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Octonions and the Leech lattice

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ABSTRACT

We give a new, elementary, description of the Leech lattice in terms of octonions, thereby providing the first real explanation of the fact that the number of minimal vectors, 196560, can be expressed in the form $3 \times 240 \times (1 + 16 + 16 \times 16)$. We also give an easy proof that it is an even self-dual lattice.

1. Introduction

The Leech lattice occupies a special place in mathematics. It is the unique 24-dimensional even self-dual lattice with no vectors of norm 2, and defines the unique densest lattice packing of spheres in 24 dimensions. Its automorphism group is very large, and is the double cover of Conway's group Co_1 [2], one of the most important of the 26 sporadic simple groups. This group plays a crucial role in the construction of the Monster [14,4], which is the largest of the sporadic simple groups, and has connections with modular forms (so-called 'Monstrous Moonshine') and many other areas, including theoretical physics. The book by Conway and Sloane [5] is a good introduction to this lattice and its many applications.

It is not surprising therefore that there is a huge literature on the Leech lattice, not just within mathematics but in the physics literature too. Many attempts have been made in particular to find simplified constructions (see for example the 23 constructions described in [3] and the four constructions described in [16]). In the latter there are described a 12-dimensional complex Leech lattice, whose symmetry group is a sextuple cover of the sporadic Suzuki group [17]; a 6-dimensional quaternionic Leech lattice [18], whose symmetry group is a double cover of an exceptional group of Lie type, namely $G_2(4)$; and a 3-dimensional quaternionic version, known as the icosian Leech lattice [15,1], which exhibits the double of cover of the Hall–Janko group as a group generated by quaternionic re-

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flections. This last is based on the ring of icosians discovered by Hamilton, which is an algebra whose norm 1 elements form a copy of the H_4 (non-crystallographic) root system.

In this paper I show how to construct a 3-dimensional *octonionic* Leech lattice, based on the Coxeter–Dickson non-associative ring of integral octonions [8], which is an algebra whose norm 1 elements form a copy of the E_8 root system.

2. Octonions and E_8

The book by Conway and Smith [6] gives a great deal of useful background on octonions, much of it based on Coxeter's paper [8]. The (real) octonion algebra is an 8-dimensional (non-associative) division algebra with an orthonormal basis $\{1 = i_\infty, i_0, \dots, i_6\}$ labelled by the projective line $PL(7) = \{\infty\} \cup \mathbb{F}_7$, with product given by $i_0 i_1 = -i_1 i_0 = i_3$ and images under the subscript permutations $t \mapsto t+1$ and $t \mapsto 2t$. The norm is $N(x) = x\bar{x}$, where $\bar{x}$ denotes the octonion conjugate of x, and satisfies $N(xy) = N(x)N(y)$.

The E_8 root system embeds in this algebra in various interesting ways. For example, we may take the 240 roots to be the 112 octonions $\pm i_t \pm i_u$ for any distinct $t, u \in PL(7)$, and the 128 octonions $\frac{1}{2}(\pm 1 \pm i_0 \pm \cdots \pm i_6)$ which have an odd number of minus signs. Denote by L the lattice spanned by these 240 octonions, and write R for $\bar{L}$. Let $s = \frac{1}{2}(-1 + i_0 + \cdots + i_6)$, so that $s \in L$ and $\bar{s} \in R$. Recall that the roots in a lattice are those of its non-zero primitive vectors such that reflection in the vector preserves the lattice. The various lattices L, R and so on are all scaled copies of E_8 so have 240 roots, which are the vectors of minimal norm in each case, and which span the lattice.

It is well known (see for example [6]) that $\frac{1}{2}(1 + i_0)L = \frac{1}{2}R(1 + i_0)$ is closed under multiplication, and forms a copy of the Coxeter–Dickson integral octonions. Denote this non-associative ring by A, so that $L = (1 + i_0)A$ and $R = A(1 + i_0)$. It follows immediately from the Moufang law $(xy)(zx) = x(yz)x$ that $LR = (1 + i_0)A(1 + i_0)$. Writing $B = \frac{1}{2}(1 + i_0)A(1 + i_0)$, we have $LR = 2B$, and the other two Moufang laws imply that $BL = L$ and $RB = R$.

Note however that B is *not* closed under multiplication. The common but incorrect assumption that it is is known as Kirmse's mistake. More explicitly, it is easy to show (and well known) that the roots of B are $\pm i_t$ for $t \in PL(7)$ together with $\frac{1}{2}(\pm 1 \pm i_t \pm i_{t+1} \pm i_{t+3})$ and $\frac{1}{2}(\pm i_{t+2} \pm i_{t+4} \pm i_{t+5} \pm i_{t+6})$ for $t \in \mathbb{F}_7$. But

$$\frac{1}{2}(1 + i_0 + i_1 + i_3) \,.\, \frac{1}{2}(1 + i_1 + i_2 + i_4) = \frac{1}{2}(i_1 + i_3 + i_4 + i_6)$$

which is not in B. The roots of A are obtained from those of B by swapping 1 with i_0. Indeed, as Coxeter [8] first pointed out, there are seven non-associative rings $A_t = \frac{1}{2}(1 + i_t)B(1 + i_t)$, obtained from B by swapping 1 with i_t.

Since $\bar{s} \in R$ we have $L\bar{s} \subseteq LR = 2B$, but all the roots of $2B$ lie in $L\bar{s}$, so $L\bar{s} = 2B$. Indeed, the same argument shows that if ρ is any root in R, then $L\rho = 2B$. The explicit description of the roots of B above shows that $2L \subset 2B \subset L$, that is $2L \subset L\bar{s} \subset L$, from which we deduce also $2L \subset Ls \subset L$. Moreover, $L\bar{s} + Ls = L$, so by self-duality of L we have $L\bar{s} \cap Ls = 2L$.

3. The octonion Leech lattice

Using L as our basic copy of E_8 in the octonions, we define the *octonionic Leech lattice* $\Lambda = \Lambda_{\mathbb{O}}$ as the set of triples (x, y, z) of octonions, with norm $N(x, y, z) = \frac{1}{2}(x\bar{x} + y\bar{y} + z\bar{z})$, such that

1. $x, y, z \in L$;
2. $x + y, x + z, y + z \in L\bar{s}$;
3. $x + y + z \in Ls$.

It is clear that the definition of Λ is invariant under permutations of the three coordinates. We show now that it is invariant under the map $r_t : (x, y, z) \mapsto (x, yi_t, zi_t)$ which multiplies two coordinates on the right by i_t. Certainly $Li_t = L$, so the first condition of the definition is preserved. Then

$y(1-i_t) \in LR = 2B = L\bar{s}$, so the second condition is preserved. Finally $(y+z)(1-i_t) \in 2BL = 2L \subset Ls$, so the third condition is preserved also. It follows that the definition is invariant under sign-changes of any of the three coordinates.

Suppose that λ is any root in L. Then the vector $(\lambda s, \lambda, -\lambda)$ lies in Λ, since $Ls \subseteq L$ and $\lambda s + \lambda = \lambda(s+1) = -\lambda\bar{s}$. Therefore Λ contains the vectors $(\lambda s, \lambda, \lambda) + (\lambda, \lambda s, -\lambda) = -(\lambda\bar{s}, \lambda\bar{s}, 0)$, that is, all vectors $(2\beta, 2\beta, 0)$ with β a root in B. Hence Λ also contains

$$\big(\lambda(1+i_0), \lambda(1+i_0), 0\big) + \big(\lambda(1-i_0), -\lambda(1+i_0), 0\big) = (2\lambda, 0, 0).$$

[Alternatively, using the fact that $L\bar{s} \cap Ls = 2L$ it follows immediately from the definition that $(2\lambda, 0, 0) \in \Lambda$; and since $Ls + L\bar{s} = L$, also $(2\beta, 2\beta, 0) \in \Lambda$.]

Applying the above symmetries it follows at once that Λ contains the following 196560 vectors of norm 4, where λ is a root of L and $j, k \in J = \{\pm i_t \mid t \in PL(7)\}$, and all permutations of the three coordinates are allowed:

$$\begin{array}{lll}
(2\lambda, 0, 0) & \text{Number:} & 3 \times 240 = 720, \\
\big(\lambda\bar{s}, \pm(\lambda\bar{s})j, 0\big) & \text{Number:} & 3 \times 240 \times 16 = 11520, \\
\big((\lambda s)j, \pm\lambda k, \pm(\lambda j)k\big) & \text{Number:} & 3 \times 240 \times 16 \times 16 = 184320.
\end{array}$$

4. Identification with the real Leech lattice

We show first that the 196560 vectors listed above span Λ. For if $(x, y, z) \in \Lambda$, then by adding suitable vectors of the third type, we may reduce z to 0. Then we know that $y \in L\bar{s}$, so by adding suitable vectors of the second type we may reduce y to 0 also. Finally we have that $x \in L\bar{s} \cap Ls = 2L$ so we can reduce x to 0 also.

At this stage it is easy to identify Λ with the Leech lattice in a number of different ways. First, let us label the coordinates of each brick of the MOG (see [9] or [7]) as follows.

$$\begin{array}{|cc|}
\hline
-1 & i_0 \\
i_4 & i_5 \\
i_2 & i_6 \\
i_1 & i_3 \\
\hline
\end{array}$$

Then it is well known (see for example [7, p. 95]) that the map $i_t \mapsto i_{t+1}$ is a symmetry of the standard Leech lattice. We double the vectors in Λ to obtain the Leech lattice on the usual scale. Now L is spanned by $1 \pm i_t$ and s, and it is trivial to verify that the vectors $(1 \pm i_0)(s, 1, 1)$ and $s(s, 1, 1)$ correspond to the following Leech lattice vectors.

$$\begin{array}{|cc|cc|cc|}
\hline
2 & & -2 & 2 & -2 & 2 \\
 & 2 & & & & \\
 & 2 & & & & \\
 & 2 & & & & \\
\hline
\end{array}
\qquad
\begin{array}{|cc|cc|cc|}
\hline
 & 2 & -2 & -2 & -2 & -2 \\
2 & & & & & \\
2 & & & & & \\
2 & & & & & \\
\hline
\end{array}
\qquad
\begin{array}{|cc|cc|cc|}
\hline
3 & -1 & 1 & 1 & 1 & 1 \\
-1 & -1 & 1 & 1 & 1 & 1 \\
-1 & -1 & 1 & 1 & 1 & 1 \\
-1 & -1 & 1 & 1 & 1 & 1 \\
\hline
\end{array}$$

Therefore all vectors in $L(s, 1, 1)$ are in the standard Leech lattice. These together with permutations and sign-changes on the three coordinates, and addition, are enough to give all the minimal vectors, which span the lattice.

An alternative approach is to show directly from our definition that Λ is an even self-dual lattice with no vectors of norm 2, whence it is the Leech lattice by Conway's characterisation [5, Chapter 12]. For if (x, y, z) is in the dual of Λ, then in particular its (real) inner product with $(2\lambda, 0, 0)$ is integral, and since L is self-dual this implies $x \in L$. Similarly its inner product with $(\lambda\bar{s}, \lambda\bar{s}, 0)$ is integral, and since the dual of B is $2B$ this implies $x + y \in L\bar{s}$. Also $(\lambda s, -\lambda, -\lambda) + (0, -\lambda\bar{s}, -\lambda\bar{s}) = (\lambda s, \lambda s, \lambda s) \in \Lambda$ and the dual of Ls is $\frac{1}{2}Ls$, so $x + y + z \in Ls$. Thus Λ contains its dual.

Conversely, if $(x, y, z) \in \Lambda$, then

$$2N(x, y, z) = N(x + y) + N(x + z) + N(y + z) - N(x + y + z)$$

and all the terms on the right-hand side are divisible by 4, so Λ is an even lattice, and in particular is contained in its own dual. That Λ has no vectors of norm 2 is easy to see: if $(x, y, z) \in \Lambda$ has norm 2, then at least one coordinate is 0, so the other coordinates lie in $L\bar{s}$; therefore there is only one non-zero coordinate, which lies in $2L$, so the vector has norm at least 4.

5. Applications

The above construction is deceptively simple. In fact, however, finding the correct definition (beginning of Section 3) was not at all easy. Over the years, many people have noticed the suggestive fact that

$$196560 = 3 \times 240 \times (1 + 16 + 16 \times 16),$$

and tried to build the Leech lattice from triples of integral octonions (see for example [10,11,13]), but until now no-one has provided a convincing explanation for this numerology. The key to the simple proofs above is the observation that $LR = 2B$ and $BL = L$: these remarkable facts appear not to have been noticed before.

[Note: After this paper was submitted, I became aware of some work in progress by Geoffrey Dixon [12], which comes much closer to the description of the minimal vectors given above, at the end of Section 3. However, as far as I can see his conjectured formula for the last set of vectors needs modification: in effect, his formula is equivalent to $(\lambda s, \pm\lambda k, \pm(\lambda j)k)$, that is, omitting the (necessary) multiplication by j in the first coordinate.]

An alternative definition of an octonion Leech lattice using the 'natural' norm $N(x, y, z) = x\bar{x} + y\bar{y} + z\bar{z}$ may be obtained by a change of basis. Re-writing the original coordinates x, y, z as xs, ys, zs we obtain the following defining conditions for the new coordinates:

1. $x, y, z \in B$;
2. $x + y, y + z \in B\bar{s}$;
3. $x + y + z \in Bs = L$.

Possibly this variant of the definition will turn out to be more natural, although it is less convenient for exhibiting the relationship to the standard copies of the Leech lattice and Conway's group.

In a further paper [20] I shall show how to generate the automorphism group of the lattice in terms of 3×3 matrices with octonion entries, and give nice descriptions of many of its maximal subgroups [19]. This will include elementary constructions of all the Suzuki-chain subgroups, which up till now have not been easy to describe directly in terms of the lattice.

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