ELSEVIER

Systems & Control Letters 44 (2001) 333–337

SYSTEMS & CONTROL LETTERS

www.elsevier.com/locate/sysconle

Homomorphisms of modules associated with polynomial matrices with infinite elementary divisors

Pudji Astuti[a, 1], Harald K. Wimmer[b, *]

[a] Departemen Matematika, Institut Teknologi Bandung, Bandung 40132, Indonesia
[b] Mathematisches Institut, Universität Würzburg, D-97074 Würzburg, Germany

Received 17 November 1999; received in revised form 29 January 2001; accepted 22 June 2001

Abstract

If the inverse of a nonsingular polynomial matrix L has a polynomial part then one can associate with L a module over the ring of proper rational functions, which is related to the structure of L at infinity. In this paper we characterize homomorphisms of such modules. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Polynomial matrices; Proper rational functions; Module homomorphisms; Duality; Infinite elementary divisors; Coprimeness

1. Introduction

According to Rosenbrock [6] a transfer matrix $G \in K^{m\times p}(s)$ of rational functions over a field K admits a *generalized state space realization*

$$G(s) = (C_1 \quad C_2)\begin{pmatrix} sI - A_1 & 0 \\ 0 & sN_2 - I \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

such that

$$G_1(s) = C_1(sI - A_1)^{-1}B_1 \tag{1.1}$$

is the strictly proper part and

$$G_2(s) = C_2(sN_2 - I)^{-1}B_2, \tag{1.2}$$

where N_2 is nilpotent, is the polynomial part of G. It is well known that the realizations (1.1) and (1.2) can be constructed by module theoretic approaches. In the case of (1.1) a construction is due to Fuhrmann [2]. For a realization theory of anticausal input output maps we refer to Conte and Perdon [1]. To describe the polynomial models that serve as state spaces for (1.1) and (1.2) we use the following notation. A rational function $f \in K(s)$ is called *proper* or *causal* (resp. *strictly proper* or *strictly causal*) if $f = 0$ or if $f \neq 0$ and $f = p/q$, $p, q \in K[s]$, $q \neq 0$, and $\deg p \leqslant \deg q$ (resp. $\deg p < \deg q$). Let $K_\infty(s)$ denote the ring of proper rational functions over K. Then

$$K(s) = K[s] \oplus s^{-1}K_\infty(s). \tag{1.3}$$

To (1.3) correspond projection operators

$$\pi_- : K(s) \to s^{-1}K_\infty(s)$$

and

$$\pi_+ = (I - \pi_-) : K(s) \to K[s].$$

* Corresponding author. Tel.: +49-931-888-5007; fax: +49-931-888-4611.

E-mail address: wimmer@mathematik.uni-wuerzburg.de (H.K. Wimmer).

[1] Supported by Deutscher Akademischer Austauschdienst under Award No. A/98/25636.

PII: S0167-6911(01)00151-7

Put

$$(f)_0 = (\pi_+ f)(0), \quad f \in K(s). \tag{1.4}$$

The decomposition (1.3), the projections π_- and π_+, and definition (1.4) extend naturally from $K(s)$ to $K^n(s)$ and $K^{m\times p}(s)$.

Let $G \in K^{m\times p}(s)$ have a realization

$$G = W_1 + P_1 D_1^{-1} Q_1, \tag{1.5}$$

where W_1, P_1, Q_1, D_1 are polynomial matrices, with D_1 of size $n_1 \times n_1$. In Fuhrmann's theory [4] a state space for a realization (1.1) of $\pi_- G$ is provided by

$$V_{D_1} = K_1^n[s]/D_1 K_1^n[s].$$

Obviously, V_{D_1} is a $K[s]$-module and thus also a vector space over K. The counterpart of (1.5) is a realization

$$G = W_2 + P_2 D_2^{-1} Q_2, \tag{1.6}$$

where P_2 and Q_2 are proper rational matrices, W_2 is strictly proper rational and D_2 is a polynomial matrix, $D_2 \in K^{n_2\times n_2}[s]$. Define

$$U^{D_2} = K_\infty^{n_2}(s)/(K_\infty^{n_2}(s) \cap D_2 s^{-1} K_\infty^{n_2}(s)). \tag{1.7}$$

Then U^{D_2} is a $K_\infty(s)$-module and at the same time a K-vector space. At the end of this section we shall indicate why U^{D_2} can be taken as a state space of a realization (1.2) of $\pi_+ G$. Let us mention that the finite and infinite *pole modules* (see [9]) of $G(s)$ are given by V_{D_1} and U^{D_2}, if (1.5) is an irreducible realization and (1.6) satisfies coprimeness conditions of the form (3.14).

We note that a nonsingular polynomial matrix $L \in K^{n\times n}[s]$ gives rise to two types of modules, namely the $K[s]$-module

$$V_L = K^n[s]/LK^n[s]$$

and the $K_\infty(s)$-module

$$U^L = K_\infty^n(s)/(K_\infty^n(s) \cap Ls^{-1} K_\infty^n(s)). \tag{1.8}$$

Besides realizations there is a wide range of issues such as similarity of state space models, system equivalence or simulation of restricted input output maps which involve two polynomial matrices L and L_1 and homomorphisms from V_L to V_{L_1} and from U^L to U^{L_1}. The $K[s]$-module homomorphisms from V_L to V_{L_1} are well understood. According to Fuhrmann [4] their description is based on intertwining relations between L and L_1. In this note we study $K_\infty(s)$-module homomorphisms from U^L to U^{L_1}. Our characterizations will be in correspondence with Fuhrmann's results in Ref. [2,4]. Comparing the definitions of V_L and U^L we observe that $LK^n[s]$ is a submodule of $K^n[s]$ whereas in general $Ls^{-1}K_\infty^n(s)$ is not contained in $K_\infty^n(s)$. Hence it is not surprising that U^L is less easy to handle than V_L and that in our study technical obstacles have to be removed which do not appear in the case of the module V_L.

To obtain a concrete representation of U^L we define a map

$$\rho^L : K_\infty^n(s) \to K^n[s]$$

by

$$\rho^L x = L\pi_+ L^{-1} x, \quad x \in K_\infty^n(s).$$

Put $\bar{x} = \rho^L x$. For $q \in K_\infty(s)$ and $\bar{x} \in \operatorname{Im} \rho^L$ we set $q \cdot \bar{x} = \overline{qx}$. This product is well defined since

$$\operatorname{Ker} \rho^L = (K_\infty^n(s) \cap s^{-1} L K_\infty^n(s)).$$

Therefore $\operatorname{Im} \rho^L$ is a $K_\infty(s)$-module, isomorphic to the quotient module U^L in (1.8). From now on we identify both modules such that

$$U^L = \operatorname{Im} \rho^L = L\pi_+ L^{-1} K_\infty^n(s).$$

Clearly, $U^L = 0$ if sL^{-1} is proper rational. A shift operator $S_-(L)$ on U^L is given by

$$S_-(L)\bar{x} = s^{-1} \cdot \bar{x}, \quad \bar{x} \in U^L.$$

Clearly, $S_-(L)$ is a nilpotent endomorphism of U^L.

Let us now give a concrete example for the use of a $K_\infty(s)$-module U^L. Based on the representation (1.6) of G we derive a realization of $\pi_+ G$ having U^{D_2} as its state space. We adapt a construction of [3]. Assume $\pi_+ G(s) = \sum_{v=0}^t G_v s^v$. Define the map $B_2 : K^p \to U^{D_2}$ by

$$B_2 \xi = \rho^{D_2} Q_2 \xi, \quad \xi \in K^p.$$

Put $N_2 = S_-(D_2)$ and define $C_2 : U^{D_2} \to K^m$ by

$$C_2 \bar{x} = -(P_2 D_2^{-1} \bar{x})_0, \quad \bar{x} \in U^{D_2}.$$

Then a straightforward computation yields

$$G_v = -C_2 N_2^v B_2, \quad v = 0, 1, \ldots, t$$

such that

$$\sum_{v=0}^t G_v s^v = C_2 (sN_2 - I)^{-1} B_2.$$

2. Basic facts of U^L

For a nonzero proper rational function $f = p/q$, $p, q \in K[s]$, let a degree function be defined by $\delta(p/q) = \deg q - \deg p$. It is well known that

$(K_\infty(s), \delta)$ is a euclidean domain. The units $K_\infty^*(s)$ are the proper rational functions f with $\delta f = 0$. The ideal (s^{-1}) is the unique maximal ideal of $K_\infty(s)$. Let us call a matrix $P \in K_\infty^{n\times n}(s)$ *bicausal* if $\det P \in K_\infty^*(s)$, i.e. if P is invertible in $K_\infty^{n\times n}(s)$. If $W \in K^{m\times r}(s)$ has rank n then there exist bicausal matrices P and Q such that

$$W = P\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} Q$$

with

$$\Sigma = \operatorname{diag}(s^{-\alpha_1}, \dots, s^{-\alpha_t}, s^{\beta_{t+1}}, \dots, s^{\beta_n}),$$

$$-\alpha_1 \leqslant \cdots \leqslant -\alpha_t < 0 \leqslant \beta_{t+1} \leqslant \cdots \beta_n. \tag{2.1}$$

The integers $-\alpha_1, \dots, \beta_n$ are uniquely determined by W. In particular, if $L \in K^{n\times n}[s]$ is nonsingular then

$$s^{-1}L = P\Sigma Q \tag{2.2}$$

for some $P, Q \in K_\infty^{n\times n}(s)^*$ and Σ as in (2.1). In the case of a linear pencil $L(s) = A_0 - A_1 s$ the polynomials $s^{\alpha_1}, \dots, s^{\alpha_t}$ are the elementary divisors of $A_0 s - A_1$ belonging to the characteristic root 0. According to [7] the matrix Σ in (2.2) and (2.1) provides information on the structure of U^L. We have

$$U^L \cong \bigoplus \{K_\infty(s)/s^{-\alpha_j} K_\infty(s),\ j = 1, \dots, t\}$$

such that U^L is a finitely generated torsion module over $K_\infty(s)$ with elementary divisors

$$s^{-\alpha_1}, \dots, s^{-\alpha_t}. \tag{2.3}$$

We call (2.3) the *infinite elementary divisors* of L. Then $s^{\alpha_1}, \dots, s^{\alpha_t}$ are the elementary divisors of the shift $S_-(L)$, and $\dim_K U^L = \alpha_1 + \cdots + \alpha_t$. To describe a dual pairing [8] between the K-linear spaces $U^{L^{\mathrm{T}}}$ and U^L we note that

$$\langle \bar{y}, \bar{x} \rangle = (y^{\mathrm{T}} L^{-1} x)_0, \quad \bar{y} \in U^{L^{\mathrm{T}}},\ \bar{x} \in U^L, \tag{2.4}$$

is a well-defined nondegenerate bilinear form on $U^{L^{\mathrm{T}}} \times U^L$.

3. Homomorphisms

Our main result is Theorem 3.3 below. Its proof will be based on the subsequent two lemmas. In the following $L \in K_\infty^{n\times n}(s)$ and $L_1 \in K_\infty^{n_1\times n_1}(s)$ will be fixed nonsingular polynomial matrices.

Lemma 3.1. *A map*

$$\Phi : K_\infty^n(s) \to U^{L_1} \tag{3.1}$$

is a $K_\infty(s)$-module homomorphism if and only if there exists a matrix $\Theta \in K_\infty^{n_1\times n}(s)$ such that

$$\Phi x = \rho^{L_1}(\Theta x), \quad x \in K_\infty^n(s). \tag{3.2}$$

Proof. Let $e_1, \dots, e_n$ be the standard basis of K^n. Assume that Φ in (3.1) is a $K_\infty(s)$-module homomorphism. Then $\Phi e_i = \rho^{L_1}\theta_i$ for some $\theta_i \in K_\infty^{n_1}(s)$ and (3.2) holds with $\Theta = (\theta_1, \dots \theta_n)$. The converse is obvious. □

Condition (3.3) below together with a somewhat technical equivalent condition will be crucial.

Lemma 3.2. *We have*

$$\Theta \operatorname{Ker} \rho^L \subseteq \operatorname{Ker} \rho^{L_1}. \tag{3.3}$$

with $\Theta \in K_\infty^{n_1\times n}(s)$ if and only if there exist a matrix $\Theta_1 \in K_\infty^{n_1\times n}(s)$ and a matrix Ψ satisfying

$$\Psi \in s^{-1}K_\infty^{n_1\times n}(s) \quad \text{and} \quad L_1\Psi \in K_\infty^{n_1\times n}(s) \tag{3.4}$$

such that

$$(\Theta + L_1\Psi)L = L_1\Theta_1. \tag{3.5}$$

Proof. It is evident that (3.5) implies (3.3). To prove the converse implication we note that (3.3) is equivalent to $\Theta \operatorname{Ker} \rho^L \subseteq s^{-1}L_1 K_\infty^{n_1}(s)$. If $s^{-1}L$ is factorized as in (2.2),

$$\begin{aligned} &s^{-1}L = P\Sigma Q, \quad \Sigma = \operatorname{diag}(A, B), \\ &A = \operatorname{diag}(s^{-\alpha_1}, \dots, s^{-\alpha_t}), \quad B = \operatorname{diag}(s^{\beta_{t+1}}, \dots, s^{\beta_n}), \end{aligned} \tag{3.6}$$

then $\operatorname{Ker} \rho^L = P \operatorname{diag}(A, I) K_\infty^n(s)$. Hence if

$$G = L_1^{-1}\Theta P \operatorname{diag}(A, I),$$

then (3.3) is equivalent to $G \in s^{-1}K_\infty^{n_1\times n}(s)$. From (3.6) and

$$\Sigma = \operatorname{diag}(A, 0) + \operatorname{diag}(0, B),$$

we obtain

$$L_1^{-1}\Theta L = G \operatorname{diag}(I, 0)Q + L_1^{-1}\Theta P \operatorname{diag}(0, I)P^{-1}L.$$

Now choose

$$\Psi = -G \operatorname{diag}(I, 0)Q.$$

Then Ψ satisfies (3.4) and if we put $\Theta_1 = L_1^{-1}\Theta L + \Psi L$ then we have $\Theta_1 \in K_\infty^{n_1\times n}(s)$, which proves (3.5). □

We extend the map ρ^{L_1} to $K^n(s)$ and define

$$\rho_e^{L_1} = L_1\pi_+ L_1^{-1}w, \quad w \in K^n(s).$$

Theorem 3.3. *The map $\phi : U^L \to U^{L_1}$ is a $K_\infty(s)$-module homomorphism if and only if there exist matrices $\Theta, \Theta_1 \in K_\infty^{n_1\times n}(s)$ such that*

$$\Theta L = L_1 \Theta_1 \tag{3.7}$$

and

$$\phi \bar{x} = \rho_e^{L_1} \Theta \bar{x}, \quad \bar{x} \in U^L. \tag{3.8}$$

If (3.7) *holds then we have*

$$\rho_e^{L_1} \Theta \bar{x} = \rho^{L_1} \Theta x \tag{3.9}$$

for all $x \in K_\infty^n(s)$.

Proof. Let us show first that (3.7) implies (3.9). We have

$$\rho_e^{L_1}\Theta\bar{x} = L_1\pi_+ L_1^{-1}\Theta\bar{x} = L_1\pi_+\Theta_1 L^{-1}\bar{x}$$
$$= L_1\pi_+\Theta_1 L^{-1}x = L_1\pi_+ L_1^{-1}\Theta x = \rho^{L_1}\Theta x. \tag{3.10}$$

Now let $\phi : U^L \to U^{L_1}$ be a $K_\infty(s)$-module homomorphism. Define $\Phi = \phi\rho^L$ such that

$$\Phi x = \phi\bar{x}, \quad x \in K_\infty^n(s). \tag{3.11}$$

Then $\Phi : K_\infty^n(s) \to U^{L_1}$ is also a $K_\infty(s)$-module homomorphism. Because due to Lemma 3.1 there exists a $\tilde{\Theta} \in K_\infty^{n_1\times n}(s)$ such that

$$\Phi x = \rho^{L_1}\tilde{\Theta}x. \tag{3.12}$$

It follows from (3.11) that $x, v \in K_\infty^n(s)$ and $\bar{x} = \bar{v}$ imply $\rho^{L_1}\tilde{\Theta}x = \rho^{L_1}\tilde{\Theta}v$. Therefore we obtain

$$\tilde{\Theta}\,\mathrm{Ker}\,\rho^L \subseteq \mathrm{Ker}\,\rho^{L_1}. \tag{3.13}$$

We can replace $\tilde{\Theta}$ in (3.12) and (3.13) by $\Theta = \tilde{\Theta} + L_1\Psi$ if $\Psi \in s^{-1}K_\infty^{n_1\times n}(s)$ and $L_1\Psi \in K_\infty^{n_1\times n}(s)$. From Lemma 3.2 we know that starting from (3.13) we can find a Ψ which yields (3.7) with $\Theta_1 \in K_\infty^{n_1\times n}(s)$. Thus we have shown that

$$\phi\bar{x} = \rho^{L_1}\Theta x = \rho_e^{L_1}\Theta\bar{x}$$

with Θ satisfying a relation (3.7).

Conversely, if a map $\phi : U^L \to U^{L_1}$ is defined by (3.7) and (3.8) then it is easy to verify that ϕ is a $K_\infty(s)$-module homomorphism. □

We remark that Theorem 3.3 remains true if condition (3.7) is replaced by

$$\pi_+ L_1^{-1}\Theta = \pi_+\Theta_1 L^{-1}.$$

Given the duality (2.4) between U^L and $U^{L^{\mathrm{T}}}$ it is not difficult to obtain the dual map of ϕ. We set $\bar{\bar{w}} = \rho^{L_1^{\mathrm{T}}}w$, $w \in K_\infty^{n_1}(s)$.

Theorem 3.4. *Let $\Theta, \Theta_1 \in K_\infty^{n_1\times n}(s)$ be such that $\Theta L = L_1\Theta_1$. Let $\phi : U^L \to U^{L_1}$ be defined by* (3.8). *Then the dual map*

$$\phi^* : U^{L_1^{\mathrm{T}}} \to U^{L^{\mathrm{T}}}$$

is given by

$$\phi^*\bar{\bar{w}} = \rho^{L^{\mathrm{T}}}\Theta_1^{\mathrm{T}}w, \quad \bar{\bar{w}} \in U^{L_1^{\mathrm{T}}}.$$

We now turn to surjectivity and injectivity. For a pair $\Theta \in K_\infty^{n_1\times n}(s)$ and $L_1 \in K^{n_1\times n_1}$ we set $(\Theta, s^{-1}L_1)_l = I$ if there exist proper rational matrices C and D such that

$$\Theta C + s^{-1}L_1 D = I. \tag{3.14}$$

Similarly, for $\Theta_1 \in K_\infty^{n_1\times n}(s)$ and $L \in K^{n\times n}$ we write $(\Theta_1, s^{-1}L)_r = I$ if $(\Theta_1^{\mathrm{T}}, s^{-1}L^{\mathrm{T}})_l = I$.

Theorem 3.5. *Let $\phi : U^L \to U^{L_1}$ be defined by* (3.9) *and* (3.7). *Then*

(i) *ϕ is surjective if and only if $(\Theta, s^{-1}L_1)_l = I$,*
(ii) *ϕ is injective if and only if $(\Theta_1, s^{-1}L)_r = I$.*

Proof. (i) Assume first that ϕ is surjective. Let $w \in K_\infty^{n_1}(s)$ be given. Then $\rho^{L_1}w = \rho^{L_1}\Theta v$ for some $v \in K_\infty^n(s)$. We have $w - \Theta v \in \mathrm{Ker}\,\rho^{L_1}$, which implies

$$w \in \Theta K_\infty^n(s) + s^{-1}L_1 K_\infty^n(s)$$

or equivalently $(\Theta, s^{-1}L_1)_l = I$. Conversely, suppose that (3.14) holds. To show that $w = \rho^{L_1}x$ is in ϕU^L we note that (3.14) implies $x = \Theta v + s^{-1}L_1 x_2$ for some $v \in K_\infty^n(s)$, $x_2 \in K_\infty^{n_1}(s)$. Because of $s^{-1}L_1x_2 \in \mathrm{Ker}\,\rho^{L_1}$ we obtain $w = \rho^{L_1}\Theta v = \phi\bar{v}$.

(ii) By duality the statement follows at once from (i). □

If M is a finitely generated p-module over a principal ideal domain and S is a submodule and Q is a quotient module of M then the relations between the invariants of M and those of S and Q are well known (see e.g. [5, p. 92, 93]). We complete our note with a corresponding observation on the existence of surjective and injective homomorphisms. Let

$$s^{-\alpha_1}, \ldots, s^{-\alpha_t}, \quad \alpha_1 \geqslant \cdots \geqslant \alpha_t$$

and

$$s^{-\gamma_1}, \ldots, s^{-\gamma_p}, \quad \gamma_1 \geqslant \cdots \geqslant \gamma_p$$

be the infinite elementary divisors of L and L_1, respectively. Then there exists a surjective $K_\infty(s)$-module homomorphism $\phi : U^L \to U^{L_1}$ if and only if

$$t \geqslant p \quad \text{and} \quad \alpha_1 \geqslant \gamma_1, \ldots, \alpha_p \geqslant \gamma_p$$

and there exists an injective ϕ if and only if

$$t \leqslant p \quad \text{and} \quad \alpha_1 \leqslant \gamma_1, \ldots, \alpha_t \leqslant \gamma_t.$$

References

[1] G. Conte, A.M. Perdon, Generalized state-space realizations for non-proper rational transfer functions, Systems Control Lett. 1 (1981) 270–276.

[2] P.A. Fuhrmann, Algebraic system theory: an analyst's point of view, J. Franklin Inst. 301 (1976) 521–540.

[3] P.A. Fuhrmann, On strict system equivalence, Internat. J. Control 25 (1977) 5–10.

[4] P.A. Fuhrmann, Linear Systems and Operators in Hilbert Space, McGraw-Hill, New York, 1981.

[5] P. Ribenboim, Rings and Modules, Interscience, New York, 1969.

[6] H.H. Rosenbrock, Structural properties of linear dynamical systems, Internat. J. Control 20 (1974) 191–202.

[7] H.K. Wimmer, The structure of nonsingular polynomial matrices, Math. Systems Theory 14 (1981) 367–379.

[8] H.K. Wimmer, Polynomial matrices and dualities, Systems Control Lett. 1 (1981) 200–203.

[9] B.F. Wyman, M.K. Sain, G. Conte, A.M. Perdon, Poles and zeros of matrices of rational functions, Linear Algebra Appl. 157 (1991) 113–139.