Cobordism of combinatorial manifolds

By Robert E. Williamson, Jr.*

Introduction

The methods developed by Thom [14] to study the cobordism groups Ω_n have been applied successfully to various classes of smooth manifolds, but not yet to piecewise linear or topological manifolds. We recall that as a basic step Thom developed the notion of transverse regularity, which is a notion of general position involving smooth manifolds and their associated vector bundles, and he proved an approximation theorem. The notion of general position is basic in the piecewise linear, if not the topological, category, but one could not apply Thom's method because piecewise linear manifolds do not have tangent or normal vector bundles. The theory of microbundles developed by Milnor [10], [11] suggests that it might be possible to apply Thom's ideas to the study of piecewise linear manifolds, and it is the object of the present paper to do this. After introducing basic facts in § 1 and § 2, we prove in § 3 a piecewise linear analogue of Thom's transverse regular approximation theorem. It is in the proof of this theorem that we are forced to deviate farthest from Thom's approach. Having established the approximation theorem, Thom uses it to prove

$$\Omega_n \cong \lim_{k\to\infty} \pi_{n+k}(M\mathrm{SO}(k)) .$$

In § 4 we establish an analogue of this theorem for the oriented cobordism group Ω_n^{PL} of PL manifolds. In § 5 we define an injection $\Omega_n \to \Omega_n^{\mathrm{PL}}$, also considered by Wall [15] and Milnor (*A survey of cobordism theory*, Enseignement Mathematique 8 (1962) 16–23), and obtain various results, which are listed in the accompanying table. Some of these have been obtained by Wall in [15] by quite different methods. We use the abbreviation '2gp' for 'a group whose order is a power of 2.' The structure of the groups thus indicated on the table is not known.

We note that for $n = 8$ the Stiefel-Whitney and Pontrjagin numbers do not determine the cobordism class of a PL manifold, although these, together with a mod 4 characteristic number defined from B_{SPL} do determine it. The author does not know of any example of a PL cobordism class that is not determined by characteristic numbers from B_{SPL}. It is a result of W. Browder,

* This research was supported in part by the National Science Foundation grant GP 812 and Air Force Office of Scientific Research grant 62-354. In addition, during the completion of the paper the author held a postdoctoral fellowship from the National Academy of Sciences, supported by the Air Force Office of Scientific Research.

to appear, that unoriented piecewise linear cobordism classes are determined by characteristic numbers defined from B_{PL}.

It will be observed that little is known about the 2 torsion of Ω_n^{PL}. It can be seen that the homomorphism

$$\Gamma_{n-1} \longrightarrow \Omega_n^{\mathrm{PL}}/\Omega_n$$

defined by Wall in [15] is neither surjective nor injective in general; it isn't known whether the subgroup $b\mathrm{P}_n = \Theta_{n-1}(\partial\pi)$ of Γ_{n-1} is always injected. We note that the ring $\Omega_*^{\mathrm{PL}} \otimes Z_3$ has zero-divisors.

The author wishes to thank E. Spanier for his advice and encouragement. Thanks are also due to M. Hirsch, W. Browder, and J. Milnor for several useful discussions.

n	Ω_n	Ω_n^{PL}	$\Omega_n^{\mathrm{PL}}/\Omega_n$	ord Γ_{n-1}
all n	...	...	finite	finite
0–3	0	0	0	1
4	Z	Z	0	1
5	Z_2	Z_2	0	1
6	0	0	0	1
7	0	0	0	1
8	$2Z$	$2Z + Z_4$	$Z_7 + Z_4$	28
9	$2Z_2$	$3Z_2$	Z_2	2
10	Z_2	$Z_2 + 2$gp	2gp	8
11	Z_2	$Z_2 + Z_3 + 2$gp	$Z_3 + 2$gp	6
12	$3Z$	$3Z + 2$gp	$Z_7 + Z_{31} + 2$gp	992
13	$4Z_2$	$4Z_2 + 2$gp	2gp	1
14	$2Z_2$	$2Z_2 + 2$gp	2gp	3
15	$3Z_2$	$3Z_2 + 2$gp	2gp	2
16	$5Z + Z_2$	$5Z + Z_2 + 2$gp	$Z_7 + Z_{49} + Z_{31} + Z_{127} + 2$gp	16256
17	$8Z_2$	$8Z_2 + 2$gp	2gp	2
18	$5Z_2$	$5Z_2 + 2$gp	2gp	16

1. Complexes and PL maps

1.1. Following Milnor [3], we shall work in the category of locally finite simplicial complexes (lfs complexes) and piecewise linear maps (briefly, PL maps). Subdivision always means rectilinear subdivision.

DEFINITION. A map $f\colon K \to L$ between lfs complexes is *piecewise linear* if there exists a subdivision K' of K such that f maps each simplex of K' linearly into a simplex of L.

If X is an lfs complex and Y is a closed subspace of it, we shall sometimes say Y is a PL *subspace* of X if Y can be triangulated so that the inclusion is PL. It follows from Lemma 1.1.5 that some subdivision of X is a subcomplex of some subdivision of Y. Given two such triangulations the identity is a PL homeomorphism from one to the other.

In the remainder of § 1 we state, mostly without proof, some basic facts about complexes and PL maps. Proofs can be found in [1], [3], and [10].

LEMMA 1.1.1. *The composite of* PL *maps is* PL.

That is, there is a category whose objects are lfs complexes and whose maps are PL maps.

LEMMA 1.1.2. *Given a* PL *map* $f: X \to Y$ *between finite simplicial complexes, there exist subdivisions* X' *of* X *and* Y' *of* Y *such that* f *is simplicial from* X' *to* Y'.

LEMMA 1.1.3. *If* X *is a subcomplex of the* lfs *complex* Y, *then any subdivision of* X *can be extended to a subdivision of* Y *in which each simplex of* Y *not in the star of* X *is unchanged.*

As an easy consequence of this, suppose we are given a locally finite family $\{K_i\}$ of compact PL subspaces in an lfs complex X. Note that $\{\mathrm{st}K_i\}$ is again locally finite. Then one can subdivide X so that each K_i can be triangulated as a subcomplex of the subdivision of X. Thus:

LEMMA 1.1.4. *Given* $\{K_i\}$ *as above with* $X = \bigcup K_i$, *and a map* $f: X \to Y$ *that is* PL *on each of the* K_i, *it follows* f *is* PL *on* X.

We say a map $f: X \to Y$ is *proper* if $f^{-1}(K)$ is compact for each compact K. Using 1.1.4, 1.1.2, and local compactness, one concludes the following slightly sharper form of 1.1.2.

LEMMA 1.1.5. *If* $f: X \to Y$ *is a proper* PL *map between* lfs *complexes, one can subdivide* X *and* Y *so* f *is simplicial.*

LEMMA 1.1.6. *If* $f, g: X \to Y$ *are homotopic* PL *maps, then there is a* PL *homotopy between them.*

Milnor's proof of 1.1.6 [3, p. 6] in fact shows that one can suppose the PL homotopy is itself homotopic to the original homotopy. From this one can prove by induction over the skeletons of X the following lemma.

LEMMA 1.1.7. *If* $f: X \to Y$ *is* PL, X *is a subcomplex of* Z, *and* f *has a continuous extension to* Z, *then* f *has a* PL *extension to* Z.

1.2. A locally finite convex cell complex can be subdivided into a simplicial complex in a manner unique up to PL homeomorphism, because any such complex has a simplicial subdivision, and any two subdivisions into cell complexes have a common refinement. If Y_1 and Y_2 are cell complexes, then $Y_1 \times Y_2$ has a natural convex cell structure, whose cells are products $\Delta \times \Sigma$ of cells Δ of Y_1 and Σ of Y_2. The complex $Y_1 \times Y_2$ is independent up to subdivision in Y_1 and Y_2. $Y_1 \times Y_2$ also has a natural simplicial structure if Y_1 and Y_2 are simplicial complexes, and the map (simplicial complex $Y_1 \times Y_2) \to$ (cell complex $Y_1 \times Y_2$) is linear on each simplex. One can also see that piecewise linear maps can be defined for cell complexes as for simplicial complexes, and for a simplicial complex, the conditions are equivalent.

LEMMA 1.2.1. *Let X, Y_1, Y_2 be complexes and let P_i: $Y_1 \times Y_2 \to Y_i$ be the projection. Then a map $f: X \to Y_1 \times Y_2$ is* PL *if and only if $P_1 f$ and $P_2 f$ are* PL.

PROOF. If f is PL then $P_1 f$ and $P_2 f$ are PL by 1.1.1. Suppose $P_1 f$ is linear on the subdivision X' of X, and $P_2 f$ is linear on the subdivision X'', so both are linear on a common refinement X_0 of X' and X''. It follows that f is linear on each cell of X_0.

We also mention that any open subset of a locally finite simplicial complex can be triangulated as an lfs complex so that the inclusion is PL, and in a manner unique to within PL homeomorphism.

1.3. The basic combinatorial fact for the transverse regularity approximation theorem (§ 3) is the linear regularity Theorem 1.3.1. Let X and Y be finite complexes and let $f: X \to Y$ be PL. For any PL subspace $Y_0 \subset Y$ (§ 1.1), $f^{-1}Y_0$ is a PL subspace of X.

DEFINITION. We shall say f is *regular* at $y \in Y$ if there is a PL subspace $U \subset Y$ that is a neighborhood of y and a PL homeomorphism h: $f^{-1}(y) \times U \to f^{-1}(U)$ such that $fh = P$, where $f^{-1}(y) \times U$ has the product structure of §1.2 and P is the projection $f^{-1}(y) \times U \to U \subset Y$. For Z a (closed) simplex, we shall write Int$Z$ for the open simplex. We will not always distinguish between a simplex or complex and its space.

THEOREM 1.3.1. *Let X be an* lfs *complex, Y a simplex, and $f: X \to Y$ a simplicial map. Then f is regular at y for any $y \in$* Int$Y$ *and we may choose h so that for any subcomplex L of X, $h^{-1}L = (L \cap f^{-1}y) \times U$.*

PROOF. If one omits the requirement that h be PL, the result is well known, as follows. Let $y = \sum a_i Y^i$ where the simplex Y has vertices $Y^0, \cdots, Y^n$ and the a_i are barycentric coordinates. Suppose for the moment that X is a single simplex. If $fx \in$ Int$Y$, say $fx = \sum t_i Y^i$, $\sum t_i = 1$, $t_i > 0$, then x can be written uniquely as $\sum t_i x_i$ where $x_i \in f^{-1}Y^i$; in particular $f^{-1}y$ consists exactly of the points $\sum a_i x_i$, $x_i \in f^{-1}Y_i$. Thus we have a PL homeomorphism

$$f^{-1}y \approx f^{-1}Y^0 \times \cdots \times f^{-1}Y^n ,$$

and we shall identify these spaces by this PL homeomorphism. Then for any closed simplex $Z \subset \text{Int } Y$ that is a neighborhood of y we also have a homeomorphism $g : f^{-1}y \times Z \approx f^{-1}Z$, which sends $(x_0, \cdots, x_n, \sum t_i Y^i) \rightarrow \sum t_i x_i$. To return to the case of an arbitrary complex X, one pieces $g : f^{-1}y \times Z \approx f^{-1}Z$ together, which can be done because g restricts properly on forces of simplexes of X. For any simplex L of X, hence for any complex, one then has $g^{-1}L = (f^{-1}y \cap L) \times Z$ so $f^{-1}y \times Z$ has a triangulation which is the product given by

$$f^{-1}Y^0 \times \cdots \times f^{-1}Y^n \times Z$$

in each simplex, so that $g^{-1}L$ is a subcomplex of $f^{-1}Z$. Unfortunately g has the quadratic terms $t_i x_i$ and is not piecewise linear. However g is a C^1-embedding in the sense of Munkres [12, § 8] and it follows that, in Munkres' terminology, any sufficiently close δ-approximation (i.e., strong C^1-approximation) is again an embedding. Suppose then that δ is a function on $f^{-1}y \times Z$ such that any δ-approximation to g is an embedding. The secant map h induced by g [12, § 9] is a linear map agreeing with g on the vertices. It will be a δ-approximation for any sufficiently fine subdivision of $f^{-1}y \times Z$ of the proper sort, and one can find such that are refinements of the subdivision we imposed on $f^{-1}y \times Z$. We have defined g so that $g^{-1}L$ is the subcomplex $(f^{-}y \cap L) \times Z$ for any simplex L of X, and it follows that the secant map h again carries $(f^{-1}y \cap L) \times Z$ into the simplex L. Since this holds for every simplex of X, we must have $h^{-1}L = (f^{-1}y \cap L) \times Z$ for every simplex L, hence every subcomplex L. In order to check that h defines a homeomorphism $f^{-1}y \times Z \approx f^{-1}Z$ and satisfies $fh = P$ it suffices to consider each simplex of X individually, so we can suppose X is a simplex. Although g is not linear, fg, which is the projection

$$f^{-1}Y^0 \times \cdots \times f^{-1}Y^n \times Z \rightarrow Z ,$$

is linear. It follows that $fh = fg = P$. We also observe that g carries the boundary of $f^{-1}Y^0 \times \cdots \times f^{-1}Y^n \times Z$ into the boundary of $f^{-1}Z$, in fact each simplex of the boundary of $f^{-1}Y^0 \times \cdots \times f^{-1}Y^n$ is carried by g into a face of $f^{-1}Z$, and it follows that the secant map h again carries the boundary of

$$f^{-1}Y^0 \times \cdots \times f^{-1}Y^n \times Z$$

into the boundary of $f^{-1}Z$. Thus for topological reasons h is a PL homeomorphism of $f^{-1}y \times Z$ onto $f^{-1}Z$. Thus h has the properties required by the theorem.

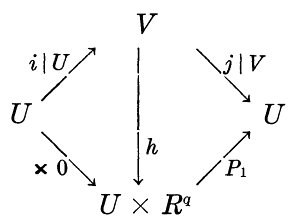
2. Microbundles

2.1. The combinatorial cobordism theory of § 4 and the transverse regularity approximation theorem on which it depends are based on Milnor's theory of microbundles, as developed in [10]. In § 2 we describe the theory as we shall use it, omitting proofs which appear in [10] or [11], generally.

DEFINITION. *A* PL *microbundle* of dimension q (briefly, a bundle) is a diagram

$$\mathfrak{x} : B \xrightarrow{\ i\ } E \xrightarrow{\ j\ } B$$

where E, B are lfs complexes and i, j are PL maps, such that the following triviality condition is satisfied. For each $b \in B$ there should exist neighborhoods U of b (in B) and V of $i(b)$ (in E) and a PL homeomorphism h: $V \approx U \times R^q$ such that the following diagram is commutative



where $\times\ 0$ means the map $b \to b \times 0$ and P_1 is the projection to U.

We shall use the following notation: $B(\mathfrak{x})$, $E(\mathfrak{x})$, $i(\mathfrak{x})$, $j(\mathfrak{x})$ for B, E, i, j, respectively, and we shall sometimes write $\mathfrak{x}$ as a subscript. We shall generally identify B with $i(B)$ since i is a PL embedding with $i(B)$ as a closed subspace.

Whether a diagram such as $\mathfrak{x}$ satisfies the local triviality property is determined by any neighborhood of $i(B)$ in E, call such a neighborhood a reduction of $\mathfrak{x}$. Because of this we use the following definition of Milnor. Let (X, A) and (Y, B) be pairs of lfs complexes.

DEFINITION. *A map germ* from (X, A) to (Y, B) is an equivalence class of mappings f, each defined on some neighborhood U_f of A in X, and mapping $(U_f, A) \to (Y, B)$. Two such maps f, f' are equivalent if and only if $f \mid V = f' \mid V$ for some neighborhood V of A in X. We write F: $(X, A) \Rightarrow (Y, B)$ for a map germ.

Composition of map germs can readily be defined, and a map germ is called a *homeomorphism germ* if it has a two sided inverse.

Let $\mathfrak{x}$ be a microbundle, then, $j(\mathfrak{x})$ defines a *projection germ*

$$J(\mathfrak{x}) \colon \big(E(\mathfrak{x}), B(\mathfrak{x})\big) \Rightarrow \big(B(\mathfrak{x}), B(\mathfrak{x})\big) .$$

If $\mathfrak{y}$ is another microbundle over the same base space $B = B(\mathfrak{y}) = B(\mathfrak{x})$, then a *bundle equivalence* is a PL homeomorphism germ F: $(E(\mathfrak{y}), B) \Rightarrow (E(\mathfrak{x}), B)$ such that $J(\mathfrak{x})F = J(\mathfrak{y})$.

Now let $\mathfrak{y}$, $\mathfrak{x}$ be q-dimensional microbundles with bases not necessarily the same. A *bundle map germ* $F: \mathfrak{y} \Rightarrow \mathfrak{x}$ is a map germ $(E(\mathfrak{y}), B(\mathfrak{y})) \Rightarrow (E(\mathfrak{x}), B(\mathfrak{x}))$ containing a representative map $f: U \rightarrow E(\mathfrak{x})$, called a *bundle map*, that maps each fiber in a 1–1 way. Thus $j(\mathfrak{x})f = fj(\mathfrak{y})$.

Bundle map germs and induced bundles are related as is to be expected.

LEMMA 2.1.1. *Let $F: \mathfrak{y} \Rightarrow \mathfrak{x}$ be a bundle map germ such that $F \mid B(\mathfrak{y})$ is a* PL *homeomorphism. Then* F *is a homeomorphism germ.*

For proof, see [10]. Given a bundle map germ $F: \mathfrak{y} \Rightarrow \mathfrak{x}$, let $f: B(\mathfrak{y}) \rightarrow B(\mathfrak{x})$ be $F \mid B(\mathfrak{y})$, which is well defined. There is a natural map germ $\mathfrak{y} \Rightarrow f^*\mathfrak{x}$ defined by $e \rightarrow (j_{\mathfrak{y}}(e), F(e))$, where we recall that $f^*\ \mathfrak{x}$ has total space contained in $B(\mathfrak{y}) \times E(\mathfrak{x})$, consisting of (e, e') such that $f(e) = j_{\mathfrak{x}}(e')$. From the lemma one can conclude

THEOREM 2.1.2. *If $f: B \rightarrow B(\mathfrak{x})$ there is a bundle map germ $f^*\mathfrak{x} \Rightarrow \mathfrak{x}$, and if F is a bundle map germ $\mathfrak{y} \Rightarrow \mathfrak{x}$, then the natural map germ is an equivalence.*

Furthermore, homotopic maps induce the same bundle.

THEOREM 2.2.1. *Let f and g be homotopic maps $B \rightarrow B(\mathfrak{x})$, then $f^*\mathfrak{x}$ and $g^*\mathfrak{x}$ are isomorphic. If $\mathfrak{x}$ is oriented, the isomorphism preserves orientation.*

2.2. *Universal microbundles.* By a universal microbundle for fiber dimension n, we mean an n-microbundle $\mathfrak{u}$ with base space $B_{\mathrm{PL}(n)}$ such that for any lfs complex X, the homotopy classes $[X, B_{\mathrm{PL}(n)}]$ correspond 1–1 with the equivalence classes of microbundles by associating to $f: X \rightarrow B_{\mathrm{PL}(n)}$ the bundle $f^*\mathfrak{u}$. Equivalently, for any bundle $\mathfrak{x}$ over a complex X and bundle map germ F of $\mathfrak{x}$ restricted to a subcomplex A, there is a bundle map germ that extends F to $\mathfrak{x}$. Then

THEOREM 2.2.1. *For each n, there is a universal microbundle for fiber dimension n,*

$$\mathfrak{u}(\mathrm{PL}(n)): B_{\mathrm{PL}(n)} \rightarrow E_{\mathrm{PL}(n)} \rightarrow B_{\mathrm{PL}(n)} .$$

Milnor proves this by constructing a complete semi-simplicial group $\mathrm{PL}(n)$ whose k-simplexes are bundle map germs $\mathfrak{e}^q_{\Delta_k} \Rightarrow \mathfrak{e}^q_{\Delta_k}$. The same argument applies to the group $\mathrm{SPL}(n)$ of orientation preserving germs and shows the existence of an oriented bundle $\mathfrak{u}(\mathrm{SPL}(n))$ that is universal for oriented bundles and orientation preserving bundle maps. Indeed, let G be any subgroup complex of $\mathrm{PL}(n)$ that satisfies these two conditions:

(a) whenever F is a k-simplex of G and $\lambda: \Delta_r \rightarrow \Delta_k$ is a PL map between standard simplexes, then λ^*F is an r-simplex of G, and

(b) if $F: \mathfrak{e}^q_{\Delta_k} \Rightarrow \mathfrak{e}^d_{\Delta_k}$ is an isomorphism germ, and if Δ_k can be subdivided so that for any map $\lambda: \Delta_r \rightarrow \Delta_k$ that is simplicial relative to Δ_r and the subdivision of Δ_k, one knows that $\lambda^* F$ is an r-simplex of G; then one should have F as a k-simplex of G. For such a G, one can readily define the notions of G-bundle, G-bundle map, etc., and if G has countable homotopy groups, Milnor's construction produces a universal G-bundle $\mathfrak{u}(G)$. Furthermore the theorem is both meaningful and true for such G. However, we shall only be concerned with oriented and non-oriented bundles, so we will not give such a generalization in detail.

2.3. Let $\mathfrak{x}$ be a bundle, X a complex, and suppose $E(\mathfrak{x})$ is contained in X so that $B(\mathfrak{x})$ is a closed subspace of X. Then we say *X contains the bundle* $\mathfrak{x}$. If $E(\mathfrak{x})$ is a neighborhood of $B(\mathfrak{x})$, we say $\mathfrak{x}$ is a *normal bundle* for $B(\mathfrak{x})$ in X.

In [3, § 3] and [4, § 5], Milnor shows that, for M and N PL manifolds such that M is PL embedded in N, for sufficiently large q, $M \times 0$ has a normal bundle in $N \times R^q$, and any two normal bundles are stably equivalent. It is not known however that any two normal bundles are equivalent. There are PL embeddings not locally flat, so there exist examples in which M has no normal bundle at all.

Let Y be a PL subspace of X, and let X contain $\mathfrak{x}$. Then we say Y is *compatible with $\mathfrak{x}$ in X* if for some neighborhood U of $B(\mathfrak{x})$ in $E(\mathfrak{x})$, for any $x \in U$, $j_{\mathfrak{x}}(x) \in Y$, if and only if $x \in Y$. If this occurs, the following diagram defines a bundle, the *inherited bundle*:

$$Y \cap B(\mathfrak{x}) \rightarrow Y \cap E(\mathfrak{x}) \rightarrow Y \cap B(\mathfrak{x}) .$$

Clearly the bundle is a reduction of $\mathfrak{x} \mid (Y \cap B(\mathfrak{x}))$. Note that if $\mathfrak{x}$ is a normal bundle, then the inherited bundle is a normal bundle.

LEMMA 2.3.1. *Suppose C is a closed subspace of X, and Y contains the bundle $\mathfrak{x}$. Suppose U is an open neighborhood of C in X. Then there is a closed* PL *subspace $Y \subset U$ of X that is a neighborhood of C and is compatible with $\mathfrak{x}$ in X.*

PROOF. One can choose a closed PL subspace $C' \subset U$ that is a neighborhood of C. There is then a closed PL subspace $B_0 \subset B(\mathfrak{x})$ that is a neighborhood of $B(\mathfrak{x}) \cap C'$ in $B(\mathfrak{x})$ and lies in U. For V a sufficiently small neighborhood of B_0 in $j^{-1}B_0$ $j = j(\mathfrak{x})$, one can then take $Y = C' \cup V$.

3. Transverse regularity

3.1. In this section, we introduce transverse regularity in the piecewise linear sense. There is no satisfactory microbundle analogue of a quotient bundle, so our definition cannot copy Thom's too closely. Furthermore it may be that normal bundles are not unique, so they appear explicitly in the definition.

DEFINITION. Let S and T be locally finite simplicial complexes and let $\mathfrak{x}$ be a normal bundle for B in T. Let f: $S \rightarrow T$ be a PL map. If $f^{-1}(B)$ has a normal bundle $\mathfrak{y}$ in S such that $f \mid E(\mathfrak{y})$ is a bundle map $\mathfrak{y} \to \mathfrak{x}$, then we shall say f is *transverse regular* for $(\mathfrak{y}, \mathfrak{x})$ (in the PL sense), or briefly, f is *t-regular*.

As usual, we shall not always distinguish between $\mathfrak{y}$ and a reduction of $\mathfrak{y}$; in effect we require that the map germ of f (§ 2.1), $(S, f^{-1}(B)) \Rightarrow (T, B(\mathfrak{x}))$ coincide with a bundle map germ $\mathfrak{y} \Rightarrow \mathfrak{x}$.

3.2. It will be convenient for the proof of Theorem 3.3.1. to assign a topology to the space $M(X, Y)$ of PL maps $X \to Y$, for complexes X and Y. For X compact, the compact open topology will do; for S compact in Theorem 3.3.1 this paragraph may be ignored. We define a base for our topology as follows. Let C_i, $i = 1, 2, \cdots$ be a locally finite family of closed subsets of X, and U_i a family of open subsets of Y. Let W be the set of all PL maps f such that $f(C_i) \subset U_i$, $i = 1, 2, \cdots$. The base consists of all such W. One can easily verify the following elementary facts which are all that we need.

For X compact this topology coincides with the compact open topology.

If $X \xrightarrow{f} Y \xrightarrow{g} Z$, then the map $g^*(h) = gh$ is a continuous map $M(X, Y) \to M(X, Z)$; if f is proper then f^*, defined by $f^*(h) = hf$, is continuous.

If $X = X_1 \cup X_2$, with inclusion $i_n : X_n \subset X$, $n = 1, 2$, then the map $M(X, Z) \to M(X_1, Z) \times M(X_2, Z)$ by $f \to (i_1^* f, i_2^* f)$ is open.

The map $M(X, Y_1 \times Y_2) \to M(X, Y_1) \times M(X, Y_2)$ by $f \to (P_1 f, P_2 f)$, where P_i: $Y_1 \times Y_2 \to Y_1$ is the projection, is a homeorphism.

3.3. We make the convention that whenever H: $Y \times I \to X$ is a PL homotopy, H_t is the composite $Y \to Y \times t \to X$, and whenever we introduce H_t: $Y \to X$ first it is understood that the corresponding H is PL.

THEOREM 3.3.1. *Let S and T be locally finite simplicial complexes and let f: $S \to T$ be a* PL *map. Let Q and X be* PL *subspaces of S and let X be a neighborhood of Q such that $f \mid X$ is t-regular for $(\mathfrak{y}_X, \mathfrak{x})$ and Q inherits $\mathfrak{y}_Q$ from $\mathfrak{y}_X$. Then there is a* PL *homotopy H_t of f that is constant on Q such that H_1 is t-regular for $(\mathfrak{y}, \mathfrak{x})$ for a $\mathfrak{y}$ such that Q inherits $\mathfrak{y}_Q$ from $\mathfrak{y}$. In adddition we may suppose that all the H_t lie in any given neighborhood of f and given any neighborhood U of $B(\mathfrak{x})$ in T, we may suppose $H_t = f$ outside $f^{-1}(U)$.*

The relative nature of this theorem allows one to show the following.

COROLLARY 3.1.2. *If g_0 and g_1 are t-regular maps $S \to T$ and $g_0 \simeq g_1$, say $g_i^{-1}(B(\mathfrak{x}))$ has normal bundle $\mathfrak{y}_i$, then we can choose the homotopy to be t-regular so that each $S \times i$ inherits $\mathfrak{y}_i$ from $S \times I$, $i = 0, 1$.*

PROOF OF 3.1.2. By 1.1.6 we can suppose the homotopy is PL, and it can be supposed constant on $[0, 1/4)$ and on $(3/4, 1]$, then $X = S \times ([0, 1/4) \cup (3/4, 1])$ can be taken as the neighborhood of $S \times (0 \cup 1) = Q$ required by the theorem with the bundle whose projection is

$$E(\mathfrak{y}_0) \times \left[0, \frac{1}{4}\right) \cup E(\mathfrak{y}_1) \times \left(\frac{3}{4}, 1\right] \longrightarrow B(\mathfrak{y}_0) \times \left[0, \frac{1}{4}\right) \cup B(\mathfrak{y}_1) \times \left(\frac{3}{4}, 1\right].$$

3.4. We shall prove the approximation theorem 3.3.1 by considering simple cases, then more complicated ones.

By a PL isotopy of a complex X, we mean a PL map $A: X \times I \rightarrow X$ such that for each $t \in I$ the composite $A_t: X \rightarrow X \times t \rightarrow X$ is a homeomorphism, and A_0 is the identity.

LEMMA 3.4.1. (Case 1) *Theorem* 3.3.1 *holds if* $\mathfrak{x}$ *is the bundle* $\mathfrak{e}_0^q: 0 \rightarrow R^q \rightarrow 0$, *$S$ is compact, and* $Q = \varnothing$. *In addition, in this case we can conclude that* $H_t = A_t f$ *where A is a* PL *isotopy of* R^q, *and that each simplex of S is compatible with* $\mathfrak{x}$.

PROOF. Let s^q be a simplex in R^q that is a neighborhood of 0, chosen so that $s^q \subset U$, where f is permitted to vary on $f^{-1}(U)$ only. Let s_0^q be the complex $\dot{s}^q * 0$ as a subdivision of s^q. For a given point $y \in \operatorname{Int} s^q$, let $A_t: s^q \rightarrow s^q$ be the homeomorphism given by the canonical isomorphism of $\dot{s}^q * 0$ with $\dot{s}^q * ty$. Then $A: s^q \times I \rightarrow s^q$ by $(x, t) \rightarrow A_t(x)$ is easily seen to be linear relative to the product convex cell triangulation of $s_0^q \times I$. We can extend A_t to R^q as the identity outside s^q Then $A_0 =$ identity and $A_1(y) = 0$. Given any neighborhood of the projection $R^q \times I \rightarrow R^q$ we can confine A to that neighborhood by confining y to some neighborhood of 0, thus confining $A_t f$ to any given neighborhood of f (§ 3.2).

Now let $t: K \rightarrow R^q$ be a PL homeomorphism. We can suppose S is subdivided so that $t^{-1}f$ is simplicial. Let $y \in R^q$ be such that $\tau^{-1}(y)$ is in the interior of some q simplex; we can choose y to be so close to 0 that $A_t f$ is in the neighborhood of f specified in the theorem. Now $A_1\tau: K \rightarrow R^q$ is again a PL homeomorphism, $(A_1\tau)^{-1}(A_1 f) = \tau^{-1}f$ is simplicial, and $(A_1\tau)^{-1}(0)$ is an interior point of a q simplex.

Let $H_t = A_t f$. Now we can apply the linear regularity Theorem 1.3.1 to conclude that there is a neighborhood E of $H_1^{-1}(0) = B$ in S and a neighborhood V of 0 in s^q such that there is a PL homeomorphism $F: B \times V \rightarrow E$ satisfying $H_1 F = P$ where $P: B \times V \rightarrow V$ is the projection. So for $j: B \times V \rightarrow B$ the projection, the bundle

$$\mathfrak{y}: B \xrightarrow{\text{incl}} E \xrightarrow{jF^{-1}} B$$

satisfies the conclusion of the theorem, for $H_1 F = P$ implies each fiber is mapped isomorphically onto V. It is also a consequence of the linear regularity theorem that each simplex of S inherits a bundle.

3.5. We next consider the relative case, Q not necessarily empty; this introduces an extension problem. We still consider $\mathfrak{x} = \mathfrak{e}_0^q$.

LEMMA 3.5.1. *Theorem* 3.3.1 *holds if* $\mathfrak{x}$ *is the bundle* $\mathfrak{e}_0^q: 0 \to R^q \to 0$ *and* S *is compact.*

PROOF. We shall divide the proof into two parts, of which the second is an extension problem. We first introduce some notation applying to both parts and do some subdividing. We use the notation introduced in the statement of Theorem 3.3.1, but we abbreviate $B(\mathfrak{y}_X)$ to B_X, $E(\mathfrak{y}_X)$ to E_X, $j(\mathfrak{y}_X)$ to j_X, and similarly for Q. Subdivide S so that X is a subcomplex of S and E_Q is a subcomplex of E_X. Subdivide B_X so B_Q is a subcomplex of it. Refine S so that j_X is linear on E_X.

PART 1 OF LEMMA 3.5.2. *There is an* H_t *satisfying the conclusion of Theorem* 3.3.1, *with* H_1 *t-regular for* $(\mathfrak{z}, \mathfrak{e}_0^q)$ *say, except* Q *may not inherit* $\mathfrak{y}_Q$ *from* $\mathfrak{z}$. *However, for each simplex* s *of* B_X, $j_X^{-1}(s)$ *is compatible with* $\mathfrak{z}$.

PROOF. According to Theorem 2.1.2, there is a bundle map germ $F_X: \mathfrak{z}_{B_X}^q \Rightarrow \mathfrak{y}_X$, and we may suppose $fF_X = P$ (as germs) where $P: B_X \times R^q \to R^q$ is the projection. We may suppose E_X so chosen that F_X contains a representative $f_X : (B_X \times V_X, B_X \times 0) \approx (E_X, B_X)$, where V_X is a PL subspace of R^q that is a neighborhood of 0. Since $\mathfrak{y}_X$ is compatible with Q, we may suppose $j_X(x) \in Q$ if and only if $x \in Q$ for $x \in E_X$.

According to Lemma 3.4.1, there is a PL isotopy A of R^q such that $A_1 f$ is t-regular, for $(\mathfrak{p}, e_0^q)$ say. By Theorem 2.1.2, there is a bundle isomorphism germ F_p, F: $\mathfrak{e}_{B(p)}^q \Rightarrow \mathfrak{p}$ such that $A_1 fF = P$ (as germs), where P is the projection $B(\mathfrak{p}) \times R^q \to R^q$.

In general $A_t f$ is not constant on Q, so we modify $A_t f$ and $\mathfrak{p}$ accordingly. Let V_A be the closure of $\{x \in R^q \mid A_t(x) \neq x$ for some $t\}$. According to Lemma 3.4.1, we can suppose V_A is arbitrarily small, in particular that $V_A \subset \operatorname{Int} E_X$. Observe that, if we define D_t: $E_X \to E_X$ by $D_t F_X(b, x) = F_X(b, A_t(x))$, then D_t is a PL isotopy of E_X such that $j_X D_t = j_X$, and

$$A_t f(x) = fD_t(x) \qquad \text{if } x \in E_X . \tag{1}$$

We assert that there is a PL isotopy G_t of S such that

(a) in $E_X \cap Q$, $G_t = D_t^{-1}$;

(b) in E_X, $j_X G_t = j_X$; and

(c) in S-Int X, $G_t =$ identity.

PROOF. There is a PL map μ: $S \to I$ such that $\mu = 1$ on Q, $\mu = 0$ on S-Int X, and on E_X, $\mu j_X = \mu$. One can construct μ on B_X, extend to E_X by $\mu j_X = \mu$, then extend to S. Then define G_t on E_X by $G_t = D_{\mu(x)\times t}^{-1}$ for $x \in E_X$. By Lemmas 1.1.1 and 1.2.1, G_t is PL on E_X. Since $V_A \subset V_X$, $G_t(x) = x$ except for

x in the interior of E_X in X, so one can extend G_t to X as the identity outside E_X; then G_t is PL on X. Since $\mu = 0$ except in Int X, G_t can be extended over S as the identity outside X and is a PL isotopy of S. Then (a) follows from $\mu = 1$ on Q, (b) follows from $j_X D_t = j_X$, and $\mu j_X = \mu$, and (c) follows from $\mu = 0$ outside Int E_X.

Now define $H_t = A_t f G_t$. From (1) and (a), it follows that $H_t \mid Q = f \mid Q$, as required by Theorem 3.3.1. Now we assert that H_1 is t-regular. We define a bundle $\mathfrak{z}$ as follows. Its base is $B(\mathfrak{z}) = H_1^{-1}(0) = G_1^{-1}(B(\mathfrak{p}))$, recall that $B(\mathfrak{p}) = (A_1 f)^{-1}(0)$. Its bundle space is $D_1(E(\mathfrak{p}))$, and we define $j_{\mathfrak{z}}$ by $j_{\mathfrak{z}} D_1 = D_1 j_{\mathfrak{p}}$. Then $\mathfrak{z}$ is a bundle and H_1 is t-regular for $(\mathfrak{z}, \mathfrak{e}_0^q)$. Also H_t may be made arbitrarily close to f by choosing V_A sufficiently small, and of course $H_t = f$ except on $f^{-1}(V_A)$.

Now according to Lemma 3.4.1, each simplex of S is compatible with $\mathfrak{p}$, therefore each subcomplex is. Since j_X is linear on E_X, for any simplex s of B_X, $j_X^{-1}(s)$ is a subcomplex, thus compatible with $\mathfrak{p}$. By definition and choice of E_X this means $j_X j_{\mathfrak{p}}(x) \in s$ if and only if $j_X(x) \in s$ for $x \in E$ (allowing $\mathfrak{p}$ to be reduced if necessary). From (a) and $j_{\mathfrak{z}} G_1 = G_1 j_{\mathfrak{p}}$, this implies

$$j_X j_{\mathfrak{z}} G_1(x) \in s \longleftrightarrow j_X G_1 j_{\mathfrak{p}}(x) \in s \longleftrightarrow j_X j_{\mathfrak{p}}(x) \in s \longleftrightarrow j_X(x) \in s \longleftrightarrow j_X G_1(x) \in s \ .$$

This completes Part 1

PART 2. *Let H_t be as in the conclusion of Part* 1. *Then there is a bundle $\mathfrak{y}$ such that H_1 is t-regular for* $(\mathfrak{y}, \mathfrak{e}_0^q)$, *and Q does inherit $\mathfrak{y}_Q$ from $\mathfrak{y}$.*

PROOF OF PART 2. By Theorem 2.1.2, there is a bundle map germ $F_S\colon e^q_{B(\mathfrak{z})} \Rightarrow \mathfrak{z}$ and a germ $F_X\colon e^q_{B_X} \Rightarrow \mathfrak{y}_X$, such that as germs, $H_1 F_S = P$ and $H_1 F_X = P$, where P is the projection to R^q. Let V_S be such a small neighborhood of 0 in R^q that F_S has a representative $f_S\colon B(\mathfrak{z}) \times V_S \approx W_S$, and F_X has a representative $f_X\colon B_X \times V_S \approx W_X$. Let V_S be so small that for $x \in W_S$, one has $j_{\mathfrak{z}}(x) \in j_X^{-1}(s)$ if and only if $x \in j_X^{-1}(s)$ for each simplex s of B_X, by compatibility, and so small that $H_1 f_X = P$ and $H_1 f_S = P$. Let $\varphi = f_S^{-1} f_X$, so $\varphi\colon B_X \times V_S \approx B_X \times V_S$, and restricts to a homeomorphism $\varphi\colon B_Q \times V_S \approx B_Q \times V_S$. Let

$$j\colon B(\mathfrak{z}) \times V_S \longrightarrow B(\mathfrak{z})$$

be the projection of the trivial bundle. We do not know that $j\varphi = j$; if this were true, we would have $j_Q = j_{\mathfrak{z}}$ on W_X, so Q would inherit $\mathfrak{y}_Q$ from $\mathfrak{z}$, and we would be done.

Our object now is to extend φ on $B_Q \times V_S$ to $\tilde{\varphi}$, $\tilde{\varphi}\colon B_{\mathfrak{z}} \times V_S \approx B_{\mathfrak{z}} \times V_S$; we can then define $\mathfrak{y}$ as the bundle with base B, bundle space W_S, and projection imposed by the isomorphism $f_S \tilde{\varphi}\colon B_{\mathfrak{z}} \times V_S \approx W_S$ in S. On $B_Q \times V_S$, $f_S \varphi = f_S f_S^{-1} f_X$, so Q would inherit $\mathfrak{y}_Q$ from $\mathfrak{y}$, since f_X is a bundle map germ to $\mathfrak{y}_X$, of which $\mathfrak{y}_Q$ is a restriction.

We have arranged for φ to have the following properties. Since $H_1 f_S = P = H_1 f_X$ on $B_X \times V_S$,

(a) $P\varphi = P$.

Both f_S and f_X cover the identity map, so

(b) $\varphi(b, 0) = (b, 0)$.

By a previous argument, for each simplex s of B_X,

(c) φ maps $s \times V_S \approx s \times V_S$.

The last condition permits us to reduce the extension problem to a single simplex. The object of the subdivisions introduced at the beginning of the proof of Lemma 3.5.1 was just to obtain this condition. It is not true in general that an isomorphism φ satisfying (a) and (b) can be extended to an isomorphism $\tilde{\varphi}$ (nor even to an isomorphism not satisfying (a) and (b)).

Note that by choosing for V_S a still smaller neighborhood of 0 in R^q, we can suppose $V_S \approx R^q$, so that the following lemma suffices to complete the proof of Lemma 3.5.1.

LEMMA 3.5.2. *For any complex* B_X, PL *embedded in the complex* B *as a neighborhood of the subcomplex* B_Q *of* B_X, *and* PL *homeomorphism* $\varphi\colon B_X \times R^q \approx B_X \times R^q$, *conditions* (a), (b), (c), *above, ensure that* $\varphi | B_Q \times R^q$ *has an extension to an isomorphism* $\tilde{\varphi}\colon B_{\mathfrak{z}} \times R^q \approx B_{\mathfrak{z}} \times R^q$ *also satisfying* (a) *and* (b).

PROOF. Let B^n be the n-skeleton of B_X together with all of B_Q. For $n = 0$ let $\varphi^0 = \varphi$ on $B_Q \times R^q$, and for a vertex v of B not in B^q, define $\varphi^0 =$ identity. Then $\varphi^0\colon B^0 \times R^q \approx B^0 \times R^q$. Suppose then φ has been extended to

$$\varphi^n\colon B^n \times R^q \approx B^n \times R^q$$

which is the identity outside $\operatorname{st} B_q$ in B_X, and for each simplex s sends $s \times R^q \approx s \times R^q$. Now let s be an $n + 1$ simplex of B_X. If s is not in the star of B_Q, then $\varphi^n =$ identity on $\dot{s} \times R^q$, so we can extend φ^n as the identity on $s \times R^q$. Otherwise let a neighborhood W of $\dot{s}$ in s be represented by a PL homeomorphism $h\colon \dot{s} \times I \to W$, $h(x, 0) = x$. Let $g\colon R^q \times I \to R^q$ be a PL contraction to 0. Extend φ^n over W by defining its components relative to h as follows: for $((a, t), x) \in (\dot{s} \times I) \times R^q$,

$$j\varphi^{n+1}\big(h(a, t), x\big) = \big(\varphi^n\big(a, g(x, t)\big), t\big)$$
$$P\varphi^{n+1}\big((a, t), x\big) = x \,,$$

where $j\colon B_{\mathfrak{z}} \times R^q \to B$ and $P\colon B_{\mathfrak{z}} \times R^q \to R^q$ are the projections. One extends φ^{n+1} over $s - W$ as the identity; note that $\varphi^{n+1}((a, 1), x) = ((a, 1), x)$. By Lemmas 1.1.1 and 1.2.1, φ^{n+1} is PL. One notes that φ^{n+1} is again the identity outside $\operatorname{st} B_Q \times R^q$ and sends $s \times R^q \approx s \times R^q$. The φ^n thus define a PL homeomorphism

$\tilde{\varphi}$: $B_X \times R^q \approx B_X \times R^q$ that is the identity outside $\mathrm{st}B_Q \times R^q$. We ought to note that we can not assume B_X is a subcomplex of $B_{\mathfrak{z}}$, for if we subdivided B_X condition (c) would be lost. Thus we can not assume the open star of B_Q in B_X is open in B. However, since B_X is a neighborhood of B_Q in $B_{\mathfrak{z}}$, by suitably choosing h we can still be sure that $\tilde{\varphi} \neq$ identity only on the interior of $B_X \times R^q$ in $B_{\mathfrak{z}} \times R^q$. Thus $\tilde{\varphi}$ can be extended over the rest of $B_{\mathfrak{z}} \times R^q$ as the identity. This completes the proof of Lemma 3.5.2, and thus of Lemma 3.5.1 also.

3.6. We next allow $\mathfrak{x}$ to have a base that is not a single point.

LEMMA 3.6.1. *Theorem* 3.3.1 *holds if* $\mathfrak{x}$ *is the bundle,* $\mathfrak{e}^q_B$: $B \to B \times R^q \to B$, *where B is contractible and S is compact.*

PROOF. We use the notation introduced in the statement of Theorem 3.3.1. Let P: $B \times R^q \to R^q$ be the projection; we will also think of it as the bundle map $\mathfrak{e}^q_B \Rightarrow \mathfrak{e}^q_0$. We can assume that S is just $f^{-1}P^{-1}(s^q)$ for some q simplex s^q that is a neighborhood of 0 in R^q, and $T = B \times R^q$, for we will construct H so that $H_t = f$ except on $f^{-1}(\mathrm{Int}\, s^q)$; we can then extend H_t over the given S by $H_t = f$ outside $f^{-1}(s^q)$. Apply now Lemma 3.5.1 with Pf: $S \to R^q$ for f. We conclude that there is a PL homotopy of Pf, $G_t: S \to R^q$ such that G_1 is t-regular, for $(\mathfrak{y}, \mathfrak{e}^q_0)$ say. Abbreviate $j(\mathfrak{e}^q_B)$ to j, that is, j is the projection $B \times R^q \to B$. Now we define a PL homotopy of f, h_t: $S \to B \times R^q$ by $h_t(x) = (jf(x), G_t(x))$, so $h_0 = f$. Since G_1 defines a bundle map to $\mathfrak{e}^q_0$, there is a bundle isomorphism germ F, F: $\mathfrak{e}^q_{B(\mathfrak{y})} \Rightarrow \mathfrak{y}$ such that G_1F: $\mathfrak{e}^q_{B(\mathfrak{y})} \Rightarrow \mathfrak{e}^q_0$ is P, so $Ph_1F = P$. Thus, for some neighborhood V of 0 in R^q and neighborhood W of $B(\mathfrak{y})$ in S, F contains a representative which we will also denote by F that is a PL homeomorphism F: $B(\mathfrak{y}) \times V \approx W$, such that $G_1F = P$. Let h_B: $B(\mathfrak{y}) \times V \to B$ be $jfF = jh_tF$.

We are going to deform h_1 to a bundle map by moving $h_B(b, x)$ to $h_B(b, 0)$; the new map will then carry fibers into fibers. To do this we need a family of paths from each point of B to each other point. Precisely, let $N \subset B \times B \times I$ be the PL subspace $B \times B \times 0 \cup B \times B \times I \cup D \times I$ where $D = \{(b, b) \mid b \in B\}$ is the diagonal. Define A: $N \to B$ by

$$A(b_1, b_2, 0) = b_1, \quad A(b_1, b_2, 1) = b_2, \quad A(b, b, t) = B .$$

Then A is PL on N and has a continuous extension to $B \times B \times I$, thus a PL extension, which we again denote by A. Let k: $R^q \to I$ be 0 outside a neighborhood V of 0 in R^q and 1 on some neighborhood $V' \subset V$ of 0. Theorem 3.3.1 requires that $H_t = f$, except on $f^{-1}(U)$, for a given neighborhood U of $B \times 0$ in $B \times R^q$; let V be so small that $f^{-1}(B \times V) \subset f^{-1}(U)$. Using the PL homeomorphism F: $B(\mathfrak{y}) \times V \approx W$, we define a homotopy H of h_1 as follows.

$$jH\big(F(b, x), t\big) = A\big(h_B(b, x), h_B(b, 0), \min\big(t, k(x)\big)\big)$$
$$PH\big(F(b, x), t\big) = x .$$

We extend H over the rest of S as the constant homotopy, $H(s, t) = h_1(s)$. As always we write $H_t(s) = H(s, t)$. We see that $H_0 = h_1$ and $jH_1 = H_1 j_{\mathfrak{y}}$ on $B(\mathfrak{y}) \times V'$, so H_1 sends fibers into fibers. The definition of the R^q component of H_1 shows that H_1 in fact maps each fiber one-to-one. Thus H_1 defines a bundle map germ $\mathfrak{y} \Rightarrow \mathfrak{e}^q_B$. Wherever $h_B(b, x) = h_B(b, 0)$—that is, wherever h_1 is already a bundle map—$H_t = h_1$. In particular, $H_t \mid Q = h_t \mid Q = f \mid Q$. In order to show that the composition of the homotopies h_t and H_t satisfies the conclusion of Theorem 3.3.1, we have only to show that we can choose h_t and H_t arbirarily close to f. One can choose h_t close to f because $jh_t = jf$ and $Ph_t = G_t$ can, according to Lemma 3.5.1, be chosen arbitrarily close to Pf. Thus it suffices to show that we can choose H_t arbitrarily close to h_1 We shall see that we can make H_t close to h_1 by restricting the set on which $k \neq 0$ to a sufficiently small neighborhood of 0. For H_t to be close to h_1, we must have $A(h_B(b, x), h_B(b, 0), -)$ close to $h_B(b, x)$. Since $B(\mathfrak{y}) \times V$ can be supposed compact, for any neighborhood of the diagonal D in $B \times B$ there is a neighborhood V' of 0 in R^q such that all the pairs $(h_B(b, x), h_B(b, 0))$ for $x \in V'$ lie in the neighborhood of D. Since $A(b, b, t) = b$, in order to make $A(b_1, b_2, t)$ close to b_1, it suffices to confine (b_1, b_2) to a sufficiently small neighborhood of D in $B \times B$. Thus by making V' small enough we can make H_t arbitrarily close to h_1 on $F(B(\mathfrak{y}) \times V) = W$, and $H_t = h_1$ outside W, so H_t can be confined to any given neighborhood of h_1.

3.7. *The general case of Theorem* 3.3.1. We now prove the transverse regularity approximation theorem as stated in § 3.3. The argument consists of piecing together homotopies whose existence is asserted by Lemma 3.6.1.

For any $x \in f^{-1}(B(\mathfrak{x}))$, there is a compact neighborhood W_X of x and an open set E_X contained in $E(\mathfrak{x})$ that is bundle isomorphic to $\mathfrak{e}^q_B$ where B is an open contractible set of $B(\mathfrak{x})$. We may therefore choose a locally finite family $\{W_i\}$ of compact PL subspaces (§ 1.1) of S whose interiors cover S. We choose them so that if W_i meets $f^{-1}(B(\mathfrak{x}))$, then there is an open set $E_i \subset E(\mathfrak{x})$ of T containing $f(W_i)$ for which there is a bundle isomorphism t_i: $B_i \times R^q \approx E_i$ where B_i is an open contractible subspace of $B(\mathfrak{x})$. We also require that, if W_i meets Q, then $W_i \subset \operatorname{Int} Y$, where Y is a closed neighborhood of Q in $\operatorname{Int} X$. We also require that, if W_i meets $f^{-1}(B(\mathfrak{x}))$, then $W_i \subset f^{-1}(U)$, where U is the neighborhood of $B(\mathfrak{x})$ specified in the assumptions of Theorem 3.3.1.

We shrink the $\{W_i\}$ to a family of compact sets $\{W'_i\}$ as follows. We shall want $W'_i \subset \operatorname{Int} W_i$ and $S = \bigcup \operatorname{Int} W'_i$. $W_0 - \bigcup_{i \neq 0} \operatorname{Int} W_i$ is a compact set in $\operatorname{Int} W_0$, so it has a compact neighborhood $W'_0 \subset \operatorname{Int} W_0$. Then

$$S = \operatorname{Int} W'_0 \cup \operatorname{Int} W_1 \cup \cdots .$$

We apply the same argument to W_1 relative to W'_0, W_2, W_3, $\cdots$ to get W'_1 so that $\operatorname{Int} W_0' \cup \operatorname{Int} W_1' \cup \operatorname{Int} W_2 \cup \cdots = S$. Let $\{W_i'\}$ be the family thus defined inductively. Then $S = \bigcup \operatorname{Int} W_i'$, for any $x \in S$ has a neighborhood meeting only finitely many W_i, say $W_0, \cdots, W_n$, and $x \in \operatorname{Int} W_0' \cup \cdots \cup \operatorname{Int} W_n' \cup \operatorname{Int} W_{n+1} \cup \cdots$ so $x \in \bigcup \operatorname{Int} W_i'$.

LEMMA 3.7.1. *There is a partition of unity on S by* PL *maps $p_0, p_1, \cdots$ with supports* $\sup(p_i) \subset \operatorname{Int} W_i$ *such that $p_i > 0$ on W_i'.*

PROOF. There is a family of PL maps $h_i: S \to I$ such that $h_i = 1$ on a neighborhood of W_i' and $\sup(h_i) \subset \operatorname{Int} W_i$. Define $q_0(x) = \min((1 - 1/2), h_0(x))$, and having defined $q_0, \cdots, q_{n-1}$, PL maps satisfying $q_0 + \cdots + q_{n-1} < 1 - (1/2)^n$, define

$$q_n(x) = \min\left(\left(1 - \left(\frac{1}{2}\right)^{n+1} - q_0(x) - \cdots - q_{n-1}(x)\right), h_n(x)\right).$$

Then $\sup(q_n) \subset \operatorname{Int} W_n$, and $q_0 + \cdots + q_n \leqq 1 - (1/2)^{n+1}$. Thus $h_n(x) \neq 0$ implies $q_n(x) \neq 0$, so $q_n > 0$ on W_n'. Also $\sum_0^\infty q_n(x) < 1$ for $x \in S$. Let $g(x) = 1 - \sum_0^\infty q_n(x)$ so g is PL, and partition $g(x)$ as follows. Define $r_0(x) = h_0(x) \times g(x)$, and having defined $r_0, \cdots, r_{n-1}$, define r_n by $r_n(x) = h_n(x) \times (g(x) - r_0(x) - \cdots - r_{n-1}(x))$. Then $\sup(r_n) \subset \operatorname{Int} W_n$ and $\sum_0^\infty r_n = 1 - \sum_0^\infty q_n$. Then the PL maps $p_n = r_n + q_n$ satisfy the lemma.

To prove Theorem 3.3.1, we now make an inductive argument, with the following *inductive assumption*. We suppose that $X_n \subset S$ is a closed neighborhood of $Y \cup W_0' \cup \cdots \subset W_n'$ in S and that H_t^n is a PL homotopy of $H_1^{n-1}(H_0^0 = f)$ such that $H_1^n \mid X_n$ is t-regular for $(\mathfrak{y}_n, \mathfrak{x})$. We assume that in $(Y \cup W_0' \cup \cdots \cup W_{n-1}') - \operatorname{Int} W_n$ the two bundle projections $j(\mathfrak{y}_n)$ and $j(\mathfrak{y}_{n-1})$ are defined on the same sets and coincide. We also suppose that H_t^n lies in the neighborhood of f specified by Theorem 3.3.1, and that H_t^n is so close to f that $H_t^n(W_k)$ meets $B(\mathfrak{x})$ only if $f(W_k)$ meets $B(\mathfrak{x})$. We assume that, if $f(W_k)$ meets $B(\mathfrak{x})$, then $H_t^n(W_k) \subset E_k$. Furthermere we suppose that there is a compact neighborhood V_n of W_n' such that $P_n > 0$ on V_n and $H_t^n(x)$ does not change with t except for $x \in \operatorname{Int} V_n$, and then only if W_n meets $f^{-1}(B(\mathfrak{x}))$ and does not meet Q.

Before completing the induction we shall show that the theorem can be derived from it. We define H, a sort of infinite composition of the homotopies H^n as follows. Let $P_n = p_0 + \cdots + p_n$ where p_k is the partition function of Lemma 3.7.1. Define $D_k \subset S \times I$ as

$$D_k = \{(x, t) \mid x \in \sup(p_k) \quad \text{and} \quad P_{k-1}(x) \leqq t \leqq P_k(x)\}.$$

It can be seen that each D_k is a PL subspace, and $\{D_k\}$ is a locally finite cover of $S \times I$. There exists a PL map $A_n: V_n \times I \to I$ such that on the set $\{(x, t) \mid x \in V_n$ and $t \leqq P_{n-1}(x)\}$ A_n is 0, while on the set $\{(x, t) \mid x \in V_n$ and $t \geqq P_n(x)\}$ A_n is 1, because $p_n > 0$ on V_n by the inductive assumption. Now we define $H(x, t)$ for $(x, t) \in D_n$ by

$$H(x, t) = H^n(x, A_n(x, t))$$

if $x \in V_n$, and for $x \notin V_n$, $H(x, t) = H^n(x, t')$, which is independent of t' by inductive assumption. Now H is well defined. Suppose $(x, t) \in D_n \cap D_m$. Let $n < m$. Then $t = P_n(x) = P_{n+1}(x) = \cdots = P_{m-1}(x)$; and, if $x \in V_n \cap V_m$, then $A_n(x, t) = 1$, while $A_{m-1}(x, t) = 0$ and

$$H^n(x, A_n(x, t)) = H^n(x, 1) = H^{n+1}(x, 0) = H^{n+1}(x, 1) = \cdots$$
$$H^{m-1}(x, A_{m-1}(x, t)) = H^{m-1}(x, 0) .$$

If $x \notin V_n \cap V_m$, then the equation holds because of the constancy of the homotopies. Now H is PL on each member of the locally finite cover $\{D_k\}$, therefore H is PL by Lemma 1.1.4. One can now see that H_t satisfies every requirement of the theorem except possible t-regularity. For the t-regularity define $j(\mathfrak{y}) = j(\mathfrak{y}_n)$ for large n; i.e., for each k there is an N such that $n \geqq N$ implies $W_n \cap W_k = \varnothing$; then define $j(\mathfrak{y}) = j(\mathfrak{y}_n) = j(\mathfrak{y}_{n+1}) = \cdots$. Thus $j(\mathfrak{y})$ is PL on each $E(\mathfrak{y}_n) \cap W'_n$, and these form a locally finite family, so $j(\mathfrak{y})$ is PL, and of course it is a bundle projection since it is one locally. Also in Q $j(\mathfrak{y}_n) = j(\mathfrak{y}_Q)$ for we can begin the induction with $\varnothing$ for W'_0; we may suppose X_0 is compatible with $\mathfrak{y}_X$ in X and take the inherited bundle for $\mathfrak{y}_0$. Then Q inherits $\mathfrak{y}_Q$ from $\mathfrak{y}$. Since H^n_1 is a bundle map on $\mathfrak{y}_n$, it follows that H_1 is a bundle map on $\mathfrak{y}$.

To complete the proof, we have only to complete the inductive argument, which we now do. If W_{n+1} does not meet $f^{-1}(B(\mathfrak{x}))$, then define H^n to be the constant homotopy, which satisfies the inductive assumption with $X_{n+1} = X_n \cup W_{n+1}$. If W_{n+1} meets Q, then $W_{n+1} \subset Y$ (by choice of the W_i) so we may take $X_{n+1} = X_n$ and again the constant homotopy satisfies the inductive assumption. So suppose W_{n+1} meets $(H^n_1)^{-1}(B(\mathfrak{x}))$ (thus also $f^{-1}(B(\mathfrak{x}))$) but not Q. Then $H^n_1(W_{n+1}) \subset E_{n+1}$, which we assumed to be a trivial neighborhood of $\mathfrak{x}$ PL homeomorphic to $B_{n+1} \times R^q$. So we can apply Lemma 3.6.1, as follows.

(a) For S, we take a compact PL subspace V_{n+1} of W_{n+1} that is a neighborhood of W'_{n+1}. By Lemma 2.4.1, we may suppose V_{n+1} is compatible with $\mathfrak{y}_n$, and we may suppose $p_n > 0$ on V_{n+1}.

(b) For X, we take $X_n \cap V_{n+1}$; and for $\mathfrak{y}_X$, the inherited bundle.

(c) For Q, we find a closed PL subspace $X'_n \subset \operatorname{Int} X_n$ that is a neighborhood of $Y \cup W'_0 \cup \cdots \cup W'_n$ and compatible with $\mathfrak{y}_n$, and take $X'_n \cap V_{n+1}$ to be Q. For $\mathfrak{y}_Q$ we take the inherited bundle from $\mathfrak{y}_n$.

(d) For f we take $H^n_1 \mid V_{n+1}$. For the neighborhood of $H^n_1 \mid V_{n+1}$ we take one so small that any map $S \to T$ whose restriction to V_{n+1} lies in the neighborhood, and whose restriction to $S - V_{n+1}$ is the same as that of H^n_1, is itself in the neighborhood of f given for the general case of Theorem 3.3.1, which is also a neighborhood of H_1^n by the inductive assumption.

We conclude that there is a PL homotopy $G^{n+1}: V_{n+1} \times I \to V_{n+1}$ of $H_1^n \mid V_{n+1}$ such that G_1^{n+1} is t-regular, for $(\mathfrak{p}, \mathfrak{x})$ say, so that $j(\mathfrak{p})$ is an extension of $j(\mathfrak{y}_n)$ in V_{n+1}, with $j(\mathfrak{y}_n)$ perhaps restricted to a reduced bundle. Now $E(\mathfrak{p}) \cup E(\mathfrak{y}_n)$ is a neighborhood of $B(\mathfrak{p}) \cup B(\mathfrak{y}_n)$ in $X_n \cup V_{n+1}$, and we can define a projection j to $B(\mathfrak{p}) \cup B(\mathfrak{y}_n)$ as $j(\mathfrak{y}_n)$ on $E(\mathfrak{y}_n)$ and $j(\mathfrak{p})$ on $E(\mathfrak{p})$. These are consistent, at least on some neighborhood of $B(\mathfrak{p}) \cup B(\mathfrak{y}_n)$, because $X_n \cap V_{n+1}$ inherits the same bundle from $\mathfrak{y}_n$ and $\mathfrak{p}$. Since j is locally the projection of a bundle, it is the projection of a bundle, say $\mathfrak{y}'$.

There exists a PL subspace W''_{n+1} of Int V_n that is a compact neighborhood of W'_{n+1}, and compatible with $\mathfrak{p}$. Then $X'_n \cup W''_{n+1}$ is compatible with $\mathfrak{y}'$ so it inherits a bundle, which we call $\mathfrak{y}_{n+1}$. We define $X_{n+1} = X'_n \cup W''_{n+1}$, so $\mathfrak{y}_{n+1}$ is a normal bundle in X_{n+1}. Let $m: S \to I$ be a PL map that is 1 on W''_{n+1} and 0 outside V_{n+1}. Define

$$H^{n+1}(x, t) = G^{n+1}(x, \min(m(x), t))$$

for $x \in V_{n+1}$, and $H^{n+1}(x, t) = H^n(x, t)$ for $x \notin V_{n+1}$. This is consistent since $m = 0$ unless $x \in V_{n+1}$. G^{n+1} is the constant homotopy on $X'_n \cap V_{n+1}$, so H_1^{n+1} is a bundle map on $\mathfrak{y}_n$, and $m = 1$ on W''_{n+1}, so H_1^{n+1} is a bundle map on $\mathfrak{p}$, therefore H^{n+1} is a bundle map on $\mathfrak{y}_{n+1}$.

We can confine H_t^{n+1} to an arbitrary neighborhood of H_t^n by confining G_t^{n+1} to a sufficiently small neighborhood of H_t^n (see (d) above), and in particular one can satisfy the requirements of the inductive assumption. It is not hard to see that we can suppose $j(\mathfrak{y}_n)$ and $j(\mathfrak{y}_{n+1})$ are defined on the same points outside V_{n+1}.

This completes the inductive argument and the proof of the t-regularity approximation Theorem 3.3.1.

4. Cobordism and the Thom theorem

4.1. We shall prove a theorem relating the PL cobordism groups to the homotopy groups of a certain spectrum. We will use this to get specific information about the structure of the oriented PL cobordism group Ω_n^{PL}. This group is defined exactly as in the C^∞ case. One considers all compact, unbounded, oriented, n-dimensional PL manifolds M (each $x \in M$ has a neighborhood PL isomorphic to an open set in R^n). One says two such manifolds M, N are cobordant if there is a compact oriented PL manifold W such that $\partial W = M \cup (-N)$, the disjoint union of M and N with the orientation of N reversed. The equivalence classes form a group under disjoint union; this is the cobordism group Ω_n^{PL} in which we shall be interested; see § 4.11. The unoriented case is similar.

We first must consider the PL analogue of Thom's L-equivalence classes. In this context the analogue of a C^∞ manifold is generally a locally finite simplicial complex rather than the more special PL manifold. The definitions below will apply equally in the oriented and unoriented cases, but we will state everything in terms of the oriented case for concreteness. Now let X be an lfs complex and let $L_k(X, \text{SPL})$, or $L_k(X)$, be the set of all pairs $(Y, \mathfrak{n})$ where Y is an lfs complex that is PL embedded in X and $\mathfrak{n}$ is an oriented normal bundle for Y in X of fiber dimension k (so Y is closed in X). We introduce an equivalence relation $\sim$ in $L_k(X)$ by setting $(Y, \mathfrak{n}) \sim (Y', \mathfrak{n}')$ if and only if there is a pair $(Z, \mathfrak{m}) \in L_k(X \times I)$ such that $Y = Z \cap (X \times 0)$, $Y' = Z \cap (X \times 1)$ and the oriented bundle $\mathfrak{m}$ restricts to the oriented bundles $\mathfrak{n}, \mathfrak{n}'$ over Y, Y' in $X \times 0$ and $X \times 1$ respectively. We denote the set of equivalence classes by $\Lambda_k(X)$ and call this the set of *oriented L-equivalence* classes. Note that the normal bundle appears explictly in the definitions.

4.2. If X is an oriented PL $n + k$-dimensional manifold, then $L_k(X)$ can be thought of as consisting of pairs $(M, \mathfrak{n})$ where M is a PL submanifold of X, oriented by the condition that $\mathfrak{n} \oplus \tau_M$ be oriented like $\tau_X \mid M$ and $\mathfrak{n}$ is a normal bundle, with orientation determined reciprocally by that on M. If $(M, \mathfrak{n}_1)$ and $(N, \mathfrak{n}_2)$ are equivalent, by $(W, \mathfrak{m})$ say, then W is a PL manifold such that $\partial W = M \cup N$, and if we orient W by the natural orientation of $I \times X$, then W induces the given orientation on M and the reverse orientation N. The only difficulty is the following.

LEMMA. *Suppose V is a* PL *$n + q$-dimensional manifold and M is a* PL *subspace of V, with normal q-dimensional bundle. Then M is a* PL *submanifold of dimension n and its boundary is exactly $M \cap \partial V$.*

This fact is well known. Each point $x \in M$ has a neighborhood W in V PL homeomorphic to a product $U \times R^q$, U a neighborhood of x in M. It is enough to show U is a PL manifold. $U \times R^q$ can be triangulated as a product, and since it is a manifold, the link of a q-simplex of R^q (which is the same as the link of a point in U) must be a combinatorial $(n - 1)$-sphere or ball, by results of Alexander.

4.3. We now want to consider a stable group obtained from $L_k(X)$ by something like suspension. Let X now be a sequence of lfs complexes $X_0, X_1, X_2, \cdots$ such that for each i, $R \times X_i$ is an open subset of X_{i+1} so that the inclusion is PL relative to the product PL structure on $R \times X_i$, and $0 \times X_i$ is closed. We will call this an S-sequence. We have in mind mainly the sequence of spheres S^n, given C^1-triangulations so that the usual inclusion $S^n \subset S^{n+1}$ is PL. There is no particularly natural way to choose a normal bundle, but one certainly exists.

There is a function $L_k(X_i) \to L_{k+1}(X_{i+1})$ which assigns to the pair $(Y, \mathfrak{n})$ the pair $(Y, \mathfrak{e}^1_Y \oplus \mathfrak{n})$, where Y is embedded by $X_i \to 0 \times X_i$, and $\mathfrak{e}^1_Y \oplus \mathfrak{n}$ is the normal bundle $R \times E(\mathfrak{n})$, note $\mathfrak{e}^1_Y$ is here oriented, so $\mathfrak{e}^1_Y \oplus \mathfrak{n}$ is also. One can make a similar definition for the triples that define the equivalence relation, so so the function is well defined. We define $\Lambda_i\{X, \text{SPL}\}$, or briefly $\Lambda_i\{X\}$, by:

$$\Lambda_i\{X\} = \lim_{k\to\infty} \Lambda_k(X_{k+i}) \ .$$

We can now define a composition on $\Lambda_i\{X\}$ which makes it into an abelian group. Given $\alpha, \beta \in \Lambda_i\{X\}$, one can suppose they are represented by $(Y, \mathfrak{n})$, $(Y', \mathfrak{n}') \in L_k(X_{k+i})$; their sum is represented by the pair in $L_{k+1}(X_{k+i+1})$ whose space is the disjoint union $Y \cup Y'$ embedded by $y \to (-1, y) \in R \times X_{k+i} \subset X_{k+i+1}$ and $y' \to (1, y')$, with normal bundles $\mathfrak{e}^1_Y \oplus \mathfrak{n}$ and $\mathfrak{e}^1_{Y'} \oplus \mathfrak{n}'$. One checks that this is commutative and associative, with the triple whose space is $\varnothing$ for an identity. To see that inverses exist one can construct them as in the C^∞ case; their existence also follows from Theorem 4.8.1.

4.4. The stable group we have defined includes in particular the cobordism groups. Let S be the sequence of spheres $S^0, S^1, S^2, \cdots$ of § 4.3, with orientations chosen so that the inclusion $R \times S^n \to S^{n+1}$ preserves orientation.

Theorem 4.4.1. $\Lambda_n\{S\} = \Omega_n^{\text{PL}}$.

PROOF. We remark that the passage to the limit seems to be necessary; it is not clear that $\Lambda_k(S^{n+k})$ is isomorphic to Ω_n^{PL} for any k, however large. The proof is carried out as in the C^∞ case, but there are technical difficulties which we shall have to dispose of. A function $\Lambda_i(S^{n+i}) \to \Omega_n^{\text{PL}}$ is defined by assigning to the pair $(M, \mathfrak{n})$ the space M; in § 4.2 we showed that M is a PL manifold, and has a natural orientation. If $(M_1, \mathfrak{n}_1) \sim (M_2, \mathfrak{n}_2)$ by $(W, \mathfrak{m})$ then $\partial W = M_1 - M_2$ and M_1, M_2 are cobordant. So the homomorphism is defined on $\Lambda_i(S^{n+i})$, and one can easily check that it induces a homomorphism $\Lambda_n\{S\} \to \Omega_n^{\text{PL}}$. The homomorphism is onto, for if M is a PL n-manifold one can certainly PL embed it in a sphere S^{n+i} of high enough dimension, and according to [10, Th. 5.8] or [11, Th. 4] there is a j such that $M \times 0$ has a normal bundle $\mathfrak{n}$ in S^{n+i+j}, so $(M, \mathfrak{n}) \in \Lambda_{i+j}(S^{n+i+j})$.

Our difficulties arise as we show that the kernel is 0. So suppose $(M, \mathfrak{n}) \in \Lambda_i(S^{n+i})$ and $M = \partial B$. We need to construct a pair $(B, \mathfrak{m}) \in \Lambda_i(I \times S^{n+i})$ so $B \cap 0 \times S = M$, $B \cap 1 \times S = \varnothing$, and $\mathfrak{m} \,|\, M = \mathfrak{n}$, perhaps moving to large i by $\Lambda_i(S^{n+1}) \to \Lambda_{i+1}(S^{n+i+1})$. We can certainly find a suitable embedding $W \to I \times S^n$, but it is less clear that we can construct a suitable normal bundle. Now let H^m be euclidean half space $R^{m-1} \times R$, so $\partial H^m = R^{m-1}$.

LEMMA 4.4.2. *For any embedding $i: M \to R^q$ there is an $m > q$ such that the embedding $i \times 0: M \to R^{m-1} = \partial H^m$ can be extended to an embedding $i': B \to H^m$ that possesses a normal microbundle* $\mathfrak{m}$.

We write $\operatorname{Int} H^m = H^m - \partial H^m$. For topological reasons it follows that $i: \operatorname{Int} B \to \operatorname{Int} H^m$ and $\mathfrak{m} \mid M$ is a normal bundle for M in R^{m-1}.

PROOF. For m large enough one can surely extend i to an embedding $i': B \to H^m$. Let B_0, H_0^m be copies of B and H^m and B' the doubled manifold $B \cup_j B_0$ formed by identifying along the boundary. Since R^m is the double of H^m, the double of i' is an embedding $B' \to R^m$. By increasing m if necessary, we can suppose B' has a normal bundle $\mathfrak{x}$ in R^n, according to Milnor [11]. Unfortunately $\mathfrak{x} \mid B$ need not be a normal bundle for B in H^m. We therefore make the following construction.

In B' there is a neighborhood of M that is PL homeomorphic to $M \times R$ so that $x \leftrightarrow (x, 0)$. Now we lift $E(\mathfrak{x} \mid B)$ into $H^{m+1} = R^m \times H^1$ as follows. There exists a PL map $\mu: E(\mathfrak{x} \mid B) \to H^1$ with the following properties: on $M \times I \subset B'$, where $\mathfrak{x} \mid M \times I \approx (\mathfrak{x} \mid M) \times I$, $\mu(x, t) = t$, and for $x \in E(\mathfrak{x} \mid B' - M \times I)$, $\mu(x) > 1$. We define a PL embedding $g: E(\mathfrak{x} \mid B) \to H^{m+1}$ by $g(x) = (x, \mu x)$. Now g puts $E(\mathfrak{x} \mid M)$ in R^m and $E(\mathfrak{x} \mid \operatorname{Int} B)$ into $\operatorname{Int} H^{m+1}$; we have only to construct a normal bundle $\mathfrak{w}$ for $E(\mathfrak{x} \mid B)$ in H^{m+1} to be done, for the composite bundle of this with $\mathfrak{x}$ will be a normal bundle for B in H^{m+1}. We construct the projection j, of $\mathfrak{w}$ as follows. Let $\bar{B} = B - \operatorname{Int} M \times I$ and let $P: R^{m+1} \to R^m$ be the projection on the first m coordinates. Now $\bar{B} \subset R^m$, in fact $\bar{B} \subset \operatorname{Int} H^m$, and on a neighborhood of $g\bar{B}$ in $P^{-1}\bar{B}$ we can define $j = gP$. Note that $Pg =$ identity. Next we define j on a neighborhood of M in R^m. As we remarked earlier, M has a neighborhood in R^m PL homeomorphic to $E(\mathfrak{x} \mid M) \times R$. We define $j: E(\mathfrak{x} \mid M) \times R \to E(\mathfrak{x} \mid M)$ to be the projection.

It is finally necessary to define j over a neighborhood of the remainder of B; i.e., of $M \times I \subset B$. Let $S \subset R^2$ be defined by $S = \{(x, y) \mid |x| \leqq 1 \text{ and } y \geqq x - 1/2\}$, let E_1 be the segment from $E_1 = \{(x, 0) \mid -1 < x < 1/2\}$ and E_2 the half line $E_2 = \{(1, y) \mid y > 1/2\}$. Map

$$M \times S \to H^{m+1} = R^m \times H^1$$

thus; $(m, x, t) \to (g(m, x), t)$ where we identify $E(\mathfrak{x} \mid M \times I)$ with $E(\mathfrak{x} \mid M) \times I$. This is an embedding since g is, and its image covers a neighborhood of $M \times I$ in $P^{-1}(M \times I)$. So we have now to define a PL projection in $M \times S$ this is compatible with j where j is already defined, that is on the image of $M \times (E \cup E_2)$. The induced projection on $M \times (E_1 \cup E_2)$ is id $\times j'$ where j' sends E_1 onto $(0, 0)$ and E_2 onto $(1, 1)$. Now j' can clearly be extended over S, so id $\times y'$ can be extended over $M \times S$, and j can be defined as required to give a normal bundle for $g(E(\mathfrak{x} \mid B))$ and this completes the lemma.

We return to the proof that $\Lambda_n\{S\} \to \Omega_n$ has kernel 0. We now have a pair $(B, \mathfrak{m}) \in L_i(S^{n+i})$ so we know $(M, \mathfrak{m} \mid M)$ represents 0 in $\Lambda_n\{S\}$, but it is not necessarily true that $\mathfrak{m} \mid M = \mathfrak{e}^r \oplus \mathfrak{n}$. We can suppose, according to [11] that there is an isomorphism $\mathfrak{m} \mid M \cong \mathfrak{e}^r \oplus \mathfrak{n}$, but it does not follow that $(M, \mathfrak{m}) \sim (M, \mathfrak{e}^k \oplus \mathfrak{n})$. For this we need a PL "isotopy of tubular neighborhoods theorem". R. Lashoff and M. Rothenberg have proved such a theorem [8, Th. 5.3]; a weak form of it states:

THEOREM (of L. and R.). *Let $\mathfrak{x}$, $\mathfrak{y}$ be normal bundles for M in a* PL *manifold N. Let $\mathfrak{w}$ be the bundle $I \times (\mathfrak{e}^k \oplus \mathfrak{x})$ over $I \times M$. Then for k large enough there is a* PL *homeomorphism $H\colon E(\mathfrak{w}) \to I \times N \times R^k$ such that $H \mid E(\mathfrak{w} \mid 0 \times M)$ is a bundle isomorphism onto $0 \times E(\mathfrak{e}^k \oplus \mathfrak{x})$ and $H \mid E(\mathfrak{w} \mid 1 \times M)$ is a bundle isomorphism onto $1 \times E(\mathfrak{y} \oplus \mathfrak{e}^k)$.*

According to this we can construct a triple $(I \times M, \mathfrak{w})$ such that (stably) $\mathfrak{w} \mid 0 \times M$ is $\mathfrak{m} \mid M$, $\mathfrak{w} \mid 1 \times M$ is $\mathfrak{n}$, and this proves that $(M, \mathfrak{n}) \sim (M, \mathfrak{m} \mid M)$ which is equivalent to 0 in $\Lambda_n\{S\}$. Thus $\Lambda_n\{S\} = \Omega_n^{\mathrm{PL}}$.

4.5. We now construct an analogue of the Thom complex [4, p. 28–29]. Let $\mathfrak{x}$ be a microbundle. Let E be an open neighborhood of $B(\mathfrak{x})$ in $E(\mathfrak{x})$ such that $E(\mathfrak{x}) - E$ is a PL subspace of $E(\mathfrak{x})$. If $E(\mathfrak{x}) - E$ is a strong deformation retract of $E(\mathfrak{x}) - B(\mathfrak{x})$ we shall say E is admissible. Then we call the quotient space formed by collapsing $E(\mathfrak{x}) - E$ to a point $*$ a *Thom complex* of $\mathfrak{x}$ (although it may not be locally finite at $*$), and denote it by $T(\mathfrak{x})$ or $T_E(\mathfrak{x})$. If $E(\mathfrak{x})$ is finite dimensional, one can construct an lfs complex homotopically equivalent to $T_E(\mathfrak{x})$ by adjoining to $E(\mathfrak{x})$ a complex that is contractible and contains $E(\mathfrak{x}) - E$. We point out that $T_E(\mathfrak{x}) - B(\mathfrak{x})$ is contractible. For $T(\mathfrak{u}(G_q))$, we write $M(G_q)$.

LEMMA 4.5.1. *Let U be any neighborhood of $B(\mathfrak{x})$ in $E(\mathfrak{x})$. There is an admissble neighborhood $E \subset U$.*

PROOF. One can subdivide $E(\mathfrak{x})$ so finely that closed star of $B(\mathfrak{x})$ lies in U, and $B(\mathfrak{x})$ is a full subcomplex of $E(\mathfrak{x})$. If one then barycentrically subdivides $E(\mathfrak{x})$ again, the open star of $B(\mathfrak{x})$ is admissible.

4.6. If a map $f\colon T(\mathfrak{y}) \to T(\mathfrak{x})$ sends $B(\mathfrak{y}) \to B(\mathfrak{x})$, then by restriction it defines a map germ $(E(\mathfrak{y}), B(\mathfrak{y})) \Rightarrow (E(\mathfrak{x}), B(\mathfrak{x}))$. We shall say f and this germ are associated. If $f\colon T(\mathfrak{y}) \to T(\mathfrak{x})$ is associated to the germ F and $f'\colon T(\mathfrak{x}) \to T(\mathfrak{z})$ to the germ F', then $f'f$ is associated to $F'F$.

LEMMA 4.6.1. Let *$\mathfrak{y}$, $\mathfrak{x}$ be microbundles with E admissible for $\mathfrak{y}$ and E' for $\mathfrak{x}$, and let $F\colon \mathfrak{y} \Rightarrow \mathfrak{x}$ be a bundle map germ. Then then there is a map $f\colon T_E(\mathfrak{y}) \to T_{E'}(\mathfrak{x})$ that is associated to F, and any two maps $T_E(\mathfrak{y}) \to T_{E'}(\mathfrak{x})$*

that are associated to F are homotopic by a homotopy that is constant on a neighborhood of $B(\mathfrak{y})$.

If F is an isomorphism germ with inverse F^{-1} associated to $g: T_{E'}(\mathfrak{y}) \rightarrow T_E(\mathfrak{y})$, then gf is associated to the identity germ $F^{-1}F$ so it is homotopic to the identity map. Similarly fg is homotopic to the identity so f is a homotopy equivalence. In particular the homotopy type of $T(\mathfrak{x})$ does not depend on the particular choice of admissible neighborhood.

PROOF OF 4.6.1. If X is a subspace of $E(\mathfrak{y})$ we shall write $\mathrm{bd}_E X$ for the boundary of X in $T_E(\mathfrak{y})$.

We may suppose F has a representative $f: U \rightarrow E(\mathfrak{x})$ where U is a neighborhood of $B(\mathfrak{y})$ in $E(\mathfrak{y})$, and we may suppose U is closed, $U \subset E$, and f embeds fibers. It follows that f sends $\mathrm{bd}_E U$ into $E(\mathfrak{x}) - B(\mathfrak{x})$. Thus the composite of f $\mathrm{bd}_E U$ and the quotient map $E(\mathfrak{x}) - B(\mathfrak{x}) \rightarrow T_{E'}(\mathfrak{x}) - B(\mathfrak{x})$, which is contractible, can be extended to a map $T_E(\mathfrak{y}) - B(\mathfrak{y}) \rightarrow T_{E'}(\mathfrak{x}) - B(\mathfrak{x})$. Now f and this extension of it define a map $T_E(\mathfrak{x}) \rightarrow T_{E'}(\mathfrak{x})$ which is clearly associated to the germ F in the above sense. Similarly, two maps f and f' that are associated to F provide, with the constant homotopy on some sufficiently small neighborhood V of $B(\mathfrak{y})$, a map $T_E(\mathfrak{y}) \times \{0, 1\} \cup V \times I \rightarrow T_{E'}(\mathfrak{x})$ that can be extended to a map $T_E(\mathfrak{y}) \times I \rightarrow T_{E'}(\mathfrak{x})$. This completes the proof.

4.7. We shall abbreviate the universal oriented bundle, which is $\mathfrak{u}(\mathrm{SPL}_q)$ or $\mathfrak{u}(\mathrm{PL}_q)$ according to whether we consider the oriented or unoriented case, to $\mathfrak{u}_q$, $B(\mathrm{SPL}_q)$ or $B(\mathrm{PL}_q)$ to B_q, etc. According to § 4.6, a classifying map germ $\mathfrak{u}_q \oplus \mathfrak{e}'_{B_q} \Rightarrow \mathfrak{u}_{q+1}$ induces a map $T(\mathfrak{u}_q \oplus \mathfrak{e}'_B) \rightarrow M(G_{q+1})$, and its homotopy class does not depend on the choice of classifying map. Let SM_q be the suspension of the Thom complex of $\mathfrak{u}_q$, which we can take to be $[-2, 2] \times M_q$, with $2 \times M_q$ identified to a point and $-2 \times M_q$ to another. If E is an admissible neighborhood for $\mathfrak{u}_q$, then $[-2, 2] \times E$ is admissible for $\mathfrak{u}_q \oplus \mathfrak{e}'_{B_q}$, so there is a (quotient) map $SM_q \rightarrow T(\mathfrak{u}_q \oplus e'_{B_q})$ which can be composed with the map above to give a map $SM_q \rightarrow M_{q+1}$. With these maps the M_q form a spectrum, which we shall call the PL *Thom spectrum* (oriented or unoriented as the case may be). Let X be an S-sequence as in § 4.3, and $f: X_k \rightarrow M_q$ a PL map. Then, one can define the suspension $Sf: X_{k+1} \rightarrow SM_q$ which on a neighborhood of X_k sends $(t, x) \rightarrow (t, fx)$ and satisfies $(Sf)^{-1}B_{q+1} = 0 \times X_k$. This induces $[X_k, M_q] \rightarrow [X_{k+1}, M_{q+1}]$, and we denote the direct limit by

$$\{X, M(\mathrm{SPL})\}_q = \lim_{i\to\infty} [X_{q+i}, M(\mathrm{SPL}_i)] .$$

If X is the sequence of spheres, we write

$$\pi_q(M\mathrm{SPL}) = \{S, M\mathrm{SPL}\}_q = \lim \pi_{q+i}(M(\mathrm{SPL}_i)) .$$

The sets $\{X, M(\mathrm{SPL})\}_q$ form a group, as is well known. The addition may be described as follows: if α, β are represented by $f, g\colon X_{k+i} \to M(\mathrm{SPL}_i)$ then $\alpha + \beta$ is represented by $F\colon X_{k+i+1} \to M(\mathrm{SPL}_{i+1})$ such that

(a) $F(t, x) = (2(t-1), f(x))$ on $[0, 2] \times X_{k+i}$,

(b) $F(t, x) = (2(t+1)g(x))$ on $[-2, 0] \times X_{k+i}$, and

(c) $F^{-1}(B_{i+1})$ is exactly $1 \times X_k \cup -1 \times X_k$.

4.8. Now we can construct the Thom map. Let X be any lfs complex and $M_q = M(\mathrm{SPL}_q)(= M(\mathrm{PL}_q)$ in the unoriented case). Let $f\colon X \to M_q$ represent some homotopy class in $[X, M_q]$. By the simplicial approximation theorem applied relative to $M_q - *$, we may suppose f is PL to a neighborhood of B_q in M_q. By Theorem 3.3.1, f is homotopic to a t-regular map, say g. By definition, $g^{-1}B_q$ then has an oriented normal bundle say $\mathfrak{n}$. Then the pair $(g^{-1}B_q, \mathfrak{n})$ defines an element of $L_q(X)$. This element does not depend on which t-regular map homotopic to f is used, for if h is another t-regular map homotopic to f, then there is a t-regular homotopy F between them, and the pair defined by F in $L_q(I \times X)$ defines a relation between the pairs defined by h and g. Thus there is a well defined function

$$T\colon [X, M_q] \to \Lambda_q(X) .$$

One easily checks that T is compatible with the homotopy identification $M_E(G_q) \to M_{E'}(G_q)$ for E, E' admissible neighborhoods. The next theorem corresponds to Thom's theorem [13, IV. 6].

THEOREM 4.8.1. *For any locally finite simplex X, the function T is a one-to-one correspondence* $[X, M(\mathrm{SPL}_q)] \to \Lambda_q(X, \mathrm{SPL})$. *Furthermore, if $X_0, X_1, \cdots$ is an s-sequence, then T induces an isomorphism* $\{X, \mathrm{SPL}\}_q \cong \Lambda_q\{X, \mathrm{SPL}\}$.

COROLLARY. $\pi_n(M\mathrm{SPL}) \cong \Omega_n^{PL}$.

The same theorem holds for unoriented L-equivalence, with SPL replaced by PL. Indeed the theorem holds for any subgroup of PL_q of the type described in § 2.2, except that for the stable group $\Lambda_q\{X, G\}$ to be defined, one must suppose that the injection $\mathrm{PL}_q \to \mathrm{PL}_{q+1}$ carries G_q into G_{q+1}.

PROOF OF 4.8.1. T is onto. If $(Y, \mathfrak{n}) \in L_q(X, \mathrm{SPL})$, then there is an orientation preserving germ $\mathfrak{n} \Rightarrow \mathfrak{u}(\mathrm{SPL}_q)$. Thus there is a PL map $f\colon U \to M(\mathrm{SPL}_q)$, where U is a neighborhood of Y in X, and since f sends the boundary of U into $M(\mathrm{SPL}_q) - B(\mathrm{SPL}_q)$ it extends to a map $X \to M(\mathrm{SPL}_q)$: this map is t-regular and clearly determines the pair $(Y, \mathfrak{n})$.

T is one-to-one. Suppose f, f' are T-regular maps such that $T[f] = T[f']$.

Suppose f defines the pair $(Y, \mathfrak{n})$ and f' the pair $(Y', \mathfrak{n}')$. Since these pairs are equivalent there is a pair $(Z, \mathfrak{m})$ as in § 4.1. Then the bundle map germs $\mathfrak{n} \Rightarrow \mathfrak{u}(\mathrm{SPL}_q)$, $\mathfrak{n}' \Rightarrow \mathfrak{u}(\mathrm{SPL}_q)$ determined by f, f' extend to a bundle map germ $\mathfrak{m} \Rightarrow \mathfrak{u}(\mathrm{SPL}_q)$, and a map $g: U \rightarrow M(\mathrm{SPL}_q)$, U a neighborhood of Z in $I \times X$, that embeds fibers extends to a t-regular map $I \times X \rightarrow M(\mathrm{SPL}_q)$ that is a homotopy between f and f'. This completes the first part of the theorem. One can easily verify from the definitions of § 4.3 and § 4.7 that the Thom map T is compatible with suspension and addition, so that it does define a group isomorphism in the limit. This completes Theorem 4.8.1.

Note that if X is finite dimensional, of dimension $n + q$ say, then it is not necessary that $\mathfrak{u}(\mathrm{SPL}_q)$ be universal; it would suffice for it to be $n + 1$-universal. Then $\mathrm{B}(\mathrm{SPL}_q)$, hence $E(\mathrm{SPL}_q)$, can be supposed finite dimensional and $M(\mathrm{SPL}_q)$ can be taken to be a finite dimensional complex.

It is an immediate consequence of the theorem that $\pi_r(M(\mathrm{SPL}_q)) = \pi_r(M(\mathrm{PL}_q)) = 0$ for $r < q$ since the r-sphere cannot contain a non-empty subspace with a normal q-dimensional bundle. If $r = q$, then the subspace could only be a finite union of points, so $\pi_q(M(\mathrm{PL}_q)) = Z_2$ and $\pi_q M(\mathrm{SPL}_q)) = Z$.

5.1. Now we want to consider the relation between Ω_n^{PL} and the corresponding group Ω_n of C^∞-manifolds. According to J.H.C. Whitehead [12], [16], every C^1-manifold admits a compatible triangulation as a PL manifold, and this PL manifold is unique to within a PL homeomorphism. This defines a homomorphism $\Omega_n \rightarrow \Omega_n^{\mathrm{PL}}$, which we call the Whitehead homomorphism. Both Stiefel-Whitney and rational Pontrjagin classes can be defined for microbundles [11, p. 6] so the corresponding numbers can be defined for PL manifolds, and will be cobordism invariants, by the same argument that shows them to be invariants in the C^∞ case. Since these numbers characterize a class in Ω^n completely,

THEOREM 5.1. *$\Omega_n \rightarrow \Omega_n^{\mathrm{PL}}$ has kernel* 0.

Let $M\mathrm{SO}(k)$ be the Thom complex for $\mathrm{SO}(k)$. According to Lashoff and Rothenberg [8], or in the stable case, which is all we need, Milnor [11], the universal $\mathrm{SO}(k)$ vector bundle γ_k can be triangulated, that is, there is a PL microbundle $\mathfrak{w}$ and a fiber preserving homeomorphism $E(\gamma_k) \approx E(\mathfrak{w})$, at least for neighborhoods of the respective base spaces, and this even provides a compatible triangulation of $E(\gamma_k)$. A classifying map $\mathfrak{w} \rightarrow \mathfrak{u}(\mathrm{SPL}_k)$ can be composed with the homeomorphism to give a map $h: M\mathrm{SO}(k) \rightarrow M\mathrm{SPL}_k$.

LEMMA 5.2. *The following diagram is commutative:*

$$\begin{array}{ccc} \Omega_n & \overset{T}{\cong} & \pi_n(MSO) \\ {\scriptstyle w}\big\downarrow & & \big\downarrow \\ \Omega_n^{\mathrm{PL}} & \overset{T_{\mathrm{PL}}}{\cong} & \pi_n(MSPL)\ . \end{array}$$

The rows are Thom isomorphisms, the top one is that of [14] and the lower one has been established in this paper. w is the Whitehead homomorphism.

PROOF. Let M^n represent an element of Ω^n. One can embed M^n in some S^{n+k}. Let ν be the normal bundle; then relative to some Riemann metric the exponential map is a homeomorphism $e: E(\nu) \to S^{n+k}$ onto an open neighborhood of M^n. The element of $\pi_{n+k}(MSO(k))$ associated to M^n by T is an extension over S^{n+k} of e^{-1} followed by a classifying map for ν. According to [1, p. 143] any open set of S^{n+k} can be triangulated so that the inclusion is linear, so there is a complex K and a homeomorphism $\psi: K \to E(\nu)$ such that $e\psi$ is linear. One can see that ψ is a C^1-triangulation of the manifold $E(\nu)$. Let M^n be given a C^1-triangulation. According to Lashof and Rothenberg, there is a C^1-triangulation $\varphi: E(\mathrm{n}) \to E(\nu)$ for some mircobundle n, and according to Whitehead (see Munkres [12, Th. 10.5]), there are C^1-triangulations $\varphi_0: E(\mathrm{n}) \to E(\nu)$, $\psi_0: K \to E(\nu)$ such that $\psi_0^{-1}\varphi_0$ is PL. Then $e\psi\psi_0^{-1}\varphi_0 = g: E(\mathrm{n}) \to S^{n+k}$ is a PL homeomorphism onto a neighborhood of $g(M^n)$ and therefore defines a normal microbundle. According to our definition the class of the PL manifold M^n in Ω^n_{SPL} is associated by T_{PL} to the homotopy class of an extension over S^{n+k} of g^{-1} followed by a classifying map. Thus, in order to prove the theorem, it suffices to prove that the following diagram is homotopy commutative. The maps on the left are actually extensions of the maps indicated.

$$\begin{array}{ccccc} S^{n+k} & \xrightarrow{e^{-1}} & T(\nu) & \xrightarrow{\text{class.}} & MSO(k) \\ & {\scriptstyle g^{-1}}\searrow & \big\downarrow{\scriptstyle \varphi^{-1}} & & \big\downarrow \\ & & T(\mathrm{n}) & \xrightarrow{\text{class.}} & MSPL_k\ . \end{array}$$

Commutativity of the right square holds because triangulation is functorial. According to Munkres [12, Th. 10.5], one can choose φ_0 and ψ_0 arbitrarily close to φ and ψ respectively. So we can suppose $\varphi_0 \simeq \varphi$, $\psi_0 - \psi$, furthermore that the homotopies are proper maps, and similarly for the inverses. It follows that $g^{-1} \simeq \varphi^{-1}e^{-1}$, and the homotopy can be chosen so that it can be extended over S^{n+k}.

5.2. Let $\{M_k\}$ be one of the spectra $\{MPL_k\}$, $\{MSPL\}$ of § 4.7. We can define, as in § 4.7

$$(1) \qquad \begin{aligned} \pi_n(M) &= \lim \pi_{n+k}(M_k) \\ H_n(M) &= \lim H_{n+k}(M_k)\ . \end{aligned}$$

We need to know that theorems such as that of Hurewicz hold for these groups. Ordinarily one considers a k so large that $\pi_{n+k}(M_k) \cong \pi_{n+k+1}(M_{k+1})$, and $H_{n+k}(M_k) \cong H_{n+k+1}(M_{k+1})$ for all n under consideration. Unfortunately, for our spectra we do not know that such a k exists for any $n > 0$. We can overcome this difficulty as follows, using some work of Kan [7].[1]

We might as well transfer the problem to the category of complete semi-simplicial complexes by means of the singular complex, so we shall use M_k to mean the singular complex of the appropriate space. Kan calls the sequence $M\colon k \longrightarrow M_k$, together with the maps $SM_k \longrightarrow M_{k+1}$ a *prespectrum*, and associates to it an object SpM much like a c.s.s. complex called a *spectrum*. Kan defines appropriate categories $\mathcal{P}_s$ and $\mathcal{S}_p$. We shall use this terminology for the remainder of this section. Since the results we want are not stated explicitly in [7] we shall describe briefly how they can be obtained.

We can define the homology groups of a spectrum $X \in \mathcal{S}_p$ as follows. Let AX be the free abelian group spectrum generated by X, as in definition 5.2 of [7] with "abelian" added. This becomes in a natural way a function $A\colon \mathcal{S}_p \longrightarrow \mathcal{S}_{p_A}$. We define

$$H_n(X) = \pi_n(AX) .$$

LEMMA. *The homology groups satisfy* (1), i.e.,

$$H_n(M) = \lim H_{n+k}(M_k) = H_n(\mathrm{Sp}M) .$$

PROOF. According to Proposition 4.6 and Definition 10.1 of [7], the left side is $\pi_n(\mathrm{Sp} \circ A(M))$, since $\pi_n(AX) = H_n(X)$ holds for c.s.s. complexes. The right side is $\pi_n(A \circ \mathrm{Sp}(M))$, and one checks from the definition that $A \circ \mathrm{Sp} = \mathrm{Sp} \circ A$.

Kan also defines a function Ps: $\mathcal{S}_p \longrightarrow \mathcal{P}_s$.

LEMMA. *If X is a spectrum*, $H_n(X) = H_n(\mathrm{Ps}X)$.

The proof is as above, using $\mathrm{Ps} \circ A = A \circ \mathrm{Ps}$.

As a result we have a functor $\mathrm{Ps} \circ \mathrm{Sp}\colon \mathcal{P}_s \longrightarrow \mathcal{P}_s$ which carries M into a prespecturm $\tilde{M}$ that is weakly equivalent to M as regards both homotopy and homology and has the following stability property: for any $N > 0$, there is a k such that $\pi_{n+k}(M_k) \cong \pi_{n+k+1}(M_{k+1})$ is an isomorphism for $n < N$ and $k > K$ (so the same holds for homology). The stability property follows from the fact that $\tilde{M}_k$ is the loop space on $\tilde{M}_{k+1}$, as one can verify directly. Now we can state a theorem that is well known [13, Th. 1] in a slightly different context.

THEOREM. *Let C be a class of abelian groups, and suppose $\pi_i(M) \in C$ for $i < N$. Suppose further that the stable group $\pi_{n+k}(S^n)$ is in C for $k \leqq K$. Then the Hurewize map $\pi_k(M) \longrightarrow H_k(M)$ is a C-isomorphism for $k \leqq N + k$.*

[1] *Added in proof.* Wall and Haefliger have recently proved stability, so § 5.2 is unnecessary.

PROOF. By the preceding remarks, it suffices to prove this for a stable spectrum, and therefore for a single c.s.s. complex in the stable range. The corresponding theorem for finite CW complexes follows from [13, Th. 1] when it is noted that the assumption on C made there can be weakened. The case of a c.s.s. complex can be reduced to this by using the realization functor.

5.3. We can now obtain some specific information about Ω_n^{PL}. This depends on the work of Kervaire and Milnor on the group Γ of differential structures on S^n [6] and on a theorem of Cerf that $\Gamma_4 = 0$, the proof of which has not yet appeared in full detail (see [3]). The following theorem of Hirsch [4] and Mazur (unpublished) is essential: there is a short exact sequence:

$$0 \longrightarrow \pi_n(B_{\mathrm{SO}}) \xrightarrow{p_\#} \pi_n(B_{\mathrm{SPL}}) \longrightarrow \Gamma_{n-1} \longrightarrow 0 ,$$

where $p\colon B_{\mathrm{SO}} \to B_{\mathrm{SPL}}$ is the classifying map of [11]. Hirsch and Mazur state the result for B_{PL} and B_{O} but the proof is the same. In the introduction we have listed some results from this, and the rest of this section will be given over to establishing them.

It follows from our results, [4] and [14], that there is a commutative diagram with exact rows, (A).

$$\begin{array}{ccccccccc}
 & & & & & & \Gamma_{n-1} & & \\
 & & & & & & \wr\Vert & & \\
0 & \longrightarrow & \pi_n(B_{\mathrm{SO}}) & \xrightarrow{p_\#} & \pi_n(B_{\mathrm{SPL}}) & \longrightarrow & \pi_n(B_{\mathrm{SPL}}, B_{\mathrm{SO}}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \text{Hurewicz}} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_n(B_{\mathrm{SO}}) & \xrightarrow{p^*} & H_n(B_{\mathrm{SPL}}) & \longrightarrow & H_n(B_{\mathrm{SPL}}, B_{\mathrm{SO}}) & \longrightarrow & 0 \\
 & & \wr\Vert\ \text{Thom} & & \wr\Vert & & \wr\Vert & & \\
0 & \longrightarrow & H_n(M\mathrm{SO}) & \longrightarrow & H_n(M\mathrm{SPL}) & \longrightarrow & H_n(M\mathrm{SPL}, M\mathrm{SO}) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \pi_n(M\mathrm{SO}) & \longrightarrow & \pi_n(M\mathrm{SPL}) & \longrightarrow & \pi_n(M\mathrm{SPL}, M\mathrm{SO}) & \longrightarrow & 0 \\
 & & \wr\Vert & & \wr\Vert & & \wr\Vert & & \\
0 & \longrightarrow & \Omega_n & \longrightarrow & \Omega_n^{\mathrm{PL}} & \longrightarrow & \Omega_n^{\mathrm{PL}}/\Omega_n & \longrightarrow & 0 .
\end{array}$$

It follows immediately from the relative Hurewicz theorem modulo the class of finite group that $\Omega_n^{\mathrm{PL}}/\Omega_n$ is finite, and 0 for $n \leqq 7$. One also concludes that $H_8(B_{\mathrm{SPL}}, B_{\mathrm{SO}}) \cong Z_7 + Z_4$. Recall that $H_8(B_{\mathrm{SO}}) \cong Z + Z + Z_2 + \cdots + Z_2$, where $Z + Z$ is generated by classes dual to p_1^2 and p_2, the universal Pontrjagin classes.

LEMMA (Milnor). *$H_8(B_{\mathrm{SPL}}) \cong Z + Z + Z_4 + Z_2 + Z_2 + Z_2 + Z_2$, where p_* is an isomorphism on $Z_2 + Z_2 + Z_2 + Z_2$ and the subgroup generated by the dual of p_1^2, and sends the dual of p_2 onto 7 times a generator.*

PROOF. That p_* is an isomorphism on $Z_2 + Z_2 + Z_2 + Z_2$ which is generated by classes that reduce to duals of Stiefel-Whitney classes is clear, since these are defined on B_{SPL}. According to [10, § 8], there is for each k a $4k$-dimensional smooth manifold W whose boundary is PL homeomorphic to a sphere. One forms the cone $C(\partial W)$ over this sphere to obtain a closed PL manifold $M = W \cup C(\partial W)$. Milnor shows that p_k, which is the only non-vanishing Pontrjagin class, is given by

$$p_k[M] = \frac{k(2k-1)!}{2^{2k-4}B_n(2^{2k-1}-1)},$$

where B_n is the n^{th} Bernoulli number. For $k = 2$, $p_2[M] = 360/7$. Thus p_* carries the dual of p_2 into an element divisible by 7. Once we know that the Z_4 of $H_8(B_{\mathrm{PL}}, B_{\mathrm{O}})$ splits off we will know that this element is 7 times a generator and we will be done. Milnor demonstrates the splitting as follows. There is a classifying space $B_{H(n)}$ for fiber spaces whose fiber has the homotopy type of S^n and a stable classifying space $B_H = \lim B_{H(n)}$. One can associate a fiber space over $B_{\mathrm{PL}(n)}$ to the universal microbundle as follows. The fiber over a point $x \in B_{\mathrm{PL}_n}$ consists of paths $\lambda\colon I \to E(\mathrm{PL}_n)$ such that $\lambda(x) = 0$, $\lambda(t) \notin B_{\mathrm{PL}_n}$ for $t > 0$. Similarly for $B_{\mathrm{O}(n)}$ and the fiber space has the fiber homotopy type of the universal sphere bundle. There is then a commutative diagram induced by classifying maps,

$$\begin{array}{ccccccc}
\pi_{k-1}(\mathrm{SO}(n)) \cong \pi_k(B_{\mathrm{SO}(n)}) & \longrightarrow & \pi_k(B_{\mathrm{SPL}_n}) & \longrightarrow & \Gamma_{n-1} \\
\searrow \qquad\qquad \searrow & & \downarrow & & \\
\pi_{n-k-1}(S^{n-1}) & \cong & \pi_k(B_{H(n-1)}) . & &
\end{array}$$

The classifying map for $B_{\mathrm{SO}(n)}$ is essentially the J-homomorphism J, which in the stable range for $k = 8$ is $Z \to Z_{240}$. Since 4 divides 240, one can conclude by applying $\otimes Z_4$ to the above diagram that Z_4 splits off in the sequence

$$0 \longrightarrow \pi_8(B_{\mathrm{SO}}) \longrightarrow \pi_8(B_{\mathrm{SPL}}) \longrightarrow Z_7 + Z_4 \longrightarrow 0 .$$

The lemma follows from an application of the Hurewicz homomorphism.

THEOREM. $\Omega_8^{\mathrm{PL}} \cong Z + Z + Z_4$.

PROOF. By (A) and the Hurewicz theorem $\Omega_8^{\mathrm{SPL}}/\Omega_8 \cong Z_7 + Z_4$, and by the existence of the M described earlier, or by the lemma above, the Z_7 does not split. The Z_4 does split, as we now show. Consider the homomorphism $\Omega_8 \to \mathrm{Hom}(H^8(B_{\mathrm{SO}}), Z)$ that assigns to the cobordism class of M the homomorphism sending a characteristic class into the corresponding number of the tangent bundle of M. By considering $P_2(\mathrm{C}) \times P_2(\mathrm{C})$, and $P_4(\mathrm{C})$, one sees this induces an isomorphism $\Omega_8 \otimes Z_4 \cong \mathrm{Hom}\,(H^8(B_{\mathrm{SO}}), Z) \otimes Z_4$. The composite of the homomorphism with the injection $\mathrm{Hom}\,(H^8(B_{\mathrm{SO}}), Z) \to H_8(B_{\mathrm{SO}})$ sends M into $f_*[M]$, where $[M]$ is the fundamental class and $f\colon M \to B_{\mathrm{SO}}$ is a classifying map for the tangent bundle. Thus we have a commutative diagram

$$\begin{array}{ccccc} \Omega_8 \otimes Z_4 & \longrightarrow & \Omega_8^{\mathrm{PL}} \otimes Z_4 & & \\ \downarrow & & \downarrow & \searrow & \\ & & & & Z_4 \longrightarrow 0 \, . \\ & & & \nearrow & \\ H_8(B_{\mathrm{SO}}) \otimes Z_4 & \longrightarrow & H_8(B_{\mathrm{SPL}}) \otimes Z_4 & & \end{array}$$

From this we see that $\Omega_8 \otimes Z_4 \rightarrow \Omega_8 \otimes Z_4$ has kernel 0, and it follows that Z_4 splits off.

Now we consider the 7 torsion of $\Omega_n^{\mathrm{PL}}/\Omega_n$. For G a group and p a prime we shall denote by pG the subgroup of elements whose order is a power of p.

An examination of the Pontrjagin numbers of the products $P_2(\mathrm{C}) \times M$, $P_4(\mathrm{C}) \times M$ and $M \times M$ shows that these generate subgroups isomorphic to Z_7 in $\Omega_{12}^{\mathrm{PL}}/\Omega_{12}$ and $Z_7 \oplus Z_{49}$ in $\Omega_{16}^{\mathrm{PL}}/\Omega_{16}$, and these groups do not split off. We shall show that there is no other 7 torsion in $\Omega_n^{\mathrm{PL}}/\Omega_n$; this will prove:

THEOREM. ${}_7(\Omega_8^{\mathrm{PL}}/\Omega_8) \cong {}_7(\Omega_{12}^{\mathrm{PL}}/\Omega_{12}) \cong Z_7$ *and* ${}_7(\Omega_{16}^{\mathrm{PL}}/\Omega_{16}) \cong Z_7 \oplus Z_{49}$. *For* $n \leqq 18$, ${}_7(\Omega_n^{\mathrm{PL}}/\Omega_n) \cong 0$ *otherwise. None of this* 7 *torsion splits*, i.e., ${}_7\Omega_n^{\mathrm{PL}} \cong 0$ *for* $n \leqq 18$.

PROOF. Applying a theorem of § 4.10, we observe that the Hurewicz homomorphism of stable groups $\pi_{8+k}(M\mathrm{SPL}, M\mathrm{SO}) \rightarrow H_{8+k}(M_{\mathrm{SPL}}, M\mathrm{SO})$ is an isomorphism modulo groups of order prime to 7 for $k \leqq 10$ since the stable k-stem of the sphere has no torsion for $k \leqq 10$. Thus for $n \leqq 18$, ${}_7(\Omega_n^{\mathrm{PL}}/\Omega_n) \cong {}_7H_n(B_{\mathrm{SPL}}, B_{\mathrm{SO}})$, using (A), so we determine this group.

We can regard $p: B_{\mathrm{SO}} \rightarrow B_{\mathrm{SPL}}$ as a fiber space projection whose fiber we may denote by Γ, see [8]. It follows from (A) that $\pi_n(\Gamma) \cong \Gamma_n$. Since Z_7 in Γ_7 is the only 7 torsion in Γ_n for $n \leqq 18$, $\pi_n(\Gamma) \cong \pi_n(K(Z_7, 7))$ modulo the class of finite groups whose orders are prime to 7, for $n \leqq 18$. Thus $H^n(\Gamma; Z_7) \cong H^n(Z_7, 7; Z_7)$ for $n \leqq 18$, and these groups are described in [2]. Now consider the spectral sequence of the fibering with coefficients Z_7, so $E_2^{p,q} \cong H^p(B_{\mathrm{SPL}}, H^q(\Gamma; Z_7))$. Now one can verify by a standard spectral sequence argument that $H^n(B_{\mathrm{SPL}}, B_{\mathrm{SO}}; Z_7)$ has rank (over Z_7) 1, 1, 1, 1, 2, 2 for $n = 8, 9, 12, 13, 16, 17$ and is 0 otherwise. It follows immediately that ${}_7H^n(B_{\mathrm{SPL}}, B_{\mathrm{SO}}) \otimes Z_7$ has rank 1, 1, 2 for $n = 9, 13$, 17 and is 0 otherwise, always considering $n \leqq 18$.

Now consider the 7 torsion that appears in the spectral sequence of the fibering with integral coefficients. Using the information obtained from the mod 7 sequence, one concludes that ${}_7H^{13}(B_{\mathrm{SPL}}, B_{\mathrm{SO}})$ must be Z_7 and ${}_7H^{17}(B_{\mathrm{SPL}}, B_{\mathrm{SO}})$ must be $Z_7 \oplus Z_{49}$. So ${}_7H_k(B_{\mathrm{SPL}}, B_{\mathrm{SO}})$ is as required by the theorem.

We have now proved the assertions made in the list concerning 2 and 7 torsion. The torsion for the primes 31 and 127 is determined in the same way.

In each case, Milnor has constructed manifolds whose Pontrjagin numbers must be multiplied by the corresponding prime to be integral, and products give all the torsion involving the prime, as one shows by a corresponding spectral sequence argument.

A slightly longer argument is required to obtain the results concerning the 3 torsion. Let $\mathcal{C}$ be the class of finite groups whose orders are not divisible by 3; all the isomorphisms in this section will hold mod $\mathcal{C}$. According to [6], $\Gamma_{10} \cong Z_3$, $\Gamma_{13} \cong Z_3$ and $\Gamma_n \cong 0$ for other $n \leqq 18$, (mod $\mathcal{C}$). Since B_{SPL} is 3-connected, we can immediately conclude from diagram (A) that $\Omega_{11}^{\mathrm{PL}}/\Omega_{11} \cong Z_3$, $\Omega_{12}^{\mathrm{PL}}/\Omega_{12} \cong 0$, $\Omega_{13}^{\mathrm{PL}}/\Omega_{13} \cong 0$, but we will not use this fact.

The results on Γ_n show that Γ has the cohomology of a two stage Postnikov system up to dimension 18, so we must determine the k-invariant $k \in H^{14}(Z_3, 10; Z_3)$ associated to it. Since $\pi_{10}\mathrm{SO} \cong \pi_{13}\mathrm{SO} \cong 0$ and $bP_{11} \cong bP_{14} \cong 0$, we have a diagram

$$\begin{array}{ccccc}
\pi_{10}\Gamma & \xrightarrow{\cong} & \Gamma_{10} & \xrightarrow{\cong} & \pi_{10} \\
\alpha_1 \downarrow & & & & \downarrow \alpha_1 = \\
\pi_{13}\Gamma & \xrightarrow{\cong} & \Gamma_{13} & \xrightarrow{+} & \pi_{13}
\end{array}$$

where α_1 refers to the homotopy operation got by composing with a generator of π_3. The diagram commutes, although I do not think that is obvious. This fact will follow from the results of a paper in preparation by the author. Given this, it follows that $\alpha_1 \colon \pi_{10}\Gamma \cong \pi_{13}\Gamma$. Now Γ has the homotopy type of a fiber space over $K(Z_3, 10)$ with fiber $K(Z_3, 13)$, and the k invariant determining the space must yield a non-trivial homotopy operation α_1, so $k = \pm P_3^1\omega_1$, if we let ω_1 be the fundamental class in $H^{10}(Z_3, 10; Z_3)$, where P_3^n is the Steenrod cube of degree $4n$. The two possibilities yield spaces of the same homotopy type, so we may suppose $k = P_3^1\omega_1$. One can now examine the cohomology spectral sequence for the fibering. Using the fact that transgression commutes with the P_3^i, and using the Adem relations $P^1P^1 = -P^2$ and $P_3^1 = \beta P_3^1 = \beta P_3^2 + P_3^2\beta$, where β is the Bockstein coboundary, one obtains the following results.

LEMMA. *$H^*(\Gamma; Z_3)$ has as a vector space basis through dimension* 18 *the following elements*: $\omega_1 \in H^{10}\Gamma$, $\beta\omega_1 \in H^{11}\Gamma$, $P_3^1\beta\omega_1 \in H^{15}\Gamma$, $\beta P_3^1\beta\omega_1 \in H^{16}\Gamma$.

It is clear the groups $H^n(\Gamma, Z)$ will be (mod $\mathcal{C}$) $H^{11}(\Gamma, Z) \cong Z_3$, $H^{16}(\Gamma, Z) \cong Z_3$, $H^n(\Gamma, Z) \cong 0$ for other $n \leqq 18$.

Next we use the results on $H^*\Gamma$ to examine the spectral sequence for the fibering $B_{\mathrm{SO}} \to B_{\mathrm{SPL}}$, with fiber Γ. We first observe that the Wu classes for $p = 3$ coincide with the reductions mod 3 of the Pontrjagin classes. Since these are defined in B_{SPL} as well as in B_{SO}, it follows that $H^*(B_{\mathrm{SPL}}, Z_3) \to H^*(B_{\mathrm{SO}}, Z_3)$

is surjective and injective on the space spanned by the Wu classes.

The reader can verify that an examination of the spectral sequence for $B_{SO} \longrightarrow B_{SPL}$, using the standard properties of cohomology spectral sequences, that $H^*(B_{SPL}; Z_3)$ is as follows.

LEMMA. *$H^*(B_{SPL}; Z_3)$ has the following vector space basis up to dimension 18: Wu classes, which are the reductions of corresponding Pontrjagin classes defined in $H^*(B_{SPL}, Z)$, and classes*

$$\begin{aligned} \mu &\in H^{11}(B_{SPL}, Z_3) \\ \beta\mu &\in H^{12}(B_{SPL}, Z_3) \\ p_1 \cup \mu &\in H^{15}(B_{SPL}, Z_3) \\ p_1 \cup \beta\mu &\in H^{16}(B_{SPL}, Z_3) \\ P_3^1\beta\mu &\in H^{16}(B_{SPL}, Z_3) \\ \beta P_3^1\beta\mu &\in H^{17}(B_{SPL}, Z_3) . \end{aligned}$$

It follows that $H^*(B_{SPL}, Z)$ has exactly the following 3-torsion through dimension 18: ${}_3H^{12}(B_{SPL}) \cong Z_3$, ${}_3H^{16}(B_{SPL}) \cong Z_3$, ${}_3H^{17}(B_{SPL}) \cong Z_3$.

We turn to the determination of Ω_n^{PL}. Let φ: $H^n(B_{SPL}) \cong H^n(M\text{SPL})$ be the Thom isomorphism; we shall usually have in mind Z_3 coefficients. We have to determine the action of the Sq_3^i on $H^*(M\text{SPL})$. This we can do using the fact that the Pontrjagin classes coincide with the Wu classes $\varphi^{-1}P_3^i\varphi(1)$ for the prime 3 together with the formula $\varphi(x) = \pi^*x \cup U$, where π: $E_{SPL} \longrightarrow B_{SPL}$ is the projection of the universal bundle and U is the Thom class of the universal microbundle. Using these facts and applying the Adem relations and Cartan formula one verifies:

LEMMA. *$H^*(M\text{SPL}, Z_3)$ has the following vector space basis through dimension 18:*

$$\begin{aligned} \varphi\mu &\in H^{11}M\text{SPL} \\ \beta\varphi\mu &\in H^{12}M\text{SPL} \qquad \textit{and } \varphi \textit{ (Pontrjagin classes)} \\ \varphi(p_1 \cup \mu) &\in H^{15}M\text{SPL} \\ \beta\varphi(p_1 \cup \mu),\ \varphi P_3^1\beta\mu &\in H^{16}M\text{SPL} \end{aligned}$$

From this, we conclude that the fibering $M\text{SO} \longrightarrow M\text{SL}$ is the product fibering with fiber $K(Z_3, 11)$, in the stable sense. We recall that this means that $M\text{SO}(n) \longrightarrow M\text{SPL}(n) \times K(Z_3, 11 + n)$ induces maps on the $n + r^{\text{th}}$ homotopy and homology groups which in the limit as n increases, are isomorphisms for $r \leqq 18$. It follows that Ω_n^{PL}/Ω_n has no 3 torsion for $n \leqq 18$ except for $\Omega_n^{PL}/\Omega_{11} \cong Z_3$.

We next determine Ω_n^{PL}, which we know must be a 2 group. We proceed just as in the determination of the 3 torsion. It is first necessary to determine the effect of composition with $\eta \in \pi_1$ on $\pi_7\Gamma$. There is a homomorphism $\pi_7/\text{Im } J \to \pi_8/\text{Im } J$ induced by η and we have a diagram

$$\begin{array}{ccc} \pi_7\Gamma & \longrightarrow & \pi_7/\text{Im } J = 0 \\ \eta\big\downarrow & & \big\downarrow\eta \\ \pi_8\Gamma & \xrightarrow{\cong} & \pi_8/\text{Im } J \end{array}$$

which is commutative. See the paragraph on 3-torsion above. It follows that the composition with η is the 0 homomorphism $\pi_7\Gamma \to \pi_8\Gamma$. It follows that the first k invariant of Γ is 0. One then concludes from an examination of the spectral sequences involved the following results (they could be obtained somewhat more directly in this case). There is a basis for $H^8(B_{\text{SPL}}; Z_2)$ consisting of Stiefel-Whitney classes, and a PL class m_3^1, and there is a basis for $H^9(B_{\text{SPL}}; Z_2)$ consisting of Stiefel-Whitney classes, a class related to m_3^1 by a Bockstein operation of higher order, and a new class m_3^2, $\beta m_3^2 \neq 0$. One concludes from this that $\Omega_9^{\text{PL}}/\Omega_9 = Z_2$, corresponding to the class m_3^2.

INSTITUTE FOR ADVANCED STUDY

REFERENCES

1. P. ALEXANDROFF and H. HOPF, Topologie, Springer, Berlin, 1935.
2. H. CARTAN, Seminaire 1954–1955, E. N. S., Paris.
3. ———, Seminaire 1961–63, E. N. S., Paris.
4. M. HIRSCH, *Obstruction theories for smoothing manifolds and maps,* Bull. Am. Math. Soc., 69 (1963).
5. ———, and B. MAZUR, Smoothings of piecewise linear manifolds, Cambridge Univ., 1964 (mimeographed).
6. M. KERVAIRE and J. MILNOR, *Groups of homotopy spheres:* I, Ann. of Math., 77 (1963), 504–536.
7. D. KAN, *Semi-simplicial spectra,* Illinois J. Math., 7 (1963), 463–478.
8. R. LASHOF and M. ROTHENBERG, Microbundles and smoothing (mimeographed and to appear).
9. J. MILNOR, Lectures on characteristic classes, Princeton Univ., 1957 (mimeographed).
10. ———, *Microbundles:* I, Topology, 3 supp. 1 (1964), 53–80.
11. ———, Microbundles and differentiable structures, Princeton University, 1961 (mimeographed).
12. J. MUNKRES, Elementary Differential Topology, Princeton Univ. Press, Princeton, N. J., 1963.
13. E. SPANIER, Symposium Internacional de Topologia Algebraica, Universidad Nacional Autonoma de Mexico, 1958 (259–272).
14. R. THOM, *Quelque propriétés globales des variétés differentiables,* Comment. Math. Helv., 28 (1954), 17–86.
15. C. T. C. WALL, *Cobordism of combinatorial n-manifolds for* $n \leqq 8$, Proc. Cambridge Philos. Soc., 60 (1964), 807–812.
16. J. H. C. WHITEHEAD, *On* C^1*-complexes,* Ann. of Math., 41 (1940), 809–814.

(Received December 9, 1964)
(Revised February 5, 1965)