Equivariant S-Duality

By

Klaus Wirthmüller

0. We generalize the concept of S-duality introduced in [6] to the equivariant case. Our presentation follows closely that suggested in [5] Exercise 8 F.

Let $C^0(G)$ denote the category of finite G-equivariant pointed CW complexes (the base point a 0-cell) and G-maps (G is a compact Lie group). Let $C^0(G)\,h$ be the homotopy category. We need consider only cellular maps as any map in $C^0(G)$ is G-homotopic to a cellular one by [2] Prop. 2.4. Note that $C^0(G)$ contains all compact smooth G-manifolds. For details concerning CW complexes see [2].

Throughout this paper, we employ the notations of [7].

1. Let $\alpha \in RO(G)$, $X, X^* \in \operatorname{ob} C^0(G)$. We define an α-*duality* to be an element $u \in \tilde{\omega}_G^\alpha(X^* \wedge X)$ (stable equivariant cohomotopy [4]) with the property that for all (closed) subgroups $H < G$, the maps

$$(1.1)\qquad \begin{aligned} u/ &: \tilde{\omega}_*^H X \to \tilde{\omega}_H^* X^*, \\ u^*/ &: \tilde{\omega}_*^H X^* \to \tilde{\omega}_H^* X \end{aligned}$$

are isomorphisms (u^* corresponds to u under the transposition $X^* \wedge X \approx X \wedge X^*$). X^* is called an α-*dual* of X.

(1.2) Proposition. *Let $u \in \tilde{\omega}_G^\alpha(X^* \wedge X)$ be a duality, $Y, Z \in \operatorname{ob} C^0(G)$. There are isomorphisms*

$$\begin{aligned} u/ &: \tilde{\omega}_H^*(Y \mid X \wedge Z) \approx \tilde{\omega}_H^*(X^* \wedge Y \mid Z), \\ u^*/ &: \tilde{\omega}_H^*(Y \mid X^* \wedge Z) \approx \tilde{\omega}_H^*(X \wedge Y \mid Z) \end{aligned}$$

generalizing (1.1).

Proof. The groups under consideration are complete (co-)homology theories in Z (resp. Y), cf. [7] 1. The slant products define morphisms between these theories, bijective for $X = Y = S^0$ by (1.1). The proposition now follows from the comparison theorem.

Let $u \in \tilde{\omega}_G^\alpha(X^* \wedge X)$ and $v \in \tilde{\omega}_G^\beta(Y^* \wedge Y)$ be dualities. There are isomorphisms

$$D(u, v): \tilde{\omega}^*(X \mid Y) \approx \tilde{\omega}^*(Y^* \mid X^*),$$

given by the diagram

$$\begin{tikzcd} \tilde{\omega}^*(X \mid Y) \arrow[r, dotted, "D(u,v)"'] \arrow[d, "\approx"', "v/"] & \tilde{\omega}^*(Y^* \mid X^*) \arrow[d, "\approx"', "u^*/"] \\ \tilde{\omega}^*(Y^* \wedge X) \arrow[r, phantom, "\approx"] & \tilde{\omega}^*(X \wedge Y^*). \end{tikzcd}$$

Clearly, $D(u, v)^{-1} = D(v^*, u^*)$. If $w \in \tilde{\omega}_G^{\gamma}(Z^* \wedge Z)$ is another duality and $f \in \tilde{\omega}^*(X \mid Y)$, $g \in \tilde{\omega}^*(Y \mid Z)$ then $D(u, w)(g\, f) = [D(u, v)\, f][D(v, w)\, g]$.

2. Duality is compatible with sums and smashed products in $C^0(G)$, more precisely (compare [6] (6.5), (6.8)):

(2.1) Proposition. *Let $u \in \tilde{\omega}^{\alpha}(X^* \wedge X)$, $v \in \tilde{\omega}^{\beta}(Y^* \wedge Y)$ be dualities.*

(a) *If $\alpha = \beta$, let c be the map $(X^* \vee Y^*) \wedge (X \vee Y) \to (X^* \wedge X) \vee (Y^* \wedge Y)$ collapsing $(X^* \wedge Y) \vee (Y^* \wedge X)$. Then $c^*(u \oplus v) \in \tilde{\omega}^{\alpha}((X^* \vee Y^*) \wedge (X \vee Y))$ is a duality.*

(b) *The image $u * v$ of $u \times v$ under the transposition*

$$\tilde{\omega}^{\alpha+\beta}(X^* \wedge X \wedge Y^* \wedge Y) \approx \tilde{\omega}^{\alpha+\beta}((X^* \wedge Y^*) \wedge (X \wedge Y))$$

is a duality.

Proof. (a) follows immediately from the additivity property of homology theories, (b) from (1.2) and the fact that $u * v/$ factors into

$$\tilde{\omega}_*(X \wedge Y) \xrightarrow{u/} \tilde{\omega}^*(X^* \mid Y) \xrightarrow{v/} \tilde{\omega}^*(Y^* \wedge X^*) \approx \tilde{\omega}^*(X^* \wedge Y^*)\,.$$

3. Let $K < G$ be a subgroup and embed the homogeneous space G/K together with a tubular neighbourhood $T \approx G \times_K V^{\gamma}$ ($\gamma \in RO(K)$) into a suitable G-representation: $G/K \subset T \subset S^{\beta}$. Necessarily, $\gamma = \beta_K - L(K, G)$, see [7] 1. Let

$$\tau \in \omega_G^{\beta}(S^{\beta} \times S^{\beta}, S^{\beta} \times S^{\beta} - \Delta S^{\beta})$$

be a Thom class. From the duality theorem [7] Th. 4.1 we obtain isomorphisms

$$\begin{aligned} \gamma_{\tau}\colon\ & \omega_*^H(S^{\beta}, S^{\beta} - G/K) \approx \omega_H^*(G/K)\,,\\ \gamma_{\tau}\colon\ & \omega_*^H\, T \approx \omega_H^*(S^{\beta}, S^{\beta} - T)\,. \end{aligned}$$

Thus $\tau \mid [(S^{\beta}, S^{\beta} - T) \times G/K]$, which may be considered an element

$$v \in \tilde{\omega}_G^{\beta}(T^c \wedge G^+/K)\,,$$

is a β-duality, i.e. we have proved:

(3.1) Proposition. *$G^+ \wedge_K S^{\beta - L(K,G)}$ is a β-dual of G^+/K.*

Remark. Simultaneously, we obtain a duality $u \in \tilde{\omega}_K^{\beta}(S^{\beta} \wedge \{1K\}^+)$. If

$$\lambda_{S^0}\colon\ \{1K\}^+ \subset G^+/K$$

is the inclusion then $D(u, v)\, \lambda \in \tilde{\omega}_K^0(G^+ \wedge_K S^{\gamma}, S^{\beta})$ is represented by the collapsing map

$$l_{S^{\gamma}}\colon\ G^+ \wedge_K S^{\gamma} \to S^L S^{\gamma} = S^{\beta} \quad \text{(see [7])}.$$

Remark. This method of constructing duals applies to any compact smooth manifold and embedded submanifolds. We mention just one result (compare [1] Prop. 3.2):

(3.2) Proposition. *Let M be a compact smooth G-Manifold, ν the normal bundle of M in some G-module V^{β}. Then $M/\partial M$ and the Thom space of ν are β-duals.*

4. Let $X, Y, X^*, Y^* \in \operatorname{ob} C^0(G)$, $u \in \tilde{\omega}_G^\alpha(X^* \wedge X)$, $v \in \tilde{\omega}_G^\alpha(Y^* \wedge Y)$ α-dualities, $f: X \to Y$ a map, and set $f^* = D(u, v) f$. We use f^* also to denote a representative map in the class f^*.

(4.1) Proposition. *There exists an* $(\alpha + 1)$*-duality* $w \in \tilde{\omega}_G^{\alpha+1}(C_{f^*} \wedge C_f)$, *compatible with the Puppe sequences of* f *and* f^*, *i.e.*

$$\begin{tikzcd}
\tilde{\omega}_* X \arrow[r, "f"] \arrow[d, "u/"] & \tilde{\omega}_* Y \arrow[r, "f^1"] \arrow[d, "v/"] & \tilde{\omega}_* C_f \arrow[r, "f^2"] \arrow[d, "w/"] & \tilde{\omega}_* SX \arrow[r, "Sf"] \arrow[d, "u/"] & \tilde{\omega}_* SY \arrow[d, "v/"] \\
\tilde{\omega}^* SX^* \arrow[r, "Sf^*"'] & \tilde{\omega}^* SY^* \arrow[r, "f^{*2}"'] & \tilde{\omega}^* C_{f^*} \arrow[r, "f^{*1}"'] & \tilde{\omega}^* X^* \arrow[r, "f^*"'] & \tilde{\omega}^* Y^*
\end{tikzcd} \tag{4.2}$$

and the dual diagram for u^*, v^*, w^* *commute.*

Proof. Since everything involved is compatible with suspensions we may assume there are representatives $u: X^* \wedge X \to S^\alpha$, $v: Y^* \wedge Y \to S^\alpha$, $f^*: Y^* \to X^*$ and a homotopy $H: u(f^* \wedge X) \simeq v(Y^* \wedge f)$. Remember the functorial property [3] 2.2 of the mapping cone and define r by

$$\begin{tikzcd}
Y^* \wedge X \arrow[r, "f^*"] \arrow[d, "f"] \arrow[dr, phantom, "H"] & X^* \wedge X \arrow[r, "f^{*1}"] \arrow[d, "u"] & C_{f^*} \wedge X \arrow[d, dashed, "r"] \\
Y^* \wedge Y \arrow[r, "v"'] & S^\alpha \arrow[r, "v^1"'] & C_v ,
\end{tikzcd}$$

similarly w by

$$\begin{tikzcd}
C_{f^*} \wedge X \arrow[r, "f"] \arrow[d, "r"] \arrow[dr, phantom, "H'"] & C_{f^*} \wedge Y \arrow[r, "f^1"] \arrow[d, "f^{*2}"] & C_{f^*} \wedge C_f \arrow[d, dashed, "w"] \\
C_v \arrow[r, "v^2"'] & SY^* \wedge Y \arrow[r, "Sv"'] & S^{\alpha+1}
\end{tikzcd}$$

(both definitions depend on the choice of homotopies H resp.

$$H': (f^{*2} \wedge Y)(C_{f^*} \wedge f) \simeq v^2 r) .$$

w makes (4.2) commute, so its duality property follows from exactness and the five lemma.

5. Recall that the (reduced) mapping cylinder $Z(i, j)$ of a diagram $X \xleftarrow{i} A \xrightarrow{j} Y$ in $C^0(G)$ is the quotient of $X \vee I^+ \wedge A \vee Y$ by the identifications $[0, a] = [i\, a]$, $[1, a] = [j\, a]$. Up to suspension, $Z(i, j)$ is the mapping cone of a single map: let $p: X \to Z(i, j)$ and $q: Y \to Z(i, j)$ be the inclusions and consider the Puppe sequence

$$X \vee Y \xrightarrow{p \vee q} Z(i, j) \xrightarrow{(p \vee q)^1} C_{p \vee q} \xrightarrow{(p \vee q)^2} SX \vee SY \xrightarrow{Sp \vee Sq} SZ(i, j) \to \cdots \tag{5.1}$$

of $p \vee q$. As $p \vee q$ is a cofibration $C_{p \vee q}$ is homotopy equivalent with

$$Z(i, j) / (X \vee Y) = SA ,$$

so $SZ(i, j)$ is the mapping cone of a map $SA \to SX \vee SY$.

Remark. (5.1) is, of course, just the Mayer-Vietoris sequence of the proper triad $(Z(i, j); X \cup [0, \frac{1}{2}], Y \cup [\frac{1}{2}, 1])$.

We are now ready to prove:

(5.2) Proposition. *Every space $X \in \operatorname{ob} C^0(G)$ has a γ-dual for some $\gamma \in RO(G)$.*

Proof. Attaching an equivariant n-cell $e^n \times G/K$ to a subcomplex $Y \subset X$ means forming the mapping cylinder of the diagram

$$G^+/K \leftarrow (\partial e^n)^+ \wedge G^+/K \to Y$$

where the left hand map is the projection and the right hand one is the attaching map. From the discussion above we know that stably, $Y \cup e^n \times G/K$ is the mapping cone of a morphism $(\partial e^n)^+ \wedge G^+/K \to G^+/K \vee Y$. By (3.1), (3.2) (applied to the standard embedding of ∂e^n into $\mathbb{R}^n$) and (2.1 b), G^+/K and $(\partial e^n)^+ \wedge G^+/K$ have duals $G^+ \wedge_K S^{\beta - L(K,G)}$ and $S(\partial e^n)^+ \wedge G^+ \wedge_K S^{\beta - L(K,G)}$ resp. The proof is completed by (2.1 a) and induction on the number of cells in X.

6. Let $SC^0(G)h$ be the stable homotopy category, i.e. mor $(X, Y) = \tilde{\omega}_G^0(X \mid Y)$. From $SC^0(G)h$ we obtain another category $\hat{S}C^0(G)h$ by adjoining formal desuspensions $S^\alpha X$ ($\alpha \in RO(G)$, $\alpha < 0$, $X \in \operatorname{ob} C^0(G)$) and the obvious morphisms. Its advantage over $SC^0(G)h$ is that all suspensions S^γ, $\gamma \in RO(G)$, are defined and invertible.

We define the *duality cofunctor*

$$D_G\colon\ \hat{S}C^0(G)h \to \hat{S}C^0(G)h$$

as follows: for every object $S^\alpha X$ fix a duality $u_X \in \tilde{\omega}_G^\gamma(X^* \wedge X)$ and set

$$D_G(S^\alpha X) = S^{-\alpha-\gamma} X^*.$$

On morphisms let D_G be

$$D(u_X, u_Y)\colon\ \tilde{\omega}_G^*(X \mid Y) \approx \tilde{\omega}_G^*(Y^* \mid X^*).$$

Different choices of duals lead to equivalent cofunctors D_G. We also have $D_G{}^2 \simeq \mathrm{Id}$, the equivalences in both cases being given by the Yoneda lemma. Note that

$$S^\beta D_G S^\beta \simeq D_G \quad \text{for all} \quad \beta \in RO(G).$$

The cofunctor D allows us to construct the homology theory "corresponding to" any given cohomology and vice versa. Just note that (co-)homology theories $\tilde{t}_*$ ($\tilde{t}^*$) on $C^0(G)$ are naturally defined on $\hat{S}C^0(G)h$ too, and form $\tilde{t}_G^* \circ D_G$. This is readily seen to be a homology theory; the exactness axiom follows from the fact that D respects Puppe sequences. If $\tilde{t}^*$ is complete (see [7] 1), so is $\tilde{t}^* \circ D$: according to the remark after (3.1), $(l_X)_* \varrho_*^{KH} = (\lambda_{X^*})^* \varrho_{KH}^*$ is isomorphic for $X = S^0$ and all $K < H < G$, and this suffices by the comparison theorem and "Lie group induction" (compare $\tilde{t}_K^* \circ D_K$ and $\tilde{t}_H^* \circ D_H(H^+ \wedge_K \cdot)$).

Conversely, assume a multiplicative pair (t_*, t^*) of complete G-theories is given and let $\theta\colon \tilde{\omega}_G^* \to \tilde{t}_G^*$ be the Hurewicz homomorphism. If $u \in \tilde{\omega}_G^\alpha(X^* \wedge X)$ is a duality then the homomorphisms

$$\begin{aligned} &\theta u/\colon\ \tilde{t}_*^H X \to \tilde{t}_H^* X^* \\ &\theta u^*/\colon\ \tilde{t}_*^H X^* \to \tilde{t}_H^* X \end{aligned} \tag{6.1}$$

are bijective for all $H < G$. This follows since $\theta u/$ is a morphism $\bar{t}_* \to \bar{t}^* \circ D$, isomorphic on the coefficients.

7. As an application of the last result, consider the following pair of theories: fix a normal subgroup $H \lhd G$ and define t_* by

$$\bar{t}^K_\alpha X = \begin{cases} \tilde{\omega}^{K/H}_{\alpha^H}(X^H) & \text{if} \quad H < K, \\ 0 & \text{otherwise} \end{cases}$$

(α^H and X^H denote H-fixed point sets; these carry a K/H-action since $H \lhd K$). Define t^* correspondingly. A straightforward verification shows that t_* and t^* form a multiplicative pair of complete G-equivariant theories (when checking for completeness, note that for $H < K' < K$, $L(K/H, K'/H) = L(K, K')$ has trivial H-action, and that $(K \times_{K'} X)^H = \emptyset$ unless $H < K'$). By (6.1), t_* and t^* are duals, so we obtain:

(7.1) Proposition. *Let H be a subgroup of G. If $u \in \tilde{\omega}^\alpha_G(X^* \wedge X)$ is a duality then $u^H \in \tilde{\omega}^{\alpha^H}_{NH/H}(X^{*H} \wedge X^H)$ is a duality where NH is the normalizer of H in G.*

References

[1] M. F. Atiyah, Thom complexes. Proc. London Math. Soc. (3) **11**, 291–310 (1961).
[2] S. Illman, Equivariant singular homology and cohomology for actions of compact Lie groups. In: Lect. Notes Math. **298**, pp. 403–415. Berlin-Heidelberg-New York 1972.
[3] D. Puppe, Homotopiemengen und ihre induzierten Abbildungen. I. Math. Z. **69**, 299–344 (1958).
[4] G. B. Segal, Equivariant stable homotopy theory. In: Actes Congrès intern. Math. 1970, Tome 2, pp. 59–63. Paris 1971.
[5] E. H. Spanier, Algebraic Topology. New York 1966.
[6] E. H. Spanier, Function spaces and duality. Ann. of Math. (2) **70**, 338–378 (1959).
[7] K. Wirthmüller, Equivariant homology and duality. Manuscripta math. **11**, 373–390 (1974).

Eingegangen am 14. 8. 1974

Anschrift des Autors:

Klaus Wirthmüller
Mathematisches Institut
der Universität
D-66 Saarbrücken 11

zur Zeit:

Department of Pure Mathematics
University of Liverpool
P. O. Box 147
Liverpool
L69 3BX