Math. Systems Theory 14, 367–379 (1981)

Mathematical Systems Theory

The Structure of Nonsingular Polynomial Matrices

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Abstract. Let K be a field and let $L \in K^{n \times n}[z]$ be nonsingular. The matrix L can be decomposed as $L(z)=\hat{Q}(z)(Rz+S)\hat{P}(z)$ so that the finite and (suitably defined) infinite elementary divisors of L are the same as those of $Rz+S$, and $\hat{Q}(z)$ and $\hat{P}(z)^T$ are polynomial matrices which have a constant right inverse. If

$$Rz+S=\begin{pmatrix} zI-A & 0 \\ 0 & I-zN \end{pmatrix}$$

and K is algebraically closed, then the columns of $\hat{Q}$ and $\hat{P}^T$ consist of eigenvectors and generalized eigenvectors of shift operators associated with L.

1. Introduction

In this note we extend Kronecker's theory of nonsingular matric pencils to nonsingular polynomial matrices. Let K be a field, $L \in K^{n \times n}[z]$ and $\det L \neq 0 \in K[z]$. Using the matrix version of Kalman's state space isomorphism theorem it can be shown [12] that L^{-1} can be written as $L^{-1}(z)=Q(Rz+S)^{-1}P$. If the size of R and S is minimal, then the pencil $Rz+S$ is determined up to strict equivalence by L. We define in an appropriate way infinite elementary divisors of polynomial matrices. Then L and $Rz+S$ have the same finite and infinite elementary divisors. Because of strict equivalence, $Rz+S$ can be assumed to be of the form $\begin{pmatrix} zI-J & 0 \\ 0 & I-zN \end{pmatrix}$ where N is nilpotent and, if K is algebraically closed, J is in Jordan normal form. If Q is partitioned accordingly, $Q=(C \quad H)$, then it will be proved that the columns of H consist of Jordan chains corresponding to infinite elementary divisors of L. The columns of C are Jordan chains corresponding to the finite spectrum of L [16]. Let $K^n(z)$ and $z^{-1}K^n_-(z)$ be the set of rational and proper rational n-vectors respectively and let π_+ and π_- be the projections of $K^n(z)$ on $K^n[z]$ and $z^{-1}K^n_-(z)$. We shall consider the mappings $\pi_- L^{-1}: K^n[z] \to z^{-1}K^n_-(z)$ and $\pi_+ zL^{-1}: z^{-1}K^n_-(z) \to K^n[z]$. Then the columns

0025/5661/81/0014-0367$02.60

of the polynomial matrix $\hat{C}(z)=L(z)C(zI-J)^{-1}$ form a basis for $\operatorname{Im} L\pi_- L^{-1}$. With respect to this basis the right shift operator z on $\operatorname{Im} L\pi_- L^{-1}$ is given by the matrix J. The columns of the matrix $\hat{H}(z)=L(z)H(I-zN)^{-1}$ are a basis for $L \operatorname{Im} \pi_+ z L^{-1}$, the left shift operator z^{-1} is represented in this basis by the matrix N. This leads to a factorization of L, $L(z)=\hat{Q}(z)(Rz+S)\hat{P}(z)$ where $\hat{Q}=(\hat{C}\quad \hat{H})$. The matrix $\hat{P}^T$ has properties analogous to those of Q.

2. Notation

The rational functions over K will be denoted by $K(z)$. An element $f\in K(z)$ is called *proper rational*, if $f=f_1/f_2, f_i\in K[z], f_2\neq 0$, and $f=0$ or $\deg f_1<\deg f_2$. For the set of proper rational functions we write $z^{-1}K_-(z)$. Clearly $z^{-1}K_-(z)=\{f \mid f \in K(z),\ f(z)=\sum_{\nu=0}^{\infty} f_{-\nu} z^{-\nu-1}\}$. Let $W=(w_{ij})$ be a rational $n\times m$ matrix, i.e. $W\in K^{n\times m}(z)$, and let $p_{ij}\in z^{-1}K_-(z)$ be the principal part of the Laurent expansion of w_{ij} at $z=0$. Then we set

$$[W]_0=(p_{ij})$$

and call it the *principal part of W at $z=0$*. Each $f\in K(z)$ can be uniquely decomposed into $f=g+h$, $g\in z^{-1}K_-(z)$, $h\in K[z]$. We define the projections $\pi_+ : K(z)\to K[z]$ by $\pi_+ f=h$ and $\pi_- : K(z)\to z^{-1}K_-(z)$ by $\pi_- f=g$. In a natural way these definitions of π_+ and π_- will be extended to $K^n(z)$ corresponding to the direct sum $K^n(z)=z^{-1}K^n_-(z)\oplus K^n[z]$. Let $D=(d_{ij})$ be an $n\times m$ matrix such that $d_{ii}=\gamma_i$ for $1\leqslant i\leqslant r$ and $d_{ij}=0$ otherwise. Then we write

$$D=\{\gamma_1,\ldots,\gamma_r\}.$$

The matrix R_j will denote the $j\times j$ nilpotent Jordan block, $R_j=(\delta_{i,i+1})$. For $L\in K^{n\times n}[z]$ and $L(z)=\sum_{\sigma=0}^{s} L_\sigma z^\sigma$, $L_s\neq 0$, we define $\deg L:=s$. Whenever the context allows it, we omit the indeterminate z in rational matrices or vectors.

3. Realizations of Rational Matrices

Definition 1. A *realization* of a rational matrix W, $W\in K^{n\times m}(z)$, $W\neq 0$, is a factorization of the form

$$W(z)=Q(Rz+S)^{-1}P \tag{3.1}$$

with R and S in $K^{r\times r}$, $P\in K^{r\times m}$ and $Q\in K^{n\times r}$. The *dimension* of the realization (3.1) is the size r of R and S. The realization (3.1) is said to be *minimal*, if its dimension r is minimal.

In the case where W is a proper rational matrix it is known that there exists a realization with $R=I$ (see [2]). Minimal realizations of such matrices are characterized by Kalman's theorem (see e.g. [2]).

Theorem 1 [8]. *If* $W(z)=C_j(zI-A_j)^{-1}B_j$, $j=1,2$, *are two minimal realizations of dimension* r *of* W, $W\in z^{-1}K_{-}^{n\times m}(z)$, *then there is a nonsingular matrix* $T\in K^{r\times r}$ *such that* $A_2=T^{-1}A_1T$, $C_2=C_1T$, $B_2=T^{-1}B_1$.

The similarity class of A_j is given by the determinantal denominators of W.

Definition 2 [3]. For a given $W\in K^{n\times m}(z)$ let $\varphi_k(W)$ denote the monic least common denominator of all minors of W of order at most k. Set $\varphi_0(W)=1$. The polynomials $\varphi_k(W)$ are called the *determinantal denominators* of W. The *width* of W is defined to be the least non-negative integer g such that $\varphi_k(W)=\varphi_{k+1}(W)$ for all $k\geqslant g$.

Theorem 2 [3]. *Let* $W(z)=C(zI-A)^{-1}B$ *be a minimal realization of* W. *Then the elementary divisors of* $zI-A$ *are the polynomials* $\varphi_g/\varphi_{g-1},\dots,\varphi_1/\varphi_0$. *The minimal polynomial is the least common denominator of all elements of* W.

Kalman's theorem can be extended to realizations of general rational matrices [12]. We note Rosenbrock's result here in a different language and with a proof which to some extent differs from his and which will be more convenient for our purposes. From Theorem 1 and Theorem 2 we obtain immediately a realization for polynomial matrices.

Lemma 1. *Any polynomial matrix* $M\in K^{n\times m}[z]$ *of degree* s, *has a realization of the form*

$$M(z)=H(I-zN)^{-1}G. \tag{3.2}$$

If the realization (3.2) *is minimal, then* N *is nilpotent and is determined up to similarity by* M. *The degree of the minimal polynomial of* N *is* $s+1$.

Proof. Since $\xi^{-1}M(\xi^{-1})$ is proper rational, we have $\xi^{-1}M(\xi^{-1})=H(\xi I-N)^{-1}G$ or $M(\xi^{-1})=H(I-\xi^{-1}N)^{-1}G$. Now $\xi^{-1}=z$ yields (3.2). The least common denominator of all elements of $\xi^{-1}M(\xi^{-1})$ is ξ^{s+1}. Hence according to Theorem 2 the minimal polynomial of N is z^{s+1}. Theorem 1 implies that N is determined up to similarity. □

Theorem 3. (a) *Any matrix* $W\in K^{n\times m}(z)$, $W\neq 0$, *has a realization.* (b) *If* $Q_i(R_iz+S_i)^{-1}P_i$, $i=1,2$, *are two minimal realizations of* W *of dimension* r, *then the linear pencils* R_iz+S_i, $i=1,2$, *are strictly equivalent, i.e. there exist nonsingular matrices* U *and* V *in* $K^{r\times r}$ *such that* $R_1z+S_1=V(R_2z+S_2)U$. *Furthermore,* $Q_1=Q_2U$ *and* $P_1=VP_2$.

Proof. The matrix $W\in K^{n\times m}(z)$ can be written as a sum $W=T+M$ with $T\in z^{-1}K_{-}^{n\times m}(z)$ and $M\in K^{n\times m}[z]$. Let $T(z)=C(zI-A)^{-1}B$ and $M(z)=H(I-zN)^{-1}G$ be realizations of T and M. Then

$$W(z)=\begin{pmatrix} C & H\end{pmatrix}\begin{pmatrix} zI-A & 0\\ 0 & I-zN\end{pmatrix}^{-1}\begin{pmatrix} B\\ G\end{pmatrix}$$

is a realization of W, which proves (a).

Assume now $Q(Rz+S)P$ to be a minimal realization of W. Then

$$Rz+S=U\begin{pmatrix} zI-A & 0 \\ 0 & I-zN \end{pmatrix}V \tag{3.3}$$

where U and V are suitable nonsingular matrices (see [5]) and N is nilpotent. If QU and VP are partitioned according to (3.3) into $QU=(C \quad H)$ and $VP=\begin{pmatrix} B \\ G \end{pmatrix}$, then $W(z)=C(zI-A)^{-1}B+H(I-zN)^{-1}G$. Thus $C(zI-A)^{-1}B$ and $H(I-zN)^{-1}G$ are minimal realizations of T and M respectively. Combining Theorem 1 and Lemma 1 and taking (3.3) into account yields the proof of (b). □

4. Finite and Infinite Elementary Divisors

Let $L\in K^{n\times n}[z]$ be nonsingular and $L^{-1}(z)=Q(Rz+S)^{-1}P$ be a minimal realization of its inverse. In this section we determine the finite and infinite elementary divisors of $Rz+S$.

Using the Smith–McMillan form we can define infinite elementary divisors for polynomial matrices. Recall that for any matrix $W\in K^{m\times n}(z)$ of rank r there are unimodular matrices $U\in K^{m\times m}[z]$ and $V\in K^{n\times n}[z]$ such that $W=UDV$ where

$$D=\{\varepsilon_1/\psi_1,\ldots,\varepsilon_r/\psi_r\}$$

and ε_i and ψ_i are relatively prime polynomials, and $\varepsilon_i|\varepsilon_k$ and $\psi_k|\psi_i$ if $i\leqslant k$. The matrix D which is uniquely determined by these properties (see [3] or [11]) is called the Smith–McMillan form of W.

Definition 3. A matrix $X\in K^{n\times m}[z]$ is said to have the *infinite elementary divisors* $y^{s_1},\ldots,y^{s_p}$, if $yX(y^{-1})$ has Smith–McMillan form

$$\{y^{-c_1}d_1(y),\ldots,y^{-c_q}d_q(y),y^{s_1}d_{q+1}(y),\ldots,y^{s_p}d_{p+q}(y)\}$$

such that $-c_1\leqslant\cdots-c_q\leqslant 0<s_1<\cdots<s_p$ and $d_\rho\in K[z]$, $d_\rho(0)\neq 0$, $\rho=1,\ldots,r$. If X has infinite elementary divisors, we say $\lambda=\infty$ is a *characteristic root* of X and define $s_1+\cdots+s_p$ to be its *multiplicity*.

If X has Smith form $\{1,\ldots,1,g_1,\ldots,g_t\}$, $g_i|g_{i+1}$, $g_1\neq 1$, then the polynomials g_i are products of elementary divisors of X. We call those divisors here the *finite elementary divisors* and their roots *finite characteristic roots* of X. In the special case of $X(z)=Rz+S$ we have $yX(y^{-1})=R+Sy$ and our definition yields the familiar one for matrix pencils.

Lemma 2 [3, p. 103]. *Let $W\in K^{n\times m}(z)$ be of width g and let $\{e_1/d_1,\ldots,e_s/d_s\}$ be the Smith–McMillan form of W. If $W=T+M$, $M\in K^{n\times m}[z]$, and $T(z)=C(zI-A)^{-1}B$ is a minimal realization, then $zI-A$ has Smith form $\{I_{r-g},d_g,\ldots,d_1\}$. In the special case of $W=L^{-1}$ the finite elementary divisors of L and of $zI-A$ are the same.*

Theorem 4. *Let $L^{-1}(z)=Q(Rz+S)^{-1}P$ be a minimal realization. Then the finite and infinite elementary divisors of $Rz+S$ and of L are the same.*

Proof. Without loss of generality we can assume

$$Rz+S=\begin{pmatrix} zI-A & 0 \\ 0 & I-zN \end{pmatrix}, \ N \text{ nilpotent}, \tag{4.1}$$

and

$$L^{-1}(z)=C(zI-A)^{-1}B+H(I-zN)^{-1}G. \tag{4.2}$$

Then $T(z)=C(zI-A)^{-1}B$ and $M(z)=H(I-zN)^{-1}G$ are minimal realizations of the proper rational part T and the polynomial part M of L^{-1}. The preceding lemma implies that the finite elementary divisors of L and of $Rz+S$ are equal. We now consider the infinite elementary divisors of $Rz+S$. Because of (4.1) they are the (finite) elementary divisors of $Iy-N$. From (4.2) we obtain

$$y^{-1}L^{-1}(y^{-1})=C(I-yA)^{-1}B+H(Iy-N)^{-1}G.$$

Without loss of generality we assume $A=\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ where A_1 is nilpotent and A_2 is non-singular. Then $y^{-1}L^{-1}(y^{-1})$ can be written as

$$y^{-1}L^{-1}(y^{-1})=C_1(I-yA_1)^{-1}B_1+\begin{pmatrix} C_2 & H \end{pmatrix}\begin{pmatrix} I-A_2y & 0 \\ 0 & Iy-N \end{pmatrix}^{-1}\begin{pmatrix} B_2 \\ G \end{pmatrix}. \tag{4.3}$$

The second term in the sum (4.3) is a minimal realization of the proper rational part of $y^{-1}L^{-1}(y^{-1})$. Let $y^{-1}L^{-1}(y^{-1})$ have the Smith–McMillan form

$$\{1/y^{s_p}h_1(y),\dots,1/y^{s_1}h_p(y),y^{e_1}/h_{p+1}(y),\dots,y^{e_t}h_n(y)\}, \tag{4.4}$$

$s_p\geqslant\cdots s_1>0$, $0\leqslant e_1\leqslant\cdots\leqslant e_t$, $h_i(0)\neq 0$. Then according to Lemma 2, the elementary divisors of $Iy-N$ are $y^{s_1},\dots,y^{s_p}$. The Smith–McMillan form of $yL(y^{-1})$ is the inverse of (4.4) and because of Definition 3 the proof is complete. □

The infinite elementary divisors of a nonsingular polynomial matrix L can be determined from its minors. We use the relation between determinantal denominators and the Smith McMillan form.

Lemma 3 [3]. *Let $W\in K^{n\times m}(z)$ have the Smith–McMillan form $\{e_1/d_1,\dots,e_r/d_r\}$. Then*

$$\varphi_k(W)=d_1\dots d_k \qquad \text{for } 1\leqslant k\leqslant r.$$

If $V(y):=y^{-1}L^{-1}(y^{-1})$ has the Smith–McMillan for (4.4), then the preceding lemma yields

$$\varphi_k(V)=y^{i_k}r_k(y), \qquad k=1,\dots,n$$

and $r_k(0)\neq 0$, $i_k=s_p+\cdots+s_{p-k}$ for $1\leqslant k\leqslant p$, $i_k=i_p$ for $k>p$. Hence the numbers i_k contain the desired information.

The rth compound of a matrix A (i.e., the matrix containing all $r\times r$ minors of A in a suitable ordering) will be denoted by $A^{[r]}$. If the $n\times n$ matrix A is invertible, then (see e.g. [7])

$$(A^{-1})^{[r]}=1/\det A\cdot A^{[n-r]}$$

holds. Hence $\det L(y^{-1})V^{[r]}=y^{-r}L(y^{-1})^{[n-r]}$ and

$$i_k=\max_{j\leqslant k}\{j-\deg\det L+\deg\delta\,|\,\delta \text{ is an } (n-j)\times(n-j) \text{ minor of } L\}.$$

5. Jordan Chains

We assume now that the field K is algebraically closed. Let L be given by $L(z)=\sum_{\mu=0}^{s}L_\mu z^\mu$ and L^{-1} by (4.2) and assume that the corresponding realization is minimal. We will investigate the matrices C and H and show that their columns consist of Jordan chains of L belonging to finite and infinite characteristic roots.

Definition 4 (see e.g. [1]). The vectors $x_0,\dots,x_p$ are said to form a *finite Jordan chain* of L corresponding to the finite characteristic root λ, $\lambda\neq\infty$, if $x_0\neq 0$ and

$$\sum_{j=0}^{k}\frac{1}{j!}L^{(j)}(\lambda)x_{k-j}=0, \qquad k=0,\dots,p-1 \tag{5.1}$$

hold.

Lemma 3. *Set* $X=(x_0,\dots,x_p)$, $x(z)=\sum_{i=0}^{p}x_i(z-\lambda)^i$, $J=\lambda I+R_{p+1}$. *Then* (5.1) *is equivalent to any of the following conditions*:

$$\frac{d^k}{dz^k}L(z)x(z)\bigg|_{z=\lambda}=0, \qquad k=0,\dots,p-1, \tag{5.2}$$

or

$$\sum_{\mu=0}^{s}L_\mu XJ^\mu=0. \tag{5.3}$$

Proof. (5.2) can be checked easily, (5.3) can be found in [9] or [15]. □

To each finite elementary divisor of L corresponds a finite Jordan chain [10]. If all those chains are put together we speak of a *full system of chains* [16] or if we consider them together with the associated Jordan matrix we shall use the term *eigenpair* [6].

Definition 5. The matrices $C \in K^{n\times r}$ and $J \in K^{r\times r}$ are said to be a *finite eigenpair* (C, J) of L, if J is a Jordan matrix, $r = \deg\det L$,

$$\sum_{\mu=0}^{s} L_\mu C J^\mu = 0 \tag{5.4}$$

and the columns of the matrix $C(zI-J)^{-1}$ are linearly independent over K.

For $J = \text{block diag}(\lambda_1 I + R_{t_1}, \ldots, \lambda_p I + R_{t_p})$ and $C = (C_1, \ldots, C_p)$ the condition (5.4) implies that the columns of C_ρ are Jordan chains to the characteristic root λ_ρ. The condition on $C(zI-J)^{-1}$ is equivalent to (J, C) being observable. We note further [15] that $U(t) = C\exp(Jt)$ is a fundamental system of $\sum_{\mu=0}^{s} L_\mu x^{(\mu)}(t) = 0$.

Minimal realizations and coprime factorizations of rational matrices can be related to eigenpairs.

Lemma 4 [16]. *If $C(zI-J)^{-1}B$ is a minimal realization of $W \in z^{-1}K_-^{n\times m}(z)$ and $W = D^{-1}F$ is a factorization such that the polynomial matrices D and F are left coprime, then (C, J) is a finite eigenpair of D.*

One of the conditions for matrices to be left coprime is the following one.

Lemma 5 (see e.g. [11]). *Two matrices $D \in K^{n\times n}[z]$ and $F \in K^{n\times r}[z]$ are left coprime, if and only if*

$$\operatorname{rank}(D(\lambda), F(\lambda)) = n \qquad \textit{for all } \lambda \in K.$$

We now define Jordan chains and eigenpairs for $\lambda = \infty$.

Definition 6. If $w_m, \ldots, w_0, \ldots, w_{-s}$ are $m+1+s$ vectors in K^n such that $m \geq 0$ and $w_m \neq 0$ and

$$\sum_{\rho=0}^{m+s} L_{j+\rho} w_{m-\rho} = 0, \qquad j = s+1, \ldots, -m+1,$$

hold, where $L_i = 0$ for $i > s$ and $i < 0$, then $w_m, \ldots, w_0$ are said to form a *Jordan chain of L at $\lambda = \infty$* (or by abuse of language an *infinite Jordan chain*).

The definition also includes the case $s = 0$. In the case $s \geq 1$ one can set $w_{-s} = 0$. Because of the difference in our definition of infinite elementary divisors and infinite eigenvalues in [13] our concept of infinite chains is slightly different from the one used in [14]. The following equivalent conditions for infinite chains can be derived from (5.1) and (5.2).

Lemma 6. *The vectors $w_m, \ldots, w_0, \ldots, w_{-s}$ satisfy the conditions of the preceding definition, if and only if they form a Jordan chain of*

$$\hat{L}(y) := y^{s+1} L(y^{-1}) \tag{5.5}$$

for the characteristic root $y=0$ or equivalently if $w_m \neq 0$ and

$$L(z)\left(\sum_{i=-s}^{m} w_i z^i\right) = p(z^{-1}) \tag{5.6}$$

with $p \in K^n[z]$.

Definition 7. The matrices $H \in K^{n \times a}$ and $N \in K^{a \times a}$ are said to form an *infinite eigenpair* (H, N) of L, if the following conditions are satisfied:

(a) N is nilpotent and in Jordan form, $N = \text{block diag}(R_{a_1}, \ldots, R_{a_t})$,

(b) a is equal to the multiplicity of the characteristic root $\lambda = \infty$ of L,

(c) if H is partitioned according to N, $H = (H_1, \ldots, H_t)$, then the columns of H_τ are an infinite Jordan chain of L, and

(d) the columns of $H(I - zN)^{-1}$ are linearly independent over K.

Theorem 5. *Let*

$$L(z)^{-1} = (C \quad H)\begin{pmatrix} zI - J & 0 \\ 0 & I - zN \end{pmatrix}^{-1}\begin{pmatrix} B \\ G \end{pmatrix} \tag{5.7}$$

be a minimal realization where J and the nilpotent matrix N are in Jordan form. Then (C, J) is a finite and (H, N) an infinite eigenpair of L.

Proof. As before let $T(z) = C(zI - A)^{-1}B$ be the proper rational and $M(z) = H(I - zN)^{-1}G$ be the polynomial part of L^{-1}. From $L^{-1} = T + M$ we obtain a factorization of T, $T = L^{-1}(I - LM)$, which is left coprime because of Lemma 5. Thus Lemma 4 implies the result on (C, J).

In order to investigate the columns of H we rewrite (5.7) as

$$y^{-1}L^{-1}(y^{-1}) = C(I - yJ)^{-1}B + H(Iy - N)^{-1}G.$$

Let $\hat{L}$ be defined by (5.5). Then

$$H(Iy - N)^{-1}G \tag{5.8}$$

is the principal part of $y^s \hat{L}(y)^{-1}$ at $y = 0$. If U and V are two unimodular matrices such that $y^{-1}L^{-1}(y^{-1}) = U(y)^{-1}S(y)V(y)$ and S is the Smith–McMillan form given by (4.4), then a left coprime factorization $D^{-1}F$ of $y^{-1}L^{-1}(y^{-1})$ is given by $D(y) = \text{diag}(y^{s_p}h_1(y), \ldots, y^{s_1}h_p(y), h_{p+1}(y), \ldots, h_n(y))U(y)$ and $F(y) = \text{diag}(1, \ldots, 1, y^{e_1}, \ldots, y^{e_t})V(y)$. Thus

$$\hat{L}(y)^{-1} = [y^s D(y)]^{-1} F(y)$$

is a left coprime factorization of $\hat{L}^{-1}$. Each Jordan chain $x_b,\dots,x_0$ of D corresponding to $y=0$ can be extended to a Jordan chain $x_b,\dots,x_0,\dots,x_{-s}$ of $y^sD(y)$ at $y=0$. It is therefore an infinite Jordan chain of L. Since the elementary divisors of the form y^k are the same in D and $Iy-N$, condition (b) of Definition 7 is satisfied and (5.8) being a minimal realization yields (d). □

6. Two Module Homomorphisms Associated with L

Let L^{-1} be given by $L^{-1}=T+M$, where M is a polynomial and T is a proper rational matrix. As it is customary in linear system theory, one can associate to T the mapping

$$\pi_-L^{-1}\colon K^n[z]\to z^{-1}K^n_-(z). \tag{6.1}$$

It is a $K[z]$ module homomorphism, if the $K[z]$ module structure on $z^{-1}K^n_-(z)$ is given by $p\cdot v=\pi_-pv$, $p\in K[z]$, $v\in z^{-1}K^n_-(z)$. It is maybe less common to consider the mapping

$$\pi_+zL^{-1}\colon z^{-1}K^n_-(z)\to K^n[z] \tag{6.2}$$

which is determined by M and for which T is irrelevant. Let $K_-(z)$ be the set of rational functions f such that $f(z)=\sum_{\nu=0}^{\infty} f_{-\nu}z^{-\nu}$. Define $f\cdot w=\pi_+ fw$ for $f\in K_-(z)$ and $w\in K^n[z]$. Then π_+zL^{-1} is a $K_-(z)$ module homomorphism.

Following Fuhrmann [4] we define a projection π_L on $K^n[z]$ by

$$\pi_Ly:=L\pi_-L^{-1}y,\qquad y\in K^n[z].$$

Let V_L be the range of π_L, $V_L:=\operatorname{Im}\pi_L$. Since $\operatorname{Ker}\pi_L=\operatorname{Ker}\pi_-L^{-1}=LK^n[z]$ we have

$$V_L\cong K^n[z]/LK^n[z]$$

and the factorization given below is canonical

$$\begin{tikzcd} K^n[z] \arrow[rr, "\pi_-L^{-1}"] \arrow[dr, "\pi_L"'] & & z^{-1}K^n_-(z) \\ & V_L \arrow[ur, "L^{-1}"'] & \end{tikzcd}$$

V_L is a vector space of dimension r, $r=\deg\det L$ and a $K[z]$ module with $p\cdot x=\pi_Lpx$, $p\in K[z]$, $x\in V_L$. A shift operator S^+ is defined on V_L by

$$S^+f=\pi_Lzf(z),\qquad f\in V_L.$$

Theorem 6 [16]. *If $C(zI-A)^{-1}B$ is a minimal realization of T, then the columns of the polynomial matrix*

$$\hat{C}(z)=L(z)C(zI-A)^{-1}$$

form a basis for the vector space V_L such that

$$S^{+}\hat{C}=\hat{C}A.$$

In the special case of $A=J$ we have a finite eigenpair (C,J) and the columns of $\hat{C}$ consist of eigenvectors and generalized eigenvectors of S^{+}.

The way we dealt with the mapping (6.1) suggests a similar approach for (6.2). Define $\pi^{L}: z^{-1}K^{n}_{-}(z)\to K^{n}[z]$ by

$$\pi^{L}v:=L\pi_{+}zL^{-1}v \qquad v\in z^{-1}K^{n}_{-}(z)$$

and U^{L} by $U^{L}:=\operatorname{Im}\pi^{L}$. The product $f\cdot w=L\pi_{+}fL^{-1}w$, $f\in K_{-}(z)$, $w\in U^{L}$, yields a $K_{-}(z)$ module structure on U^{L}. We have a canonical factorization

$$\begin{tikzcd} z^{-1}K^{n}_{-}(z) \arrow[rr, "\pi_{+}zL^{-1}"] \arrow[dr, "\pi^{L}"'] & & K^{n}[z] \\ & U^{L} \arrow[ur, "L^{-1}"'] & \end{tikzcd}$$

On U^{L} multiplication by z^{-1} is possible. We define a left shift operator S^{-} on U^{L} by

$$S^{-}w=L\pi_{+}z^{-1}L^{-1}w, \qquad w\in U^{L}. \tag{6.3}$$

Lemma 7. *The dimension of U^{L} is equal to the multiplicity of the characteristic root $\lambda=\infty$ of L.*

Proof. Let X and Y be two unimodular matrices which transform $y^{-1}L^{-1}(y^{-1})$ into Smith McMillan form,

$$y^{-1}L^{-1}(y^{-1})=X(y)\tilde{F}(y)Y(y),$$

where $\tilde{F}$ is the matrix (4.4). Since $Y(z^{-1})$ is an isomorphism of $z^{-1}K^{n}_{-}(z)$ we can assume without loss of generality $Y=I$. Write $\tilde{F}$ as a sum $\tilde{F}=F+F_2$ with

$$F=\operatorname{diag}\left(1/y^{s_p}h_1(y),\dots,1/y^{s_1}h_p(y),0,\dots,0\right)$$

and

$$F_2=\operatorname{diag}\left(0,\dots,0,\,y^{e_1}/h_{p+1}(y),\dots,\,y^{e_t}/h_n(y)\right).$$

Since $\pi_+X(z^{-1})F_2(z^{-1})v(z)=0$ for $v\in z^{-1}K_-^n(z)$, it remains to consider $X(z^{-1})F(z^{-1})$. Let x_i be the ith column of X, $i=1,\dots,p$. Then X unimodular and $h_i(0)\neq 0$ imply $g_i:=1/h_i(z^{-1})x_i(z^{-1})=\sum_{\nu=0}^{\infty}g_{-\nu}^i z^{-\nu}$, $g_0^i\neq 0$. Thus we can focus on a vector space

$$U_g:=\{w\,|\,w(z)=\pi_+z^mf(z)g(z),\qquad f\in z^{-1}K_-(z)\}$$

with $g=\sum_{\nu=0}^{\infty}g_{-\nu}z^{-\nu}$, $g_0\neq 0$. The dimension of U_g is equal to m, since the vectors $\{g_0, g_0z+g_{-1},\dots,g_0z^{m-1}+\cdots+g_{-m+1}\}$ form a basis for U_g. □

We note without proof that the vectors $g_0,\dots,g_{m-1}$ form an infinite Jordan chain and that the given construction, if performed for each infinite elementary divisor, yields an infinite eigenpair of L.

Theorem 7. *Let $H(I-zN)^{-1}G$ be a minimal realization of the polynomial part M of L^{-1}. Then the columns of*

$$\hat{H}(z)=L(z)H(I-zN)^{-1}$$

form a basis for U^L and

$$S^-\hat{H}=\hat{H}N.$$

Proof. Because of Theorem 1 we can assume N to be in Jordan form. Theorem 5 and Definition 7 imply that the columns of $\hat{H}$ are linearly independent over K and that their number is equal to the multiplicity of $\lambda=\infty$ which is—according to Lemma 7—equal to $\dim U^L$.

In order to show that the columns of $\hat{H}$ are in U^L we can restrict ourselves to the case $N=R_{b+1}$. Because of Theorem 5 the columns of $H=(h_b,\dots,h_0)$ form an infinite Jordan chain of L and because of (5.6) there exist vectors $h_{-1},\dots,h_{-s}$ such that

$$L(z)(h_b,\dots,h_{-s})z^{-i}(z^b,\dots,z^{-s})^T=z^{-i}p(z^{-1}),\qquad p\in K[z],$$
$$i=0,\dots,b.$$

Then

$$H(z^{b-i},\dots,1,0,\dots,0)^T=zL^{-1}(z)v_i(z)+(h_b,\dots,h_{-s})$$
$$\times(0,\dots,0,z^{-1},\dots,z^{-s-i})^T$$

with $v_i(z)=z^{-(i+1)}p(z^{-1})\in z^{-1}K_-^n(z)$, which means that each column of $H(I-zN)^{-1}$ is in $\operatorname{Im}\pi_+zL^{-1}$. From (6.3) we obtain $S^-\hat{H}=L\pi_+z^{-1}H(I-zN)^{-1}=\hat{H}N$. □

Our investigation has lead us to the main result of this paper.

Theorem 8. *Let $L \in K^{n\times n}[z]$ be nonsingular and*

$$L^{-1}(z) = Q(Rz+S)^{-1}P$$

be a minimal realization such that

$$Rz+S = \begin{pmatrix} zI-A & 0 \\ 0 & I-zN \end{pmatrix},$$

$Q=(C \quad H)$ *and* $P=\begin{pmatrix} B \\ G \end{pmatrix}$. *Then L can be factored into*

$$L(z) = \hat{Q}(z)(Rz+S)\hat{P}(z) \tag{6.4}$$

where $\hat{Q}=(\hat{C} \quad \hat{H})$ *and* $\hat{P}=\begin{pmatrix} \hat{B} \\ \hat{G} \end{pmatrix}$ *are given by* $\hat{C}(z)=L(z)C(zI-A)^{-1}$, $\hat{B}(z)=(zI-A)^{-1}BL(z)$, $\hat{H}(z)=L(z)H(I-zN)^{-1}$ *and* $\hat{G}=(I-zN)^{-1}GL(z)$. *The columns of* $\hat{C}$ *form a basis for*

$$V_L = \operatorname{Im} L\pi_- L^{-1}|_{K^n[z]} \cong K^n[z]/LK^n[z]$$

and

$$S^+\hat{C} = \hat{C}A$$

where S^+ *is the right shift operator on* V_L. *The columns of* $\hat{H}$ *form a basis for*

$$U^L = \operatorname{Im} L\pi_+ zL^{-1}|_{z^{-1}K^n_-(z)}$$

and

$$S^-\hat{H} = \hat{H}N$$

where S^- *is the left shift operator on* U^L. *The columns of* $\hat{B}^T$ *and* $\hat{G}^T$ *have analogous properties with respect to* L^T.

The finite and infinite elementary divisors of L and $Rz+S$ *are the same.*

Proof. The factorization (6.4) follows from the definition of $\hat{Q}$ and $\hat{P}$. The other statements are contained in Theorems 4, 6 and 7. □

Note that the polynomial matrix $\hat{Q}$ has a constant right inverse, $\hat{Q}(z)P=I$, and $\hat{P}$ has a constant left inverse, $Q\hat{P}(z)=I$.

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Received Sept 26, 1979 and in revised form August 27, 1980.