Systems & Control Letters 2 (1983) 301–306
North-Holland Publishing Company

February 1983

Hermitian polynomial matrices, isometries and inertial signs

Harald K. WIMMER

Mathematisches Institut, Universität Würzburg, D-8700 Würzburg, West Germany

Received 22 August 1982
Revised 28 November 1982

Nonsingular hermitian polynomial matrices are described by two inner product spaces and selfadjoint operators. The associated hermitian linear pencils are used to characterize isometries. As an application a new proof of a factorization result on polynomial matrices with constant signature is given.

Keywords: Polynomial matrix structure, Hermitian realizations, Hermitian pencils, Inner product spaces, Inertial signs, Polynomial matrices with constant signature.

1. Introduction

Linear dynamical systems with internal constraints give rise to transfer functions which reflect these internal properties, e.g. positive real matrices or symmetric matrices of rational functions. The role of such matrices and their realizations in different areas of applications (such as electrical network synthesis) is described in [7].

In this note we study hermitian polynomial matrices. Our approach is module theoretic as in [9] and [3]. We continue our work [9] and investigate the relation between hermitian polynomial matrices and their associated inner product spaces and selfadjoint operators. As an application we will give a new proof of a factorization result [5] on polynomial matrices with constant signature, which is relevant to the theory of filtering.

2. Notation

Let $\mathbb{C}((z^{-1}))$ denote the vector space of truncated Laurent series of the form

$$f(z)=\sum_{j=-\infty}^{n_f} f_j z^j,\quad f_j\in\mathbb{C}, \tag{2.1}$$

and let $z^{-1}\mathbb{C}[[z^{-1}]]$ be the space of formal power series in z^{-1} with vanishing constant term. The decomposition

$$\mathbb{C}((z^{-1}))=\mathbb{C}[z]\oplus z^{-1}\mathbb{C}[[z^{-1}]]$$

induces projections π_+ and π_- of $\mathbb{C}((z^{-1}))$ on $\mathbb{C}[z]$ and $z^{-1}\mathbb{C}[[z^{-1}]]$ respectively. If $f\in\mathbb{C}((z^{-1}))$ is given by (2.1) we put

$$(f)_i:=f_i.$$

In a natural way the preceding definitions are extended to $\mathbb{C}^n((z^{-1}))$ and $\mathbb{C}^{m\times n}((z^{-1}))$.

For

$$G(z)=\sum_{i=-\infty}^{n_G} G_i z^i,\quad G_i\in\mathbb{C}^{m\times n},$$

we define G^* by $G^*(z)=\Sigma G_i^* z^i$. We call G *hermitian* if $G=G^*$ holds.

If $L\in\mathbb{C}^{n\times n}[z]$ is non-singular then a complex number α is a *characteristic root* of L if $\det L(\alpha)=0$ holds. We say $\lambda=\infty$ is a characteristic root of L if L^{-1} has a polynomial part.

The following matrices will appear:

$E_n:=(\delta_{i,n+1-i})\in\mathbb{C}^{n\times n}$, i.e.

$$E_n=\begin{pmatrix} 0 & & \cdot^{\cdot^{\cdot}} 1\\ & 1 & \\ 1 & & 0\end{pmatrix}_{n\times n},$$

$R_n:=(\delta_{i,n-i+2})\in\mathbb{C}^{n\times n}$, i.e.

$$R_n:=\begin{pmatrix} 0 & & & \cdot^{\cdot^{\cdot}} 0\\ & & \cdot^{\cdot^{\cdot}} & 1\\ & 0 & 1 & \\ 0 & 1 & & 0\end{pmatrix}_{n\times n},$$

$$S_k(z):=\begin{pmatrix} E_k-zR_k\\ -z\ 0\ \dots\ 0\end{pmatrix}_{(k+1)\times k}. \tag{2.2}$$

3. Hermitian polynomial matrices and inner products

Our note is based on [9]. In order to make it self-contained we review the results of [9]. Let $L \in \mathbb{C}^{n\times n}[z]$ be non-singular. The two maps

$$\pi_L : \mathbb{C}^n[z] \to \mathbb{C}^n[z]$$

and

$$\rho^L : \mathbb{C}^n[[z^{-1}]] \to \mathbb{C}^n[z]$$

which we associate with L are defined by

$$\pi_L f := L\pi_- L^{-1} f, \quad f \in \mathbb{C}^n[z],$$

and

$$\rho^L v := L\pi_+ L^{-1} v, \quad v \in \mathbb{C}^n[[z^{-1}]].$$

Let V_L and U^L denote the respective ranges

$$V_L := \operatorname{Im} \pi_L, \qquad U^L := \operatorname{Im} \rho^L.$$

Define

$$p \cdot f := \pi_L pf, \quad p \in \mathbb{C}[z], f \in V_L, \tag{3.1}$$

and

$$q \cdot v := L\pi_+ L^{-1} qv, \quad q \in \mathbb{C}[[z^{-1}]], v \in U^L. \tag{3.2}$$

Then V_L with scalar multiplication (3.1) is a $\mathbb{C}[z]$-module [2] and U^L with (3.2) is a $\mathbb{C}[[z^{-1}]]$-module [8]. The dimension of the $\mathbb{C}$-vector space V_L is equal to deg det L and dim U^L is equal to the multiplicity of the characteristic root $\lambda = \infty$ of L. If L is hermitian then

$$[f, g] := (f^* L^{-1} g)_{-1}, \quad (f, g) \in V_L \times V_L, \tag{3.3}$$

is an indefinite inner product on V_L. On U^L an inner product can be defined by

$$\langle u, v\rangle := (x^* L^{-1} y)_0 \tag{3.4}$$

where x and y are in $\mathbb{C}^n[[z^{-1}]]$ such that $u = \rho^L x$ and $v = \rho^L y$ holds. The right shift S^+,

$$S^+ f := z \cdot f, \quad f \in V_L,$$

is a linear operator which is selfadjoint with respect to the inner product (3.3). Similarly the left shift S^-,

$$S^- v := z^{-1} \cdot v, \quad v \in U^L,$$

is selfadjoint on U^L.

Any hermitian rational matrix $W \in \mathbb{C}^{n\times n}(z)$, $W \neq 0$, can be written as

$$W(z) = M(z)(zP + Q)^{-1} M^*(z) \tag{3.5}$$

where $zP + Q \in \mathbb{C}^{r\times r}[z]$ is a non-singular hermitian pencil and $M \in \mathbb{C}^{n\times r}[z]$. We call (3.5) a *hermitian realization* of W. (3.5) is a *minimal realization* if the size r of the pencil $zP + Q$ is minimal.

Lemma 3.1. *Any hermitian $W \in \mathbb{C}^{n\times n}(z)$, $W \neq 0$, admits a hermitian realization. If (3.5) is a minimal realization then the hermitian pencil $zP + Q$ is determined by W up to congruence.*

Theorem 3.2. *Let $L = L^* \in \mathbb{C}^{n\times n}[z]$ be non-singular and let*

$$\pi_- L^{-1}(z) = D(zA_1 - A_0)^{-1} D^*,$$

$$\pi_+ L^{-1}(z) = R(N_0 - N_1 z)^{-1} R^*$$

be minimal hermitian realizations. Then A_1 and N_0 are non-singular, $N_1^{-1} N_0$ is nilpotent. The columns of

$$\hat{D}(z) := L(z) D(A_1 z - A_0)^{-1}$$

are a basis of V_L, the columns of

$$\hat{R}(z) := L(z) R(N_0 - N_1 z)^{-1}$$

are a basis of U^L. With respect to these bases the matrices of the inner products (3.3) and (3.4) are given by A_1^{-1} and N_0^{-1}, the matrices of S^+ and S^- are $A_0 A_1^{-1}$ and $N_1 N_0^{-1}$ respectively.

Furthermore

$$\pi_-\left(\hat{D}(z)^* L^{-1}(z) \hat{D}(z)\right) = (A_1 z - A_0)^{-1}. \tag{3.6}$$

4. Hermitian pencils

In this section we recall the normal form of hermitian non-singular pencils and prove some auxiliary lemmas.

Theorem 4.1. (See e.g. [6] for references.) *Let $zP + Q \in \mathbb{C}^{n\times n}[z]$ be a hermitian pencil which is non-singular, i.e.*

$$\det(zP + Q) \neq 0 \in \mathbb{C}[z].$$

Then there exists a non-singular matrix $T \in \mathbb{C}^{n\times n}$ such that $T(zP + Q)T^*$ is the direct sum of blocks of *the following types* I, II, III:

$$\varepsilon D_r(z,a) := \varepsilon((z-a)E_r - R_r) \tag{I}$$

with $a \in \mathbb{R}$ *and* $\varepsilon = \pm 1$.

$$\begin{pmatrix} 0 & D_s(z;b) \\ D_s(z;\bar b) & 0 \end{pmatrix}, \tag{II}$$

a $2s \times 2s$ *matrix with* $b \notin \mathbb{R}$.

$$\varepsilon F_t(z) = \varepsilon(E_t - zR_t), \quad \varepsilon = \pm 1. \tag{III}$$

A block of type I corresponds to an elementary divisor $(z-a)^r$, $a \in \mathbb{R}$, of $zP+Q$. To each conjugate complex pair $(z-b)^s$, $(z-\bar b)^s$ of non-real elementary divisors of $zP+Q$ there is associated a block of type II. Each infinite elementary divisor y^t (i.e. each elementary divisor of $P+yQ$ of the form y^t) contributes a block of type III. Up to ordering of the blocks the direct sum is uniquely determined by $zP+Q$.

To each real or infinite characteristic root of $zP+Q$ belongs a set of signs ε which we call the *inertial signs*. The terminology is not uniform. In [3] elementary divisors and signs are called the Cauchy characteristic, in [4] the term sign characteristic is used in connection with real characteristic roots.

We now consider blocks of type III and I.

Lemma 4.2. *A block* F_t *can be factored as*

$$F_t(z) = M(z)KM^*(z)$$

with $M \in \mathbb{C}^{t\times t}[z]$ *and* $K \in \mathbb{C}^{t\times t}$.

Proof. For t even, $t = 2k$, write

$$F_{2k} = \begin{pmatrix} 0 & F_k \\ F_k & V \end{pmatrix}$$

where $V = (v_{ij})$ has zero entries except $v_{11} = -z$. Then

$$F_{2k} = \begin{pmatrix} 0 & F_k \\ 1 & \frac{1}{2}V \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ F_k & \frac{1}{2}V \end{pmatrix}.$$

For t odd, $t = 2k+1$, we partition F_t as

$$F_{2k+1}(z) = \begin{pmatrix} U & S_k(z) \\ S_k^*(z) & 0 \end{pmatrix}$$

where U is a $(k+1)\times(k+1)$ matrix with a 1 entry at the bottom of the main diagonal and where S_k is given by (2.2). Then

$$F_{2k+1}(z) = \begin{pmatrix} U & S_k(z) \\ S_k^*(0) & 0 \end{pmatrix} \times E_{2k+1}\begin{pmatrix} U & S_k(0) \\ S_k^*(0) & 0 \end{pmatrix}.$$

The following observation can easily be verified.

Lemma 4.3. *Put*

$$Y_r(z;a) := \begin{pmatrix} 1 & (z-a) & \cdots & (z-a)^{r-1} \\ 0 & & & \\ \vdots & & I_{r-1} & \\ 0 & & & \end{pmatrix}.$$

Then

$$Y_r(z;a)D_r(z;a)Y_r^*(z;a) = \begin{pmatrix} z^r & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & F_{r-1}(z) & \\ 0 & & & \end{pmatrix}. \tag{4.1}$$

5. Inertial signs

Let $\lambda = a$ be a real characteristic root of the non-singular hermitian matrix $L \in \mathbb{C}^{n\times n}[z]$. We consider the principal part $H_a(z)$ of L^{-1} at the pole a. We can assume without loss of generality that in a minimal hermitian realization

$$H_a(z) = C(A_1 z - A_0)^{-1}C^* \tag{5.1}$$

the $r \times r$ pencil $zA_1 - A_0$ is in normal form

$$zA_1 - A_0 = \text{block diag}\big(\varepsilon_1 D_{n_1}(z;a), \ldots, \varepsilon_k D_{n_k}(z;a)\big). \tag{5.2}$$

We partition C corresponding to (5.2) into $C = (C_1, \ldots, C_k)$ and denote the first column of C_i by γ_i. As the realization (5.1) is assumed to be minimal we have

$$\text{rank}\begin{pmatrix} A_1 a - A_0 \\ C \end{pmatrix} = r$$

or equivalently

$$\operatorname{rank}(\gamma_1, \dots, \gamma_k) = k. \tag{5.3}$$

The following result is a special case of Theorem 9.9 in [3]. It is obtained here as an immediate consequence of Lemma 3.1.

Theorem 5.1. *Let a be a real characteristic root of the hermitian non-singular matrix $L \in \mathbb{C}^{n \times n}[z]$. The principal part H_a of L^{-1} at a can be written as*

$$H_a(z) = \tilde{C}(z)\operatorname{diag}\big(\varepsilon_1(z-a)^{-n_1}, \dots, \varepsilon_k(z-a)^{-n_k}\big)\tilde{C}^*(z) + M(z) \tag{5.4}$$

with $\varepsilon_i \in \{1, -1\}$, $M \in \mathbb{C}^{n \times n}[z]$ and $\tilde{C} \in \mathbb{C}^{n \times k}[z]$ such that $\tilde{C}(a)$ has full column rank. The numbers n_i, ε_i, $i = 1, \dots, k$, are uniquely determined by L. The polynomials $(z-a)^{n_i}$ are the elementary divisors and the integers ε_i are the inertial signs corresponding to the characteristic root a of the pencil $A_1 z - A_0$ in (5.1).

Proof. Recall (4.1) and let H_a be given by (5.1) and (5.2). Put

$$\tilde{c}_i := C_i\big(1, (z-a), \dots, (z-a)^{r-1}\big)^T, \quad i = 1, \dots, k,$$

and $\tilde{C} := (\tilde{c}_1, \dots, \tilde{c}_k)$. Then (5.4) holds and $\tilde{C}(a) = (\gamma_1, \dots, \gamma_n)$. Hence (5.3) implies rank $\tilde{C}(a) = k$.

We show that the numbers n_i and ε_i in (5.4) are uniquely determined. Let $e = \operatorname{col}(1, 0, \dots, 0)$ be an r-vector. Then

$$\varepsilon(z-a)^{-r} = e^* \varepsilon D_r(z; a)^{-1} e.$$

With this observation it is easy to reverse the preceding steps and construct a minimal hermitian realization for H_a of the form (5.1) and (5.2). According to Lemma 3.1 the pencil $zA_1 - A_0$ in (5.1) is determined up to congruence, which completes the proof.

If L is a non-singular pencil then the numbers ε_i in (5.4) are the inertial signs corresponding to a. This suggests to use (5.4) to define inertial signs for general polynomial matrices.

Definition 5.2. Let $a \in \mathbb{R}$ be a characteristic root of the non-singular matrix $L = L^* \in \mathbb{C}^{n \times n}[z]$. Let the principal part H_a of L^{-1} at the pole a be given by (5.4) such that rank $\tilde{C}(a) = k$ holds. We call the numbers ε_i, $i = 1, \dots, k$, the inertial signs of the elementary divisors $(z-a)^{n_i}$ of L.

In [4] results on analytic perturbations have been used to describe the inertial signs of polynomial matrices with invertible leading coefficient.

We now turn to infinite elementary divisors of L. Recall [8] that a non-singular $L \in \mathbb{C}^{n \times n}[z]$ has the infinite elementary divisors $y^{s_1}, \dots, y^{s_p}$, if $yL(y^{-1})$ has Smith–McMillan form

$$\operatorname{diag}\big(y^{-c_1}d_1(y), \dots, y^{-c_q}d_q(y), y^{s_1}d_{q+1}(y), \dots, y^{s_p}d_{p+q}(y)\big)$$

such that

$$-c_1 \le \cdots \le -c_q \le 0 < s_1 \le \cdots \le s_p$$

and

$$d_\rho \in \mathbb{C}[z], \quad d_\rho(0) \neq 0, \quad \rho = 1, \dots, p+q,$$

holds. The following statement can be proved along the lines of Theorem 5.1.

Theorem 5.3. *Let $L = L^* \in \mathbb{C}^{n \times n}[z]$ be non-singular and let G_0 denote the principal part of $y^{-1}L^{-1}(y^{-1})$ at 0. Then H_0 can be written as*

$$H_0 = V(y)\operatorname{diag}\big(\varepsilon_1 y^{-s_1}, \dots, \varepsilon_p y^{s_p}\big)V^*(y) + R(z) \tag{5.5}$$

with $Z \in \mathbb{C}^{n \times p}[z]$, $R \in \mathbb{C}^{n \times n}[z]$, $\varepsilon_i \in \{1, -1\}$ such that rank $V(0) = p$ *holds. The polynomials y^{s_i}, $i = 1, \dots, p$, are the infinite elementary divisors of L, the corresponding signs ε_i are uniquely determined by L.*

Definition 5.4. We call the numbers $\varepsilon_i \in \{1, -1\}$, $i = 1, \dots, p$, given by (5.5) the inertial signs of the infinite elementary divisors y^{s_i} of L.

6. Isometries

Let $L_i \in \mathbb{C}^{n_i \times n_i}[z]$ be hermitian and non-singular. We call a map $\alpha: V_{L_2} \to V_{L_2}$ an $F[z]$-*isometry*, if it is an $F[z]$-module isomorphism which preserves the inner product (3.3). Similarly $F[[z^{-1}]]$-isometries between U^{L_1} and U^{L_2} are defined. In the case of linear pencils isometry is related to congruence.

Lemma 6.1. *Let $zP_i - Q_i \in \mathbb{C}^{n_i \times n_i}[s]$, $i = 1, 2$, be hermitian pencils such that P_i, $i = 1, 2$, is non-singu-*

lar. There is an $F[z]$-isometry

$$\alpha: V_{zP_1-Q_1} \to V_{zP_2-Q_2}$$

if and only if the pencils are congruent, i.e. $n_1 = n_2$ *and*

$$zP_2 - Q_2 = T(zP_1 - Q_1)T^* \tag{6.1}$$

for some non-singular constant T.

Proof. From $\dim V_L = \deg \det L$ follows $n_1 = n_2 = n$. We take the canonical basis of $\mathbb{C}^n$ as a basis of $V_{zP_i-Q_i}$. Let T be the matrix of α. As the matrices corresponding to the respective shift operators are similar and the matrices of the inner products are congruent, we have

$$TQ_1P_1^{-1}T^{-1} = Q_2P_2^{-1} \quad \text{and} \quad P_1^{-1} = T^*P_2^{-1}$$

which is (6.1). The converse is obvious.

Theorem 6.2. *Let* $L_i \in \mathbb{C}^{n_i \times n_i}[z]$, $i = 1, 2$, *be hermitian and non-singular. The following statements are equivalent.*

(i) *There exists an* $F[z]$-*isometry between* V_{L_1} *and* V_{L_2}.

(ii) *There exists polynomial matrices* X *and* Y *such that*

$$XL_1 = L_2Y \tag{6.2}$$

and

$$\pi_- X^*L_2^{-1}X = \pi_- L_1^{-1}, \qquad \pi_- YL_1^{-1}Y^* = \pi_- L_2^{-1} \tag{6.3}$$

hold.

(iii) L_1 *and* L_2 *have the same finite elementary divisors and their real characteristic roots have the same inertial signs.*

Proof. (iii) $\Rightarrow$ (ii) Let

$$\pi_- L_i^{-1}(z) = C_i(zP_i - Q_i)^{-1}C_i^*, \quad i = 1, 2, \tag{6.4}$$

be minimal hermitian realizations and let $\hat{C}_i \in \mathbb{C}^{n_i \times n_i}[z]$, $i = 1, 2$, be defined by

$$L_i(z)C_i = \hat{C}_i(z)(zP_i - Q_i). \tag{6.5}$$

From (3.6) we know

$$\pi_- \hat{C}_i^*L_i^{-1}\hat{C}_i = (zP_i - Q_i)^{-1}.$$

According to Theorem 5.1 we can work with the same pencil $zP - Q = zP_i - Q_i$ for both L_1 and L_2.

Put $X := \hat{C}_2C_1^*$ and $Y := C_2\hat{C}_1^*$. It can easily be verified that X and Y satisfy (6.2) and (6.3).

(ii) $\Rightarrow$ (i) We use the matrices X and Y in (6.2) and (6.3) to define

$$\alpha v := \pi_{L_2}Xv, \quad v \in V_{L_1},$$

and

$$\beta w := \pi_{L_1}Y^*w, \quad w \in V_{L_2}.$$

The mappings

$$\alpha: V_{L_1} \to V_{L_2} \quad \text{and} \quad \beta: V_{L_2} \to V_{L_1}$$

are $F[z]$-module homomorphisms [2]. Let u and v be in V_{L_1}, then

$$\pi_-[(\alpha u)^*L_2^{-1}\alpha v] = \pi_-(u^*L_1^{-1}v).$$

Similarly we have

$$\pi_-[(\beta w)^*L_1^{-1}\beta y] = \pi_-(\omega^*L_2^{-1}y)$$

for any w and y in V_{L_2}. Hence α and β are injective $\mathbb{C}$-linear maps between the vector spaces V_{L_i}, $i = 1, 2$, they are $F[z]$-isometries and $\beta = \alpha^{-1}$.

(i) $\Rightarrow$ (iii) Let $\pi_- L_i^{-1}$, $i = 1, 2$, be given by (6.4). By the preceding argument we obtain from (6.5) an $F[z]$-isometry between V_{L_i} and $V_{zP_i-Q_i}$, $i = 1, 2$.

Therefore we can assume $L_i(z) = zP_i - Q_i$, $i = 1, 2$, and Lemma 6.1 completes the proof.

The corresponding result on $F[[z^{-1}]]$-isometries between U^{L_1} and U^{L_2} would require a description of $F[[z^{-1}]]$-module homomorphisms between U^{L_1} and U^{L_2}. Since those homomorphisms will be discussed elsewhere, we note only an equivalence of two statements.

Theorem 6.3. *The non-singular hermitian polynomial matrices* L_1 *and* L_2 *have the same infinite elementary divisors and corresponding inertial signs if and only if there is an* $F[[z^{-1}]]$-*isometry between* U^{L_1} *and* U^{L_2}.

7. Hermitian polynomial matrices with constant signature

A non-singular hermitian matrix $L \in \mathbb{C}^{n \times n}[z]$ is said to have *constant signature* if for all $\lambda \in \mathbb{R}$ which are not characteristic roots of L, the matrices $L(\lambda)$ have the same signature. Such matrices are characterized in [5]. As an application of the preceding results we give a new proof of the following factorization theorem.

Theorem 7.1 [5]. *$L \in \mathbb{C}^{n\times n}[z]$ has constant signature if and only if L can be factored as*

$$L(z) = M(z)KM^*(z) \tag{7.1}$$

such that $M \in \mathbb{C}^{n\times n}[z]$ and $K \in \mathbb{C}^{n\times n}$.

Proof. It is the '$\Rightarrow$' part of the theorem which is not obvious. Let

$$L^{-1}(z) = T(zH_1 + H_0)^{-1}T^* \tag{7.2}$$

be a minimal hermitian realization of L^{-1}. We reduce the problem to the pencil $zH_1 + H_0$. Let a be a real characteristic root of L and let the principal part of L^{-1} at the pole a be given by (5.4). For $\varepsilon > 0$ the signatures of $L^{-1}(a+\varepsilon)$ and $L^{-1}(a-\varepsilon)$ are equal. Recall that $\tilde{C}(a)$ in (5.4) has full column rank. Hence for each odd n_i the diagonal entry $\varepsilon_i(z-a)^{n_i}$ has to be matched by a corresponding entry $-\varepsilon_i(z-a)^{-m_i}$ with odd m_i. This implies that the normal form of the pencil $zH_1 + H_0$ in (7.2) contains the pair of blocks

$$B := \begin{pmatrix} D_{n_i}(z;a) & 0 \\ 0 & -D_{m_i}(z;a) \end{pmatrix}$$

or $-B$. From (4.1) we see that B can be written as

$$B = G \operatorname{block\,diag}\left(z^{n_i}, -z^{m_i}, F_{n_i}, -F_{m_i}\right) G^*$$

where G is a suitable polynomial matrix. According to Lemma 4.2 a block F_i can be factored in the form (7.1). Put

$$R(z) := \begin{pmatrix} z^k & 0 \\ 0 & z^l \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1+z & 1-z \\ 1-z & 1+z \end{pmatrix}$$

with $n_i = 2k+1$, $m_i = 2l+1$. Then

$$\begin{pmatrix} z^{n_i} & 0 \\ 0 & -z^{m_i} \end{pmatrix} = R^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R$$

and B is of the form (7.1). Using (4.1) again we obtain a desired factorization of $D_r(z;a)$ in the case of even r. Finally,

$$\begin{pmatrix} 0 & D_s \\ \bar{D}_s & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & D_s \end{pmatrix}^* \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & D_s \end{pmatrix}$$

takes the blocks of type II into account. As we have considered all possible block types in the normal form of $zH_1 + H_0$, the proof is complete.

References

[1] B.D. Anderson and J.B. Moore, *Optimal Filtering* (Prentice-Hall, Englewood Cliffs, NJ, 1979).

[2] P.A. Fuhrmann, *Linear Operators and Systems in Hilbert Space* (McGraw-Hill, New York, 1981).

[3] P.A. Fuhrmann, On symmetric rational transfer functions, to appear.

[4] I. Gohberg, P. Lancaster, L. Rodman, Spectral analysis of selfadjoint matrix polynomials, *Annals of Math.* 112 (1980) 37–41.

[5] I. Gohberg, P. Lancaster, L. Rodman, Factorization of selfadjoint matrix polynomials with constant signature, *Linear and Multilinear Algebra* 11 (1982) 209–224.

[6] R.C. Thompson, The characteristic polynomial of a principal subpencil of a hermitian matrix pencil, *Linear Algebra Appl.* 14 (1976) 137–177.

[7] J.C. Willems, Realizations of systems with internal passivity and symmetry constraints, *J. Franklin Inst.* 301 (1976) 605–621.

[8] H.K. Wimmer, The structure of non-singular polynomial matrices, *Math. Systems Theory* 14 (1981) 367–379.

[9] H.K. Wimmer, Indefinite inner product spaces associated with hermitian polynomial matrices, *Linear Algebra Appl.*, to appear.