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INERTIAL SIGNATURES OF HERMITIAN POLYNOMIAL MATRICES*

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Abstract. Signs associated with real characteristic roots of a hermitian polynomial matrix $L(s)$ are determined from signatures of Hankel matrices of $L(s)^{-1}$.

Key words. hermitian matrix pencil, polynomial matrix, Weierstraß canonical form, inertial signs, signature, Hankel matrix

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1. Introduction.

We first review the notion of inertial signs of a hermitian pencil $P(s) = A_0 + A_1 s$. Suppose $A_0, A_1 \in \mathbb{C}^{n\times n}$ are hermitian, and A_1 is nonsingular. Then $\det P \neq 0$ (zero polynomial) and P^{-1} is strictly proper rational. Let $\sigma(P) = \{\lambda \mid \det P(\lambda) = 0\}$ denote the set of *characteristic values* of P. The following result goes back to Weierstraß (see [7], [5], [4]).

LEMMA 1.1. *Let $A_0 + A_1 s \in \mathbb{C}^{n\times n}[s]$ be a hermitian pencil and let A_1 be nonsingular. Then there exists a nonsingular matrix $T \in \mathbb{C}^{n\times n}$ such that $T(A_0 + A_1 s)T^*$ is the direct sum of blocks of types* I *and* II *as follows:*

$$\text{(I)}\qquad \epsilon\, D_r(s,\alpha) = \epsilon \begin{pmatrix} 0 & 0 & . & . & . & -1 & s-\alpha \\ 0 & 0 & . & . & . & s-\alpha & 0 \\ . & . & . & . & . & . & . \\ -1 & s-\alpha & . & . & . & 0 & 0 \\ s-\alpha & 0 & . & . & . & 0 & 0 \end{pmatrix}_{r\times r}$$

with $\alpha \in \mathbb{R}$ and $\epsilon \in \{1, -1\}$, and

$$\text{(II)}\qquad G_{2k}(s,\beta) = \begin{pmatrix} 0 & D_k(s,\beta) \\ D_k(s,\bar\beta) & 0 \end{pmatrix}_{2k\times 2k}$$

with $\beta \notin \mathbb{R}$. The direct sum $T(A_0 + A_1 s)T^* =$

$$\operatorname{diag}\big(\dots, \epsilon\, D_r(s,\alpha), \dots, G_{2k}(s,\beta), \dots\big) \tag{1.1}$$

is uniquely determined up to ordering of blocks.

The block diagonal matrix in (1.1) is the *Weierstraß canonical form* of the pencil $A_0 + A_1 s$. We observe that a number $\epsilon = \pm 1$ is attached to each block of type I. Thus a sign can be associated with each elementary divisor corresponding to a real characteristic root α.

DEFINITION 1.2. *Let $A_0 + A_1 s$ have π_i elementary divisors of the form $(s-\alpha)^i$ and let $\epsilon_{i1}, \dots, \epsilon_{i\pi_i}$ be the corresponding signs. Then*

$$(\dots, \epsilon_{i1}, \dots, \epsilon_{i\pi_i}, \dots) \tag{1.2}$$

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will be called the inertial signs of α*. Set*

$$\eta_i = \epsilon_{i1} + \cdots + \epsilon_{i\pi_i}, \tag{1.3}$$

such that $\eta_i = 0$ *if* $\pi_i = 0$*. We call the numbers* η_i *the inertial signatures of* α*.*

If $A \in \mathbb{C}^{n\times n}$ is a hermitian matrix with ℓ positive and ν negative eigenvalues (counting multiplicities), then the difference $\ell - \nu$ is the *signature* of A. It will be denoted by $\operatorname{sgn} A$. If $\pi_i \neq 0$, then

$$\eta_i = \operatorname{sgn} \operatorname{diag}(\epsilon_{i1}, \ldots, \epsilon_{i\pi_i}).$$

Definition 1.2 is motivated by [6]. The terminology is not uniform. The Cauchy characteristic in [2] includes elementary divisors and signs, and the term sign characteristic can be found in [3].

Now consider a nonsingular hermitian polynomial matrix $L \in \mathbb{C}^{n\times n}[s]$ such that

$$L(s) = A_0 + A_1\, s + \cdots + A_t\, s^t$$

and $A_i = A_i^*$, $i = 0, \ldots, t$. Assume that L^{-1} is strictly proper rational. Set $m = \deg\det L$. A factorization

$$L^{-1}(s) = C(A_0 + A_1 s)^{-1} C^* \tag{1.4}$$

is a *hermitian minimal realization* of L^{-1} if $P(s) = A_0 + A_1 s \in \mathbb{C}^{m\times m}[s]$ is a hermitian pencil and $C \in \mathbb{C}^{n\times m}$. The following observation (see, e.g., [9]) is an immediate consequence of Kalman's state space isomorphism theorem.

Lemma 1.3. *Let* $L \in \mathbb{C}^{n\times n}[s]$ *be a nonsingular hermitian polynomial matrix with* $\deg\det L = m$*. Suppose* L^{-1} *is strictly proper rational. Then* L^{-1} *admits a hermitian minimal realization. If* (1.4) *and*

$$L^{-1}(s) = \tilde{C}(\tilde{A}_0 + \tilde{A}_1 s)^{-1} \tilde{C}^*$$

are two hermitian minimal realizations, then A_1 *and* $\tilde{A}_1$ *are nonsingular, and there exists a nonsingular matrix* $T \in \mathbb{C}^{m\times m}$ *such that*

$$\tilde{A}_0 + \tilde{A}_1 s = T(A_0 + A_1 s)T^* \quad \text{and} \quad C = \tilde{C} T^*. \tag{1.5}$$

Set $\sigma(L) = \{\lambda \mid \det L(\lambda) = 0\}$. If (1.4) is a minimal hermitian realization, then $\sigma(L) = \sigma(A_0 + A_1 s)$, and (see, e.g., [1]) the elementary divisors of the pencil $A_0 + A_1 s$ are the same as those of the polynomial matrix L. Moreover the preceding lemma shows that the pencil $A_0 + A_1 s$ is determined by L up to congruence. This leads to the following definition of inertial signs and signatures of polynomial matrices.

Definition 1.4. *Let* $L \in \mathbb{C}^{n\times n}[s]$ *be nonsingular and hermitian. Suppose* L^{-1} *is strictly proper rational, and* $L^{-1}(s) = C(A_0 + A_1 s)^{-1} C^*$ *is a hermitian minimal realization. Let* $\alpha \in \sigma(L)$ *and* $\alpha \in \mathbb{R}$*. The inertial signs and signatures of* α *are defined to be those of the characteristic value* α *of the pencil* $A_0 + A_1 s$*.*

It is the purpose of this paper to determine inertial signatures of real characteristic values of L using Laurent expansions of L^{-1}. Without loss of generality we may assume $0 \in \sigma(L)$. We focus on the inertial signatures at $\alpha = 0$. Let

$$W_{L^{-1}}(s) = s^{-1}[W_0 + s^{-1} W_1 + \cdots + s^{-(k-1)} W_{k-1}] \tag{1.6}$$

be the principal part of the Laurent expansion of $L^{-1}(s)$ at $\alpha = 0$. Define the Hankel matrices

$$H(L^{-1}) = \begin{pmatrix} W_0 & W_1 & . & . & . & W_{k-2} & W_{k-1} \\ W_1 & W_2 & . & . & . & W_{k-1} & \\ . & . & . & . & . & & \\ W_{k-2} & W_{k-1} & & & & & \\ W_{k-1} & & & & & & \end{pmatrix}, \tag{1.7}$$

$$H(sL^{-1}) = \begin{pmatrix} W_1 & . & . & . & W_{k-2} & W_{k-1} \\ W_2 & . & . & . & W_{k-1} & \\ . & . & . & . & & \\ W_{k-1} & & & & & \end{pmatrix}, \ldots, \; H(s^{k-1}L^{-1}) = W_{k-1}.$$

Our main result is the following.

THEOREM 1.5. *Let $L \in \mathbb{C}^{n \times n}[s]$ be a nonsingular hermitian polynomial matrix with a strictly proper rational inverse. Let $\alpha = 0$ be a characteristic root of L. Assume that 0 is a pole of order k of L^{-1}. Let η_i be the inertial signatures of L at $\alpha = 0$. Then*

$$\begin{pmatrix} \eta_1 \\ . \\ . \\ . \\ \eta_{k-1} \\ \eta_k \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & 0 & . & . & . & . & . \\ & 1 & 0 & -1 & 0 & & . & . & . \\ & & 1 & 0 & -1 & 0 & . & . & . \\ & & & . & . & . & . & . & . \\ & & & & . & . & . & . & . \\ & & & & & & 0 & -1 & 0 \\ & & & & & & 1 & 0 & -1 \\ & & & & & & & 1 & 0 \\ & & & & & & & & 1 \end{pmatrix} \begin{pmatrix} \operatorname{sgn} H(s^{k-1}L^{-1}) \\ . \\ . \\ . \\ \operatorname{sgn} H(sL^{-1}) \\ \operatorname{sgn} H(L^{-1}) \end{pmatrix}. \tag{1.8}$$

The proof of the theorem will be given in section 3. It is based on a result of Turnbull [8].

2. Turnbull's signature test. In this section we deal with a hermitian pencil $P(s) = A_0 + A_1 s$. We describe a modified form of Turnbull's signature test [8]. The following notation will be used. We set $D_r(s) = D_r(s, 0)$, and define $r \times r$ matrices

$$E_r = (\delta_{i,r+1-i}) = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix},$$

$$N_r = (\delta_{i+1,i}) = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

Then $D_r(s) = (sI - N_r^T)E_r$.

LEMMA 2.1. *Let $P(s) = A_0 + A_1 s$ have an elementary divisor s^k and suppose $\pi_i = 0$ for $i > k$. Let the inertial signs and signatures of $\alpha = 0$ be given by* (1.2) *and* (1.3). *If*

$$W_{P^{-1}}(s) = s^{-1}[M_0 + s^{-1}M_1 + \cdots + s^{k-1}M_{k-1}] \tag{2.1}$$

is the principal part of the Laurent expansion of $P^{-1}(s)$ at $\alpha = 0$, then

$$\begin{pmatrix} \operatorname{sgn} M_0 \\ \operatorname{sgn} M_1 \\ . \\ . \\ . \\ \operatorname{sgn} M_{k-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & . & . & . \\ & 1 & 0 & 1 & 0 & 1 & . & . & . \\ & & 1 & 0 & 1 & 0 & . & . & . \\ & & & . & . & . & . & . & . \\ & & & & . & . & . & . & . \\ & & & & & . & . & . & . \\ & & & & & & 1 & 0 & 1 \\ & & & & & & & 1 & 0 \\ & & & & & & & & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ . \\ . \\ . \\ \eta_{k-1} \\ \eta_k \end{pmatrix}. \tag{2.2}$$

Proof. We may assume

$$P(s) = \operatorname{diag}\big(\hat{P}(s),\, Q(s)\big),\ \ \sigma(\hat{P}) = \{0\},\ \ 0 \notin \sigma(Q). \tag{2.3}$$

Let $\hat{P}(s)$ be in Weierstraß canonical form such that

$$\hat{P}(s) = \operatorname{diag}\big(\dots, \epsilon_{i1} D_i(s), \dots, \epsilon_{i\pi_i} D_i(s), \dots\big). \tag{2.4}$$

Then

$$W_{P^{-1}}(s) = \hat{P}^{-1}(s) = \operatorname{diag}(\dots, \epsilon_{i1} D_i^{-1}(s), \dots, \epsilon_{i\pi_i} D_i^{-1}(s), \dots).$$

We first deal with the case $\hat{P}(s) = \epsilon\, D_k(s)$ and proceed as in [8]. From

$$\epsilon\, D_k^{-1}(s) = \epsilon \sum_{i=0}^{k} N_k^i E_k s^{-i-1} = \epsilon \begin{pmatrix} 0 & 0 & 0 & . & . & . & 0 & s^{-1} \\ 0 & 0 & 0 & . & . & . & s^{-1} & s^{-2} \\ . & . & . & . & . & . & . & . \\ 0 & s^{-1} & s^{-2} & . & . & . & s^{-(k-2)} & s^{-(k-1)} \\ s^{-1} & s^{-2} & s^{-3} & . & . & . & s^{-(k-1)} & s^{-k} \end{pmatrix}$$

follows

$$M_i = \epsilon\, N_k^i E_k = \epsilon\, \operatorname{diag}\big(0_{i\times i}, E_{k-i}\big).$$

Therefore $\operatorname{sgn} E_k = 0$ if k is even, and $\operatorname{sgn} E_k = 1$ if k is odd. Hence

$$\operatorname{sgn} M_{k-1} = \epsilon, \quad \operatorname{sgn} M_{k-2} = 0, \quad \operatorname{sgn} M_{k-3} = \epsilon, \dots, \tag{2.5}$$

and (2.2) holds with $(\eta_k, \eta_{k-1}, \dots, \eta_1) = (\epsilon, 0, \dots, 0)$. In the general case, with $\hat{P}(s)$ given as in (2.4), we obtain (2.2) by inspecting $\hat{P}^{-1}(s)$ and using (2.5). $\square$

3. Proof of the theorem. We shall need a generalization of Sylvester's law of inertia [2, p. 200].

LEMMA 3.1. *Let $A \in \mathbb{C}^{n\times n}$ be hermitian. If $Y \in \mathbb{C}^{t\times n}$ has full column rank, then the matrices A and YAY^* have the same rank and the same signature.*

The proof of Theorem 1.5 starts from a minimal hermitian realization

$$L^{-1}(s) = C(A_0 + A_1 s)^{-1} C^*, \tag{3.1}$$

where $P(s) = A_0 + A_1 s$ is given by (2.3). Then $\hat{P}(s) = \hat{A}_0 + \hat{A}_1 s$, and $\hat{A}_1$ is nonsingular, and $\hat{N} = -\hat{A}_1^{-1}\hat{A}_0$ is nilpotent with $\hat{N}^k = 0$. Let $C = (\hat{C}, D)$ be partitioned in accordance with (2.3). Then

$$L^{-1}(s) = \hat{C}\hat{P}^{-1}(s)\hat{C}^* + DQ(s)D^*$$

and

$$W_{L^{-1}}(s) = \hat{C}\hat{P}^{-1}(s)\hat{C}^* = \hat{C}[\hat{A}_1(-\hat{N}+sI)]^{-1}\hat{C}^*.$$

Hence we have (1.6) with

$$W_i = \hat{C}\hat{N}^i\hat{A}_1^{-1}\hat{C}^*, \quad i = 0, \ldots, k-1.$$

Let $H(\hat{P}^{-1})$ be the Hankel matrix associated with $\hat{P}^{-1}(s)$. Then

$$H(\hat{P}^{-1}) = \begin{pmatrix} \hat{A}_1^{-1} & \hat{N}\hat{A}_1^{-1} & . & . & . & \hat{N}^{k-2}\hat{A}_1^{-1} & \hat{N}^{k-1}\hat{A}_1^{-1} \\ \hat{N}\hat{A}_1^{-1} & \hat{N}^2\hat{A}_1^{-1} & . & . & . & \hat{N}^{k-1}\hat{A}_1^{-1} & \\ . & . & . & . & . & & \\ \hat{N}^{k-1}\hat{A}_1^{-1} & & & & & & \end{pmatrix}.$$

Because of $\hat{N}^k = 0$ and $\hat{N}\hat{A}_1^{-1} = \hat{A}_1^{-1}\hat{N}^T$ we obtain

$$H(\hat{P}^{-1}) = \begin{pmatrix} I \\ \hat{N} \\ \vdots \\ \hat{N}^{k-1} \end{pmatrix} \hat{A}_1^{-1} \begin{pmatrix} I & \hat{N}^T & \ldots & (\hat{N}^{k-1})^T \end{pmatrix}. \tag{3.2}$$

Let

$$\mathcal{O} = \mathcal{O}(\hat{N}, \hat{C}) = \begin{pmatrix} \hat{C} \\ \hat{C}\hat{N} \\ \vdots \\ \hat{C}\hat{N}^{k-1} \end{pmatrix}$$

be the observability matrix of the pair $(\hat{N}, \hat{C})$. Then $H(L^{-1}) = \mathcal{O}\hat{A}_1^{-1}\mathcal{O}^*$, and similarly

$$H(s^iL^{-1}) = \mathcal{O}\hat{N}^i\hat{A}_1^{-1}\mathcal{O}^*, \quad i = 1, \ldots, k-1.$$

The realization (3.1) is minimal. Hence $\mathcal{O}$ has full column rank [1], and Lemma 3.1 implies

$$\operatorname{sgn} H(s^iL^{-1}) = \operatorname{sgn} \hat{N}^i\hat{A}_1^{-1}, \quad i = 0, \ldots, k-1. \tag{3.3}$$

Recall $W_{P^{-1}}(s) = \hat{P}^{-1}(s)$. Therefore the matrices M_i in (2.1) are given by $M_i = \hat{N}^i\hat{A}_1^{-1}$. Thus we have $\operatorname{sgn} M_i = \operatorname{sgn} H(s^iL^{-1})$. Then (2.1) yields

$$\big(\operatorname{sgn} H(L^{-1}), \ldots, \operatorname{sgn} H(s^iL^{-1})\big) = (\eta_1, \ldots, \eta_k)(I + N_k^2 + N_k^4 + \cdots),$$

and because of

$$(I + N_k^2 + N_k^4 + \cdots)^{-1} = I - N_k^2$$

the proof is complete.

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