A USEFUL FUNCTOR AND THREE FAMOUS EXAMPLES IN TOPOLOGY(1)

BY

R. F. WILLIAMS

The purpose of this note is to describe a functor which provides a framework for certain constructions in topology. It is related to the sets (E, π, B, X, q) described in [7] and is particularly adapted to discussing the limit of repeated modifications of triangulable spaces. Roughly speaking, one forms a space $X\Delta K$ by replacing each top dimensional simplex of a complex K with a copy of a space X. If in addition there are mappings on the spaces X, K, these induce a mapping on the new space $X \Delta K$.

It has been called to my attention that several authors have considered analogous functors (though not as far as I know, in written form). This is not surprising inasmuch as $X\Delta K$ is defined just as the Whitney sum of two bundles.

Though the principal applications of this functor are to be found elsewhere, in a paper by Frank Raymond and the author [8; 7] and a forthcoming paper by the author, three famous examples, Pontrjagin [6], Boltyanskii [2], and Kolmogoroff [5] are given as applications in the last section. Two of these are in dimension theory proper, but the third is essentially about transformation groups. This was pointed out by Professor Deane Montgomery whom the author would like to thank for his considerable aid and encouragement.

It is hoped that the reader will find our description of Boltyanskii's example easier than the original, as a simpler, more homogeneous version is given. In addition, in our version of Kolmogoroff's example, the group acts without fixed points. This answers a question raised by Anderson [1].

1. **Definitions.** Throughout this section n is a fixed integer $\geqq 0$ and s is the standard n-simplex with vertices $v_0, v_1, \cdots, v_n$. s will be regarded as a complex and as being closed when thought of as a space. Otherwise all simplexes, σ, τ, ρ are taken as open and $\sigma \in K$ means σ is a simplex of K. The dimension of a complex K is the maximum of the dimensions of the simplexes of K.

For an n-complex K there is a natural map $\phi_K : K' \to s$, where K' is the barycentric subdivision of K and ϕ_K sends $b(\sigma^i)$ (the barycenter of an i-simplex of K) into the ith vertex, $v_i \in s$. This extends uniquely to the simplicial map ϕ_K.

Received by the editors February 15, 1962.

(1) National Science Foundation Fellow, 1959–1961 This research was supported in part by contract No. NSF G-10369.

Note that $\phi_K \,|\, \bar{\sigma}$ is a homeomorphism for each $\sigma \in K$. Also that if L is another n-complex and $t: K \to L$ is a map which is

(M2) *simplicial and collapses no simplex*

then

$$\begin{tikzcd} K \arrow[rr, "t"] \arrow[dr, "\phi_K"'] & & L \arrow[dl, "\phi_L"] \\ & s & \end{tikzcd}$$

is commutative.

Now suppose

(01) *X is a topological space and $g: X \to s$.*

Define $X\Delta_q K = \{(x,k) \in X \times K : qx = \phi_K k\}$. This will usually be written simply $X\Delta K$. Then for $t: K \to L$ satisfying (M2) we define

$$1\Delta t : X\Delta K \to X\Delta L$$

by $(1\Delta t)\,(x,k) = (x, tk)$. Note that $(1\Delta u)\,(1\Delta t) = 1\Delta ut$, where defined.

The identity map $1: s \to s$ gives $s\Delta K \approx K'$ and we will identify these spaces via the map $(x,k) \to k$. Now consider the diagram

$$\begin{tikzcd} X \arrow[dd, "f"'] \arrow[dr, "q"] & X\Delta K \arrow[l] \arrow[r] & K \arrow[dl, "\phi_K"'] \arrow[dd, "t"] \\ & s & \\ Y \arrow[ur, "r"'] & Y\Delta L \arrow[l] \arrow[r] & L \arrow[ul, "\phi_L"] \end{tikzcd}$$

If the triangle on the left is commutative then there will clearly be an induced map $f\Delta t: X\Delta K \to Y\Delta L$. But this is too great a restriction upon the map f and we require only that

(M1) *$rfq^{-1}(\bar{\sigma}) \subset \bar{\sigma}$, for each simplex $\sigma \in s$.*

Given this, define

$$f\Delta 1 : X\Delta K \to Y\Delta K,$$

by

$$(f\Delta 1)(x,k) = (fx, (\phi \,|\, \bar{\sigma})^{-1}\, rfx), \quad \text{where } \sigma \in K',\ k \in \bar{\sigma}.$$

There are four possible difficulties with this definition. In order that $f\Delta 1$ be

(1) defined at all, we need to know that $rfx \in \phi(\bar{\sigma}) =$ domain of $(\phi \,|\, \bar{\sigma})^{-1}$;

(2) well defined, we need $k \in \bar{\sigma}$, $\bar{\tau}$ to imply $(\phi \,|\, \bar{\sigma})^{-1} rfx = (\phi \,|\, \bar{\tau})^{-1} rfx$;

(3) continuous, we need only check (1) and (2);

(4) a map into $Y\Delta K$, we need only note that $r(fx) = \phi(\phi \,|\, \bar{\sigma})^{-1} rfx$.

Thus we check (1,2): first $qx = \phi k \in \phi(\bar{\sigma})$ so that $x \in q^{-1}\phi(\bar{\sigma})$ and hence $rfx \in rfq^{-1}(\phi\bar{\sigma}) \subset \phi\bar{\sigma}$. Secondly, suppose $k \in \bar{\sigma}, \bar{\tau}$ and $k \in \rho$, say. Then ρ is a face of both σ and τ. Hence

$$(\phi\,|\,\bar{\sigma})^{-1}\,|\,\phi(\bar{\rho}) = (\phi\,|\,\bar{\rho})^{-1} = (\phi\,|\,\bar{\tau})^{-1}\,|\,\phi(\bar{\rho}),$$

so that $(\phi\,|\,\bar{\sigma})^{-1}rfx = (\phi\,|\,\bar{\tau})^{-1}rfx$.

REMARK. $X\Delta K$ may be interpreted as follows. Let $X_\sigma = \{(x,k) \in X\Delta K : k \in \bar{\sigma}\}$, for $\sigma \in K'$. For an n-simplex σ, the map $X \to X_\sigma$ which sends $x \to (x,(\phi\,|\,\bar{\sigma})^{-1}qx)$ is a homeomorphism. Similarly for ρ an i-simplex in $K', X_\rho \approx q^{-1}(\phi\bar{\rho}) = q^{-1}(\bar{\rho}')$ and ρ' is an i-simplex of s. Hence for two n-simplexes σ, $\tau \in K', \rho$ their common face, then X_σ and X_τ are copies of X identified along their respective copies of X_ρ.

Among the immediate consequences of the definitions are

(1.1) $(1\Delta u)(1\Delta t) = 1\Delta ut$, where u, t satisfy (M2).

(1.2) $(g\Delta 1)(f\Delta 1) = gf\Delta 1$, where the following triangles satisfy (M1),

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "q"'] & Y \arrow[r, "g"] \arrow[d, "r"] & Z \arrow[dl, "p"] \\ & s & \end{tikzcd} .$$

(1.3) the following diagram is commutative:

$$\begin{tikzcd} X\Delta K \arrow[r, "f\Delta 1"] \arrow[d, "1\Delta t"] & Y\Delta K \arrow[d, "1\Delta t"] \\ X\Delta L \arrow[r, "f\Delta 1"'] & Y\Delta L \end{tikzcd} .$$

We will verify (1.3); going over and down:

$$(x,k) \to (fx,(\phi\,|\,\bar{\sigma})^{-1}rfx) \;\to\; (fx, t(\phi\,|\,\bar{\sigma})^{-1}rfx);$$

down and over:

$$(x,k) \to (x,tk) \to (fx,(\theta\,|\,t\bar{\sigma})^{-1}rfx),$$

where $\phi = \phi_K$ and $\theta = \phi_L$ are the maps described above. Then these two maps agree as all maps in the following commutative triangle are 1-1:

$$\begin{tikzcd} \bar{\sigma} \arrow[r, "t\,|\,\bar{\sigma}"] \arrow[dr, "\phi\,|\,\bar{\sigma}"'] & t\bar{\sigma} \arrow[d, "\theta\,|\,(t\bar{\sigma})"] \\ & \phi(\bar{\sigma}) \end{tikzcd} .$$

Then in general define $f\Delta t = (f\Delta 1)(1\Delta t) = (1\Delta t)(f\Delta 1)$.

(1.4) $q\Delta 1 : X\Delta K \to K'$ is defined by $(q\Delta 1) : X\Delta K \to s\Delta K \approx K'$ by our identification above. Note $q\Delta 1$ is the restriction of the projection $X \times K \to K$ to $X\Delta K$.

2. **Simplicial theory and chain maps.** An additional hypothesis is needed to insure that $X\Delta K$ be a complex. One that suffices is

(S01) X is a complex, $q^{-1}(\bar{\sigma})$ is a subcomplex for all $\sigma \in s$, and

(SM1) $f: X \to Y$ is simplicial and satisfies (M1).

REMARK. One may always assume that $q: X \to s$ is simplicial relative to some subdivision of s. For given X, q satisfying (S01), there is a simplicial approximation q' (relative to some subdivision of s) to q such that $q'q^{-1}(\sigma) \subset \sigma$ for all $\sigma \in s$. Such a q' exists by induction on the skeleta $s_0, s_1, \cdots, s_n$ of s.

Then $q^{-1}(\sigma) = q'^{-1}(\sigma)$ for all $\sigma \in s$ and hence there are the maps

$$1\Delta 1 : X\Delta_q K \to X\Delta_{q'} K, \quad 1\Delta 1 : X\Delta_{q'} K \to X\Delta_q K$$

and their composition in either order is the identity. Also if $f: X \to Y$, $r: Y \to s$, satisfy (SM1) then $r'fq'^{-1}(\bar{\sigma}) = rfq^{-1}(\bar{\sigma}) \subset \bar{\sigma}$, for all $\sigma \in s$ so that the replacement $q \to q'$ is natural.

However the weaker requirement (S01) seems more "natural" and will be used. We remain in this simplicial category in all that follows.

(2.1) LEMMA. *If $q: X \to s$ satisfies* (S01) *then $X\Delta K$ has a natural simplicial structure.*

Proof. The proof is in four parts. In (1,2) we triangulate $X\Delta K$ and in (3,4) show that the induced maps are simplicial.

(1) For $\sigma \in K'$ define $X_\sigma = \{(x,k) \in X\Delta K : k \in \bar{\sigma}\}$. Then $h_\sigma(x) = (x, (\phi|\bar{\sigma})^{-1}qx)$ defines a homeomorphism $h_\sigma : q^{-1}\phi\bar{\sigma} \to X_\sigma$. h_σ is clearly 1-1 and if $(x,k) \in X_\sigma$, then $qx = \phi k \in \phi\bar{\sigma}$ so that $h_\sigma(x) = (x, (\phi|\bar{\sigma})^{-1}qx) = (x, (\phi|\bar{\sigma})^{-1}\phi k) = (x,k)$. But by hypothesis $q^{-1}\phi\bar{\sigma}$ is a complex so that h_σ triangulates X_σ.

(2) If τ is a face of $\sigma \in K'$, then $(h_\sigma | q^{-1}\phi\bar{\tau})x = (x, (\phi | \bar{\tau})^{-1}qx) = (x, (\phi|\bar{\tau})^{-1}qx) = h_\tau x$, for all $x \in q^{-1}\phi(\bar{\tau})$. Thus the triangulations h_σ defined in part (1) agree where they intersect and taken together triangulate $X\Delta K$. If X and K are finite complexes so is $X\Delta K$.

(3) Now suppose in addition that $r: Y \to s$ satisfies (S01) and $f: X \to Y$ satisfies (SM1). In order to show that $f\Delta 1 : X\Delta K \to Y\Delta K$ is simplicial suppose $\rho \in X\Delta K$. Then ρ is a simplex of X_σ for some $\sigma \in K'$. That is, there is a simplex $\rho_1 \in q^{-1}\phi\bar{\sigma} \subset X$ such that $h_\sigma(\rho_1) = \rho$. Then $\rho_2 = f\rho_1$ is a simplex in $r^{-1}\phi\bar{\sigma}$ because $rfq^{-1}(\bar{\sigma}) \subset \bar{\sigma}$. Now by definition $(f\Delta 1)(\rho_1) = (f\rho_1, (\phi|\bar{\sigma})^{-1}rf\rho_1) = (\rho_1, (\phi|\bar{\sigma})^{-1}r\rho_2)$ and this last is a simplex in the natural triangulation of $Y\Delta K$.

(4) Varying the second factor suppose $t: K \to L$ satisfies (M1). Then $(1\Delta t)(\rho, (\phi_K|\bar{\sigma})^{-1}q\rho) = (\rho, t(\phi_K|\bar{\sigma})^{-1}q\rho) = (\rho, (\phi_L|t\bar{\sigma})^{-1}q\rho)$ which is a simplex of $X\Delta L$.

(2.2) NOTATION. A point $(x,k) \in X\Delta K \subset X \times K$ will be denoted at times by by $x\Delta k$. A simplex $\tau \in X_\sigma \subset X\Delta K$ has the form $\tau = h_\sigma(\rho) = (\rho, (\phi|\bar{\sigma})^{-1}q\rho)$ where $\sigma \in K'$. We introduce the notation $(\rho,\sigma) = \tau$; note $(\rho,\sigma) \neq \rho\Delta\sigma$ as this last involves a number of copies of ρ depending upon the dimensions of ρ and σ. The notation (ρ,σ) is far from unique, for if σ is a face of σ', then $(\rho,\sigma)=(\rho,\sigma')$.

If $C\in C_*(q^{-1}\phi(\bar{\sigma}),A)$, i.e., $C=\sum\alpha_i\cdot\rho_i$ where $\rho_i\in q^{-1}\phi(\bar{\sigma})$, we let (c,σ) denote $\sum\alpha_i(\rho_i,\sigma)$. One easily verifies the fact that $\partial(c,\sigma)=(\partial c,\sigma)$. Also one can check the formula $(q\Delta 1)_{\#}(c,\sigma)=(q_{\#}c,\sigma)=(\phi\,|\,\bar{\sigma})^{-1}q_{\#}c$. The last $=$ follows from our identification $s\Delta K=K'$.

Chain maps. $C_*(K,A)\to C_*(X\Delta K,A)$.

Now suppose $\gamma:C_*(s;A)\to C_*(X;A)$ is a chain map carried by q^{-1}. Then

(2.3) *γ induces a chain map $\gamma\Delta 1:C_*(s\Delta K;A)\to C_*(X\Delta K;A)$ carried by $(q\Delta 1)^{-1}$. If $q_{\#}\gamma=1$ then $(q\Delta 1)_{\#}(\gamma\Delta 1)=1$.*

Proof. Recall that $s\Delta K=K'$. Then $\gamma\Delta 1$ is defined on elementary chains $\alpha\cdot\sigma$ of $C_*(K',A)$ by $(\gamma\Delta 1)(\alpha\cdot\sigma)=(\gamma(\alpha\cdot\phi\sigma),\sigma)$. Then clearly $\gamma\Delta 1$ is carried by $(q\Delta 1)^{-1}$. To check that $\gamma\Delta 1$ is a chain map:

$$\begin{aligned}\partial((\gamma\Delta 1)(\alpha\cdot\sigma)) &= \partial(\gamma(\alpha\phi\sigma),\sigma) = (\partial\gamma(\alpha\cdot\phi\sigma),\sigma)\\ &= (\gamma(\alpha\cdot\partial\phi\sigma),\sigma) = (\gamma(\alpha\cdot\phi\partial\sigma),\sigma)\\ &= (\gamma\Delta 1)(\partial(\alpha\sigma)).\end{aligned}$$

If $q_{\#}\gamma=1$, then

$$\begin{aligned}(q\Delta 1)_{\#}(\gamma\Delta 1)(\alpha\sigma) &= (q\Delta 1)_{\#}(\gamma(\alpha\cdot\phi\sigma),\sigma) = (\phi\,|\,\bar{\sigma})^{-1}q_{\#}\gamma(\alpha\cdot\phi\sigma)\\ &= (\phi\,|\,\bar{\sigma})^{-1}\ 1(\alpha\cdot\phi\sigma) = \alpha\cdot\sigma.\end{aligned}$$

(2.4) Corollary. *If there is a chain map $\gamma:C_*(s;A)\to C_*(X;A)$ such that $q_{\#}\gamma=1$, then $(q\Delta 1)_*:H_*(X\Delta K;A)\to H_*(K;A)$ is onto.*

Proof. For then the chain map $(\gamma\Delta 1):C_*(K;A)\to C_*(X\Delta K;A)$ induces a map $(\gamma\Delta 1)_*:H_*(K;A)\to H_*(X\Delta K;A)$, such that $(q\Delta 1)_*(\gamma\Delta 1)_*=1$. Hence $(q\Delta 1)_*$ is onto.

3. **Applications.** Throughout this section s is the standard 2-simplex. We need the following results for compact metric spaces X. They can be found in Hurewicz-Wallman [4, p. 72] and [4, p. 152].

A. $\dim X\leqq n \Leftarrow$ for each $\varepsilon>0$ there exists an ε-map f of X into a complex Y_ε of dimension $\leqq n$.

B. $\dim X\geqq n\Leftarrow H_n(X;G)\neq 0$ for some coefficient group G. ($f:X\to Y$ is an ε-map if diameter $f^{-1}(y)<\varepsilon$ for all $y\in Y$.)

Now suppose X is the inverse limit of a sequence

$$X_1\xleftarrow{f_1} X_2\xleftarrow{f_2} X_3\longleftarrow \cdots$$

where X_i is a complex (simplicial or cellular) and f_i is simplicial or cellular. Let $\sigma\in X_i$, set $X_\sigma=\{x\in X:x_i\in\sigma\}$, and define $d_i=\max_{\sigma\in K_i}$ (diameter X_σ). Assume that $d_i\to 0$ as $i\to\infty$. Then A, B can be reformulated as follows.

A′. $\dim X \leqq n \Leftarrow$ for each i there is a simplicial (or cellular) approximation $g_i : X_{i+1} \to X_i$ to f_i such that $g_i(X_{i+1}) \subset$ (n-skeleton of X_i).

B′. $\dim X \geqq n \Leftarrow$ for each i, $f_{i*} : H_n(X_{i+1}; G) \to H_n(X_i; G)$ is onto and nontrivial for at least one i.

For then the Čech homology $H_n(X; G)$ is not zero.

I. (Pontrjagin [6]). There exist compact metric spaces Φ_p, one for each prime p, such that $\dim \Phi_p = 2$ and

$$\dim \Phi_p \times \Phi_{p'} = \begin{cases} 3, & \text{if } p \neq p', \\ 4, & \text{if } p = p'. \end{cases}$$

Proof. We consider only the case $p \neq p'$. Let $X_0 = S^1 \times [0,1]$ and let X_p be the identification space formed from X_0 by identifying $(\theta, 1) \sim (\theta + 2\pi/p, 1)$ for all $\theta \in S^1$. (Note: X_2 = Möbius band.) Let B_p correspond to $S' \times 0$ and C_p to $S^1 \times 1$. Then $(X_p, B_p \cup C_p)$ is a relative, orientable manifold and choosing orientations, $\partial X_p = B_p + p \cdot C_p$.

Define $q' : X_0 \to s$ by

$$q' : S^1 \times [0,1] \to S^1 \times [0, 1/2] \to S^1 * b = s,$$

in which the first map retracts $[1/2, 1]$ to $1/2$, the second collapses $S^1 \times 1/2$ to b, and $S^1 * b$, the cone over S^1 is identified with s in the natural way. Now q' factors through X_p so that we have the commutative diagram

$$\begin{tikzcd} X_0 \arrow[rr] \arrow[dr, "q'"'] & & X_p \arrow[dl, "q"] \\ & s & \end{tikzcd}.$$

(I.1) $q^{-1} : C_*(s; Z_p) \to C_*(X_p; Z_p)$ *is a chain map.*

We must show that $\partial q^{-1}(\alpha\sigma^i) = q^{-1}(\partial\alpha\sigma^i)$, $\sigma^i \in s$, $\alpha \in Z_p$. This is clear for $i = 0, 1$, as $q^{-1} \mid \partial s$ is 1-1. For the only 2-simplex $\sigma^2 \in s$ we get

$$\partial q^{-1}(\alpha \cdot \sigma^2) = \alpha(\partial X_p) = \alpha(B_p + p \cdot C_p) = \alpha B_p, \text{ as } p\alpha = 0.$$

Finally, $q^{-1}(\partial\alpha\sigma^2) = \alpha \cdot q^{-1}(\partial\sigma^2) = \alpha \cdot B_p$.

Next suppose $p \neq p'$. Then there are the maps $q : X_p \to s$, $q' : X_{p'} \to s$ and hence the product map $q \times q' : X_p \times X_{p'} \to s \times s$. Let $\hat{B} = (q \times q')^{-1}(\partial(s \times s)) = B_p \times X_{p'} \cup X_p \times B_{p'}$. Then

(I.2) *There is a map* $r : X_p \times X_{p'} \to \partial(s \times s) = S^3$ *such that* $r \mid \hat{B} = q \times q' \mid \hat{B}$.

Proof. We need only check the hypothesis of the Hopf extension theorem [4]. As $(X_p \times X_{p'}, (B_p \cup C_p) \times X_{p'} \cup (X_p \times (B_{p'} \cup C_{p'}))$ is a relative oriented manifold each homology class $v \in H_4(X_p \times X_{p'}, \hat{B}; S^1)$ has a representative of the form $\alpha \cdot X_p \times X_{p'}$, $\alpha \in S^1$. Then ∂v is represented by

$$\alpha\partial(X_p \times X_{p'}) = \alpha(\hat{B} + pC_p \times X_{p'} + p' \cdot X_p \times C_{p'});$$

hence we must have $\alpha(pC_p \times X_{p'} + p'X_p \times C_{p'}) = 0$, and as $p \neq p', \alpha = 0$. Hence $H_4(X_p \times X_{p'}, \hat{B}; S^1) = 0$ and the map r exists.

Let K be a triangulated two-sphere and define K_i inductively by $K_{i+1} = X_p \Delta K_i$. Define $\pi_i : K_{i+1} \to K_i$ by $q\Delta 1 : X_p \Delta K_i \to K_i$. Let Φ_p be the inverse limit of the sequence $K_1 \xleftarrow{\pi_1} K_2 \xleftarrow{\pi_2} K_3 \leftarrow \cdots$.

(I.3) $\dim \Phi_{p'} = 2$, *all primes* p.

Proof. The inequality $\dim \Phi_p \leqq 2$ is easy. To check the other it suffices (by A′) to show that $\pi_{i*} : H_2(K_{i+1}; Z_p) \to H_2(K_i; Z_p)$ is onto for $i = 1, 2, \cdots$, because $H_2(K_1; Z_p) = Z_p$. But this follows from (I.1) and Corollary (2.4).

(I.4) $\dim \Phi_p \times \Phi_{p'} = 3$, *for* $p \neq p'$.

Proof. The inequality $\geqq 3$ follows from general principles [2, p. 34]. To prove the other one, note that $\Phi_p \times \Phi_{p'}$ is the inverse limit of the sequence

$$K_1 \times K_1' \xleftarrow{\pi_1 \times \pi_1'} K_2 \times K_2' \longleftarrow \cdots$$

where K_i, π_i are as above, and K_i', π_i' are the corresponding spaces and maps for $\Phi_{p'}$. Regarding $K_i \times K_i'$ as a CW-complex with cells $\sigma \times \sigma'$, $\sigma \in K_i$, $\sigma' \in K_i'$, we can map each product of the form $(X_p, \sigma) \times (X_p, \sigma')$ into $\partial(\sigma \times \sigma')$ using the map r of (I.2). These are consistent throughout $K_{i+1} \times K_{i+1}'$ because on the intersection of two such products, r agrees with π_i which is consistent. This yields a map $\psi: K_{i+1} \times K_{i+1}' \to$ (3-skeleton of $K_i \times K_i'$) which is a cellular approximation to $\pi_i \times \pi_i'$ so that by (A′), $\dim \Phi_p \times \Phi_{p'} \leqq 3$.

II. (Boltyanskii [2]). There is a compact metric space X of dimension 2 such that $\dim X \times X = 3$.

Proof. Fix a prime p and let $n_i = p^{2^i}$, $i = 1, 2, 3, \cdots$. Let $X_0 = S^1 \times [0,1]$ and let X_i be the space formed from X_0 by identifying $(\theta, 0) \sim (\theta + 2\pi/n_i, 0)$ and $(\theta, 1) \sim (\theta + 2\pi/n_{i+1}, 1)$, for all $\theta \in S^1$. Let $B_i \subset X_i$ correspond to $S^1 \times 0$ and C_i to $S^1 \times 1$. Then $(X_i, B_i \cup C_i)$ is a relative, orientable manifold and $\partial X_i = n_i B_i + n_{i+1} C_i$. Define $q' : X_0 \to s$ by

$$q' : S^1 \times [0,1] \xrightarrow[n_i]{} S^1 \times [0,1] \to S^1 * b = s$$

in which the first map sends $S^1 \to S^1$ by $\theta \to n_i\theta$ for all $\theta \in S^1$, and the second is as in (I) above.

Then q' factors through X_i defining q_i:

$$\begin{tikzcd} X_0 \arrow[rr] \arrow[dr, "q'"'] & & X_i \arrow[dl, "q_i"] \\ & s & \end{tikzcd} .$$

Below we consider $Z_{n_i} \subset Z_{n_{i+1}} \subset S^1$, the reals mod 1, and choose generators $\alpha_i \in Z_{n_i}$, such that $n_i \alpha_{i+1} = \alpha_i$.

(II.1) *There is a chain map* γ_i *such that the following diagram*

$$\begin{tikzcd}
C_*(s;Z) \arrow[rr, "\gamma_i"] \arrow[dr, "\eta_\#"'] & & C_*(X_i;Z_{n_{i+1}}) \arrow[dl, "q_{i_\#}"] \\
& C_*(s;Z_{n_i}) &
\end{tikzcd}$$

is commutative, in which $\eta_\#$ is induced by the natural map $\eta: Z \to Z_{n_i}$.

Proof. For a j-simplex $\sigma^j \in s$, $n \in Z$, define

$$\gamma_i(n \cdot \sigma^j) = \begin{cases} n\alpha_i q_i^{-1}(\sigma^j), & j = 0, 1; \\ n\alpha_{i+1} q_i^{-1}(\sigma^j), & j = 2. \end{cases}$$

The relation $q_{i_\#}\gamma_i = \eta_i$ follows from the fact that q_i is n_i-to-1 on $X_i - B_i \cup C_i$, 1-1 on B_i and n_{i+1}-to-1 on C_i. As in (I) we need only check $\partial\gamma_i(n \cdot \sigma^2) = \gamma_i(n\partial\sigma^2)$ for the unique 1-simplex $\sigma^2 \in s$. But $\partial\gamma_i(n\sigma^2) = n\alpha_{i+1}\partial X_i = n\alpha_{i+1}(n_i B_i + n_{i+1}C_i)$ $= n\alpha_i B_i$ and $\gamma_i(\partial n \cdot \sigma^2) = n\alpha_i q_i^{-1}(\partial\sigma^2) = n\alpha_i B_i$.

Now let $\hat{B}_i = B_i \times X_i \cup X_i \times B_i = (q_i \times q_i)^{-1}(\partial(s \times s))$. Then

(II.2) *There is a map $r_i: X_i \times X_i \to \partial(s \times s) \approx S^3$ such that $r_i|\hat{B}_i = (q_i \times q_i)|\hat{B}_i$.*

Proof. As before, every homology class $v \in H_4(X_i \times X_i, \hat{B}_i; S^1)$ has a representative of the form $z = \alpha \cdot X_i \times X_i$, $\alpha \in S^1$. Then $\partial z = \partial(\alpha \cdot X_i \times X_i)$ $= \alpha(n_i B_i + n_{i+1}C_i) \times X_i + \alpha \cdot X_i \times (n_i B_i + n_{i+1}C_i)$. Hence $\alpha n_{i+1}C_i \times X_i$ $+ \alpha \cdot n_{i+1}X_i \times C_i = 0$ and $n_{i+1}\alpha = 0$ (and thus $H_4(X_i \times X_i, \hat{B}_i; S') = Z_{n_{i+1}}$). Then $q_{i_\#}\partial z = q_{i_\#}(\alpha n_i B_i \times X_i + \alpha n_i X_i \times B_i) = \alpha n_i(\partial s) \times (n_i s) + \alpha n_i(n_i s) \times (\partial s)$ $= \alpha n_{i+1} \cdot \partial(s \times s) = 0$, and r_i exists by the Hopf extension theorem.

As in (I) define X as the inverse limit of

$$K_1 \xleftarrow{\pi_1} K_2 \xleftarrow{\pi_2} K_3 \longleftarrow \cdots$$

where $K_1 = S^2$ and $K_{i+1} = X_{i+1}\Delta K_i$. $\pi_i: K_{i+1} \to K_i$ is given by $q_{i+1}\Delta 1: X_{i+1}\Delta K_i \to K_i$.

(II.3) $\dim X \times X = 3$ *follows just as in* I *from* (II.2).

(II.4) $\dim X = 2$.

Proof. That $\dim X \geqq 2$ follows just as in (I), but the other inequality is a bit different because of the change of coefficients in (III.1). The claim is that $H_2(X; S^1) \neq 0$ and this will follow from

$$\text{(4a)} \quad \begin{cases} \pi_{1*}: H_2(K_2; S^1) \to H_2(K_1 S^1) \text{ is nontrivial,} \\ \pi_{i*}: H_2(K_{i+1}; S^1) \to H_2(K_i; S^1) \text{ is onto,} \quad i = 2, 3, \cdots. \end{cases}$$

We need only consider the cycle groups $Z_2(K_i; S^1)$ as we are in the top dimension. But as in the proof of (II.2), $Z_2(K_i; S^1) = Z_2(K_i; Z_{n_{i+1}})$. Now it follows from (II.1) and Corollary (2.4) that $\pi_{i_\#}: C_2(K_{i+1}; Z_{n_{i+2}}) \to Z_2(K_i; Z_{n_{i+1}})$ is onto. But any 2-chain which maps onto a cycle under $\pi_{i\#}$ must be a cycle, as $\pi_i|$ π_1^{-1} (1-skeleton of K_i) is a homeomorphism.

III. (Kolmogoroff [5]). *There is a compact* 1-*dimensional space* Y *and a compact abelian group* A *acting freely on* Y *such that* $\dim Y/A = 2$.

Proof. This is based on example I and we let the notation of (I) stand. Let $\gamma_t : S^1 \to S^1$ be the identity for $0 \leqq t \leqq 1/2$, and a rotation for $1/2 \leqq t \leqq 1$, γ_1 being a rotation of period p. Let $Y_0 = S^1 \times [0,1] \times Z_p$ and define $f_0, g_0 \colon Y_0 \to Y_0$ by

$$f_0(x,t,\alpha) = (x,t,\alpha+\omega), \qquad \omega \text{ a generator of } Z_p,$$

$$g_0(x,t,\alpha) = \begin{cases} (\gamma_t x, t, \alpha+\omega), & \alpha \neq 0, \\ (\gamma_t^{-p+1} x, t, \omega), & \alpha = 0. \end{cases}$$

Let $A = S^1 \times 1 \times Z_p$. Then on A, $g_0(x,t,\alpha) = (\gamma_t x, t, \alpha+\omega)$ so that $f_0 \mid A$ and $g_0 \mid A$ commute.

Let $Y_p = Y_0/g_0 \mid A$. Then f_0, g_0 induce maps $f_p, g_p : Y_p \to Y_p$ both of period p and f_p has no fixed point. Define $q_1' : Y_0 \to s$ by

$$q_1' : Y_0 \to Y_0/g_0 = X_0 \to s$$

in which the first map is the orbit map and the second is q', defined in (I). Then q_1' collapses all orbits of g_0.

Furthermore q_1' collapses all orbits of f_0. For this there are two cases: $0 \leqq t \leqq 1/2$ for which $g_0(x,t,\alpha) = f_0(x,t,\alpha)$, and the f_0-orbits $= g_0$-orbits; for $1/2 \leqq t \leqq 1$, $q_1'(x,t,\alpha) = b$ and $q_1' f_0(x,t,\alpha) = q_1'(x,t,\alpha+\omega) = b$ as well.

Hence there are the induced maps

$$\begin{tikzcd} Y_0 \arrow[r] \arrow[dr, "q_1'"'] & Y_p \arrow[r] \arrow[d, "q_1"] & Y_p/f_p \arrow[dl, "\bar{q}_1"] \\ & s & \end{tikzcd} .$$

(III.1) $Y_p/f_p \approx X_p$ *and identifying these* $\bar{q}_1 = q : X_p \to s$.

Proof. First $Y_p/f_p = (Y_0/g_0 \mid A)/f_p = (Y_0/f_p)/g_p \mid A' = X_0/(\theta,1) \sim (\theta + 2\pi/p, 1) = X_p$. The second statement follows because q_1' was defined in terms of q'.

(III.2) *There is a map* $r_1 : Y_p \to \partial s$ *such that*

$$r_1 q^{-1}(\bar{\sigma}) \subset \bar{\sigma} \textit{ for all } \sigma \in s.$$

Proof. Define $r_1' : Y_0 \to \partial s$ by $r_1' \colon Y_0 \to Y_0/g_0 = S^1 \times [0,1] \to S^1 = \partial s$. Then as r_1' collapses all orbits of g_0, r_1' factor through Y_p:

$$\begin{tikzcd} Y_0 \arrow[r] \arrow[dr, "r_1'"'] & Y_p \arrow[d, "r_1"] \\ & s \end{tikzcd} .$$

Note that $r_1q^{-1}(\bar\sigma)\subset\bar\sigma$, for $\sigma=\sigma^0,\sigma^1\in s$, as here $r_1=q_1$. For $\sigma=\sigma^2$ this condition is trivial.

Let $L_1=S^2$ and define L_i inductively by $L_{i+1}=Y_p\Delta L_i$. Define $\pi_{i1}:L_{i+1}\to L_i$ by $q\Delta 1: Y_p\Delta L_i\to L_i$, and $Y=$ inverse limit of (L_i,π_{i1}). Let $F_{i0}:L_i\to L_i$ be the identity and define $F_{i1},\cdots,F_{ii}:L_i\to L_i$ inductively by

$$F_{i+1,j}=f_p\Delta F_{i,j-1},\qquad j=1,2,\cdots,i+1.$$

Note that $F_{i+1,j}$ has no fixed point because f_p has none.

Then $\{F_{ij}\}_{j=1}^{i}$ generates a free action of $Z_p+\cdots+Z_p$ (i summands) upon L_i, and

(III.3) $\pi_{i1}:L_{i+1}\to L_i$ *is equivariant. That is*

$$\pi_{i1}F_{i+1,j} = F_{i,j-1}\pi_{i1},\qquad j=1,2,\cdots,i.$$

For

$$\begin{aligned}\pi_{i1}F_{i+1,j} &= (q_1\Delta 1)(f_p\Delta F_{i,j-1}) = q_1f_p\Delta F_{i,j-1}\\ &= q_1\Delta F_{i,j-1} = (1\Delta F_{i\,j-1})(q_1\Delta 1) = F_{i,j-1}\pi_{i1}.\end{aligned}$$

Let $[f,\,g,\cdots]$ denote the group generated by $f,\,g,\cdots$. Then define the abelian group A as the inverse limit

$$[F_{10}]\xleftarrow{\pi_{11}}[F_{2j}]_{j=1}^{1}\xleftarrow{\pi_{21}}[F_{3j}]_{j=1}^{2}\xleftarrow{\pi_{31}}\cdots.$$

Then A acts on Y, coordinatewise and thus freely. This uses the equivariance, (III.3).

(III.4) dim $Y=1$.

Proof. We prove only dim $Y\leqq 1$, and hence need only check the hypothesis of (A′). There is the map

$$1\Delta 1: Y_p\Delta_{q_1}L_i\to Y_p\Delta_{r_1}L_i$$

because $r_1q^{-1}(\bar\sigma) = rq^{-1}(\bar\sigma)\subset\bar\sigma$ for all $\sigma\in s$. Now define ψ by

$$\psi: Y_p\Delta_{q_1}L_i\xrightarrow{1\Delta 1}Y_p\Delta_{r_1}L_i\xrightarrow{r_1\Delta 1}(\partial s)\Delta L_i=\text{1-skeleton of } L_i.$$

This is clearly a simplicial approximation to π_{i1} and (A′) is verified.

(III.5) $Y/A=\Phi_p$ *so that* dim $Y/A=2$.

Proof. By induction; we note that $L_1/F_{10}=S^2/1=K_1$. Now

$$\begin{aligned}L_{i+1}/[F_{i+1,1},\cdots,F_{i+1,i+1}] &= Y_p/f_p\Delta(L_i/[F_{i,0},\cdots,F_{i,i}])\\ &=(Y_p/f_p)\,\Delta(L_i/[F_{i,1},\cdots,F_{i,i}])=(Y_p\ f_p)\Delta K_i\end{aligned}$$

by the induction hypothesis. But by (III.1) this last is $X_p\Delta K_i=K_{i+1}$. Hence Y/A is the inverse limit of

$$K_0\longleftarrow K_1\longleftarrow\cdots$$

in which the connecting maps are $\bar q_1\Delta 1=q\Delta 1: X_p\Delta K_i\to K_i$. But this is precisely the definition of Φ_p.

REMARKS. (1) Kolmogoroff, writing in 1937, did not express his ideas in terms of transformation groups. His concern was to construct an open map which raised dimension and the map he gives is precisely the orbit map of an action of a group A, $Y' \to Y'/A = \Phi_p$, for $p = 2$. A notable difference is that in his example the (implicit) action of A is far from free.

Y' can be described (for any p) just as Y, except that in place of Y_p one uses Y_p' defined by $Y_p = Y_0/g_0 | A'$, where $A' = S^1 \times \{0,1\} \times Z_p$. On Y' one can equally well define the action of the p-adic group A_p (given as the inverse limit of the sequence $Z_p \leftarrow Z_{p^2} \leftarrow Z_{p^3} \leftarrow \cdots$) so that $Y'/A_p = \Phi_p$. This version of Kolmogoroff's is then closer to that given in [7; 8], in which A_p acts (not freely) on a 2-dimensional space X and $\dim X/A_p = 4$.

Finally, one can define a 1-dimensional space Y'' and a free action of A_p on Y'' such that dim $Y''/A_p = 2$. This differs from the example above in two ways. First, instead of Y_p one uses $Y_p'' = Y_0''/g_0'' | A''$, where $Y_0'' = D_p \times Z_p$. D_p is a 2-dimensional disk from which have been deleted p open disks with boundaries $A_0, \cdots, A_{p-1}$; $A'' = \bigcup A_i$. The map $g_0'' : Y_0'' \to Y_0''$ takes $A_a \times b \to A_{a+\omega} \times (b + \omega)$, whereas $f_0'' : A_a \times b \to A_a \times (b + \omega)$, for $a, b \in Z_p$. The crucial difference is that the co-index [3] (or B-index, [9]) of (Y_p'', f_p'') is 1; that of (Y_p, f_p) is 2.

Next, instead of defining $L_i'' = Y_p'' \Delta L_i''$, one uses a more complicated inductive procedure, just as in [8], making use of the fact that co-index $(Y_p'', f_p'') = 1$. Generally speaking, the higher the co-index, the "deeper" the example. This topic will be pursued in a future paper.

BIBLIOGRAPHY

1. R. D. Anderson, *Zero dimensional compact groups of homeomorphisms*, Pacific J. Math. **7** (1957), 799.

2. V. Boltyanskii, *An example of a two-dimensional compactum whose topoligical square is three-dimensional*, Amer. Math. Soc. Transl. No. 48, (1951), 3–6; [Dokl. Akad. Nauk SSSR **67** (1949), 597–599].

3. P. E. Conner and E. E. Floyd, *Fixed point free involutions and equivariant maps*, Bull. Amer. Math. Soc. **66** (1960), 420.

4. Hurewicz and Wallman, *Dimension theory*, Princeton Univ. Press, Princeton, N. J., revised edition, 1948.

5. A. Kolmogoroff, *Über offene Abbildungen*, Ann. of Math. (2) **38** (1937), 36–38.

6. L. Pontrjagin, *Sur un hypothèse fondamental de la théorie de la dimension*, C. R. Acad. Sci. Paris **130** (1930), 1405–1407.

7. Frank Raymond and R. F. Williams, *Examples of p-adic transformation groups*, Bull. Amer. Math. Soc. **66** (1960), 392–394.

8. ———, *Examples of p-adic transformation groups*, (to appear).

9. C. T. Yang, *Continuous functions from spheres to Euclidian to spaces*, Ann. of Math. (2) **62** (1955), 284–292.

THE UNIVERSITY OF CHICAGO,
CHICAGO, ILLINOIS