducible maps for then by taking sums of the corresponding even maps λ we would have the result in general. For the irreducible maps b given by Proposition 1.7 we exhibit therefore a symmetric rational matrix $\hat{A}$ of determinant $\pm \frac{1}{N}$, where N is the order of $G = G^*$, which modulo 1 represents b and such that the symmetric integral matrix A , the inverse of $\hat{A}$, has even diagonal elements. The map λ corresponding to A will then be even and so give the result of the theorem.

For case (7) we take the matrix

$$\hat{A} = \begin{bmatrix} 0 & 2^{-k} \\ 2^{-k} & 0 \end{bmatrix} \quad \text{when } A = \begin{bmatrix} 0 & 2^k \\ 2^k & 0 \end{bmatrix}$$

obviously has even diagonal elements.

For case (8) we take the matrix

$$\hat{A} = \begin{bmatrix} 2^{1-k} & 2^{-k} & 0 & 0 \\ 2^{-k} & 2^{1-k} & 1 & 0 \\ 0 & 1 & 2c & 1 \\ 0 & 0 & 1 & 2d \end{bmatrix} \quad \begin{aligned} \text{where } 3c &= 2^k + (-1)^k \\ d &= (-1)^{k-1}. \end{aligned}$$

It can be easily checked that the inverse A has even diagonal elements.

For cases (1) and (2) we have $b: \mathbb{Z}_p^k \times \mathbb{Z}_p^k \rightarrow S$ with $b(x,x) = n p^{-k}$, where p is an odd prime, n is prime to p , $k \geq 1$, and x is a generator of $\mathbb{Z}_p^k$. We can suppose $0 < |n| < p^k$ and n even for if n were odd we could replace it by either $n - p^k$ or $n + p^k$ which would be even. We use the Euclidean algorithm to construct the matrix

$$\hat{A} = \begin{pmatrix} n p^{-k} & 1 & 0 & \dots & 0 \\ 1 & a_1 & 1 & \dots & \\ 0 & 1 & a_2 & \dots & \\ \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & a_{r-1} & 1 \\ 0 & \dots & 0 & 1 & a_r \end{pmatrix}$$

Since n and p are coprime there exist integers d_1 and d_2 such that

$$1 = n d_1 - p^k d_2$$

d_2 must be odd and we can suppose that d_1 is even (otherwise we could replace d_1 by $(d_1 + p^k)$ which would be even and d_2 by $(d_2 + n)$). Since $0 < |n| < p^k$ we have $0 < |d_2| < |d_1|$. We continue inductively setting

$$d_i = a_i d_{i+1} - d_{i+2}$$

where $a_i d_{i+1}$ is the closest even multiple of d_{i+1} to d_i with remainder d_{i+2} . We continue factoring until

$$d_{r-1} = a_{r-1} d_r + 1$$

and we set $\bar{+} d_r = a_r$.

Since $1 = (n, p^k) = (d_1, d_2) = (d_2, d_3) = \dots = (d_{r-1}, d_r)$ it follows inductively that d_i is even for i odd and odd for i even. r must therefore be odd and $a_r = \bar{+} d_r$ will be even.

If $d_r = a_r$ then $\det \hat{A} = p^{-k}$ and d_i is the determinant of the minor consisting of the last $r + 1 - i$ rows and columns. If $-d_r = a_r$ then $\det \hat{A} = -p^{-k}$ and $-d_i$ is the determinant of the minor consisting of the last $r + 1 - i$ rows and columns. In either case the integers $a_1, \dots, a_r$ are even by construction.

If D_i is the determinant of the minor consisting of the first i rows and columns of $\hat{A}$ then by induction we can see that D_i is an even multiple of p^{-k} for i odd and an odd multiple of p^{-k} for i even.

The diagonal element of A corresponding to the position of a_i in $\hat{A}$ is $p^k \cdot D_i \cdot d_{i+1}$ and since either i or $i+1$ is even it follows that this is an even integer. The first diagonal element of A is $p^k d_1$ which is even since d_1 is even.

For cases (3) - (6) we have $b: Z_2^k \times Z_2^k \rightarrow S$ with $b(x,x) = n 2^{-k}$ where $k \geq 1$ and n is odd, x being a generator of Z_2^k . We can suppose $0 < |n| < 2^k$ (in fact $n = +1, -1, +5$ or -5). We use the Euclidean algorithm to construct the matrix

$$\hat{A} = \begin{pmatrix} n 2^{-k} & 1 & 0 & \dots\dots\dots 0 \\ 1 & a_1 & 1 & \dots\dots\dots \\ 0 & 1 & a_2 & \dots\dots\dots \\ \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & 0 \\ \dots\dots\dots & \dots\dots\dots & a_{r-1} & 1 \\ 0 & \dots\dots\dots & 0 & 1 & a_r \end{pmatrix}$$

Since n and 2 are coprime there exist integers d_1 and d_2 such that

$$1 = n d_1 - 2^k d_2.$$

d_1 must be odd and we can suppose that d_2 is even (otherwise we could replace d_2 by $(d_2 + n)$ which would be even and d_1 by $(d_1 + 2^k)$). Since $0 < |n| < 2^k$ we have $0 < |d_2| < |d_1|$. We continue inductively setting

$$d_i = a_i d_{i+1} - d_{i+2}$$

where $a_i d_{i+1}$ is the closest even multiple of d_{i+1} to d_i with remainder d_{i+2} . We continue factorising until

$$d_{r-1} = a_{r-1} d_r \pm 1$$

and we set $\bar{d}_r = a_r$.

Since $1 = (n, 2^k) = (d_1, d_2) = (d_2, d_3) = \dots = (d_{r-1}, d_r)$ it follows inductively that d_i is even for i even and odd for i odd. r must therefore be even and $a_r = \bar{+} d_r$ will be even.

If $d_r = a_r$ then $\det \hat{A} = 2^{-k}$ and d_i is the determinant of the minor consisting of the last $r + 1 - i$ rows and columns. If $-d_r = a_r$ then $\det \hat{A} = -2^{-k}$ and $-d_i$ is the determinant of the minor consisting of the last $r + 1 - i$ rows and columns. In either case the integers $a_1, \dots, a_r$ are even by construction.

If D_i is the determinant of the minor consisting of the first i rows and columns then it can easily be seen by induction that D_i is an even multiple of 2^{-k} for $i \neq 1$ and $D_1 = n 2^{-k}$.

The diagonal element of A corresponding to the position of a_i in $\hat{A}$ is $2^k \cdot D_i \cdot d_{i+1}$ and this is even for $i \neq 1$ since $2^k D_i$ is even and for $i = 1$ since d_2 is even. The first diagonal element of A is $2^k d_1$ which is of course even. § .

Theorem 2.2

Two symmetric bilinear maps λ_1 and λ_2 are stably equivalent if and only if the induced pairs (G_1, b_1) and (G_2, b_2) are equivalent.

The proof of the theorem is preceded by three preparatory lemmas but before stating and proving these we state the following corollary which follows immediately from the theorem and Lemma 2.1.

Corollary 2.3

Any map λ is stably equivalent to an even map.

Lemma 2.4

If a non-singular map λ induces $b_1 + b_2$ where $b_1: Z_p^k \times Z_p^k \rightarrow S$ is irreducible of type (1) - (6) then λ is stably equivalent to $\lambda_1 + \lambda_2$ where λ_2 induces b_2 and λ_1 , which induces b_1 , is given by the matrix:

$$\hat{C} = \begin{bmatrix} n p^{-k} & 1 & 0 & \dots & 0 \\ 1 & a_1 & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & a_{r-1} & 1 \\ 0 & \dots & 0 & 1 & a_r \end{bmatrix} \quad \text{for } \hat{\lambda}_1, \text{ constructed by the Euclidean algorithm.}$$

Proof

Suppose the matrix $\hat{A}$ represents $\hat{\lambda}$ then the idea of the proof is to show that there exists a symmetric unimodular integral matrix $\hat{B}$ such that the matrix

$$\begin{bmatrix} \hat{A} & | & 0 \\ \hline 0 & | & \hat{B} \end{bmatrix}$$

is integrally equivalent to a matrix

$$\begin{bmatrix} \hat{C} & | & 0 \\ \hline 0 & | & \hat{D} \end{bmatrix}$$

Suppose the matrix D represents a map λ_2 then since λ_1 induces b_1 λ_2 must induce b_2 . B since it is unimodular will represent a unimodular map λ^u and so we will have $\lambda + \lambda^u$ equivalent to $\lambda_1 + \lambda_2$ which shows λ stably equivalent to $\lambda_1 + \lambda_2$. If $\lambda: H \times H \rightarrow Z$ and $\lambda^u: H^u \times H^u \rightarrow Z$ then $\hat{\lambda} + \hat{\lambda}^u$ has matrix

$$\begin{bmatrix} \hat{A} & | & 0 \\ \hline 0 & | & \hat{B} \end{bmatrix}$$

with respect to a basis for $\hat{H} \oplus \hat{H}^u$ given by bases for $\hat{H}$ and $\hat{H}^u$.

We show that by another choice of basis for $\hat{H} \oplus \hat{H}^u$ $\hat{\lambda} + \hat{\lambda}^u$ has matrix

$$\begin{bmatrix} \hat{C} & | & 0 \\ \hline 0 & | & \hat{D} \end{bmatrix}$$

If M is the integral matrix of determinant ± 1 representing the change of basis we have

$$M^t \begin{bmatrix} \hat{A} & | & 0 \\ \hline 0 & | & \hat{B} \end{bmatrix} M = \begin{bmatrix} \hat{C} & | & 0 \\ \hline 0 & | & \hat{D} \end{bmatrix}$$

or in terms of the integral matrices representing λ , λ^u , λ_1 and λ_2

$$M^{-1} \begin{bmatrix} A & | & 0 \\ \hline 0 & | & B \end{bmatrix} (M^{-1})^t = \begin{bmatrix} C & | & 0 \\ \hline 0 & | & D \end{bmatrix}$$

For a generator x of Z_p^k b_1 is given by $b_1(x, x) = n p^{-k}$ for some integer n prime to p and $b_2: G_2 \times G_2 \rightarrow S$ say. The map λ , at least up to adding on a unimodular map, is such that there exists a basis $\hat{e}_1, \dots, \hat{e}_m$ for $\hat{H}$ which in the exact sequence

$$0 \rightarrow H \xrightarrow{\Pi} \hat{H} \xrightarrow{\partial} Z_p^k \oplus G_2 \rightarrow 0$$

has $\partial(\hat{e}_1) = x$ and $\partial(\hat{e}_j) \in G_2$ for $j \neq 1$. If this were not so we would replace λ by $\lambda + \lambda^u$ where $\lambda^u: H^u \times H^u \rightarrow Z$ is unimodular

with $\hat{H}^u \approx Z$ having a generator $\hat{e}$, then if $\partial(c) = x$ for an element $c \in \hat{H}$ in the corresponding sequence for $\lambda + \lambda^u$ we have $\partial(\hat{e}_1) = x$ where $\hat{e}_1 = \hat{e} + c$ is an element that forms part of a basis for $\hat{H} \oplus \hat{H}^u$. The other elements of the basis can be modified so that they map to G_2 . Thus we can assume that the matrix $\hat{A}$ with respect to the basis $\hat{e}_1, \dots, \hat{e}_m$ is of the form

$$\hat{A} = \begin{array}{c} \begin{array}{ccc} & \hat{e}_1 & \hat{e}_2 & & \hat{e}_m \end{array} \\ \left[\begin{array}{c|cccc} d_1 + n p^{-k} & d_2 & \cdot & \cdot & \cdot & d_m \\ \hline d_2 & \dots & \dots & \dots & \dots & \dots \\ \cdot & \dots & \dots & \dots & \dots & \dots \\ \cdot & \dots & \dots & \dots & \dots & \dots \\ \cdot & \dots & \dots & \dots & \dots & \dots \\ d_m & \dots & \dots & \dots & \dots & \dots \end{array} \right] \end{array}$$

where $d_1, \dots, d_m$ are integers since we have for $j \neq 1$

$$\hat{\lambda}(\hat{e}_1, \hat{e}_j) \text{ modulo } 1 = b(\partial(\hat{e}_1), \partial(\hat{e}_j)) = 0 \text{ since } \partial(\hat{e}_1) = x \in Z_p^k$$

$$\text{and } \partial(\hat{e}_j) \in G_2 \text{ and } \hat{\lambda}(\hat{e}_1, \hat{e}_1) \text{ modulo } 1 = b(\partial(\hat{e}_1), \partial(\hat{e}_1)) = b_1(x, x) = n p^{-k}$$

By adding to $\hat{A}$ the $r \times r$ unimodular matrix

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}$$

and by changing the order of the basis we obtain the matrix

$$\left[\begin{array}{c|ccc|cccc} d_1 + n p^{-k} & 0 & \dots & 0 & d_2 & \dots & d_m \\ \hline 0 & 1 & 0 & \dots & 0 & & \\ \vdots & 0 & \ddots & \ddots & & & \\ \vdots & \vdots & \ddots & \ddots & & & \\ \vdots & \vdots & \ddots & \ddots & 0 & & \\ \vdots & 0 & \dots & 0 & 1 & & \\ \hline d_2 & & & & & \dots & \\ \vdots & & & & 0 & \dots & \\ d_m & & & & & \dots & \end{array} \right]$$

Now by adding a unimodular matrix (± 1) and by adding the basis element to the i th and j th elements of the basis for the matrix above we will increase the (i,j) th element of the matrix by ± 1 . By a sequence of steps like this we can change any element of the minor of the first $r + 1$ rows and columns by any integer. We thus obtain a matrix:

$$\begin{array}{c}
 \hat{f} \quad \hat{f}_1 \dots \dots \hat{f}_r \quad \hat{g}_1 \quad \hat{g}_2 \quad \dots \hat{g}_t \\
 \left[\begin{array}{cccc|cccc}
 n p^{-k} & 1 & 0 & \dots & 0 & c_1 & c_2 & \dots c_t \\
 1 & a_1 & & & & c_{11} & c_{21} & \dots c_{t1} \\
 0 & & & & & & & & & \\
 & & & & 0 & & & & & \\
 & & & & & & & & & \\
 & & & & a_{r-1} & 1 & & & & \\
 0 & \dots & 0 & 1 & a_r & c_{1r} & c_{2r} & \dots c_{tr} \\
 \hline
 c_1 & c_{11} & \dots & c_{1r} & & & & & & \\
 c_2 & c_{21} & \dots & c_{2r} & & & & & & \\
 \vdots & \vdots & & \vdots & & & & & & \\
 c_t & c_{t1} & \dots & c_{tr} & & & & & &
 \end{array} \right]
 \end{array}$$

with respect to a suitable choice of basis $\hat{f}, \hat{f}_1, \dots, \hat{f}_r, \hat{g}_1, \hat{g}_2, \dots, \hat{g}_t$ where the elements $c_i, c_{i1}, \dots, c_{ir}$ for $i = 1, \dots, t$ are integers.

Now add on the unimodular matrix

$$\begin{array}{c}
 \hat{h} \quad \hat{\ell} \quad \hat{h}_1 \quad \hat{\ell}_1 \dots \dots \hat{h}_r \quad \hat{\ell}_r \\
 \left[\begin{array}{cccc|cccc}
 0 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\
 -1 & 0 & & & & & & \\
 \hline
 0 & & 0 & -1 & & & & \\
 & & -1 & 0 & & & & \\
 & & & & & & & \\
 & & & & & & & 0 \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & 0 & -1 \\
 0 & \dots & 0 & & -1 & 0 & &
 \end{array} \right]
 \end{array}$$

with respect to
the basis

$$\hat{h}, \hat{\ell}, \hat{h}_1, \hat{\ell}_1, \dots, \hat{h}_r, \hat{\ell}_r$$

By the change of basis indicated we obtain a matrix in the required form.

[illegible]

Proof

The steps in the proof of this lemma and the next lemma, Lemma 2.6, are the same as those used in Lemma 2.4 and so we produce just the main points here.

By adding a suitably large unimodular map λ^u to λ and by taking a suitable basis $\hat{e}_1, \hat{e}_2, \hat{f}_1, \hat{f}_2, \dots, \hat{f}_r$ for $\hat{H} \oplus \hat{H}^u$ the matrix for $\hat{\lambda} + \hat{\lambda}^u$ will be of the form

$$\begin{array}{c} \begin{array}{cccccc} \hat{e}_1 & \hat{e}_2 & \hat{f}_1 & \hat{f}_2 & \dots & \hat{f}_r \end{array} \\ \left[\begin{array}{cc|cccc} 0 & 2^{-k} & a_1 & a_2 & \dots & a_r \\ 2^{-k} & 0 & c_1 & c_2 & \dots & c_r \\ \hline a_1 & c_1 & & & & \\ a_2 & c_2 & & & & \\ \dots & & & & & \\ \dots & & & & & \\ a_r & c_r & & & & \end{array} \right] \end{array}$$

where $a_1, a_2, \dots, a_r$ and $c_1, c_2, \dots, c_r$ are integers.

Now add onto this matrix the unimodular matrix

$$\begin{array}{c} \begin{array}{cccc} \hat{g}_1 & \hat{h}_1 & \hat{g}_2 & \hat{h}_2 \end{array} \\ \left[\begin{array}{cccc} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{array} \right] \end{array}$$

with respect to a basis $\hat{g}_1, \hat{h}_1, \hat{g}_2, \hat{h}_2$.

The following change of basis indicated produces a matrix in the required form to give the result of the lemma.

$$\begin{array}{cccccccc}
 \hat{e}_1 & \hat{e}_2 & \hat{f}_1 & & \hat{f}_r & & 2^k \hat{e}_1 & 2^k \hat{e}_2 \\
 +1 & +2 & +1 & \dots & +r & & + & + \\
 g_1 & g_2 & c_1 h_1 & & c_r h_1 & & h_2 & h_1 \\
 & & c_1 h_2 & & c_r h_r & & &
 \end{array}
 \left[\begin{array}{cc|cccccc}
 0 & 2^{-k} & 0 & \dots & \dots & \dots & 0 & 0 \\
 2^{-k} & 0 & 0 & \dots & \dots & \dots & 0 & 0 \\
 \hline
 0 & 0 & & & & & & \\
 \vdots & \vdots & & & & & & \\
 \vdots & \vdots & & & & & & \\
 0 & 0 & & & & & &
 \end{array} \right]
 \begin{array}{c}
 \hat{D} \\
 .s.
 \end{array}$$

Lemma 2.6

If a non-singular map λ induces $b_1 + b_2$ where $b_1: (Z_2^k \oplus Z_2^k) \times (Z_2^k \oplus Z_2^k) \rightarrow S$ is irreducible of type (8) then λ is stably equivalent to $\lambda_1 + \lambda_2$ where λ_2 induces b_2 and λ_1 , which induces b_1 , is given by the matrix

$$\hat{C} = \begin{bmatrix} 2^{1-k} & 2^{-k} & 0 & 0 \\ 2^{-k} & 2^{1-k} & 1 & 0 \\ 0 & 1 & 2c & 1 \\ 0 & 0 & 1 & 2d \end{bmatrix} \quad \text{for } \hat{\lambda}_1 \text{ where } 3c = 2^k + (-1)^k \text{ and } d = (-1)^{k-1}$$

Proof

By adding a suitably large unimodular map λ^u to λ and by taking a suitable basis $\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4, \hat{f}_1, \hat{f}_2, \dots, \hat{f}_r$ for $\hat{H} \oplus \hat{H}^u$ the matrix for $\lambda + \lambda^u$ will be of the form

$$\begin{array}{c|cccccc} \begin{array}{c} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \\ \hat{e}_4 \\ \hline \hat{a}_1 \\ \hat{a}_2 \\ \vdots \\ \hat{a}_r \end{array} & \begin{array}{c} \hat{e}_2 \\ \hat{e}_3 \\ \hat{e}_4 \\ \hline \hat{c}_1 \\ \hat{c}_2 \\ \vdots \\ \hat{c}_r \end{array} & \begin{array}{c} \hat{e}_3 \\ \hat{e}_4 \\ \hline \hat{d}_1 \\ \hat{d}_2 \\ \vdots \\ \hat{d}_r \end{array} & \begin{array}{c} \hat{e}_4 \\ \hline \hat{m}_1 \\ \hat{m}_2 \\ \vdots \\ \hat{m}_r \end{array} & \begin{array}{c} \hat{f}_1 \\ \hat{f}_2 \\ \dots \\ \hat{f}_r \end{array} & \begin{array}{c} \hat{f}_2 \\ \dots \\ \hat{f}_r \end{array} & \begin{array}{c} \hat{f}_r \end{array} \\ \hline \begin{array}{c} 2^{1-k} \\ 2^{-k} \\ 0 \\ 0 \\ \hline \hat{a}_1 \\ \hat{a}_2 \\ \vdots \\ \hat{a}_r \end{array} & \begin{array}{c} 2^{-k} \\ 2^{1-k} \\ 1 \\ 0 \\ \hline \hat{c}_1 \\ \hat{c}_2 \\ \vdots \\ \hat{c}_r \end{array} & \begin{array}{c} 0 \\ 0 \\ 2c \\ 1 \\ \hline \hat{d}_1 \\ \hat{d}_2 \\ \vdots \\ \hat{d}_r \end{array} & \begin{array}{c} 0 \\ 0 \\ 1 \\ 2d \\ \hline \hat{m}_1 \\ \hat{m}_2 \\ \vdots \\ \hat{m}_r \end{array} & \begin{array}{c} a_1 \\ a_2 \\ \dots \\ a_r \\ \hline \dots \\ \dots \\ \dots \\ \dots \end{array} & \begin{array}{c} a_2 \\ \dots \\ a_r \\ \hline \dots \\ \dots \\ \dots \\ \dots \end{array} & \begin{array}{c} \dots \\ a_r \\ \hline \dots \\ \dots \\ \dots \\ \dots \end{array} \end{array}$$

where $a_1, a_2, \dots, a_r, c_1, c_2, \dots, c_r, d_1, d_2, \dots, d_r$, and $m_1, m_2, \dots, m_r$ are all integers.

Now add onto this matrix the unimodular matrix

$$\begin{array}{c|cccccc} \hat{g}_1 & \hat{h}_1 & \hat{g}_2 & \hat{h}_2 & \hat{g}_3 & \hat{h}_3 & \hat{g}_4 & \hat{h}_4 \\ \hline 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{array}$$

with respect to the basis $\hat{g}_1, \hat{h}_1, \hat{g}_2, \hat{h}_2, \hat{g}_3, \hat{h}_3, \hat{g}_4, \hat{h}_4$.

The following change of basis indicated produces a matrix in the required form to give the result of the lemma.

Proof of Theorem 2.2

A unimodular map λ^u induces the zero map and so for any map λ the sum $\lambda + \lambda^u$ induces the same pair (G, b) as the map λ . It is clear then that it is necessary for the induced pairs (G_1, b_1) and (G_2, b_2) of two maps λ_1 and λ_2 to be equivalent for λ_1 and λ_2 to be stably equivalent, and it remains therefore to show that this condition is also sufficient.

Suppose a map $\lambda: H \times H \rightarrow Z$ induces (G, b) and suppose that the rank of $\hat{G}$ is r . In the exact sequence

$$0 \rightarrow \hat{G} \rightarrow H \xrightarrow{\Pi} \hat{H} \rightarrow G \rightarrow 0$$

let $H_F = \ker \Pi$. Then H_F has rank r and H will split as a direct sum $H = H_F \oplus H_T$ for some subgroup H_T of H . For $a \in H_F$ since $\Pi(a) = 0$ we have $\lambda(a, b) = 0$ for any $b \in H$ and so λ splits as a sum $\lambda_F + \lambda_T$ of the zero map

$$\lambda_F: H_F \times H_F \rightarrow Z \quad \text{and a map}$$

$$\lambda_T: H_T \times H_T \rightarrow Z \quad \text{which induces } b.$$

Suppose that b is the sum of the irreducible maps $b_1, b_2, \dots, b_s$ then by applications of Lemmas 2.4, 2.5 and 2.6 it follows that λ_T is stably

equivalent to a sum $\lambda_1 + \lambda_2 + \dots + \lambda_s + \lambda^u$ where $\lambda_1, \dots, \lambda_s$ are given by the specific matrices quoted in the lemmas and depending on $b_1, \dots, b_s$ and λ^u is unimodular (λ^u arises since in the statement of the lemmas if b_2 is zero then λ_2 introducing b_2 must be a unimodular map say λ^u). Hence λ_T is stably equivalent to $\lambda_1 + \lambda_2 + \dots + \lambda_s$ and λ is thus stably equivalent to $\lambda_F + \lambda_1 + \lambda_2 + \dots + \lambda_s$.

Now if λ' is a map inducing (G', b') which is equivalent to (G, b) then $\text{rank } \hat{G}' = \text{rank } \hat{G} = r$ and b' will be the sum of $b_1, b_2, \dots, b_s$ and so it will follow that λ' is also stably equivalent to $\lambda_F + \lambda_1 + \lambda_2 + \dots + \lambda_s$ and hence λ and λ' will be stably equivalent. § .

Decompositions

We now return to the manifolds P and L . We recall that a closed $(s-1)$ -connected $(2s+1)$ -manifold P , where $s = 3$ or 7 , has homology invariants (G, b) where $b: G^* \times G^* \rightarrow S$ is a non-singular symmetric bilinear form and a manifold $L \in \mathcal{H}(s+1)$ has homology invariants (H, λ) where $\lambda: H \times H \rightarrow Z$ is a symmetric bilinear map.

Theorem 2.7

Suppose a symmetric bilinear map $\lambda: H \times H \rightarrow Z$ induces the pair (G, b) . If a closed $(s-1)$ -connected $(2s+1)$ -manifold P , where $s = 3$ or 7 , has homology invariants (G, b) then $P \# \Sigma = \partial L$ for a manifold $L \in \mathcal{H}(s+1)$ with homology invariants (H, λ) , where $\Sigma \in \bigoplus_{2s+1} H$ is a homotopy sphere.

Proof

By Theorem 1.6 $P \# \Sigma_1 = \partial L_1$ for some manifold $L_1 \in \mathcal{H}(s+1)$ and some homotopy sphere $\Sigma_1 \in \bigoplus_{2s+1} H$; and if L_1 has homology invariants (H_1, λ_1) then λ_1 induces the homology invariants (G, b) of P . By Theorem 2.2 since λ also induces (G, b) it follows that λ and λ_1 are stably equivalent i.e. there exist unimodular maps λ^u and λ_1^u such that $\lambda + \lambda^u$ and $\lambda_1 + \lambda_1^u$ are equivalent. Suppose that $V_1 \in \mathcal{H}(s+1)$ is a manifold with invariant λ_1^u then since λ_1^u is unimodular it follows that $\partial V_1 = \Sigma_2$ for some homotopy sphere $\Sigma_2 \in \bigoplus_{2s+1} H$. Then $P \# \Sigma_1 \# \Sigma_2 = \partial(L_1 + V_1)$ where $L_1 + V_1$ will have invariant $\lambda_1 + \lambda_1^u$. Now by Propositions 1.4 and 1.5 since $\lambda_1 + \lambda_1^u$ can be split as $\lambda + \lambda^u$ it follows that $L_1 + V_1$ splits as a boundary connected sum accordingly i.e.

$L_1 + V_1 = L + V$ for manifolds $L, V \in \mathcal{H}(s+1)$ where L has invariant λ and V has invariant λ^u , and since λ^u is unimodular $\partial V = \Sigma_3$ for some homotopy sphere $\Sigma_3 \in \textcircled{H}_{2s+1}$. Thus $P \# \Sigma_1 \# \Sigma_2 = \partial(L+V)$ and hence $P \# \Sigma = \partial L$ where $\Sigma = \Sigma_1 \# \Sigma_2 \# (-\Sigma_3) \in \textcircled{H}_{2s+1}$ is a homotopy sphere. § .

Before using Theorem 2.7 to decompose the manifolds P we prove two corollaries that will be used later.

Corollary 2.8

If P is a closed $(s-1)$ -connected $(2s+1)$ -manifold then for $s = 3$ $P = \partial L$ for a manifold $L \in \mathcal{H}(4)$ and for $s = 7$ $P \# \Sigma = \partial L$ for a manifold $L \in \mathcal{H}(8)$ and a homotopy sphere $\Sigma \in \textcircled{H}_{15}$ where the invariant λ of L is even.

Proof

By Lemma 2.1 the homology invariants (G, b) of P , for $s = 3$ or 7 are induced by an even map λ and so by the theorem $P \# \Sigma = \partial L$ for a manifold $L \in \mathcal{H}(s+1)$ and $\Sigma \in \textcircled{H}_{2s+1}$ where L has invariant λ and this gives the result for $s = 7$.

For $s = 3$, by (21), a generator of the group of homotopy spheres $\mathbb{H}_7$ is given by the boundary of the parallelisable manifold $W \in \mathcal{H}(4)$ whose intersection matrix is

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

By taking boundary connected sums of the manifold W we can thus obtain a parallelisable manifold $V \in \mathcal{H}(4)$ with $\partial V = -\Sigma$ and intersection invariant λ^u even. Thus $P = P \# \Sigma \# (-\Sigma) = \partial(L+V)$ and the invariant $\lambda + \lambda^u$ of $L + V$ is even. § .

Corollary 2.9

If P is a closed $(s-1)$ -connected $(2s+1)$ -manifold then for $s = 3$ $P = \partial L$ for a manifold $L \in \mathcal{H}(4)$ and for $s = 7$ $P \# \Sigma = \partial L$ for a manifold $L \in \mathcal{H}(8)$ and a homotopy sphere $\Sigma \in \mathbb{H}_{15}$ where the Stiefel - Whitney class $w_{s+1}(L) = 0$.

Proof

For a manifold $L \in \mathcal{H}_{(s+1)}$ the invariant $\hat{\alpha} \in H^{s+1}(L)$ is the obstruction to a cross-section of the tangent bundle and the Stiefel-Whitney class $w_{s+1}(L)$ is the obstruction to a cross-section of the associated SO/SO_s - bundle both over the $(s+1)$ - skeleton of L . Thus if

$$p: SO \rightarrow SO/SO_s$$

is the natural projection inducing

$$\begin{array}{ccc} p_*: \Pi_s(SO) & \rightarrow & \Pi_s(SO/SO_s) \\ \wr & & \wr \\ \mathbb{Z} & & \mathbb{Z}_2 \end{array}$$

which induces

$$p_{**}: H^{s+1}(L) \rightarrow H^{s+1}(L; \mathbb{Z}_2)$$

then $p_{**}(\hat{\alpha}) = w_{s+1}(L)$; and so if $\hat{\alpha}$ is an even element then $w_{s+1}(L) = 0$.

Now for the invariant λ of L we have $\lambda(x, x) = \hat{\alpha}(x) \bmod 2$ for any $x \in H$ and hence $\hat{\alpha}$ is an even element if and only if the map λ is even and so the result follows by the previous corollary. § .

We say that a closed $(s-1)$ - connected $(2s+1)$ - manifold P , for $s = 3$ or 7 , is indecomposable if either $G = H_s(P) \simeq \mathbb{Z}$, or

$G = H_s(P)$ is finite and the linking invariant $b: G \times G \rightarrow S$ is irreducible when $G \simeq Z_p^k$, for a prime p , or $G \simeq Z_2^k \oplus Z_2^k$, or of course $G \simeq 0$ when P is a homotopy sphere.

A homotopy sphere $\Sigma \in \mathcal{H}_{2s+1}$ has zero invariants (G, b) and so if a manifold P is indecomposable then $P \# \Sigma$ is also indecomposable.

Corollary 2.10

Any closed $(s-1)$ -connected $(2s+1)$ -manifold P , where $s = 3$ or 7 , is a connected sum of indecomposable manifolds.

Proof

Suppose P has homology invariants (G, b) and suppose that $\hat{G}$ has rank r (we will take $G \neq 0$). If $\bar{\lambda}: H \times H \rightarrow Z$ is the zero map where $H \simeq Z$ then the sum $\underbrace{\bar{\lambda} + \bar{\lambda} + \dots + \bar{\lambda}}_{r \text{ times}}$ induces $(\hat{G}, 0)$.

If $b = b_1 + b_2 + \dots + b_t$ is a sum of the irreducible forms $b_1, b_2, \dots, b_t$ which are induced by non-singular maps $\lambda_1, \lambda_2, \dots, \lambda_t$. Then $\lambda_1 + \lambda_2 + \dots + \lambda_t$ induces $b_1 + b_2 + \dots + b_t = b$ and so $\lambda = \bar{\lambda} + \bar{\lambda} + \dots + \bar{\lambda} + \lambda_1 + \lambda_2 + \dots + \lambda_t$ induces (G, b) .

By Theorem 2.7 there exists a manifold $L \in \mathcal{H}^{(s+1)}$ with intersection invariant $\lambda = \bar{\lambda} + \bar{\lambda} + \dots + \bar{\lambda} + \lambda_1 + \lambda_2 + \dots + \lambda_t$ where $P \# \Sigma = \partial L$ for some $\Sigma \in \mathcal{H}_{2s+1}$. By Propositions 1.4 and 1.5 L splits as a boundary connected sum $L = L'_1 + L'_2 + \dots + L'_r + L_1 + L_2 + \dots + L_t$ of manifolds $L'_i, L_j \in \mathcal{H}^{(s+1)}$ for $i = 1, \dots, r$ and $j = 1, \dots, t$ where $L'_1, L'_2, \dots, L'_r$ each have intersection invariant $\bar{\lambda}$ and $L_1, L_2, \dots, L_t$ have intersection invariants $\lambda_1, \lambda_2, \dots, \lambda_t$ respectively. If $P'_i = \partial L'_i$ and $P_j = \partial L_j$ then $H_s(P'_i) \cong \mathbb{Z}$ and so the manifolds P'_i are indecomposable, and $H_s(P_j)$ is finite with irreducible linking invariant b_j and so the manifolds P_j are indecomposable. Thus

$$P \# \Sigma = P'_1 \# P'_2 \# \dots \# P'_r \# P_1 \# P_2 \# \dots \# P_t \text{ and so}$$

$$P = P'_1 \# P'_2 \# \dots \# P'_r \# P_1 \# P_2 \# \dots \# (P_t \# (-\Sigma))$$

a connected sum of indecomposable manifolds. § .

CHAPTER 3

CLASSIFICATION OF THE
INDECOMPOSABLE MANIFOLDS P

The indecomposable manifolds P are now considered separately and the classification results obtained are all modulo the addition of homotopy spheres to the manifolds P . For the free case when $G = H_s(P) \approx \mathbb{Z}$ Theorem 3.7 shows that the manifolds P are classified by the invariants $(G \approx \mathbb{Z}, b = 0, \hat{\beta})$. For the torsion case when there are no elements of order 2 in G Theorem 3.4 shows that the manifolds P are still classified by our invariants $(G, b, \hat{\beta})$, but when elements of order 2 occur in G then the invariants may represent two distinct manifolds and this situation is dealt with in Theorems 3.5, 3.6 and 3.7. The results are obtained by considering the manifolds P as boundaries of the manifolds L and using Theorem 2.7 of the previous chapter, and also in the torsion case using Theorem 3.2 proved in this chapter.

The Free Case

We consider now indecomposable manifolds P with $G = H_s(P) \approx \mathbb{Z}$. Since there are no torsion elements in G the linking invariant b of

P is of course zero. The tangential invariant $\hat{\beta} \in H^{s+1}(P)$ we regard as an element of G by duality.

Theorem 3.1

If P_1 and P_2 are closed $(s-1)$ -connected $(2s+1)$ -manifolds, where $s = 3$ or 7 , such that $G_1 = H_s(P_1) \simeq Z$, $G_2 = H_s(P_2) \simeq Z$ and there exists an isomorphism $h: G_1 \rightarrow G_2$ with $h(\hat{\beta}_1) = \hat{\beta}_2$, where $\hat{\beta}_1 \in G_1$ and $\hat{\beta}_2 \in G_2$ are the tangential invariants of P_1 and P_2 ; then there exists an orientation preserving diffeomorphism $f: P_1 \rightarrow P_2 \amalg \Sigma$, for some $\Sigma \in \bigoplus_{2s+1} H$, with $f_* = h$.

Proof

By the method described in chapter 2 the zero map $\lambda: Z \times Z \rightarrow Z$ induces the pair $(Z, 0)$ and so by Theorem 2.7 there exist $\Sigma_1, \Sigma_2 \in \bigoplus_{2s+1} H$ such that $P_1 \amalg \Sigma_1 = \partial L_1$ where $L_1 \in \mathcal{H}(s+1)$ has $H_1 = H_{s+1}(L_1) \simeq Z$ and intersection invariant λ_1 zero, and $P_2 \amalg \Sigma_2 = \partial L_2$ where $L_2 \in \mathcal{H}(s+1)$ has $H_2 = H_{s+1}(L_2) \simeq Z$ and invariant λ_2 zero.

In the exact cohomology sequence for (L_i, P_i) $i = 1, 2$

$$0 \rightarrow \begin{array}{c} \hat{G}_i \\ \cong \\ Z \end{array} \rightarrow \begin{array}{c} H_i \\ \cong \\ Z \end{array} \xrightarrow{0} \begin{array}{c} \hat{H}_i \\ \cong \\ Z \end{array} \rightarrow \begin{array}{c} G_i \\ \cong \\ Z \end{array} \rightarrow 0$$

$\hat{H}_i$ maps isomorphically to G_i and so by Theorem 1.6 the tangential invariant $\hat{\alpha}_i$ of L_i maps isomorphically onto $\hat{\beta}_i$. Since there exists an isomorphism $h: G_1 \rightarrow G_2$ with $h(\hat{\beta}_1) = \hat{\beta}_2$ there exists a corresponding isomorphism from $\hat{H}_1$ to $\hat{H}_2$ sending $\hat{\alpha}_1$ to $\hat{\alpha}_2$. Thus by Proposition 1.4 there exists an orientation preserving diffeomorphism $g: L_1 \rightarrow L_2$ whose induced homomorphism corresponds to the isomorphism above. The restriction of g to $P_1 \amalg \Sigma_1$ will then give an orientation preserving diffeomorphism from $P_1 \amalg \Sigma_1$ to $P_2 \amalg \Sigma_2$ whose induced homomorphism will be h . The statement of the theorem now follows where $\Sigma = \Sigma_2 \amalg (-\Sigma_1)$. § .

The Torsion Case

We now consider indecomposable manifolds P where $G = H_s(P)$ is a finite group (when $G \cong \mathbb{Z}_p^k$, for a prime p , or $G \cong \mathbb{Z}_2^k \oplus \mathbb{Z}_2^k$). The approach here is set by the following result.

Theorem 3.2

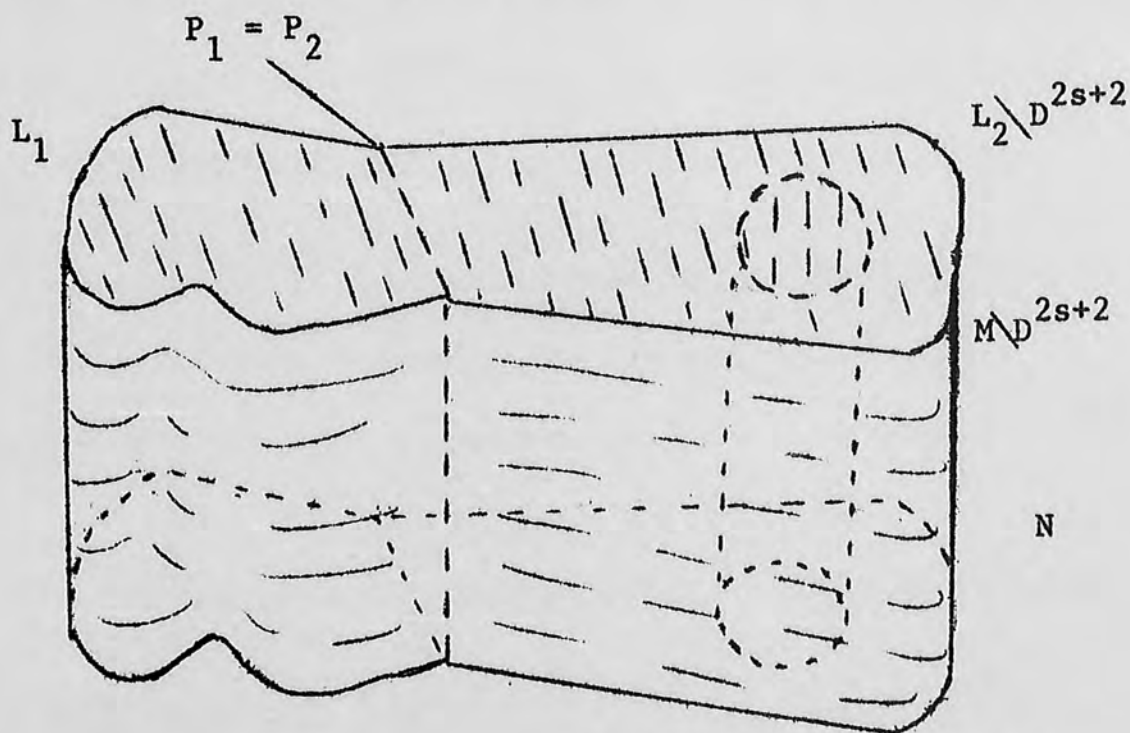
Suppose manifolds $L_1, L_2 \in \mathcal{H}^{(s+1)}$ with $P_1 = \partial L_1$, $P_2 = \partial L_2$ and suppose that there exists an orientation preserving diffeomorphism $f: P_1 \rightarrow P_2$; then there exist manifolds $V_1, V_2 \in \mathcal{H}^{(s+1)}$

with $\partial V_1 = \partial V_2 = S^{2s+1}$ and an orientation preserving diffeomorphism

$g: L_1 + V_1 \rightarrow L_2 + V_2$ with $g|_{P_1} = f$.

Proof

Let $M = L_1 \cup_f (-L_2)$ where points of $\partial L_1 = P_1$ and $\partial L_2 = P_2$ are identified under f . Then M is a closed s -connected $(2s+2)$ -manifold (this can be checked from the Mayer-Vietoris sequence for the homology of M). If D^{2s+2} is an embedded disc in M which we take to be contained in L_2 then $M \setminus D^{2s+2} \in \mathcal{H}(s+1)$



Consider the manifold $N = (M \setminus D^{2s+2}) \times I$. The boundary $\partial N = M \# (-M)$ and since the embedded disc lies inside $L_2 \subset N$ $\partial N = L_1 \cup_f (-L_2 \# (-M))$. N is therefore a cobordism with corner of L_1 to $-(L_2 \# M)$.

Now $H_q(N, L_1) \approx H_q(M \setminus D^{2s+2}, L_1)$ since $N = (M \setminus D^{2s+2}) \times I$ is homotopically equivalent to $M \setminus D^{2s+2}$. Thus $H_{2s+2}(N, L_1) \approx 0$ and

$$H_q(N, L_1) \approx H_q(M, L_1) \text{ for } q \neq 2s+2 \\ \approx H_q(L_2, P_2) \text{ by excision.}$$

Thus $H_q(N, L_1) \approx 0$ for $q \neq s+1$ and $H_{s+1}(N, L_1) \approx H_{s+1}(L_2, P_2)$ which is free abelian of rank k say.

By the Handle Decomposition Theorem, stated in chapter 0, N has a handle decomposition on $L_1 \times I$ with $k(s+1)$ - handles attached to $L_1 \times 1$. If the attaching maps of the handles are $f_i: S^s \times D^{s+2} \rightarrow L_1 \times 1$ for $1 \leq i \leq k$ we can assume that the images lie inside a $(2s+2)$ - disc D^{2s+2} in $L_1 \times 1$ since the embeddings $f_i|_{S^s \times 0}$ are isotopic to zero by Haefliger (9), or equally Theorem 1 of (8), and so can be deformed to lie within D^{2s+2} and then by the Isotopy Extension Theorem and the Tubular Neighbourhood Theorem we can suppose that the f_i lie inside D^{2s+2} .

With all the attaching maps inside D^{2s+2} N is of the form

$$N = (L_1 \times I) + (D^{2s+3} \cup h_1^{s+1} \cup h_2^{s+1} \cup \dots \cup h_k^{s+1})$$

where the boundary connected sum takes place along the embedded disc D^{2s+2} in $L_1 \times 1$.

$D^{2s+3} \cup h_1^{s+1} \cup \dots \cup h_k^{s+1} \in \mathcal{H}(2s+3, k, s+1)$ and so

$M_1 = \partial(D^{2s+3} \cup h_1^{s+1} \cup \dots \cup h_k^{s+1})$ is a closed s -connected $(2s+2)$ -manifold.

Then from above there is an orientation preserving diffeomorphism

from $L_1 \# M_1$ to $L_2 \# M$ which necessarily sends P_1 to P_2 . Let

$V_1 = M_1 \setminus D^{2s+2}$ and $V_2 = M \setminus D^{2s+2}$ then $V_1, V_2 \in \mathcal{H}(s+1)$ with

$\partial V_1 = \partial V_2 = S^{2s+1}$ and $L_1 + V_1 = L_1 \# M_1, L_2 + V_2 = L_2 \# M$

giving the result of the theorem. $\square$

Corollary 3.3

Suppose manifolds $L_1, L_2 \in \mathcal{H}(s+1)$ with $P_1 = \partial L_1,$

$P_2 = \partial L_2$ and suppose that there exists an orientation preserving

diffeomorphism $f: P_1 \rightarrow P_2 \# \Sigma$ for some $\Sigma \in \mathcal{H}_{2s+1}$; then

there exist manifolds $V_1, V_2 \in \mathcal{H}(s+1)$ with ∂V_1 and ∂V_2

homotopy spheres and an orientation preserving diffeomorphism

$g: L_1 + V_1 \rightarrow L_2 + V_2.$

Proof

We claim that $\Sigma = \partial V$ for some manifold $V \in \mathcal{H}(s+1)$. This

is clear for $s = 3$ since any homotopy 7-sphere occurs as a

boundary of a manifold in $\mathcal{H}(4)$ and the case $s = 7$ will be

justified later in Chapter 5.

The corollary now follows by applying the theorem to the manifolds $L_1 + V$ and L_2 when in fact in the statement of the corollary we would have $\partial V_1 = \Sigma$ and $\partial V_2 = S^{2s+1}$. § .

We can now state the classification theorems for the indecomposable manifolds P with invariants $(G, b, \hat{\beta})$ where G is finite and the form b is one of the irreducible types given in Proposition 1.7. To say that a collection of manifolds P are classified, up to the addition of homotopy spheres, by the invariants $(G, b, \hat{\beta})$ we mean that if there exists a manifold P_1 with invariants $(G_1, b_1, \hat{\beta}_1)$ and an isomorphism $h: G \rightarrow G_1$ preserving the linking invariants and with $h(\hat{\beta}) = \hat{\beta}_1$ then there exists an orientation preserving diffeomorphism $f: P \rightarrow P_1 \# \Sigma$, for some $\Sigma \in \bigoplus_{2s+1} \mathbb{H}$, where $f_* = h$. To say that there are two distinct manifolds, up to the addition of homotopy spheres, corresponding to a particular value of the invariants $(G, b, \hat{\beta})$ we mean that there exist two manifolds P_1 and P_2 both with invariants $(G, b, \hat{\beta})$ and such that P_1 is not diffeomorphic to $P_2 \# \Sigma$ for any $\Sigma \in \bigoplus_{2s+1} \mathbb{H}$, moreover if P is any other manifold with invariants $(G, b, \hat{\beta})$ P is diffeomorphic to either $P_1 \# \Sigma$ or $P_2 \# \Sigma$ for some $\Sigma \in \bigoplus_{2s+1} \mathbb{H}$.

Theorem 3.4

The manifolds P with $G \approx Z_p k$, p an odd prime, and linking invariant b of type (1) or (2) are classified, up to the addition of homotopy spheres, by the invariants $(G \approx Z_p k, b, \hat{\beta})$.

Theorem 3.5

Suppose manifold P has $G \approx Z_2 k$ and linking invariant b of type (3), (4), (5), or (6) then:

- (i) If $k \geq 3$ the manifolds P with tangential invariant $\hat{\beta}$ divisible by 4 are classified, up to the addition of homotopy spheres, by the invariants $(G \approx Z_2 k, b, \hat{\beta})$.

There are two distinct manifolds P , up to the addition of homotopy spheres, for each value of the invariants if $\hat{\beta}$ is not divisible by 4.

- (ii) If $k = 2$, when b must be of type (3) or (4), the manifolds P with $\hat{\beta}$ not divisible by 4 (equivalently $\hat{\beta} \neq 0$) are classified, up to the addition of homotopy spheres, by the invariants $(G \approx Z_4, b, \hat{\beta})$.

There are two distinct manifolds P , up to the addition of homotopy spheres, for each value of the invariants if $\hat{\beta}$ is divisible by 4 (when $\hat{\beta}$ must be zero).

- (iii) If $k = 1$, when b must be of type (3) and $\hat{\beta}$ must be zero, there are two distinct manifolds P , up to the addition of homotopy spheres.

Theorem 3.6

Suppose manifold P has $G \approx Z_2^k \oplus Z_2^k$ and linking invariant b of type (7) then:

- (i) If $k \geq 2$ the manifolds P with $\hat{\beta}$ divisible by 4 are classified, up to the addition of homotopy spheres, by the invariants $(G \approx Z_2^k \oplus Z_2^k, b, \hat{\beta})$.

There are two distinct manifolds P , up to the addition of homotopy spheres, for each value of the invariants if $\hat{\beta}$ is not divisible by 4.

- (ii) If $k=1$, when $\hat{\beta}$ must be zero, there are two distinct manifolds P , up to the addition of homotopy spheres.

Theorem 3.7

Suppose manifold P has $G \approx Z_2^k \oplus Z_2^k$ and linking invariant b of type (8), when of necessity $k \geq 2$; then the manifolds P with $\hat{\beta}$ divisible by 4 are classified, up to the addition of homotopy spheres, by the invariants $(G \approx Z_2^k \oplus Z_2^k, b, \hat{\beta})$.

There are two distinct manifolds, up to the addition of homotopy spheres, for each value of the invariants if $\hat{\beta}$ is not divisible by 4.

General Method of Proof

The methods of proof of Theorem 3.4 - 3.7 are basically the same and we give the main steps in the argument here; the details we fill in separately later.

In all of the theorems an indecomposable manifold P with invariants $(G, b, \hat{\beta})$ is considered, where the linking invariant b is irreducible. As described in chapter 2 a non-singular bilinear

map $\lambda: H \times H \rightarrow Z$ induces a non-singular form b , and suppose that λ is the particular map inducing the irreducible map b , given in the proof of Lemma 2.1 or equally in the statements of Lemmas 2.4, 2.5 or 2.6; then λ is even. By Theorem 2.7 P, up to the addition of a homotopy sphere, is the boundary of a manifold $L \in \mathcal{H}(s+1)$ with homology invariants (H, λ) . If $\hat{\alpha} \in \hat{H}$ is the tangential invariant of L then since $\lambda(x, x) = \hat{\alpha}(x) \bmod 2$ for all $x \in H$ and since λ is even $\hat{\alpha}$ must be an even element of $\hat{H}$.

If P_1, P_2 are any two manifolds having the same invariants $(G, b, \hat{\beta})$ (by P_1 and P_2 having the same invariants we mean that there exists an isomorphism between their groups G preserving their linking invariants b and sending their tangential invariants $\hat{\beta}$ to each other) then, up to the addition of homotopy spheres, P_1, P_2 are the boundaries of manifolds $L_1, L_2 \in \mathcal{H}(s+1)$ whose invariants are $(H, \lambda, \hat{\alpha}_1), (H, \lambda, \hat{\alpha}_2)$ respectively. The manifolds $V \in \mathcal{H}(s+1)$ whose boundaries are homotopy spheres are those manifolds with intersection maps λ^u unimodular. If the manifolds $V_1, V_2 \in \mathcal{H}(s+1)$ with $\partial V_1, \partial V_2$ homotopy spheres have invariants $(H_1^u, \lambda_1^u, \hat{\alpha}_1^u), (H_2^u, \lambda_2^u, \hat{\alpha}_2^u)$ then by Proposition 1.5 the invariants of the manifolds $L_1 + V_1$ and $L_2 + V_2$ are $(H \oplus H_1^u, \lambda + \lambda_1^u, \hat{\alpha}_1 \oplus \hat{\alpha}_1^u)$ and $(H \oplus H_2^u, \lambda + \lambda_2^u, \hat{\alpha}_2 \oplus \hat{\alpha}_2^u)$.

If it can be shown that there exists an isomorphism $k: H \oplus H_1^u \rightarrow H \oplus H_2^u$ preserving the intersection numbers $\lambda + \lambda_1^u$, $\lambda + \lambda_2^u$ and with $(\hat{\alpha}_2 \oplus \hat{\alpha}_2^u) \circ k = \hat{\alpha}_1 \oplus \hat{\alpha}_1^u$ then by Proposition 1.4 there exists an orientation preserving diffeomorphism $f: L_1 + V_1 \rightarrow L_2 + V_2$ with $f_* = k$. The restriction of f to the boundary of $L_1 + V_1$ then gives an orientation preserving diffeomorphism from P_1 to P_2 , up to the addition of homotopy spheres.

We therefore consider all possible values of the tangential invariant $\hat{\alpha} \in \hat{H}$ that can occur for a manifold $L \in \mathcal{H}(s+1)$ with invariants $(H, \lambda, \hat{\alpha})$ to have as its boundary a manifold P with invariants a particular value of $(G, b, \hat{\beta})$. There are now two cases:

- (i) If it can be shown that for any two of these possible values $\hat{\alpha}_1, \hat{\alpha}_2$ for the tangential invariant, which correspond to manifolds L_1, L_2 with invariants $(H, \lambda, \hat{\alpha}_1), (H, \lambda, \hat{\alpha}_2)$, there exists invariants $(H_1^u, \lambda_1^u, \hat{\alpha}_1^u), (H_2^u, \lambda_2^u, \hat{\alpha}_2^u)$, which correspond to manifolds V_1, V_2 with $\partial V_1, \partial V_2$ homotopy spheres, such that there exists an isomorphism $k: H \oplus H_1^u \rightarrow H \oplus H_2^u$ as above; then all the manifolds P with invariants the particular value of $(G, b, \hat{\beta})$

are diffeomorphic, up to the addition of homotopy spheres: i.e. the manifolds P , up to the addition of homotopy spheres, are classified by the invariants $(G, b, \hat{\beta})$.

- (ii) If however it can be shown that the possible values of $\hat{\alpha}$ fall into two classes such that for any two values $\hat{\alpha}_1, \hat{\alpha}_2$ from the same class we have the situation just described above; then for manifolds L corresponding to values of $\hat{\alpha}$ from the same class the boundaries ∂L , up to the addition of homotopy spheres, are all diffeomorphic. And if $\hat{\alpha}_1, \hat{\alpha}_2$ are values from the two distinct classes and for any possible value of the invariants $(H_1^u, \lambda_1^u, \hat{\alpha}_1^u), (H_2^u, \lambda_2^u, \hat{\alpha}_2^u)$, which correspond to manifolds V_1, V_2 with $\partial V_1, \partial V_2$ homotopy spheres, there does not exist any isomorphism $k: H \oplus H_1^u \rightarrow H \oplus H_2^u$ which preserves intersection numbers $\lambda + \lambda_1^u, \lambda + \lambda_2^u$ and has $(\hat{\alpha}_2 \oplus \hat{\alpha}_2^u) \circ k = \hat{\alpha}_1 \oplus \hat{\alpha}_1^u$; then it follows that $L_1 + V_1$ is not diffeomorphic to $L_2 + V_2$ for any manifolds V_1, V_2 with $\partial V_1, \partial V_2$ homotopy spheres. By Corollary 3.3 therefore the boundaries of L_1 and L_2 , up to the addition of homotopy spheres, cannot be diffeomorphic. Thus in this situation there are two distinct manifolds P , up to the addition of homotopy

spheres, corresponding to the value of the invariants $(G, b, \hat{\beta})$.

In practice rather than work with λ it is more convenient to use the associated symmetric bilinear map $\hat{\lambda} : \hat{H} \times \hat{H} \rightarrow Q$, described in chapter 2. If in the situation above an isomorphism $\hat{k} : \hat{H} \oplus \hat{H}_1^u \rightarrow \hat{H} \oplus \hat{H}_2^u$ is constructed which preserves the maps $\hat{\lambda} + \hat{\lambda}_1^u$, $\hat{\lambda} + \hat{\lambda}_2^u$ and with $\hat{k}(\hat{\alpha} \oplus \hat{\alpha}_1^u) = \hat{\alpha} \oplus \hat{\alpha}_2^u$, this is equivalent to constructing a required isomorphism $k : H \oplus H_1^u \rightarrow H \oplus H_2^u$.

For some basis $\hat{e}_1, \dots, \hat{e}_r$ of $\hat{H}$ $\hat{\lambda}$ is represented by the symmetric rational matrix

$$\hat{A} = \begin{bmatrix} \hat{e}_1 & \dots & \hat{e}_r \\ \hat{\lambda}(\hat{e}_1, \hat{e}_1) & \dots & \hat{\lambda}(\hat{e}_1, \hat{e}_r) \\ \dots & \dots & \dots \\ \hat{\lambda}(\hat{e}_r, \hat{e}_1) & \dots & \hat{\lambda}(\hat{e}_r, \hat{e}_r) \end{bmatrix}$$

given in the proof of Lemma 2.1 where the symmetric integral matrix A , the inverse of $\hat{A}$, has even diagonal elements. The tangential invariant $\hat{\alpha} = \alpha_1 \hat{e}_1 + \alpha_2 \hat{e}_2 + \dots + \alpha_r \hat{e}_r$ for some integers $\alpha_1, \alpha_2, \dots, \alpha_r$ and since $\hat{\alpha}$ is an even element of $\hat{H}$ the integers $\alpha_1, \dots, \alpha_r$ must be even. Of interest now are the possible values of the even integers $\alpha_1, \dots, \alpha_r$ for which the

boundary of a manifold L with invariants $(H, \lambda, \hat{\alpha})$ has invariants a particular value of $(G, b, \hat{\beta})$.

A unimodular map $\hat{\lambda}_1^u$ has the associated unimodular map $\hat{\lambda}_1^u$ and with respect to a basis $\hat{f}_1, \hat{f}_2, \dots, \hat{f}_t$ of $\hat{H}_1^u$ $\hat{\lambda}_1^u$ is represented by the symmetric integral unimodular matrix $\hat{C} = (\hat{\lambda}_1^u(\hat{f}_i, \hat{f}_j))$. Then $\hat{\lambda} + \hat{\lambda}_1^u$ has the matrix

$$\begin{array}{c} \hat{e}_1 \dots \hat{e}_r \quad \hat{f}_1 \dots \hat{f}_t \\ \left[\begin{array}{c|c} \hat{A} & 0 \\ \hline 0 & \hat{C} \end{array} \right] \end{array} \quad \begin{array}{l} \text{with respect to the basis} \\ \hat{e}_1, \dots, \hat{e}_r, \hat{f}_1, \dots, \hat{f}_t \\ \text{of } \hat{H} \oplus \hat{H}_1^u \end{array}$$

If $\hat{\lambda} + \hat{\lambda}_2^u$ is represented by the matrix

$$\begin{array}{c} \hat{E}_1 \dots \hat{E}_r \quad \hat{F}_1 \dots \hat{F}_t \\ \left[\begin{array}{c|c} \hat{A} & 0 \\ \hline 0 & \hat{D} \end{array} \right] \end{array} \quad \begin{array}{l} \text{with respect to the basis} \\ \hat{E}_1, \dots, \hat{E}_r, \hat{F}_1, \dots, \hat{F}_t \\ \text{of } \hat{H} \oplus \hat{H}_2^u, \end{array}$$

where $\hat{D}$ is the matrix of $\hat{\lambda}_2^u$; then an isomorphism $\hat{k}: \hat{H} \oplus \hat{H}_1^u \rightarrow \hat{H} \oplus \hat{H}_2^u$ which preserves the maps $\hat{\lambda} + \hat{\lambda}_1^u, \hat{\lambda} + \hat{\lambda}_2^u$ must have $\hat{k}^{-1}(\hat{E}_1), \dots, \hat{k}^{-1}(\hat{E}_r), \hat{k}^{-1}(\hat{F}_1), \dots, \hat{k}^{-1}(\hat{F}_t)$ a basis of $\hat{H} \oplus \hat{H}_1^u$ such that with respect to this basis $\hat{\lambda} + \hat{\lambda}_1^u$ has matrix:

$$\left[\begin{array}{ccc|ccc} \hat{k}^{-1}(\hat{E}_1) & \dots & \hat{k}^{-1}(\hat{E}_r) & \hat{k}^{-1}(\hat{F}_1) & \dots & \hat{k}^{-1}(\hat{F}_t) \\ \hline & \hat{A} & & & 0 & \\ \hline & 0 & & & \hat{D} & \end{array} \right]$$

If $\hat{\alpha} \oplus \hat{\alpha}^u = (\alpha_1 \hat{E}_1 + \dots + \alpha_r \hat{E}_r) \oplus (\alpha_1^u \hat{F}_1 + \dots + \alpha_t^u \hat{F}_t)$ is

the tangential invariant in $\hat{H} \oplus \hat{H}_2^u$ then if $\hat{k}$ sends the tangential

invariants to each other the tangential invariant in $\hat{H} \oplus \hat{H}_1^u$ is

$$\alpha_1 \hat{k}^{-1}(\hat{E}_1) + \dots + \alpha_r \hat{k}^{-1}(\hat{E}_r) + \alpha_1^u \hat{k}^{-1}(\hat{F}_1) + \dots + \alpha_t^u \hat{k}^{-1}(\hat{F}_t).$$

If this element with respect to the original basis is

$$\hat{\alpha}' \oplus \hat{\alpha}^{u'} = (\alpha'_1 \hat{e}_1 + \dots + \alpha'_r \hat{e}_r) + (\alpha_1^{u'} \hat{f}_1 + \dots + \alpha_t^{u'} \hat{f}_t) \text{ and if in}$$

terms of the original basis

$$\hat{k}^{-1}(\hat{E}_i) = a_{i1} \hat{e}_1 + \dots + a_{ir} \hat{e}_r + b_{i1} \hat{f}_1 + \dots + b_{it} \hat{f}_t$$

$$\hat{k}^{-1}(\hat{F}_j) = c_{j1} \hat{e}_1 + \dots + c_{jr} \hat{e}_r + d_{j1} \hat{f}_1 + \dots + d_{jt} \hat{f}_t$$

then $\alpha'_i = a_{i1} \alpha_1 + \dots + a_{ir} \alpha_r + c_{i1} \alpha_1^u + \dots + c_{it} \alpha_t^u$, for

$i = 1, \dots, r$ and $j = 1, \dots, t$; i.e. the integers $\alpha'_1, \dots, \alpha'_r$

are given as sums of the integers $\alpha_1, \dots, \alpha_r, \alpha_1^u, \dots, \alpha_t^u$.

Therefore to prove the theorems we consider matrices

associated with $\hat{\lambda} + \hat{\lambda}^u$ for a basis $\hat{e}_1, \dots, \hat{e}_r, \hat{f}_1, \dots, \hat{f}_t$ of

$\hat{H} \oplus \hat{H}^u$ where $\hat{\lambda}^u$ is any unimodular map i.e. we consider matrices

$$\begin{matrix} \hat{e}_1, \dots, \hat{e}_r, & \hat{f}_1, \dots, \hat{f}_t \\ \left[\begin{array}{c|c} \hat{A} & 0 \\ \hline 0 & \hat{C} \end{array} \right] \end{matrix}$$

where $\hat{C}$ is any unimodular matrix.

We consider all possible bases $\hat{E}_1, \dots, \hat{E}_r, \hat{F}_1, \dots, \hat{F}_t$ of $\hat{H} \oplus \hat{H}^u$ such that with respect to this basis the matrix of $\hat{\lambda} + \hat{\lambda}^u$ is of the form

$$\begin{matrix} \hat{E}_1 \dots \hat{E}_r & \hat{F}_1 \dots \hat{F}_t \\ \left[\begin{array}{c|c} \hat{A} & 0 \\ \hline 0 & \hat{D} \end{array} \right] \end{matrix}$$

when it follows that $\hat{D}$ must be unimodular.

The tangential invariant $\hat{\alpha} \oplus \hat{\alpha}^u$ in terms of the two bases is given as:

$$\begin{aligned} \hat{\alpha} \oplus \hat{\alpha}^u &= \alpha_1 \hat{E}_1 + \dots + \alpha_r \hat{E}_r + \alpha_1^u \hat{F}_1 + \dots + \alpha_t^u \hat{F}_t \\ &= \alpha'_1 \hat{e}_1 + \dots + \alpha'_r \hat{e}_r + \bar{\alpha}_1^u \hat{f}_1 + \dots + \bar{\alpha}_t^u \hat{f}_t \end{aligned}$$

for integers $\alpha_1, \dots, \alpha_r, \alpha_1^u, \dots, \alpha_t^u$ and $\alpha'_1, \dots, \alpha'_r, \bar{\alpha}_1^u, \dots, \bar{\alpha}_t^u$.

The expressions for the basis $\hat{E}_1, \dots, \hat{E}_r, \hat{F}_1, \dots, \hat{F}_t$ in terms of the original basis $\hat{e}_1, \dots, \hat{e}_r, \hat{f}_1, \dots, \hat{f}_t$ provide expressions for the integers $\alpha'_1, \dots, \alpha'_r$ in terms of the integers $\alpha_1, \dots, \alpha_r, \alpha_1^u, \dots, \alpha_t^u$.

to $\hat{C}$ and $\hat{D}$ the matrices

$$\begin{bmatrix} \hat{C} & 0 \\ \hline \begin{matrix} \pm 1 & & 0 \\ 0 & \ddots & \\ & 0 & \ddots \\ & & \pm 1 \end{matrix} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \hat{D} & 0 \\ \hline \begin{matrix} \pm 1 & & 0 \\ 0 & \ddots & \\ & 0 & \ddots \\ & & \pm 1 \end{matrix} \end{bmatrix}$$

will be odd unimodular matrices with signature one ('odd' means at least one diagonal element is odd). Now integral congruence classes of odd unimodular matrices are determined by the rank and signature alone and so the matrices with respect to suitable bases will be of the required form.

Finally before giving the details for the separate proofs it should be recalled that by Lemma 1.2 the tangential invariant $\hat{\beta} \in G$ for a manifold P is always an even element.

Proof of Theorem 3.4

$G \simeq \mathbb{Z}_p^k$ and $b(x, x) = np^{-k}$ for a generator x of $\mathbb{Z}_p^k$, where n is prime to p . Suppose the tangential invariants $\hat{\beta} = \beta x$ for the integer β . If $h: G \rightarrow G$ is an isomorphism which preserves b and if $h(x) = mx$ then we must have:

$$\begin{aligned}
 b(mx, mx) &= b(x, x) \\
 \text{i.e. } m^2 np^{-k} &= np^{-k} \pmod{1} \\
 (m^2 - 1) np^{-k} &= 0 \pmod{1} \\
 \text{and thus } m^2 &= 1 \pmod{p^k}.
 \end{aligned}$$

The image $h(\beta x) = m\beta x$.

By Lemma 2.1 we have the matrix for $\hat{\lambda}$, where λ induces b , given by:

$$\hat{A} = \begin{matrix} & \hat{e} & \hat{e}_1 & & \hat{e}_r \\ \begin{bmatrix} np^{-k} & 1 & 0 & \dots & 0 \\ 1 & a_1 & \dots & \dots & \dots \\ 0 & & \dots & \dots & 0 \\ \dots & \dots & \dots & a_{r-1} & 1 \\ 0 & \dots & 0 & 1 & a_r \end{bmatrix} & \text{with respect to the basis} \\ & \hat{e}, \hat{e}_1, \dots, \hat{e}_r \text{ of } \hat{H}. & \end{matrix}$$

In the exact sequence

$$0 \rightarrow H \rightarrow \hat{H} \xrightarrow{\partial} \underset{\substack{\mathbb{Z}_p^k \\ \mathbb{Z}_p^k}}{G} \rightarrow 0$$

$\partial(\hat{e}_i) = 0$ for $i = 1, \dots, r$ and $\partial(\hat{e}) = x$, a generator of Z_p^k as above. Suppose that α is an integer such that for the tangential invariant $\alpha \hat{e} \in \hat{H}$ $\partial(\alpha \hat{e}) = \beta x$ (α is even and if β is even take $\alpha = \beta$ and if β is odd take $\alpha = \beta + p^k$) then possible values for the tangential invariants $\hat{\alpha} \in \hat{H}$, which must be even, that map to βx or its isomorphic image $m\beta x$ are $\hat{\alpha} = (m\alpha + 2rp^k)\hat{e} + \alpha_1 \hat{e}_1 + \dots + \alpha_r \hat{e}_r$ for any integer r , any even integers $\alpha_1, \dots, \alpha_r$ and any integer m such that $m^2 = 1 \pmod{p^k}$.

To $\hat{A}$ add the unimodular matrix

$$\begin{array}{cccccc} \hat{f} & \hat{g} & \hat{f}_1 & \hat{g}_1 & \dots & \hat{f}_r & \hat{g}_r \\ \left[\begin{array}{cccccc|cc} 0 & -1 & & & & & & 0 \\ -1 & 0 & & & & & & \\ \hline & & 0 & -1 & & & & \\ & & -1 & 0 & & & & \\ \hline & & & & \dots & & 0 & -1 \\ & 0 & & & & & -1 & 0 \end{array} \right] \end{array}$$

Then with respect to the basis $\hat{E}, \hat{E}_1, \dots, \hat{E}_r, \hat{F}, \hat{F}_1, \dots, \hat{F}_r,$

$\hat{g}, \hat{g}_1, \dots, \hat{g}_r$ indicated we have the following matrix (the change of basis employed here is the same as was used in the proof of Lemma 2.4).

$$\begin{array}{c}
 \begin{array}{cccccccccccc}
 \hat{E} & \hat{E}_1 & & & \hat{E}_r & p^k \hat{e} & \hat{F}_1 & & \hat{F}_{r-1} & & & \\
 \hat{e} & \hat{e}_1 & & & \hat{e}_r & \hat{f} & \hat{e}_1 & \dots & \hat{e}_{r-1} & & \hat{F}_r & \\
 \hat{g} & \hat{g}_1 & \dots & & \hat{g}_r & \hat{f}_1 & \hat{f}_2 & \dots & \hat{f}_r & & \hat{a}_r \hat{f}_r & \hat{g} & \hat{g}_1 & \dots & \hat{g}_r
 \end{array} \\
 \left[\begin{array}{cccc|cccccccccccc}
 np^{-k} & 1 & 0 & \dots & 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
 1 & a_1 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 0 & . & \dots & \dots & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 . & . & . & a_{r-1} & 1 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 0 & \dots & 0 & 1 & a_r & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
 \hline
 0 & . & . & . & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 . & . & . & . & . & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 . & . & . & . & . & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 . & . & . & . & . & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 0 & . & . & . & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
 \end{array} \right]
 \end{array}$$

If tangential invariant $\hat{\alpha} \oplus \hat{\alpha}^u = \alpha \hat{E} + \alpha_1 \hat{E}_1 + \dots + \alpha_r \hat{E}_r + \alpha^u \hat{F} + \alpha_1^u \hat{F}_1 + \dots + \alpha_r^u \hat{F}_r + \text{terms in } \hat{g}, \hat{g}_i;$ and for other basis $\hat{\alpha} \oplus \hat{\alpha}^u = \alpha' \hat{e} + \alpha'_1 \hat{e}_1 + \dots + \alpha'_r \hat{e}_r + \text{terms in } \hat{f}, \hat{f}_i, \hat{g}, \hat{g}_i$ then we have

$$\begin{aligned}
 \alpha' &= \alpha + p^k \alpha^u \\
 \alpha'_1 &= \alpha_1 + \alpha_1^u \\
 &\dots \dots \dots \\
 \alpha'_r &= \alpha_r + \alpha_r^u
 \end{aligned}$$

$\hat{D}$ must be an even unimodular matrix and so $\alpha^u, \alpha_1^u, \dots, \alpha_r^u$ can be any even integers. Thus the coefficients $\alpha_1, \dots, \alpha_r$ of $\hat{\alpha}$ can be any even integers and the coefficient of $\hat{e}$ can take values $\alpha + 2rp^k$ for any integer r . It remains to show that the coefficient $m\alpha$ for $\hat{e}$ can occur.

Add to $\hat{A}$ the unimodular matrix

$$\begin{matrix} \hat{f} & \hat{g} \\ \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \end{matrix}$$

Then with respect to the basis $\hat{E}, \hat{e}_1, \dots, \hat{e}_r, \hat{F}, \hat{g}$ indicated we have the following matrix, where $m^2 = t p^k + 1$.

$$\begin{array}{c} \hat{E} \\ " \\ \hat{me} \\ + \\ f \end{array} \quad \begin{array}{c} \hat{e}_1 \dots\dots\dots \hat{e}_r \\ m \quad 0 \dots\dots\dots 0 \\ a_1 \dots\dots\dots 0 \\ \dots\dots\dots 1 \\ \dots\dots\dots a_{r-1} \\ \dots\dots\dots 0 \quad 1 \quad \dots\dots\dots a_r \end{array} \quad \begin{array}{c} \hat{F} \\ tp^k \hat{e} \\ + \\ \hat{mf} \\ \hat{g} \end{array} \quad \begin{array}{c} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}$$

If the tangential invariant with respect to the new basis has coefficients α of $\hat{E}$ and α^u of $\hat{F}$ and with respect to the original basis coefficient α' of $\hat{e}$ then

$$\alpha' = m\alpha + p^k \alpha^u$$

where α^u can take on the value of any even integer since the inverse of the matrix above has even diagonal elements.

The trouble now is that the matrix is not in the form

$$\begin{bmatrix} \hat{A} & 0 \\ 0 & \hat{D} \end{bmatrix}$$

required, however it would be in this form if we could change certain

elements in the matrix by integers. The proof of Lemma 2.4 shows how a unimodular matrix can be added to the matrix and a new basis chosen to accomplish this. It can be seen from the change of basis used that the leading coefficient in an expression for $\hat{\alpha}$ can only be changed by a multiple of p^k . Thus allowing for this process we obtain $\alpha' = m\alpha + \text{some multiple of } p^k$; and since α is even and α' must be even the multiple of p^k must be an even one. §

Proof of Theorem 3.5

$G \simeq Z_2^k$ and $b(x,x) = n2^{-k}$ for a generator x of Z_2^k , where n is odd. As in the previous proof if the tangential invariant $\hat{\beta} = \beta x$ an isomorphism of G preserving b maps this to $m\beta x$ where $m^2 = 1 \pmod{2^k}$. Since $\hat{\beta}$ is even the integer β must be even.

By Lemma 2.1 the matrix for $\hat{\lambda}$, where λ induces b , is given by

$$\hat{A} = \begin{matrix} & \hat{e} & \hat{e}_1 & \dots\dots\dots & \hat{e}_r \\ \left[\begin{array}{cccc} n 2^{-k} & 1 & 0 & \dots\dots\dots 0 \\ 1 & a_1 & \dots\dots\dots & \\ 0 & \dots\dots\dots & & 0 \\ \dots\dots\dots & & a_{r-1} & 1 \\ 0 & \dots\dots\dots 0 & 1 & a_r \end{array} \right] \end{matrix}$$

In the exact sequence

$$0 \rightarrow H \rightarrow \hat{H} \xrightarrow{\partial} \underset{\substack{\mathbb{Z} \\ 2^k}}{G} \rightarrow 0$$

$\partial(\hat{e}_i) = 0$ $i = 1, \dots, r$ and $\partial(\hat{e}) = x$. Suppose that α is an even integer such that $\partial(\alpha\hat{e}) = \beta x$ (since β is even we could take $\alpha = \beta$) then possible values of $\hat{\alpha} \in \hat{H}$ that map to βx or its isomorphic image $m\beta x$ are $\hat{\alpha} = (m\alpha + r2^k) \hat{e} + \alpha_1 \hat{e}_1 + \dots + \alpha_r \hat{e}_r$ for any integer r , any even integers $\alpha_1, \dots, \alpha_r$ and any integer m such that $m^2 \equiv 1 \pmod{2^k}$.

By the proof of the previous theorem with $p = 2$ $\alpha_1, \dots, \alpha_r$ can be any even integers and the leading coefficient of $\hat{\alpha}$ can be $\alpha + r 2^{k+1}$, for any integer r , or $m\alpha +$ some multiple of 2^k (whether the multiple is odd or even depends on m as we shall see) and so this shows that the possible values split into at most two classes.

Consider now a general situation where

$$\hat{C} = \begin{array}{c} \begin{array}{cccccc} \hat{g}_1 & \hat{g}_2 & \dots & \dots & \dots & \hat{g}_{2s-1} & \hat{g}_{2s} & \hat{f} \end{array} \\ \left[\begin{array}{cccccc|cc} 0 & 1 & & & & & & 0 \\ 1 & 0 & & & & & & \\ \hline & & 0 & 1 & & & & \\ & & 1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & 0 & 1 & \\ & & & & & 1 & 0 & \\ & & & & & & & 1 \end{array} \right] \end{array}$$

and $\hat{D}$ is similarly of this form.

Consider any possible change of basis $\hat{E}, \hat{E}_1, \dots, \hat{E}_r, \hat{G}_1, \dots, \hat{G}_{2s}, \hat{F}$ to give the matrix (which is just the matrix $\left[\begin{array}{c|c} \hat{A} & \hat{O} \\ \hline \hat{O} & \hat{D} \end{array} \right]$)

$$\begin{array}{c}
 \hat{E} \\
 \text{" " } \hat{me} \\
 + \\
 \vdots \\
 + \\
 \hat{uf}
 \end{array}
 \begin{array}{c}
 \hat{E}_1 \quad \dots \quad \hat{E}_r \quad \hat{G}_1 \quad \dots \quad \hat{G}_{2s} \\
 n2^{-k} \quad 1 \quad 0 \quad \dots \quad 0 \quad | \quad 0 \quad \dots \quad 0 \\
 1 \quad a_1 \quad \dots \quad | \quad \dots \\
 0 \quad \dots \quad 0 \quad | \quad \dots \\
 \dots \quad \dots \quad a_{r-1} \quad 1 \quad | \quad \dots \\
 0 \quad \dots \quad 0 \quad 1 \quad a_r \quad | \quad 0 \quad \dots \quad 0 \\
 \hline
 0 \quad \quad \quad 0 \quad | \quad 0 \quad 1 \quad \quad 0 \\
 \vdots \quad \quad \quad \vdots \quad | \quad 1 \quad 0 \quad \quad \vdots \\
 \vdots \quad \quad \quad \vdots \quad | \quad \quad \quad \quad \quad \quad \vdots \\
 \vdots \quad \quad \quad \vdots \quad | \quad \quad \quad \quad \quad \quad \vdots \\
 \vdots \quad \quad \quad \vdots \quad | \quad \quad \quad \quad \quad \quad \vdots \\
 \vdots \quad \quad \quad \vdots \quad | \quad \quad \quad \quad \quad \quad \vdots \\
 \vdots \quad \quad \quad \vdots \quad | \quad \quad \quad \quad \quad \quad \vdots \\
 \vdots \quad \quad \quad \vdots \quad | \quad \quad \quad \quad \quad \quad \vdots \\
 \vdots \quad \quad \quad \vdots \quad | \quad \quad \quad \quad \quad \quad \vdots \\
 0 \quad \quad \quad 0 \quad | \quad 0 \quad \quad \quad \quad \quad \quad \vdots
 \end{array}
 \begin{array}{c}
 \hat{F} \\
 \text{" " } 2^k \hat{ve} \\
 + \\
 \vdots \\
 + \\
 \hat{w} \hat{f}
 \end{array}$$

To give the diagonal element $n2^{-k}$ we must have $m^2 = t2^k + 1$,
for some integer t , and since all other elements of the matrix
are integers $\hat{E}_1, \dots, \hat{E}_r, \hat{G}_1, \dots, \hat{G}_{2s}, \hat{F}$ expressed in terms
of the original basis start with elements which are multiples of

$2^k e$. Let the tangential invariant have coefficients α of $\hat{E}$, α^u of $\hat{F}$ in terms of new basis and coefficient α' of e in terms of the original basis, then α^u is odd since it corresponds to the odd diagonal element 1. The coefficients of $\hat{E}_1, \dots, \hat{E}_r, \hat{G}_1, \dots, \hat{G}_{2s}$ for the tangential invariant are even and so in an expression for α' the contribution here would be a multiple of 2^{k+1} . Hence

$$\begin{aligned}\alpha' &= m\alpha + 2^k v \alpha^u \pmod{2^{k+1}} \\ &= m\alpha + 2^k v \pmod{2^{k+1}} \text{ since } \alpha^u \text{ is odd}\end{aligned}$$

The integer w must be odd to give the diagonal element 1 and since for the tangential invariant the coefficients of $\hat{E}, \hat{E}_1, \dots, \hat{E}_r, \hat{G}_1, \dots, \hat{G}_{2s}$ and $\hat{e}_1, \dots, \hat{e}_r, \hat{g}_1, \dots, \hat{g}_{2s}$, with respect to the two bases, are even only even multiples of $\hat{e}_1, \dots, \hat{e}_r, \hat{g}_1, \dots, \hat{g}_{2s}$ can occur in the expression for $\hat{F}$.

Now if t is $\begin{cases} \text{odd} \\ \text{even} \end{cases}$ then u must be $\begin{cases} \text{odd} \\ \text{even} \end{cases}$ and then v is $\begin{cases} \text{odd} \\ \text{even} \end{cases}$.

This follows since any odd contribution to $\hat{E}^2 = \hat{\lambda}(\hat{E}, \hat{E})$ can only come from $m^2 n 2^{-k} = t n + n 2^{-k}$ and u^2 and any odd contribution to $\hat{E} \cdot \hat{F}$, which is zero, comes from $mnv + uw$. Thus

$$\alpha' = m\alpha + t 2^k \pmod{2^{k+1}}$$

The only possible values for m where $m^2 \equiv 1 \pmod{2^k}$ are
 $m = (q \cdot 2^{k-1} + 1)$ or $(q \cdot 2^{k-1} - 1)$, for any integer q , when
 $k \neq 1$ and any odd m when $k = 1$. We now have three cases.

(i) $k \geq 3$.

If q is even then t is even giving

$$\alpha' = \pm \alpha \pmod{2^{k+1}} \text{ since } \alpha \text{ is even}$$

and if q is odd t is odd giving

$$\alpha' = \pm \alpha + \alpha 2^{k-1} + 2^k \pmod{2^{k+1}}.$$

Thus for α divisible by 4 (which corresponds to β divisible by 4) we have

$$\alpha' = \pm \alpha + 2^k \pmod{2^{k+1}}$$

and we therefore obtain all possible values. If α is not divisible by 4 (when β is not divisible by 4) we have

$$\alpha' = \pm \alpha \pmod{2^{k+1}}$$

and so the possible values here split into two classes.

(ii) $k = 2$.

t is always even for any value of q and so if q is even we have

$$\alpha' = \pm \alpha \pmod{8}$$

and if q is odd we have

$$\alpha' = \underline{+} \alpha + 2\alpha \pmod{8}$$

Thus if α is not divisible by 4 (β not divisible by 4) this gives

$$\alpha' = \underline{+} \alpha + 4 \pmod{8}$$

and we therefore obtain all possible values. If α is divisible by 4 (β divisible by 4) then

$$\alpha' = \underline{+} \alpha \pmod{8}$$

and so the possible values split into two classes.

(iii) $k = 1$.

t is always even and for m odd and α even $m\alpha = \alpha \pmod{4}$ so that we obtain

$$\alpha' = \underline{+} \alpha \pmod{4}$$

and so the possible values here split into two classes. $\S$.

Proof of Theorem 3.6

$G \simeq Z_2^k \oplus Z_2^k$ and $b(x,x) = b(y,y) = 0$, $b(x,y) = 2^{-k}$ for generators x,y of $Z_2^k \oplus Z_2^k$. If $a_1x + b_1y$, $a_2x + b_2y$ are

also generators of $Z_2^k \oplus Z_2^k$ which give the same values under b as x, y then it follows that $a_1 b_1 = 0 \pmod{2^{k-1}}$, $a_2 b_2 = 0 \pmod{2^{k-1}}$ and $a_1 b_2 + a_2 b_1 = 1 \pmod{2^k}$. Suppose the tangential invariant $\beta = \beta_1 x + \beta_2 y$ then since $\hat{\beta}$ is even β_1 and β_2 are even integers. Under an isomorphism of G preserving b $\hat{\beta}$ has image

$$\beta_1(a_1 x + b_1 y) + \beta_2(a_2 x + b_2 y) = (a_1 \beta_1 + a_2 \beta_2)x + (b_1 \beta_1 + b_2 \beta_2)y$$

for integers a_1, a_2, b_1, b_2 as above.

By Lemma 2.1 the matrix for $\hat{\lambda}$, where λ induces b , is given by

$$\hat{A} = \begin{bmatrix} \hat{e}_1 & \hat{e}_2 \\ 0 & 2^{-k} \\ 2^{-k} & 0 \end{bmatrix}$$

In the exact sequence

$$0 \rightarrow H \rightarrow \hat{H} \xrightarrow[\substack{\partial \\ R}]{\partial} G \rightarrow 0$$

$$Z_2^k \oplus Z_2^k$$

$\partial(\hat{e}_1) = x$, $\partial(\hat{e}_2) = y$ for generators x, y of $Z_2^k \oplus Z_2^k$ as above.

Suppose α_1 and α_2 are even integers such that

$$\partial(\alpha_1 \hat{e}_1 + \alpha_2 \hat{e}_2) = \beta_1 x + \beta_2 y \text{ (we could take } \alpha_1 = \beta_1, \alpha_2 = \beta_2)$$

then possible values for tangential invariants $\hat{\alpha} \in \hat{H}$ that map to

$$\beta_1 x + \beta_2 x \text{ or its isomorphic image } (a_1 \beta_1 + a_2 \beta_2)x + (b_1 \beta_1 + b_2 \beta_2)y$$

$$\text{are } \hat{\alpha} = (a_1\alpha_1 + a_2\alpha_2 + r_12^k) e_1 + (b_1\alpha_1 + b_2\alpha_2 + r_22^k) \hat{e}_2$$

for any integers r_1, r_2 and any integers a_1, a_2, b_1, b_2 which satisfy the conditions $a_1b_1 = 0 \bmod 2^{k-1}$, $a_2b_2 = 0 \bmod 2^{k-1}$, $a_1b_2 + a_2b_1 = 1 \bmod 2^k$.

To $\hat{A}$ add the unimodular matrix

$$\begin{matrix} \hat{f}_1 & \hat{g}_1 & \hat{f}_2 & \hat{g}_2 \\ \left[\begin{array}{cccc} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{array} \right] \end{matrix}$$

Then with respect to the basis indicated the following matrix is obtained.

$$\begin{array}{cccccc}
 \hat{E}_1 & \hat{E}_2 & \hat{F}_1 & \hat{F}_2 & & \\
 \hat{e}_1 & \hat{e}_2 & 2^k \hat{e}_1 & 2^k \hat{e}_2 & & \\
 \hat{g}_1 & \hat{g}_2 & \hat{f}_2 & \hat{f}_1 & \hat{g}_1 & \hat{g}_2 \\
 \hline
 0 & 2^{-k} & | & 0 & 0 & 0 \\
 2^{-k} & 0 & | & 0 & 0 & 0 \\
 \hline
 0 & 0 & | & & & \\
 0 & 0 & | & & \hat{D} & \\
 0 & 0 & | & & & \\
 0 & 0 & | & & &
 \end{array}$$

If the tangential invariant has coefficients α_1 of $\hat{E}_1$, α_2 , of $\hat{E}_2$, α_1^u of $\hat{F}_1$, α_2^u of $\hat{F}_2$ with respect to the new basis and α'_1 of $\hat{e}_1$, α'_2 of $\hat{e}_2$ with respect to the original basis then

$$\alpha'_1 = \alpha_1 + 2^k \alpha_1^u$$

$$\alpha'_2 = \alpha_2 + 2^k \alpha_2^u$$

$\hat{D}$ must be an even unimodular matrix and so α_1^u and α_2^u can be any even integers and hence we can obtain

$$\alpha'_1 = \alpha_1 + r_1 2^{k+1}$$

$$\alpha'_2 = \alpha_2 + r_2 2^{k+1}$$

for any integers r_1, r_2 .

For integers a_1, a_2, b_1, b_2 as above since $a_1x + b_1y,$

$a_2x + b_2y$ are generators of $\mathbb{Z}_2^k \oplus \mathbb{Z}_2^k$ there exist integers

p_1, p_2, q_1, q_2 such that

$$p_1(a_1x + b_1y) + p_2(a_2x + b_2y) = x$$

$$q_1(a_1x + b_1y) + q_2(a_2x + b_2y) = y$$

$$\text{i.e. } \begin{cases} p_1a_1 + p_2a_2 = 1 + t_12^k \\ p_1b_1 + p_2b_2 = u_12^k \end{cases} \quad \text{and} \quad \begin{cases} q_1a_1 + q_2a_2 = t_22^k \\ q_1b_1 + q_2b_2 = 1 + u_22^k \end{cases}$$

for some integers t_1, t_2, u_1, u_2 . Now adding to $\hat{A}$ the unimodular matrix

$$\begin{pmatrix} \hat{f}_1 & \hat{f}_2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

the matrix with respect to the new basis

$$\hat{E}_1 = a_1\hat{e}_1 + b_1\hat{e}_2 + \hat{f}_1, \hat{E}_2 = a_2\hat{e}_1 + b_2\hat{e}_2 + \hat{f}_2,$$

$$\hat{F}_1 = t_12^k\hat{e}_1 + u_12^k\hat{e}_2 + p_1\hat{f}_1 + p_2\hat{f}_2, \hat{F}_2 = t_22^k\hat{e}_1 + u_22^k\hat{e}_2 + q_1\hat{f}_1 +$$

$$q_2\hat{f}_2 \quad \text{would be of the form} \quad \left[\begin{array}{c|c} \hat{A} & \hat{O} \\ \hline 0 & \hat{D} \end{array} \right], \text{ where}$$

$\hat{D}$ is unimodular, if we could change elements in the matrix by integer values. The proof of Lemma 2.5 shows how a unimodular matrix could be added and a change of basis employed to achieve this. It can be seen that this would only change the coefficients of $\hat{E}_1, \hat{E}_2$ for $\hat{\alpha}$ by some multiples of 2^k . Thus we obtain possible coefficients α'_1 of $\hat{e}_1, \alpha'_2$ of $\hat{e}_2$ where

$$\alpha'_1 = a_1 \alpha_1 + a_2 \alpha_2 + \text{some multiple of } 2^k$$

$$\alpha'_2 = b_1 \alpha_1 + b_2 \alpha_2 + \text{some multiples of } 2^k.$$

To discover which multiples of 2^k are possible consider a general situation where

$$\hat{C} = \begin{bmatrix} \hat{g}_1 & \hat{g}_2 & \dots & \dots & \dots & \hat{g}_{2s-1} & \hat{g}_{2s} & \hat{f} \\ 0 & 1 & & & & & 0 & \\ 1 & 0 & & & & & & \\ & & \boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}} & & & & & \\ & & & \dots & & \boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}} & & \\ & & & & & & & 1 \end{bmatrix}$$

and $\hat{D}$ is also of this form.

Consider any possible change of basis $\hat{E}_1, \hat{E}_2, \hat{G}_1, \dots, \hat{G}_{2s}, \hat{F}$

to give the matrix

$$\begin{array}{c}
 \begin{array}{c} \hat{E}_1 \\ \hat{E}_2 \\ a_1 \hat{e}_1 + b_1 \hat{e}_2 + \dots + u_1 \hat{f} \\ 2^{-k} \\ 0 \\ \vdots \\ 0 \end{array}
 \begin{array}{c} \hat{E}_2 \\ a_2 \hat{e}_1 + b_2 \hat{e}_2 + \dots + u_2 \hat{f} \\ 2^{-k} \\ 0 \\ \vdots \\ 0 \end{array}
 \begin{array}{c} \hat{G}_1 \dots \hat{G}_{2s} \\ \vdots \\ 0 \end{array}
 \begin{array}{c} \hat{F} \\ v_1 2^k \hat{e}_1 + v_2 2^k \hat{e}_2 + \dots + w \hat{f} \\ \vdots \\ 1 \end{array}
 \end{array}
 \left[\begin{array}{cccc}
 0 & 2^{-k} & 0 & \dots & 0 \\
 2^{-k} & 0 & 0 & \dots & 0 \\
 0 & 0 & 0 & 1 & 0 \\
 \vdots & \vdots & 1 & 0 & \vdots \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 \vdots & \vdots & \vdots & 0 & 1 \\
 \vdots & \vdots & 0 & 1 & 0 \\
 0 & 0 & \vdots & \vdots & 1
 \end{array} \right]$$

To produce this matrix we must have $a_1 b_1 = m_1 2^{k-1}$,

$a_2 b_2 = m_2 2^{k-1}$ and $a_1 b_2 + a_2 b_1 = 1 + n 2^k$ for some integers

m_1, m_2, n . Also in terms of the original basis $\hat{e}_1, \hat{e}_2, \hat{g}_1, \dots, \hat{g}_{2s}, \hat{f}$ only multiples of $2^k \hat{e}_1, 2^k \hat{e}_2$ can occur in expressions for $\hat{G}_1, \dots, \hat{G}_{2s}, \hat{F}$ and only even multiples of $\hat{g}_1, \dots, \hat{g}_{2s}$ can occur in the expression for $\hat{F}$.

Let the tangential invariant have coefficients α_1 of $\hat{E}_1$, α_2 of $\hat{E}_2$, α^u of $\hat{F}$ in terms of the new basis and α'_1 of $\hat{e}_1$, α'_2 of $\hat{e}_2$ in terms of the original basis. In expressions for α'_1 , α'_2 any contribution from the coefficients of $\hat{G}_1, \dots, \hat{G}_{2s}$ will be multiples of 2^{k+1} thus

$$\alpha'_1 = a_1\alpha_1 + a_2\alpha_2 + v_1 2^k \alpha^u \text{ mod } 2^{k+1}$$

$$\alpha'_2 = b_1\alpha_1 + b_2\alpha_2 + v_2 2^k \alpha^u \text{ mod } 2^{k+1}$$

α^u is odd since it corresponds to diagonal element 1 so

$$\alpha'_1 = a_1\alpha_1 + a_2\alpha_2 + v_1 2^k \text{ mod } 2^{k+1}$$

$$\alpha'_2 = b_1\alpha_1 + b_2\alpha_2 + v_2 2^k \text{ mod } 2^{k+1}$$

The integer w must be odd in the expression for $\hat{F}$ (to give diagonal element 1) and so:

If m_1 is $\begin{cases} \text{odd} \\ \text{even} \end{cases}$ then u_1 must be $\begin{cases} \text{odd} \\ \text{even} \end{cases}$ and so $a_1 v_2 + b_1 v_1$ must be $\begin{cases} \text{odd} \\ \text{even} \end{cases}$.

If m_2 is $\begin{cases} \text{odd} \\ \text{even} \end{cases}$ then u_2 must be $\begin{cases} \text{odd} \\ \text{even} \end{cases}$ and so $a_2 v_2 + b_2 v_1$ must be $\begin{cases} \text{odd} \\ \text{even} \end{cases}$.

Consider now two cases.

(i) $k \geq 2$

$a_1 b_1 + a_2 b_1 = 1 + n2^k$ and so either a_1 and b_2 are odd or a_2 and b_1 are odd. We assume a_1 and b_2 are odd when since $a_1 b_1 = m_1 2^{k-1}$, $a_2 b_2 = m_2 2^{k-1}$ it follows that $b_1 = m_1 2^{k-1} \pmod{2^k}$ and $a_2 = m_2 2^{k-1} \pmod{2^k}$ and $a_1 b_2 \equiv 1 \pmod{2^k}$ when from above v_2 and m_1 have the same parity and v_1 and m_2 have the same parity. Thus

$$\alpha'_1 = a_1 \alpha_1 + m_2 \alpha_2 2^{k-1} + m_2 2^k \pmod{2^{k+1}}.$$

$$\alpha'_2 = b_2 \alpha_2 + m_1 \alpha_1 2^{k-1} + m_1 2^k \pmod{2^{k+1}}.$$

If α_2 is divisible by 4 this gives

$$\alpha'_1 = a_1 \alpha_1 + m_2 2^k \pmod{2^{k+1}}$$

. . if α_2 is not divisible by 4

$$\alpha'_1 = a_1 \alpha_1 \pmod{2^{k+1}}$$

Similarly if α_1 is divisible by 4

$$\alpha'_2 = b_2 \alpha_2 + m_1 2^k \pmod{2^{k+1}}$$

and if α_1 is not divisible by 4

$$\alpha'_2 = b_2 \alpha_2 \bmod 2^{k+1}.$$

Now consider possible values for a_1 and b_2 to give

$$\alpha'_1 = \alpha_1 + \text{a multiple of } 2^k$$

$$\alpha'_2 = \alpha_2 + \text{a multiple of } 2^k$$

for if we obtain all possible multiples of 2^k , by taking other values of a_1 and b_2 all possible values for α'_1, α'_2 can be obtained.

For α_1 and α_2 both divisible by 4 (when equivalently β_1 and β_2 are both divisible by 4 i.e. $\hat{\beta}$ is divisible by 4) and $a_1 = b_2 = 1$ we obtain by taking odd or even values for m_1, m_2

$$\begin{cases} \alpha'_1 = \alpha_1 \bmod 2^{k+1} \\ \alpha'_2 = \alpha_2 \bmod 2^{k+1} \end{cases}$$

$$\begin{cases} \alpha'_1 = \alpha_1 \bmod 2^{k+1} \\ \alpha'_2 = \alpha_2 + 2^k \bmod 2^{k+1} \end{cases}$$

$$\begin{cases} \alpha'_1 = \alpha_1 + 2^k \bmod 2^{k+1} \\ \alpha'_2 = \alpha_2 \bmod 2^{k+1} \end{cases}$$

$$\begin{cases} \alpha'_1 = \alpha_1 + 2^k \bmod 2^{k+1} \\ \alpha'_2 = \alpha_2 + 2^k \bmod 2^{k+1} \end{cases}$$

i.e. all possible multiples of 2^k occur and so all possible values of α'_1, α'_2 are obtained.

For at least one of α_1, α_2 not divisible by 4 for $a_1\alpha_1 = \alpha_1 \bmod 2^k$ and $b_2\alpha_2 = \alpha_2 \bmod 2^k$ we must have either a_1 or $b_2 = 1 \bmod 2^{k-1}$ and then since $a_1 b_2 = 1 \bmod 2^k$ it follows that possible values for a_1 and b_2 are $a_1 = 1 + s_1 2^{k-1}$ and $b_2 = 1 + s_2 2^{k-1}$ where s_1 and s_2 are either both odd or both even. Substituting these values above we obtain the following.

If α_1 divisible by 4, α_2 not

$$\begin{cases} \alpha'_1 = \alpha_1 \bmod 2^{k+1} \\ \alpha'_2 = \alpha_2 \bmod 2^{k+1} \end{cases} \quad \begin{cases} \alpha'_1 = \alpha_1 \bmod 2^{k+1} \\ \alpha'_2 = \alpha_2 + 2^k \bmod 2^{k+1} \end{cases}$$

If α_2 divisible by 4, α_1 not

$$\begin{cases} \alpha'_1 = \alpha_1 \bmod 2^{k+1} \\ \alpha'_2 = \alpha_2 \bmod 2^{k+1} \end{cases} \quad \begin{cases} \alpha'_1 = \alpha_1 + 2^k \bmod 2^{k+1} \\ \alpha'_2 = \alpha_2 \bmod 2^{k+1} \end{cases}$$

If α_1 and α_2 both not divisible by 4.

$$\begin{cases} \alpha'_1 = \alpha_1 \bmod 2^{k+1} \\ \alpha'_2 = \alpha_2 \bmod 2^{k+1} \end{cases} \quad \begin{cases} \alpha'_1 = \alpha_1 + 2^k \bmod 2^{k+1} \\ \alpha'_2 = \alpha_2 + 2^k \bmod 2^{k+1} \end{cases}$$

The possible multiples of 2^k split into two classes in these cases but since we have assumed a_1 and b_2 odd we still have to consider the case where a_2 and b_1 are odd or equivalently we can consider odd values of a_1 and b_2 such that

$$\alpha'_1 = \alpha_2 + \text{a multiple of } 2^k$$

$$\alpha'_2 = \alpha_1 + \text{a multiple of } 2^k$$

i.e. we must have

$$a_1 \alpha_1 = \alpha_2 + r_1 2^k \quad \text{--- --- --- --- --- --- ---} \quad (1)$$

$$b_2 \alpha_2 = \alpha_1 + r_2 2^k \quad \text{--- --- --- --- --- --- ---} \quad (2)$$

for some integers r_1, r_2 . For this to occur since a_1 and b_2 are odd α_1 and α_2 must be divisible by the same

power of 2 and so we must have α_1 and α_2 both not divisible by 4. $(2) + b_2 \cdot (1)$ gives

$$\alpha_1(a_1 b_2 - 1) = (r_1 b_2 + r_2)2^k$$

$a_1 b_2 - 1 = 0 \pmod{2^k}$ and α_1 is even so $r_1 b_2 + r_2$ is even and since b_2 is odd r_1 and r_2 have the same parity.

We therefore obtain values

$$\begin{cases} \alpha'_1 = \alpha_2 \pmod{2^{k+1}} \\ \alpha'_2 = \alpha_1 \pmod{2^{k+1}} \end{cases} \quad \begin{cases} \alpha'_1 = \alpha_2 + 2^k \pmod{2^{k+1}} \\ \alpha'_2 = \alpha_1 + 2^k \pmod{2^{k+1}} \end{cases}$$

which correspond to those we have already obtained.

Thus for α_1 or α_2 not divisible by 4 (when $\hat{\beta}$ is not divisible by 4) the possible multiples of 2^k split into two classes and thus the possible values of α'_1, α'_2 split into two classes.

(ii) $k = 1$.

In this case $\hat{\beta} = 0$ and so we can take $\alpha_1 = 0, \alpha_2 = 0$ and so going back to the matrix $\alpha'_1 = v_1 2 \pmod{4}, \alpha'_2 = v_2 2 \pmod{4}$.

We have $a_1 b_2 + a_2 b_1 = 1 \pmod{2}$ and assuming a_1 and b_2 are odd there are 3 possibilities.

- (a) a_2 and b_1 are both even when v_1 and v_2 are both even and hence

$$\alpha'_1 = 0 \pmod{4}$$

$$\alpha'_2 = 0 \pmod{4}$$

- (b) a_2 is odd and b_1 is even when v_2 is even and v_1 is odd and hence

$$\alpha'_1 = 2 \pmod{4}$$

$$\alpha'_2 = 0 \pmod{4}$$

- (c) a_2 is even and b_1 is odd when v_1 is even and v_2 is odd and hence

$$\alpha'_1 = 0 \pmod{4}$$

$$\alpha'_2 = 2 \pmod{4}.$$

Assuming that a_2 and b_1 are odd, instead of a_1 and b_2 will only give these values again and so the possible values for α'_1, α'_2 split into two classes; one class contains even pairs α'_1, α'_2 both of which are not divisible by 4 and the other class contains all other even pairs. § .

Proof of Theorem 3.7

$G \cong Z_2^k \oplus Z_2^k$, $k \geq 2$, and $b(x,x) = b(y,y) = 2^{1-k}$,
 $b(x,y) = 2^{-k}$ for generators x,y of $Z_2^k \oplus Z_2^k$. If $a_1x + b_1y$,
 $a_2x + b_2y$ are also generators which give the same values under
 b as x,y then $a_1^2 + b_1^2 + a_1 b_1 = 1 \pmod{2^{k-1}}$, $a_2^2 + b_2^2 + a_2 b_2 = 1 \pmod{2^{k-1}}$,
and $2(a_1 a_2 + b_1 b_2) + a_1 b_2 + a_2 b_1 = 1 \pmod{2^k}$. Suppose the
tangential invariant $\hat{\beta} = \beta_1 x + \beta_2 y$ then since $\hat{\beta}$ is even β_1 and β_2
are even integers. Under an isomorphism of G preserving b $\hat{\beta}$
has image

$(a_1 \beta_1 + a_2 \beta_2) x + (b_1 \beta_1 + b_2 \beta_2) y$ for integers
 a_1, a_2, b_1, b_2 as above.

By Lemma 2.1 the matrix for $\hat{\lambda}$, where λ induces b , is
given by:

$$\hat{A} = \begin{matrix} & \begin{matrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 & \hat{e}_4 \end{matrix} \\ \begin{bmatrix} 2^{1-k} & 2^{-k} & 0 & 0 \\ 2^{-k} & 2^{1-k} & 1 & 0 \\ 0 & 1 & 2c & 1 \\ 0 & 0 & 1 & 2d \end{bmatrix} \end{matrix}$$

In the exact sequence

$$0 \rightarrow H \rightarrow \hat{H} \xrightarrow{\partial} \underset{\mathbb{R}}{G} \rightarrow 0$$

$$Z_2^k \oplus Z_2^k$$

$\partial(\hat{e}_1) = x, \partial(\hat{e}_2) = y, \partial(\hat{e}_3) = \partial(\hat{e}_4) = 0$, for generators x, y of $Z_2^k \oplus Z_2^k$ as above. Suppose α_1 and α_2 are even integers such that $\partial(\alpha_1 \hat{e}_1 + \alpha_2 \hat{e}_2) = \beta_1 x + \beta_2 y$ (we could take $\alpha_1 = \beta_1, \alpha_2 = \beta_2$) then possible values for tangential invariants $\hat{\alpha} \in \hat{H}$ that map to $\beta_1 x + \beta_2 y$ or its isomorphic image $(a_1 \beta_1 + a_2 \beta_2) + (b_1 \beta_1 + b_2 \beta_2)y$ are $\hat{\alpha} = (a_1 \alpha_1 + a_2 \alpha_2 + r_1 2^k) \hat{e}_1 + (b_1 \alpha_1 + b_2 \alpha_2 + r_2 2^k) \hat{e}_2 + \alpha_3 \hat{e}_3 + \alpha_4 \hat{e}_4$ for any integers r_1, r_2 , any even integers α_3, α_4 , and any integers a_1, a_2, b_1, b_2 which satisfy the conditions $a_1^2 + b_1^2 + a_1 b_1 = 1 \pmod{2^{k-1}}$, $a_2^2 + b_2^2 + a_2 b_2 = 1 \pmod{2^{k-1}}$ and $2(a_1 a_2 + b_1 b_2) + a_1 b_2 + a_2 b_1 = 1 \pmod{2^k}$.

To $\hat{A}$ add the unimodular matrix

$$\begin{array}{c}
 \hat{f}_1 \quad \hat{g}_1 \quad \hat{f}_2 \quad \hat{g}_2 \quad \hat{f}_3 \quad \hat{g}_3 \quad \hat{f}_4 \quad \hat{g}_4 \\
 \left[\begin{array}{cc|cc|cc|cc}
 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 \hline
 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
 \hline
 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
 \hline
 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
 \end{array} \right]
 \end{array}$$

Then with respect to the basis indicated the following matrix is obtained:-

$$\begin{array}{cccccccccccc}
 \hat{E}_1 & \hat{E}_2 & \hat{E}_3 & \hat{E}_4 & \hat{F}_1 & \hat{F}_2 & \hat{F}_3 & \hat{F}_4 & & & & \\
 \hat{e}_1 & \hat{e}_2 & \hat{e}_3 & \hat{e}_4 & 2^k \hat{e}_1 & 2^k \hat{e}_2 & 2^k \hat{e}_3 & 2^k \hat{e}_4 & & & & \\
 \hat{g}_1 & \hat{g}_2 & \hat{g}_3 & \hat{g}_4 & \hat{f}_1 & \hat{f}_2 & \hat{f}_3 & \hat{f}_4 & \hat{g}_1 & \hat{g}_2 & \hat{g}_3 & \hat{g}_4 \\
 \hline
 2^{1-k} & 2^{-k} & 0 & 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
 2^{-k} & 2^{1-k} & 1 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 0 & 1 & 2c & 1 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 0 & 0 & 1 & 2d & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
 \hline
 0 & 0 & 0 & 0 & & & & & & & & \\
 \vdots & \vdots & \vdots & \vdots & & & & & & & & \\
 \vdots & \vdots & \vdots & \vdots & & & & & & & & \\
 \vdots & \vdots & \vdots & \vdots & & & & & & & & \\
 \vdots & \vdots & \vdots & \vdots & & & & & & & & \\
 \vdots & \vdots & \vdots & \vdots & & & & & & & & \\
 0 & 0 & 0 & 0 & & & & & & & &
 \end{array}
 \left[\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right]$$

If the tangential invariant has coefficients α_i of $\hat{E}_i$ and α_i^u of $\hat{F}_i$ in terms of the new basis and α_i' of $\hat{e}_i$ in terms of the original basis then:

$$\alpha_1' = \alpha_1 + 2^k \alpha_1^u$$

$$\alpha_2' = \alpha_2 + 2^k \alpha_2^u$$

$$\alpha_3' = \alpha_3 + \alpha_3^u$$

$$\alpha_4' = \alpha_4 + \alpha_4^u$$

$\alpha_1^u, \alpha_2^u, \alpha_3^u, \alpha_4^u$ can be any even integers and so α_3', α_4' can be any even integers as required and

$$\alpha_1' = \alpha_1 + r_1 2^{k+1}$$

$$\alpha_2' = \alpha_2 + r_2 2^{k+1}$$

for any integers r_1, r_2 .

For generators $a_1x + b_1y, a_2x + b_2y$ of $Z_2^k \oplus Z_2^k$, as above, there exist integers p_1, p_2, q_1, q_2 giving x, y in terms of these generators exactly as described in the proof of Theorem 3.6. Now adding to $\hat{A}$ the unimodular matrix

$$\begin{matrix} \hat{f}_1 & \hat{f}_2 \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{matrix}$$

the matrix with respect to the new basis $\hat{E}_1 = a_1\hat{e}_1 + b_1\hat{e}_2 + \hat{f}_1$, $\hat{E}_2 = a_2\hat{e}_1 + b_2\hat{e}_2 + \hat{f}_2, \hat{e}_3, \hat{e}_4, \hat{F}_1 = t_1 2^k \hat{e}_1 + u_1 2^k \hat{e}_2 + p_1 \hat{f}_1 + p_2 \hat{f}_2$, $\hat{F}_2 = t_2 2^k \hat{e}_1 + u_2 2^k \hat{e}_2 + q_1 \hat{f}_1 + q_2 \hat{f}_2$ can be reduced to the form

$$\left[\begin{array}{c|c} \hat{A} & 0 \\ \hline 0 & \hat{D} \end{array} \right]$$

where $\hat{D}$ is unimodular, by changing elements of the matrix by integer values. The proof of Lemma 2.6 shows how this can be achieved by adding a unimodular matrix and changing the basis. This process would only change the coefficients of $\hat{E}_1, \hat{E}_2$ for $\hat{\alpha}$ by some multiples of 2^k . Thus possible coefficients α'_1 of $\hat{e}_1$, α'_2 of $\hat{e}_2$ are obtained where

$$\alpha'_1 = a_1 \alpha_1 + a_2 \alpha_2 + \text{some multiple of } 2^k$$

$$\alpha'_2 = b_1 \alpha_1 + b_2 \alpha_2 + \text{some multiple of } 2^k.$$

Now consider a general situation where

$$\hat{C} = \begin{bmatrix} \hat{g}_1 & \hat{g}_2 & & \hat{g}_{2s-1} & \hat{g}_{2s} & \hat{f} \\ 0 & 1 & & & 0 & \\ 1 & 0 & & & & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ & & & 0 & 1 & \\ & & & 1 & 0 & \\ & 0 & & & & 1 \end{bmatrix}$$

and $\hat{D}$ is also of this form.

Consider any possible change of basis $\hat{E}_1, \hat{E}_2, \hat{E}_3, \hat{E}_4, \hat{G}_1, \dots, \hat{G}_{2s}, \hat{F}$ to give the matrix

$$\begin{array}{c}
 \begin{array}{c} \hat{E}_1 \\ \hat{E}_2 \end{array} \quad \begin{array}{c} \hat{E}_3 \\ \hat{E}_4 \\ \hat{G}_1 \end{array} \quad \dots \quad \begin{array}{c} \hat{G}_{2s} \end{array} \quad \begin{array}{c} \hat{F} \\ \vdots \\ \hat{F} \end{array} \\
 \begin{array}{c} a_1 e_1 \\ + a_2 e_1 \\ b_1 e_2 \\ + b_2 e_2 \\ \vdots \\ + u_1 f \\ + u_2 f \end{array} \quad \begin{array}{c} a_2 e_1 \\ + a_2 e_2 \\ b_2 e_2 \\ + b_2 e_2 \\ \vdots \\ + u_2 f \\ + u_2 f \end{array} \quad \begin{array}{c} v_1 2^k e_1 \\ + v_2 2^k e_2 \\ \vdots \\ + w f \end{array} \\
 \left[\begin{array}{cccc|cccc}
 2^{1-k} & 2^{-k} & 0 & 0 & 0 & \dots & 0 & 0 \\
 2^{-k} & 2^{1-k} & 1 & 0 & 0 & \dots & 0 & 0 \\
 0 & 1 & 2c & 1 & 0 & \dots & 0 & 0 \\
 0 & 0 & 1 & 2d & 0 & \dots & 0 & 0 \\
 \hline
 0 & 0 & 0 & 0 & 0 & 1 & & 0 \\
 \vdots & \vdots & \vdots & \vdots & 1 & 0 & & \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 1 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 0 & 0 & 0 & 0 & 0 & \vdots & \vdots & 1
 \end{array} \right]
 \end{array}$$

To produce this matrix we must have $a_1^2 + b_1^2 + a_1 b_1 = 1 + m_1 2^{k-1}$,
 $a_2^2 + b_2^2 + a_2 b_2 = 1 + m_2 2^{k-1}$, and $2(a_1 a_2 + b_1 b_2) + a_1 b_2 + a_2 b_1 = 1 + n 2^k$
 for some integers m_1, m_2, n . Also in terms of the original basis

only multiples of $2^k \hat{e}_1, 2^k \hat{e}_2$ can occur in expressions for $\hat{E}_3, \hat{E}_4, \hat{G}_1, \dots, \hat{G}_{2s}, \hat{F}$ and only even multiples of $\hat{e}_3, \hat{e}_4, \hat{g}_1, \dots, \hat{g}_{2s}$ can occur in the expression for $\hat{F}$. Let the tangential invariant have coefficients α_1 of $\hat{E}_1, \alpha_2$ of $\hat{E}_2$ in terms of the new basis and α'_1 of $\hat{e}_1, \alpha'_2$ of $\hat{e}_2$ in terms of the original basis then by the usual argument

$$\alpha'_1 = a_1 \alpha_1 + a_2 \alpha_2 + v_1 2^k \pmod{2^{k+1}}$$

$$\alpha'_2 = b_1 \alpha_1 + b_2 \alpha_2 + v_2 2^k \pmod{2^{k+1}}$$

The integer w must be odd and so:

If m_1 is $\begin{cases} \text{odd} \\ \text{even} \end{cases}$ then u_1 must be $\begin{cases} \text{odd} \\ \text{even} \end{cases}$ and so $a_1 v_2 + b_1 v_1$ must be $\begin{cases} \text{odd} \\ \text{even} \end{cases}$ (which comes from the fact that $2(a_1 v_1 + b_1 v_2) + a_1 v_2 + b_1 v_1$ must be $\begin{cases} \text{odd} \\ \text{even} \end{cases}$).

If m_2 is $\begin{cases} \text{odd} \\ \text{even} \end{cases}$ then u_2 must be $\begin{cases} \text{odd} \\ \text{even} \end{cases}$ and so $a_2 v_2 + b_2 v_1$ must be $\begin{cases} \text{odd} \\ \text{even} \end{cases}$.

We now consider possible values for a_1, a_2, b_1, b_2 to give

$$\alpha'_1 = \alpha_1 + \text{a multiple of } 2^k$$

$$\alpha'_2 = \alpha_2 + \text{a multiple of } 2^k.$$

If all multiples of 2^k occur then by taking other values of a_1, a_2, b_1, b_2 all possible values of α'_1, α'_2 can be obtained.

The values $a_1 = 1 + s_1 2^{k-1}$, $b_2 = 1 + s_2 2^{k-1}$, $a_2 = m_2 2^{k-1} \bmod 2^k$, $b_1 = m_1 2^{k-1} \bmod 2^k$, where s_1 and s_2 are either both odd or both even, satisfy these requirements and from above v_2 and m_1 have the same parity and v_1 and m_2 have the same parity. Thus we have exactly the same situation as in the proof of Theorem 3.6 and we therefore obtain the same results. If α_1 and α_2 are divisible by 4 (when equivalently $\hat{\beta}$ is divisible by 4) we thus obtain

$$\alpha'_1 = \alpha_1 + \text{any multiple of } 2^k$$

$$\alpha'_2 = \alpha_2 + \text{any multiple of } 2^k$$

and so in this case all possible values of α'_1, α'_2 can occur and we obtain the result of the theorem.

For α_1 and α_2 not both divisible by 4 these values of a_1, a_2, b_1, b_2 give only half the possible multiples of 2^k , for example for α_1 and α_2 both not divisible by 4 we obtain

$$\begin{cases} \alpha'_1 &= \alpha_1 \bmod 2^{k+1} \\ \alpha'_2 &= \alpha_2 \bmod 2^{k+1} \end{cases} \quad \begin{cases} \alpha'_1 &= \alpha_1 + 2^k \bmod 2^{k+1} \\ \alpha'_2 &= \alpha_2 + 2^k \bmod 2^{k+1} \end{cases}$$

(We consider $\alpha'_1, \alpha'_2 \pmod{2^{k+1}}$ since we know we can add any multiple of 2^{k+1} to either α_1 or α_2). To complete the proof we need only show that these are the only multiples of 2^k that can be obtained for other values of a_1, a_2, b_1, b_2 , when the possible values of α'_1, α'_2 will fall into two classes.

First we note that we can take α_1 and α_2 both not divisible by 4, for if $\hat{\beta} = \beta_1 x + \beta_2 y$ for generators x, y of $Z_2^k \oplus Z_2^k$ where β_1 is divisible by 4 and β_2 not divisible by 4 then $\hat{\beta} = (\beta_1 + \beta_2) x + (-\beta_2)(x-y)$ when $\beta_1 + \beta_2$ and $-\beta_2$ are both not divisible by 4 and $x, x-y$ are generators of $Z_2^k \oplus Z_2^k$ giving the same values under the map b as x, y .

Since $2(a_1 a_2 + b_1 b_2) + a_1 b_2 + a_2 b_1 = 1 + n 2^k$ either a_1 and b_2 are odd or a_2 and b_1 are odd. We assume that a_1 and b_2 are odd.

Consider possible values of a_1, a_2, b_1, b_2 to give:

$$\begin{cases} a_1 \alpha_1 + a_2 \alpha_2 = \alpha_1 + r_1 2^k & \text{---} & \text{---} & \text{---} & \text{---} & (1) \\ b_1 \alpha_1 + b_2 \alpha_2 = \alpha_2 + r_2 2^k & \text{---} & \text{---} & \text{---} & \text{---} & (2) \end{cases}$$

or

$$\begin{cases} a_1 \alpha_1 + a_2 \alpha_2 = \alpha_2 + r_1 2^k & \text{---} & \text{---} & \text{---} & \text{---} & (3) \\ b_1 \alpha_1 + b_2 \alpha_2 = \alpha_1 + r_2 2^k & \text{---} & \text{---} & \text{---} & \text{---} & (4) \end{cases}$$

which we must consider since we have assumed a_1 and b_2 odd.

Since α_1 and α_2 are both not divisible by 4 to give solutions above both a_2 and b_1 must be even and in this case v_2 and m_1 will have the same parity and v_1 and m_2 will have the same parity. Therefore we will obtain

$$\begin{cases} \alpha'_1 = \alpha_1 + (r_1 + m_2)2^k \bmod 2^{k+1} \\ \alpha'_2 = \alpha_2 + (r_2 + m_1)2^k \bmod 2^{k+1} \end{cases}$$

or
$$\begin{cases} \alpha'_1 = \alpha_2 + (r_1 + m_2)2^k \bmod 2^{k+1} \\ \alpha'_2 = \alpha_1 + (r_2 + m_1)2^k \bmod 2^{k+1} \end{cases}$$

If we can show that $(r_1 + m_2)$ and $(r_2 + m_1)$ have the same parity in the two cases then only those multiples of 2^k that we have already obtained can occur and we will have finished.

$(1)^2 + (2)^2 + (1) \cdot (2)$ gives

$$\begin{aligned} \alpha_1^2 m_1 2^{k-1} + \alpha_2^2 m_2 2^{k-1} + \alpha_1 \alpha_2 n 2^k &= \alpha_1 r_2 2^k + \alpha_2 r_1 2^k \\ + \alpha_1 r_1 2^{k+1} + \alpha_2 r_2 2^{k+1} + 2^{2k} (r_1^2 + r_2^2 + r_1 r_2) \end{aligned}$$

Putting $\alpha_1 = 2\eta_1$, $\alpha_2 = 2\eta_2$ and dividing by 2^{k+1} gives:

$$\begin{aligned} m_1 n_1^2 + m_2 n_2^2 + 2n n_1 n_2 &= r_2 n_1 + r_1 n_2 + 2(r_1 n_1 + r_2 n_2) \\ &+ 2^{k-1}(r_1^2 + r_2^2 + r_1 r_2) \end{aligned}$$

$$\text{Hence } m_1 n_1^2 - r_2 n_1 = r_1 n_2 - m_2 n_2^2 \pmod{2}$$

and as n_1 and n_2 are odd

$$m_1 + r_2 = m_2 + r_1 \pmod{2}$$

Similarly $(3)^2 + (4)^2 + (3) \cdot (4)$ also gives

$$m_1 + r_2 = m_2 + r_1 \pmod{2}. \quad \S .$$

The indecomposable manifolds P , where G contains no elements of order 2, are classified by the invariants $(G, b, \hat{\beta})$, and by Lemma 1.3 the invariants of the connected sum of manifolds P are the direct sums of the invariants of the component manifolds and so by Theorems 3.1 and 3.4 the following result follows.

Theorem 3.8

If P_1 and P_2 are closed $(s-1)$ -connected $(2s+1)$ -manifolds, where $s = 3$ or 7 , with invariants $(G_1, b_1, \hat{\beta}_1)$ and $(G_2, b_2, \hat{\beta}_2)$ such

that G_1 and G_2 have no elements of order 2 and there exists an isomorphism $h: G_1 \rightarrow G_2$ which preserves the linking invariants b_1 and b_2 and with $h(\hat{\beta}_1) = \hat{\beta}_2$; then there exists an orientation preserving diffeomorphism $f: P_1 \rightarrow P_2 \neq \Sigma$, for some $\Sigma \in \textcircled{H}_{2s+1}$, with $f_* = h$.

CHAPTER 4

EMBEDDINGS AND IMMERSIONS

The immersion problem for closed 2 - connected 7 - manifolds in euclidean space is determined by Theorem 4.1. Embeddings of the handlebodies $\mathcal{H}(s+1)$ in euclidean space are then considered; Theorem 4.2 giving results for general values of the integer s excluding certain low values of s which are dealt with in a more complete way by Theorem 4.3. These results give embedding theorems for the boundaries of the manifolds of $\mathcal{H}(s+1)$ and in particular for closed $(s-1)$ - connected $(2s+1)$ - manifolds, where $s = 3$ or 7 , in Theorem 4.5.

Immersion

Theorem 4.1

Suppose P is a closed 2- connected 7- manifold then

P immerses in R^{10} .

P immerses in R^9 if and only if $\hat{\beta} = 0$.

P immerses in R^8 if and only if $\hat{\beta} = 0$.

Proof

We use the immersion result of Hirsch (12) which states that an m -manifold M immerses in $\mathbb{R}^{m+r}$ if and only if the SO/SO_r -bundle associated with the tangent bundle of M has a cross-section.

For $r = 1, 2$ $\pi_2(SO/SO_r) \approx \pi_6(SO/SO_r) \approx 0$ and $\pi_3(SO/SO_r) \approx \mathbb{Z}$ and so the only obstruction to a cross-section of the SO/SO_r -bundle of P lies in $H^4(P; \pi_3(SO/SO_r)) \approx H^4(P)$.

The fibration $SO_r \rightarrow SO \xrightarrow{P} SO/SO_r$ induces the exact homotopy sequence

$$\begin{array}{ccccccc} \dots \rightarrow \pi_3(SO_r) & \rightarrow & \pi_3(SO) & \xrightarrow{P_*} & \pi_3(SO/SO_r) & \rightarrow & \pi_2(SO_r) \rightarrow \dots \\ \downarrow \mathbb{R} & & \downarrow \mathbb{R} & & \downarrow \mathbb{R} & & \downarrow \mathbb{R} \\ 0 & & \mathbb{Z} & & \mathbb{Z} & & 0 \end{array}$$

P_* is therefore an isomorphism and induces an isomorphism

$$\begin{array}{ccc} P_{**}: H^4(P; \pi_3(SO)) & \rightarrow & H^4(P; \pi_3(SO/SO_r)) \\ \downarrow \mathbb{R} & & \downarrow \mathbb{R} \\ H^4(P) & & H^4(P) \end{array}$$

$\hat{\beta}$ is the obstruction to a cross-section of the tangent bundle of M and so the isomorphic image $P_{**}(\hat{\beta})$ will be the obstruction to

a cross-section of the associated SO/SO_r - bundle and hence the obstruction to immersing P in R^{7+r} . P thus immerses in R^8 or R^9 if and only if $\hat{\beta} = 0$.

To show that P immerses in R^{10} we use an argument similar to that employed by E. Thomas (42) to show that the span of certain 7 - manifolds is ≥ 4 .

By Lemma 1.1 the Stiefel - Whitney classes $w_r(P) = 0$ ($r \neq 0$) and hence the normal Stiefel - Whitney classes $\bar{w}_r = 0$. If $\bar{\beta} \in H^4(P)$ is the obstruction to triviality of the normal bundle of an immersion of P in R^{15} then by the argument of Lemma 1.2 $\bar{\beta}$ is an even element.

Suppose $\bar{\beta} = 2x$ for some $x \in H^4(P)$ then by Hopf (15) there exists a map $f: P^4 \rightarrow S^4$, where P^4 is the 4-skeleton of P , such that $f^*(a) = x$ where $f^*: H^4(S^4) \rightarrow H^4(P^4) \approx H^4(P)$ and a is a generator of $H^4(S^4) \approx Z$.

The obstruction to extending f to a map $f': P \rightarrow S^4$ consists of a class of elements $\phi(x) \in H^7(P)$, where ϕ has defining relation $S_q^2 S_q^2 = 0$. But by Theorem 5.1 of Thomas (41) $0 \in \phi(x)$ and so an extension f' exists.

The 3 - plane bundles over S^4 have Pontrjagin classes corresponding to the elements divisible by 4 in $H^4(S^4) \approx \mathbb{Z}$ and there therefore exists a 3 - plane bundle δ over S^4 with $p_1(\delta) = 4a$. $\delta \oplus 5\theta$, where θ is the trivial line bundle, is then an 8 - plane bundle over S^4 with $p_1(\delta \oplus 5\theta) = 4a$ and so by Kervaire, Lemma 1.1 (19), the obstruction to triviality of $\delta \oplus 5\theta$ is the element $2a$.

Let $\gamma = f'*(\delta)$ be the induced 3 - plane bundle over P then the obstruction to triviality of the 8 - plane bundle $\gamma \oplus 5\theta = f'*(\delta \oplus 5\theta)$ is $f^*(2a) = 2x = \bar{\beta}$. Hence the normal bundle $\nu = \gamma \oplus 5\theta$.

ν has therefore a transversal 5 - field and so by Theorem 6.4 of Hirsch (12) P can be immersed in 5 fewer dimensions i.e. P immerses in R^{10} . §.

Embeddings

To embed handlebodies of $\mathcal{H}(s+1)$ we employ the notion of a spine of a manifold first used by Hirsch and which is described and used by Wall in (46). The general principle is as follows.

Suppose a manifold M is immersed in R^k , and that M has a subcomplex N of dimension n such that M can be embedded in any neighbourhood of N , and $k \geq 2n + 1$. Then M can be embedded in R^k .

This holds for since embeddings are dense in the space of maps $N \rightarrow R^k$ and immersions are open in the space of maps $M \rightarrow R^k$, we may suppose that the immersion embeds N . It then embeds some neighbourhood of N and hence the result.

A subcomplex N of M as above is called a spine of M .

The following result is due to Wall (46).

Let W be a manifold with a handle presentation with all handles of dimension $\leq r$. Then W has an r - dimensional spine.

It follows from this that a manifold $L \in \mathcal{H}(s+1)$ has an $(s+1)$ - dimensional spine and so if L immerses in R^k for $k \geq 2s+3$ then L embeds in R^k . Using the immersion results of Hirsch (12) we now state and prove Theorems 4.2 and 4.3. In the proofs of these theorems the homotopy groups of SO are due to Bott (2), the homotopy groups of SO_r are due to Kervaire (20),

and the homotopy groups of the Stiefel manifolds SO/SO_r are due to Paechter (32).

Theorem 4.2

Suppose $L \in \mathcal{H}(s+1)$.

If $s = 2, 4, 5, 6 \pmod 8$ then L embeds in R^{2s+3} .

If $s = 0, 1, 3 \pmod 8$ $s \neq 0, 1, 3, 8, 9, 11$ then L embeds in R^{3s-3} .

If $s = 7 \pmod 8$ $s \neq 7$ then L embeds in R^{3s} .

Proof

Suppose that the obstruction to a cross-section of the tangent bundle of L is the element $\hat{\alpha} \in H^{s+1}(L; \Pi_s(SO))$, $\hat{\alpha}$ is therefore the obstruction to parallelisability of L . L has dimension $2s+2$ and so by Hirsch (12) the obstruction to immersing L in $R^{(2s+2)+r}$, and hence by above to embedding L , is the obstruction to a cross-section of the SO/SO_r - bundle associated with the tangent bundle of L . The obstruction then lies in the group $H^{s+1}(L; \Pi_s(SO/SO_r))$.

The fibration $SO_r \xrightarrow{P} SO \rightarrow SO/SO_r$ induces the exact homotopy sequence

$$\dots \rightarrow \Pi_s(SO_r) \rightarrow \Pi_s(SO) \xrightarrow{p_*} \Pi_s(SO/SO_r) \rightarrow \Pi_{s-1}(SO_r) \rightarrow \dots$$

and p_* induces the homomorphism

$$p_{**}: H^{s+1}(L; \Pi_s(SO)) \rightarrow H^{s+1}(L; \Pi_s(SO/SO_r))$$

and the obstruction to embedding L in $R^{(2s+2)+r}$ is thus $p_{**}(\hat{\alpha})$.

We prove the theorem in each case by showing that for the appropriate value of r $p_* = 0$ and so $p_{**} = 0$ and the obstruction vanishes.

For $s = 2, 4, 5, 6 \pmod{8}$ $\Pi_s(SO) \approx 0$ and so trivially $\hat{\alpha} = 0$ and $p_* = 0$ for any value of $r \geq 1$ and so L embeds in R^{2s+3} .

In this case the manifolds L are parallelisable and the result follows also by the theorem that any m -dimensional parallelisable manifold M with non-vacuous boundary embeds in R^{m+1} .

For $s = 0 \pmod{8}$ $s \neq 0, 8$ taking $r = s - 5$ $\Pi_s(SO) \approx Z_2$, $\Pi_{s-1}(SO) \approx Z$, $\Pi_{s-1}(SO_{s-5}) \approx Z \oplus Z_2$, and $\Pi_s(SO/SO_{s-5}) \approx Z_2$ and since the sequence

$$\dots \rightarrow \Pi_s(SO) \xrightarrow{p_*} \Pi_s(SO/SO_{s-5}) \rightarrow \Pi_{s-1}(SO_{s-5}) \rightarrow \Pi_{s-1}(SO) \rightarrow \dots$$

$$\quad \quad \quad \begin{matrix} \mathbb{Z}_2 \\ \uparrow \\ \mathbb{Z} \end{matrix} \quad \quad \quad \begin{matrix} \mathbb{Z}_2 \\ \uparrow \\ \mathbb{Z} \end{matrix} \quad \quad \quad \begin{matrix} \mathbb{Z} \oplus \mathbb{Z}_2 \\ \uparrow \\ \mathbb{Z} \end{matrix} \quad \quad \quad \begin{matrix} \mathbb{Z} \\ \uparrow \\ \mathbb{Z} \end{matrix}$$

is exact $p_* = 0$ and hence L embeds in $R^{(2s+2)+s-5} = R^{3s-3}$.

For $s = 1 \bmod 8$ $s \neq 1, 9$ $\Pi_s(SO/SO_{s-5}) \approx 0$ and so $p_* = 0$.

For $s = 3 \bmod 8$ $s \neq 3, 11$ $\Pi_s(SO) \approx \mathbb{Z}$, $\Pi_{s-1}(SO) \approx 0$, $\Pi_{s-1}(SO_{s-5}) \approx \mathbb{Z}_8$, $\Pi_s(SO/SO_{s-5}) \approx \mathbb{Z}_8$ and since the sequence

$$\begin{array}{ccccccc} \dots & \rightarrow & \Pi_s(SO) & \xrightarrow{p_*} & \Pi_s(SO/SO_{s-5}) & \rightarrow & \Pi_{s-1}(SO_{s-5}) \rightarrow \Pi_{s-1}(SO) \rightarrow \dots \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ & & \mathbb{Z} & & \mathbb{Z}_8 & & \mathbb{Z}_8 \quad 0 \end{array}$$

is exact $p_* = 0$.

For $s = 7 \bmod 8$ $s \neq 7$ taking $r = s-2$ $\Pi_s(SO) \approx \mathbb{Z}$, $\Pi_{s-1}(SO) \approx 0$, $\Pi_{s-1}(SO_{s-2}) \approx \mathbb{Z}_8$ and $\Pi_s(SO/SO_{s-2}) \approx \mathbb{Z}_8$ and since the sequence

$$\begin{array}{ccccccc} \dots & \rightarrow & \Pi_s(SO) & \xrightarrow{p_*} & \Pi_s(SO/SO_{s-2}) & \rightarrow & \Pi_{s-1}(SO_{s-2}) \rightarrow \Pi_{s-1}(SO) \rightarrow \dots \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ & & \mathbb{Z} & & \mathbb{Z}_8 & & \mathbb{Z}_8 \quad 0 \end{array}$$

is exact $p_* = 0$ so L embeds in $R^{(2s+2)+s-2} = R^{3s}$.

In fact in this case $\pi_s(SO/SO_{s-5}) \cong Z_{16}$ and by Kervaire $\pi_{s-1}(SO_{s-5}) \cong Z_{8d}$ where d is ambiguously 1 or 2. If d were 2 then we would have $p_* = 0$ here which would show that L embeds in R^{3s-3} . § .

Hoo and Mahowald (14) give tables of homotopy groups of Stiefel manifolds SO/SO_r not covered by Paechter (32) and these can be used to give further embedding theorems for the manifolds L in terms of conditions on $\hat{\alpha}$, particularly in the cases $s = 3, 7 \pmod{8}$.

We now consider the cases not covered by Theorem 4.2.

Theorem 4.3

If $L \in \mathcal{H}(2)$ then

L embeds in R^6

L embeds in R^5 if and only if L is parallelisable.

If $L \in \mathcal{H}(4)$ then

L embeds in R^{12}

L embeds in R^{11} if and only if $\hat{\alpha}$ is divisible by 2 i.e. if and only if $w_4(L) = 0$.

L embeds in R^9, R^{10} if and only if L is parallelisable.

If $L \in \mathcal{H}(8)$ then

L embeds in R^{24}

L embeds in R^{23} if and only if $\hat{\alpha}$ is divisible by 2. i.e. if and only if $w_8(L) = 0$.

L embeds in R^{22} if and only if $\hat{\alpha}$ is divisible by 4.

L embeds in R^{21} if and only if $\hat{\alpha}$ is divisible by 8.

L embeds in $R^{17}, R^{18}, R^{19}, R^{20}$ if and only if L is parallelisable.

If $L \in \mathcal{H}(9)$ then

L embeds in R^{24}

L embeds in $R^{19}, \dots, R^{23}$ if and only if L is parallelisable.

If $L \in \mathcal{H}(10)$ then

L embeds in R^{26} .

L embeds in $R^{21}, \dots, R^{25}$ if and only if L is parallelisable.

If $L \in \mathcal{H}(12)$ then

L embeds in R^{31}

L embeds in $R^{25}, \dots, R^{30}$ if and only if L is parallelisable.

Proof

As in the proof of Theorem 4.2 the obstruction to embedding

$L \in \mathcal{H}(s+1)$ in $R^{(2s+2)+r}$ is $p_{**}(\hat{\alpha})$ where

$$p_{**}: H^{s+1}(L; \Pi_s(SO)) \rightarrow H^{s+1}(L; \Pi_s(SO/SO_r))$$

is induced by:

$$p_*: \Pi_s(SO) \rightarrow \Pi_s(SO/SO_r).$$

For $r > s$ $\Pi_s(SO/SO_r) \approx 0$ and so it follows that L embeds in R^{3s+3} , thus as in the statement of the theorem L embeds in R^6 for $s=1$, L embeds in R^{12} for $s=3$, and L embeds in R^{24} for $s=7$.

If L is parallelisable i.e. $\hat{\alpha} = 0$ then $p_{**}(\hat{\alpha}) = 0$ for all $r \geq 1$ and so L embeds in R^{2s+3} .

Between these two cases we consider the relation between the homomorphisms p_* and p_{**} . $H^{s+1}(L)$ is a free abelian group of rank r say and by the Universal Coefficient Theorem

$$H^{s+1}(L; \Pi_s(SO)) \approx \underbrace{\Pi_s(SO) \oplus \dots \oplus \Pi_s(SO)}_{r \text{ times}} \quad \text{and}$$

$$H^{s+1}(L; \Pi_s(SO/SO_r)) \approx \underbrace{\Pi_s(SO/SO_r) \oplus \dots \oplus \Pi_s(SO/SO_r)}_{r \text{ times}}$$

and p_{**} is given by the map $\underbrace{p_* \oplus \dots \oplus p_*}_{r \text{ times}}$.

It follows that if p_* is a monomorphism so is p_{**} and so $p_{**}(\hat{\alpha})$ vanishes if and only if $\hat{\alpha}$ vanishes, while if for example $p_*: \mathbb{Z} \rightarrow \mathbb{Z}_n$ were the mod n map then $p_{**}(\hat{\alpha})$ would vanish if and only if $\hat{\alpha}$ were divisible by n .

We now consider the cases separately.

If $L \in \mathcal{H}(2)$ then as $SO(1)$ is a single point the obstruction to embedding L in R^5 is easily identified with $\hat{\alpha}$.

If $L \in \mathcal{H}(4)$ then from the exact sequence

$$\dots \rightarrow \Pi_3(SO) \xrightarrow{p_*} \Pi_3(SO/SO_3) \rightarrow \Pi_2(SO_3) \rightarrow \dots$$

$$\begin{array}{ccccc} & \mathbb{R} & & \mathbb{R} & \\ & \downarrow & & \downarrow & \\ & \mathbb{Z} & & \mathbb{Z}_2 & \\ & & & & \mathbb{O} \end{array}$$

p_* is the mod 2 map and so the obstruction $p_{**}(\hat{\alpha})$ to embedding L in R^{11} , which by definition is $w_4(L)$, vanishes if and only if $\hat{\alpha}$ is divisible by 2.

In the exact sequence

$$\dots \rightarrow \Pi_3(SO_2) \xrightarrow{p_*} \Pi_3(SO) \rightarrow \Pi_3(SO/SO_2) \rightarrow \Pi_2(SO_2) \rightarrow \dots$$

$$\begin{array}{ccccccc} & \mathbb{R} & & \mathbb{R} & & \mathbb{R} & \\ & \downarrow & & \downarrow & & \downarrow & \\ & \mathbb{O} & & \mathbb{Z} & & \mathbb{Z} & \\ & & & & & & \mathbb{O} \end{array}$$

p_* is an isomorphism and so L embeds in R^{10} if and only if $\hat{\alpha} = 0$.

If $L \in \mathcal{H}(8)$ then from the exact sequence

$$\dots \rightarrow \Pi_7(SO) \xrightarrow{p_*} \Pi_7(SO/SO_7) \rightarrow \Pi_6(SO) \rightarrow \dots$$

$$\begin{array}{ccccc} & \mathbb{R} & & \mathbb{R} & \\ & \downarrow & & \downarrow & \\ & \mathbb{Z} & & \mathbb{Z}_2 & \\ & & & & \mathbb{O} \end{array}$$

p_* is the mod 2 map and so the obstruction $p_{**}(\hat{\alpha}) = w_8(L)$ to embedding L in R^{23} vanishes if and only if $\hat{\alpha}$ is divisible by 2.

In the exact sequence

$$\begin{array}{ccccccc} \dots & \rightarrow & \Pi_7(SO) & \xrightarrow{p_*} & \Pi_7(SO/SO_6) & \rightarrow & \Pi_6(SO_6) \rightarrow \dots \\ & & \wr & & \wr & & \wr \\ & & \mathbb{Z} & & \mathbb{Z}_4 & & 0 \end{array}$$

p_* is the mod 4 map so L embeds in R^{22} if and only if $\hat{\alpha}$ is divisible by 4.

In the exact sequence

$$\begin{array}{ccccccc} \dots & \rightarrow & \Pi_7(SO) & \xrightarrow{p_*} & \Pi_7(SO/SO_5) & \rightarrow & \Pi_6(SO_5) \rightarrow \dots \\ & & \wr & & \wr & & \wr \\ & & \mathbb{Z} & & \mathbb{Z}_8 & & 0 \end{array}$$

p_* is the mod 8 map and so L embeds in R^{21} if and only if $\hat{\alpha}$ is divisible by 8.

In the exact sequence

$$\begin{array}{ccccccc} \dots & \rightarrow & \Pi_7(SO) & \xrightarrow{p_*} & \Pi_7(SO/SO_4) & \rightarrow & \Pi_6(SO_4) \rightarrow \Pi_6(SO) \rightarrow \dots \\ & & \wr & & \wr & & \wr \\ & & \mathbb{Z} & & \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{12} & & \mathbb{Z}_{12} \oplus \mathbb{Z}_{12} \\ & & & & & & 0 \end{array}$$

p_* is a monomorphism and so L embeds in R^{20} if and only if $\hat{\alpha} = 0$.

If $L \in \mathcal{H}(9)$ then since $\Pi_8(SO/SO_6) \approx 0$ L embeds in R^{24} .

In the exact sequence

$$\begin{array}{ccccccc} \dots & \rightarrow & \Pi_8(SO) & \xrightarrow{p_*} & \Pi_8(SO/SO_5) & \rightarrow & \Pi_7(SO_5) \rightarrow \dots \\ & & \downarrow \mathbb{Z}_2 & & \downarrow \mathbb{Z}_2 & & \downarrow \mathbb{Z} \end{array}$$

p_* is an isomorphism and so L embeds in R^{23} if and only if $\hat{\alpha} = 0$.

If $L \in \mathcal{H}(10)$ then from the exact sequence

$$\begin{array}{ccccccc} \dots & \rightarrow & \Pi_9(SO) & \xrightarrow{p_*} & \Pi_9(SO/SO_6) & \rightarrow & \Pi_8(SO_6) \rightarrow \Pi_8(SO) \rightarrow \dots \\ & & \downarrow \mathbb{Z}_2 & & \downarrow \mathbb{Z}_{12} & & \downarrow \mathbb{Z}_{24} & & \downarrow \mathbb{Z}_2 \end{array}$$

$p_* = 0$ and so L embeds in R^{26} .

In the exact sequence

$$\begin{array}{ccccccc} \dots & \rightarrow & \Pi_9(SO) & \xrightarrow{p_*} & \Pi_9(SO/SO_5) & \rightarrow & \Pi_8(SO_5) \rightarrow \dots \\ & & \downarrow \mathbb{Z}_2 & & \downarrow \mathbb{Z}_2 & & \downarrow 0 \end{array}$$

p_* is an isomorphism and so L embeds in R^{25} if and only if $\hat{\alpha} = 0$.

If $s = 2, 4, 5, 6 \pmod 8$ then P almost embeds in R^{2s+3} .

If $s = 0, 1, 3 \pmod 8$ $s \neq 0, 1, 3, 8, 9, 11$ then P almost embeds in R^{3s-3} .

If $s = 7 \pmod 8$ $s \neq 7$ then P almost embeds in R^{3s} .

Proof

The method of proof of Theorem 1.6 shows that closed s -parallelisable $(s-1)$ -connected $(2s+1)$ -manifolds, up to the addition of homotopy $(2s+1)$ -spheres, occur as boundaries of manifolds in $\mathcal{H}(s+1)$ and the theorem now follows from Theorem 4.2. § .

A similar result follows for those manifolds occurring as boundaries of the handlebodies discussed in Theorem 4.3; however we restrict ourselves to the cases $s = 3$ and 7 and by also using a theorem of De Sapio (5) obtain the following result.

Theorem 4.5

Suppose P is a closed 2 -connected 7 -manifold, then P embeds in R^{11}

P embeds in R^9 if and only if $\hat{\beta} = 0$.

Suppose P is a closed 6- connected 15 manifold, then
 P embeds in R^{24} .

P almost embeds in R^{23} .

P almost embeds in R^{22} if $\hat{\beta}$ is divisible by 4.

P almost embeds in R^{21} if $\hat{\beta}$ is divisible by 8.

P almost embeds in R^{17} if and only if $\hat{\beta} = 0$.

Proof

Theorem 4.1 shows that $\hat{\beta} = 0$ is necessary for a 2- connected 7- manifold to be immersed in R^9 and so certainly this is necessary for an embedding. It equally follows since $SO(2) = S^1$ that $\Pi_7(SO) \approx \Pi_7(SO/SO_2)$ and $\Pi_{14}(SO/SO_2) \approx 0$ and so $\hat{\beta}$ is the obstruction to immersing a 6- connected 15- manifold in R^{17} .

That $\hat{\beta} = 0$, when P is stably parallelisable, implies that P almost embeds in R^9 for P a 7- manifold and in R^{17} for P a 15- manifold follows from Theorem A of De Sapio (5), which states that if M^m is a k - connected Π - manifold, $m \geq 5$ and $k < [m/2]$ (with an extra condition if $m = 6 \bmod 8$), then M almost embeds in $R^{2m-2k-1}$. Now every homotopy 7- sphere bounds

a parallelisable manifold of $\mathcal{H}(4)$ which by Theorem 4.3 embeds in R^9 and so in this case "almost embeddable" implies "embeddable".

A closed 2- connected 7- manifold P embeds in R^{11} by Theorem 4.3 since by Corollary 2.9 $P = \partial L$ where $L \in \mathcal{H}(4)$ with $w_4(L) = 0$.

If P is a closed 6- connected 15- manifold then by Corollary 2.9 $P \stackrel{H}{=} \Sigma = \partial L$, for some $\Sigma \in \mathcal{H}_{15}$, where $L \in \mathcal{H}(8)$ with $w_8(L) = 0$ and so by Theorem 4.3 P almost embeds in R^{23} . That P embeds in R^{24} now follows since by Haefliger (9) every homotopy 15- sphere embeds in R^{24} .

By Corollary 2.10 P splits as a connected sum of indecomposable manifolds and if 4 or 8 divides $\hat{\beta}$ then 4 or 8 must divide the tangential invariants of each of the indecomposable manifolds, and so it is sufficient to prove the remaining parts of the theorem for indecomposable manifolds.

From Theorems 3.1, 3.4 - 3.7 the indecomposable manifolds P with $\hat{\beta}$ divisible by 4 or 8 either have $\hat{\beta} = 0$ or the manifolds are classified, up to the addition of homotopy spheres, by the

invariants $(G, b, \hat{\beta})$. The case $\hat{\beta} = 0$ has been dealt with already and so suppose $\hat{\beta}$ is non-zero and divisible by 4 or 8 i.e. $\hat{\beta} = 4x$ or $8x$ for some $x \in G$. If $\lambda: H \times H \rightarrow Z$ is any even bilinear map that induces (G, b) then in the exact sequence

$$0 \rightarrow \hat{G} \rightarrow H \xrightarrow{\Pi} \hat{H} \xrightarrow{\partial} G \rightarrow 0$$

if $\partial(a) = x$ set $\hat{\alpha} = 4a$ or $8a$. Then the manifold $L \in \mathcal{H}(8)$

with invariants $(H, \lambda, \hat{\alpha})$ has boundary ∂L with invariants

$(G, b, \hat{\beta})$ and so $\partial L = P \amalg \Sigma$ for some $\Sigma \in \mathcal{H}_{15}$. Since $\hat{\alpha}$ is divisible

by 4 or 8 the corresponding manifold P almost embeds in R^{22}

or R^{21} by Theorem 4.3. § .

CHAPTER 5

INERTIA GROUPS OF THE

MANIFOLDS P

For an m -manifold M the inertia group $I(M)$ is the subgroup of $(H)_m$ consisting of those homotopy spheres Σ for which $M \# \Sigma$ is diffeomorphic to M by an orientation preserving diffeomorphism.

Results concerning inertia groups $I(M)$ for certain manifolds M are given by Browder (3), Brown and Steer (4), De Sapia (6), Eels and Kuiper (7), Kosinski (22), Levine (23), Novikov (30), Tamura (39) and Wall (44) and (VI).

This chapter is concerned with determining the inertia groups $I(P)$ of closed $(s-1)$ -connected $(2s+1)$ -manifolds P , where $s = 3$ or 7 , although we consider briefly more general values of the integer s . Corollary 5.2 shows that only homotopy spheres that bound parallelisable manifolds occur in $I(P)$ and Theorem 5.6 determines $I(P)$ for certain manifolds P . These results include the cases where $G = H_s(P)$ has finite order when $I(P) = 0$ and

where G is torsion - free when $I(P)$ depends on the divisibility of the tangential invariant $\hat{\beta}$ by powers of 2 and by the prime 7 for $s = 3$ and 127 for $s = 7$. As a contribution to the case where G contains elements of finite and infinite order we give examples of manifolds P_1, P_2 where $H_s(P_1)$ is torsion - free, $H_s(P_2)$ is finite and where $I(P_1 \# P_2) \cong I(P_1) \oplus I(P_2)$.

We consider first general values of the integer s .

Theorem 5.1

Suppose P is a closed s -parallelisable $(s-1)$ - connected $(2s+1)$ - manifold and $\Sigma \in I(P)$, then $\Sigma = \partial V$ for some manifold $V \in \mathcal{H}(s+1)$.

Proof

Recall the proof of Theorem 1.6 which applies to any closed s -parallelisable $(s-1)$ - connected $(2s+1)$ - manifold P .

By Wall (45) P is formed by surgeries of type $(s+1, s+1)$ from some homotopy sphere Σ and thus $P \# (-\Sigma)$ is formed by surgeries

of type $(s+1, s+1)$ from S^{2s+1} . Now attaching $(s+1)$ - handles to D^{2s+2} by the embeddings which define the surgeries gives $P \# (-\Sigma) = \partial L$ for some $L \in \mathcal{H}(s+1)$.

To consider the inertia group of P since $I(P) = I(P \# \Sigma)$ for any $\Sigma \in \mathcal{H}_{2s+1}$ we can assume without loss of generality that P is formed by surgeries of type $(s+1, s+1)$ from S^{2s+1} , when for any homotopy sphere Σ it follows that $P \# \Sigma$ is formed by surgeries of type $(s+1, s+1)$ from Σ .

Now suppose $\Sigma \in I(P)$ then combining our surgeries above we have Σ is formed by surgeries of type $(s+1, s+1)$ from $P \# \Sigma = P$ which in turn is formed by surgeries of type $(s+1, s+1)$ from S^{2s+1} . Thus Σ is formed by surgeries of type $(s+1, s+1)$ from S^{2s+1} and so as before by attaching $(s+1)$ - handles to D^{2s+2} to correspond to these surgeries we obtain $\Sigma = \partial V$ for some $V \in \mathcal{H}(s+1)$. § .

If a homotopy sphere $\Sigma = \partial W$ where W is parallelisable then if W is not s -connected it could be made so by surgery when we would have $W \in \mathcal{H}(s+1)$. So it is clear that elements of $\mathcal{H}_{2s+1}(\partial \Pi)$ occur as boundaries of manifolds of $\mathcal{H}(s+1)$, but in

general it is not known which other homotopy spheres, if any, also occur as boundaries. However for the particular case $s = 7$ Theorem 4 of (43) shows that the elements of $\bigoplus_{15} (\partial \Pi)$ are the only homotopy 15 - spheres that occur as boundaries of manifolds of $\mathcal{H}(8)$. Since every closed 6- connected 15- manifold is 7- parallelisable the following result follows from Theorem 5.1.

Corollary 5.2

If P is a closed 6- connected 15- manifold then $I(P) \subset \bigoplus_{15} (\partial \Pi)$.

We note that the proof of Theorem 5.1 justifies the claim made in the proof of Corollary 3.3. The situation was that $P_2 \# \Sigma = P_1 = \partial L_1$ and $P_2 = \partial L_2$ for $L_1, L_2 \in \mathcal{H}(s+1)$. Thus P_2 is formed by surgeries of type $(s+1, s+1)$ from S^{2s+1} and likewise P_2 is formed by surgeries of type $(s+1, s+1)$ from S^{2s+1} whence $P_2 \# \Sigma$ is formed by surgeries of type $(s+1, s+1)$ from Σ . It follows now by the argument used in the proof of the theorem that $\Sigma = \partial V$ for some $V \in \mathcal{H}(s+1)$.

Now consider closed $(s-1)$ - connected $(2s+1)$ - manifolds P for certain values of the integer s .

$$\underline{s = 5 \bmod 8}$$

For $s = 5 \bmod 8$ P is a Π -manifold and Theorem 6.1 of Kosinski (22) gives restrictions on the size of inertia groups of $(2s+1)$ -dimensional $(s-1)$ -connected Π -manifolds and for the case $s = 5 \bmod 8$ he shows that $I(P) \approx 0$.

By Theorem 7 of (IV) the manifolds P are classified, up to the addition of homotopy spheres, by the invariants $G = H_s(P)$, the linking invariant $b: G^* \times G^* \rightarrow S$ and a quadric map $q: G^* \rightarrow Q/2Z$. These are all homology invariants and therefore topological invariants and so since $I(P) = 0$ it follows that each manifold P admits exactly $|\bigoplus_{2s+1} H|$ non-diffeomorphic differential structures.

$$\underline{s = 6 \bmod 8}$$

Here the manifolds P are again Π -manifolds and Kosinski's result in this case gives $I(P) \subset \bigoplus_{2s+1} H(\partial\Pi)$. Since s is even $\bigoplus_{2s+1} H(\partial\Pi) \approx 0$ or Z_2 by (21) and so $I(P)$ is determined up to a factor of 2.

$$\underline{s = 3 \text{ or } 7}$$

For an m -manifold M the signature $\tau(M)$ is the signature of the intersection matrix of M for m divisible by 4 and is defined to be zero otherwise.

Lemma 5.3 below is stated and proved for general values of the integer s ; after this we restrict to $s = 3$ or 7 .

Lemma 5.3

Suppose $L_1, L_2 \in \mathcal{H}(s+1)$, $f: \partial L_1 \rightarrow \partial L_2$ an orientation preserving diffeomorphism and $M = L_1 \cup_f (-L_2)$; then

$$\tau(M) = \tau(L_1) - \tau(L_2)$$

Proof

For s even $\tau(L_1) = \tau(L_2) = \tau(M) = 0$ by definition and so the result is trivially true here.

For s odd suppose $P = \partial L_1 = \partial L_2$ then taking cohomology with coefficient group the rationals $\mathbb{Q}$ the signatures $\tau(L_1)$, $\tau(L_2)$, $\tau(M)$

are the signatures of the symmetric bilinear maps given by the cup product.

$$H^{s+1}(L_1, P) \times H^{s+1}(L_1, P) \rightarrow H^{2s+2}(L_1, P) \simeq Q$$

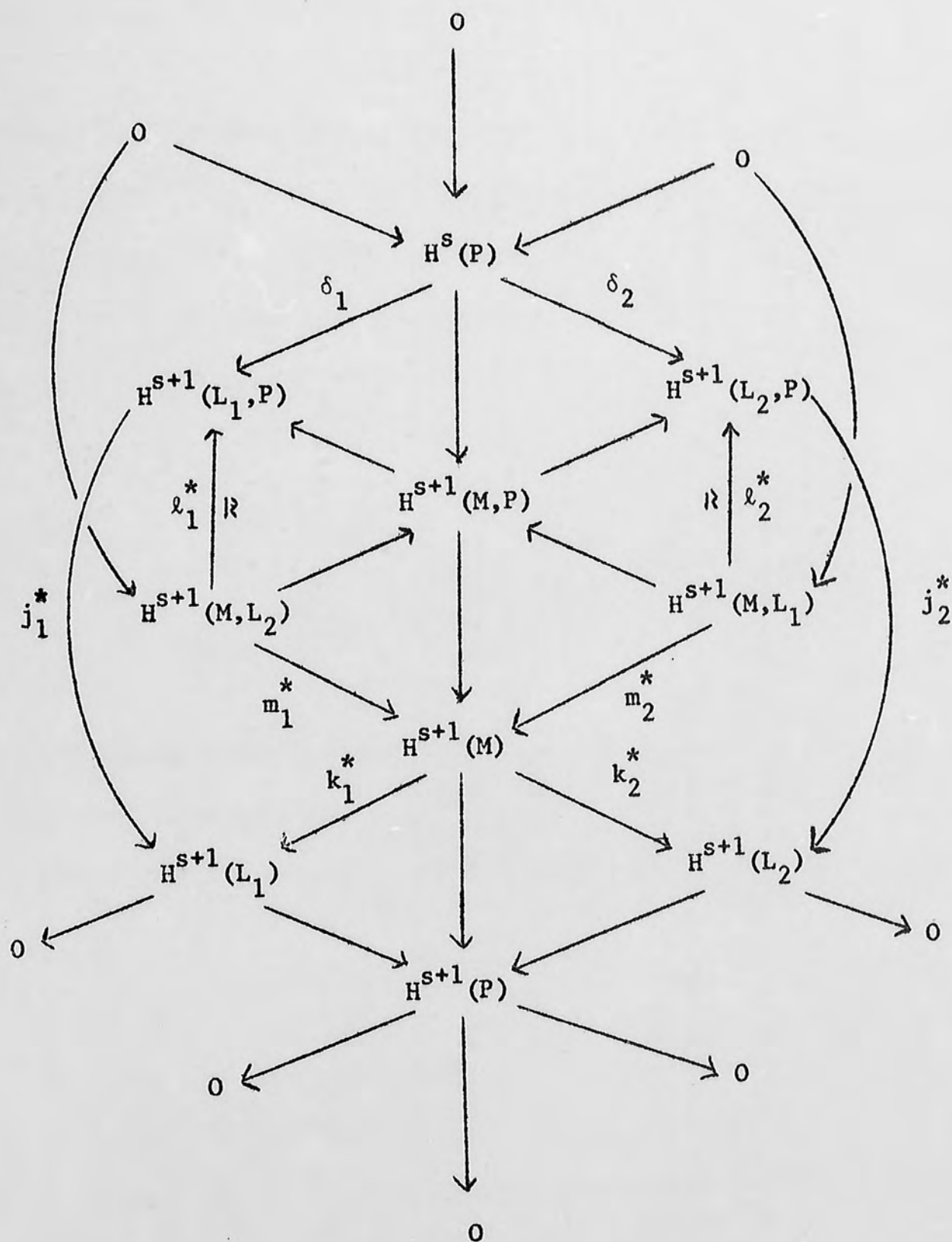
$$H^{s+1}(L_2, P) \times H^{s+1}(L_2, P) \rightarrow H^{2s+2}(L_2, P) \simeq Q$$

$$H^{s+1}(M) \times H^{s+1}(M) \rightarrow H^{2s+2}(M) \simeq Q$$

where the isomorphisms between $H^{2s+2}(L_1, P)$, $H^{2s+2}(L_2, P)$, $H^{2s+2}(M)$ and Q are given by cap product with the fundamental cycles of L_1 , L_2 and M .

Suppose $H^{s+1}(L_1, P)$ has rank r_1 and $H^{s+1}(L_2, P)$ has rank r_2 then from the Mayer-Vietoris sequence for M the non-zero cohomology groups of M are in dimensions 0 , $s+1$, $2s+2$ with $H^{s+1}(M)$ of rank $r_1 + r_2$.

Cohomology sequences associated with M , L_1 , L_2 , P give rise to the following commutative diagram, again where the coefficient group is Q .



ℓ_1^* and ℓ_2^* are isomorphisms by excision.

Suppose $H^s(P)$ has rank q then $\delta_1(H^s(P))$ and $\delta_2(H^s(P))$ have rank q in $H^{s+1}(L_1, P)$ and $H^{s+1}(L_2, P)$ respectively. Since $x \cup y = x \cup j_1^*(y)$ or $x \cup j_2^*(y)$ for elements $x, y \in H^{s+1}(L_1, P)$ or $H^{s+1}(L_2, P)$ it follows that the intersection matrices for L_1 and L_2 respectively are

$$\begin{bmatrix} e_1 & \dots & e_q & e_{q+1} & \dots & e_{r_1} \\ \hline & 0 & & & 0 & \\ \hline & 0 & & & B & \end{bmatrix} \text{ and } \begin{bmatrix} f_1 & \dots & f_q & f_{q+1} & \dots & f_{r_2} \\ \hline & 0 & & & 0 & \\ \hline & 0 & & & C & \end{bmatrix}$$

for some matrices B and C where bases $e_1, \dots, e_q$ and $f_1, \dots, f_q$ are taken for $\delta_1(H^s(P))$ and $\delta_2(H^s(P))$ and extended to bases $e_1, \dots, e_{r_1}$ and $f_1, \dots, f_{r_2}$ of $H^{s+1}(L_1, P)$ and $H^{s+1}(L_2, P)$ respectively.

Consider the elements $E_i = m_1^* \ell_1^{*-1}(e_i)$ for $i = 1, \dots, r_1$ and $F_j = m_2^* \ell_2^{*-1}(f_{q+j})$ for $j = 1, \dots, s_2 = (r_2 - q)$. $m_1^* \ell_1^{*-1}$ and $m_2^* \ell_2^{*-1}$ are monomorphisms and the elements E_i and F_j are orthogonal since $k_2^*(F_j) = j_2^*(f_{q+j}) \neq 0$ while $k_2^*(E_i) = 0$ by exactness and so a basis for $H^{s+1}(M)$ is given by $E_1, \dots, E_{r_1},$

$F_1, \dots, F_{s_2}$ together with another q elements. Now

$$E_i \cup E_j = m_1^* \ell_1^{*-1}(e_i) \cup m_1^* \ell_1^{*-1}(e_j) = e_i \cup e_j$$

$$F_i \cup F_j = m_2^* \ell_2^{*-1}(f_{q+i}) \cup m_2^* \ell_2^{*-1}(f_{q+j}) = -f_{q+i} \cup f_{q+j}$$

(the $-$ sign occurs since M is formed from $-L_2$ i.e. L_2 with the opposite orientation)

$$E_i \cup F_j = m_1^* \ell_1^{*-1}(e_i) \cup m_2^* \ell_2^{*-1}(f_{q+j}) = 0$$

since the product factors through $H^{2s+2}(M, L_1 \cup L_2) \approx 0$. The intersection matrix for M is therefore of the form:

$$A = \left[\begin{array}{ccc|ccc|ccc} E_1 & \dots & E_q & E_{q+1} & \dots & E_{r_1} & F_1 & \dots & F_{s_2} & \dots \\ \hline & 0 & & & 0 & & & 0 & & D \\ \hline & 0 & & & B & & & 0 & & U \\ \hline & 0 & & & 0 & & & -C & & V \\ \hline D^t & & & U^t & & & V^t & & & W \end{array} \right]$$

for some matrices D, U, V, W .

Since M is a closed manifold A is non-singular.

For $0 \leq \epsilon \leq 1$ let A_ϵ denote the matrix:

$$A_{\varepsilon} = \begin{bmatrix} 0 & | & 0 & | & 0 & | & D \\ \hline 0 & | & B & | & 0 & | & \varepsilon U \\ \hline 0 & | & 0 & | & -C & | & \varepsilon V \\ \hline D^t & | & \varepsilon U^t & | & \varepsilon V^t & | & \varepsilon W \end{bmatrix}$$

A_{ε} has the same non-zero determinant $-\det B \cdot \det C \cdot (\det D)^2$ as $A = A_1$ for each ε and so the signature $\tau(A) = \tau(A_1) = \tau(A_0)$.

$$\begin{aligned} \text{Thus } \tau(A) &= \tau \begin{bmatrix} 0 & | & 0 & | & 0 & | & D \\ \hline 0 & | & B & | & 0 & | & 0 \\ \hline 0 & | & 0 & | & -C & | & 0 \\ \hline D^t & | & 0 & | & 0 & | & 0 \end{bmatrix} \\ &= \tau \begin{bmatrix} B & | & 0 \\ \hline 0 & | & -C \end{bmatrix} + \tau \begin{bmatrix} 0 & | & D \\ \hline D^t & | & 0 \end{bmatrix} = \tau(B) - \tau(C). \end{aligned}$$

But $\tau(A) = \tau(M)$, $\tau(B) = \tau(L_1)$ and $\tau(C) = \tau(L_2)$ and so

$$\tau(M) = \tau(L_1) - \tau(L_2). \quad \S .$$

Now take $s = 3$ or 7 and for $L \in \mathcal{H}(s+1)$ with $\partial L = P$ suppose that there exists an orientation preserving diffeomorphism

$f: P \rightarrow P \amalg \Sigma$ for $\Sigma \in \mathcal{H}_{2s+1}(\partial\Pi)$. $\Sigma = \partial W$ for some manifold $W \in \mathcal{H}(s+1)$ which is parallelisable and define $M = (L+W) \cup_f (-L)$ by identifying $P \amalg \Sigma$ and P by f .

Theorem 5.4

With M as above and $\gamma \in H^{s+1}(M)$ the obstruction to a cross-section of the tangent bundle over the $(s+1)$ - skeleton of M , then

$$\frac{\tau(W)}{8} = \frac{\tau(M)}{8} = \frac{\gamma^2}{8} \mod \begin{cases} 28 & \text{for } s = 3 \\ 2^6 \cdot 127 & \text{for } s = 7. \end{cases}$$

Proof

$\tau(L+W) = \tau(L) + \tau(W)$ and so it follows immediately from Lemma 5.3 that $\tau(W) = \tau(M)$. Hirzebruch's Index Theorem and the integrality of the $\hat{A}$ genus are now used to relate $\tau(M)$ and γ^2 .

If $s = 3$ when M is an 8- manifold the $\hat{A}$ genus is given by

$$\hat{A}(M) = \hat{A}_2(p_1, p_2)$$

where, as in Chapter 1 of Hirzebruch (13), $\{A_j\}$ is the m -sequence associated with the power series $\frac{1}{2} \sqrt{z} / \sinh \frac{1}{2} \sqrt{z}$ and p_1, p_2 are the Pontrjagin classes of M .

$$\text{i.e. } \hat{A} = \frac{1}{2 \cdot 7.45} (-4 p_2 + 7 p_1^2)$$

Hirzebruch's Index Theorem (see (13)) gives

$$\tau(M) = L_2(p_1, p_2)$$

where $\{L_j\}$ is the m -sequence associated with the power series $\sqrt{z} / \tanh \sqrt{z}$.

$$\text{i.e. } \tau(M) = \frac{1}{4 \cdot 5} (7 p_2 - p_1^2)$$

Eliminating p_2 from these two equations gives

$$\tau(M) = \frac{p_1^2}{4} - 8.28 \hat{A}$$

Now M is a spin manifold and so $\hat{A}$ is an integer and we therefore have

$$\frac{\tau(M)}{8} = \frac{p_1^2}{32} \pmod{28}$$

By Lemma 1.1 of Kervaire (19) $p_1 = 2\gamma$ and hence

$$\frac{\tau(M)}{8} = \frac{\gamma^2}{8} \pmod{28}.$$

If $s = 7$ when M is a 16-manifold the non-zero Pontrjagin classes are p_2 and p_4 and the $\hat{A}$ genus of M is given by

$$\hat{A}(M) = \hat{A}_4(0, p_2, 0, p_4)$$

$$\text{i.e. } \hat{A} = \frac{1}{2^{15} \cdot 3^4 \cdot 5^2 \cdot 7} (-192 p_4 + 208 p_2^2)$$

Hirzebruch's Index Theorem gives

$$\tau(M) = L_4(0, p_2, 0, p_4)$$

$$\text{i.e. } \tau(M) = \frac{1}{3^4 \cdot 5^2 \cdot 7} (381 p_4 - 19 p_2^2)$$

Eliminating p_4 from the two equations gives

$$\tau(M) = \frac{p_2^2}{36} - 2^9 \cdot 127 \hat{A}$$

M is a spin-manifold and $\hat{A}$ therefore an integer and hence

$$\frac{\tau(M)}{8} = \frac{p_2^2}{8 \cdot 36} \mod 2^6 \cdot 127.$$

By Kervaire (19) $p_2 \neq 6\gamma$ giving

$$\frac{\tau(M)}{8} = \frac{\gamma^2}{8} \mod 2^6 \cdot 127. \quad \S.$$

By Kervaire and Milnor (21)

$$\frac{\tau(W)}{8} \mod \begin{cases} 28 \text{ for } s = 3 \\ 2^6 \cdot 127 \text{ for } s = 7 \end{cases} \text{ determines the homotopy sphere } \Sigma = \partial W \text{ as}$$

$$\text{an element of } \bigoplus_{2s+1} (\partial \Pi) \simeq \begin{cases} \mathbb{Z}_{28} \text{ for } s = 3 \\ \mathbb{Z}_{2^6 \cdot 127} \text{ for } s = 7. \end{cases}$$

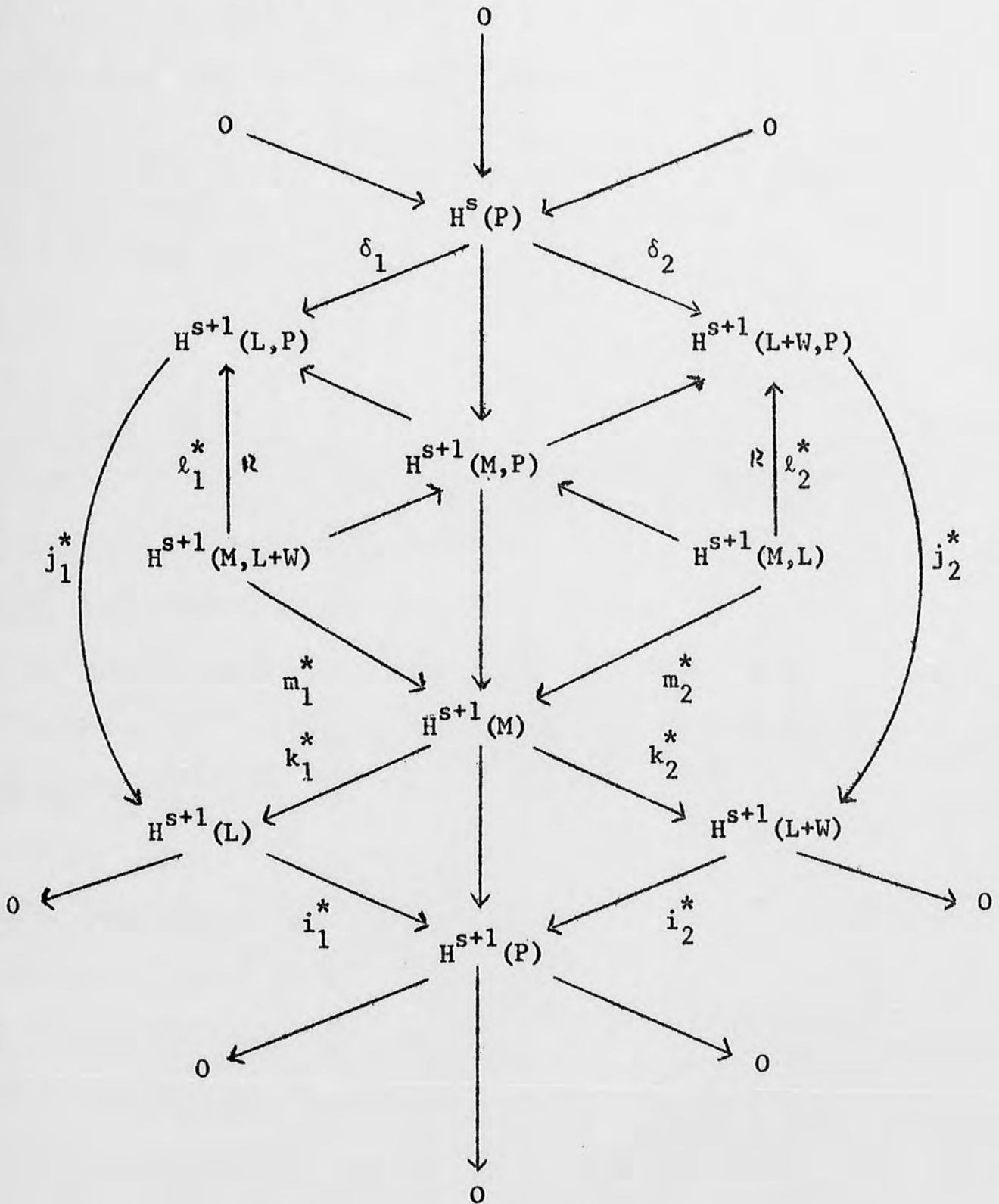
To obtain necessary conditions for $\Sigma \in I(P)$ we relate γ^2 to the value of the tangential invariant $\hat{\beta} \in H^{s+1}(P)$ of P .

Lemma 5.5

- (i) If $\hat{\beta}$ has finite order then $\gamma^2 = 0$.
- (ii) If $\hat{\beta}$ is divisible by the integer r then r divides γ^2 .
- (iii) If $H^{s+1}(P) \simeq G$ is torsion - free and $\hat{\beta}$ is divisible by r then $2r$ divides γ^2 .

Proof

Consider the following commutative diagram of cohomology groups with coefficient group the integers $\mathbb{Z}$.



The diagram gives rise to the Mayer - Vietoris sequence

$$0 \rightarrow H^s(P) \xrightarrow{\Delta} H^{s+1}(M) \xrightarrow{\Phi} H^{s+1}(L) \oplus H^{s+1}(L+W) \xrightarrow{\Psi} H^{s+1}(P) \rightarrow 0$$

$$\text{where } \Delta = \begin{pmatrix} m_1^* & \ell_1^* \\ 0 & 0 \end{pmatrix}^{-1} \delta_1 = - \begin{pmatrix} m_2^* & \ell_2^* \\ 0 & 0 \end{pmatrix}^{-1} \delta_2, \quad \Phi(x) = k_1^*(x) \oplus k_2^*(x)$$

$$\text{for } x \in H^{s+1}(M), \quad \Psi(y \oplus z) = i_1^*(y) - i_2^*(z) \text{ for}$$

$$y \in H^{s+1}(L), \quad z \in H^{s+1}(L+W).$$

$$H^{s+1}(L+W) \simeq H^{s+1}(L) \oplus H^{s+1}(W) \text{ and}$$

$H^{s+1}(L+W, P) \simeq H^{s+1}(L, P) \oplus H^{s+1}(W, \Sigma)$ and in this way j_2^* is identified

with $j_1^* \oplus t^*$ where the isomorphism $t^*: H^{s+1}(W, \Sigma) \rightarrow H^{s+1}(W)$ is

induced from the inclusion $t: W \subset (W, \Sigma)$. $\partial(L+W) = P \xrightarrow{\parallel} \Sigma$ and

$H^{s+1}(P \xrightarrow{\parallel} \Sigma) \simeq H^{s+1}(P) \oplus H^{s+1}(\Sigma) \simeq H^{s+1}(P)$ and so the induced

homomorphism $H^{s+1}(L+W) \rightarrow H^{s+1}(P \xrightarrow{\parallel} \Sigma)$ of the inclusion $P \xrightarrow{\parallel} \Sigma \subset L + W$

can be identified with the homomorphism given by i_1^* on $H^{s+1}(L)$

with $H^{s+1}(W)$ being mapped trivially. In forming $M \xrightarrow{\parallel} P \xrightarrow{\parallel} \Sigma$

is identified with P by the diffeomorphism $f: P \rightarrow P \xrightarrow{\parallel} \Sigma$ and so

i_2^* is identified with the homomorphism given by $f^* i_1^*$ on $H^{s+1}(L)$

with $H^{s+1}(W)$ being mapped trivially. For convenience in what

follows we write $n_1 = \begin{pmatrix} m_1^* & \ell_1^* \\ 0 & 0 \end{pmatrix}^{-1}$ and $n_2 = \begin{pmatrix} m_2^* & \ell_2^* \\ 0 & 0 \end{pmatrix}^{-1}$.

The tangential invariant of L is an element $\hat{\alpha} \in H^{s+1}(L)$ and since W is parallelisable the tangential invariant of $L + W$ is $\hat{\alpha} \in H^{s+1}(L+W) \simeq H^{s+1}(L) \oplus H^{s+1}(W)$. $\gamma, \hat{\alpha}$ and $\hat{\beta}$ are all obstructions to a cross-section of the tangent bundle over the $(s+1)$ - skeletons of the corresponding manifolds and it therefore follows that $k_1^*(\gamma) = \hat{\alpha}$, $k_2^*(\gamma) = \hat{\alpha}$, $i_1^*(\hat{\alpha}) = i_2^*(\hat{\alpha}) = \hat{\beta}$.

(i) If $\hat{\beta}$ is of finite order say N then $i_1^*(N\hat{\alpha}) = N\hat{\beta} = 0$ and $i_2^*(N\hat{\alpha}) = 0$. By exactness $N\hat{\alpha} = j_1^*(a)$ for some element $a \in H^{s+1}(L, P)$ and since $j_2^* = j_1^* \oplus t^*$ we also have $j_2^*(a) = N\hat{\alpha}$ for $\alpha \in H^{s+1}(L, P) \oplus H^{s+1}(W, \Sigma) \simeq H^{s+1}(L + W, \Sigma)$

$$\begin{aligned} \text{Now } \phi(N\gamma - n_1(a) - n_2(a)) &= k_1^* \oplus k_2^*(N\gamma - n_1(a) - n_2(a)) \\ &= (N\hat{\alpha} - k_1^* n_1(a)) \oplus (N\hat{\alpha} - k_2^* n_2(a)) \\ &= (N\hat{\alpha} - j_1^*(a)) \oplus (N\hat{\alpha} - j_2^*(a)) \\ &= 0 \end{aligned}$$

So by exactness of the Mayer - Vietoris sequence

$$N\gamma - n_1(a) - n_2(a) = \Delta(d)$$

for some element $d \in H^s(P)$.

$$\text{i.e. } N\gamma = n_1(a) + n_2(a) + \Delta(d)$$

In squaring this equation by the cup product $n_1(a)^2 = -a^2$, since in forming $M \cdot -L$, i.e. L with the opposite orientation, is used, and $n_2(a)^2 = a^2$. The product of an element in the image of n_1 and an element in the image of n_2 is zero since it factors through $H^{2s+2}(M, L \cup L+W) \approx 0$. Since $\Delta(d)$ is in the image of both n_1 and n_2 it follows that

$$N^2 \quad \gamma^2 = -a^2 + a^2 = 0$$

$$\text{i.e.} \quad \gamma^2 = 0.$$

(ii) If $\hat{\beta}$ is divisible by r then $\hat{\beta} = r x$ for some $x \in H^{s+1}(P)$. Then $\hat{\alpha} = r X + Y$ for $X, Y \in H^{s+1}(L)$ where $i_1^*(X) = x$ and $i_2^*(Y) = 0$. By exactness therefore $Y = j_1^*(a) = j_2^*(a)$ for some $a \in H^{s+1}(L, P)$.

Suppose $f^*: H^{s+1}(P) \rightarrow H^{s+1}(P)$ has $f^*(x) = x + z$ then as f is a diffeomorphism $f^*(\hat{\beta}) = \hat{\beta}$ and so $r z = 0$. If $i_1^*(Z) = z$ for $Z \in H^{s+1}(L)$ then $i_1^*(r Z) = 0$ and by exactness $r Z = j_1^*(b)$ for some $b \in H^{s+1}(L, P)$

$$\begin{aligned} \text{Now} \quad \psi(X + Z \oplus X) &= i_1^*(X + Z) - i_2^*(X) \\ &= i_1^*(X + Z) - f^* i_1^*(X) \\ &= x + z - (x + z) \\ &= 0 \end{aligned}$$

By exactness of the Mayer - Vietoris sequence $X + Z \oplus X = \Phi(c)$
for some $c \in H^{s+1}(M)$.

$$\begin{aligned}\text{Now } \Phi(\gamma - rc) &= (\hat{\alpha} - r(X + Z)) \oplus (\hat{\alpha} - rX) \\ &= (Y - rZ) \oplus Y \\ &= j_1^*(a - b) \oplus j_2^*(a) \\ &= k_1^* n_1(a - b) \oplus k_2^* n_2(a) \\ &= \Phi(n_1(a - b) + n_2(a))\end{aligned}$$

Therefore by exactness

$$\gamma - rc = n_1(a - b) + n_2(a) + \Delta(d) \text{ for some } d \in H^s(P).$$

Squaring this equation since $n_1(a - b)^2 = -(a - b)^2$ and $n_2(a)^2 = a^2$ and all other products of right hand side are zero we obtain

$$\begin{aligned}\gamma^2 - 2rc.\gamma + r^2c^2 &= 2a.b - b^2 \\ \text{i.e. } \gamma^2 &= 2rc.\gamma - r^2c^2 + b.(2a - b) \dots (*)\end{aligned}$$

Now $b.(2a - b) = j_1^*(b).(2a - b) = rZ.(2a - b)$ and obviously r divides $2rc.\gamma$ and r^2c^2 and so r divides γ^2 .

(iii) If G is torsion - free and r divides $\hat{\beta}$ then we can proceed exactly as in (ii) except that since there is no torsion $z = 0$ and so we can take $Z = 0$ and $b = 0$ when equation (*) above gives

$$\gamma^2 = 2rc.\gamma - r^2 c^2.$$

Now by Lemma 1.2 $\hat{\beta}$ is even and so r can be taken to be even when $2r$ divides r^2 and it follows that $2r$ divides γ^2 . §.

Theorem 5.6

Suppose P is a closed $(s-1)$ - connected $(2s+1)$ - manifold, where $s = 3$ or 7 , then

(i) If $\hat{\beta} \in H^{s+1}(P)$ has finite order then $I(P) \simeq 0$.

(ii) If $\hat{\beta}$ is divisible by the integer r and $\Sigma \in I(P)$ then Σ is divisible by $\frac{r}{8}$.

(iii) If $H^{s+1}(P) \simeq G$ is torsion - free and r is the largest integer dividing $\hat{\beta}$ then $I(P)$ consists exactly of those elements of $\bigoplus_{2s+1} (H)_{2s+1}(\partial\Pi)$ divisible by $\frac{r}{4}$.

$$(\mathbb{H})_7 = (\mathbb{H})_7 (\partial\Pi) \simeq \mathbb{Z}_{28} \text{ and } (\mathbb{H})_{15} (\partial\Pi) \simeq \mathbb{Z}_{2^6 \cdot 127} \text{ and}$$

so all elements of $(\mathbb{H})_7 (\partial\Pi)$ are divisible by all primes but for 2 and 7 and all elements of $(\mathbb{H})_{15} (\partial\Pi)$ are divisible by all primes but for 2 and 127. Thus in practice to apply the results of the theorem above for $I(P)$ it is only necessary to know whether 7 divides $\hat{\beta}$ for $s = 3$ or 127 divides $\hat{\beta}$ for $s = 7$ and to know which powers of 2 divide $\hat{\beta}$ for both $s = 3$ and 7).

Proof

By Theorem 1.6 $P \neq \Sigma = \partial L$ for some $L \in \mathcal{H}(s+1)$ and some $\Sigma \in (\mathbb{H})_{2s+1}$ and since $I(P \stackrel{H}{=} \Sigma) = I(P)$ the boundary of L can be used to evaluate the inertia group of P . By Corollary 5.2 $I(P) \subset (\mathbb{H})_{2s+1} (\partial\Pi)$ and so necessary conditions for a homotopy sphere Σ to belong to $I(P)$ are given by Theorem 5.4 and Lemma 5.5.

If $\Sigma = \partial W$ where W is parallelisable by Kervaire and

$$\text{Milnor (21) } \frac{\tau(W)}{8} \pmod{\begin{cases} 28 \text{ for } s = 3 \\ 2^6 \cdot 127 \text{ for } s = 7 \end{cases}}$$

determines Σ as an element of $(H)_{2s+1}(\partial\Pi) \simeq \begin{cases} Z_{28} & \text{for } s = 3 \\ Z_{2^6 \cdot 127} & \text{for } s = 7. \end{cases}$

By Theorem 5.4 therefore $\frac{\gamma^2}{8} \bmod \begin{cases} 28 & \text{for } s = 3 \\ 2^6 \cdot 127 & \text{for } s = 7 \end{cases}$

determines Σ as an element of $(H)_{2s+1}(\partial\Pi)$.

Now by Lemma 5.5:

- (i) If $\hat{\beta}$ has finite order $\gamma^2 = 0$ and so $\Sigma = 0$ and $I(P) \simeq 0$.
- (ii) If $\hat{\beta}$ is divisible by r then $\frac{\gamma^2}{8}$ is divisible by $\frac{r}{8}$ and so Σ is divisible by $\frac{r}{8}$.
- (iii) If G is torsion - free and $\hat{\beta}$ is divisible by r then $\frac{\gamma^2}{8}$ is divisible by $\frac{r}{4}$ and so Σ is divisible by $\frac{r}{4}$.

Thus parts (i) and (ii) of the theorem are proved and it remains to show that in case (iii) if r is the largest integer dividing $\hat{\beta}$ then the fact that Σ is divisible by $\frac{r}{4}$ is a sufficient condition for Σ to belong to $I(P)$.

We now recall Theorem 4 of Wall (43). If $V \in \mathcal{H}(s+1)$ has

$\partial V = \Sigma$ a homotopy sphere then $\Sigma \in \mathcal{H}_{2s+1}(\partial\Pi)$ and the intersection

map $\lambda: H \times H \rightarrow Z$ of V is unimodular. If we write $\lambda(x,y) = x \cdot y$

then the tangential invariant $\hat{\alpha} \in \hat{H}$ determines a unique element

$\chi \in H$ by the rule $\chi \cdot x = \hat{\alpha}(x)$ for all $x \in H$. If τ is the signature

of V , i.e. the signature of λ , then Σ as an element of $\mathcal{H}_{2s+1}(\partial\Pi)$ is

determined by

$$\frac{\chi^2 - \tau}{8} \bmod \begin{cases} 28 & \text{for } s = 3 \\ 2^6 \cdot 127 & \text{for } s = 7. \end{cases}$$

(Under the notation of Chapter 2 since λ is unimodular it

induces an isomorphism $\Pi: H \rightarrow \hat{H}$ and χ is therefore the unique

element such that $\Pi(\chi) = \hat{\alpha}$. λ induces a unimodular map

$\hat{\lambda}: \hat{H} \times \hat{H} \rightarrow Q$ and so by the definition of $\hat{\lambda}$ $\chi^2 = \hat{\lambda}(\chi, \chi) = \hat{\lambda}(\hat{\alpha}, \hat{\alpha})$.

Before using this result we observe that it is enough

to prove the sufficiency of part (iii) of the theorem assuming

$G \simeq Z$. If G is free of rank k and if r is the largest integer

dividing $\hat{\beta}$ then $\hat{\beta} = r e_1$ for some indivisible element $e_1 \in G$.

Now for choosing a basis $e_1, e_2, \dots, e_k$ for G it follows by the

general decomposition results of chapter 2 that $P = P_1 \overset{11}{\parallel} P_2 \overset{11}{\parallel} \dots \overset{11}{\parallel} P_k$

(possibly ~~≠~~ a homotopy sphere but this can be ignored in considering the inertia group of P) where $H_s(P_i) \simeq \mathbb{Z}$ with generator e_i . The tangential invariant of P_1 is thus $\hat{\beta} = r e_1$. Now $I(P) \supset I(P_1) \oplus I(P_2) \oplus \dots \oplus I(P_k) \supset I(P_1)$ and so if $\Sigma \in I(P_1)$ then certainly $\Sigma \in I(P)$.

By Theorem 3.1 if P has $G \simeq \mathbb{Z}$ then, up to the addition of a homotopy sphere, $P = \partial L$ with $L \in \mathcal{H}(s+1)$ having $H_{s+1}(L) = H \simeq \mathbb{Z}$ with generator e , intersection matrix of L $\begin{bmatrix} e \\ 0 \end{bmatrix}$ and the tangential invariant $\hat{\alpha} \in \hat{H}$ of L maps isomorphically onto the tangential invariant $\hat{\beta}$ of P . If we can show that for r the largest integer dividing $\hat{\beta}$, and therefore dividing $\hat{\alpha}$ (we take $\hat{\alpha}(e) = r$ i.e. $\hat{\alpha} = r \hat{e}$), and Σ a homotopy sphere divisible by $\frac{r}{4}$ then $L + V = L + V'$ for $V, V' \in \mathcal{H}(s+1)$ with $\partial V = S^{2s+1}$, $\partial V' = \Sigma$ then

$\partial(L+V) = P = P \neq \Sigma = \partial(L+V')$ and so $\Sigma \in I(P)$, the result we want.

Let V have intersection matrix $\begin{bmatrix} f \\ 1 \end{bmatrix}$ where f is a generator of $H_{s+1}(V) \simeq \mathbb{Z}$ and $\hat{\alpha}(f) = \pm 1$ then $\chi = \pm f$ and the signature $\tau = 1$ so $\frac{\chi^2 - \tau}{8} = 0$ and therefore $\partial V = S^{2s+1}$.

$L + V$ has intersection matrix $\begin{bmatrix} e & f \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

with respect to the basis e, f of $H_{s+1}(L+V) \approx H_{s+1}(L) \oplus H_{s+1}(V) \approx \mathbb{Z} \oplus \mathbb{Z}$.

Consider the change of basis

$$\begin{array}{c} F \\ \text{"} \\ \begin{array}{cc} e & e + f \\ \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right] \end{array} \end{array}$$

By Propositions 1.4 and 1.5 $L + V = L + V'$ where V' has intersection matrix $\begin{bmatrix} F \\ 1 \end{bmatrix}$. Now $\hat{\alpha}(F) = \hat{\alpha}(e+f) = \hat{\alpha}(e) + \hat{\alpha}(f) = r + 1$ and so $\chi = (r+1)F$ and

$$\frac{\chi^2 - \tau}{8} = \frac{(r+1)^2 - 1}{8} = \frac{r(r+2)}{8}$$

Thus $\partial V' = \Sigma$ as an element of $(H)_{2s+1}(\partial \Pi)$ is given by

$$\frac{r(r+2)}{8} \bmod \begin{cases} 28 & \text{for } s = 3 \\ 2^6 \cdot 127 & \text{for } s = 7. \end{cases}$$

Now each element of $(H)_{2s+1}(\partial \Pi)$ is divisible by powers of all primes except for the primes 2 and 7 for $s = 3$ and 2 and 127 for $s = 7$.

Thus to give the result we only have to check that if r is not divisible

by 7 or 127 then either $\frac{r(r+2)}{8}$ or $\frac{r(r-2)}{8}$ is not divisible by

7 or 127, and if r is divisible by 2^k but not divisible by

2^{k+1} then either $\frac{r(r+2)}{8}$ or $\frac{r(r-2)}{8}$ is divisible by 2^{k-2} and

not by 2^{k-1} for $k \geq 2$ and either

$\frac{r(r+2)}{8}$ or $\frac{r(r-2)}{8}$ is odd for $k=1$. It is clear that once we have an element in $(H)_{2s+1}(\partial\Pi)$ not divisible by 7 or 127 and divisible by 2^{k-2} and not 2^{k-1} we can obtain all elements of this form by taking multiples of this element.

If 7 or 127 does not divide r then 7 or 127 does not divide $(r+2)$ or $(r-2)$ and so 7 or 127 does not divide $\frac{r(r+2)}{8}$ or $\frac{r(r-2)}{8}$.

For $k \geq 2$ if 2^k divides r and 2^{k+1} does not divide r then $(r+2)$ is divisible by 2 but not by 4 and hence $\frac{r(r+2)}{8}$ is divisible by 2^{k-2} and not by 2^{k-1} .

If 2 divides r but 4 does not divide r then either $(r+2)$ or $(r-2)$ is divisible by 4 and not by 8 and hence $\frac{r(r+2)}{8}$ or $\frac{r(r-2)}{8}$ is odd. §.

By a suitable choice of the invariant $\hat{\beta}$ for a manifold P with $H_s(P) \simeq \mathbb{Z}$ it follows from part (iii) of Theorem 5.6 that any subgroup of $(H)_{2s+1}(\partial\Pi)$ occurs as the inertia group of some manifold P with $H_s(P) \simeq \mathbb{Z}$.

The manifolds P with G torsion - free by Theorem 3.8 are classified, up to the addition of homotopy spheres, by the invariants $(G, b, \hat{\beta})$. G and b are homology invariants and by Kervaire (19) $2\hat{\beta} = p_1(P)$ for $s = 3$ and $6\hat{\beta} = p_2(P)$ for $s = 7$. By Novikov (31) rational Pontrjagin classes are topological invariants and so since G is torsion - free $p_1(P)$ for $s = 3$ and $p_2(P)$ for $s = 7$ are topological invariants and so $\hat{\beta}$ is a topological invariant. Thus for G torsion - free the manifolds P are classified, up to the addition of homotopy spheres, by the topological invariants $(G, b, \hat{\beta})$ and therefore a manifold P here admits exactly $|\hat{H}_{2s+1} / I(P)|$ differential structures. Now $\hat{H}_7 = \hat{H}_7(\partial\Pi)$ and $\hat{H}_{15} = \hat{H}_{15}(\partial\Pi) \oplus \mathbb{Z}_2$ and so from above:

If N is any divisor of 28 for $s = 3$ or of $2^7 \cdot 127$ for $s = 7$ such that $N \geq 2$ for $s = 7$ then there exists a manifold P with $H_s(P) \simeq \mathbb{Z}$ which admits exactly N differential structures.

The Inertia Groups and Connected Sums

For any two manifolds M_1 and M_2 it is clear that $I(M_1 \# M_2) \supset I(M_1) \oplus I(M_2)$.

For closed $(s-1)$ - connected $(2s+1)$ - manifolds P_1 and P_2 , where $s = 3$ or 7 , from part (iii) of Theorem 5.6 $I(P_1 \amalg P_2) = I(P_1) \oplus I(P_2)$ when $H_s(P_1)$ and $H_s(P_2)$ are both torsion - free. This equality also trivially holds when $H_s(P_1)$ and $H_s(P_2)$ are both finite groups since by part (i) of Theorem 5.6 $I(P_1) = I(P_2) = I(P_1 \amalg P_2) \approx 0$. However the examples below show that this equality is no longer true in general when the free and torsion cases are combined.

Let manifolds $L_1, L_2, L_3 \in \mathcal{J}^{\ell(s+1)}$ whose boundaries are P_1, P_2, P_3 be given as follows:

L_1 has $H_1 = H_{s+1}(L_1) \approx \mathbb{Z}$ with generator e_1 , intersection matrix $\begin{bmatrix} e_1 \\ 0 \end{bmatrix}$ and tangential invariant given by $\hat{\alpha}(e_1) = 2^k p$ (i.e. $\hat{\alpha} = 2^k p \hat{e}_1 \in \hat{H}_1$) where $p = 7$ for $s = 3$, $p = 127$ for $s = 7$ and $k \geq 5$ for $s = 3$, $k \geq 9$ for $s = 7$.

$P_1 = \partial L_1$ then has $H_s(P_1) \approx \mathbb{Z}$ with a generator e'_1 where the tangential invariant $\hat{\beta}_1 = 2^k p e'_1$.

L_2 has $H_2 = H_{s+1}(L_2) \approx \mathbb{Z}$ with generator e_2 , intersection matrix $\begin{bmatrix} e_2 \\ p \end{bmatrix}$ and tangential invariant given by $\hat{\alpha}(e_2) = 1$ (i.e. $\hat{\alpha} = \hat{e}_2 \in \hat{H}_2$).

$P_2 = \partial L_2$ then has $H_s(P_2) \simeq Z_p$ with a generator e'_2 such that the linking invariant is given by $b(e'_2, e'_2) = p^{-1}$ and tangential invariant $\hat{\beta}_2 = e'_2$.

L_3 has $H_3 = H_{s+1}(L_3) \simeq Z$ with generator e_3 , intersection matrix $\begin{bmatrix} e_3 \\ 2^{k-1} \end{bmatrix}$ and tangential invariant given by $\hat{\alpha}(e_3) = 2$ (i.e. $\hat{\alpha} = 2e_3 \in \hat{H}_3$).

$P_3 = \partial L_3$ then has $H_s(P_3) \simeq Z_{2^{k-1}}$ with a generator e'_3 such that the linking invariant is given by $b(e'_3, e'_3) = 2^{1-k}$ and tangential invariant $\hat{\beta}_3 = 2e'_3$.

In addition let $V \in \mathcal{H}(s+1)$ have $H_{s+1}(V) \simeq Z \oplus Z$ with basis f_1, f_2 , corresponding intersection matrix

$$\begin{bmatrix} f_1 & f_2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and tangential invariant given by $\hat{\alpha}(f_1) = 2$, $\hat{\alpha}(f_2) = 0$. The signature τ of V is zero and $\chi = 2f_2$ so that $\chi^2 = 4f_2^2 = 0$ and hence $\partial V = S^{2s+1}$.

$L_1 + L_2 + V$ has intersection matrix

$$\begin{array}{c} \begin{array}{cccc} e_1 & e_2 & f_1 & f_2 \end{array} \\ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right] \end{array}$$

Consider the change of basis

$$\begin{array}{c} \begin{array}{c} E_2 \\ e_1 \\ + \\ e_2 \\ - \end{array} \end{array} \quad \begin{array}{c} \begin{array}{c} F_2 \\ 2^{k-1} e_2 \\ - \\ 2^{2k-3} p f_1 \\ + \end{array} \end{array}$$

$$\begin{array}{c} \begin{array}{cccc} e_1 & 2^{k-1} p f_1 & f_1 & f_2 \end{array} \\ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right] \end{array}$$

$$\begin{aligned} \hat{\alpha}(E_2) &= \hat{\alpha}(e_1) + \hat{\alpha}(e_2) - 2^{k-1} p \hat{\alpha}(f_1) \\ &= \hat{\alpha}(e_2) + 2^k p - 2^{k-1} p \cdot 2 \\ &= \hat{\alpha}(e_2) = 1 \end{aligned}$$

and so by Propositions 1.4 and 1.5 $L_1 + L_2 + V = L_1 + L_2 + V'$
and taking boundaries $P_1 \equiv P_2 = P_1 \equiv P_2 \equiv \Sigma$ where $\Sigma = \partial V'$ and
 V' has the intersection matrix

$$\begin{array}{cc} f_1 & F_2 \\ \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \end{array}$$

$$\hat{\alpha}(F_2) = 2^{k-1} - 2^{2k-3} p \cdot 2 = 2^{k-1} (1 + 2^{k-1} p)$$

$$\hat{\alpha}(f_1) = 2 \text{ and so}$$

$$\chi = 2^{k-1} (1 - 2^{k-1} p) f_1 + 2F_2$$

The signature of V' is zero and so Σ as an element of $(H)_{2s+1}(\partial H)$ is given by

$$\frac{\chi^2}{8} = \frac{2 \cdot 2^{k-1} (1 - 2^{k-1} p) \cdot 2}{8} \mod \begin{cases} 28 \text{ for } s = 3 \\ 2^6 \cdot 127 \text{ for } s = 7 \end{cases}$$

Now $p = 7$ for $s = 3$ and $p = 127$ for $s = 7$ and k is large enough
so that this reduces to

$$2^{k-2} \mod \begin{cases} 28 \text{ for } s = 3 \\ 2^6 \cdot 127 \text{ for } s = 7. \end{cases}$$

$$\textcircled{H}_7(\partial\Pi) \simeq Z_{28} \simeq Z_4 \oplus Z_7 \text{ and } \textcircled{H}_{15}(\partial\Pi) \simeq Z_{2^6 \cdot 127} \simeq Z_{2^6} \oplus Z_{127}$$

and so Σ is a generator of a cyclic group Z_7 for $s = 3$ and

Z_{127} for $s = 7$. Now 2^k divides the tangential invariant

$\hat{\beta}_1 \oplus \hat{\beta}_2$ of $P_1 \overset{\parallel}{\underset{\parallel}{\dashv}} P_2$ and so by part (ii) of Theorem 5.6 any

$\Sigma \in I(P_1 \overset{\parallel}{\underset{\parallel}{\dashv}} P_2)$ is divisible by 2^{k-3} . Since $k-3 \geq 2$ for $s = 3$

and $k-3 \geq 6$ for $s = 7$ we must therefore have

$$I(P_1 \overset{\parallel}{\underset{\parallel}{\dashv}} P_2) \simeq \begin{cases} Z_7 & \text{for } s = 3 \\ Z_{127} & \text{for } s = 7. \end{cases}$$

$L_1 + L_3 + V$ has intersection matrix

$$\begin{matrix} & e_1 & e_3 & f_1 & f_2 \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2^{k-1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{matrix}$$

Consider the change of basis

$$\begin{array}{cccc}
 & E_3 & & F'_2 \\
 & e_1 & & p e_3 \\
 & + & & - \\
 & e_3 & & 2^{k-2} p^2 f_1 \\
 & - & & + \\
 e_1 & 2^{k-1} p f_1 & f_1 & f_2 \\
 \left[\begin{array}{cccc}
 0 & 0 & 0 & 0 \\
 0 & 2^{k-1} & 0 & 0 \\
 0 & 0 & 0 & 1 \\
 0 & 0 & 1 & 0
 \end{array} \right]
 \end{array}$$

$\hat{\alpha}(E_3) = \hat{\alpha}(e_3) = 2$ and so by Propositions 1.4 and 1.5 $L_1 + L_3 + V = L_1 + L_3 + V''$ and taking boundaries $P_1 \frac{11}{14} P_3 = P_1 \frac{11}{14} P_3 \frac{11}{11} \Sigma$ where $\Sigma = \partial V''$ and V'' has intersection matrix

$$\begin{array}{cc}
 f_1 & F'_2 \\
 \left[\begin{array}{cc}
 0 & 1 \\
 1 & 0
 \end{array} \right]
 \end{array}$$

$$\hat{\alpha}(F'_2) = 2p - 2^{k-2} p^2 \cdot 2 = 2p(1 - 2^{k-2} p)$$

$$\hat{\alpha}(f_1) = 2 \text{ and so}$$

$$\chi = 2p(1 - 2^{k-2} p) f_1 + 2F'_2$$

ζ as an element of $(H)_{2s+1}(\partial \Pi)$ is given by

$$\frac{\chi^2}{8} = \frac{2 \cdot 2p(1 - 2^{k+2}p) \cdot 2}{8} \mod \begin{cases} 28 \text{ for } s = 3 \\ 2^6 \cdot 127 \text{ for } s = 7 \end{cases}$$

$$= p \mod \begin{cases} 28 \text{ for } s = 3 \\ 2^6 \cdot 127 \text{ for } s = 7. \end{cases}$$

Σ is therefore a generator for a cyclic group Z_4 for $s = 3$ and Z_2^6 for $s = 7$. Now p divides the tangential invariant $\hat{\beta}_1 \oplus \hat{\beta}_3$ of $P_1 \amalg P_3$ and so by part (ii) of Theorem 5.6 any $\Sigma \in I(P_1 \amalg P_3)$ is divisible by p . We therefore have

$$I(P_1 \amalg P_3) \simeq \begin{cases} Z_4 & \text{for } s = 3 \\ Z_2^6 & \text{for } s = 7. \end{cases}$$

$$I(P_1 \amalg P_2 \amalg P_3) \text{ contains } I(P_1 \amalg P_2)$$

and $I(P_1 \amalg P_3)$ and so we have

$$I(P_1 \amalg P_2 \amalg P_3) \simeq \begin{cases} Z_4 \oplus Z_7 & \text{for } s = 3 \\ Z_2^6 \oplus Z_{127} & \text{for } s = 7. \end{cases}$$

$$\text{i.e. } I(P_1 \amalg P_2 \amalg P_3) = \bigoplus_{2s+1} (H) \quad (\partial \Pi)$$

Since the homology of P_1 is torsion - free and $2^k p$ divides $\hat{\beta}_1$ it follows from part (iii) of Theorem 5.6 that $I(P_1) \approx 0$.

Since the homology groups $H_s(P_2)$, $H_s(P_3)$, $H_s(P_2 \amalg P_3)$ are all finite it follows from part (i) of Theorem 5.6 that $I(P_2) = I(P_3) = I(P_2 \amalg P_3) \approx 0$.

To summarize the inertia groups of P_1 , P_2 , P_3 and their connected sums are as follows:

$$I(P_1) = I(P_2) = I(P_3) = I(P_2 \amalg P_3) \approx 0.$$

$$I(P_1 \amalg P_2) \approx \begin{cases} Z_7 & \text{for } s = 3 \\ Z_{127} & \text{for } s = 7 \end{cases}$$

$$I(P_1 \amalg P_3) \approx \begin{cases} Z_4 & \text{for } s = 3 \\ Z_2^6 & \text{for } s = 7 \end{cases}$$

$$I(P_1 \amalg P_2 \amalg P_3) = \bigoplus_{2s+1} (H_{2s+1}(\partial \Pi)) \approx \begin{cases} Z_{28} & \text{for } s = 3 \\ Z_2^{6.127} & \text{for } s = 7 \end{cases}$$

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