manuscripta math. 11, 373 - 390 (1974)

EQUIVARIANT HOMOLOGY AND DUALITY

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This note is concerned with stable G-equivariant homology and cohomology theories (G a compact Lie group). In important cases, when H-equivariant theories are defined naturally for all closed subgroups H of G, we show that the G-(co)homology groups of $G \times_H X$ are isomorphic with H-(co)homology groups of X. We introduce the concept of orientability of G-vector bundles and manifolds with respect to an equivariant cohomology theory and prove a duality theorem which implies an equivariant analogue of Poincaré - Lefschetz duality.

The ideas developed here partly originate from suggestions made by T. tom Dieck, who introduced me to the subject.

1 Equivariant homology

G is a compact Lie group, $GTop^o$ the category of pointed G-spaces.

Let RO(G) be the real representation ring of G and identify every element of $RO(G)^+$ (the semi-group of isomorphism classes of real representations) with one of its representatives in a suitable manner (cf. e.g. [5] I.1). Fix a subgroup A of RO(G) consisting of even-dimensional virtual G-modules. For $V \in RO(G)^+$ let $[V]$ denote the coset of V in RO(G)/A, let S^V be the one-point compactification of V (or, ambiguously, suspension by it) and $|V|$ the real dimension of V.

A G-<u>homology theory</u> indexed by RO(G)/A consists of

(1) a collection $(\tilde{t}_\alpha)_{\alpha\in RO(G)/A}$ of functors $\tilde{t}_\alpha\colon GTop^o \longrightarrow Ab$

(2) a collection $(\sigma_{V\alpha})_{V\in RO(G)^+,\ \alpha\in RO(G)/A}$ of natural transformations $\sigma_{V\alpha}\colon \tilde{t}_\alpha \longrightarrow \tilde{t}_{\alpha+[V]}\cdot S^V$

such that

(a) $\tilde{t}_\alpha$ is homotopy invariant and half-exact, i.e. for $f\colon X \longrightarrow Y$ the sequence $\tilde{t}_\alpha X \longrightarrow \tilde{t}_\alpha Y \longrightarrow \tilde{t}_\alpha C_f$ is exact (C_f denotes the mapping cone of f)

(b) $T_*\sigma_V\sigma_W = (-1)^{|V||W|}\sigma_W\sigma_V$ where $T\colon S^V S^W \longrightarrow S^W S^V$ interchanges the factors

(c) all $\sigma_{V\alpha}$ are equivalences of functors.

The corresponding unreduced theory is defined by $t_\alpha(X,Y) = \tilde{t}_\alpha C(X,Y) = \tilde{t}_\alpha C_{Y^+\subset X^+}$. Cohomology is defined and denoted in the obvious way.

$GTop^o$ may be replaced by a suitable subcategory, e.g. a category of G-equivariant CW complexes ([6]).

Consider theories t, u, v indexed by RO(G)/A, RO(G)/B, RO(G)/C respectively, with $A + B \subset C$. We shall use the four external products ([9] §6)

(1)	homology	cross	$\times$:	$\tilde{t}_*X$	$\otimes$ $\tilde{u}_*Y$	$\longrightarrow$	$\tilde{v}_*(X\wedge Y)$
(2)	cohomology	cross	$\times$:	$\tilde{t}^*X$	$\otimes$ $\tilde{u}^*Y$	$\longrightarrow$	$\tilde{v}^*(X\wedge Y)$
(3)	homology	slant	$\backslash$:	$\tilde{t}^*Y$	$\otimes$ $\tilde{u}_*(X\wedge Y)$	$\longrightarrow$	$\tilde{v}_*X$
(4)	cohomology	slant	$/$:	$\tilde{t}^*(X\wedge Y)$	$\otimes$ $\tilde{u}_*Y$	$\longrightarrow$	$\tilde{v}^*X$

as well as the cup and cap products induced by (2) resp. (3). Our sign conventions are consistent with [8]. Of particular interest are the cases t = u = v (multiplicative theory; $\tilde{t}^\alpha S^o = \tilde{t}_{-\alpha}S^o$ etc. is always understood) and u = v (t multiplicative and acting on u). In these cases we assume a unit in t having the usual properties.

Now consider (closed) subgroups $K \subset H \subset G$. Let Γ be the semi-direct product of $K \times K$, acting on H by both-sided translation, with $\mathbb{Z}_2$, the non-trivial element being inversion on H. Thus H is a Γ-space, and K (being stable under Γ) has a linear Γ-tube in H ([2] II.4, we identify a tube with its image in H). Denote the bundle projection $T \longrightarrow H$ by π and let $L = L(K,H)$ be the fibre at 1 with K acting orthogonally upon it by conjugation. We identify L and its class in $RO(K)$.

There is a left and right K-equivariant map $l\colon H^+ \longrightarrow T^c \approx S^L K^+$ where the arrow collapses $H - T$ into the base point, and the homeomorphism sends h to $[h \cdot \pi h^{-1}, \pi h]$. Note that for any K-space X, l defines a K-map $l_X\colon H^+ \wedge_K X \longrightarrow S^L X$. Similarly, from the inclusion $\lambda\colon K \subset H$ we obtain maps $\lambda_X\colon X \longrightarrow H^+ \wedge_K X$.[1]

Let us assume that for every subgroup $H \subset G$ a subgroup A_H of $RO(H)$ is given such that the forgetful functors $RO(H) \longrightarrow RO(K)$, $K \subset H \subset G$, map A_H into A_K. Denote the homomorphism $RO(H)/A_H \longrightarrow RO(K)/A_K$ by $\alpha \longmapsto \alpha_K$.

A <u>complete</u> <u>G-homology</u> <u>theory</u> $\tilde{t}_*$ consists of one H-homology theory $\tilde{t}^H_*$ for every subgroup $H \subset G$ together with natural transformations (<u>restrictions</u>)

$$\rho^{KH}_*\colon \tilde{t}^H_* \longrightarrow \tilde{t}^K_{*(H)}$$

of homology theories such that $\rho^{MK}_* \rho^{KH}_* = \rho^{MH}_*$ ($M \subset K \subset H \subset G$). Here $\tilde{t}^K_{*(H)}$ means the graded group $(\tilde{t}^K_{\alpha_K})_{\alpha \in RO(H)/A_H}$.
The following axiom is to be satisfied: the composition

$$\tilde{t}^H_* \,(H^+ \wedge_K \cdot) \xrightarrow[\rho^{KH}_*]{} \tilde{t}^K_{*(H)} \,(H^+ \wedge_K \cdot) \xrightarrow[l_{X*}]{} \tilde{t}^K_{*(H)} \,(S^L X)$$

respectively

$$\tilde{t}^*_H \,(H^+ \wedge_K \cdot) \xrightarrow[\rho^*_{KH}]{} \tilde{t}^{*(H)}_K \,(H^+ \wedge_K \cdot) \xrightarrow[\lambda_X{}^*]{} \tilde{t}^{*(H)}_K$$

[1] In terms of pairs, l is (essentially) the inclusion $H \subset (H, H-K)$ dual to λ (compare Theorem 4.1 below).

is isomorphic.

There is a comparison theorem for complete theories defined on the category of finite CW complexes: a morphism of complete theories is isomorphic if it induces isomorphisms on the coefficients $\tilde{t}^H_* S^0$, all $H \subset G$.

Products in complete theories are assumed to commute with the restriction homomorphisms ρ.

2 Spectra

We construct equivariant homology theories by means of spectra. In order to avoid signs we give the construction of the functors t_α for only those α with $\alpha = [2V]$ for some representation V. The definition is readily completed then by use of suspensions.

Recall that RO(G) is a directed set and that every representation $V \in 2RO(G)$ has a canonical complex G-module structure. For these V all complex automorphisms of S^V are G-homotopic (see [5] I.1).

A G-spectrum $\underline{E}$ over RO(G)/A consists of

(1) a final subset $\epsilon \subset 2RO(G)^+$, closed under addition and subtraction (as far as possible in $RO(G)^+$)

(2) a family $(E_\alpha)_{\alpha \in [\epsilon]}$ of pointed G-spaces

(3) G-maps $e_{V\alpha}\colon S^V E_\alpha \longrightarrow E_{[V]+\alpha}$, $\alpha \in [\epsilon]$, $V \in \epsilon$

such that $e_{V,[W]+\alpha} \cdot e_{W\alpha}$ is pointed G-homotopy equivalent to $e_{V+W,\alpha}$.

For any G-spectrum $\underline{E}$ we define associated contra-co-variant bifunctors $\tilde{t}^\alpha(X \mid Y \mid \underline{E})$ to be the colimits over pointed G-homotopy sets

$$[S^V X \;,\; E_{[V]+\alpha} \wedge Y]^0_G, \; V \in \epsilon$$

as usual (cf. [4]). It is well-known ([5] I.1) that these functors are actually homology and cohomology theories, for fixed X respectively Y. In particular we have theories $\tilde{t}_\alpha(Y \mid \underline{E}) := \tilde{t}^{-\alpha}(S^0 \mid Y \mid \underline{E})$ and $\tilde{t}^\alpha(X \mid \underline{E}) := \tilde{t}^\alpha(X \mid S^0 \mid \underline{E})$.

Let $(A_H)_{H \subset G}$ be a family as above. A complete G-spectrum is a family of H-spectra $\underline{E}^H$ such that the forgetful functor maps ϵ^H into ϵ^K $(K \subset H)$ and, secondly, $E^H_{[\alpha]}$ and $E^K_{[\alpha_K]}$ are K-homotopy equivalent by equivalences commuting with the maps of the spectra.

A complete spectrum defines a family of functors $(\tilde{t}^*_H(X \mid Y))_{H \subset G}$. There are obvious restriction morphisms $\rho^*\colon \tilde{t}^*_H \longrightarrow \tilde{t}^{*}_K{}^{(H)}$.

THEOREM 2.1. *$\tilde{t}_*$ and $\tilde{t}^*$ are complete G-theories.*

Proof. Let $K \subset H \subset G$. Note at first that the forgetful functor maps $RO(H)^+$ onto a final subset of $RO(K)^+$ (see [2] 0.4.2). Then the cohomology assertion follows immediately from

$$[S^V H^+ \wedge_K X \,,\, E \wedge Y]^0_H \approx [H^+ \wedge_K S^V X \,,\, E \wedge Y]^0_H$$
$$\approx [S^V X \,,\, E \wedge Y]^0_K.$$

Choose an H-module $V \in \epsilon_H$ such that K is the stabilizer of some point $x \in V$ (cf. [2] 0.5.2) and let $W \subset V$ be a linear slice at x. We have an H-map $\psi\colon S^V \longrightarrow H^+ \wedge_K S^W$ and a K-map $\varphi\colon S^V \longrightarrow S^W S^L$ so that $\psi^{-1}\colon H \times_K W \approx HW \subset V$ and $\varphi^{-1}\colon W \times L \approx T \times_K W \approx TW \subset V$ are the canonical maps. φ is an H-homotopy equivalence (it is homotopic to its differential at x). Let ω be a homotopy inverse of $(\mathrm{id} \wedge -\mathrm{id}) \cdot \varphi$. We assert that the following composition $\varkappa$ inverts $1_* \rho^{KH}_*$:

$$\tilde{t}^{*}_K{}^{(H)}(X \mid S^L Y) \xrightarrow[S^W]{} \tilde{t}^{*}_K{}^{(H)}(S^W X \mid S^W S^L Y) \xrightarrow[\omega_*]{}$$

$$\tilde{t}_K^{*(H)}(S^W X \mid S^V Y) \xrightarrow[H^+\wedge_K]{} \tilde{t}_H^*(H^+ \wedge_K S^W X \mid H^+ \wedge_K S^V Y) \approx$$

$$\tilde{t}_H^*(H^+ \wedge_K S^W \wedge X \mid S^V H^+ \wedge_K Y) \xrightarrow[\psi^*]{}$$

$$\tilde{t}_H^*(S^V X \mid S^V H^+ \wedge_K Y) \xrightarrow[(S^V)^{-1}]{} \tilde{t}_H^*(X \mid H^+ \wedge_K Y).$$

Let $f\colon S^U X \longrightarrow E \wedge S^L Y$ represent an element in $\tilde{t}_K^{*(H)}(X \mid S^L Y)$, and let $\overline{f}\colon S^V S^U X \longrightarrow S^V E \wedge S^L Y$ be the corresponding representative of $1_* \rho_* \varkappa(f)$. The following K-homotopy commutative diagram shows that f and $\overline{f}$ represent the same class:

$$\begin{tikzcd}
S^V S^U X \arrow[rr, "\overline{f}"'] \arrow[dr, "(1)"'] \arrow[ddd, equal] & & S^V E \wedge S^L Y \arrow[r, "\approx", "(5)"'] & S^W S^L E \wedge S^L Y \arrow[ddd, "\approx"', "(6)"] \\
& S^W S^L S^U X \arrow[r, "(2)"'] \arrow[d, equal] & S^L S^W L^+ \wedge S^U X \arrow[u, "(3)"'] \arrow[d, equal] & \\
& S^W S^L S^U X \arrow[r, "(4)"'] & S^L S^W L^+ \wedge S^U X & \\
S^V S^U X \arrow[rr, "S^V f"'] & & S^V E \wedge S^L Y \arrow[r, "\approx", "(5)"'] & S^W S^L E \wedge S^L Y
\end{tikzcd}$$

Explanation: (1) is $\varphi \wedge S^U X$, (2) is the diagonal on L. It is readily verified that there is a unique map (3) that is the identity on S^L and makes the upper left hand part of the diagram commute. (4) maps $x \in S^L$ to $[x,o] \in S^L L^+$ and is homotopic with (2). (5) is $\varphi \wedge E \wedge S^L Y$, and finally (6), mapping $[x,y] \in S^L S^L$ to $[y,-x] \in S^L S^L$, is homotopic to the identity. The lower part of the diagram commutes up to a homotopy $\varphi\omega \simeq \mathrm{id} \wedge -\mathrm{id}\colon S^W S^L \longrightarrow S^W S^L$.

Now assume that $F\colon S^U X \longrightarrow E \wedge H^+ \wedge_K Y$ is an H-map and let $\overline{F}\colon S^V S^U X \longrightarrow S^V E \wedge H^+ \wedge_K Y$ be its transformed representing $\varkappa 1_* \rho_*(F)$. We abbreviate

$S = S^V S^U X$, $R = S^V E \wedge H^+ \wedge_K Y$,

$P = W \times F|[S^U X - F^{-1}(*)]^{-1} p^{-1}(T/K) \subset S$ and

$Q = V \times F|[S^U X - F^{-1}(*)]^{-1} p^{-1}(1K) \subset S$ (p is the projection to H/K),

$P^* = P \cup \{*\} \subset S/(S - HP)$, $Q^* = Q \cup \{*\} \subset S/(S - HQ)$.

The canonical maps $H^+ \wedge_K P^* \longrightarrow S/(S - HP)$ and $H^+ \wedge_K Q^* \longrightarrow S/(S - HQ)$ are homeomorphisms.

Look at the following diagrams, which we explain below:

$$\begin{tikzcd}
S \arrow[rr, "\overline{F}"'] \arrow[d, equal] & & R \arrow[d, equal] \\
 & H^+\wedge_K P^* \arrow[r, "(3)"'] & R \arrow[dd, equal] \\
S \arrow[ur, "(1)"] \arrow[dr] \arrow[ddd, equal] & & \\
 & H^+\wedge_K P^* \arrow[r, "(7)"'] & R \arrow[d, equal] \\
 & H^+\wedge_K Q^* \arrow[r, "(8)"'] & R \arrow[dd, equal] \\
S \arrow[ur, "(2)"] \arrow[dr] \arrow[d, equal] & & \\
S \arrow[rr, "S^V F"'] & H^+\wedge_K Q^* \arrow[r, "(9)"'] & R \arrow[d, equal] \\
 & & R
\end{tikzcd}$$

$$\begin{tikzcd}
P^* \arrow[r, "(4)"'] \arrow[d, equal] & S^W S^L E \wedge H^+ \wedge_K Y \arrow[r, "(5)"'] \arrow[d, equal] & R \arrow[d, equal] \\
P^* \arrow[r, "(4)"'] & S^W S^L E \wedge H^+ \wedge_K Y \arrow[r, "(6)"'] & R
\end{tikzcd}$$

$$\begin{tikzcd}
I^+ \wedge S \arrow[rr, "(1o)"'] \arrow[d, "(12)"] & & R \\
I^+ \wedge H^+ \wedge_K Q^* \arrow[urr, "(11)"'] & & I^+ \wedge Q^* \arrow[ll, "(13)"] \arrow[u, "(14)"']
\end{tikzcd}$$

(1) and (2) are quotient maps, (3) is the unique map making the top part of the left diagram commutative. (3) corresponds to a K-map $P^* \longrightarrow R$, which factors into (5)(4) as indicated on the right, (5) being $\omega \wedge \mathrm{id}$. (6) sends $[x,g,h] \in S^W S^L H^+$ to $[g \cdot \omega(x,g), gh] \in S^V H^+$ and is K-homotopic with (5). (7) is the H-extension of (6)(4). There are unique maps (8) and (9) such that (8)(2)=(7)(1) and (9)(2)=$S^V F$. It remains to prove that (8) and (9) are H-homotopic: if Φ is a K-homotopy from $\omega \cdot (\mathrm{id} \wedge -\mathrm{id}) \cdot \varphi$ to the identity of S^V, then (1o) = $\Phi \wedge F$ factors into (11)(12) where (12) is the quotient map. The inclusion (13) followed by (11) is a K-homotopy (14) of maps $Q^* \longrightarrow R$. The H-extension of (14) joins (8) and (9).

This completes the proof.

If $\underline{E}^i$, $i = 1,2,3$, are complete G-spectra over $(RO(H)/A_H^i)_{H\subset G}$ with $A_H^1 + A_H^2 \subset A_H^3$, a <u>pairing</u> ϵ of $\underline{E}^1 \wedge \underline{E}^2$ into $\underline{E}^3$ consists of one pairing ϵ^H: $\underline{E}^{1,H} \wedge \underline{E}^{2,H} \longrightarrow \underline{E}^{3,H}$ of H-spectra for each H, commuting with the restriction maps up to pointed equivariant homotopy. A pairing induces products in homology (cf. [9] §6).

Examples of multiplicative complete G-spectra are the sphere spectrum ([7]) leading to equivariant stable homotopy, and the Thom spectra ([5]) leading to bordism theories.

<u>3 Orientation</u>

Let $\pi: E \longrightarrow B$ be a G-vector bundle and let $M(\pi)$ denote its Thom space. Assume that $\tilde{t}^*$ is a multiplicative complete cohomology theory. $\tilde{t}^*M(\pi)$ is a t^*B module by means of the cup product.

<u>PROPOSITION 3.1</u>. <u>If</u> B <u>is</u> <u>a</u> <u>homogeneous</u> G-<u>space</u> <u>then</u> $\tilde{t}^*M(\pi)$ <u>is</u> <u>free</u> <u>cyclic</u> <u>over</u> t^*B.

Proof. We may assume $B = G/H$, $E = G \times_H V$ with $V \in RO(H)$, thus $M(\pi) = G^+ \wedge_H S^V$. The composition

$$(\sigma^V)^{-1}\lambda^*\rho^*_{HG}\colon \tilde{t}^*_G\,(G^+ \wedge_H S^V) \longrightarrow t^*_H(G)$$

is a module isomorphism over $\lambda^*\rho^*_{HG}\colon t^*_G\,(G/H) \longrightarrow t^*_H(G)$, hence the assertion.

Return to the general case. A <u>Thom class</u> for π in t^* is a homogeneous element $\tau \in \tilde{t}^*_G M(\pi)$ such that for every orbit $b \subset B$, $\tau|M(\pi|b)$ is a free module generator of $\tilde{t}^*_G M(\pi|b)$ over $t^*_G b$. A (t^*_G -) <u>orientable</u> G-bundle is one admitting a Thom class.

<u>PROPOSITION 3.2</u>. <u>If</u> τ_i <u>is</u> <u>a</u> <u>Thom</u> <u>class</u> <u>for</u> π_i: $E_i \longrightarrow B_i$ $(i = 1,2)$, <u>then</u> $\tau_1 \times \tau_2$ <u>is</u> <u>a</u> <u>Thom</u> <u>class</u> <u>for</u> $\pi_1 \times \pi_2$.

Proof. By naturality we may assume $B_i = G/H_i$, $M(\pi_i) = G^+ \wedge_{H_i} S_i$ and need consider only the orbit of $z = (1H_1, 1H_2)$. The inclusion $Gz \subset B_1 \times B_2$ corresponds to the diagonal $d\colon G/(H_1 \cap H_2) \longrightarrow G/H_1 \times G/H_2$. The assertion follows now from commutativity of

$$
\begin{tikzcd}
\tilde t^*_G(G^+\wedge_{H_1}S_1) \arrow[dd, "\approx"] & \otimes & \tilde t^*_G(G^+\wedge_{H_2}S_2) \arrow[r, "\times"'] \arrow[dd, "\approx"] & \tilde t^*_G(G^+\wedge_{H_1}S_1 \wedge G^+\wedge_{H_2}S_2) \arrow[d, "d^*"] \\
 & & & \tilde t^*_G(G^+ \wedge_{H_1\cap H_2} S_1S_2) \arrow[d, "\approx"] \\
\tilde t^*_{H_1}(G)S_1 & \otimes & \tilde t^*_{H_2}(G)S_2 \arrow[r, "\times"'] & \tilde t^*_{H_1\cap H_2}(G)\,S_1S_2 \\
t^*_{H_1}(G) \arrow[u, "\approx"] & \otimes & t^*_{H_2}(G) \arrow[u, "\approx"] \arrow[r, "\times"'] & t^*_{H_1\cap H_2}(G) \arrow[u, "\approx"]
\end{tikzcd}
$$

(the restrictions $\rho^*_{H_1\cap H_2, H_i}$ involved in the lower part of the diagram are not indicated).

A similar argument shows

PROPOSITION 3.3. *A t^*_G-orientable bundle is t^*_H-orientable for every* $H \subset G$.

By a G-*manifold* we mean a (paracompact) topological manifold with boundary together with a locally smooth G-action upon it ([2] IV). Recall that any G-manifold is an equivariant ANR (if its topology has a countable base) and has an equivariant collaring ([2] V.1.5). The *tangent bundle* τX of a G-manifold X with empty boundary is the $(X, X - *)$ bundle

$$pr_1\colon (X\times X\ ,\ X\times X - \Delta X) \longrightarrow X$$

(ΔX denotes the diagonal in $X\times X$). The notion of Thom class and orientability clearly applies to τX though this need not be a vector bundle. If ∂X is not empty we define a Thom class to be a homogeneous element in $t^*(X\times X, X\times X - \Delta X)$ which restricts to a Thom class of $X - \partial X$.

PROPOSITION 3.4. *Let X be an H-manifold.* $G \times_H X$ *is* t^*_G *- orientable iff X has a Thom class in* $t^{*(G)-[L(H,G)]}_H$.

Proof. This follows because

$$((G\times_H X) \times (G\times_H X),(G\times_H X) \times (G\times_H X) - \Delta(G\times_H X)) \approx$$

$$G \times_H (X \times G\times_H X, X \times G\times_H X - \Delta X) \xrightarrow[G \times_H (X\times 1_X)]{} \frac{G \times_H (S^L,S^L - o) \times (X\times X, X\times X - \Delta X)}{G \times_H \quad * \quad \times \quad X\times X}$$

induces an isomorphism in cohomology ($(x,x) \in \Delta X$ is identified with $[x,1,x] \in X \times G \times_H X$).

Clearly a G-manifold with stably trivial tangent bundle has Thom classes in any complete theory. It is also obvious that there are larger classes of manifolds with orientations in the various cobordism theories.

4 Duality

Let X be a G-manifold without boundary, and let $\tau \in t^\xi(X\times X, X\times X - \Delta X)$ be a Thom class. Let (u_*,u^*) be a pair of complete G-theories such that t acts on u. Suppose that u_* has compact supports ([8] 4.8.11). For every compact pair (A,B) in X we define a *duality map*

$$\gamma_\tau : u_\alpha(X-B,X-A) \longrightarrow u^{\xi-\alpha}(A,B)$$

which sends z to $[\tau|(A,B) \times (X-B,X-A)]/z$.[1] Set $\bar{u}^*(A,B) = \operatorname{colim} u^*(U,V)$, with (U,V) varying over pairs of

[1] The slant product need not be defined for arbitrary pairs. This difficulty can be avoided as follows: let (U,V) be a closed neighbourhood pair of (A,B). By means

compact neighbourhoods of (A,B) in X, and let

$$\bar{\gamma}_\tau : u_*(X-B,X-A) \longrightarrow \bar{u}^*(A,B)$$

be the colimit homomorphism.

<u>THEOREM 4.1</u>. $\bar{\gamma}_\tau$ *is an isomorphism*.

Proof. Consider the special case $A = Gx$, $B = \emptyset$ first. The situation is displayed in the commutative diagrams

$$\begin{tikzcd}
t_G^*(A\times X\ ,\ A\times X - \Delta A) \arrow[rr] \arrow[d, "\approx"', "(1)^*"] & & t_G^*(A\times(X,X-A)) \arrow[d, "(2)^*"] \\
t_G^*(G\times_H(G\times_H V, G\times_H V - H\times_H o)) \arrow[rr, "(3)^*"'] \arrow[dr, "\rho^*"] \arrow[dd, "\approx"'] & & t_G^*(G/H\times G\times_H(V,V-o)) \arrow[d, "\rho^*"] \\
& t_H^*(G\times_H(G\times_H V, G\times_H V - H\times_H o)) \arrow[r, "(3)^*"'] \arrow[dl, "(4)^*"] & t_H^*(G/H\times G\times_H(V,V-o)) \arrow[d, "(5)^*"] \\
t_H^*(G\times_H V\ ,\ G\times_H V - H\times_H o) \arrow[rr] & & t_H^*(G\times_H(V,V-o)) \\
\tilde{t}_H^*(S^L S^V) \arrow[u, "\approx", "1^*"'] \arrow[rr, equal] & & \tilde{t}_H^*(S^L S^V) \arrow[u, "1^*"'] \\
t_H^* \arrow[u, "\approx", "\sigma^{L+V}"'] \arrow[rr, equal] & & t_H^* \arrow[u, "\sigma^{L+V}"']
\end{tikzcd}$$

and

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of a function that separates A and X − U we construct a map

$$C(A,B) \wedge (X-V)/(X-U) \longrightarrow C((A,B) \times (X-B,X-A))$$

and obtain a product

$$t^*((A,B) \times (X-B,X-A)) \otimes \tilde{t}_*((X-V)/(X-U)) \longrightarrow t^*(A,B).$$

Now take the colimit over (U,V) and apply the axiom of compact supports.

$$
\begin{tikzcd}
t^*_G(A\times(X,X-A)) \arrow[d, "(2)^*"] & \otimes & u^G_*(X,X-A) \arrow[r] & u^*_G A \arrow[d, "(7)^*"', "\approx"] \\
t^*_G(G/H\times G\times_H(V,V-o)) \arrow[d, "\rho^*"] & \otimes & u^G_*(G\times_H(V,V-o)) \arrow[u, "(6)_*", "\approx"'] \arrow[d, "\rho_*"'] \arrow[r] \arrow[ddd, bend left, "\approx"] & u^*_G(G/H) \arrow[d, "\rho^*"'] \arrow[dd, bend left, "\approx"] \\
t^*_H(G/H\times G\times_H(V,V-o)) \arrow[d, "(5)^*"] & \otimes & u^H_*(G\times_H(V,V-o)) \arrow[d, equal] \arrow[r] & u^*_H(G/H) \arrow[d, "(8)^*"'] \\
t^*_H(G\times_H(V,V-o)) & \otimes & u^H_*(G\times_H(V,V-o)) \arrow[d, "1_*"'] \arrow[r] & u^*_H \arrow[d, equal] \\
\tilde{t}^*_H(S^LS^V) \arrow[u, "1^*"'] & \otimes & \tilde{u}^H_*(S^LS^V) \arrow[r] & u^*_H \arrow[d, equal] \\
t^*_H \arrow[u, "\sigma^{L+V}"'] & \otimes & u^H_* \arrow[u, "\sigma_{L+V}", "\approx"'] \arrow[r] & u^*_H
\end{tikzcd}
$$

Explanation: H is the stabilizer of x, V a linear slice at x. (1): $[g,h,v]\longmapsto(gx,ghv)$, (2): $[g,h,v]\longmapsto(gx,hv)$, (3): $[g,h,v]\longmapsto(g,g^{-1}h,v)$, (4),(5): $[h,v]\longmapsto[1,h,v]$, (6): $[h,v]\longmapsto hv$, (7): $[h]\longmapsto hx$, (8): $*\longmapsto 1H$. The unlabelled horizontal arrows of the first diagram are induced by inclusions, those of the second one are slant products.

The Thom class $\tau|A \in t^*_G(A\times X\ ,\ A\times X - \Delta A)$ corresponds to a unit in t^*_H by the vertical isomorphism. From this fact the assertion follows (clearly $u^*A = \bar{u}^*A$).

In the general case we may assume $B = \emptyset$ (by the five lemma), further that X is compact (by excising the complement of a compact neighbourhood of A). Sometimes we shall not distinguish between a G-subset of X and its image in X/G.

We set up a spectral sequence along the lines of [1] 3. Let $Q = (Q_j)_{j\in J}$ be an open G-covering of X with the properties

(a) J is finite

(b) $\dim \text{nerve}\ Q \leq \dim X$

(such coverings are final in the set of all open coverings). Choose $l \geq |J|$ and realise $N := \text{nerve}\ Q$ as a linear subcomplex $|N|$ of $\mathbb{R}^l$ such that the vertices of N are affine independent in $\mathbb{R}^l$. We abbreviate:

$\bar{\sigma}$ the closed simplex σ, $\dot{\sigma}$ the boundary, $\mathring{\sigma} = \bar{\sigma} - \dot{\sigma}$,

$Q_\sigma = \bigcap_{j\in\sigma} Q_j$ if $\bigcap_{j\in\sigma} Q_j$ meets A, $Q_\sigma = \emptyset$ otherwise, $C_\sigma = Q_\sigma^-$,

$Z = \mathbb{R}^l \times X$, $W = \bigcup_{\text{all}\ \sigma} \bar{\sigma} \times C_\sigma \subset Z$, $C = pr_2 W$.

W is filtered by $\emptyset = W^{-1} \subset W^o \subset \ldots \subset W^{\dim X} = W$ with $W^p = pr_1^{-1}|N^p|$ (N^p the p-skeleton of N). It follows ([1] 3.2, [3] XV §7) that there is a strongly convergent spectral sequence E^* with $E_2^* = H^*(Q|P^*)$ and termination u^*C; the coefficient presheaf P^* on X/G (see [8] 6.7) sends an open subset $U \subset X/G$ to $u^*(U^-)$ if U meets A, to $\{o\}$ otherwise. The spectral sequence is functorial with respect to refinements of coverings ([1] 3.2). Taking the colimit over open coverings, we obtain a spectral sequence $\bar{E}^*$, with $\bar{E}_2^* = \check{H}^*(X/G|P^*)$ (Čech cohomology, see [8] 6.7), converging strongly to $\bar{u}^*A$.

We imitate this procedure in homology. Consider the (cohomology) spectral sequence E_* with $E^1_{p\alpha} = u_{l-p-\alpha}(Z-W^{p-1}, Z-W^p)$ set up by the Cartan - Eilenberg method ([3] XV §7). The sets $\mathring{\sigma}$ ($\dim \sigma = p$) are closed in $\mathbb{R}^l - |N^{p-1}|$, hence we can choose pairwise disjoint neighbourhoods N_σ. There are isomorphisms

$$u_*(Z-W^{p-1}, Z-W^p) \approx \sum_{\dim\sigma=p} u_*(N_\sigma\times X, N_\sigma\times X - \mathring{\sigma}\times C_\sigma) \approx$$
$$u_*((\mathbb{R}^l-\dot{\sigma}, \mathbb{R}^l-\bar{\sigma}) \times (X, X-C_\sigma))$$

(excise $Z - (W^{p-1} \cup \bigcup_\sigma N_\sigma\times X)$ on the left and $(\mathbb{R}^l-N_\sigma) \times X$ on the right). Choose a sequence $\sigma_o \subset \sigma_1 \subset \ldots \subset \sigma_p = \sigma$, with σ_j a j-simplex. We have isomorphisms

$$u_*((\mathbb{R}^1-\dot{\sigma}_j\ ,\ \mathbb{R}^1-\overline{\sigma}_j) \times (X,X-C_\sigma)) \approx$$
$$u_*((\mathbb{R}^1-\dot{\sigma}_{j-1}\ ,\ \mathbb{R}^1-\overline{\sigma}_{j-1}) \times (X,X-C_\sigma))$$

dual to those used in the computation of E_2^* (to prove bijectivity, approximate the simplexes by suitable neighbourhoods and apply the axiom of compact supports). Finally we have

$$u_*((\mathbb{R}^1,\mathbb{R}^1-\overline{\sigma}_o) \times (X,X-C_\sigma)) \approx u_*(X,X-C_\sigma)$$

by suspension.

Proceeding further like in the cohomology case we obtain an isomorphism $E_*^2 \approx H^*(Q|P_*)$, with $P_*U = u_*(X,X-U^-)$ if U meets A, $P_*U = \{o\}$ otherwise ($U \subset X/G$ open).

E_* has termination $u_*(X,X-C)$: clearly E_* converges to $u_*(Z,Z-W)$. Choose an open neighbourhood V of C in X and a G-function $\psi: X \longrightarrow I$ such that $X - V = \psi^{-1}\{o\}$ and $C \subset \psi^{-1}\{1\}$. Let $(\varphi_j)_{j\in J}$ be a partition of unity subordinate to Q. The formula

$$(s,x) \longmapsto (s - \psi x\cdot \sum_{j\in J} \varphi_j x\cdot\{\overline{j}\}\ ,\ x)$$

defines a homeomorphism h of Z onto itself, homotopic to the identity and carrying iC onto $o \times C$ where $i: C \longrightarrow W$ is the homotopy inverse of pr_2 sending x to $(\sum_{j\in J} \varphi_j x\cdot\{\overline{j}\}\ ,\ x)$. With $r = \sup\{|s|\ |\ s \in |N|\}$ and $R = \{(s,x) \in Z\ |\ |s| < 2r\cdot\psi x\}$ the inclusions $Z - R \subset Z - o{\times}V$ and $Z - R \subset h(Z - W) \cap (Z - o{\times}V)$ are homotopy equivalences. Taking colimits over neighbourhoods V we obtain

$$u_*(Z,Z-W) \approx u_*((\mathbb{R}^1,\mathbb{R}^1-o) \times (X,X-C)) \approx u_*(X,X-C)$$

as asserted above.

The spectral sequence E_* does not depend on the chosen realisation of N as a subspace of some $\mathbb{R}^l$: suppose $|N| \subset \mathbb{R}^l$ and $|N|' \subset \mathbb{R}^{l'}$ are different choices, $l \leq l'$. Then there is an affine orientation-preserving isomorphism $h: \mathbb{R}^{l'-l} \times \mathbb{R}^l \longrightarrow \mathbb{R}^{l'}$ sending each vertex of $o \times |N|$ to the corresponding vertex in $|N|'$. $(l'-l)$-fold trivial suspension followed by $(h \times id_X)_*$ is an isomorphism

$$u_*(Z-W^p,Z-W^q) \approx u_*(Z'-W'^p,Z'-W'^q)$$

for every p,q $(p \leq q)$. Furthermore on $H^*(Q|P_*)$ and $u_*(X,X-C)$ the identities are induced.

We make E_* functorial with respect to refinements of coverings: let $Q \leq Q'$, $\lambda: J' \longrightarrow J$ a refinement function. Define a third covering Q'' by $J'' = J + J'$, $Q''_j = Q_j$, $Q''_{j'} = Q_{\lambda j'}$, and factor λ into

$$J' \xrightarrow[\lambda']{} J'' \underset{\lambda''}{\overset{\mu}{\leftrightarrows}} J$$

where $\lambda''j = j$, $\lambda''j' = \lambda j'$, $\lambda'j' = j'$, $\mu j = j$ $(j \in J, j' \in J')$. Note that the nerve of Q'' is that of Q with every vertex j blown up into a k-simplex if $\lambda^{-1}j$ has k elements. By means of the injective functions λ' and μ we may identify N' resp. N with subcomplexes of N''. By embedding $|N''|$ into $\mathbb{R}^l$ we obtain inclusions $Z - W'' \subset Z - W'$ and $Z - W'' \subset Z - W$, which induce morphisms $\lambda'_*: E''_* \longrightarrow E'_*$ and $\mu_*: E''_* \longrightarrow E_*$. As both λ'' and μ are refinement functions of coverings they are contiguity inverse to each other. This implies that $\mu^*: H^*(Q''|P_*) \longrightarrow H^*(Q|P_*)$ is isomorphic, so $\mu_*: E''_* \longrightarrow E_*$ is an isomorphism of E^2 spectral sequences. We define $\lambda_* = \lambda'_* \cdot \mu_*^{-1}: E_* \longrightarrow E'_*$. This does not depend on the choice of λ, and it is functorial.

Now form the colimit spectral sequence $\overline{E}_*$ with $\overline{E}^2_* = \check{H}^*(X/G|P_*)$ and $\overline{E}_* \Rightarrow u_*(X,X-A)$.

Let η be the canonical Thom class of $\mathbb{R}^l$. The duality maps

$\gamma_{\eta\times\tau}$ induce a morphism $\overline{E}_* \longrightarrow \overline{E}^*$. The map induced on the 2-level is $\check{H}^*(X/G|\gamma_\tau)$ where γ_τ is considered as a homomorphism of presheaves on X/G. By the first part of the proof, γ_τ is a local isomorphism, hence $\check{H}^*(X/G|\gamma_\tau)$ is isomorphic (cf. [8] 6.8.17). Thus we have an isomorphism of E^2 spectral sequences. Since the map induced in the termination is a filtration of $\overline{\gamma}_\tau$: $u_*(X,X-A) \longrightarrow \overline{u}^*A$, $\overline{\gamma}_\tau$ is isomorphic. This proves Theorem 4.1.

By the same technique we can prove a Thom isomorphism theorem for $\overline{u}^*$ (see [8] 5.7.1o) (which, of course, is true in more general circumstances). In order to construct the spectral sequence for $u^*(C\times X\ ,\ C\times X - \Delta C)$ start from the $(X,X - *)$ bundle that the projection $W \longrightarrow X$ induces from τX.

We outline briefly that orientability of G-manifolds can be described alternatively by fundamental classes.

Let X and t be as above. The cap product turns $t_*(X,X-b)$ into a free cyclic module over t^*b ($b \subset X$ an orbit). Set $t_*^c X = \lim_A t_*A$, taken over all compact $A \subset X$. $\zeta \in t_\xi^c X$ is a <u>fundamental class</u> for X if, for every orbit $b \subset X$, its image under $t_*^c X \longrightarrow t_*(X,X-b)$ is a free generator of $t_*(X,X-b)$ over t^*b.

<u>THEOREM 4.2</u>. <u>There is a one-to-one correspondence between compatible families of Thom classes for</u> $\tau X|A$ ($A \subset X$ <u>compact</u>) <u>and fundamental classes for</u> X.

Proof. Let (τ_A) be such a family of Thom classes. For each compact $A \subset X$ we have $\overline{\gamma}_\tau$: $t_\xi(X,X-A) \approx \overline{t}^o A$. If A is an orbit a look at the diagrams in the proof of Theorem 4.1 shows that γ_τ is an isomorphism of t^*A modules. Hence the family $(\overline{\gamma}_\tau^{-1}(1_A)) \in t_\xi^c X$, where 1_A is the unit in $\overline{t}^o A$, is a fundamental class.

Conversely, suppose $\zeta \in t_\xi^c X$ is a fundamental class. We apply the following version of the Thom isomorphism:

$$\bar{\kappa}_\zeta\colon \bar{t}^*(A\times X\ ,\ A\times X\ -\ \Delta A) \longrightarrow \bar{t}^*A$$

is isomorphic, with κ_ζ sending u to $[u|A\times(X,X-A)]/\zeta$ and the bar indicating approximation of A by its neighbourhoods in X. The proof is similar to that of Theorem 4.1 (the local part has actually been proved there). It follows that $(\bar{\kappa}_\zeta{}^{-1}(1_A))$ is a compatible family of Thom classes.

COROLLARY 4.3. Cap product with a fundamental class is an isomorphism $\bar{u}^*(A,B) \longrightarrow u_*(X-B,X-A)$.

Proof. By naturality and exactness of Mayer - Víetoris sequences it suffices to prove this for linear tubes X. To these the proof of [8] 6.3.11-12 applies after minor modifications.

The results of this paragraph can be reformulated for relative G-manifolds. In particular there is a Poincaré - Lefschetz duality for compact G-manifolds (with boundary). The procedure is quite formal, and we refer to [8] 6.2.18-2o.

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(Received September 7, 1973)