Polynomial matrices and dualities

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With a non-singular polynomial matrix $L \in F^{n\times n}[z]$ over a field F we associate two vector spaces V_L and U^L with respective shift operators S^+ and S^-. A duality between V_L and V_{L^T} and between U^L and U^{L^T} is established. The spaces V_L and U^L and the maps S^+ and S^- determine L in the following way. Let the columns of the polynomial matrices $\hat{C}$ and $\hat{B}^T$ form a pair of dual bases of V_L and V_{L^T} and let A be the matrix of S^+ with respect to the basis $\hat{C}$, $S^+\hat{C} = \hat{C}A$, furthermore let the columns of $\hat{H}$ and $\hat{G}^T$ be dual bases of U^L and U^{L^T} and let N be the matrix of S^- with respect to $\hat{H}$, $S^-\hat{H} = \hat{H}N$, then L is completely determined by these matrices. L can be factored as

$$L(z) = \begin{pmatrix} \hat{C}(z) & \hat{H}(z) \end{pmatrix} \begin{pmatrix} zI - A & 0 \\ 0 & I + zN \end{pmatrix} \begin{pmatrix} \hat{B}(z) \\ \hat{G}(z) \end{pmatrix}.$$

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1. Introduction, notation, preliminaries

In this note we continue our investigation [3] and will show that a non-singular polynomial matrix is completely determined by two pairs of dual vector spaces and by two shift operators.

Let F be a field. $F((z^{-1}))$ shall denote the set of all truncated Laurent series with coefficients in F, i.e. the set of all formal series of the form

$$f(z) = \sum_{-\infty < i \le k} f_i z^i, \quad f_i \in F, k \in \mathbb{Z}. \tag{1.1}$$

$F[[z^{-1}]]$ is the set of all formal power series in z^{-1} and $z^{-1}F[[z^{-1}]]$ is the subset of those with vanishing constant term. If f is given by (1.1) then the projection π_- of $F((z^{-1}))$ onto $z^{-1}F[[z^{-1}]]$ is defined by $\pi_- f(z) = \sum_{i<0} f_i z^i$ and the projection $\pi_+ : F((z^{-1})) \to F[z]$ by $\pi_+ f(z) = \sum_{i \ge 0} f_i z^i$. The set of rational functions $F(z)$ can be imbedded into $F((z^{-1}))$ and we call an element of $\pi_- F(z)$ *proper rational.* $(f)_i$ denotes the coefficient f_i in (1.1). The preceding definitions will be extended in a natural way to $F^n((z^{-1}))$ and $F^{n\times m}((z^{-1}))$.

Throughout this note $L \in F^{n\times n}[z]$ will be a non-singular polynomial matrix, i.e. det $L \ne 0 \in F[z]$. We associate the following two mappings with L. Define $\pi_L : F^n[z] \to F^n[z]$ by

$$\pi_L b := L\pi_- L^{-1} b, \quad b \in F^n[z]$$

and put

$$V_L := \operatorname{Im} \pi_L .$$

Then [1] V_L is an $F[z]$-module with $p \cdot v = \pi_L pv$, $p \in F[z]$, $v \in V_L$. Moreover, V_L is a vector space over F with $\dim V_L = \deg\det L$. A right shift operator S^+ on V_L is given by $S^+ v := z \cdot v$. The second mapping $\rho^L : F^n[[z^{-1}]] \to F^n[z]$ is defined by [3]

$$\rho^L y := L\pi_+ L^{-1} y, \quad y \in F^n\big[[z^{-1}]\big].$$

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Its range $U^L := \operatorname{Im} \rho^L$ is an $F[[z^{-1}]]$-module with a scalar multiplication $q \cdot w := \rho^L q w$, $q \in F[[z^{-1}]]$, $w \in U^L$. As a vector space U^L has a dimension which is equal to the multiplicity of the characteristic root $\lambda = \infty$ of L. The left shift S^- on U^L, given by $S^- w = z^{-1} \cdot w$, is a nilpotent operator on U^L.

Let $W \in F^{m \times n}(z)$, $W \neq 0$, be a matrix of rational functions. A factorization

$$W(z) = Q(Rz + S)^{-1} P \tag{1.2}$$

with R and S in $F^{l \times l}$, $P \in F^{l \times n}$ and $Q \in F^{m \times l}$ is called a *realization* of W. The realization (1.2) is *minimal*, if the size of R and S is minimal. We recall that a minimal realization of L^{-1} yields bases for V_L and U^L.

Theorem 1.1 [3]. *Let $L^{-1}(z) = Q(Rz + S)^{-1} P$ be a minimal realization such that*

$$Rz + S = \begin{pmatrix} zI - A & 0 \\ 0 & I - zN \end{pmatrix} \tag{1.3}$$

and N is nilpotent. If $Q = (C\ \ H)$ and $P = (B\ \ G)^{\mathrm{T}}$ are partitioned according to (2.3), *then the columns of $\hat{C}(z) := L(z)C(zI - A)^{-1}$ form a basis of V_L and $S^+\hat{C} = \hat{C}A$, i.e. A is the matrix of the shift operator S^+ with respect to this basis. The columns of $\hat{H}(z) := L(z)H(I - zN)^{-1}$ are a basis of U^L and $S^-\hat{H} = \hat{H}N$. Define $\hat{B}(z) := (zI - A)^{-1} BL(z)$ and $\hat{G}(z) := (I - zN)^{-1} GL(z)$. Then the columns of $\hat{B}^{\mathrm{T}}$ and $\hat{G}^{\mathrm{T}}$ have analogous properties with respect to L^{T}. Furthermore, L can be factored as*

$$L(z) = \hat{Q}(z)(Rz + S)\hat{P}(z) \tag{1.4}$$

with $\hat{Q} = (\hat{C}\ \ \hat{H})$ and $\hat{P} = (\hat{B}\ \ \hat{G})^{\mathrm{T}}$. L and $Rz + S$ have the same finite and infinite elementary divisors.

2. Dual pairings

In [2] Fuhrmann introduced the following dual pairing of V_L and $V_{L^{\mathrm{T}}}$.

Lemma 2.1 [2]. *For $f \in V_L$ and $g \in V_{L^{\mathrm{T}}}$ let $[f, g]$ be defined by*

$$[f, g] = (g^{\mathrm{T}} L^{-1} f)_{-1}. \tag{2.1}$$

Then (2.1) *is a scalar product on $V_L \times V_{L^{\mathrm{T}}}$.*

The preceding pairing of V_L and $V_{L^{\mathrm{T}}}$ produces dual bases in Theorem 1.1.

Theorem 2.1. *Let $\pi_- L^{-1}(z) = C(zI - A)^{-1} B$ be a minimal realization of the proper rational part of L^{-1}. The bases of V_L and $V_{L^{\mathrm{T}}}$ formed by the columns of $\hat{C}(z) := L(z)C(zI - A)^{-1}$ and $[\hat{B}(z)]^{\mathrm{T}} = [(zI - A)^{-1} BL(z)]^{\mathrm{T}}$ are dual ones with respect to the scalar product* (2.1).

Proof. We have to show

$$(\hat{B} L^{-1} \hat{C})_{-1} = I_r \tag{2.2}$$

where $r = \dim V_L$ and $A \in F^{r \times r}$. Put $M := \pi_-(\hat{B} L^{-1} \hat{C})$. We will prove $M(z) = (zI - A)^{-1}$ which implies (2.2).

We have $M(z) = \pi_-(\hat{B}(z) C(zI - A)^{-1})$ and

$$M(z) A^i B = \pi_-\left(\hat{B}(z) C(zI - A)^{-1} B z^i\right) = \pi_-\left(\hat{B}(z) L^{-1}(z) z^i\right) = (zI - A)^{-1} A^i B.$$

The realization $C(zI - A)^{-1} B$ is minimal, therefore rank $(B, AB, \ldots, A^{r-1}B) = r$ and $M(z) = (zI - A)^{-1}$.

We now turn to U^L.

Lemma 2.2. *Let $u \in U^L$ and $w \in U^{L^{\mathrm{T}}}$ be given and let $x=(\rho^L)^{-1}(u)$ and $y=(\rho^{L^{\mathrm{T}}})^{-1}(w)$ be pre-images of u and w. Then*

$$\langle u, w\rangle = \left(y^{\mathrm{T}}L^{-1}x\right)_0 \tag{2.3}$$

defines a scalar product on $U^L \times U^{L^{\mathrm{T}}}$.

Proof. Obviously (2.3) is a bilinear form. It is well defined, since $\rho^L d=0$ implies $(y^{\mathrm{T}}L^{-1}d)_0=0$. The proof of the next theorem will show that (2.3) is non-degenerate.

Theorem 2.2. *Let $\pi_+ L^{-1}(z)=H(I-zN)^{-1}G$ be a minimal realization of the polynomial part of L^{-1}. The columns of $\hat{H}(z)=L(z)H(I-zN)^{-1}$ and of $[\hat{G}(z)]^{\mathrm{T}}=[(I-zN)^{-1}GL(z)]^{\mathrm{T}}$ are bases of U^L and $U^{L^{\mathrm{T}}}$ which are dual with respect to* (2.3).

Proof. Put $s:=\dim U^L$. Let X and Y^{T} be two matrices in $F^{n\times s}[[z^{-1}]]$ such that $\rho^L X=\hat{H}$ and $(\rho^{L^{\mathrm{T}}})Y^{\mathrm{T}}=\hat{G}^{\mathrm{T}}$ holds. Our target equation $(YL^{-1}X)_0=I_s$ follows from $\pi_+(Y(z)L^{-1}(z)X(z))=(I-zN)^{-1}$. To show this put $K:=\pi_+(YL^{-1}X)$. Then

$$K=\pi_+\left[\pi_+(YL^{-1})X\right]=\pi_+(\hat{G}L^{-1}X).$$

The proper rational part of L^{-1} can be ignored in the following calculations. For $i\geqslant 0$ we have

$$\begin{aligned} HN^iK(z) &= \pi_+\left(H(I-zN)^{-1}z^{-i}GX(z)\right)=\pi_+\left(z^{-i}L^{-1}(z)X(z)\right)\\ &= \pi_+\left[z^{-i}\pi_+\left(L^{-1}(z)X(z)\right)\right]=HN^i(I-zN)^{-1}. \end{aligned}$$

Because $H(I-zN)^{-1}G$ is a minimal realization of π_+L^{-1} the equations

$$HN^iK(z)=HN^i(I-zN)^{-1},\quad i=0,1,2,\ldots$$

imply $K(z)=(I-zN)^{-1}$.

The preceding observations lead to the following result.

Theorem 2.3. *Let the matrices $\hat{M}$ and $\hat{K}^{\mathrm{T}}$ be in $F^{n\times r}[z]$ such that their columns make up dual bases for V_L and $V_{L^{\mathrm{T}}}$ and let E be the matrix of the shift S^+ with respect to the basis $\hat{M}$. Furthermore, let the columns of $\hat{T}\in F^{n\times s}[z]$ and $\hat{R}^{\mathrm{T}}\in F^{n\times s}[z]$ be dual bases of U^L and $U^{L^{\mathrm{T}}}$ and let D be the matrix of S^- with respect to the basis $\hat{T}$. Then*

$$L(z)=\begin{pmatrix}\hat{M}(z) & \hat{T}(z)\end{pmatrix}\begin{pmatrix} zI-E & 0\\ 0 & I-zD\end{pmatrix}\begin{pmatrix}\hat{K}(z)\\ \hat{R}(z)\end{pmatrix}. \tag{2.4}$$

Proof. We shall work with the minimal realization of L^{-1} and the bases which are given by Theorem 1.1. Since $\hat{C}(z)=L(z)C(zI-A)^{-1}$ provides a basis of V_L we have $\hat{C}(z)=\hat{M}(z)X$ for some non-singular $X\in F^{r\times r}$. Therefore $\hat{M}(z)=L(z)CX(zI-XAX^{-1})^{-1}$ and the matrix E of S^+ with respect to the basis $\hat{M}$ is equal to XAX^{-1}. It follows from Theorem 2.1 that the bases given by $\hat{C}$ and $\hat{B}^{\mathrm{T}}$ are dual ones. Hence $\hat{B}=X^{-1}\hat{K}$. Similarly $\hat{H}=\hat{T}Y$, $\hat{G}=X^{-1}\hat{R}$ and $D=YNY^{-1}$ for a suitable non-singular Y. Now (1.4) implies

$$L(z)=\begin{pmatrix}\hat{M}(z) & \hat{T}(z)\end{pmatrix}\begin{pmatrix} X & 0\\ 0 & Y\end{pmatrix}\begin{pmatrix} zI-A & 0\\ 0 & I-zN\end{pmatrix}\begin{pmatrix} X^{-1} & 0\\ 0 & Y^{-1}\end{pmatrix}\begin{pmatrix}\hat{K}(z)\\ \hat{R}(z)\end{pmatrix}$$

which is (2.4).

References

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