Math. Z. 177, 187–192 (1981)

Mathematische
Zeitschrift

Spherical Fibrations and Manifolds

Cornelia Wissemann-Hartmann
Institut für Mathematik, Ruhr-Universität Bochum, D-4630 Bochum, Federal Republic of Germany

0. Introduction

Let η be a spherical fibration over a Poincaré space. We prove a formula for the Spivak normal fibration of its total space. As a consequence we get: Let the base space be a closed connected manifold. Then the total space has the homotopy type of a manifold if η is stably equivalent to a sphere bundle. The converse is true if η additionally has a cross section.

1. Formulae for Spivak Normal Fibrations of Total Spaces

Let $\eta=(E,p,X)$ be a spherical fibration over a Poincaré space X. Here a Poincaré space is a space of the homotopy type of a finite complex such that $\bigcap\mu\colon H^k(X;\mathbb{Z})\cong H_{n-k}(X;\mathbb{Z}^\tau)$, where $\mu\in H_n(X;\mathbb{Z}^\tau)$ is the fundamental class with respect to some orientation homomorphism $\tau\colon \pi_1(X)\to\mathbb{Z}_2$ (so nonorientable manifolds are included).

We now prove formulae for the Spivak normal fibration of the Poincaré space E and the Poincaré pair (M_p, E) where M_p denotes the mapping cylinder of p.

Theorem 1. *Let $\eta=(E,p,X)$ be a spherical fibration over a Poincaré space X. Then the following holds for the Spivak normal fibrations ν_E of E and ν_X of X:*

$$\nu_E\sim p^*(\nu_X+\eta^{-1}),$$

where '$\sim$' denotes stable equivalence.

Proof. We use the notion of Poincaré embedding as defined in [1] which carries over to the nonorientable case ([10]).

Proposition 1.1. *Let $\eta=(E,p,X)$ be a spherical fibration over a Poincaré space X, and let N be a large integer. Then there exists a Poincaré embedding $i\colon E\to X\times S^N$ with normal fibration $p^*(\eta^{-1})$ such that $E\xrightarrow{i} X\times S^N\to X$ is homotopic to p.*

0025-5874/81/0177/0187/$01.00

From this we get Theorem 1 as follows:

$$\nu_E \sim i^* \nu_{X\times S^N} + p^* \eta^{-1} \sim p^*(j^* \nu_{X\times S^N} + \eta^{-1}) \sim p^*(\nu_X + \eta^{-1}).$$

The first and last equivalence is 1.3 in [1], applied to the Poincaré embedding i: $E \to X\times S^N$ and j: $X \to X \times S^N$, respectively (the normal fibration of j is trivial). The middle equivalence follows from $i \simeq pj$ which is true by 1.1, since N is large ($N > \dim \mathrm{E}$, the CW-dimension). □

Proposition 1.1 is a special case of the following fact (take $\xi = \eta^{-1}$):

Proposition 1.2. *Let ξ and η be spherical fibrations over a Poincaré space X, and let i be the canonical embedding in the commutative diagram*

$$\begin{tikzcd}
E(\eta) \arrow[rr, "i"] \arrow[dr, "p"'] & \arrow[d, phantom, "+"] & E(\eta+\xi) \arrow[dl] \\
& X &
\end{tikzcd}$$

Then i is a Poincaré embedding with normal fibration $p^* \xi$.

Proof. The Whitney sum $\eta + \xi$ is the fibre join of $E(\eta) \to X$ and $E(\xi) \to X$ with the coordinate topology ([8]). Poincaré embeddings are defined with the aid of certain mapping cylinders which carry the quotient topology. The only difficulty in the proof of 1.2 is to compare these topologies. We proceed as follows: If 1.2 is true for ξ and η, then it is true for all fibrations of the same fibre homotopy type. Therefore, using [2] or [3], we may assume that ξ and η are locally trivial (the fibres, of course, are not spheres, but they have the homotopy type of spheres).

The fibre join of locally trivial fibrations $E(\eta) \to X$ and $E(\xi) \to X$ with the quotient topology is denoted by $E(\eta * \xi) \to X$; it is again a locally trivial fibration. The identity $E(\eta * \xi) \to E(\eta + \xi)$ is a continuous fibre map which is a homotopy equivalence on the fibres (as they have the homotopy type of compact spaces, and the quotient topology coincides with the coordinate topology on the join of compact spaces). Therefore, $\eta * \xi$ is fibre homotopically equivalent to $\eta + \xi$, and we may replace the latter by the first in 1.2.

Now $E(\eta * \xi) = T$ is just the double mapping cylinder of the canonical diagram

$$E(\eta) \leftarrow E(p^* \xi) \to E(\xi),$$

and the inclusion i: $E(\eta) \to E(\eta * \xi)$ corresponds to the canonical embedding of $E(\eta)$ in T. Then, by definition, i is a Poincaré embedding with normal fibration $p^* \xi$. □

From Theorem 1, we get the formula for the Spivak normal fibration of (M_p, E):

Corollary 2. *Let $\eta = (E, p, X)$ be a spherical fibration over a Poincaré space X and let r: $M_p \to X$ be the retraction of the mapping cylinder to the base space. It holds:*

$$\nu_{M_p, E} \sim r^*(\nu_X + \eta^{-1}).$$

Proof. Having once proven the formula in Theorem 1, Corollary 2 follows by standard considerations on doubles as

$$E(\eta+\varepsilon^1)\simeq M_p \bigcup_E M_p, \qquad (\simeq \text{means homotop}).$$

Knowing $\nu_{E(\eta+\varepsilon^1)}$ by Theorem 1, one gets the formula for (M_p, E) by restriction. $\square$

Remark. If η is not a fibration but a bundle over a manifold, similar formulae follow from geometrical considerations of tangent spaces and can be found in literature.

Corollary 2 is proved by Spivak in his proof of uniqueness of Spivak normal fibrations (see [8], 5.4) but - compared with our situation - under two rigorous restrictions:

firstly he requires the fibration to be in the stable range ($d>n+1$), which e.g. does not apply to the Hilton-Roitberg examples, which will be investigated in Sect. 3, secondly his class of base spaces is the class of spaces whose universal covering spaces satisfy Poincaré-duality; that excludes spaces like $S^p \vee A$, where A is an acyclic finite CW-complex with finite fundamental group which is included in our class of base spaces.

2. Smoothing the Total Space of a Spherical Fibration

According to a result of Hilton and Roitberg (see [4]), there exist spherical fibrations with a sphere as base space which are not fibre equivalent to bundles but whose total spaces have the homotopy type of manifolds. (All manifolds are closed here.)

Work of Smith (see [7]) and Stöcker (see [9]) shows that such phenomena can not occur in the stable range: the total space has the homotopy type of a manifold iff the fibration is fibre equivalent to a bundle. We now can prove:

Theorem 3. *Let $\eta=(E,p,M)$ be a spherical fibration over a CAT manifold. If η is stably equivalent to a CAT sphere bundle, then E has the homotopy type of a CAT manifold, where CAT is one of the categories DIFF, PL or TOP.*

Theorem 4. *Let $\eta=(E,p,M)$ be as in Theorem 3 but with a cross section. Then the converse holds: If E is a homotopy CAT manifold, then η is stably equivalent to a CAT sphere bundle.*

In [7] and [9] all fibrations have a cross section as they are in the stable range. So 3 and 4 generalize the results obtained there to any manifold as base space and to nonorientable fibrations.

Proof of Theorem 4. If E is a homotopy manifold, its Spivak normal fibration is stably a sphere bundle. So with Theorem 1: $\nu_E \sim p^*(\nu_M+\eta^{-1}) \sim$ sphere bundle. Applying the section s and making use of the group structure gives: $\eta \sim s^*(\nu_E^{-1}) + \nu_M \sim$ sphere bundle. $\square$

For the proof of Theorem 3 we need Corollary 2 and a surgery result, namely the $\pi-\pi$ theorem of C.T.C. Wall.

Proof of Theorem 3. By the statement of the $\pi-\pi$ theorem, (M_p, E) (which is even a Poincaré space in the sense of Wall, as the base space is a manifold) has the homotopy type of a manifold with boundary iff $v_{M_p,E}$ is stably a sphere bundle, provided $\pi_1(M_p) \cong \pi_1(E)$ and the dimension of (M_p, E) is not less than 6.

All cases where these two conditions are not fulfilled follow by elementary considerations, as we proceed under the assumption that these conditions are fulfilled.

If η is stably equivalent to a sphere bundle, it follows from Corollary 2 that $v_{M_p,E}$ is it too. So (M_p, E) has the homotopy type of a manifold with boundary, which means that E is a homotopy manifold.

So Theorem 3 is proved. □

Remarks. (1) Up to now we didn't succeed in showing that the existence of the cross section is more than a technical assumption. Suppose Theorem 4 becomes wrong without cross section, then there exists a spherical fibration $\eta=(E, p, M)$ where E and M are homotopy manifolds but $E(\eta+\varepsilon^1)$, ε^1 trivial fibration with fibre S^0, is a Poincaré space but no homotopy manifold.

(2) Simultaneously we proved under the same conditions:

Lemma. *E is a homotopy manifold* iff *v_E is stably equivalent to a sphere bundle.*

(3) An analogous result might be valid for fibrations with any fibre having the homotopy type of a manifold. Possibly the lemma above could hold in general.

(4) An immediate consequence from the formulae and the $\pi-\pi$ theorem is the following:

Let E be the total space of the Spivak normal fibration v_X of a Poincaré space X. Then E has the homotopy type of a smooth manifold.

3. Spherical Fibrations over Spheres

Reformulating Theorem 3 and 4 for spherical fibrations over spheres provides a method for constructing new examples of manifolds from spherical fibrations. Given a spherical fibration with cross section with base space S^p and fibre S^q where $p, q \geqq 2$, the total space E is known to have the homotopy type of the following cell complex: $E \simeq S^p \vee S^q \bigcup_{[i_p, i_q]+i_{q*}[f]} e^{p+q}$, where $[i_p, i_q]$ denotes the Whitehead product of the inclusions i_k: $S^k \to S^k \vee S^j$ and $[f] \in \pi_{p+q-1}(S^p)$ (see [6], Sect. 3). As f determines the corresponding fibration uniquely up to fibre homotopy type, we write η_f and E_f for this fibration and its total space.

From the theorems above, we get the following corollaries (compare [7], [9]; we now restrict to the differentiable case although the results remain valid in the topological and PL-case):

Corollary 3.1. *E_f has the homotopy type of a differentiable manifold* iff $\{f\}\in\operatorname{Im}(J: \pi_{p-1}(O)\to\pi_{p-1})$, *where $\{f\}$ denotes the stable homotopy class of f, J the stable Hopf-Whitehead homomorphism and O the stable orthogonal group.* □

We are mainly interested in cases where η_f itself is not fibre equivalent, but is stably equivalent to a sphere bundle, so that E_f is a new manifold (in the sense that it is not the total space of a sphere bundle). In the notation of [9] let $C[f]:=\{[f]+[\rho, \mathrm{id}_{S^q}]\ \rho\in\pi_p(S^q)\}$ and $W_f:=C[f]\cup C[-((-\mathrm{id}_{S^q})\circ f)]$. Then

Corollary 3.2. *Given $[f]\in\pi_{p+q-1}(S^p)$ such that $\{f\}\in\operatorname{Im} J$ and $W_f\cap J_q=\phi$, where J_q: $\pi_{p-1}(O(q))\to\pi_{p+q-1}(S^q)$. Then η_f is not fibre equivalent to a sphere bundle but E_f is a homotopy manifold.* □

As in the metastable range fibre equivalence is the same as stable equivalence (see [5], Theorem 3.2), new manifolds only occur in the range $p>2q-3$. Now we are able to clarify the Hilton-Roitberg examples which are

Examples with fibre S^2 over S^p: As $\pi_{p-1}(O(2))=0$ for $p>2$, every nonzero $[f]\in\pi_{p+1}(S^2)$ determines a fibration which is not fibre equivalent to a bundle. So any nonzero $[f]$ which under suspension is mapped into $\operatorname{Im} J$, provides an example for a fibration as in 3.2. Lemma 2.5 in [4], which states that for $p>2$ E_f is homotopy equivalent to E_g iff $[f]=\pm[g]$, completes the determination of new manifolds.

So e.g. when $p=4$, the generator $[f]\in\pi_5(S^2)\cong\mathbb{Z}_2$ is mapped nontrivially into the image $\mathbb{Z}_8\subset\mathbb{Z}_{24}$ of J. That gives: $S^2\vee S^4\bigcup_{[i_2,i_4]+i_{2*}[f]} e^6$ is a homotopy manifold.

When $p=5$, similar observations show: there are 6 new seven-dimensional manifolds, one for each pair $([f],[-f])$, where $0\neq[f]\in\pi_6(S^2)\cong\mathbb{Z}_{12}$. Much more examples may be constructed this way.

Thanks to R. Stöcker.

Literature

1. Browder, W.: Poincaré spaces, their normal fibration and surgery. Inventiones math. **17**, 191–202 (1972)
2. Casson, A., Gottlieb, D.H.: Fibrations with compact fibres. Amer. J. Math. **99**, 159–189 (1977)
3. Fadell, E.: The equivalence of fiber spaces and bundles. Bull. Amer. Math. Soc. **66**, 50–53 (1960)
4. Hilton, P., Roitberg, J.: Note on quasifibrations and fibre bundles. Illinois J. Math. **15**, 1–8 (1971)
5. James, I.M.: On the iterated suspension. Quart. J. Math. Oxford Ser. **5**, 1–10 (1954)
6. James, I.M., Whitehead, J.H.C.: The homotopy theory of sphere bundles over spheres I. Proc. London math. Soc., **III**, Ser. 4, 198–218 (1954)
7. Smith, L.: Manifolds with few cells and the stable homotopy of spheres. Proc. Amer. math. Soc. **31**, 279–284 (1972)
8. Spivak, M.: Spaces satisfying Poincaré duality. Topology **6**, 77–101 (1967)
9. Stöcker, R.: Note on quasifibrations and manifolds. Proc. Amer. math. Soc. **43**, 219–225 (1974)
10. Wall, C.T.C.: Surgery on compact manifolds. London-New York: Academic Press 1970

Received July 4, received in final form September 26, 1980

Added in Proof

Concerning Remark (1) we lately found an example that the cross section is really a necessary condition (see: Sutherland, W.A.: Homotopy-smooth sphere fibrings. Bol. Soc. Mat. Mexicana **11**, 73–79 (1966); our results contain parts of Sutherland's).

His Example 3.3 is a special case of the following: Let $2<p\equiv 2 \bmod 4$ and let the stable stem π_{p-1} contain an element β of odd order (this implies $\beta\notin \mathrm{Im}J$), take $\beta'\in\pi_{2p-2}(S^{p-1})$ such that $S^2\beta'=\beta$. Set $\alpha=[i_p, i_p]+S\beta'$.

α uniquely determines a spherical fibration η with fiber S^{p-1} over S^p (η has no cross section as the Hopf invariant of $\alpha\neq 0$, and is not stably equivalent to a sphere bundle as $\beta\notin \mathrm{Im}\,J$).

So we get - for every p as above - an example we searched for in Remark (1).