GENERALIZED ARF INVARIANTS AND REDUCED POWER OPERATIONS IN CYCLIC HOMOLOGY

GENERALIZED ARF INVARIANTS AND REDUCED POWER OPERATIONS IN CYCLIC HOMOLOGY

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Contents

Introduction.

The subject of this thesis belongs to the branch of mathematics named algebraic K-theory of forms, also called algebraic L-theory. The main objective of the theory is to classify quadratic and hermitian forms over rings endowed with an (anti-) involution up to some notion of similarity defined on the forms.

The L-groups were originally designed in a geometrical context by C.T.C. Wall in [22] as 'surgery obstruction groups': In order to classify compact manifolds up to (simple) homotopy equivalence, one performs 'surgery', a certain process of cutting, replacing and glueing. Whether this surgery succeeds depends on an obstruction in some L-group of the group ring of the fundamental group of the manifold. Conversely, every element of the L-group represents a surgery problem. Thus the geometric question is translated into an algebraic one. We refer to §9 and particularly to corollary 9.4.1 of *loc. cit.* for the details.

In his paper [24] Wall defines groups $L_i^s(R)$ and $L_i^h(R)$ for arbitrary rings R with anti-structure in a purely algebraic way. These L-groups, the ones we will work with, are in essence Grothendieck- and Whitehead groups of categories of (non-singular) quadratic modules. We refer to chapter I for the actual definition. In view of the preceding it is clear that one is particularly interested in the case where R is a group ring. Very little is known about L-groups of infinite groups. The computations in the literature are either concerned with L-groups of finite groups (see e.g. [15, 10]) or, when infinite groups are considered, the results obtained yield but very limited information (see e.g. [8]). Our main aim is to compute L-groups by constructing sufficiently good invariants. Moreover, it is important to have such invariants at our disposal to evaluate a particular element in an L-group.

To give a detailed description of our achievements in this direction, we need the following facts.
Let R be a ring equipped with an anti-structure.

1. The anti-structure on R induces involutions on the algebraic K-groups $K_i(R)$, which in turn give rise to the Tate cohomology groups $H^*(K_i(R))$ $(i = 1, 2)$.

2. ([24]) There is an invariant $L_i^h(R) \longrightarrow H^i(K_1(R))$, called discriminant.

3. ([27]) If I is a two-sided ideal of R which is invariant under the anti-structure and R is complete with respect to the I-adic topology, then $L_i^h(R)$ is isomorphic to $L_i^h(R/I)$ and there is an exact sequence

$$\cdots \to L_i^s(R) \to L_i^s(R/I) \to H^i(V) \to L_{i-1}^s(R) \to L_{i-1}^s(R/I) \to \cdots .$$

   Here V denotes the kernel of the map $K_1(R) \to K_1(R/I)$.

4. ([9]) There is a homomorphism $L_i^s(R) \longrightarrow H^{i+1}(K_2(R))$, called Hasse-Witt invariant.

The basic idea in our construction of new invariants is to extend the anti-structure on the ring R in a rather exotic way to the ring of formal power series in one variable $R[[T]]$. By combining 1 to 4 in this situation, we construct invariants for the L-groups $L_i^s(R)$ and $L_i^h(R)$ taking values in the Tate cohomology group $H^i(K_1(R[[T]]))$ *viz.*

$$\omega_1^s\colon L_i^s(R) \to H^i(K_1(R[[T]])) \qquad \omega_1^h\colon L_i^h(R) \to H^i(K_1(R[[T]])).$$

Further we obtain a secondary invariant ω_2 which lives on the kernel of ω_1^s and takes values in some quotient of the cohomology group $H^{i+1}(K_2(R[[T]]))$. The invariant ω_1^h is a generalization of the discriminant homomorphism. To be precise, the discriminant homomorphism of 2 coincides with the composition of ω_1^h and the natural projection $H^i(K_1(R[[T]])) \to H^i(K_1(R))$.

In some of the cases where the results of [4, 21, 18, 14, 6] on the computation of K-groups can be applied, we are able to compute the value groups of our invariants. For the value group of ω_1^h we do this for commutative rings and for group rings. We compute the value group of ω_2 by assuming R to carry a structure of partial λ-ring, by using the techniques of [6] to compute K-groups of λ-rings. The notion of partial λ-ring occurs more or less implicit in [13, 12]. In the cases we are interested in, this notion is much weaker than the notion of λ-ring.

We confine our investigations to the 'Arf-part' of $L_{2i}^*(R)$. Roughly speaking, the 'Arf-part' of $L_{2i}^*(R)$ is the subgroup consisting of differences of similarity classes of quadratic forms whose underlying bilinear forms are standard forms. We refer to section 2 of chapter I for a precise formulation of the definition. Whereas the discriminant and the Hasse-Witt invariant merely reveal information on the underlying bilinear form of a quadratic form, the new invariants ω_1^s, ω_1^h and ω_2 detect the structure of the Arf-groups.

It turns out that ω_1^h is not unfamiliar to us in the sense that its restriction to the Arf-group is essentially the Arf invariant constructed by F. Clauwens in his paper [5]. Clauwens proved that this Arf invariant is injective for group rings of finite groups. We show that ω_1^h is an isomorphism whenever R is the group ring of a finite group. The restriction of ω_2 to the Arf-part of the kernel of ω_1^s yields what we call the secondary Arf invariant. As an example we use these invariants to compute an L-group of the polynomial ring in two variables over the integers. Further we determine the Arf-group of the group presented by $\langle X, Y, S \mid S^2 = (XS)^2 = (YS)^2 = 1, \quad XY = YX \rangle$. This example is typical in so far as it demonstrates how a suitable representation of the group under consideration enables us to compute the Arf-group.

The fact that an instance of cyclic homology [17], endowed with some kind of squaring operation, emerges from the computations on the value group of ω_2, gives the impulse to search for generalized Arf invariants, taking values in cyclic homology theory. We conceived the idea that some quotient of the quaternionic homology groups J.-L. Loday [16] came up with, might serve as suitable value groups for new Arf invariants. In order to find legitimate value groups for these nascent invariants, some effort is needed to construct appropriate operations for

the various homology theories. The theory concerned with what we call 'reduced power operations' on (low-dimensional) Hochschild- cyclic- and quaternionic homology groups, is rather interesting in its own right.
Ultimately we obtain an Arf invariant called Υ which takes values in a quotient of the quaternionic homology group HQ_1. It is a generalization of all Arf invariants mentioned so far: when we specialize to the case of commutative rings, we retain both the ordinary and the secondary Arf invariant at the same time. Based upon the techniques exposed in [16] we can compute the group HQ_1 for group rings. This enables us to comprehend the value group of Υ to a certain extent. This group turns out to be a rather complicated direct limit of groups, nevertheless it is quite manageable in concrete situations and we are able to prove that Υ is injective for all groups that possess an infinite cyclic subgroup of finite index. Thus we have augmented the class of groups for which we are able to determine the Arf-group. It is not yet clear how good this invariant really is: on one hand there are examples of groups which do not belong to the above-mentioned class, for which Υ is injective, while on the other hand there are examples of groups for which we cannot determine whether Υ is injective.

We touch upon the contents of each chapter now. The first section of chapter I is the reflection of an attempt to familiarize the reader with some of the notations, definitions and results from algebraic K- and L-theory, needed to define and study the Arf-groups. In the second section we define the Arf-groups and the Arf invariant. Then we give a presentation of the Arf-groups due to F. Clauwens [5]. Further we mention the fact that the Arf invariant is injective for finite groups. We conclude the first chapter with a few examples of Arf-groups of (infinite) groups.

In the second section of chapter II we construct the new invariants ω_1^s, ω_1^h and ω_2 by extending a given anti-structure to the ring of formal power series $R[[T]]$ as described above. In section 3 we compute the value group $H^0(K_1(R[[T]]))$ of ω_1^s and ω_1^h for commutative rings, and group rings. Then we show that the restriction of ω_1^h to the Arf-groups coincides with the Arf invariant in these cases. Further we compute the cohomology group $H^1(K_2(R[[T]]))$ for rings R that possess a structure of partial λ-ring, in section 4. The final section of this chapter is devoted to the example of the polynomial ring in two variables over the integers, $\mathbf{Z}[X,Y]$ and the example of the group we mentioned earlier.

As we mentioned before, the surfacing of operations on cyclic homology groups in the computations on ω_2, is the motivation for studying operations on Hochschild, cyclic and quaternionic homology in chapter III. We give the definitions of the various homologies and mention the most important examples in the first section. In the next section we construct operations on these homologies needed to produce well-defined Arf invariants. Section 3 treats Morita invariance as a preparation for the definition of the invariant Υ in the final section.

In chapter IV we apply all of the preceding to the case where R is a group ring. In the first section we compute the quaternionic homology group HQ_1 of group algebras. We use this in the second section to get an idea of what the value group of Υ looks like in this case. In section 3 we characterize groups having two ends (or equivalently groups having an infinite cyclic subgroup of finite index), in a for our purposes convenient way, as pull-backs of finite groups and infinite cyclic or dihedral groups. In section 4 we show that Υ is injective for groups with two ends.

Most of the results in chapter II are already contained in [31].

List of some frequently used notations

Chapter I

Algebraic K-theory of quadratic forms and the Arf invariant.

1 Collecting the relevant K- and L-theory.

The material in this first section related to quadratic forms and L-groups has primarily been extracted from [9] and [24], while the facts concerning the algebraic K-groups K_0, K_1 and K_2 have been taken from [3] and [19].

First of all we need the notion of ring with anti-structure. The word ring will always mean associative ring with identity, written 1.

1.1 Definition. An anti-automorphism α of a ring R is a ring isomorphism $\alpha: R \to R^\circ$, where R° denotes the opposite ring of R. A ring with anti-structure (R, α, u), consists of a ring R, equipped with an anti-automorphism α of R and a unit $u \in R$ such that $\alpha(u)u = 1$ and $\alpha^2(r) = uru^{-1}$ for every $r \in R$.

Remark. Let (R, α, u) be a ring with anti-structure.

- If u is central in R, then α is an anti-involution, i.e. an anti-automorphism of order at most 2.
- If α is the identity, then R must be commutative and $u^2 = 1$. The converse is not necessarily true.

We give a few examples of rings with anti-structure including the most important ones.

- R commutative, α the identity, $u = \pm 1$.

Let (R, α, u) be a ring with anti-structure. Then the anti-structure on R can be extended to

- the group algebra $R[G]$, for every group G, by the formula
$$\sum r_i g_i \mapsto \sum \alpha(r_i) g_i^{-1}.$$
- the ring $M_n(R)$ of $(n \times n)$-matrices over R by
$$A \mapsto A^\alpha,$$
where $(A^\alpha)_{ij} := \alpha(A_{ji})$. Thus A^α is the conjugate transpose of A.

· the polynomial ring in one variable $R[T]$ by

$$\sum r_i T^i \mapsto \sum \alpha(r_i)(1-T)^i.$$

· the polynomial ring in one variable $R[T]$ by

$$\sum r_i T^i \mapsto \sum \alpha(r_i)(-T)^i.$$

1.2 Definition. Denote by $\mathcal{P}(R)$ the category of finitely generated projective right R-modules and R-homomorphisms. The anti-automorphism α enables us to define a contravariant functor $D_\alpha\colon \mathcal{P}(R) \to \mathcal{P}(R)$ as follows.
For every $P \in \text{Obj } \mathcal{P}(R)$ we define
$D_\alpha P := \text{Hom}_R(P, R)$ equipped with a right R-module structure by
$(gr)(p) := \alpha^{-1}(r)g(p)$, for every $g \in \text{Hom}_R(P, R)$, $p \in P$ and $r \in R$.
For every $f \in \text{Hom}_R(P, Q)$ we define
$D_\alpha f \in \text{Hom}_R(D_\alpha Q, D_\alpha P)$ by $(D_\alpha f)(h) := h \circ f$ for all $h \in D_\alpha Q$.

1.3. We make the following observations.

1. If M is free with basis $e_1, \ldots, e_m$, then $D_\alpha M$ is free with basis $e_1^*, \ldots, e_m^*$. Here $e_i^* \in D_\alpha M$ is determined by $e_i^*(e_j) = \delta_{ij}$ (Kronecker delta). One calls $e_1^*, \ldots, e_m^*$ the basis dual to $e_1, \ldots, e_m$.

2. If M is free with basis $e_1, \ldots, e_m$, N is free with basis $f_1, \ldots, f_n$ and $\phi \in \text{Hom}_R(M, N)$ has $(n \times m)$-matrix A with respect to these bases, then A^α is the matrix of $D_\alpha \phi$ with respect to the dual bases. Just as in the case of square matrices $(A^\alpha)_{ij} = \alpha(A_{ji})$ Note that $(AB)^\alpha = B^\alpha A^\alpha$.

3. Suppose $e_1, \ldots, e_m$ and $f_1, \ldots, f_m$ are both bases of M and X is the base-change matrix. If A is the matrix of $\phi \in \text{Hom}_R(M, D_\alpha M)$ with respect to $e_1, \ldots, e_m$ and its dual, then $X^\alpha A X$ is the matrix of ϕ with respect to $f_1, \ldots, f_m$ and its dual.

1.4 Lemma. [9, section 1] The map $\eta_{\alpha,u}\colon 1_{\mathcal{P}(R)} \to D_\alpha^2$ defined by
$(\eta_{\alpha,u}P)(p)(g) := u^{-1}\alpha(g(p))$ for every $P \in \text{Obj } \mathcal{P}(R)$, $p \in P$ and $g \in D_\alpha P$ is a natural equivalence.

Proof. Although the proof is rather straightforward we give some of the arguments because they might be instructive.

· $(\eta_{\alpha,u}P)(p) \in D_\alpha^2 P$: for every $r \in R$ we have

$$\begin{aligned}(\eta_{\alpha,u}P)(p)(gr) &= u^{-1}\alpha(gr(p)) \\ &= u^{-1}\alpha(\alpha^{-1}(r)g(p)) \\ &= u^{-1}\alpha(g(p))r \\ &= (\eta_{\alpha,u}P)(p)(g)r\end{aligned}$$

· $\eta_{\alpha,u}P \in \mathrm{Hom}_R(P, D^2_\alpha P)$: for every $r \in R$ we have

$$\begin{array}{rcl} (\eta_{\alpha,u}P)(pr)(g) & = & u^{-1}\alpha(g(pr)) \\ & = & u^{-1}\alpha(g(p)r) \\ & = & u^{-1}\alpha(r)\alpha(g(p)) \\ & = & \alpha^{-1}(r)u^{-1}\alpha(g(p)) \\ & = & \alpha^{-1}(r)(\eta_{\alpha,u}P)(p)(g) \\ & = & ((\eta_{\alpha,u}P)(p)r)(g) \end{array}$$

· $\eta_{\alpha,u}$ is natural: for every $\phi \in \mathrm{Hom}_R(P, Q)$ and $h \in D_\alpha Q$ the diagram

$$\begin{array}{ccc} P & \xrightarrow{\eta_{\alpha,u}P} & D^2_\alpha P \\ \downarrow f & & \downarrow D^2_\alpha f \\ Q & \xrightarrow{\eta_{\alpha,u}Q} & D^2_\alpha Q \end{array}$$

commutes since

$$\begin{array}{rcl} (D^2_\alpha f)((\eta_{\alpha,u}P)(p))(h) & = & (\eta_{\alpha,u}P)(p)(D_\alpha f(h)) \\ & = & u^{-1}\alpha(h(f(p))) \\ & = & (\eta_{\alpha,u}Q)(f(p))(h) \end{array}$$

· $\eta_{\alpha,u}P$ is an isomorphism: there exists a canonical isomorphism $D_\alpha(P \oplus Q) \cong D_\alpha P \oplus D_\alpha Q$, so we may assume that P is free with basis $e_1, \ldots, e_m$ say. From the definition of η we deduce $(\eta_{\alpha,u}P)(e_i) = e_i^{**}u$.

The rest is clear. •

Corollary. If M is free with basis $e_1, \ldots, e_m$, then $\eta_{\alpha,u}(M): M \to D^2_\alpha M$ has matrix uI_m with respect to $e_1, \ldots, e_m$ and $e_1^{**}, \ldots, e_m^{**}$.

Notation. From now on we write P^α instead of $D_\alpha P$ and f^α instead of $D_\alpha f$.

1.5 Proposition. *The map $T_{\alpha,u} = T_{\alpha,u}(P, Q): \mathrm{Hom}_R(Q, P^\alpha) \to \mathrm{Hom}_R(P, Q^\alpha)$ defined by*

$$T_{\alpha,u}(f) := f^\alpha \circ \eta_{\alpha,u}P$$

is a natural isomorphism and $T_{\alpha,u}(P, Q) \circ T_{\alpha,u}(Q, P) = 1_{\mathrm{Hom}_R(P,Q^\alpha)}$. In other words $T_{\alpha,u}$ defines a self-adjunction of the functor D_α.

Proof. As in [9, Proposition 1.2] •

1.6 Lemma. If M is free with basis $e_1, \ldots, e_m$ and $\phi \in \mathrm{Hom}_R(M, M^\alpha)$ has matrix A with respect to this basis and its dual, then $T_{\alpha,u}(\phi)$ has matrix $A^\alpha u$ with respect to the same bases.

Proof. Immediate by the corollary to definition 1.4 and the second observation of 1.3. •

We are now in a position to introduce the notion of quadratic module.

1.7 Definition. In the case that $P = Q$ in proposition 1.5 we obtain a group endomorphism $T_{\alpha,u}\colon \mathrm{Hom}_R(P, P^\alpha) \longrightarrow \mathrm{Hom}_R(P, P^\alpha)$ satisfying $T_{\alpha,u}^2 = 1$.
A quadratic, to be precise (α, u)-quadratic, R-module is a pair $(P, [\phi])$ consisting of a module $P \in \mathrm{Obj}\ \mathcal{P}(R)$ and the class $[\phi] \in \mathrm{Coker}(1 - T_{\alpha,u})$ of an element $\phi \in \mathrm{Hom}_R(P, P^\alpha)$.
The quadratic module $(P, [\phi])$ is called non-singular if the image $b_{[\phi]}$ of $[\phi]$ under the 'bilinearization-map' $b\colon \mathrm{Coker}(1 - T_{\alpha,u}) \to \mathrm{Ker}(1 - T_{\alpha,u})$, induced by the homomorphism $1+T_{\alpha,u}\colon \mathrm{Hom}_R(P, P^\alpha) \to \mathrm{Hom}_R(P, P^\alpha)$, is an isomorphism.

Remark.

· If 2 is invertible in R, then b is an isomorphism, with inverse determined by $\phi \mapsto [\frac{1}{2}\phi]$. Thus there is a 1-1 correspondence between non-singular quadratic forms and symmetric non-singular bilinear forms, i.e. elements of $\mathrm{Iso}(P, P^\alpha) \cap \mathrm{Ker}(1 - T_{\alpha,u})$.

· In the literature one denotes by $\mathrm{Sesq}(P)$ the additive group of sesquilinear forms on P i.e. biadditive maps $\phi\colon P \times P \longrightarrow R$ satisfying $\phi(p_1 r_1, p_2 r_2) = \alpha^{-1}(r_1)\phi(p_1, p_2)r_2$ for every $p_1, p_2 \in P$ and $r_1, r_2 \in R$. In the case that R is commutative and α is the identity, $\mathrm{Sesq}(P)$ is the group of R-bilinear maps. There is a bijective correspondence $\mathrm{Sesq}(P) \longleftrightarrow \mathrm{Hom}_R(P, P^\alpha)$ by associating to an element $\phi \in \mathrm{Sesq}(P)$ the map $f \in \mathrm{Hom}_R(P, P^\alpha)$ defined by $f(p_1)(p_2) := \phi(p_1, p_2)$ for every $p_1, p_2 \in P$.

We proceed to define the various categories of quadratic modules. Along the way we shall briefly recall the relevant definitions and facts from algebraic K-theory.
The following categories and functors will all be 'categories with product' as in [3, Ch.VII, §1].

1.8 Definition.

- Let $Q(R, \alpha, u)$ denote the category with
  objects: non-singular quadratic (right) R-modules,
  morphisms: $(P, [\phi]) \to (Q, [\psi])$ are the isomorphisms $f\colon P \to Q$ satisfying $[f^\alpha \psi f] = [\phi]$,
  product:

$$(P, [\phi]) \perp (Q, [\psi]) := (P \oplus Q, [(\pi_P)^\alpha \phi \pi_P + (\pi_Q)^\alpha \psi \pi_Q]),$$

  where $\pi_P\colon P \oplus Q \to P$ and $\pi_Q\colon P \oplus Q \to Q$ are the natural projections.

- Let $\overline{\mathcal{P}(R)}$ denote the category with
  objects: objects of $\mathcal{P}(R)$,
  morphisms: isomorphisms of $\mathcal{P}(R)$,
  product: product of $\mathcal{P}(R)$.

- Now on one hand we have the forgetful functor $F: Q(R,\alpha,u) \to \overline{\mathcal{P}(R)}$, which is of course product preserving. While on the other hand there is the so-called hyperbolic functor $H: \overline{\mathcal{P}(R)} \to Q(R,\alpha,u)$ defined by

$$H(P) := (P \oplus P^\alpha, [\upsilon]), \quad H(f) := f \oplus (f^\alpha)^{-1},$$

  where $\upsilon: P \oplus P^\alpha \to (P \oplus P^\alpha)^\alpha$ is determined by $(\upsilon(p,g))(p',g') := g(p')$. H is product preserving as well. The objects $H(P)$ are called hyperbolic.

- A product preserving functor $G: \mathcal{C} \to \mathcal{D}$ is called cofinal if for each object A of $\mathcal{D}$ there exist objects B of $\mathcal{D}$ and C of $\mathcal{C}$, such that $A \perp B \cong G(C)$. A subcategory $\mathcal{C}$ of a category $\mathcal{D}$ is called cofinal if the inclusion functor is cofinal.

1.9 Lemma. [23, theorem 3]. For every $(P,[\phi]) \in \mathrm{Obj}\ Q(R,\alpha,u)$ there exists an isomorphism $(P,[\phi]) \perp (P,-[\phi]) \cong H(P)$. Consequently H is cofinal.

Proof. It is not hard to verify that the morphism $\xi: P \oplus P \to P \oplus P^\alpha$ defined by $\xi(p_1,p_2) := (p_1 - b_{[\phi]}^{-1}(\phi(p_1-p_2)), b_{[\phi]}(p_1-p_2))$ does the job. We refer to *loc. cit.* for a detailed proof. •

1.10 Definition. As usual $\mathrm{GL}(R)$ denotes the direct limit of the general linear groups $\mathrm{GL}_n(R)$ consisting of invertible $n \times n$-matrices over R, with respect to the embeddings $\mathrm{GL}_n(R) \hookrightarrow \mathrm{GL}_{n+1}(R)$ defined by

$$(A) \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for all} \quad (A) \in \mathrm{GL}_n(R).$$

A matrix is called elementary if it differs from the identity matrix at no more than one off-diagonal position. Denote by $E_n(R)$ resp. $E(R)$ the subgroup of $\mathrm{GL}_n(R)$ resp. $\mathrm{GL}(R)$ generated by all elementary matrices. According to the Whitehead lemma [19, §3] $E(R)$ coincides with the commutator subgroup of $\mathrm{GL}(R)$. By definition $K_1R := \mathrm{GL}(R)/E(R)$. We use the additive notation in the abelian group K_1R.

There is a general procedure for defining the Whitehead group $K_1\mathcal{C}$ of a category $\mathcal{C}$ with product, but we do not need it for our purposes. It follows from lemma 1.9 that the $H(R^n)$ are cofinal in $Q(R,\alpha,u)$. According to [3, Ch.VII, §2.3] we may just as well define $K_1Q(R,\alpha,u)$ as follows under these circumstances.

1.11 Definition. $K_1Q(R,\alpha,u)$ is the commutator quotient of the direct limit

$$\varinjlim \mathrm{Aut}(H(R^n))$$

where the limit is taken with respect to the canonical embeddings
$\mathrm{Aut}(H(R^n)) \longrightarrow \mathrm{Aut}(H(R^n) \perp H(R)) \cong \mathrm{Aut}(H(R^{n+1}))$.

1.12 Remark. Analogously $K_1(R)$ is the Whitehead group of both $\mathcal{P}(R)$ and $\overline{\mathcal{P}(R)}$. Since the free modules R^n are cofinal in both categories, the groups $K_1(\mathcal{P}(R))$ and $K_1(\overline{\mathcal{P}(R)})$ both coincide with the commutator quotient of the direct limit

$$\varinjlim \operatorname{Aut}(R^n)$$

where the limit is taken with respect to the canonical embeddings $\operatorname{Aut}(R^n) \longrightarrow \operatorname{Aut}((R^n) \perp (R)) \cong \operatorname{Aut}(R^{n+1})$. Upon choosing a basis for R^n we may identify $\operatorname{Aut}(R^n)$ with $\mathrm{GL}_n(R)$ and consequently $K_1\mathcal{P}(R) \cong K_1(\overline{\mathcal{P}(R)}) \cong K_1 R$.

1.13. Let us return to $Q(R, \alpha, u)$. We choose a basis for R^n and the dual basis for $(R^n)^\alpha$. Since the matrix of υ with respect to these bases, takes the form

$$\Sigma_{2n} := \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix}$$

we may identify $\operatorname{Aut}(H(R^n))$ with the subgroup of $\mathrm{GL}_{2n}(R)$ consisting of all matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{2n}(R) \qquad \text{(here } A, B, C \text{ and } D \text{ are } n \times n\text{-matrices)}$$

satisfying

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^\alpha \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} - \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix} = X - X^\alpha u$$

for some $(2n \times 2n)$-matrix X. This subgroup of $\mathrm{GL}_{2n}(R)$ is called the general quadratic group and is denoted by $\mathrm{GQ}_{2n}(R)$. As a consequence $K_1(Q(R, \alpha, u))$ can be identified with the commutator quotient of the group

$$\mathrm{GQ}(R) := \varinjlim \mathrm{GQ}_{2n}(R),$$

where the limit is taken with respect to the embeddings

$$\mathrm{GQ}_{2n}(R) \hookrightarrow \mathrm{GQ}_{2(n+1)}(R) \text{ defined by } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & B & 0 \\ 0 & 1 & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

1.14 Definition. For every $n \in N$ we define $t_{\alpha,u}\colon \mathrm{GL}_{2n}(R) \to \mathrm{GL}_{2n}(R)$ by

$$t_{\alpha,u}(X) = U_{2n}^{-1} X^\alpha U_{2n} \text{ for every } X \in \mathrm{GL}_{2n}(R), \text{ here } U_{2n} := \begin{pmatrix} 0 & I_n \\ uI_n & 0 \end{pmatrix}.$$

Explicitly: for every

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{2n}(R)$$

we have

$$t_{\alpha,u}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} D^{\alpha^{-1}} & u^{-1}B^{\alpha} \\ C^{\alpha}u & A^{\alpha} \end{pmatrix}.$$

Note that $D^{\alpha^{-1}} = u^{-1}D^{\alpha}u$ since $\alpha^2(r) = uru^{-1}$ for every $r \in R$. Furthermore, $t_{\alpha,u}$ is an anti-involution since

$$\begin{aligned} t^2_{\alpha,u}(X) &= U_{2n}^{-1}(U_{2n}^{-1}X^{\alpha}U_{2n})^{\alpha}U_{2n} \\ &= U_{2n}^{-1}U_{2n}^{\alpha}X^{\alpha\alpha}(U_{2n}^{-1})^{\alpha}U_{2n} \\ &= U_{2n}^{-2}uXu^{-1}U_{2n}^{2} \\ &= X \end{aligned}$$

1.15 Proposition. *The following statements are equivalent:*

(a)

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{2n}(R)$$

belongs to $\mathrm{GQ}_{2n}(R)$

(b)

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{2n}(R)$$

and

$$\begin{pmatrix} A^{\alpha}C & A^{\alpha}D - 1 \\ B^{\alpha}C & B^{\alpha}D \end{pmatrix} = X - X^{\alpha}u \quad \textit{for some } X$$

(c)

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{2n}(R)$$

and

$$\begin{cases} A^{\alpha}D + C^{\alpha}uB = 1 \\ A^{\alpha}C + C^{\alpha}uA = 0 \\ B^{\alpha}D + D^{\alpha}uB = 0 \\ \textit{the diagonal entries of } A^{\alpha}C \textit{ and } B^{\alpha}D \textit{ belong to} \\ \{x - \alpha(x)u \mid x \in R\} \end{cases}$$

(d)

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{2n}(R)$$

and

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = t_{\alpha,u}\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and the diagonal entries of $A^{\alpha}C$ *and* $B^{\alpha}D$ *belong to* $\{x - \alpha(x)u \mid x \in R\}$

(e)

$$\begin{cases} A^\alpha D + C^\alpha uB = 1 \\ A^\alpha C + C^\alpha uA = 0 \\ B^\alpha D + D^\alpha uB = 0 \\ DA^\alpha + Cu^{-1}B^\alpha = 1 \\ DC^\alpha + Cu^{-1}D^\alpha = 0 \\ BA^\alpha + Au^{-1}B^\alpha = 0 \\ \textit{the diagonal entries of } A^\alpha C \textit{ and } B^\alpha D \textit{ belong to} \\ \{x - \alpha(x)u \mid x \in R\} \end{cases}$$

Proof.
(a)⇔ (b):
Immediate by writing out the condition in 1.13.
(b)⇔ (c):
From

$$\begin{pmatrix} A^\alpha C & A^\alpha D - 1 \\ B^\alpha C & B^\alpha D \end{pmatrix} = X - X^\alpha u \text{ for some } X.$$

it follows that

$$\begin{cases} A^\alpha D - 1 = -(B^\alpha C)^\alpha u = -C^\alpha uB \\ 0 = A^\alpha C + (A^\alpha C)^\alpha u = A^\alpha C + C^\alpha uA \\ \text{the diagonal entries of } A^\alpha C \text{ belong to } \{x - \alpha(x)u \mid x \in R\} \\ 0 = B^\alpha D + (B^\alpha D)^\alpha u = B^\alpha D + D^\alpha uB \\ \text{the diagonal entries of } B^\alpha D \text{ belong to } \{x - \alpha(x)u \mid x \in R\} \end{cases}$$

and vice versa.
(c)⇔ (d):
The identity $A^\alpha D + C^\alpha uB = 1$ holds if and only if $D^{\alpha^{-1}} A + u^{-1}B^\alpha C = 1$. Combined with the other equations of statement **(c)** this reads

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} D^{\alpha^{-1}} & u^{-1}B^\alpha \\ C^\alpha u & A^\alpha \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ &= \left(t_{\alpha,u} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \end{aligned}$$

The rest is obvious.
(d)⇔ (e):
Immediate by writing out the equations

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \left(t_{\alpha,u} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right)$$

and

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \left(t_{\alpha,u} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

•

1.16. The product preserving functors

$$F\colon Q(R,\alpha,u)\to\overline{\mathcal{P}(R)},$$

$$H\colon\overline{\mathcal{P}(R)}\to Q(R,\alpha,u)$$

and

$$D_\alpha\colon\mathcal{P}(R)\to\mathcal{P}(R)$$

of definition 1.8 and 1.2 induce homomorphisms

$$F_*\colon K_1Q(R,\alpha,u)\to K_1R,$$

$$H_*\colon K_1R\to K_1Q(R,\alpha,u)$$

and

$$t=t_\alpha\colon K_1R\to K_1R.$$

Now F_* is determined by

$$F_*([X])=[X] \text{ for every } X\in \mathrm{GQ}(R),$$

H_* by

$$H_*([X])=\left[\begin{pmatrix} X & 0\\ 0 & (X^\alpha)^{-1}\end{pmatrix}\right] \text{ for every } X\in \mathrm{GL}(R).$$

t by

$$t([X])=[X^\alpha] \text{ for every } X\in \mathrm{GL}(R).$$

Note that t is an involution since

$$t^2([X])=[X^{\alpha\alpha}]=[uXu^{-1}]=[X].$$

1.17 Lemma. $F_*\circ H_*=1-t$.

Proof. For every $X\in\mathrm{GL}(R)$ we have

$$\begin{pmatrix} X & 0\\ 0 & (X^\alpha)^{-1}\end{pmatrix}=\begin{pmatrix} X(X^\alpha)^{-1} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} X^\alpha & 0\\ 0 & (X^\alpha)^{-1}\end{pmatrix}.$$

But according to [19, §2] the class of

$$\begin{pmatrix} X^\alpha & 0\\ 0 & (X^\alpha)^{-1}\end{pmatrix}$$

is trivial in $K_1(R)$. In view of the preceding this proves the assertion. •

1.18 Definition. [24] A subgroup $\mathcal{X}$ of K_1R is called involution invariant if $t(\mathcal{X})=\mathcal{X}$. For every involution invariant subgroup $\mathcal{X}$ of K_1R define

$$L_1^{\mathcal{X}}(R,\alpha,u):=\frac{F_*^{-1}(\mathcal{X})}{H_*(\mathcal{X})}.$$

1.19 Definition.

- Let $B(R)$ denote the category with
  objects: (M, e) where M is a free right R-module and $e = [e_1, \dots, e_{2m}]$ is an equivalence class of bases of M; two bases being equivalent when the base-change-matrix belongs to $E(R)$, i.e. it represents $0 \in K_1(R)$,
  morphisms: isomorphisms preserving classes,
  product: $(M, e) \perp (N, f) := (M \oplus N, ef)$ where
  $e = [e_1, \dots, e_{2m}]$, $f = [f_1, \dots, f_{2n}]$ and $ef = [e_1, \dots, e_{2m}, f_1, \dots, f_{2n}]$.

- Let $BQ(R, \alpha, u)$ denote the category with
  objects: $(M, [\phi], e)$,
  where $(M, [\phi]) \in \mathrm{Obj}\ Q(R, \alpha, u)$ and $(M, e) \in \mathrm{Obj}\ B(R)$,
  morphisms: isomorphisms preserving both structures,
  product: obvious.

- Again there is a product-preserving functor $H_b\colon B(R) \to BQ(R, \alpha, u)$ defined by

$$H_b(M, e) := (M \oplus M^\alpha, [\upsilon], ee^*) \quad H_b(f) := f \oplus (f^\alpha)^{-1}.$$

  Here $e^* = [e_1^*, \dots, e_{2m}^*]$ and υ is as before.

1.20 Lemma. H_b is cofinal.

Proof. Let (M, θ, e) be an object of $BQ(R, \alpha, u)$. Lemma 1.9 supplies a $Q(R, \alpha, u)$-isomorphism

$$\xi\colon (M, \theta, e) \perp (M, -\theta, e) \to H_b(M, e).$$

Let γ be the element ξ determines in $K_1 R$. By choosing the class f of bases of $M \oplus M$ in such a way that

$$\xi\colon (M \oplus M, \theta \perp -\theta, f) \to H_b(M, e)$$

represents $-\gamma \in K_1 R$, we obtain a $BQ(R, \alpha, u)$-isomorphism

$$\xi \perp \xi\colon (M, \theta, e) \perp (M, -\theta, e) \perp (M \oplus M, \theta - \theta, f) \to H_b(M \oplus M, ee).$$

This proves the assertion. •

1.21 Definition. Let $(\mathcal{C}, \perp)$ be a category with product. The Grothendieck group $K_0\mathcal{C}$ of $\mathcal{C}$ is defined as the abelian group given by the following presentation:
generators: classes $[A]$ of isomorphic objects A of $\mathcal{C}$. We assume that these classes form a set.
relations: $[A] + [B] = [A \perp B]$.

1.22. Lemma 1.20 implies that

· each element of $K_0BQ(R,\alpha,u)$ can be written in the form $[A]-[B]$ where $A \in BQ(R,\alpha,u)$ and B is hyperbolic.

· the equality $[A]-[B]=[A']-[B']$ holds in $K_0BQ(R,\alpha,u)$ if and only if there exists a hyperbolic object C such that $A \perp B' \perp C \cong A' \perp B \perp C$.

1.23 Definition. Define $\widetilde{K_0}BQ(R,\alpha,u)$ as the kernel of the rank-map

$$rk\colon \widetilde{K_0}BQ(R,\alpha,u) \to \mathbf{Z}$$

induced by the map

$$BQ(R,\alpha,u) \to \mathbf{Z} \text{ given by } (M,\theta,[e_1,\dots,e_{2m}]) \mapsto 2m.$$

1.24 Definition. The map $BQ(R,\alpha,u) \to K_1R$ determined by

$$(M,\theta,e) \mapsto [\text{a 'matrix' of } b_\theta \text{ with respect to } e \text{ and } e^*]$$

induces a homomorphism $\delta\colon K_0BQ(R,\alpha,u) \to K_1R$, called discriminant.

1.25 Remark. b_θ determines a matrix with respect to e and e^* only up to elementary matrices. It is therefore legitimate to speak about the class of this 'matrix' in K_1R.

1.26 Remark. Further we ought to mention the fact that δ is a priori non-trivial on hyperbolic objects:
given a hyperbolic object $H_b(M,e)=(M \oplus M^\alpha,[v],ee^*)$ in $BQ(R,\alpha,u)$, the matrix Σ_{2m} of v actually (not only up to elementary matrices) takes the form

$$\Sigma_{2m} = \begin{pmatrix} 0 & I_m \\ 0 & 0 \end{pmatrix} \text{ (no matter what } e \text{ looks like).}$$

Hence $b_{[v]}$ has matrix $U_{2m} = \begin{pmatrix} 0 & I_m \\ uI_m & 0 \end{pmatrix}$. The class of this matrix in K_1R is not necessarily trivial.

1.27 Definition. [24, §3] Define a homomorphism $\tau\colon K_1(R) \to \widetilde{K_0}BQ(R,\alpha,u)$ as follows :
Suppose we are given an $x \in K_1(R)$. Choose $(M,\theta,e) \in BQ(R,\alpha,u)$ and $\gamma \in \mathrm{Aut}(M)$ in such a way that the matrix determined by γ represents x in $K_1(R)$. Define $\tau([x]) := [(M,\theta,\gamma(e)]-[M,\theta,e]$ where $\gamma(e)=[\gamma(e_1),\dots,\gamma(e_{2m})]$. It is not hard to check that τ is a well-defined homomorphism.

1.28 Lemma. $\delta \circ \tau = 1+t$.

Proof. Using the third observation of 1.3 we obtain

$$\delta \circ \tau([A]) = [A^\alpha BA]-[B] = [A^\alpha A] = (1+t)([A]) \quad \text{for all} \quad A \in \mathrm{GL}(R),$$

where B is a 'matrix' of b_θ and θ is as in the construction of τ. •

1.29 Definition. For every involution invariant subgroup $\mathcal{X}$ of K_1R define

$$L_0^{\mathcal{X}}(R,\alpha,u) := \frac{\delta^{-1}(\mathcal{X})}{\tau(\mathcal{X})}$$

here $\delta\colon \widetilde{K_0}BQ(R,\alpha,u) \to K_1(R)$ is the restriction of the discriminant.

Notation. Write L_ε^s instead of $L_\varepsilon^{\{0\}}$ and L_ε^h instead of $L_\varepsilon^{K_1(R)}$ for $\varepsilon = 0, 1$.

1.30. Let (R,α,u) be a ring with anti-structure and $\mathcal{X}$ an involution invariant subgroup of $K_1(R)$. Every element l of $L_0^{\mathcal{X}}(R,\alpha,u)$ can be written in the form

$$[M,[\phi],e] - [M',[\phi'],e'],$$

with $rk([M,[\phi],e]) = rk([M',[\phi'],e']) = 2m$ say. Let

$$\Gamma([M,[\phi],e]) \text{ resp. } \Gamma([M',[\phi'],e'])$$

denote the matrix of ϕ resp. ϕ' with respect to a basis in the class e resp. e'. Since the quadratic modules $(M,[\phi])$ and $(M',[\phi'])$ are non-singular, it follows from definition 1.7 that these matrices belong to $\mathcal{N}_{2m}(R)$, where

$$\mathcal{N}_k(R) := \{\Gamma \in M_k(R) \mid \Gamma + \Gamma^\alpha u \in \mathrm{GL}_k(R)\}.$$

We associate to l the difference

$$[\Gamma([M,[\phi],e])] - [\Gamma([M',[\phi'],e'])]$$

of classes with respect to the following relations:

- ⋄ For all $\Gamma_1, \Gamma_1' \in \mathcal{N}_{2m_1}(R)$ and $\Gamma_2, \Gamma_2' \in \mathcal{N}_{2m_2}(R)$,

  $$[\Gamma_1] - [\Gamma_1'] + [\Gamma_2] - [\Gamma_2'] = [\Gamma_1 \perp \Gamma_2] - [\Gamma_1' \perp \Gamma_2'].$$

  where $\perp$ is determined by

  $$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \perp \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} A & 0 & B & 0 \\ 0 & A' & 0 & B' \\ C & 0 & D & 0 \\ 0 & C' & 0 & D' \end{pmatrix}.$$

  This follows from the definition of the product in $BQ(R,\alpha,u)$.

- ⋄ For all $\Xi \in M_{2m}(R)$

  $$[\Gamma] = [\Gamma + \Xi - \Xi^\alpha u].$$

  This is clear in view of definition 1.7 and the observations of 1.3.

- ⋄ For all $\Delta \in \mathrm{GL}_{2m}(R)$ with $[\Delta] \in \mathcal{X}$

  $$[\Gamma] = [\Delta^\alpha \Gamma \Delta].$$

  This is a consequence of definition 1.8 and the observations of 1.3.

Conversely, for all $\Gamma, \Gamma' \in \mathcal{N}_{2m}(R)$ we associate to $[\Gamma] - [\Gamma']$ the element

$$[R^{2m}, [\phi], st] - [R^{2m}, [\phi'], st] \in L_0^{\mathcal{X}}(R, \alpha, u).$$

Here st denotes the standard basis of R^{2m} and ϕ resp. ϕ' is the homomorphism which has matrix Γ resp. Γ' with respect to this standard basis.

Thus we have established a bijective correspondence between elements of $L_0^{\mathcal{X}}(R, \alpha, u)$ and differences of classes of elements of $\mathcal{N}_{2m}(R)$ under the given relations. Regarding the first item of 1.22 we may thus write every element of $L_0^{\mathcal{X}}(R, \alpha, u)$ as a difference $[\Gamma] - [\Sigma_{2m}]$, with $\Gamma \in \mathcal{N}_{2m}(R)$.

Finally, we interpret the second item of 1.22 as follows. For all $\Gamma \in \mathcal{N}_{2m}(R)$ and $\Gamma' \in \mathcal{N}_{2m'}(R)$,

$$[\Gamma] - [\Sigma_{2m}] = [\Gamma'] - [\Sigma_{2m'}] \quad \text{in} \quad L_0^{\mathcal{X}}(R, \alpha, u)$$

if and only if there exist $n \in \mathbb{N}$, $\Xi \in M_{2(n+m+m')}$ and $\Delta \in \mathrm{GL}_{2(n+m+m')}$ such that

$$\Gamma \perp \Sigma_{2(n+m')} = \Delta^\alpha(\Gamma' \perp \Sigma_{2(n+m)})\Delta + \Xi - \Xi^\alpha u \quad \text{and} \quad [\Delta] \in \mathcal{X}.$$

We conclude this section by stating some definitions and facts from algebraic K- and L-theory needed in the sequel.

1.31 Theorem. [24, Theorem 3] *Given an abelian group A and an involution $t: A \to A$ the Tate-cohomology groups $H^n(A; t)$ are defined by*

$$H^n(A; t) := \frac{\mathrm{Ker}(1 - (-1)^n t)}{\mathrm{Im}(1 + (-1)^n t)}.$$

Suppose $\mathcal{X}_1 \subset \mathcal{X}_2$ are involution invariant subgroups of $K_1(R)$, then there exists an exact sequence

$$\begin{array}{ccccccc}
H^1(\mathcal{X}_2/\mathcal{X}_1) & \xrightarrow{\tilde{\tau}} & L_0^{\mathcal{X}_1}(R,\alpha,u) & \longrightarrow & L_0^{\mathcal{X}_2}(R,\alpha,u) & \xrightarrow{\tilde{\delta}} & H^0(\mathcal{X}_2/\mathcal{X}_1) \\
\uparrow & & & & & & \downarrow \\
L_1^{\mathcal{X}_2}(R,\alpha,u) & & & & & & L_1^{\mathcal{X}_1}(R,\alpha,-u) \\
\uparrow & & & & & & \downarrow \\
L_1^{\mathcal{X}_1}(R,\alpha,u) & & & & & & L_1^{\mathcal{X}_2}(R,\alpha,-u) \\
\uparrow & & & & & & \downarrow \\
H^0(\mathcal{X}_2/\mathcal{X}_1) & \longleftarrow & L_0^{\mathcal{X}_2}(R,\alpha,-u) & \longleftarrow & L_0^{\mathcal{X}_1}(R,\alpha,-u) & \longleftarrow & H^1(\mathcal{X}_2/\mathcal{X}_1)
\end{array}$$

Here $\tilde{\tau}$ resp. $\tilde{\delta}$ is induced by τ resp. δ.

1.32 Theorem. [26] Morita invariance.
If (R, α, u) is a ring with anti-structure and the matrix ring $M_n(R)$ is equipped with the conjugate transpose anti-structure, then $L_\varepsilon^*(M_n(R), \alpha, uI_n)$ is isomorphic to $L_\varepsilon^*(R, \alpha, u)$.

1.33 Theorem. [26] Scaling.
If (R,α,u) is a ring with anti-structure and v is a unit in R, then $L^*_\varepsilon(R,\alpha,u)$ is isomorphic to $L^*_\varepsilon(R,\alpha',u')$, where $\alpha'(r) := v\alpha(r)v^{-1}$ and $u' := v\alpha(v^{-1})u$.

1.34 Theorem. [27, Lemma 5] *Suppose I is a two-sided ideal of R such that R is complete in the I-adic topology. If $\alpha(I) = I$, then R/I can be equipped with an anti-structure in an obvious way and the projection $R \to R/I$ induces an isomorphism $L^h_\varepsilon(R) \longrightarrow L^h_\varepsilon(R/I)$.*

1.35 Definition. [19] Denote by $e_{ij}(a) \in E_n(R)$ the elementary matrix having the element $a \in R$ at the (i,j)-entry.
For $n \geq 3$ let $\mathrm{St}_n(R)$ be the group with the following presentation
generators: one generator $x_{ij}(a)$ for every $e_{ij}(a) \in E_n(R)$
relations:

$$x_{ij}(a)x_{ij}(b) = x_{ij}(a+b)$$

$$[x_{ij}(a), x_{kl}(b)] = \begin{cases} 1 & \text{if } i \neq l,\ j \neq k \\ x_{il}(ab) & \text{if } j = k,\ i \neq l. \end{cases}$$

The Steinberg group of R denoted by $\mathrm{St}(R)$ is by definition the direct limit

$$\varinjlim \mathrm{St}_n(R),$$

where the limit is taken with respect to the embeddings $\mathrm{St}_n(R) \hookrightarrow \mathrm{St}_{n+1}(R)$ coming from the embeddings $E_n(R) \hookrightarrow E_{n+1}(R)$ of definition 1.10. Since the relations for the x_{ij} in $\mathrm{St}_n(R)$ also hold for the e_{ij} in $E_n(R)$, there is a natural homomorphism $\phi\colon \mathrm{St}_n(R) \to E_n(R)$, taking generators $x_{ij}(a)$ to $e_{ij}(a)$, which in the limit gives rise to a homomorphism $E(R) \to \mathrm{St}(R)$. The kernel of this last homomorphism is by definition the K-group K_2R.

1.36 Lemma. [19, theorem 5.1] K_2R is the center of the Steinberg group.

1.37 Definition. Denote by $\mathrm{GL}_{2\infty}(R)$ the direct limit of the groups $\mathrm{GL}_{2n}(R)$ with respect to the embeddings

$$\mathrm{GL}_{2n}(R) \hookrightarrow \mathrm{GL}_{2(n+1)}(R) \text{ defined by } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & B & 0 \\ 0 & 1 & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Similarly one defines $E_{2\infty}(R)$ and correspondingly $\mathrm{St}_{2\infty}(R)$.

1.38. [9, corollary 1.7] The anti-involutions $t_{\alpha,u}$ on the $\mathrm{GL}_{2n}(R)$ give rise to anti-involutions on the $E_{2n}(R)$ which in turn lift to anti-involutions of $\mathrm{St}_{2n}(R)$. See definition 1.14 for formulas. These provide for the following commutative diagram with exact rows and vertical arrows (anti)-involutions:

$$\begin{array}{ccccccccc} 0 \longrightarrow & K_2(R) & \longrightarrow & \mathrm{St}_{2\infty}(R) & \longrightarrow & \mathrm{GL}_{2\infty}(R) & \longrightarrow & K_1(R) & \longrightarrow 0 \\ & \downarrow t_\alpha & & \downarrow t_{\alpha,u} & & \downarrow t_{\alpha,u} & & \downarrow t_\alpha & \\ 0 \longrightarrow & K_2(R) & \longrightarrow & \mathrm{St}_{2\infty}(R) & \longrightarrow & \mathrm{GL}_{2\infty}(R) & \longrightarrow & K_1(R) & \longrightarrow 0 \end{array}$$

1.39 Definition. Following [9] one can construct a homomorphism

$$G: L_0^s(R) \to H^1(K_2(R); t)$$

as follows:
Let

$$l = [\Gamma] - [\Sigma_{2m}] \in L_0^s(R)$$

and $X = \Gamma + \Gamma^\alpha u$. Then

$$U_{2m}^{-1} X \in E(R) \quad \text{and} \quad X^\alpha u = X.$$

Hence

$$t_{\alpha,u}(U_{2m}^{-1}X) = t_{\alpha,u}(X) \cdot t_{\alpha,u}(U_{2m}^{-1}) = U_{2m}^{-1} X^\alpha U_{2m} U_{2m} = U_{2m}^{-1} X^\alpha u = U_{2m}^{-1} X.$$

Now choose a lift $\gamma \in \mathrm{St}(R)$ of $U_{2m}^{-1}X$ and define

$$G(l) := [\gamma^{-1} t_{\alpha,u} \gamma] \in H^1(K_2(R); t).$$

It's not hard to check that G is a well-defined homomorphism.

2 The Arf invariant.

In this section we define the main object of study in this thesis: the Arf-groups.

Suppose we are given a ring with anti-structure (R, α, u) and an involution invariant subgroup $\mathcal{X}$ of $K_1(R)$. We will analyse the subgroup of $L_0^{\mathcal{X}}(R, \alpha, u)$ consisting of all differences of classes of forms whose underlying bilinear form is standard.

2.1 Definition. Recall the considerations of 1.30 and define $\mathrm{Arf}^{\mathcal{X}}(R, \alpha, u)$ as the subgroup of $L_0^{\mathcal{X}}(R, \alpha, u)$ generated by all elements

$$(A, B) := \left[\begin{pmatrix} A & I_m \\ 0 & B \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & I_m \\ 0 & 0 \end{pmatrix}\right],$$

where $A, B \in \Lambda_m(R) := \{X \in M_m(R) \mid X + X^\alpha u = 0\}$.
Note that

$$\begin{pmatrix} A & I_m \\ 0 & B \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & I_m \\ 0 & 0 \end{pmatrix}$$

both belong to $\mathcal{N}_{2m}(R)$.
Further we define $\Gamma_m(R) := \{X - X^\alpha u \mid X \in M_m(R)\}$.

2.2 Lemma. All elements (A, B) of $\mathrm{Arf}^{\mathcal{X}}(R, \alpha, u)$ can be written in the form:

$$(A, B) = \sum_{i=1}^{m} (A_{ii}, B_{ii}).$$

Proof. Recall 1.30. Since $A + A^\alpha u = B + B^\alpha u = 0$ we find

$$\begin{aligned}
(A, B) &= \left[\begin{pmatrix} A & I_m \\ 0 & B \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & I_m \\ 0 & 0 \end{pmatrix}\right] \\
&= \sum_{i=1}^{m} \left[\begin{pmatrix} A_{ii} & 1 \\ 0 & B_{ii} \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] \\
&= \sum_{i=1}^{m} (A_{ii}, B_{ii})
\end{aligned}$$

•

2.3 Proposition.
Suppose we are given $A, B \in \Lambda_m(R)$ and $A', B' \in \Lambda_{m'}(R)$. Then

$$(A, B) = (A', B') \quad \textit{in} \quad \mathrm{Arf}^{\mathcal{X}}(R, \alpha, u),$$

if and only if there exist

$$n \in \mathbb{N} \quad \textit{and} \quad \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} \in \mathrm{GL}_{2(n+m+m')}(R) \quad \textit{with} \quad \left[\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}\right] \in \mathcal{X},$$

such that

$$A' = X^\alpha AX + X^\alpha Z + Z^\alpha BZ \pmod{\Gamma_{n+m+m'}(R)},$$

$$B' = Y^\alpha AY + Y^\alpha T + T^\alpha BT \pmod{\Gamma_{n+m+m'}(R)} \quad \textit{and}$$

$$t_{\alpha,u}\begin{pmatrix} X & Y \\ Z & T \end{pmatrix} = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{-1}.$$

Here A, B, A', B' are considered to be elements of $M_{n+m+m'}(R)$, by the embeddings $M_k(R) \hookrightarrow M_{k+1}(R)$ defined by

$$(\, C\,) \longleftrightarrow \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. Regarding the final assertion of 1.30 it suffices to make the following statements. Define $k := n + m + m'$.

$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^\alpha \begin{pmatrix} A & I_k \\ 0 & B \end{pmatrix} \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} \text{ takes the form } \begin{pmatrix} A' & I_k \\ 0 & B' \end{pmatrix}$$

$\pmod{\Gamma_{2k}(R)}$ precisely when the difference

$$\begin{pmatrix} X^\alpha AX + X^\alpha Z + Z^\alpha BZ & X^\alpha AY + X^\alpha T + Z^\alpha BT \\ Y^\alpha AX + Y^\alpha Z + T^\alpha BT & Y^\alpha AY + Y^\alpha T + T^\alpha BT \end{pmatrix} - \begin{pmatrix} A' & I_k \\ 0 & B' \end{pmatrix}$$

belongs to $\Gamma_{2k}(R)$. From the fact that the matrices A, B, A', B' in this expression belong to $\Lambda_{2k}(R)$ we deduce:

$$\begin{cases} X^\alpha T + Z^\alpha uY = 1 \\ X^\alpha Z + Z^\alpha uX = 0 \\ Y^\alpha T + T^\alpha uY = 0 \end{cases}$$

This is equivalent to

$$\left(t_{\alpha,u}\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}\right)\begin{pmatrix} X & Y \\ Z & T \end{pmatrix} = 1.$$

•

We will give a presentation for the groups $\mathrm{Arf}^{\mathcal{X}}(R, \alpha, u)$ in the next theorem. Although our definition of the Arf- and L-groups is a priori quite different from the one in [5], the presentation is nearly the same. We refer to [1] for a comparison of the various L-groups. Moreover this presentation is not quite the same as the one in [5], because our u is not necessarily central in R. At least not yet.

2.4 Theorem. Compare [5].
As abelian group $Arf^{\mathcal{X}}(R, \alpha, u)$ has the following presentation:
generators: (a, b) *where* $a, b \in \Lambda_1(R)$

relations: 1) $(a, b_1 + b_2) = (a, b_1) + (a, b_2)$ *for all* $a, b_1, b_2 \in \Lambda_1(R)$
2) $(a_1 + a_2, b) = (a_1, b) + (a_2, b)$ *for all* $a_1, a_2, b \in \Lambda_1(R)$
3) $(a, b) = (b, uau^{-1})$ *for all* $a, b \in \Lambda_1(R)$
4) $(a, b) = 0$ *for all* $a \in \Lambda_1(R)$, $b \in \Gamma_1(R)$
5) $(a, \alpha(x)bx) = (xa\alpha^{-1}(x), b)$ *for all* $a, b \in \Lambda_1(R)$, $x \in R$
6) $(a, b) = (a, ba\alpha^{-1}(b))$. *for all* $a, b \in \Lambda_1(R)$
7) $\sum_{i=1}^{n}((X^\alpha Z)_{ii}, (Y^\alpha T)_{ii}) = 0$ *if* $\begin{pmatrix} X & Y \\ Z & T \end{pmatrix} \in \mathrm{GL}_{2n}(R)$,
$t_{\alpha,u}\begin{pmatrix} X & Y \\ Z & T \end{pmatrix} = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{-1}$ *and* $\left[\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}\right] \in \mathcal{X}$.

Proof. $\mathrm{Arf}^{\mathcal{X}}(R, \alpha, u)$ is generated by the (a, b) because of lemma 2.2. To prove the relations, we will now exploit proposition 2.3.
Let $A, B \in \Lambda_m(R)$.
Choosing
$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix} = \begin{pmatrix} I_m & u^{-1}B \\ 0 & I_m \end{pmatrix}$$
in proposition 2.3 yields
$$\begin{aligned} (A, B) &= (A, (u^{-1}B)^\alpha Au^{-1}B + (u^{-1}B)^\alpha + B) \\ &= (A, B^\alpha uAu^{-1}B + B^\alpha u + B) \\ &= (A, BAB^{\alpha^{-1}}). \end{aligned}$$
Taking $m = 1$ this proves *6*.
Choosing
$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ A & I_m \end{pmatrix}$$
in proposition 2.3 yields
$$(A, B) = (A + A + A^\alpha BA, B) = (A^\alpha BA, B).$$
As a consequence $(a, 0) = (0, a) = 0$ for all $a \in \Lambda_1(R)$, which proves *4*.
Let $A', B', C', D' \in \Lambda_m(R)$ and $X' \in M_m(R)$.
Choosing
$$A = \begin{pmatrix} A' & 0 \\ 0 & C' \end{pmatrix}, \qquad B = \begin{pmatrix} B' & 0 \\ 0 & D' \end{pmatrix}$$
and
$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix} = \begin{pmatrix} I_m & 0 & 0 & X' \\ 0 & I_m & -u^{-1}X'^\alpha & 0 \\ 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & I_m \end{pmatrix}$$
in proposition 2.3 yields
$$\left(\begin{pmatrix} A' & 0 \\ 0 & C' \end{pmatrix}, \begin{pmatrix} B' & 0 \\ 0 & D' \end{pmatrix}\right)$$

$$= \left(\begin{pmatrix} A' & 0 \\ 0 & C' \end{pmatrix}, \begin{pmatrix} 0 & -uX' \\ X'^{\alpha} & 0 \end{pmatrix} \begin{pmatrix} A' & 0 \\ 0 & C' \end{pmatrix} \begin{pmatrix} 0 & X' \\ -u^{-1}X'^{\alpha} & 0 \end{pmatrix} + \begin{pmatrix} 0 & -uX' \\ X'^{\alpha} & 0 \end{pmatrix} + \begin{pmatrix} B' & 0 \\ 0 & D' \end{pmatrix}\right)$$

$$= \left(\begin{pmatrix} A' & 0 \\ 0 & C' \end{pmatrix}, \begin{pmatrix} uX'C'u^{-1}X'^{\alpha} & 0 \\ 0 & X'^{\alpha}A'X' \end{pmatrix} + \begin{pmatrix} B' & 0 \\ 0 & D' \end{pmatrix}\right)$$

Hence

$$(A', B') + (C', D') = (A', B' + uX'C'u^{-1}X'^{\alpha}) + (C', X'^{\alpha}A'X' + D'). \qquad (1)$$

First choose $C' = u^{-1}B'u$, $D' = 0$ and $X' = 1$ to obtain

$$(A', B') = (u^{-1}B'u, A'),$$

which proves *3*.
Then choose $A' = D'$ and $X' = 1$ to obtain

$$(A', B') + (C', A') = (A', B' + uC'u^{-1})$$

which by *3* is equivalent to

$$(A', B') + (A', uC'u^{-1}) = (A', B' + uC'u^{-1}).$$

This proves *1*.
Note that *2* follows from and *1* and *3*.
In order to verify *5* we use *1, 2, 3* and *4* to see that equation 1 comes down to

$$(A', uX'C'u^{-1}X'^{\alpha}) = (C', X'^{\alpha}A'X').$$

But since $(A', uX'C'u^{-1}X'^{\alpha}) = (X'C'X'^{\alpha^{-1}}, A')$, this proves *5*.
Note that all choices for $\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}$ we have made so far satisfy the conditions of proposition 2.3.
Finally suppose $\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}$ agrees with the conditions of *7*. To prove the theorem it suffices to show that

$$(X^{\alpha}AX + X^{\alpha}Z + Z^{\alpha}BZ, Y^{\alpha}AY + Y^{\alpha}T + T^{\alpha}BT) = (A, B) + (X^{\alpha}Z, Y^{\alpha}T)$$

modulo the relations *1* to *6*. This is accomplished by using the relations for $X, Y.Z$ and T listed in proposition 1.15. We equate

$$(X^{\alpha}AX, Y^{\alpha}AY) = (YX^{\alpha}AXY^{\alpha^{-1}}, A) = (XY^{\alpha^{-1}}, A)$$

and in the same fashion

$$(Z^{\alpha}BZ, T^{\alpha}BT) = (ZT^{\alpha^{-1}}, B).$$

Further

$$\begin{aligned}
&(X^\alpha AX, Y^\alpha T) + (X^\alpha Z, Y^\alpha AY) \\
&= (A, X^{\alpha^2} Y^\alpha T X^\alpha) + (Y X^\alpha Z Y^{\alpha^{-1}}, A) \\
&= (u^{-1} X^{\alpha^2} Y^\alpha T X^\alpha u, A) + (Y X^\alpha Z Y^{\alpha^{-1}}, A) \\
&= (X Y^{\alpha^{-1}}, A)
\end{aligned}$$

and analogously

$$(Z^\alpha BZ, Y^\alpha T) + (X^\alpha Z, T^\alpha BT) = (Z T^{\alpha^{-1}}, B).$$

Finally we have

$$\begin{aligned}
&(X^\alpha AX, T^\alpha BT) + (Z^\alpha BZ, Y^\alpha AY) \\
&= (A, X^{\alpha^2} T^\alpha B T X^\alpha) + (Y Z^\alpha B Z Y^{\alpha^{-1}}, A) \\
&= (A, X^{\alpha^2} T^\alpha B T X^\alpha) + (A, u Y Z^\alpha B Z Y^{\alpha^{-1}} u^{-1}) \\
&= (A, X^{\alpha^2} T^\alpha B T X^\alpha) + (A, (1 - X^{\alpha^2} T^\alpha) B (1 - T X^\alpha)) \\
&= (A, B).
\end{aligned}$$

This completes the proof. •

2.5 Theorem. *There is a well-defined homomorphism, called Arf invariant*

$$\omega \colon \mathrm{Arf}^{\mathcal{X}}(R, \alpha, u) \to R/\kappa(R),$$

defined by

$$(A, B) \mapsto [Tr(A^\alpha B)] .$$

Here $\kappa(R)$ denotes the additive subgroup of R generated by

$$\{x + x^2, y + \alpha(y) \mid x, y \in R\}$$

Observe that $xy - yx, 2x \in \kappa(R)$ for all $x, y \in R$.

Proof. Analogous to the proof of [5, theorem 2]. •

2.6 Definition. For every group G we define

$$L^{s,h}(G) := L_0^{s,h}(\mathbb{F}_2[G], \alpha, 1)$$

and correspondingly

$$\mathrm{Arf}^{s,h}(G) := \mathrm{Arf}^{s,h}(\mathbb{F}_2[G], \alpha, 1),$$

where α is determined by $\alpha(g) := g^{-1}$ for all $g \in G$. Further we define

$$K(G) := \frac{\mathbb{F}_2[G]}{\kappa(\mathbb{F}_2[G])}.$$

2.7 Remark. See also [5]. From the presentation of theorem 2.4 we deduce that $\mathrm{Arf}^{s,h}(G)$ is generated by all (g,h) with $g,h \in {}_2G := \{x \in G \mid x^2 = 1\}$ and that the following relations hold:

$$\begin{array}{rcll} (g,h) & = & (h,g) & \\ (g,h) & = & (xgx^{-1}, xhx^{-1}) & \text{for all} \quad x \in G \\ (g,h) & = & (g, hgh). & \end{array}$$

The value group $K(G)$ of the Arf invariant $\mathrm{Arf}^{s,h}(G) \longrightarrow K(G)$, is in fact the $\mathbb{F}_2$-vectorspace generated by the quotient set $\mathcal{C}\ell(G) := G/\sim$, where $\sim$ denotes the equivalence relation on G generated by $g \sim g^{-1}$, $g \sim hgh^{-1}$ and $g \sim g^2$.

2.8 Theorem. *The Arf invariant* $\mathrm{Arf}^{s,h}(G) \to K(G)$ *is injective whenever* G *is a finite group.*

Proof. We refer to [5] for the proof. •

We will revert to these theorems later on.

2.9 Lemma. Let a, b and c be elements of order two in a group G and assume that c commutes with a and b. Then the relation

$$(a, bc) = (a, b)$$

holds in $\mathrm{Arf}^{s,h}(G)$.

Proof. $(a,bc) = (a, bcabc) = (a, bab) = (a,b)$. •

Example. Let G be the group with presentation

$$\langle X, S \mid X^{12} = S^2 = 1,\ SXS = X^5 \rangle.$$

So G is a semidirect product of the group of order 2 and the cyclic group of order 12.

2.10 Proposition. *The elements* $(1,1)$, (X^2S, S) *form a basis for* $\mathrm{Arf}^{s,h}(G)$.

Proof. A little computation yields $\mathcal{C}\ell(G) = \{[1],[X]\}$. The Arf invariant is injective and maps $(1,1)$ to $[1]$ and (X^2S, S) to $[X^2] = [X]$, hence the assertion is true. •

The following example is meant to illustrate how tricky manipulations with the relations in $\mathrm{Arf}^{s,h}(G)$ can be.

Example. Let G be the group with presentation

$$G := \langle X, Y, S \mid S^2 = (XS)^2 = Y^{12} = 1, \quad SYS = Y^5, \quad XY = YX \rangle.$$

This group fits into the short exact sequence

$$1 \longrightarrow C \times C_{12} \longrightarrow G \longrightarrow C_2 \longrightarrow 1,$$

where C_2 has generator S, C_{12} has generator Y and C is the infinite cyclic group generated by X. Actually G is a semidirect product of C_2 and $C \times C_{12}$. We show that the elements (S, SX^2Y^2) and (SX, SX^3Y^2) of $\mathrm{Arf}^{s,h}(G)$ coincide. The Arf invariant ω maps both elements to the class of X^2Y^2 in $K(G)$. We equate

$$\begin{aligned}
(S, SX^2Y^2) &= (S, SX^4Y^4) \\
&= (S, SX^2Y^8) \\
&= (S, SXY^4) \\
&= (SXY^2SSXY^2, SXY^2SXY^4SXY^2) \\
&= (SX^2Y^4, SX) \\
&= (SX, SX^2Y^4) \\
&= (SX, SX^3Y^8) \\
&= (SX, SX^5Y^4) \\
&= (SX, SX^3Y^2).
\end{aligned}$$

Since the Arf invariant maps both (S, SY^2) and (SX, SXY^2) to the class of Y^2 in $K(G)$, one might conjecture that these elements are equal too, but this is false.

Example. Let G be the group with presentation

$$G := \langle Y, S \mid S^2 = (YS)^4 = (Y^2S)^2 = 1\rangle.$$

This group is actually an extension of the infinite cyclic group by the dihedral group D_4:

$$\begin{array}{ccccc}
1 \longrightarrow C \longrightarrow & G \longrightarrow D_4 & \longrightarrow 1 \\
& S \longmapsto \sigma & \\
& Y \longmapsto \sigma\tau &
\end{array}$$

Here C is the infinite cyclic group generated by Y^2 and D_4 is the dihedral group with presentation

$$D_4 = \langle \sigma, \tau \mid \sigma^2 = (\sigma\tau)^2 = \tau^4 = 1\rangle.$$

2.11 Proposition. *The set*

$$\{(1,1)\} \cup \left\{(Y^{4i+2}S, S) \mid i > 0\right\}$$

constitutes a basis for $\mathrm{Arf}^{s,h}(G)$.

Proof. The elements of order 2 in G are $Y^{2i}S$, $(YS)^2$ and $Y^{2i}S(YS)^2$. Note that $(YS)^2$ is central in G. So we may use lemma 2.9 to see that $\mathrm{Arf}^{s,h}(G)$ is generated by elements of the form $(Y^{2i}S, Y^{2j}S)$. The identities

$$(Y^{2i}S, Y^{2j}S) = (Y^{-2k}Y^{2i}SY^{2k}, Y^{-2k}Y^{2j}SY^{2k})$$

$$
\begin{aligned}
& = (Y^{2i-4k}S, Y^{2j-4k}S), \\
(Y^{2i}S, Y^2S) & = (Y^{2i}S, Y^2S(YS)^2) \\
& = (Y^{2i}S, YSY^{-1}) \\
& = (Y^{2i-1}SY, S) \\
& = (Y^{2i-2}S(SY)^2, S) \\
& = (Y^{2i-2}S, S), \\
(Y^{4i}S, S) & = (Y^{2i}SSY^{2i}S, S) \\
& = (Y^{2i}S, S), \\
(Y^{2i}S, S) & = (SY^{2i}SS, S) \\
& = (Y^{-2i}S, S)
\end{aligned}
$$

show that $\{(1,1)\} \cup \{(Y^{4i+2}S, S) \mid i > 0\}$ is a set of generators for $\mathrm{Arf}^{s,h}(G)$. We use the Arf invariant $\mathrm{Arf}^{s,h}(G) \to K(G)$ to prove that these elements are independent. It is easy to verify that

$$\mathcal{C}\ell(G) = \{[1]\} \cup \{[Y^{2i+1}] \mid i > 0\}$$

by writing down all generating relations in $\mathcal{C}\ell(G)$.
The Arf invariant maps $(1,1)$ to $[1]$ and $(Y^{4i+2}S, S)$ to $[Y^{4i+2}] = [Y^{2i+1}]$. This proves the assertion. •

Example. Let G be the group with presentation

$$G := \langle X, Y, S \mid S^2 = (XS)^2 = (YS)^4 = (Y^2S)^2 = 1, \quad XY = YX \rangle.$$

This group is actually an extension of the free abelian group A of rank 2 by the dihedral group D_4:

$$1 \longrightarrow A \longrightarrow G \xrightarrow{\pi} D_4 \longrightarrow 1$$

where $\pi(S) := \sigma$, $\pi(X) := 1$, $\pi(Y) := \sigma\tau$ and A is generated by X and Y^2.

2.12 Proposition. $\mathrm{Arf}^{s,h}(G)$ *is generated by*

$$
\begin{aligned}
\{(1,1)\} & \cup \{(X^{2i+1}Y^{2j}S, S) \mid i \geq 0\} \\
& \cup \{(X^{2i}Y^{4j+2}S, S) \mid j \geq 0\} \\
& \cup \{(X^{2i+1}Y^{4j+2}S, XS) \mid j \geq 0\}.
\end{aligned}
$$

Proof. The elements of order 2 in G are $X^iY^{2j}S$, $(YS)^2$ and $X^iY^{2j}S(YS)^2$. Note that $(YS)^2$ is central in G again. So $\mathrm{Arf}^{s,h}(G)$ is generated by elements of the form $(X^iY^{2j}S, X^kY^{2l}S)$. We may assume that $k, l \in \{0,1\}$ by the identity

$$(X^iY^{2j}S, X^kY^{2l}S) = (X^{i-2m}Y^{2j-4n}S, X^{k-2m}Y^{2l-4n}S).$$

We may even assume that $l = 0$ by the relation

$$\begin{aligned}(X^iY^{2j}S, X^kY^2S) &= (X^iY^{2j}S, X^kY^2S(YS)^2) \\ &= (X^iY^{2j}S, X^kYSY^{-1}) \\ &= (X^iY^{2j-1}SY, X^kS) \\ &= (X^iY^{2j-1}(SY)^3, X^kS) \\ &= (X^iY^{2j-2}S, X^kS).\end{aligned}$$

When $k = 0$ we may assume that i or j is odd:

$$\begin{aligned}(X^{2i}Y^{4j}S, S) &= (X^iY^{2j}SSX^iY^{2j}S, S) \\ &= (X^iY^{2j}S, S)\end{aligned}$$

In this situation we may assume that one odd exponent is positive:

$$\begin{aligned}(X^iY^{2j}S, S) &= (SX^iY^{2j}SS, S) \\ &= (X^{-i}Y^{-2j}S, S)\end{aligned}$$

When $k = 1$ we may assume that i and j are odd:

$$\begin{aligned}(X^{2i}Y^{2j}S, XS) &= (Y^{2j}S, X^{-2i+1}S) \\ &= (X^{2i-1}Y^{2j}S, S) \\ (X^{2i+1}Y^{4j}S, XS) &= (X^{i+1}Y^{2j}SXSX^{i+1}Y^{2j}S, XS) \\ &= (X^{i+1}Y^{2j}S, XS)\end{aligned}$$

And finally, we may assume that j is positive:

$$\begin{aligned}(X^iY^{2j}S, XS) &= (XSX^iY^{2j}SXS, XS) \\ &= (X^{-i+2}Y^{-2j}S, XS)\end{aligned}$$

This proves the proposition. •

Example. Let G be the group with presentation

$$G := \langle X, Y, Z \mid X^2 = Y^2 = Z^2 = (XY)^3 = (YZ)^3 = (XZ)^3 = 1\rangle.$$

This group is known as the affine Weyl group $\widetilde{A_2}$.
Define $U := XYZY$, $V := YXZX$ and $W := ZXYX$. Then $UVW = 1$ and U, V and W commute. The subgroup H of G generated by U, V and W is normal since,

$$\begin{array}{ll} XUX = U^{-1} & XVX = W^{-1} \\ YUY = W^{-1} & YVY = V^{-1} \\ ZUZ = V^{-1} & ZWZ = W^{-1}. \end{array}$$

Further $G/H \cong S_3 = \langle x, y \mid x^2 = y^2 = (xy)^3 = 1\rangle$. These groups fit into the short exact sequence

$$1 \longrightarrow H \longrightarrow G \underset{\alpha}{\rightleftarrows} S_3 \longrightarrow 1,$$

which splits by $\alpha(x) := X$ and $\alpha(y) := Y$. Thus G is actually a semidirect product of S_3 and H.

2.13 Proposition. *The elements*

$$(1,1), \quad (X,Y), \quad (Y,Z), \quad (X,Z) \quad \textit{and} \quad (XU^i, X) \quad i>0 \ \textit{odd}$$

form a basis for $\mathrm{Arf}^{s,h}(G)$.

Proof. We merely sketch the proof.
The elements of order two in G are XU^i, YV^i and $XYXW^i$. So in $\mathrm{Arf}^{s,h}(G)$ one has the following types of elements.

1. (XU^i, XU^j)
2. (XU^i, YV^j)
3. $(XU^i, XYXW^j)$
4. (YV^i, YV^j)
5. $(YV^i, XYXW^j)$
6. $(XYXW^i, XYXW^j)$

We prove that all of these elements are actually of the desired type by using the relations

$$XU^i = V^iXV^{-i} = W^iXW^{-i},$$
$$YV^i = U^iYU^{-i} = W^iYW^{-i},$$
$$XYXYV^iXYX = XU^{-i}.$$

1. Conjugation by W^{-j} yields $(XU^i, XU^j) = (XU^{i-j}, X)$.
   And further
   $$(XU^i, X) = (XU^iXXU^i, X) = (XU^{2i}, X),$$
   $$(XU^i, X) = (XXU^iX, X) = (XU^{-i}, X).$$
2. Conjugation by U^{-j} yields $(XU^i, YV^j) = (XU^{i+2j}, Y)$. But because
   $$(XU^i, Y) = (U^{-1}WXU^iW^{-1}U, U^{-1}WYW^{-1}U) = (XU^{i+3}, Y),$$
   only the elements
   (X,Y),
   $(XU, Y) = (YZY, Y) = (Y,Z)$ and
   $(XU^{-1}, Y) = (XYZYX, Y) = (Z, YXYXY) = (X,Z)$ remain.
3. Conjugation by X yields $(XU^i, XYXW^j) = (XU^{-i}, YV^{-j})$.
4. Conjugation by XYX yields $(YV^i, YV^j) = (XU^{-i}, XU^{-j})$.
5. Conjugation by Y yields $(YV^i, XYXW^j) = (YV^{-i}, XU^{-j})$.

6. Conjugation by X yields $(XYXW^i, XYXW^j) = (YV^{-i}, YV^{-j})$.

We give a list of generating relations in $\mathcal{C}\ell(G)$.

- $U^iV^j \sim U^{-i}V^{-j} \sim U^{2i}V^{2j} \sim U^{j-i}V^j \sim U^iV^{i-j} \sim U^jV^i$
- $XU^iV^j \sim XV^j \sim U^jV^{2j} \sim U^jV^{-j}$
- $YU^iV^j \sim YU^i \sim U^{2i}V^i \sim U^iV^{-i}$
- $XYXU^iV^j \sim YU^{j-i}V^j \sim U^{i-j}V^{j-i}$
- $YXU^iV^j \sim XYU^{j-i}V^j$
- $XYU^iV^j \sim XYU^{i+1}V^{j-1} \sim XYU^{i+1}V^{j+2}$

The Arf invariant maps

$$\begin{cases} (1,1) & \text{to} \quad [1] \\ (X,Y) & \text{to} \quad [XY] \\ (X,Z) & \text{to} \quad [XZ] = [XYU] \\ (Y,Z) & \text{to} \quad [YZ] = [XYU^{-1}] \\ (XU^i, X) & \text{to} \quad [U^i] \quad i \text{ is positive and odd.} \end{cases}$$

From the list of relations we see that these images are independent, which proves the proposition •

We will review some of these examples in chapter IV.

Chapter II

New Invariants for L-groups.

1 Extension of the anti-structure to the ring of formal power series.

To construct new invariants we start by extending a given anti-structure on a ring R to the ring of formal power series $R[[T]]$, in a highly non-trivial manner. The fact that the projection $R[[T]] \to R$ induces an isomorphism of the associated L-groups, enables us to build new invariants.

1.1 Definition. Suppose we are given a ring with antistructure (R, α, u). For every $n \in \mathbb{N} \cup \{\infty\}$ we define

$$\begin{aligned} R_n &:= \begin{cases} R[T]/(T^{n+1})\,, & \text{the truncated polynomial ring, if } n \in \mathbb{N} \\ R[[T]]\,, & \text{the ring of formal power series, if } n = \infty, \end{cases} \\ \mathcal{I}_n &:= TR_n, \text{ the two-sided ideal of } R_n \text{ generated by the class of } T, \\ u_n &:= u(1+T). \end{aligned}$$

Note that the class of T in R_n is also denoted by T. Now we extend the anti-structure on R to an anti-structure on R_n by the formula

$$\alpha\left(\sum a_k T^k\right) := \sum \alpha(a_k)\left(\frac{-T}{1+T}\right)^k.$$

1.2 Lemma. For every $n \in \mathbb{N} \cup \{\infty\}$

1. (R_n, α, u_n) is a ring with antistructure.
2. $\mathcal{I}_n$ is an involution invariant two-sided ideal of R_n, i.e. $\alpha(\mathcal{I}_n) = \mathcal{I}_n$.
3. R_n is complete in the $\mathcal{I}_n$-adic topology.
4. The projection $R_n \to R$ splits and α respects this splitting.

Proof. The proof is trivial and therefore omitted. •

2 Construction of the invariants $\omega_1^{s,h}$ and ω_2.

2.1. In algebraic K-theory on has functors

$$K_i\colon \quad \textit{category of ideals} \to \quad \textit{category of abelian groups}$$

for every $i \in \mathbb{N}$. The *category of ideals* is the category with
objects: pairs (R, I) consisting of a ring R and a two-sided ideal I of R
morphisms: $f\colon (R, I) \to (S, J)$ are the ringhomomorphisms $f\colon R \to S$ satisfying $f(I) \subseteq J$.
The groups $K_i(R) := K_i(R, R)$ are the ones we already came across in the first chapter. For every pair (R, I) there exists a long exact sequence

$$\cdots \to K_{i+1}(R/I) \to K_i(R, I) \to K_i(R) \to K_i(R/I) \to \cdots$$

2.2. Let (R, α, u) be a ring with anti-structure and (R_n, α, u_n) the associated extension. Since the projection $R_n \to R$ splits, we have

$$K_i(R_n) \cong K_i(R) \oplus K_i(R_n, \mathcal{I}_n)$$

by the functoriality of the K_i. The involutions t_α on $K_1(R_n)$ and $K_2(R_n)$ induced by α respect this splitting. Consequently, the Tate cohomology groups split accordingly:

$$H^{0,1}(K_i(R_n)) \cong H^{0,1}(K_i(R)) \oplus H^{0,1}(K_i(R_n, \mathcal{I}_n)).$$

2.3 Theorem. *The following periodic sequence is exact.*

$$\begin{array}{ccccccc}
H^1(K_1(R_n,\mathcal{I}_n)) & \xrightarrow{\tilde\tau} & L_0^s(R_n,{}^-,u_n) & \longrightarrow & L_0^s(R,{}^-,u) & \xrightarrow{\omega_1^s} & H^0(K_1(R_n,\mathcal{I}_n)) \\
\uparrow & & & & & & \downarrow \\
L_1^s(R,{}^-,u) & & & & & & L_1^s(R_n,{}^-,-u_n) \\
\uparrow & & & & & & \downarrow \\
L_1^s(R_n,{}^-,u_n) & & & & & & L_1^s(R,{}^-,-u) \\
\uparrow & & & & & & \downarrow \\
H^0(K_1(R_n,\mathcal{I}_n)) & \longleftarrow & L_0^s(R,{}^-,-u) & \longleftarrow & L_0^s(R_n,{}^-,-u_n) & \longleftarrow & H^1(K_1(R_n,\mathcal{I}_n))
\end{array}$$

Here $\tilde\tau$ is induced by the homomorphism τ of definition 1.27 and ω_1^s is induced by the discriminant homomorphism.

Proof. From theorem 1.31 of chapter I we obtain the following commutative diagram with exact rows ($\varepsilon = 0, 1$)

$$\begin{array}{ccccccccc}
L^h_{1-\varepsilon}(R_n) & \to & H^{1-\varepsilon}(K_1R) & \to & L_\varepsilon^{K_1(R_n,\mathcal{I}_n)}(R_n) & \to & L^h_\varepsilon(R_n) & \to & H^\varepsilon(K_1R) \\
\downarrow & & \| & & \downarrow & & \downarrow & & \| \\
L^h_{1-\varepsilon}(R) & \to & H^{1-\varepsilon}(K_1R) & \to & L^s_\varepsilon(R) & \to & L^h_\varepsilon(R) & \to & H^\varepsilon(K_1R)
\end{array}$$

Theorem 1.34 of chapter I implies that $L^h_\varepsilon(R_n) \to L^h_\varepsilon(R)$ is an isomorphism. Consequently $L_\varepsilon^{K_1(R_n,\mathcal{I}_n)}(R_n)$ is isomorphic to $L^s_\varepsilon(R)$ by applying the five lemma to the diagram above. When we insert this in the sequence of theorem 1.31 of chapter I applied to the ring R_n with $\mathcal{X}_1 = 0$ and $\mathcal{X}_2 = K_1(R_n, \mathcal{I}_n)$, we obtain the desired periodic exact sequence. •

2.4 Definition. Define

$$\omega_1^h\colon L_0^h(R,\alpha,u) \longrightarrow H^0(K_1(R_n,\mathcal{I}_n);t_\alpha)$$

as the composition of homomorphisms

$$L_0^h(R,\alpha,u) \cong L_0^h(R_n,\alpha,u_n) \xrightarrow{\tilde{\delta}} H^0(K_1(R_n);t_\alpha) \longrightarrow H^0(K_1(R_n,\mathcal{I}_n);t_\alpha),$$

where $\tilde{\delta}$ is induced by the discriminant homomorphism δ. Notice that ω_1^s factors through ω_1^h.
Define d as the composition of homomorphisms

$$H^1(K_1(R_n,\mathcal{I}_n)) \xrightarrow{\widetilde{\tau}} L_0^s(R_n,{}^-,u_n) \xrightarrow{G} H^1(K_2(R_n,\mathcal{I}_n)),$$

where G denotes the homomorphism of definition 1.39 of the first chapter.

2.5 Lemma. The map d can explicitly be given by

$$d([X]) = [\gamma^{-1}t_{\alpha,u}\gamma], \text{ for all } X \in GL(R),$$

where $\gamma \in \mathrm{St}(R)$ is a lift of $(t_{\alpha,u}X)X \in E(R)$.

Proof. Immediate by the definitions of G and τ. •

2.6 Theorem. *The homomorphism G induces a homomorphism*

$$\omega_2\colon \mathrm{Ker}(\omega_1^s) \to \mathrm{Coker}(d).$$

Proof. This is clear now in view of the exact sequence of theorem 2.3 and definition 2.4. •

3 Recognition of ω_1^h.

We now proceed to analyse ω_1^h. It will turn out that ω_1^h is strongly related to the Arf invariant of the first chapter.

3.1 Proposition. *Let (R, α, u) be a ring with anti-structure. For all $(a, b) \in \mathrm{Arf}^h(R, \alpha, u)$*

$$\omega_1^h((a,b)) = \left[1 + \frac{\alpha(a)bT^2}{1+T}\right] \in H^0(K_1(R_n, \mathcal{I}_n))$$

Proof. We may take

$$\left[\begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] \in L_0^h(R_n, \alpha, u_n)$$

as a lift of $(a, b) \in \mathrm{Arf}^h(R, \alpha, u) \subseteq L_0^h(R, \alpha, u)$.

$$\begin{aligned}
\omega_1^h((a,b)) &= \left[\begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix} + \begin{pmatrix} \alpha(a) & 0 \\ 1 & \alpha(b) \end{pmatrix} u(1+T)\right] \\
&\quad - \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} u(1+T)\right] \\
&= \left[\begin{pmatrix} \alpha(a)uT & 1 \\ u(1+T) & \alpha(b)uT \end{pmatrix} \begin{pmatrix} 0 & 1 \\ u(1+T) & 0 \end{pmatrix}^{-1}\right] \\
&= \left[\begin{pmatrix} 1 & \frac{\alpha(a)T}{1+T} \\ \alpha(b)uT & 1 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} 1 & \frac{-\alpha(a)T}{1+T} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{\alpha(a)T}{1+T} \\ \alpha(b)uT & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\alpha(b)uT & 1 \end{pmatrix}\right] \\
&= \left[1 - \frac{\alpha(a)\alpha(b)uT^2}{1+T}\right] \\
&= \left[1 + \frac{\alpha(a)bT^2}{1+T}\right]
\end{aligned}$$

•

For the time being we will assume that R is commutative and write $^-$ instead of α.

3.2 Lemma.

$$q\colon H^0(R; ^-) \longrightarrow H^0(R; ^-)$$

defined by

$$[x] \longmapsto [x^2]$$

is a homomorphism.

Proof. q is well-defined:

- $x^2 = \overline{x}^2$ for all $x \in R$, satisfying $x = \overline{x}$.
- $(x+\overline{x})^2 = x^2 + x\overline{x} + \overline{x}x + \overline{x}^2 = (x^2 + x\overline{x}) + \overline{(x^2 + x\overline{x})}$, for all $x \in R$.

q is a homomorphism, since for all $x, y \in R$ satisfying $x = \overline{x}$, $y = \overline{y}$

$$\begin{aligned} q([x+y]) &= [(x+y)^2] \\ &= [x^2 + xy + yx + y^2] \\ &= [x^2 + xy + \overline{xy} + y^2] \\ &= [x^2 + y^2] \\ &= q([x]) + q([y]). \end{aligned}$$

•

3.3 Definition. Define $C(R) := \mathrm{Coker}(1+q)$.

3.4 Proposition. *If n is even $(\neq 0)$ or $n = \infty$, then*

$$\lambda\colon H^0(K_1(R_n, \mathcal{I}_n); t_\alpha) \longrightarrow C(R)$$

defined below is an isomorphism.

Proof. We denote by $1+\mathcal{I}_n$ the multiplicative group of units in R_n, which are congruent to 1 modulo $\mathcal{I}_n$. According to [4, theorem 3.2] the homomorphism $(1+\mathcal{I}_n) \to K_1(R_n, \mathcal{I}_n)$ determined by the composition

$$(1+\mathcal{I}_n) \subset (R_n)^* = \mathrm{GL}_1(R_n) \to K_1(R_n, \mathcal{I}_n)$$

is an isomorphism. Since this isomorphism respects the involutions we may and will identify $H^0(K_1(R_n, \mathcal{I}_n); t)$ and $H^0(1+\mathcal{I}_n; \overline{})$.
Define $Z := \{f \in 1+\mathcal{I}_n \mid f = \overline{f}\}$ and $B := \{g\overline{g} \mid g \in 1+\mathcal{I}_n\}$.
If

$$f \equiv 1 + aT + bT^2 \pmod{T^3}$$

for certain $a, b \in R$, then

$$\overline{f} \equiv 1 - \overline{a}T + (\overline{a} + \overline{b})T^2 \pmod{T^3}$$

and

$$f\overline{f} \equiv 1 + (a - \overline{a})T + (\overline{a} - a\overline{a} + b + \overline{b})T^2 \pmod{T^3}.$$

So $f \in Z$ implies $a = \overline{b} - b$. It is easy to verify that the map $Z \to C(R)$ defined by $f \mapsto [b\overline{b}]$ vanishes on B and induces a homomorphism

$$\lambda\colon H^0(1+\mathcal{I}_n; \overline{}) \to C(R).$$

Define

$$\mu\colon C(R) \to H^0(1+\mathcal{I}_n; \overline{})$$

by

$$[z] \mapsto [1 + zT^2/(1+T)].$$

First note that $1 + zT^2/(1+T) \in Z$. We will prove that μ is well-defined. If $[z] = 0$ in $C(R)$, there exist $x, y \in R$ with $y = \overline{y}$, such that $z = x + \overline{x} + y + y^2$. Define

$$f := 1 + zT^2/(1+T) \quad \text{and} \quad g := 1 + yT - (x+y)T^2,$$

then

$$g\overline{g} = 1 - (x + \overline{x} + y + y^2)T^2 \quad \text{and} \quad fg\overline{g} \equiv 1 \pmod{T^3}.$$

So we may assume $f \equiv 1 \pmod{T^3}$.
We assert that $[h] = 1$ for all $h \in Z$ satisfying $h \equiv 1 \pmod{T^3}$.
By induction we assume $k > 0$ and

$$h \equiv 1 + aT^{2k+1} + bT^{2k+2} \pmod{T^{2k+3}},$$

for certain $a, b \in R$. Now

$$\overline{h} \equiv 1 - \overline{a}T^{2k+1} + ((2k+1)\overline{a} + \overline{b})T^{2k+2} \pmod{T^{2k+3}}.$$

So $h \in Z$ implies $(2k+1)a = \overline{b} - b$ and $\overline{a} = -a$.
Defining

$$g := 1 + (b + ka)T^{2k+1} - (k+1)bT^{2k+2},$$

yields

$$\begin{aligned} g\overline{g} &\equiv 1 + ((ka+b) - (k\overline{a} + \overline{b}))T^{2k+1} + \\ &\quad ((2k+1)(k\overline{a} + \overline{b}) - (k+1)(b + \overline{b}))T^{2k+2} \\ &\equiv 1 - aT^{2k+1} - bT^{2k+2} \pmod{T^{2k+3}} \end{aligned}$$

and

$$hg\overline{g} \equiv 1 \pmod{T^{2k+3}}.$$

By induction we find $[h] = 1$.
Thus μ is well-defined. Finally we prove that $\mu = \lambda^{-1}$:
For all $[z] \in C(R)$,

$$\lambda\mu([z]) = \lambda(1 + zT^2/(1+T)) = [z\overline{z}] = [z^2] = [z].$$

For all $f := 1 + aT + bT^2 + \cdots \in Z$,

$$\mu\lambda([f]) = \mu([b\overline{b}]) = \left[1 + b\overline{b}T^2/(1+T)\right],$$

But since

$$f^{-1}(1 + b\overline{b}T^2/(1+T))(1 + \overline{b}T)\overline{(1 + \overline{b}T)} \equiv 1 \pmod{T^3}$$

we may apply the same argument as before to see that $\mu\lambda([f]) = [f]$. •

3.5 Theorem. *The composition of homomorphisms*

$$\mathrm{Arf}^h(R, ^-, u) \subseteq L_0^h(R, ^-, u) \xrightarrow{\omega_1^h} H^0(K_1(R_n, \mathcal{I}_n)) \xrightarrow{\lambda} C(R) \longrightarrow R/\kappa(R),$$

is just the Arf invariant $\mathrm{Arf}^h(R, ^-, u) \to R/\kappa(R)$ *defined in section 2 of the first chapter. Here* $C(R) \to R/\kappa(R)$ *is induced by inclusion.*

Proof. In view of proposition 3.1 we have

$$\begin{aligned}\lambda\omega_1^h((a,b)) &= \lambda\left(\left[1 + \frac{\overline{a}bT^2}{1+T}\right]\right)\\ &= [\overline{a}b].\end{aligned}$$

The rest is clear. •

From now on R is not necessarily commutative. Let $(R, ^-, u)$ be a ring with anti-structure. We wish to prove that the Arf invariant

$$\mathrm{Arf}^h(R, ^-, u) \to R/\kappa(R),$$

we dealt with in section 2 of chapter I, factors through the invariant

$$\omega_1^h \colon \mathrm{Arf}^h(R, ^-, u) \longrightarrow H^0(K_1(R_2, \mathcal{I}_2)).$$

Here follows an attempt to uncover the connection between

$$R/\kappa(R)$$

and the Tate cohomology group

$$H^0(K_1(R_2, \mathcal{I}_2)),$$

in the non-commutative case. Let us fix the following notations.

- A is the truncated polynomial ring R_2.
- $\mathcal{I}$ is the two-sided ideal of A generated by T,
- $^- \colon A \to A$ is the extension of $^-$ on R to A determined by

  $$T \mapsto \frac{-T}{1+T} = -T + T^2,$$

  i.e. $\overline{a + bT + cT^2} = \overline{a} - \overline{b}T + (\overline{b} + \overline{c})T^2$.
- $1 + \mathcal{I}$ denotes the multiplicative group of units in A which are congruent to 1 modulo $\mathcal{I}$.
- We write $W = W(A, \mathcal{I})$ for the subgroup of $1 + \mathcal{I}$ generated by the set $\{(1 + ax)(1 + xa)^{-1} \mid a \in A, x \in \mathcal{I}\}$. According to [21, theorem 2.1] W is the kernel of the surjection $1 + \mathcal{I} \to K_1(A, \mathcal{I})$. We will identify $K_1(A, \mathcal{I})$ and $(1 + \mathcal{I})/W$.

· For all $r, s \in R$ we define $[r,s] := rs - sr$. And $R_{\text{ab}} := R/[R,R]$ the quotient of R as an additive group by the subgroup generated by all $[r,s]$. This is actually the Hochschild homology group $H_0(R)$.

As we saw in section 2 of chapter II the anti-automorphism $\bar{\ }$ of A induces an involution t on the relative K-group $K_1(A, \mathcal{I})$. We want to investigate the structure of the Tate cohomology groups $H^0(K_1(A,\mathcal{I})) \cong H^0((1+\mathcal{I})/W)$. We proceed to take a close look at the group W.

3.6 Lemma. Every element of W has the form

$$1 + \left(\sum_i [u_i, v_i]\right) T + \left(\sum_k [r_k, s_k] + \sum_i u_i v_i [u_i, v_i] + \sum_{i<j} [u_i, v_i][u_j, v_j]\right) T^2.$$

Proof. Substituting $a = a_0 + a_1 T$ and $x = x_1 T + x_2 T^2$ in the expression $(1+ax)(1+xa)^{-1}$ yields

$$\begin{aligned}
&(1 + (a_0 + a_1 T)(x_1 T + x_2 T^2))(1 + (x_1 T + x_2 T^2)(a_0 + a_1 T))^{-1} \\
&\quad = (1 + a_0 x_1 T + (a_0 x_2 + a_1 x_1) T^2)(1 + x_1 a_0 T + (x_1 a_1 + x_2 a_0) T^2)^{-1} \\
&\quad = (1 + a_0 x_1 T + (a_0 x_2 + a_1 x_1) T^2)(1 - x_1 a_0 T + ((x_1 a_0)^2 - x_1 a_1 - x_2 a_0) T^2) \\
&\quad = 1 + [a_0, x_1] T + ([a_0, x_2] + [a_1, x_1] + [x_1, a_0] x_1 a_0) T^2.
\end{aligned}$$

When $a_0 = 0$ we obtain elements like

$$1 + [r,s] T^2$$

and modulo such elements we find expressions of the form

$$1 + [u,v] T + uv[u,v] T^2.$$

Note that

$$(1 + [u,v]T + uv[u,v]T^2)^{-1} = 1 + [v,u]T + vu[v,u]T^2.$$

Thus W is generated by

$$\left\{1 + [u,v]T + uv[u,v]T^2,\ 1 + [r,s]T^2 \ \mid r,s,u,v \in R\right\}.$$

Writing out a product of such elements yields the desired result. •

We also need the Hochschild homology group $H_1(R)$. We refer to chapter III for the definitions. The Hochschild homology group $H_1(R)$ and the cyclic homology group $HC_1(R)$ are defined as:

$$H_1(R) := \frac{\operatorname{Ker}(b\colon R \otimes R \to R)}{\operatorname{Im}(b\colon R \otimes R \otimes R \to R)}$$

$$HC_1(R) := \frac{\operatorname{Ker}(b\colon R \otimes R \to R)}{\operatorname{Im}(b\colon R \otimes R \otimes R \to R) + \operatorname{Im}(1 - x)},$$

where

$$b(u \otimes v) := [u,v], \quad b(u \otimes v \otimes w) := uv \otimes w - u \otimes vw + wu \otimes v$$

$$\text{and} \quad x\colon R \otimes R \to R \otimes R \ \text{ is defined by } \ x(u \otimes v) := -v \otimes u.$$

3.7. Define $\theta\colon R \otimes R \to R_{\mathrm{ab}}$ by

$$\theta(\sum_i u_i \otimes v_i) := \sum_{i<j} [u_i, v_i][u_j, v_j] + \sum_i u_i v_i [u_i, v_i].$$

θ is well-defined in the sense that the right-hand side does not depend on the order of summation in $\sum_i u_i \otimes v_i$. Observe that

$$\theta(x+y) = \theta(x) + \theta(y) + b(x) \cdot b(y).$$

for all $x, y \in R \otimes R$. So the restriction of θ to $\mathrm{Ker}(b)$ is a homomorphism. Furthermore it is easy to verify that θ vanishes on $\mathrm{Im}(b)$ and $\mathrm{Im}(1-x)$. Consequently θ induces a homomorphism $\theta'\colon HC_1(R) \to R_{\mathrm{ab}}$.

In view of the preceding it is clear that the sequence

$$\begin{array}{ccccccccc} HC_1(R) & \xrightarrow{\theta'} & R_{\mathrm{ab}} & \longrightarrow & K_1(A, \mathcal{I}) & \longrightarrow & R_{\mathrm{ab}} & \longrightarrow 0 \\ & & [s] & \longmapsto & [1+sT^2] & & & \\ & & & & [1+aT+bT^2] & \longmapsto & [a] & \end{array}$$

is exact.
The anti-automorphism $\overline{}\colon R \to R$ induces an involution on R_{ab}:

$$\overline{[u,v]} = [\overline{v}, \overline{u}]$$

$$[\overline{\overline{r}}] = [uru^{-1}] = [r] \quad \text{ in } R_{\mathrm{ab}}.$$

Furthermore $\mathrm{Im}(\theta')$ is invariant under this involution. When we equip R_{ab} on the left-hand side with this involution and R_{ab} on the right-hand side with the involution $[a] \mapsto [-\overline{a}]$, we obtain the short exact sequence of groups with involutions

$$0 \longrightarrow \mathrm{Coker}(\theta') \longrightarrow K_1(A, \mathcal{I}) \longrightarrow R_{\mathrm{ab}} \longrightarrow 0$$

which gives rise to the six-term exact sequence

$$\begin{array}{ccccc} H^0(\mathrm{Coker}(\theta')) & \longrightarrow & H^0(K_1(A, \mathcal{I})) & \longrightarrow & H^1(R_{\mathrm{ab}}) \\ {\scriptstyle\delta}\uparrow & & & & \downarrow \\ H^0(R_{\mathrm{ab}}) & \longleftarrow & H^1(K_1(A, \mathcal{I})) & \longleftarrow & H^1(\mathrm{Coker}(\theta')). \end{array}$$

We compute the differential map $\delta\colon H^0(R_{\mathrm{ab}}) \to H^0(\mathrm{Coker}(\theta'))$.

3.8 Lemma. If $[a] \in H^0(R_{\text{ab}})$, i.e. $\overline{a} - a = b(x)$ for some $x \in R \otimes R$, then

$$\delta([a]) = [a + a\overline{a} + \theta(x)].$$

Proof. The element $1 + aT$ is a lift of a in $K_1(A, \mathcal{I})$. And in $K_1(A, \mathcal{I})$ we have

$$\begin{aligned}
(1+aT)\overline{(1+aT)} &= (1+aT)(1-\overline{a}T+\overline{a}T^2) \\
&= 1+(a-\overline{a})T+(\overline{a}-a\overline{a})T^2 \\
&= (1+(a-\overline{a})T+(\overline{a}-a\overline{a})T^2)(1+b(x)T+\theta(x)T^2) \\
&= 1+((a-\overline{a})(\overline{a}-a)+\overline{a}-a\overline{a}+\theta(x))T^2 \\
&= 1+((a-\overline{a})(\overline{a}-a)+\overline{a}-a\overline{a}+\theta(x))T^2.
\end{aligned}$$

But this is the image of

$$\begin{aligned}
[(a-\overline{a})(\overline{a}-a)+\overline{a}-a\overline{a}+\theta(x)] &= [\overline{a}+\overline{a}a+\theta(x)] \\
&= [a+a\overline{a}+\theta(x)]
\end{aligned}$$

in $H^0(\text{Coker}(\theta'))$. •

Now we specialize to the case that R is the group ring $\mathbf{Z}[G]$ of an arbitrary group G.

3.9 Lemma. $\theta' = 0$

Proof. Every cycle of $HC_1(R)$ can be written as

$$\left[\sum_i g_i \otimes h_i\right],$$

by using the relation $g \otimes h + h \otimes g = 0$. The condition for this element to be a cycle reads $\sum g_i h_i = \sum h_i g_i$. Such an cycle can be decomposed as a sum of cycles of the form

$$[g \otimes h] \quad \text{with} \quad gh = hg$$

or of the form

$$\left[\sum_i^n g_i \otimes h_i\right] \quad \text{with} \quad g_i h_i = \begin{cases} h_{i+1}g_{i+1} & \text{for } i < n \\ h_1 g_1 & \text{for } i = n \end{cases}.$$

The homomorphism θ' is obviously zero on elements of the first type. As far as the second type is concerned we have the following identities in R_{ab}

$$\begin{aligned}
\theta'\left(\left[\sum g_i \otimes h_i\right]\right) &= \sum_{i<j}[g_i,h_i][g_j,h_j] + \sum_i g_i h_i [g_i,h_i] \\
&= \sum_{i<j} g_i h_i g_j h_j + \sum_{i<j} h_i g_i h_j g_j + \sum_i g_i h_i g_i h_i +
\end{aligned}$$

$$
\begin{aligned}
& -\sum_{i<j} h_i g_i g_j h_j - \sum_{i<j} g_i h_i h_j g_j - \sum_{i} g_i h_i h_i g_i \\
= & \sum_{i<j} g_i h_i g_j h_j + \sum_{i<j} g_i h_i g_j h_j + \sum_{i} g_i h_i g_i h_i + \\
& -\sum_{i<j} g_j h_j h_i g_i - \sum_{i<j} g_i h_i h_j g_j - \sum_{i} g_i h_i h_i g_i \\
= & \sum_{i,j} g_i h_i g_j h_j - \sum_{i,j} g_j h_j h_i g_i \\
= & \left(\sum g_i h_i\right)^2 - \left(\sum g_i h_i\right)\left(\sum h_i g_i\right) \\
= & \ 0
\end{aligned}
$$

This proves the lemma. •

The next move is to figure out what $\delta\colon H^0(R_{\mathrm{ab}}) \longrightarrow H^0(R_{\mathrm{ab}})$ looks like in this case. Suppose we are given an element $[a] \in H^0(R_{\mathrm{ab}})$. Then we may assume that $a = \sum g_i$ by using the fact that $[g + g^{-1}] = 0$ in $H^0(R_{\mathrm{ab}})$. The condition for a to be a cycle reads

$$\sum g_i - g_i^{-1} = \sum h_j - h_j',$$

where $h_j \in G$ and h_j' is a conjugate of h_j. From this we conclude that every g_i is conjugated to some g_j^{-1}. Note that $[g + h^{-1}g^{-1}h] = [g + g^{-1}] = 0$ in $H^0(R_{\mathrm{ab}})$. Thus it suffices to consider the case that $a = g$ where $g = h^{-1}g^{-1}h$. We follow lemma 3.8. Now $g^{-1} - g = [h^{-1}, gh]$, so

$$
\begin{aligned}
\delta([g]) &= [g + gg^{-1} + \theta([h^{-1} \otimes gh])] \\
&= [g + 1 + h^{-1}gh(g^{-1} - g)] \\
&= [g + 1 + g^{-1}(g^{-1} - g)] \\
&= [g + g^{-2}] \\
&= [g + g^2].
\end{aligned}
$$

As a consequence we have

$$\mathrm{Coker}(\delta) = \frac{\{a \in \mathbf{Z}[G]_{\mathrm{ab}} \mid a = \overline{a}\}}{\mathrm{Span}\{g - h^{-1}gh, g_1 + g_1^{-1}, g_2 + g_2^2 \mid g_2 \sim g_2^{-1}\}}.$$

Our main conclusion is that in the case of a group ring the invariant

$$\omega_1^h\colon \mathrm{Arf}^h(R, {}^-, u) \longrightarrow H^0(K_1(A, \mathcal{I}))$$

factors through an injective homomorphism

$$\mathrm{Coker}(\delta\colon H^0(R_{\mathrm{ab}}) \to H^0(R_{\mathrm{ab}})) \hookrightarrow H^0(K_1(A, \mathcal{I}))$$

and that there is a homomorphism

$$\mathrm{Coker}(\delta) \longrightarrow R/\kappa(R).$$

4 Computations on the invariant ω_2.

In order to study the invariant ω_2, we wish to compute the cokernel of the homomorphism

$$d\colon H^1(K_1(R_n, \mathcal{I}_n); t_\alpha) \to H^1(K_2(R_n, \mathcal{I}_n); t_\alpha).$$

We confine our inquiries to the case where R is commutative, for then we have the following theorem at our disposal.

4.1 Theorem. *Let R be a commutative ring with identity and I an ideal contained in the Jacobson radical of R. Then $K_2(R, I)$ is isomorphic to the abelian group with presentation:*

generators:	$<a, b>$	*with* $a \in I$ *or* $b \in I$
relations:	$<a, b> = -<b, a>$	*if* $a \in I$ *or* $b \in I$
	$<a, b> + <a, c> = <a, b + c - abc>$	*if* $a \in I$ *or* $b, c \in I$
	$<a, bc> = <ab, c> + <ac, b>$	*if* $a \in I$ *or* $b \in I$ *or* $c \in I$.

The isomorphism maps $<a, b>$ to the Dennis-Stein element $<a, b>_\circ \in K_2(R, I)$.

Proof. See [18, 14]. •

A little digression seems in order. We refer to [19, §9] and [7] for more background.
Let $n > 2$.
For any unit $r \in R$ one has the elements $w_{ij}(r) := x_{ij}(r)x_{ji}(-r^{-1})x_{ij}(r)$ and $h_{ij}(r) := w_{ij}(r)w_{ij}(-1)$ in $\mathrm{St}_n(R)$, where i and j are distinct integers between 1 and n.
Further, for every couple of units $r, s \in R$,

$$h_{ij}(rs)h_{ij}^{-1}(r)h_{ij}^{-1}(s) \in \mathrm{St}_n(R)$$

determines an element $\{r, s\}$ in $K_2(R)$, which does not depend on i or j.
And for all $a, b \in R$ such that $1 - ab$ is a unit of R,

$$x_{ji}(-b(1-ab)^{-1})x_{ij}(-a)x_{ji}(b)x_{ij}(a(1-ab)^{-1})h_{ij}^{-1}(1-ab) \in \mathrm{St}_n(R)$$

determines the Dennis-Stein element $<a, b>_\circ \in K_2(R)$ which does not depend on i or j either. Note the sign conventions.
In $K_2(R)$ the following relations hold, whenever the left-hand side is defined.

$$\begin{aligned}
\{r_1 r_2, s\} &= \{r_1, s\}\{r_2, s\} \\
\{r, s\} &= \{s, r\}^{-1} \\
\{r, -r\} &= 1 \\
\{r, 1-r\} &= 1 \\
<a, b>_\circ &= <b, a>_\circ{}^{-1} \\
<a, b>_\circ <a, c>_\circ &= <a, b + c - abc>_\circ
\end{aligned}$$

$$\begin{aligned} <a, bc>_\circ &= <ab, c>_\circ <ac, b>_\circ \\ <0, a>_\circ &= 1 \\ \{r, s\} &= <(1-r)s^{-1}, s>_\circ . \end{aligned}$$

Note that we used an additive notation in dealing with the symbols $<\,,\,>$ and a multiplicative notation for the corresponding Dennis-Stein elements $<\,,\,>_\circ$. Nevertheless we will often omit the $\circ$.

4.2 Proposition. *Let R be a commutative ring and $\overline{}\colon R \to R$ an involution. The involution t on $K_2(R)$ induced by $\overline{}$ satisfies*

$$t(<a, b>_\circ) = <\overline{b}, \overline{a}>_\circ .$$

Proof. We will work in $\mathrm{St}_{2n}(R)$. We drop the decorations of the anti-involution on the Steinberg group and simply write t. From definition 1.14 and 1.38 of the first chapter we deduce

$$t(x_{ij}(a)) = x_{n+j\,n+i}(\overline{a}),$$

provided that i and j do not exceed n.
Thus $t(w_{12}(r)) = w_{n+2\,n+1}(\overline{r})$ and

$$\begin{aligned} t(h_{12}^{-1}(r)) &= w_{n+2\,n+1}^{-1}(\overline{r}) w_{n+2\,n+1}^{-1}(-1) \\ &= w_{n+2\,n+1}(-\overline{r}) w_{n+2\,n+1}(1) && (1) \\ &= w_{n+1\,n+2}(\overline{r}^{-1}) w_{n+1\,n+2}(-1) && (2) \\ &= h_{n+1\,n+2}(\overline{r}^{-1}) \end{aligned}$$

In (1) we used the relation $w_{ij}(r) = w_{ij}^{-1}(-r)$ and (2) follows from the relation $w_{ij}(r) = w_{ji}(-r^{-1})$. See [19, lemma 9.5]. Hence

$$\begin{aligned} t(<a, b>_\circ) &= h_{n+1\,n+2}((1-\overline{ab})^{-1}) x_{n+2\,n+1}(\overline{a}(1-\overline{ab})^{-1}) x_{n+1\,n+2}(\overline{b}) \cdot \\ &\quad\ x_{n+2\,n+1}(-\overline{a}) x_{n+1\,n+2}(-\overline{b}(1-\overline{ab})^{-1}) \\ &= h_{n+1\,n+2}((1-\overline{ab})^{-1}) <-\overline{b}, -\overline{a}>_\circ h_{n+1\,n+2}(1-\overline{ab}) \\ &= <-\overline{b}, -\overline{a}>_\circ \{(1-\overline{ab})^{-1}, 1-\overline{ab}\} \\ &= <\overline{b}, \overline{a}>_\circ <-\overline{ab}, -1>_\circ \{(1-\overline{ab})^{-1}, -1\} \\ &= <\overline{b}, \overline{a}>_\circ \end{aligned}$$

which proves the assertion. •

To make life more congenial, we will assume R to carry some additional structure. In that way $H^1(K_2(R_n, \mathcal{I}_n))$ becomes fairly accessible for computations by the techniques of [6]. The following definition occurs implicitly in [13] and [12]. It describes a notion of what one could call 'partial λ-ring'.

4.3 Definition. Let R be a commutative ring with identity and $k \in \mathbb{N} \cup \{\infty\}$. A $k\lambda$-ring structure on R consists of operations $\theta^p\colon R \to R$, for every prime number $p \leq k$, which satisfy the following conditions

1) $\theta^p(1) = 0$ for all $p \leq k$

2) $\theta^p(a+b) = \theta^p(a) + \theta^p(b) + \sum_{k=1}^{p-1} \frac{1}{p}\binom{p}{k} a^k b^{p-k}$ for all $p \leq k$

3) $\theta^p(ab) = \theta^p(a)b^p + \theta^p(b)a^p - p\theta^p(a)\theta^p(b)$ for all $p \leq k$

4) $\theta^p(\psi^q(a)) = \psi^q(\theta^p(a))$ for all $p, q \leq k$

here ψ^q is defined by $\psi^q(a) := a^q - q\theta^q(a)$.
We then call R an $k\lambda$-ring.

4.4 Remark. It is easy to verify that multiplication by p transforms the equations 1 to 4 into

1') $\psi^p(1) = 1$ for all $p \leq k$

2') $\psi^p(a+b) = \psi^p(a) + \psi^p(b)$ for all $p \leq k$

3') $\psi^p(ab) = \psi^p(a)\psi^p(b)$ for all $p \leq k$

4') $\psi^p(\psi^q(a)) = \psi^q(\psi^p(a))$ for all $p, q \leq k$.

Thus the so called Adams operations ψ_p are ringhomomorphisms, which satisfy the compatibility conditions 4'.
Conversely, if R is a torsion-free commutative ring equipped with ψ_p satisfying 1' to 4' such that $\psi_p(a) \equiv a^p \pmod{pR}$ for all $p \leq k$, then R becomes a $k\lambda$-ring in the obvious way and the ψ_p are the associated Adams operations.
As far as the references to [13] and [12] are concerned, a few remarks are in order.

- We point out the differences in sign conventions between the definition in [12] and the one above.
- Condition 4 in our list is equivalent to what is called the permutability of θ_p and θ_q in [13].

The terminology is explained by the following theorem.

4.5 Theorem. [13, theorem 3]. *The notions λ-ring and $\infty\lambda$-ring coincide.*

4.6 Lemma. Any structure of $k\lambda$-ring on a ring R admits a unique extension to the rings $R[T]$ and R_n for all $n \in \mathbb{N} \cup \{\infty\}$, under the condition that $\theta_p(T) = 0$ for all $p \leq k$.

Proof. There exists a unique $k\lambda$-ring structure on the ring of integers $\mathbf{Z}$ defined by $\psi_p := 1$.
Since the polynomial ring $\mathbf{Z}[T]$ has no torsion and the condition $\theta_p(T) = 0$ implies $\psi_p(T) = T^p$, the formula $\psi_p(\sum a_i T^i) = \sum a_i T^{ip}$, determines a unique structure of $k\lambda$-ring on $\mathbf{Z}[T]$.
We now call upon [12, theorem 3], which reads as follows. If R_1 and R_2 are $k\lambda$-rings, then $R_1 \otimes R_2$ can be provided with a unique structure of $k\lambda$-ring, such that the canonical maps $R_1 \to R_1 \otimes R_2$ and $R_2 \to R_1 \otimes R_2$ preserve every θ_p. Applying this theorem in our situation, proves the assertion for the ring $R[T] = R \otimes \mathbf{Z}[T]$.
From condition 2 in definition 4.3 we deduce that $f \equiv g \pmod{T^l R[T]}$ implies

$\theta_p(f) \equiv \theta_p(g) \pmod{T^l R[T]}$. Consequently the $k\lambda$-ring structure on $R[T]$ extends uniquely to the rings R_n for all $n \in \mathbb{N} \cup \{\infty\}$. •

4.7 Definition. Let $\delta: R \to \Omega_R$ be the universal derivation on R and define $\Omega_{R_n,\mathcal{I}_n} := \mathrm{Ker}(\Omega_{R_n} \to \Omega_R)$.
Define recursively

$$\begin{aligned} \Omega(R,1) &:= \Omega_R, \\ \Omega(R,n+1) &:= \Omega(R,n) \oplus \frac{R \oplus \Omega_R}{\mathrm{Span}\{((n+1)a, \delta a) \mid a \in R\}}. \end{aligned}$$

Define

$$\widetilde{\Omega}(R,n) := \begin{cases} \Omega(R,n) \oplus \dfrac{R}{(n+1)R} & \text{if } n \text{ is odd} \\ \Omega(R,n) & \text{if } n \text{ is even.} \end{cases}$$

Define

$$\widetilde{K_2}(R_n,\mathcal{I}_n) := \begin{cases} K_2(R_n,\mathcal{I}_n) & \text{if } n \text{ is odd} \\ \dfrac{K_2(R_n,\mathcal{I}_n)}{\mathrm{Span}\{<aT^n, T> \mid a \in R\}} & \text{if } n \text{ is even.} \end{cases}$$

4.8 Lemma. As R-modules

$$\frac{\Omega_{R_n,\mathcal{I}_n}}{\delta\mathcal{I}_n} = \Omega(R,n) \oplus \frac{R}{(n+1)R}$$

Proof. Write J for the ideal of $R[T]$ generated by T^{n+1}. We have

$$\Omega_{R_n} = \frac{\Omega_{R[T]}}{J\Omega_{R[T]} + \delta J}$$

and as R-modules

$$\Omega_{R[T]} = (R \otimes_{\mathbf{Z}} \Omega_{\mathbf{Z}[T]}) \oplus (R[T] \otimes_R \Omega_R),$$

So

$$\Omega_{R_n,\mathcal{I}_n} = \underbrace{(R \oplus \Omega_R) \oplus \cdots \oplus (R \oplus \Omega_R)}_{n \quad \text{copies}} \oplus \frac{R}{(n+1)R}.$$

Dividing out $\delta\mathcal{I}_n$ yields the desired result. •

We are now in the position to apply the machinery of [6] to our situation. As a matter of fact, the construction in *loc. cit.* yields a homomorphism

$$\nu_n: K_2(R_n,\mathcal{I}_n) \to \frac{\Omega_{R_n,\mathcal{I}_n}}{\delta\mathcal{I}_n},$$

even when R possesses a $(n+1)\lambda$-ring structure. In view of lemma 4.8 we obtain a homomorphism

$$\nu_n: K_2(R_n,\mathcal{I}_n) \to \Omega(R,n) \oplus \frac{R}{(n+1)R}.$$

Furthermore we obtain a homomorphism

$$\widetilde{\nu_n}\colon \widetilde{K_2}(R_n,\mathcal{I}_n) \to \widetilde{\Omega}(R,n)$$

whenever R is a $n\lambda$-ring ($n > 1$).

4.9 Theorem. *ν_n and $\widetilde{\nu_n}$ are isomorphisms.*

Proof. We refer to *loc. cit.* for the definitions of the ν_n. We proceed by applying induction on n.
$n = 1$: R is a 2λ-ring and $\nu_1\colon K_2(R_1,\mathcal{I}_1) \to \Omega_R \oplus \frac{R}{2R}$ is determined by

$$\nu_1 <aT,b> = (a\delta b, [a^2\theta^2(b)]), \quad \nu_1 <cT,T> = (0,[c]).$$

It is straightforward to check that $\nu_1^{-1}\colon \Omega_R \oplus \frac{R}{2R} \to K_2(R_1,\mathcal{I}_1)$ is well defined by

$$\nu_1^{-1}(a\delta b,[c]) = <aT,b> + <a^2\theta^2(b)T,T> + <cT,T>$$

$n > 1$: Consider the diagram

$$\begin{array}{ccccccccc}
 & & & & K_2(R_n,\mathcal{I}_n^n) & & & & \\
 & & & & \big\downarrow{\scriptstyle\tau} & & & & \\
0 & \to & \dfrac{R\oplus\Omega_R}{(na,\delta a)} & \xrightarrow{\iota} & K_2(R_n,\mathcal{I}_n) & \xrightarrow{\kappa} & \Omega(R,n-1)\oplus\frac{R}{(n+1)R} & \to & 0 \\
 & & \big\downarrow{\scriptstyle\pi} & & \big\downarrow{\scriptstyle\chi} & & \big\downarrow{\scriptstyle\pi} & & \\
0 & \to & \dfrac{R}{nR} & \xrightarrow{\iota} & K_2(R_{n-1},\mathcal{I}_{n-1}) & \xrightarrow{\kappa} & \Omega(R,n-1) & \to & 0
\end{array}$$

Here χ and τ are the obvious maps and $\mathrm{Ker}(\chi) = \mathrm{Im}(\tau)$. In the top row κ is the obvious direct summand of ν_n and $\iota([a,b\delta c]) = <aT^{n-1},T> + <bT^n,c>$. In the bottom row κ is the obvious direct summand of ν_{n-1} and $\iota([a]) = <aT^{n-1},T>$. The maps denoted by π are the cononical projections. We compute

$$\nu_n <aT^n,b> = [0,a\delta b] \in \frac{R\oplus\Omega_R}{(na,\delta a)}$$
$$\nu_n <aT^{n-1},T> = [a,0] \in \frac{R\oplus\Omega_R}{(na,\delta a)}$$
$$\nu_n <aT^n,T> = [a] \in \frac{R}{(n+1)R}$$

Therefore the map ι in the top row is split by the remaining summand of ν_n; and since $\pi\nu_n = \nu_{n-1}\chi$ this implies that the map ι in the bottom row is split by the remaining summand of ν_{n-1}. The bottom row is exact by the induction hypothesis.
Suppose that $x \in K_2(R_n,\mathcal{I}_n)$ and

$$\kappa(x) = 0 \in \Omega(R,n-1) \oplus \frac{R}{(n+1)R} \quad (\star).$$

Then there exists a $y \in \frac{R\oplus\Omega_R}{(na,\delta a)}$ such that $\iota(\pi(y)) = \chi(x)$. The exactness of the column guarantees the existence of an element $z \in K_2(R_n, \mathcal{I}_n^n)$ satisfying $x - \iota(y) = \tau(z)$. Thus there exists an $r \in R$ such that $x + <rT^n, T> \in \mathrm{Im}(\iota)$. But $[r] = 0 \in \frac{R}{(n+1)R}$ because of $(\star)$. So $x \in \mathrm{Im}(\iota)$. This proves that ν_n is an isomorphism.

If n is odd and $n > 1$, then the notions $n\lambda$-ring and $(n+1)\lambda$-ring coincide and the preceding proves that $\widetilde{\nu_n}$ is an isomorphism. For n even consider the following diagram.

$$\begin{array}{ccccccccc}
0 & \to & \dfrac{R\oplus\Omega_R}{(na,\delta a)} & \xrightarrow{\tilde{\iota}} & \widetilde{K_2}(R_n,\mathcal{I}_n) & \xrightarrow{\tilde{\kappa}} & \Omega(R,n-1) & \to & 0 \\
 & & \downarrow \pi & & \downarrow \tilde{\chi} & & \downarrow 1 & & \\
0 & \to & \dfrac{R}{nR} & \xrightarrow{\iota} & K_2(R_{n-1},\mathcal{I}_{n-1}) & \xrightarrow{\kappa} & \Omega(R,n-1) & \to & 0
\end{array}$$

and proceed as before. •

Corollary. If R possesses a structure of $n\lambda$-ring, then $H^1(K_2(R_n, \mathcal{I}_n); t)$ is isomorphic to $H^1(\widetilde{\Omega}(R,n); \widetilde{\nu_n} t \widetilde{\nu_n}^{-1})$

Proof. Note that $t<aT^n, T> = <\overline{a}T^n, T>$ in $K_2(R_n, \mathcal{I}_n)$ and $<aT^n, T>$ is an odd torsion element of $K_2(R_n, \mathcal{I}_n)$ if n is even. But odd torsion elements vanish in $H^1(K_2(R_n, \mathcal{I}_n))$, so $H^1(K_2(R_n, \mathcal{I}_n)) \cong H^1(\widetilde{K_2}(R_n, \mathcal{I}_n))$. In view of the preceding theorem this yields the desired result. •

This enables us to compute these cohomology groups in the cases where $\widetilde{\nu_n}$ is manageable. The next theorem for instance, shows what these groups look like for $n = 1$ and $n = 2$. For all abelian groups A and numbers k we write ${}_kA$ to denote $\{a \in A \mid ka = 0\}$.

4.10 Theorem. *Let R be a 2λ-ring and $\overline{}: R \to R$ the identity. Then*

$$\begin{aligned}
H^1(K_2(R_1, \mathcal{I}_1)) &\cong \frac{R}{2R} \oplus {}_2(\Omega_R) \\
H^1(K_2(R_2, \mathcal{I}_2)) &\cong \{\alpha \in {}_2(\Omega_R) \mid (1+\phi^2)\alpha \in \delta({}_2R)\} \\
&\quad \oplus \frac{R}{2R} \oplus \frac{\Omega_R}{2\Omega_R + \delta R + \mathrm{Im}(1+\phi^2)}
\end{aligned}$$

where $\phi^2: \Omega_R \to \Omega_R$ is given by $\phi^2(a\delta b) = \psi^2(a)(b\delta b - \delta\theta^2(b))$.

Proof. Again we refer to [6], for more details on the operations ϕ^2. According to proposition 4.2

$$t(<aT, b>) = <b, -aT> = <aT, b>$$

and

$$t(<aT, T>) = <-T, aT> = <aT, T>$$

in $K_2(R_1, \mathcal{I}_1)$. So in view of the corollary to theorem 4.9 we have

$$H^1(K_2(R_1, \mathcal{I}_1)) \cong H^1(\frac{R}{2R} \oplus \Omega_R; 1).$$

The isomorphism

$$\widetilde{\nu_2}\colon K_2(R_2, \mathcal{I}_2) \to \Omega_R \oplus \frac{R \oplus \Omega_R}{(2a, \delta a)}$$

is given by

$$\begin{aligned}
\widetilde{\nu_2}(<aT, b>) &= (a\delta b, [a^2\theta^2(b), (a^2 - \theta^2(a))\delta\theta^2(b) + \theta^2(a)b\delta b + \theta^2(b)a\delta a]),\\
\widetilde{\nu_2}(<aT, T>) &= (0, [a, 0]),\\
\widetilde{\nu_2}(<aT^2, b>) &= (0, [0, a\delta b]).
\end{aligned}$$

Using proposition 4.2 we compute

$$\widetilde{\nu_2} t \widetilde{\nu_2}^{-1}(\alpha, [b, \gamma]) = (\alpha, [-b, -(1 + \phi^2)(\alpha) - \gamma]).$$

Hence

$$\mathrm{Ker}(1 + \widetilde{\nu_2} t \widetilde{\nu_2}^{-1}) = \{(\alpha, [b, \gamma]) \mid 2\alpha = 0 \text{ and } [0, (1 + \phi^2)(\alpha)] = [0, 0]\},$$
$$\mathrm{Im}(1 - \widetilde{\nu_2} t \widetilde{\nu_2}^{-1}) = \{(0, [2b, 2\gamma + (1 + \phi^2)(\alpha)])\}$$

and the quotient of these groups equals the right-hand-side of the second isomorphism. •

As far as stability is concerned we have:

4.11 Proposition. *Let $n \neq 0$ be even. If R is a $(n+2)\lambda$-ring and $^- = 1$, then*

$$H^1(\mathrm{Ker}(\widetilde{K_2}(R_{n+2}, \mathcal{I}_{n+2}) \to \widetilde{K_2}(R_n, \mathcal{I}_n))) \;\cong\; \frac{{}_{n+2}\mathrm{Ker}(2\delta)}{{}_{n+2}\mathrm{Ker}(\delta)} \oplus \frac{R}{2R}.$$

Proof. Consider the exact sequence

$$0 \to \frac{R \oplus \Omega_R}{((n+1)a, \delta a)} \oplus \frac{R \oplus \Omega_R}{((n+2)a, \delta a)} \xrightarrow{\widetilde{\iota}} \widetilde{K_2}(R_{n+2}, \mathcal{I}_{n+2}) \to \widetilde{K_2}(R_n, \mathcal{I}_n) \to 0,$$

where $\widetilde{\iota}$ is defined by

$$\widetilde{\iota}([a, b\delta c], [x, y\delta z]) = <aT^n, T> + <bT^{n+1}, c> + <xT^{n+1}, T> + <yT^{n+2}, z>.$$

A splitting σ of $\widetilde{\iota}$ is given by the appropriate direct summand of $\widetilde{\nu}_{n+2}$. The involution t on both $\widetilde{K_2}$-groups induces the involution $\sigma t \widetilde{\iota}$ on

$$\frac{R \oplus \Omega_R}{((n+1)a, \delta a)} \oplus \frac{R \oplus \Omega_R}{((n+2)a, \delta a)}.$$

A little computation shows that

$$\sigma t \widetilde{\iota}([a, \alpha], [b, \beta]) = ([a, \alpha], [-b, \delta a - (n+1)\alpha - \beta]).$$

Now $([a,\alpha],[b,\beta]) \in \mathrm{Ker}(1+\sigma t\widetilde{\iota})$, if and only if $([2a,2\alpha],[0,\delta a-(n+1)\alpha])=0$. Thus putting $n=2m$, there exist $r,s \in R$ satisfying the relations:

$$\begin{aligned}
2a &= (2m+1)r,\\
2\alpha &= \delta r,\\
(2m+2)s &= 0 \text{ and}\\
\delta s &= \delta a-(2m+1)\alpha.
\end{aligned}$$

Hence $[a,\alpha]=[a,\delta a-\delta s-m\delta r]=[a+(2m+1)mr,\delta(a-s)]=[0,-\delta s]=[-s,0]$ and $2\delta s=(2m+2)s=0$.
Conversely, if $[a,\alpha]=[s,0]$ for some $s\in R$ satisfying $2\delta s=(2m+2)s=0$, then $([a,\alpha],[b,\beta]) \in \mathrm{Ker}(1+\sigma t\widetilde{\iota})$.
The observation that $\mathrm{Im}(1-\sigma t\widetilde{\iota})=\{([0,0],[2b,\beta'])\}$ completes the proof. •

The final contribution to the comprehension of the value group of ω_2 comes from the following proposition.

4.12 Proposition. Compare [9, theorem 4.1.]. *Let $(R,^{-},u)$ be a commutative ring with antistructure. If n is even,*

$$d\colon H^1(K_1(R_n,\mathcal{I}_n)) \longrightarrow H^1(K_2(R_n,\mathcal{I}_n))$$

assigns to the class $[x]$ of the element $x\in 1+\mathcal{I}_n$ the class $[\{x,-u\}]$. Recall that we identified $H^1(K_1(R_n,\mathcal{I}_n))$ and $H^1(1+\mathcal{I}_n)$.

Proof. We will work in $\mathrm{GL}_{2k}(R_n)$ and $\mathrm{St}_{2k}(R_n)$.
Suppose $x\in 1+\mathcal{I}_n$ and $\overline{x}=x^{-1}$. Let X be the image of x under the map $1+\mathcal{I}_n \longrightarrow \mathrm{GL}_1(R_n) \hookrightarrow \mathrm{GL}_k(R_n)$. By definition $t_{^{-},u_n}(X)X=\begin{pmatrix} X & 0\\ 0 & X^{-1}\end{pmatrix}$ and $h_{1\,k+1}(x)$ is a lift of this element in $\mathrm{St}_{2k}(R_n)$. According to lemma 2.5

$$d([x])=d([X])=[h^{-1}_{1\,k+1}(x)\,t_{^{-},u_n}(h_{1\,k+1}(x))].$$

But from the definition of $t_{^{-},u_n}$ we compute

$$\begin{aligned}
t_{^{-},u_n}(h_{1\,k+1}(x)) &= t_{^{-},u_n}(w_{1\,k+1}(x)w_{1\,k+1}(-1))\\
&= w_{1\,k+1}(-u_n^{-1})w_{1\,k+1}(u_n^{-1}\overline{x})\\
&= w_{1\,k+1}(-u_n^{-1})w_{1\,k+1}(-1)w_{1\,k+1}(1)w_{1\,k+1}(u_n^{-1}x^{-1})\\
&= h_{1\,k+1}(-u_n^{-1})h^{-1}_{1\,k+1}(-u_n^{-1}x^{-1}).
\end{aligned}$$

Thus

$$\begin{aligned}
d([x]) &= [h^{-1}_{1\,k+1}(x)h_{1\,k+1}(-u_n^{-1})h^{-1}_{1\,k+1}(-u_n^{-1}x^{-1})]\\
&= [\{x,u_n\}]\\
&= [\{x,-u\}\{x,-(1+T)\}].
\end{aligned}$$

It remains to show that $\{x, -(1+T)\}$ vanishes in $H^1(K_2(R_n, \mathcal{I}_n))$. First note that $\{x, -u\}$ is a cycle:

$$\begin{aligned} t(\{x,-u\}) &= t(<-u^{-1}(1-x), -u>) \\ &= <-u^{-1}, -u(1-x^{-1})> \\ &= \{-u^{-1}, x^{-1}\} \\ &= \{x, -u\}^{-1}. \end{aligned}$$

Now choose $y \in R_n$ such that $1 - x^{-1} = yT$. So $1 - x = -\overline{y}T(1+T)^{-1}$. We compute

$$\begin{aligned} (1-t)(<T,y>) &= <T,y><-T(1+T)^{-1}, \overline{y}> \\ &= <T,y><T,-(1+T)^{-1}\overline{y}><-(1+T)^{-1}, \overline{y}T> \\ &= <T, y-(1+T)^{-1}\overline{y} + y\overline{y}T(1+T)^{-1}> \cdot \\ &\quad\; <-(1+T)^{-1}, (1+T)(x-1)> \\ &= <T, y-(1+T)^{-1}\overline{y} + y\overline{y}T(1+T)^{-1}>\{x, -(1+T)\}. \end{aligned}$$

But since

$$(y - (1+T)^{-1}\overline{y} + y\overline{y}T(1+T)^{-1})T = 1 - x^{-1} + 1 - x + (1-x^{-1})(x-1) = 0,$$

we have

$$y - (1+T)^{-1}\overline{y} + y\overline{y}T(1+T)^{-1} = zT^n \quad \text{for some} \quad z \in R.$$

For $\{x, -u\}$ is a cycle, so is $<T, zT^n>$. What's more $<T, zT^n>$ is an odd torsion element in $K_2(R_n, \mathcal{I}_n)$, because $0 = <T^{n+1}, z> = (n+1)<T, zT^n>$ and n is even. This finishes the proof. •

Corollary. If $u = -1$ in the situation of proposition 4.12, d is the zero map.

4.13. The composition of homomorphisms

$$\mathrm{Arf}^s(R,1,-1) \hookrightarrow L_0^s(R,1,-1) \xrightarrow{\lambda\omega_1^s} C(R) = \frac{R}{\mathrm{Span}\{x + x^2 \mid x \in R\}}$$

maps $\underbrace{(a,b)}$ to $[ab]$. This surjection splits by the homomorphism $[r] \mapsto (r, 1)$. Writing $\widetilde{\mathrm{Arf}}(R)$ for the kernel, we obtain a splitting

$$\mathrm{Arf}^s(R,1,-1) \cong \widetilde{\mathrm{Arf}}(R) \oplus \frac{R}{\mathrm{Span}\{x + x^2 \mid x \in R\}}.$$

$\widetilde{\mathrm{Arf}}(R)$ is generated by $\langle\langle a,b\rangle\rangle := (a,b) + (ab, 1)$, where $a, b \in R$. The following relations hold in $\widetilde{\mathrm{Arf}}(R)$:

$\langle\langle a, b_1 + b_2\rangle\rangle = \langle\langle a, b_1\rangle\rangle + \langle\langle a, b_2\rangle\rangle$

$\langle\langle a, b\rangle\rangle = \langle\langle b, a\rangle\rangle$

$$\begin{aligned}
&\langle\langle a,b\rangle\rangle = 0 && \text{for } a\in 2R\\
&\langle\langle ax^2,b\rangle\rangle = \langle\langle a,bx^2\rangle\rangle && \text{for every } x\in R\\
&\langle\langle a,b\rangle\rangle = \langle\langle a,ab^2\rangle\rangle &&\\
&\langle\langle a,1\rangle\rangle = 0 &&
\end{aligned}$$

The secondary Arf invariant is by definition the the restriction of ω_2 to the $\widetilde{\text{Arf}}$-part of $\text{Ker}(\omega_1^s)$:

$$\widetilde{\text{Arf}}(R) \hookrightarrow \text{Ker}(\omega_1^s) \xrightarrow{\omega_2} \text{Coker}(d) = H^1(K_2(R_2,\mathcal{I}_2)).$$

The next theorem tells us what this invariant looks like for $n=2$.

4.14 Theorem. $\omega_2(\langle\langle a,b\rangle\rangle) = [<aT^2,b>] \in H^1(K_2(R_2,\mathcal{I}_2))$.

Proof. Let $\langle\langle a,b\rangle\rangle = (a,b)+(ab,1)$ be represented by

$$\left[\begin{pmatrix} a & 0 & 1 & 0\\ 0 & ab & 0 & 1\\ 0 & 0 & b & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\right] \in L_0^s(R,1,-1).$$

A lift of this element in $L_0^s(R_2,\alpha,-(1+T))$ is given by

$$l := \left[\begin{pmatrix} a & 0 & 1 & 0\\ 0 & ab & 0 & 1\\ 0 & 0 & b & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\right].$$

To apply the map G of definition 2.4 we choose

$$\gamma := x_{24}(T^2-T)x_{13}(b(T-T^2))h_{12}(1+abT^2)x_{31}(-aT)x_{42}(-abT) \in \text{St}_4(R_2)$$

as a lift of

$$\left(\begin{pmatrix} a & 0 & 1 & 0\\ 0 & ab & 0 & 1\\ 0 & 0 & b & 0\\ 0 & 0 & 0 & -1\end{pmatrix} + u_2\begin{pmatrix} a & 0 & 0 & 0\\ 0 & ab & 0 & 0\\ 1 & 0 & b & 0\\ 0 & 1 & 0 & -1\end{pmatrix}\right)\begin{pmatrix} 0 & 0 & u_2^{-1} & 0\\ 0 & 0 & 0 & u_2^{-1}\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & b(T-T^2) & 0\\ 0 & 1 & 0 & T^2-T\\ -aT & 0 & 1 & 0\\ 0 & -abT & 0 & 1\end{pmatrix} \in E_4(R_2).$$

Using the definition of t and the calculations in the proof of proposition 4.2 we find

$$t\gamma^{-1} = x_{24}(T-T^2)x_{13}(b(T^2-T))h_{34}(1-abT^2)x_{31}(aT)x_{42}(abT).$$

A little computation shows that

$$\begin{aligned} G(l) &= [\gamma^{-1}(t\gamma)] \\ &= [<abT, T-T^2><aT, b(T^2-T)> \\ &\quad h_{12}(1+abT^2)h_{34}(1-abT^2)h_{42}(1-abT^2)h_{31}(1+abT^2)] \\ &= [<aT^2, b>] \in H^1(K_2(R_2, \mathcal{I}_2)). \end{aligned}$$

But since $\omega_2(\langle\langle a, b\rangle\rangle) = G(l)$ this finishes the proof. •

Taking the (primary) Arf invariant into account we have the following result.

4.15 Theorem. *Let R be a 2λ-ring. The invariant*

$$\mathrm{Arf}^s(R, 1, -1) \to \frac{R}{\{x + x^2\}} \oplus \frac{\Omega_R}{2\Omega_R + \delta R + \{x\delta y + x^2 y\delta y \mid x, y \in R\}}$$

maps (a, b) to $([ab], [a\delta b])$.

Proof. We compute $\phi^2(a\delta b)$ modulo $2\Omega_R + \delta R$:

$$\begin{aligned} \phi^2(a\delta b) &\equiv \psi^2(a)(b\delta b - \delta\theta^2(b)) \\ &\equiv (a^2 - 2\theta^2(a))(b\delta b - \delta\theta^2(b)) \\ &\equiv a^2 b\delta b - a^2\delta\theta^2(b) \\ &\equiv a^2 b\delta b. \end{aligned}$$

Thus

$$2\Omega_R + \delta R + \mathrm{Im}(1 + \phi^2) = 2\Omega_R + \delta R + \{x\delta y + x^2 y\delta y \mid x, y \in R\}.$$

In view of the preceding the rest is obvious. •

Let R be an arbitrary commutative ring. We recognize

$$\frac{\Omega_R}{2\Omega_R + \delta R}$$

as an instance of a cyclic homology group *viz.* $HC_1(R/2R)$.

The assignment $a \mapsto a\delta a$ determines a well-defined homomorphism

$$q' \colon R \to \frac{\Omega_R}{\delta R}.$$

Under the assumption that $2R = 0$

$$\theta \colon R \to R \qquad x \mapsto x^2$$

$$\theta' \colon \frac{\Omega_R}{\delta R} \to \mathrm{Coker}\ q' \qquad [a\delta b] \mapsto [a^2 b\delta b]$$

are well-defined homomorphisms. From this point of view

$$\frac{R}{\{x+x^2\}} = \mathrm{Coker}(1+\theta)$$

and

$$\frac{\Omega_R}{2\Omega_R + \delta R + \{x\delta y + x^2 y \delta y \mid x, y \in R\}} = \mathrm{Coker}(1+\theta').$$

We are a bit sloppy here in denoting the projection $\frac{\Omega_R}{\delta R} \to \mathrm{Coker}\, q'$ by 1.

These observations are the motivation for investigating (operations on) cyclic homology groups. In the next chapter we will construct the homomorphism

$$\mathrm{Arf}^s(R, 1, -1) \to \frac{R}{\{x+x^2 \mid x \in R\}} \oplus \frac{\Omega_R}{2\Omega_R + \delta R + \{(r + r^2\delta s)\delta s \mid r, s \in R\}}$$

without the assumption that R carries some extra structure.

It turns out that the right generalization in the non-commutative case involves the notion of quaternionic homology groups. We will enter into details in the next chapter.

5 Examples.

Example. Let $R = \mathbf{Z}[X,Y]$ be the polynomial ring in two variables.

5.1 Theorem.

$$L_0^s(R,1,-1) \cong \frac{R}{\{f+f^2\}} \oplus \frac{\Omega_R}{2\Omega_R + \delta R + \{f\delta g + f^2 g\delta g \mid f,g \in R\}}.$$

Proof. First we claim that $L_0^s(R,1,-1) = \mathrm{Arf}^s(R,1,-1)$.
Let $(M,[\phi],e) \in BQ(R,1,-1)$ be given. Then $b_{[\phi]}(m)(m) = 0$ for every $m \in M$. Choose a basis element f in M. There exists an element $g \in M$ such that $b_{[\phi]}(g) = f^*$. Thus we obtain a decomposition

$$(M,[\phi],e) \cong (N,[\phi_{|N}],[f,g]) \perp (N^\perp,[\phi_{|N^\perp}],h),$$

where $N := \mathrm{Span}(f,g)$, $N^\perp := \{m \in M \mid b_{[\phi]}(m)(N) = 0\}$ and h is some class of bases. Given the fact that $K_1(R) \cong \mathbf{Z}/2$ it may be necessary to interchange the roles of f and g to get the right class of bases at the right hand side. In this decomposition the first summand is isomorphic to

$$(R^2, \left[\begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}\right], [(1,0),(0,1)])$$

for some $a,b \in R$. An induction argument proves the claim.
Furthermore R has a structure of λ-ring by lemma 4.6. Next we claim that

$$\frac{R}{\{f+f^2\}} \oplus \frac{\Omega_R}{2\Omega_R + \delta R + \{f\delta g + f^2 g\delta g \mid f,g \in R\}} \to \mathrm{Arf}^s(R,1,-1)$$

defined by

$$\left([x], \sum [a\delta b]\right) \longmapsto (x,1) + \sum \langle\langle a,b\rangle\rangle$$

is a well defined inverse of the homomorphism in theorem 4.15. The only non-trivial point on our checklist is: show that this map respects the relation

$$a\delta bc + ab\delta c + ac\delta b = 0.$$

This amounts to showing that the relation

$$\langle\langle a,bc\rangle\rangle = \langle\langle ab,c\rangle\rangle + \langle\langle ac,b\rangle\rangle$$

holds in $\widetilde{\mathrm{Arf}}(R)$. But this follows immediately from the identity

$$\langle\langle f,g\rangle\rangle = \langle\langle f\frac{\partial g}{\partial x},x\rangle\rangle + \langle\langle f\frac{\partial g}{\partial y},y\rangle\rangle \quad \text{for every } f,g \in R.$$

It suffices to prove this for monomials by additivity. By using the relations in $\widetilde{\mathrm{Arf}}(R)$ we see that

$$\langle\langle X^iY^j, X^kY^l\rangle\rangle = \langle\langle X^iY^jkX^{k-1}Y^l, X\rangle\rangle + \langle\langle X^iY^jX^klY^{l-1}, Y\rangle\rangle$$

whenever k or l is even. By symmetry this is also true when i or j is even. In the remaining case i, j, k and l are all odd and

$$\begin{aligned}\langle\langle X^iY^j, X^kY^l\rangle\rangle &= \langle\langle XY, X^{i+k-1}Y^{j+l-1}\rangle\rangle \\ &= \langle\langle XY, XYX^{i+k-2}Y^{j+l-2}\rangle\rangle \\ &= \langle\langle XY, X^{(i+k-2)/2}Y^{(j+l-2)/2}\rangle\rangle.\end{aligned}$$

An induction argument finishes the proof. •

Example. Let G be the group with presentation

$$G := \langle X, Y, S \mid S^2 = (XS)^2 = (YS)^2 = 1, \quad XY = YX\rangle.$$

We study $\mathrm{Arf}^s(G)$ and $\mathrm{Arf}^h(G)$. Recall that we are working with the anti-involution determined by $\overline{g} = g^{-1}$ for all $g \in G$. Let H be the subgroup of G generated by X and Y. These groups fit into the split short exact sequence

$$1 \longrightarrow H \longrightarrow G \longrightarrow C_2 \longrightarrow 1,$$

where C_2 is the group of order two generated by S. Elements of order two in G have the form X^iY^jS for some $i, j \in \mathbf{Z}$. Every element $f \in \mathbb{F}_2[G]$ can be decomposed in a unique way as $f = f_- + f_+S$ with $f_-, f_+ \in \mathbb{F}_2[H]$.

5.2 Proposition. $\mathrm{Arf}^{s,h}(G)$ *is generated by the elements*

$$\begin{cases} (1,1) & \\ (X^{2i}Y^{2j+1}S, S) & \textit{with } j \geq 0 \\ (X^{2i+1}Y^{2j}S, S) & \textit{with } i \geq 0 \\ (X^{2i+1}Y^{2j+1}S, S) & \textit{with } i \geq 0 \\ (X^{2i}Y^{2j+1}S, XS) & \textit{with } j \geq 0 \\ (X^{2i+1}Y^{2j+1}S, XS) & \textit{with } j \geq 0 \\ (X^{2i+1}Y^{2j+1}S, YS) & \textit{with } i \geq 0. \end{cases}$$

5.3 Remark. We say that an element $f \in \mathbb{F}_2[H]$ fulfils condition 1 resp. 2 if all terms X^iY^j of f satisfy $i \geq 0$ resp. $j \geq 0$. Using the fact that for each $h \in \mathbb{F}_2[H]$ there exist unique $h_0, h_1, h_2, h_3 \in \mathbb{F}_2[H]$ such that

$$h = h_0^2 + h_1^2x + h_2^2y + h_3^2xy,$$

we can reformulate proposition 5.2 as follows. Every element of $\mathrm{Arf}^{s,h}(G)$ is of the form

$$(fS, S) + (gS, XS) + (hS, YS),$$

with

- f_1, f_3 satisfy condition 1, f_2 satisfies condition 2 and $f_0 \in \mathbb{F}_2$
- g_2, g_3 satisfy condition 2 and $g_0 = g_1 = 0$

· h_3 satisfies condition 1 and $h_0 = h_1 = h_2 = 0$.

5.4 Lemma. Every element of $\mathrm{Arf}^{s,h}(G)$ is a sum of elements of the form

$$(X^mY^nS, S), \quad (X^mY^nS, XS), \quad (X^mY^nS, YS).$$

Proof. It suffices to prove this for generators (X^iY^jS, X^kY^lS). Conjugation by X and Y yields

$$(X^iY^jS, X^kY^lS) = \begin{cases} (X^{i\pm 2}Y^jS, X^{k\pm 2}Y^lS) \\ (X^iY^{j\pm 2}S, X^kY^{l\pm 2}S). \end{cases}$$

This proves that our generator has the desired form whenever one of the exponents i, j, k or l is even.
If all exponents are odd, we have

$$(X^iY^jS, X^kY^lS) = (XYS, X^{k-i+1}Y^{l-j+1}S)$$

where both $k-i+1$ and $l-j+1$ are odd. But since

$$\begin{aligned}(XYS, X^{2i+1}Y^{2j+1}S) &= (XYS, X^{i+1}Y^{j+1}SXYSX^{i+1}Y^{j+1}S) \\ &= (XYS, X^{i+1}Y^{j+1}S)\end{aligned}$$

and

$$(XYS, XYS) = (XYS, 1) = (1,1) = (S,S),$$

we can use an induction argument to prove the assertion in this case. •

We turn to the proof of the proposition.

Proof. By lemma 5.4 it suffices to prove the claim for the elements

$$(X^mY^nS, S), \quad (X^mY^nS, XS), \quad (X^mY^nS, YS).$$

◇ (X^mY^nS, S)
We may assume that m or n is odd by using the relations

$$(S,S) = (1,1)$$

$$(X^{2m}Y^{2n}S, S) = (X^mY^nSSX^mY^nS, S) = (X^mY^nS, S).$$

Further we may assume that the odd exponent is positive since

$$(X^mY^nS, S) = (SX^mY^nSS, S) = (X^{-m}Y^{-n}S, S).$$

◇ (X^mY^nS, XS)
We may assume that n is odd by

$$(X^{2m}Y^{2n}S, XS) = (S, X^{-2m+1}Y^{-2n}S) = (X^{2m-1}Y^{2n}S, S)$$

$$\begin{aligned}(X^{2m+1}Y^{2n}S, XS) &= (X^{m+1}Y^nSXSX^{m+1}Y^nS, XS) \\ &= (X^{m+1}Y^nS, XS).\end{aligned}$$

And we may assume that n is positive since

$$(X^mY^nS, XS) = (XSX^mY^nSXS, XS) = (X^{-m+2}Y^{-n}S, XS).$$

⋄ (X^mY^nS, YS)
We may assume that n is odd by

$$(X^{2m}Y^{2n}S, YS) = (X^{2m}Y^{2n-1}S, S)$$

$$(X^{2m+1}Y^{2n}S, YS) = (XS, X^{-2m}Y^{-2n+1}S) = (X^{-2m}Y^{-2n+1}S, XS).$$

We may assume that m is odd by the relation

$$\begin{aligned}(X^{2m}Y^{2n+1}S, YS) &= (X^mY^{n+1}SYSX^mY^{n+1}S, YS)\\ &= (X^mY^{n+1}S, YS).\end{aligned}$$

And we may assume that m is positive since

$$(X^mY^nS, YS) = (YSX^mY^nSYS, YS) = (X^{-m}Y^{-n+2}S, YS).$$

This completes the proof. •

The Arf invariant

$$\mathrm{Arf}^s(G) \longrightarrow K(G) = \frac{\mathbb{F}_2[G]}{\mathrm{Span}\{a+\overline{a}, b+b^2 \mid a, b \in \mathbb{F}_2[G]\}}$$

which maps

$$\begin{cases}(X^iY^jS, X^kY^lS) & \text{to } [X^{i-k}Y^{j-l}]\\ (X^iY^jS, 1) = (1,1) & \text{to } [1]\end{cases}$$

splits by

$$\begin{cases}[X^iY^j] & \mapsto (X^iY^jS, S)\\ [X^iY^jS] & \mapsto (1,1).\end{cases}$$

We write $\widetilde{\mathrm{Arf}}(G)$ for the remaining summand. Thus

$$\mathrm{Arf}^s(G) \cong \widetilde{\mathrm{Arf}}(G) \oplus K(G).$$

Observe that the inclusion $\mathbb{F}_2[H] \hookrightarrow \mathbb{F}_2[G]$ induces an isomorphism

$$K(H) \hookrightarrow\!\!\!\!\rightarrow K(G),$$

with inverse $[a] \longmapsto [a_- + a_+\overline{a_+}]$.
$\widetilde{\mathrm{Arf}}(G)$ is generated by

$$\langle\langle a, b\rangle\rangle := (a_+S, b_+S) + (a_+\overline{b_+}S, S),$$

where $a = \overline{a}, b = \overline{b}$ in $\mathbb{F}_2[G]$. The following relations hold in $\widetilde{\mathrm{Arf}}(G)$:
$\langle\langle a, b\rangle\rangle = \langle\langle a_+S, b_+S\rangle\rangle$
$\langle\langle a, 1\rangle\rangle = \langle\langle 1, a\rangle\rangle = 0$
$\langle\langle a, S\rangle\rangle = \langle\langle S, a\rangle\rangle = 0$
$\langle\langle a, b_1+b_2\rangle\rangle = \langle\langle a, b_1\rangle\rangle + \langle\langle a, b_2\rangle\rangle$

$\langle\langle a, b\rangle\rangle = \langle\langle b, a\rangle\rangle$
$\langle\langle \overline{c}ac, b\rangle\rangle = \langle\langle a, cb\overline{c}\rangle\rangle$ for every $c \in \mathbb{F}_2[G]$
$\langle\langle a, b\rangle\rangle = \langle\langle a, \overline{b}ab\rangle\rangle$

Now we consider the representation $\rho{:}\mathbb{F}_2[G] \to M_2(R)$ of G over the ring $R := \mathbb{F}_2[H]$ determined by

$$\begin{aligned} X &\longmapsto \begin{pmatrix} X & 0 \\ 0 & X^{-1} \end{pmatrix} \\ Y &\longmapsto \begin{pmatrix} Y & 0 \\ 0 & Y^{-1} \end{pmatrix} \\ S &\longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

and the diagram

$$\begin{array}{ccccc} \mathrm{Arf}^s(G) & & \xrightarrow{\qquad\qquad\psi\qquad\qquad} & & \mathrm{Arf}^s(R,1,1) \\ \big\downarrow \imath & & & & \big\downarrow \jmath \\ L^s(G) & \xrightarrow{\widetilde{\rho}} & L_0^s(M_2(R),\alpha,1) & \xrightarrow{\gamma} & L_0^s(R,1,1). \end{array}$$

Here $\imath$ and $\jmath$ are inclusion maps,
$\widetilde{\rho}$ is induced by ρ,
$U := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,
$\alpha(A) := UA^tU$ for all $A \in M_2(R)$,
γ is the composition of the ‘scaling-isomorphism’

$$L_0^s(M_2(R),\alpha,1) \xrightarrow{\cong} L_0^s(M_2(R),\mathsf{transpose},1)$$

and the ‘Morita-isomorphism’

$$L_0^s(M_2(R),\mathsf{transpose},1) \xrightarrow{\cong} L_0^s(R,1,1).$$

5.5 Lemma. $(X^iY^jS, X^kY^lS) \overset{\psi}{\longmapsto} (X^{-i}Y^{-j}, X^kY^l) + (X^iY^j, X^{-k}Y^{-l})$.

Proof. $\imath$ maps (X^iY^j, X^kY^l) to

$$\left[\begin{pmatrix} X^iY^jS & 1 \\ 0 & X^kY^lS \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right],$$

$\widetilde{\rho}$ maps this element to

$$\left[\begin{pmatrix} 0 & X^iY^j & 1 & 0 \\ X^{-i}Y^{-j} & 0 & 0 & 1 \\ 0 & 0 & 0 & X^kY^l \\ 0 & 0 & X^{-k}Y^{-l} & 0 \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right],$$

γ maps this element to

$$\left[\begin{pmatrix} X^iY^j & 0 & 0 & 1 \\ 0 & X^{-i}Y^{-j} & 1 & 0 \\ 0 & 0 & X^kY^l & 0 \\ 0 & 0 & 0 & X^{-k}Y^{-l} \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right].$$

Now we apply the isometry $\begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix}$. Note that this isometry is admissible since its class in $K_1(R)$ is trivial. This yields

$$\left[\begin{pmatrix} X^{-i}Y^{-j} & 0 & 1 & 0 \\ 0 & X^iY^j & 0 & 1 \\ 0 & 0 & X^kY^l & 0 \\ 0 & 0 & 0 & X^{-k}Y^{-l} \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right].$$

This element is equal to $\jmath\left((X^{-i}Y^{-j}, X^kY^l) + (X^iY^j, X^{-k}Y^{-l})\right)$. •

Consequently

$$\psi(\langle\langle fS, gS\rangle\rangle) = \langle\langle \overline{f}, g\rangle\rangle + \langle\langle f, \overline{g}\rangle\rangle \in \widetilde{\mathrm{Arf}}(R)$$

for all $f, g \in \mathbb{F}_2[H]$. We are now in the position to apply the machinery of the previous section and in particular the secondary Arf invariant

$$\widetilde{\mathrm{Arf}}(R) \to \frac{\Omega_R}{\delta R + \{(a + a^2b)\delta b \mid a, b \in R\}}$$

of theorem 4.15.

5.6 Theorem. *The invariant*

$$\begin{array}{rcl} \mathrm{Arf}^s(G) & \longrightarrow & \dfrac{R}{\mathrm{Span}\{a + \overline{a}, b + b^2 \mid a, b \in R\}} \oplus \dfrac{\Omega_R}{\delta R + \{(a + a^2b)\delta b \mid a, b \in R\}} \\ (fS, gS) & \longmapsto & ([f\overline{g}], [\overline{f}\delta g + f\delta\overline{g}]), \end{array}$$

is injective and the elements mentioned in proposition 5.2 constitute a basis for $\mathrm{Arf}^s(G)$.

Proof. By the reformulation of proposition 5.2 in remark 5.3 it suffices to prove that

$$(f'S, S) + (gS, XS) + (hS, YS) = 0 \quad \Longrightarrow \quad f' = g = h = 0,$$

whenever $f', g, h \in \mathbb{F}_2[H]$ satisfy the conditions mentioned in remark 5.3. Suppose $\xi := (f'S, S) + (gS, XS) + (hS, YS) = 0$. Define $f := f' + gX^{-1} + hY^{-1}$. Then

$$\xi = (fS, S) + \langle\langle gS, XS\rangle\rangle + \langle\langle hS, YS\rangle\rangle$$

and f still fulfils the condition of remark 5.3. The image

$$([f], [(gX^{-1} + \overline{g}X)X^{-1}\delta X + (hY^{-1} + \overline{h}Y)Y^{-1}\delta Y])$$

of ξ vanishes in

$$\frac{R}{\mathrm{Span}\{a + \overline{a}, b + b^2 \mid a, b \in R\}} \oplus \frac{\Omega_R}{\delta R + \{(a + a^2 b)\delta b \mid a, b \in R\}}.$$

We will exploit the following facts to show that $f = g = h = 0$.

- For each $h \in R$ there are unique $h_0, h_1, h_2, h_3 \in R$ such that $h = h_0^2 + h_1^2 X + h_2^2 Y + h_3^2 XY$.
- If $h \in R$ is symmetric, i.e. $\overline{h} = h$ and the constant term of h is zero, then $h = p + \overline{p}$ for some $p \in R$.
- If $h \in R$ is symmetric, then h_0^2, $h_1^2 X$, $h_2^2 Y$ and $h_3^2 XY$ are symmetric.

The fact that $[f] = 0$ guarantees the existence of $a, b \in R$ such that

$$f = a + a^2 + b + \overline{b}.$$

This implies: $f_0 = 0$ and $a_0^2 + a^2$ is symmetric. So $a_0 + a = a_0 + a_0^2 + a_1^2 X + a_2^2 Y + a_3^2 XY$ is symmetric as well. By applying induction on

$$\max\{|i| + |j| \mid X^i Y^j \text{ is a term of } a + a^2\}$$

we conclude that $a + a^2$ is symmetric. Hence $f_1^2 X + f_2^2 Y + f_3^2 XY$ is symmetric, but the conditions on f_1, f_2, f_3 make this impossible unless $f = 0$.
Since $[(gX^{-1} + \overline{g}X)X^{-1}\delta X + (hY^{-1} + \overline{h}Y)Y^{-1}\delta Y] = 0$ there exist $a, b, c \in R$ such that

$$(gX^{-1}+\overline{g}X)X^{-1}\delta X+(hY^{-1}+\overline{h}Y)Y^{-1}\delta Y = (a+a^2)X^{-1}\delta X+(b+b^2)Y^{-1}\delta Y+\delta c$$

Since

$$\begin{aligned} \delta c &= \delta(c_0^2 + c_1^2 X + c_2^2 Y + c_3^2 XY) \\ &= c_1^2 X X^{-1}\delta X + c_2^2 Y Y^{-1}\delta Y + c_3^2 XY X^{-1}\delta X + c_3^2 XY Y^{-1}\delta Y, \end{aligned}$$

we may assume that $c_0 = 0$ and it follows that

$$gX^{-1} + \overline{g}X = a + a^2 + c_1^2 X + c_3^2 XY,$$

$$hY^{-1} + \overline{h}Y = b + b^2 + c_2^2 Y + c_3^2 XY.$$

Substituting $g = g_2^2 Y + g_3^2 XY$ and $h = h_3^2 XY$ gives us the identities

$$g_2^2 X^{-1} Y + g_3^2 Y + \overline{g_2^2 X^{-1} Y + g_3^2 Y} = a + a^2 + c_1^2 X + c_3^2 XY,$$

$$h_3^2 X + \overline{h_3^2 X} = b + b^2 + c_2^2 Y + c_3^2 XY.$$

From these equations we deduce that $a_0 = a$ and $b_0 = b$, thus $a + a^2 = b + b^2 = 0$. Hence $c_1 = c_2 = c_3 = 0$. But then the restrictions on g_2, g_3 and h_3 imply $g_2 = g_3 = h_3 = 0$. This finishes the proof. •

Chapter III

Hochschild, cyclic and quaternionic homology.

1 Definitions and notations.

In the fourth section of the previous chapter we explained why we are interested in constructing certain operations on cyclic homology groups. We start by summing up the definitions of the various homologies we need. We refer to [17, 16] for more details.

Let k denote a commutative ring with identity.

1.1 Definition. A simplicial k-module is a series of k-modules $\{M_n \mid n \in \mathbb{N}\}$, endowed with k-module homomorphisms

$$d_i\colon M_n \to M_{n-1} \quad \text{for all} \quad i \in \{0, 1, \ldots, n\}$$

$$s_i\colon M_n \to M_{n+1} \quad \text{for all} \quad i \in \{0, 1, \ldots, n\},$$

satisfying

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \quad \text{if } i < j \\ d_i s_j &= \begin{cases} s_{j-1} d_i & \text{if } i < j \\ 1 & \text{if } j \leq i \leq j+1 \\ s_j d_{i-1} & \text{if } i > j+1 \end{cases} \\ s_i s_j &= s_{j+1} s_i \quad \text{if } i \leq j. \end{aligned}$$

1.2 Definition. A cyclic k-module is a simplicial k-module $\{M_n \mid n \in \mathbb{N}\}$ equipped with homomorphisms

$$x\colon M_n \to M_n$$

satisfying

$$\begin{aligned} x^{n+1} &= 1 \\ d_i x &= -x d_{i-1} \quad \text{for all} \quad i \in \{1, \ldots, n\} \\ d_0 x &= (-1)^n d_n \\ s_i x &= -x s_{i-1} \quad \text{for all} \quad i \in \{1, \ldots, n\} \\ s_0 x &= (-1)^{n+1} x^2 s_n. \end{aligned}$$

1.3 Definition. A quaternionic k-module consists of a simplicial k-module $\{M_n \mid n \in \mathbb{N}\}$ and homomorphisms

$$\begin{cases} x\colon M_n \to M_n \\ y\colon M_n \to M_n \end{cases}$$

satisfying

$$\begin{array}{rcll} x^{n+1} & = & y^2 & \\ xyx & = & y & \\ d_i x & = & -x d_{i-1} & \text{for all} \quad i \in \{1,\dots,n\} \\ s_i x & = & -x s_{i-1} & \text{for all} \quad i \in \{1,\dots,n\} \\ d_i y & = & (-1)^n y d_{n-i} & \text{for all} \quad i \in \{0,\dots,n\} \\ s_i y & = & (-1)^{n+1} y s_{n-i} & \text{for all} \quad i \in \{0,\dots,n\}. \end{array}$$

1.4 Definition. A quaternionic k-module is called a dihedral k-module when $y^2 = 1$.

1.5 Example. Let R be a k-algebra. We write R^{n+1} as an abbreviation for the (n+1)-fold tensor product $R \otimes_k R \otimes_k \cdots \otimes_k R$. The k-modules

$$M_n := R^{n+1}$$

and the homomorphisms d_i and s_i determined by

$$\begin{array}{rcl} d_i(a_0 \otimes \cdots \otimes a_n) & := & \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & \text{for } 0 \le i < n \\ a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} & \text{for } i = n \end{cases} \\ s_i(a_0 \otimes \cdots \otimes a_n) & := & a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n \quad \text{for all } 0 \le i \le n \end{array}$$

constitute a simplicial k-module. The homomorphisms $x\colon R^{n+1} \to R^{n+1}$ determined by

$$x(a_0 \otimes \cdots \otimes a_n) := (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}$$

make this simplicial module into a cyclic module.
If in addition R is equipped with an anti-involution of k-algebras $\overline{}\colon R \to R$, it even becomes a dihedral module by defining

$$y(a_0 \otimes \cdots \otimes a_n) := (-1)^{\frac{1}{2}n(n+1)}(\overline{a_0} \otimes \overline{a_n} \otimes \cdots \otimes \overline{a_1}).$$

1.6 Example. More general, given a k-algebra R and a R-bimodule P we can turn

$$M_n := P \otimes_k R^n$$

into a simplicial k-module through the homomorphisms

$$\begin{array}{rcl} d_i(p \otimes r_1 \otimes \cdots \otimes r_n) & := & \begin{cases} p r_1 \otimes r_2 \otimes \cdots \otimes r_n & \text{for } i = 0 \\ p \otimes r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n & \text{for } 0 < i < n \\ r_n p \otimes r_1 \otimes \cdots \otimes r_{n-1} & \text{for } i = n \end{cases} \\ s_i(p \otimes r_1 \otimes \cdots \otimes r_n) & := & p \otimes r_1 \otimes \cdots \otimes r_i \otimes 1 \otimes r_{i+1} \otimes \cdots \otimes r_n \\ & & \text{for } 0 \le i \le n \end{array}$$

1.7 Definition. For every simplicial k-module M_* one constructs the chain complex $\mathcal{B}(M_*)$ called Hochschild complex as follows:

$$\cdots \xrightarrow{b} M_{n+1} \xrightarrow{b} M_n \xrightarrow{b} M_{n-1} \xrightarrow{b} \cdots \xrightarrow{b} M_0$$

where

$$b := \sum_{i=0}^{n} (-1)^i d_i.$$

The Hochschild-homology of M_* is by definition the homology of this chain complex.
In case M_* is the simplicial k-module of example 1.5 we denote this chain complex by (R^*, b) and its homology by $H_*(R)$.

1.8 Definition. If M_* is a cyclic k-module one can build a double complex $\mathcal{C}(M_*)$:

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
M_n & \xleftarrow{1-x} & M_n & \xleftarrow{L} & M_n & \xleftarrow{1-x} & M_n & \longleftarrow & \cdots \\
\Big\downarrow b & & \Big\downarrow -b' & & \Big\downarrow b & & \Big\downarrow -b' & & \\
M_{n-1} & \xleftarrow{1-x} & M_{n-1} & \xleftarrow{L} & M_{n-1} & \xleftarrow{1-x} & M_{n-1} & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where

$$\begin{aligned}
b &:= \sum_{i=0}^{n} (-1)^i d_i \\
b' &:= \sum_{i=0}^{n-1} (-1)^i d_i \\
L &:= \sum_{i=0}^{n} x^i
\end{aligned}$$

The cyclic homology $HC_n(M_*)$ of M_* is by definition the n-th homology of the total complex $\operatorname{Tot}\mathcal{C}(M_*)$ associated to $\mathcal{C}(M_*)$, i.e.

$$HC_n(M_*) := H_n(\operatorname{Tot}\mathcal{C}(M_*)).$$

In the case that M_* is the cyclic module of example 1.5 we denote this cyclic homology by

$$HC_n(R).$$

1.9 Definition. If M_* is a quaternionic module one can build a double complex $\mathcal{D}(M_*)$ as follows:

$$\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

$$
\begin{array}{ccccccccccc}
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
M_n & \overset{\alpha}{\leftarrow} & M_n \oplus M_n & \overset{\beta}{\leftarrow} & M_n \oplus M_n & \overset{\gamma}{\leftarrow} & M_n & \overset{N}{\leftarrow} & M_n & \leftarrow \cdots \\
\Big\downarrow b & & \Big\downarrow -\widetilde{B} & & \Big\downarrow \widehat{B} & & \Big\downarrow -b' & & \Big\downarrow b & & \\
M_{n-1} & \overset{\alpha}{\leftarrow} & M_{n-1} \oplus M_{n-1} & \overset{\beta}{\leftarrow} & M_{n-1} \oplus M_{n-1} & \overset{\gamma}{\leftarrow} & M_{n-1} & \overset{N}{\leftarrow} & M_{n-1} & \leftarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

where

$$
\begin{aligned}
b &:= \sum_{i=0}^{n} (-1)^i d_i \\
b' &:= \sum_{i=0}^{n-1} (-1)^i d_i \\
\widetilde{B} &:= \begin{pmatrix} b' & 0 \\ 0 & b \end{pmatrix} \\
\widehat{B} &:= \begin{pmatrix} b & 0 \\ 0 & b' \end{pmatrix} \\
L &:= \sum_{i=0}^{n} x^i \\
N &:= \sum_{i=0}^{3} L y^i \\
\alpha &:= \begin{pmatrix} 1-x & 1-y \end{pmatrix} \\
\beta &:= \begin{pmatrix} L & 1+yx \\ -1-y & x-1 \end{pmatrix} \\
\gamma &:= \begin{pmatrix} 1-x \\ yx-1 \end{pmatrix}
\end{aligned}
$$

The quaternionic homology $HQ_n(M_*)$ of M_* is by definition the n-th homology of the total complex $\operatorname{Tot}\mathcal{D}(M_*)$ associated to $\mathcal{D}(M_*)$ i.e.

$$HQ_n(M_*) := H_n(\operatorname{Tot}\mathcal{D}(M_*)).$$

In the case that M_* is the quaternionic module of example 1.5 we denote this quaternionic homology by

$$HQ_n(R).$$

2 Reduced power operations.

In this section we will construct operations on various low dimensional homology groups. These operations will be used later on to define new Arf invariants. We feel that the material in this section is interesting in its own right.

Notation. Let p be a fixed prime number for the rest of this section. For every $n \in \mathbb{N}$, I_n denotes the set $\{1, 2, \ldots, n\}$. I_n will act as a set of indices. The symmetric group of degree p, S_p acts on the p-fold cartesian product I_n^p of I_n by

$$\tau(i_1, \ldots, i_p) := (i_{\tau(1)}, \ldots, i_{\tau(p)}) \quad \text{for all} \quad (i_1, \ldots, i_n) \in I_n^p, \tau \in S_p.$$

Consider the permutation $\sigma := (1\,2 \cdots p)^{-1}$. Define $\Delta_n := \{\gamma \in I_n^p \mid \sigma\gamma = \gamma\}$. Let Γ_n denote a set of representatives for the σ-orbits of the free action of σ on $I_n^p - \Delta_n$.

Let R be an associative ring with identity. Now recall the definitions of the Hochschild homology group $H_0(R)$ and the cyclic homology group $HC_0(R)$. Observe that both groups are equal to Coker(b), where $b\colon R \otimes R \to R$ is defined by

$$b(r_1 \otimes r_2) = r_1 r_2 - r_2 r_1.$$

For all $r \in R$ we denote by $[r]$ the class of r in $H_0(R)$.

2.1 Proposition. *$\theta_p\colon H_0(R) \to H_0(R/pR)$ defined by*

$$\theta_p([r]) := [r^p],$$

is a well-defined homomorphism.

Proof. For all maps $\alpha\colon I_n \to R$ and elements $\gamma = (i_1, \ldots, i_p) \in I_n^p$, we will write $\gamma(\alpha)$ instead of $\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_p}$. We assert that

$$\sum_{k=1}^{p} \sigma^k \gamma(\alpha) = p\gamma(\alpha) - b\left(\sum_{l=1}^{p-1} \alpha_{i_1}\cdots\alpha_{i_l} \otimes \alpha_{i_{l+1}}\cdots\alpha_{i_p}\right).$$

This is easily verified by writing everything out. For all $\alpha\colon I_2 \to R$, the following identity holds in $H_0(R/pR)$:

$$\begin{aligned}
[(\alpha_1 + \alpha_2)^p] &= [\alpha_1^p + \alpha_2^p + \sum_{\gamma \in I_2^p - \Delta_2} \gamma(\alpha)] \\
&= [\alpha_1^p + \alpha_2^p + \sum_{\gamma \in \Gamma_2} \sum_{k=1}^{p} \sigma^k \gamma(\alpha)] \\
&= [\alpha_1^p + \alpha_2^p] \\
&= [\alpha_1^p] + [\alpha_2^p].
\end{aligned}$$

So it suffices to show that $[(b(\alpha_1 \otimes \alpha_2))^p] = 0$ in $H_0(R/pR)$. Now then:

$$\begin{aligned}
[(b(\alpha_1 \otimes \alpha_2))^p] &= [(\alpha_1\alpha_2 - \alpha_2\alpha_1)^p] \\
&= [(\alpha_1\alpha_2)^p + (-1)^p(\alpha_2\alpha_1)^p] \\
&= [\alpha_1\alpha_2(\alpha_1\alpha_2)^{p-1} - \alpha_2(\alpha_1\alpha_2)^{p-1}\alpha_1] \\
&= [b(\alpha_1 \otimes \alpha_2(\alpha_1\alpha_2)^{p-1})] \\
&= 0
\end{aligned}$$

This proves the proposition. •

Recall the definitions of the Hochschild homology group $H_1(R)$ and the cyclic homology group $HC_1(R)$:

$$H_1(R) := \frac{\operatorname{Ker}(b\colon R \otimes R \to R)}{\operatorname{Im}(b\colon R \otimes R \otimes R \to R)}$$

$$HC_1(R) := \frac{\operatorname{Ker}(b\colon R \otimes R \to R)}{\operatorname{Im}(b\colon R \otimes R \otimes R \to R) + \operatorname{Im}(1-x)},$$

where

$$b(r_1 \otimes r_2) = r_1 r_2 - r_2 r_1,$$

$$b(r_1 \otimes r_2 \otimes r_3) = r_1 r_2 \otimes r_3 - r_1 \otimes r_2 r_3 + r_3 r_1 \otimes r_2,$$

$$x\colon R \otimes R \to R \otimes R \text{ is defined by } x(r_1 \otimes r_2) = -r_2 \otimes r_1.$$

For all $\xi \in \operatorname{Ker}(b\colon R \otimes R \to R)$, we denote by $[\xi]$ the class of ξ in $H_1(R)$ as well as in $HC_1(R)$.
Let $\alpha, \beta\colon I_n \to R$ be set-theoretic maps. For every p-tuple $\gamma = (i_1, \ldots, i_p) \in I_n^p$ we write

$$\gamma(\alpha, \beta)$$

instead of

$$\alpha_{i_1}\beta_{i_1}\alpha_{i_2}\beta_{i_2}\cdots\alpha_{i_{p-1}}\beta_{i_{p-1}} \otimes \alpha_{i_p}\beta_{i_p} \in R \otimes R.$$

2.2 Theorem. *The map $\theta_p\colon H_1(R) \to HC_1(R/pR)$ determined by*

$$\left[\sum_{i \in I_n} \alpha_i \otimes \beta_i\right] \mapsto \left[\sum_{i \in I_n} (\alpha_i\beta_i)^{p-1}\alpha_i \otimes \beta_i + \sum_{\gamma \in \Gamma_n} \sum_{t=1}^{p-1} \left(t\sigma^t\gamma(\alpha,\beta) - t\sigma^t\gamma(\beta,\alpha)\right)\right]$$

is a well-defined homomorphism.

2.3 Remark. In the case that $p = 2$ this reads $\theta_2\colon H_1(R) \to HC_1(R/2R)$

$$\left[\sum_{i=1}^n \alpha_i \otimes \beta_i\right] \mapsto \left[\sum_{i=1}^n \alpha_i\beta_i\alpha_i \otimes \beta_i + \sum_{i<j} (\alpha_i\beta_i \otimes \alpha_j\beta_j + \beta_i\alpha_i \otimes \beta_j\alpha_j)\right].$$

We will prove this theorem with the help of a series of lemmas.

2.4 Lemma. Let $m > 1$. For all $r_1, r_2, \ldots, r_m \in R$:

$$\begin{aligned}\sum_{i=1}^{m} r_{i+1}r_{i+2}\cdots r_m r_1 r_2 \cdots r_{i-1} \otimes r_i &= (1-x)(r_1\cdots r_m \otimes 1)\\ &+ b\left(\sum_{i=1}^{m-2} r_{i+2}\cdots r_m \otimes r_1 \cdots r_i \otimes r_{i+1}\right)\\ &+ b(1 \otimes r_1 \cdots r_{m-1} \otimes r_m)\\ &- b(1 \otimes r_1 \cdots r_m \otimes 1)\end{aligned}$$

Proof. Simply a matter of writing everything out. •

Corollary. For all $\alpha, \beta\colon I_n \to R$ and $\gamma \in I_n^p$

$$\left[\sum_{t=1}^{p-1}(t\sigma^t\gamma(\alpha,\beta) - t\sigma^{t+1}\gamma(\alpha,\beta))\right] = \left[\sum_{t=1}^{p}\sigma^t\gamma(\alpha,\beta)\right] = 0.$$

Corollary. θ_p does not depend on the choice of Γ_n.

2.5 Lemma. Let $\mathcal{F}(R \times R)$ be the free abelian monoid on the set $R \times R$ and $\otimes\colon \mathcal{F}(R\times R) \to R \otimes R$ be the canonical morphism. There is a bijective correspondence between homomorphisms on $\mathrm{Ker}(b\colon R\otimes R \to R)$ and morphisms on $\mathrm{Ker}(b\otimes\colon \mathcal{F}(R\times R) \to R)$ which kill all elements of the form

$(u, 0)$	$u \in R$
$(0, u)$	$u \in R$
$(u, v+w) + (u, -v) + (u, -w)$	$u, v, w \in R$
$(u+v, w) + (-u, w) + (-v, w)$	$u, v, w \in R$.

Proof. To a homomorphism f on $\mathrm{Ker}(b)$, we associate the morphism $f\otimes$ on $\mathrm{Ker}(b\otimes)$. It is clear that this morphism meets all requirements.

Conversely suppose f is a morphism on $\mathrm{Ker}(b\otimes)$ as in the statement above. Define the homomorphism g on $\mathrm{Ker}(b)$ as follows: If $\xi = \sum_i \alpha_i \otimes \beta_i$ belongs to $\mathrm{Ker}(b)$, we choose $\eta = \sum_i(\alpha_i, \beta_i)$ as a lift of ξ in $\mathrm{Ker}(b\otimes)$, and define $g(\xi) := f(\eta)$. Let us verify that this is well-defined.

Suppose $\tilde{\eta} = \sum_i(\tilde{\alpha}_i, \tilde{\beta}_i)$ is another lift of ξ in $\mathrm{Ker}(b\otimes)$. Consider the difference $\eta - \tilde{\eta}$ in the free abelian group $\mathcal{F}g(R\times R)$. By definition of the tensor-product, this takes the form:

$$\begin{aligned}&\sum_{k_1}\{(u_{k_1}+v_{k_1}, w_{k_1}) - (u_{k_1}, w_{k_1}) - (v_{k_1}, w_{k_1})\} +\\ &\sum_{k_2}\{(u_{k_2}, w_{k_2}) + (v_{k_2}, w_{k_2}) - (u_{k_2}+v_{k_2}, w_{k_2})\} +\\ &\sum_{k_3}\{(u_{k_3}, v_{k_3}+w_{k_3}) - (u_{k_3}, v_{k_3}) - (u_{k_3}, w_{k_3})\} +\\ &\sum_{k_4}\{(u_{k_4}, v_{k_4}) + (u_{k_4}, w_{k_4}) - (u_{k_4}, v_{k_4}+w_{k_4})\}\end{aligned}$$

for certain $u_{k_i}, v_{k_i}, w_{k_i} \in R$. As a consequence we have in $\mathcal{F}(R \times R)$:

$$\begin{aligned}
\eta \quad + \quad & \sum_{k_1} \{(u_{k_1}, w_{k_1}) + (-u_{k_1}, w_{k_1}) + (0, w_{k_1})\} + \\
& \sum_{k_1} \{(v_{k_1}, w_{k_1}) + (-v_{k_1}, w_{k_1}) + (0, w_{k_1})\} + \\
& \sum_{k_2} \{(u_{k_2} + v_{k_2}, w_{k_2}) + (-u_{k_2}, w_{k_2}) + (-v_{k_2}, w_{k_2})\} + \\
& \sum_{k_2} \{2(0, w_{k_2})\} + \\
& \sum_{k_3} \{(u_{k_3}, v_{k_3}) + (u_{k_3}, -v_{k_3}) + (u_{k_3}, 0)\} + \\
& \sum_{k_3} \{(u_{k_3}, w_{k_3}) + (u_{k_3}, -w_{k_3}) + (u_{k_3}, 0)\} + \\
& \sum_{k_4} \{(u_{k_4}, v_{k_4} + w_{k_4}) + (u_{k_4}, -v_{k_4}) + (u_{k_4}, -w_{k_4})\} + \\
& \sum_{k_4} \{2(u_{k_4}, 0)\} = \\
\tilde{\eta} \quad + \quad & \sum_{k_1} \{(u_{k_1} + v_{k_1}, w_{k_1}) + (-u_{k_1}, w_{k_1}) + (-v_{k_1}, w_{k_1})\} + \\
& \sum_{k_1} \{2(0, w_{k_1})\} + \\
& \sum_{k_2} \{(u_{k_2}, w_{k_2}) + (-u_{k_2}, w_{k_2}) + (0, w_{k_2})\} + \\
& \sum_{k_2} \{(v_{k_2}, w_{k_2}) + (-v_{k_2}, w_{k_2}) + (0, w_{k_2})\} + \\
& \sum_{k_3} \{(u_{k_3}, v_{k_3} + w_{k_3}) + (u_{k_3}, -v_{k_3}) + (u_{k_3}, -w_{k_3})\} + \\
& \sum_{k_3} \{2(u_{k_3}, 0)\} + \\
& \sum_{k_4} \{(u_{k_4}, v_{k_4}) + (u_{k_4}, -v_{k_4}) + (u_{k_4}, 0)\} + \\
& \sum_{k_4} \{(u_{k_4}, w_{k_4}) + (u_{k_4}, -w_{k_4}) + (u_{k_4}, 0)\}.
\end{aligned}$$

This implies $f(\eta) = f(\tilde{\eta})$. Hence g is well-defined. The rest is obvious. •

We want to apply this lemma to the map

$$\tilde{\theta}_p \colon \mathrm{Ker}(b\otimes) \to HC_1(R/pR)$$

defined by

$$\sum_{i\in I_n}(\alpha_i,\beta_i)\mapsto\left[\sum_{i\in I_n}(\alpha_i\beta_i)^{p-1}\alpha_i\otimes\beta_i+\sum_{\gamma\in\Gamma_n}\sum_{t=1}^{p-1}\big(t\sigma^t\gamma(\alpha,\beta)-t\sigma^t\gamma(\beta,\alpha)\big)\right]$$

But first we need another lemma to show that $\tilde{\theta}_p$ is well-defined in the sense that the formula on the right-hand side defines a cycle in $HC_1(R/pR)$.

2.6 Lemma. For all $\alpha,\beta\colon I_n\to R$ with $\sum_{i\in I_n}(\alpha_i,\beta_i)\in\operatorname{Ker}(b\otimes)$

$$b\left(\sum_{i\in I_n}(\alpha_i\beta_i)^{p-1}\alpha_i\otimes\beta_i+\sum_{\gamma\in\Gamma_n}\sum_{t=1}^{p}\big(t\sigma^t\gamma(\alpha,\beta)-t\sigma^t\gamma(\beta,\alpha)\big)\right)=0.$$

Proof. Writing $\overline{\gamma}(\alpha,\beta)$ instead of $\alpha_{i_1}\beta_{i_1}\cdots\alpha_{i_p}\beta_{i_p}$, for every $\gamma=(i_1,\ldots,i_p)\in I_n^p$, the expression becomes

$$\begin{aligned}
&\sum_{\gamma\in\Delta_n}(\overline{\gamma}(\alpha,\beta)-\overline{\gamma}(\beta,\alpha))+\\
&\sum_{\gamma\in\Gamma_n}\sum_{t=1}^{p-1}(t\overline{\sigma^t\gamma}(\alpha,\beta)-t\overline{\sigma^{t+1}\gamma}(\alpha,\beta)-t\overline{\sigma^t\gamma}(\beta,\alpha)+t\overline{\sigma^{t+1}\gamma}(\beta.\alpha))\\
&\quad=\sum_{\gamma\in\Delta_n}(\overline{\gamma}(\alpha,\beta)-\overline{\gamma}(\beta,\alpha))+\sum_{\gamma\in\Gamma_n}\sum_{t=1}^{p}(\overline{\sigma^t\gamma}(\alpha,\beta)-\overline{\sigma^t\gamma}(\beta,\alpha))\\
&\quad=\sum_{\gamma\in\Delta_n}(\overline{\gamma}(\alpha,\beta)-\overline{\gamma}(\beta,\alpha))+\sum_{\gamma\in I_n^p-\Delta_n}(\overline{\gamma}(\alpha,\beta)-\overline{\gamma}(\beta,\alpha))\\
&\quad=\sum_{\gamma\in I_n^p}(\overline{\gamma}(\alpha,\beta)-\overline{\gamma}(\beta,\alpha))\\
&\quad=\left(\sum_{i\in I_n}\alpha_i\beta_i\right)^p-\left(\sum_{i\in I_n}\beta_i\alpha_i\right)^p\\
&\quad=0.
\end{aligned}$$

This proves the assertion. •

We proceed by showing that $\tilde{\theta}_p$ is a morphism on $\operatorname{Ker}(b\otimes)$.

2.7. Suppose we are given $\alpha,\beta\colon I_n\to R$ and $\alpha',\beta'\colon I_{n'}\to R$, such that

$$\eta=\sum_{i\in I_n}(\alpha_i,\beta_i)\quad\text{and}\quad\eta'=\sum_{i\in I_{n'}}(\alpha'_i,\beta'_i)$$

are in $\operatorname{Ker}(b\otimes)$. Let's say

$$r:=\sum_{i\in I_n}\alpha_i\beta_i=\sum_{i\in I_n}\beta_i\alpha_i\quad\text{and}\quad r':=\sum_{i\in I_{n'}}\alpha'_i\beta'_i=\sum_{i\in I_{n'}}\beta'_i\alpha'_i.$$

We identify the disjoint union $I_n \vee I_{n'}$ and $I_{n+n'}$. Define $\tilde{\alpha}\colon I_{n+n'} \to R$ by

$$\tilde{\alpha}(i) := \begin{cases} \alpha(i) & \text{if } i \in I_n \\ \alpha'(i) & \text{if } i \in I_{n'} \end{cases}$$

and define $\tilde{\beta}$ in the same way. The map $I_{n+n'} \to I_2$ defined by

$$i \mapsto \begin{cases} 1 & \text{if } i \in I_n \\ 2 & \text{if } i \in I_{n'} \end{cases}$$

induces a map $\pi\colon I^p_{n+n'} \to I^p_2$ which preserves the σ-action. Therefore

$$\Gamma_{n+n'} = \Gamma_n \cup \Gamma_{n'} \cup \bigcup_{\lambda \in \Gamma_2} \pi^{-1}(\lambda).$$

Using this terminology we equate

$$\begin{aligned} & \tilde{\theta}_p(\eta + \eta') - \tilde{\theta}_p(\eta) - \tilde{\theta}_p(\eta') \\ = & \left[\sum_{t=1}^{p-1} \sum_{\lambda \in \Gamma_2} \sum_{\gamma \in \pi^{-1}(\lambda)} (t\sigma^t \gamma(\tilde{\alpha}, \tilde{\beta}) - t\sigma^t \gamma(\tilde{\beta}, \tilde{\alpha})) \right] \\ = & \left[\sum_{t=1}^{p-1} \sum_{\lambda \in \Gamma_2} (t\sigma^t \lambda(\rho) - t\sigma^t \lambda(\rho)) \right] \\ = & \ 0, \end{aligned}$$

where $\rho\colon I_2 \to R$ is defined by $\rho(1) = r$ and $\rho(2) = r'$.
And $\lambda(\rho) = \rho_{i_1} \cdots \rho_{i_{p-1}} \otimes \rho_{i_p}$ if $\lambda = (i_1, \ldots, i_p) \in I^p_2$.

2.8. Now it is time to apply lemma 2.5 and show that $\tilde{\theta}_p$ induces a homomorphism θ'_p on $\operatorname{Ker}(b\colon R \otimes R \to R)$:

- ◇ It is clear that $\tilde{\theta}_p(u, 0) = \tilde{\theta}_p(0, u) = 0$, for all $u \in R$.
- ◇ $\tilde{\theta}_p((u, v+w) + (u, -v) + (u, -w)) = 0$, for all $u, v, w \in R$:
  Define $\alpha, \beta\colon I_3 \to R$ by

  $$\begin{array}{lll} \alpha(1) := u & \alpha(2) := u & \alpha(3) := u \\ \beta(1) := v + w & \beta(2) := -v & \beta(3) := -w. \end{array}$$

  The map $I_3 \to I_2$ defined by $1 \mapsto 1, \quad 2 \mapsto 2, \quad 3 \mapsto 2$ induces a map $\pi\colon I^p_3 \to I^p_2$ which preserves the σ-action. Define $\alpha', \beta'\colon I_2 \to R$ by

  $$\begin{array}{ll} \alpha'(1) := u & \alpha'(2) := u \\ \beta'(1) := -v & \beta'(2) := -w. \end{array}$$

  And finally we define

  $$\gamma_1 := u\beta'_1 u\beta'_2 \cdots u \otimes \beta'_{i_p} \qquad \gamma_2 := \beta'_1 u\beta'_2 u \cdots \beta'_{i_p} \otimes u$$

for all $\gamma = (i_1, \ldots, i_p) \in I_2^p$.

$$\tilde{\theta}_p((u, v+w) + (u, -v) + (u, -w)) = \tilde{\theta}_p \left(\sum_{i \in I_3} (\alpha_i, \beta_i) \right).$$

$$\begin{aligned}
&(u(v+w))^{p-1} u \otimes (v+w) - (uv)^{p-1} u \otimes v - (uw)^{p-1} u \otimes w \\
&\quad = -\sum_{\gamma \in I_2} \gamma_1 + \sum_{\gamma \in \Delta_2} \gamma_1 \\
&\quad = -\sum_{\gamma \in I_2 - \Delta_2} \gamma_1 \\
&\quad = -\sum_{\gamma \in \Gamma_2} \sum_{t=1}^{p} (\sigma^t \gamma)_1
\end{aligned}$$

$$\begin{aligned}
&\sum_{\gamma \in \Gamma_3} \sum_{t=1}^{p-1} (t\sigma^t \gamma(\alpha, \beta) - t\sigma^t \gamma(\beta, \alpha)) \\
&\quad = \sum_{\lambda \in \Gamma_2} \sum_{t=1}^{p-1} \sum_{\gamma \in \pi^{-1}(\lambda)} (t\sigma^t \gamma(\alpha, \beta) - t\sigma^t \gamma(\beta, \alpha)) \\
&\qquad + \sum_{\gamma \in \Gamma_2} \sum_{t=1}^{p-1} (t\sigma^t \gamma(\alpha', \beta') - t\sigma^t \gamma(\beta', \alpha'))
\end{aligned}$$

But for all $\lambda \in \Gamma_2$ we have

$$\begin{aligned}
&\left[\sum_{\gamma \in \pi^{-1}(\lambda)} (t\sigma^t \gamma(\alpha, \beta) - t\sigma^t \gamma(\beta, \alpha)) \right] \\
&= \pm \left[t((u(v+w))^{p-1} \otimes u(v+w) - ((v+w)u)^{p-1} \otimes (v+w)u \right] \\
&= 0,
\end{aligned}$$

since $[(ab)^k \otimes ab - (ba)^k \otimes ba] = 0$ in $HC_1(R)$. Further

$$\begin{aligned}
&\left[\sum_{\gamma \in \Gamma_2} \sum_{t=1}^{p-1} (t\sigma^t \gamma(\alpha', \beta') - t\sigma^t \gamma(\beta', \alpha')) \right] \\
&\quad = \left[\sum_{\gamma \in \Gamma_2} \sum_{t=1}^{p-1} (t(\sigma^{t+1} \gamma)_2 - t(\sigma^t \gamma)_2) \right] \\
&\quad = \left[-\sum_{\gamma \in \Gamma_2} \sum_{t=1}^{p} (\sigma^t \gamma)_2 \right]
\end{aligned}$$

Conclusion:

$$\begin{aligned}\tilde{\theta}_p((u,v+w)+(u,-v)+(u,-w)) &= \left[-\sum_{\gamma\in\Gamma_2}\sum_{t=1}^{p}((\sigma^t\gamma)_1+(\sigma^t\gamma)_2)\right] \\ &= \left[-\sum_{\gamma\in\Gamma_2}\sum_{t=1}^{p}\sigma^t\gamma(\beta',\alpha')\right] \\ &= 0\end{aligned}$$

according to the corollary following lemma 2.4.

- ◇ In a similar way one can prove that $\tilde{\theta}_p((u+v,w)+(-u,w)+(-v,w))=0$, for all $u,v,w\in R$.

Thus we obtain a homomorphism $\theta'_p\colon \mathrm{Ker}(b)\longrightarrow HC_1(R/pR)$.

2.9 Proposition. $\theta'_p(u\otimes v+v\otimes u)=[(uv)^{p-1}\otimes uv]$ *for all* $u,v\in R$.

Proof. Define $\alpha,\beta\colon I_2\to R$ by $\alpha(1):=u$, $\alpha(2):=v$, $\beta(1):=v$, $\beta(2):=u$. We equate

$$\begin{aligned}&\theta'_p(u\otimes v+v\otimes u)\\ &= \left[(uv)^{p-1}u\otimes v+(vu)^{p-1}v\otimes u+\sum_{\gamma\in\Gamma_2}\sum_{t=1}^{p-1}t\sigma^t\gamma(\alpha,\beta)-t\sigma^t\gamma(\beta,\alpha)\right]\end{aligned}$$

The permutation $I_2\to I_2$ determined by $1\mapsto 2$, $2\mapsto 1$, induces a permutation $\pi\colon I_2^p\to I_2^p$ which preserves the σ-action. Since $\pi\gamma(\alpha,\beta)=\gamma(\beta,\alpha)$ for every $\gamma\in\Gamma_2$, the term involving the double sum in the equation above vanishes. Adding this to the fact that $[(uv)^{p-1}u\otimes v+(vu)^{p-1}v\otimes u]=[(uv)^{p-1}\otimes uv]$ proves the proposition. •

2.10. To finish the proof of theorem 2.2 it only remains to show that

$$\theta'_p(uv\otimes w-u\otimes vw+wu\otimes v)=0 \quad\text{for all}\quad u,v,w\in R.$$

For this purpose we define $\alpha,\beta\colon I_3\to R$ by

$$\begin{array}{lll}\alpha(1):=uv & \alpha(2):=vw & \alpha(3):=wu\\ \beta(1):=w & \beta(2):=u & \beta(3):=v\end{array}$$

We use proposition 2.9 to equate

$$\begin{aligned}&\theta'_p(uv\otimes w-u\otimes vw+wu\otimes v)\\ &= \theta'_p((uv\otimes w+vw\otimes u+wu\otimes v)-(u\otimes vw+vw\otimes u))\\ &= [(uvw)^{p-1}uv\otimes w+(vwu)^{p-1}vw\otimes u+(wuv)^{p-1}wu\otimes v\\ &\quad +\sum_{\gamma\in\Gamma_3}\sum_{t=1}^{p-1}(t\sigma^t\gamma(\alpha,\beta)-t\sigma^t\gamma(\beta,\alpha))-(uvw)^{p-1}\otimes uvw].\end{aligned}$$

The permutation $I_3 \to I_3$ defined by $1 \mapsto 3, \quad 2 \mapsto 1, \quad 3 \mapsto 2$, induces a permutation π of I_3^p which respects the σ-action. Since $\pi\gamma(\alpha,\beta) = \gamma(\beta,\alpha)$ for every $\gamma \in \Gamma_3$, the term involving the double sum in the equation above vanishes. And because

$$[(uvw)^{p-1}uv \otimes w + (vwu)^{p-1}vw \otimes u + (wuv)^{p-1}wu \otimes v - (uvw)^{p-1} \otimes uvw] = 0,$$

we are done.

This completes the proof of theorem 2.2.

2.11. Let $B\colon HC_0(R) \to H_1(R)$ denote the homomorphism determined by $[r] \mapsto [r\otimes 1 + 1\otimes r] = [1\otimes r]$. The composition of B and $\theta_p\colon H_1(R) \to HC_1(R/pR)$ yields a homomorphism $q\colon HC_0(R) \to HC_1(R/pR)$, which, as a consequence of proposition 2.9, maps $[r]$ to $[r^{p-1} \otimes r]$.

2.12 Theorem. *The homomorphism $\theta_p\colon H_1(R) \to HC_1(R/pR)$ induces a homomorphism $HC_1(R) \to \mathrm{Coker}(q)$.*

Proof. This is an immediate consequence of proposition 2.9. •

2.13. Now recall the definition 1.9 of the quaternionic homology group $HQ_1(R)$. There is an isomorphism

$$HQ_1(R) \hookrightarrow\!\!\!\!\rightarrow \frac{\mathrm{Ker}((b \;\; 1-y)\colon (R\otimes R)\oplus R \to R)}{\mathrm{Span}\left\{\begin{array}{c}(r\otimes s + s\otimes r, -rs - \overline{rs}),\\ (u\otimes v + \overline{u}\otimes\overline{v}, vu - uv),\\ (0, 2(w+\overline{w})),\\ (xy\otimes z - x\otimes yz + zx\otimes y, 0)\end{array}\right\}}$$

defined by

$$[\varpi, a, b] \mapsto [\varpi, 0, b + a + \overline{a}].$$

Here

$$b\colon R\otimes R \to R \;\text{ is defined by }\; b(r_1\otimes r_2) = r_1r_2 - r_2r_1,$$

$$y\colon R \to R \;\text{ is defined by }\; y(r) = \overline{r}.$$

2.14. The correspondence $x \mapsto [1\otimes x, 0]$ obviously defines a homomorphism $\nu_R\colon R \to HQ_1(R)$.

2.15 Lemma. The map $x \mapsto [x\otimes x, 0]$ determines a well-defined homomorphism $\mu_R\colon R \to \mathrm{Coker}\,\nu_R$.

Proof. For all $x, y \in R$,
$\mu_R(x+y) - \mu_R(x) - \mu_R(y) = [x\otimes y + y\otimes x, 0] = [\nu_R(xy)] = 0$. The rest is obvious. •

2.16 Theorem. *There exists a well-defined homomorphism*

$$\vartheta = \vartheta_R \colon HQ_1(R) \to \operatorname{Coker}(\mu_{R/2R}) = \frac{HQ_1(R/2R)}{\operatorname{Span}\{[x \otimes x, 0] \mid x \in R\}}$$

defined by:

$$\left[\sum_{i \in I_n} \alpha_i \otimes \beta_i, c\right] \mapsto \left[\sum_{i \in I_n} \alpha_i\beta_i\alpha_i \otimes \beta_i + \sum_{i \in I_n} \alpha_i\beta_i \otimes \beta_i\alpha_i \right.$$
$$\left. + \sum_{i<j} (\alpha_i\beta_i + \beta_i\alpha_i) \otimes (\alpha_j\beta_j + \beta_j\alpha_j) + c \otimes \overline{c}, c^2\right]$$

The proof will come about in a few steps.

2.17 Remark.

$$\vartheta\left(\left[\sum_{i \in I_n} \alpha_i \otimes \beta_i, c\right]\right) = \left[\sum_{i \in I_n} \alpha_i\beta_i\alpha_i \otimes \beta_i + \left(\sum_{i \in I_n} \alpha_i\beta_i\right) \otimes \left(\sum_{i \in I_n} \beta_i\alpha_i\right)\right.$$
$$\left. + \sum_{\gamma \in \Gamma_2} (\gamma(\alpha, \beta) - \gamma(\beta, \alpha)) + c \otimes \overline{c}, c^2\right].$$

$\sum_{\gamma \in \Gamma_2} [\gamma(\alpha, \beta) - \sigma\gamma(\alpha, \beta)] = 0$, since $[r \otimes s + s \otimes r, 0] = 0$ in $\operatorname{Coker}(\mu_{R/2R})$. Thus it is clear that ϑ does not depend on the way the sum $\sum_{i \in I_n} \alpha_i \otimes \beta_i$, is ordered.

2.18 Lemma. If $(\sum_{i \in I_n} \alpha_i \otimes \beta_i, c) \in \operatorname{Ker}(b \;\; 1 - y)$, then

$$\left(\sum_{i \in I_n} \alpha_i\beta_i\alpha_i \otimes \beta_i + \alpha_i\beta_i \otimes \beta_i\alpha_i + \right.$$
$$\left. \sum_{i<j} (\alpha_i\beta_i + \beta_i\alpha_i) \otimes (\alpha_j\beta_j + \beta_j\alpha_j) + c \otimes \overline{c}, c^2\right)$$

is a cycle in $HQ_1(R/2R)$.

Proof. The image of this expression under the homomorphism $(b \;\; 1 - y)$ equals

$$\sum_{i \in I_n} (\alpha_i\beta_i)^2 + (\beta_i\alpha_i)^2 + \alpha_i\beta_i\beta_i\alpha_i + \beta_i\alpha_i\alpha_i\beta_i +$$
$$\sum_{i<j} (\alpha_i\beta_i + \beta_i\alpha_i)(\alpha_j\beta_j + \beta_j\alpha_j) + (\alpha_j\beta_j + \beta_j\alpha_j)(\alpha_i\beta_i + \beta_i\alpha_i) +$$
$$c\overline{c} + \overline{c}c + c^2 + \overline{c}^2 =$$
$$\sum_{i,j \in I_n} (\alpha_i\beta_i + \beta_i\alpha_i)(\alpha_j\beta_j + \beta_j\alpha_j) + c\overline{c} + \overline{c}c + c^2 + \overline{c}^2 =$$
$$(c + \overline{c})^2 + c\overline{c} + \overline{c}c + c^2 + \overline{c}^2 = 0$$

•

2.19 Lemma. As before $\mathcal{F}(R \times R)$ denotes the free abelian monoid on the set $R \times R$ and $\otimes: \mathcal{F}(R \times R) \to R \otimes R$ is the canonical mapping. Compare lemma 2.5. There is a bijective correspondence between homomorphisms on

$$\mathrm{Ker}((b\ \ 1-y): (R \otimes R) \oplus R \to R)$$

and morphisms on

$$\mathrm{Ker}\left((b\ \ 1-y)\begin{pmatrix} \otimes & 0 \\ 0 & 1 \end{pmatrix}\right) = \mathrm{Ker}((b\otimes\ \ 1-y): \mathcal{F}(R \times R) \oplus R \to R),$$

which kill all elements of the form

$$\begin{array}{ll} ((u,0),0) & u \in R \\ ((0,u),0) & u \in R \\ ((u,v+w)+(u,-v)+(u,-w),0) & u,v,w \in R \\ ((u+v,w)+(-u,w)+(-v,w),0) & u,v,w \in R. \end{array}$$

Proof. Modulo a few minor adjustments the proof of lemma 2.5 will do. •

We apply this lemma to the map $\tilde{\vartheta}: \mathrm{Ker}(b\otimes\ \ 1-y) \to \mathrm{Coker}(\mu_{R/2R})$ defined by

$$\begin{aligned} \tilde{\vartheta}\left(\sum_{i \in I_n}(\alpha_i,\beta_i), c\right) \quad = \quad & \Bigg[\sum_{i \in I_n} \alpha_i\beta_i\alpha_i \otimes \beta_i + \alpha_i\beta_i \otimes \beta_i\alpha_i + \\ & \sum_{i<j}(\alpha_i\beta_i + \beta_i\alpha_i) \otimes (\alpha_j\beta_j + \beta_j\alpha_j) + c \otimes \overline{c}, c^2\Bigg]. \end{aligned}$$

2.20 Lemma. $\tilde{\vartheta}$ is a morphism on $\mathrm{Ker}(b\otimes\ \ 1-y)$.

Proof. If

$$\eta := \left(\sum_{i \in I_n}(\alpha_i,\beta_i), c\right) \quad \text{and} \quad \eta' := \left(\sum_{i \in I_{n'}}(\alpha'_i,\beta'_i), c'\right)$$

are in $\mathrm{Ker}(b\otimes\ \ 1-y)$, then

$$\begin{aligned} & \tilde{\vartheta}(\eta+\eta') - \tilde{\vartheta}(\eta) - \tilde{\vartheta}(\eta') \\ & \quad = \quad \Bigg[\left(\sum_{i \in I_n} \alpha_i\beta_i + \beta_i\alpha_i\right) \otimes \left(\sum_{i \in I_{n'}} \alpha'_i\beta'_i + \beta'_i\alpha'_i\right) + \\ & \qquad\quad (c+c') \otimes (\overline{c} + \overline{c'}) + c \otimes \overline{c} + c' \otimes \overline{c'}, (c+c')^2 + c^2 + {c'}^2\Bigg] \\ & \quad = \quad \left[(c+\overline{c}) \otimes (c' + \overline{c'}) + c \otimes \overline{c'} + c' \otimes \overline{c}, cc' + c'c\right] \\ & \quad = \quad \left[c \otimes c' + \overline{c} \otimes c' + c' \otimes \overline{c} + \overline{c} \otimes \overline{c'}, cc' + c'c\right] \\ & \quad = \quad 0 \end{aligned}$$

which proves the lemma. •

The next step is to prove that $\tilde{\vartheta}$ induces a homomorphism ϑ' on $\mathrm{Ker}(b \ \ 1-y)$.

2.21. We use lemma 2.19:

- $\diamond$ It is clear that $\tilde{\vartheta}((u,0),0) = \tilde{\vartheta}((0,u),0) = 0$ for all $u \in R$.
- $\diamond$ For all $u, v, w \in R$,

$$\begin{aligned}
&\tilde{\vartheta}((u, v+w) + (u,-v) + (u,-w), 0) \\
&\quad = [u(v+w)u \otimes (v+w) + uvu \otimes v + uwu \otimes w + \\
&\qquad u(v+w) \otimes (v+w)u + uv \otimes vu + uw \otimes wu + \\
&\qquad (u(v+w) + (v+w)u) \otimes (uv+vu) + \\
&\qquad (u(v+w) + (v+w)u) \otimes (uw+wu) + \\
&\qquad (uv+vu) \otimes (uw+wu), 0] \\
&\quad = [uvu \otimes w + uwu \otimes v + uv \otimes wu + uw \otimes vu + \\
&\qquad (uv+vu) \otimes (uw+wu)] \\
&\quad = 0.
\end{aligned}$$

- $\diamond$ In the same fashion one proves that $\tilde{\vartheta}((u+v,w)+(-u,w)+(-v,w),0) = 0$ for all $u, v, w \in R$.

Finally we use 2.13 to verify that ϑ' induces the promised homomorphism ϑ.

2.22. For all $r, s, u, v, w, x, y, z \in R$ we have

$$\begin{aligned}
&\vartheta'(r \otimes s + s \otimes r, rs + \overline{rs}) \\
&\quad = [rsr \otimes s + rs \otimes sr + srs \otimes r + sr \otimes rs + \\
&\qquad (rs + sr) \otimes (rs + sr) + (rs + \overline{rs}) \otimes (rs + \overline{rs}), (rs + \overline{rs})^2] \\
&\quad = [0, rsrs + rs\overline{rs} + \overline{rs}rs + \overline{rsrs}] \\
&\quad = 0, \\
&\vartheta'(u \otimes v + \overline{u} \otimes \overline{v}, vu - uv) \\
&\quad = [uvu \otimes v + \overline{uvu} \otimes \overline{v} + uv \otimes vu + \overline{uv} \otimes \overline{vu} + \\
&\qquad (uv + vu) \otimes (\overline{uv} + \overline{vu}) + (uv + vu) \otimes (\overline{uv + vu}), (uv + vu)^2] \\
&\quad = [uvu \otimes v + \overline{uvu} \otimes \overline{v} + uv \otimes vu + \overline{uv} \otimes \overline{vu}, (uv + vu)^2] \\
&\quad = 0, \\
&\vartheta'(xy \otimes z + x \otimes yz + zx \otimes y, 0) \\
&\quad = [xyzxy \otimes z + xyzx \otimes yz + zxyzx \otimes y + \\
&\qquad xyz \otimes zxy + xyz \otimes yzx + zxy \otimes yzx + \\
&\qquad (xyz + zxy) \otimes (xyz + yzx) + \\
&\qquad (xyz + zxy) \otimes (zxy + yzx) + \\
&\qquad (xyz + yzx) \otimes (zxy + yzx), 0] \\
&\quad = 0, \\
&\vartheta'(0, 2(w + w')) = 0.
\end{aligned}$$

This step completes the proof of theorem 2.16.

3 Morita invariance.

3.1 Theorem. *Let A be the ring of $m \times m$-matrices over the k-algebra R. The trace-maps* $\mathrm{Tr}\colon A^n \to R^n$ *determined by*

$$\mathrm{Tr}(X_1 \otimes X_2 \otimes \cdots \otimes X_n) := \sum_{i_1,\ldots,i_n} (X_1)_{i_1 i_2} \otimes (X_2)_{i_2 i_3} \otimes \cdots \otimes (X_n)_{i_n i_1}$$

yield a chain equivalence between the Hochschild complexes (A^*, b) and (R^*, b). A chain inverse is given by the maps $\iota\colon R^n \to A^n$ defined by

$$\iota(r_1 \otimes r_2 \otimes \cdots \otimes r_n) := E_{11}(r_1) \otimes \cdots \otimes E_{11}(r_n)$$

Where $E_{ij}(r)$ denotes the $m \times m$-matrix with r in the (i,j)-entry and zeros in all other entries.

Proof. It's easy to check that Tr and ι are chain maps. We immediately see that $\mathrm{Tr} \circ \iota = 1$. We will show that $\iota \circ \mathrm{Tr} \simeq 1$ simply by giving a chain homotopy. For that purpose we proceed to introduce the following definitions:
Define

$$\gamma\colon A^{n+1} \to A^{n+1}$$

by

$$\gamma(X_0 \otimes X_1 \otimes \cdots \otimes X_n) := (-1)^{n+1} \sum_{i=1}^{m} E_{i1}(1) \otimes E_{1i}(1) X_n X_0 \otimes X_1 \otimes \cdots \otimes X_{n-1},$$

$$s\colon A^n \to A^{n+1}$$

by

$$s(X_1 \otimes \cdots \otimes X_n) := X_1 \otimes \cdots \otimes X_n \otimes 1$$

and finally

$$\chi_n\colon A^n \to A^{n+1}$$

by

$$\chi_n := (-1)^{n+1} \sum_{k=1}^{n} \gamma^k s.$$

The following relations are valid:

$$\sum_{i=1}^{m} E_{i1}(1) E_{1i}(1) = 1 \tag{1}$$

$$d_0 \gamma \stackrel{1}{=} (-1)^{n+1} d_n$$

$$d_0 \gamma^k = (-1)^{n+1} d_n \gamma^{k-1} \quad \text{if } \ k > 0 \tag{2}$$

$$d_i \gamma = -\gamma d_{i-1} \quad \text{if } \ 1 \le i < n \tag{3}$$

$$d_i \gamma^k = = (-1)^k \gamma^k d_{i-k} \quad \text{if } \ k \le i < n \tag{4}$$

$$\begin{aligned}
d_1\gamma^2 &\overset{3}{=} -\gamma d_0\gamma && (5)\\
&\overset{2}{=} (-1)^n\gamma d_n\\
&= (-1)^n\gamma d_{n-1}\\
d_i\gamma^k &\overset{4}{=} (-1)^{i-1}\gamma^{i-1}d_1\gamma^{k-i+1} && (6)\\
&\overset{5}{=} (-1)^{n+i-1}\gamma^i d_{n-1}\gamma^{k-i-1}\\
&= (-1)^{n+k}\gamma^{k-1}d_{n+i-k} \text{ if } 0<i<k\\
\gamma s d_n &= \gamma d_n s && (7)\\
E_{1i}(1)XE_{j1}(1) &= E_{11}(X_{ij}) && (8)\\
d_n\gamma^n s &\overset{8}{=} \iota\circ \mathrm{Tr} && (9)
\end{aligned}$$

Now we are in the position to prove that

$$b\chi_n + \chi_{n-1}b = 1 - \iota\,\mathrm{Tr}:$$

$$\begin{aligned}
b\chi_n &= (-1)^{n+1}\sum_{k=1}^{n}\sum_{i=0}^{n}(-1)^i d_i\gamma^k s\\
&= (-1)^{n+1}(\sum_{k=1}^{n}(d_0\gamma^k s + (-1)^n d_n\gamma^k s) + \sum_{k=1}^{n}\sum_{i=1}^{n-1}(-1)^i d_i\gamma^k s)\\
&\overset{2}{=} 1 - d_n\gamma^n s +\\
&\quad (-1)^{n+1}(\sum_{k=1}^{n-1}\sum_{i=k}^{n-1}(-1)^i d_i\gamma^k s + \sum_{k=2}^{n}\sum_{i=1}^{k-1}(-1)^i d_i\gamma^k s)\\
&\overset{4\,6}{=} 1 - \iota\,\mathrm{Tr} +\\
&\quad (-1)^{n+1}(\sum_{k=1}^{n-1}\sum_{i=k}^{n-1}(-1)^{i+k}\gamma^k d_{i-k}s + \sum_{k=2}^{n}\sum_{i=1}^{k-1}(-1)^{n+i+k}\gamma^{k-1}d_{n+i-k}s)\\
&= 1 - \iota\,\mathrm{Tr} +\\
&\quad (-1)^{n+1}(\sum_{k=1}^{n-1}\sum_{m=0}^{n-k-1}(-1)^m\gamma^k d_m s + \sum_{k=2}^{n}\sum_{m=n-k+1}^{n-1}(-1)^m\gamma^{k-1}d_m s)\\
&= 1 - \iota\,\mathrm{Tr} + (-1)^{n+1}(\sum_{k=1}^{n-1}\sum_{m=0}^{n-1}(-1)^m\gamma^k d_m s)\\
&\overset{7}{=} 1 - \iota\,\mathrm{Tr} + (-1)^{n+1}(\sum_{k=1}^{n-1}\sum_{m=0}^{n-1}(-1)^m\gamma^k s d_m)\\
&= 1 - \iota\,\mathrm{Tr} - \chi_{n-1}b.
\end{aligned}$$

•

Let $\bar{\ }: R \to R$ be an anti-involution of k-algebras. We extend this anti-involution to an anti-involution $\bar{\ }: A \to A$ by defining $(\overline{X})_{ij} = \overline{X_{ji}}$ for every $X \in$

A. According to example 1.5 we may regard both R^* and A^* as quaternionic modules.

3.2 Theorem. *The map* Tr *induces isomorphisms*

$$H_*(A) \xrightarrow{\text{Tr}} H_*(R)$$

$$HC_*(A) \xrightarrow{\text{Tr}} HC_*(R)$$

$$HQ_*(A) \xrightarrow{\text{Tr}} HQ_*(R)$$

Proof. It is clear from the definitions that both ι and Tr preserve x and y. •

3.3 Theorem. *The following diagrams commute*

◇

$$\begin{array}{ccc} HC_0(A) & \xrightarrow{B} & H_1(A) \\ \downarrow{\scriptstyle \text{Tr}} & & \downarrow{\scriptstyle \text{Tr}} \\ HC_0(R) & \xrightarrow{B} & H_1(R) \end{array}$$

◇

$$\begin{array}{ccc} H_0(A) & \xrightarrow{\theta_p} & H_0(A/pA) \\ \downarrow{\scriptstyle \text{Tr}} & & \downarrow{\scriptstyle \text{Tr}} \\ H_0(R) & \xrightarrow{\theta_p} & H_0(R/pR) \end{array}$$

◇

$$\begin{array}{ccc} HC_1(A) & \xrightarrow{\theta_p} & HC_1(A/pA)/\operatorname{Im}(q) \\ \downarrow{\scriptstyle \text{Tr}} & & \downarrow{\scriptstyle \text{Tr}} \\ HC_1(R) & \xrightarrow{\theta_p} & HC_1(R/pR)/\operatorname{Im}(q) \end{array}$$

◇

$$\begin{array}{ccc} HQ_1(A) & \xrightarrow{\vartheta_A} & \operatorname{Coker}(\mu_{(A/2A)}) \\ \downarrow{\scriptstyle \text{Tr}} & & \downarrow{\scriptstyle \text{Tr}} \\ HQ_1(R) & \xrightarrow{\vartheta_R} & \operatorname{Coker}(\mu_{(R/2R)}) \end{array}$$

Proof. A little examination of the definitions shows that B, θ_p and ϑ commute with ι. •

4 Generalized Arf invariants.

Let (R, α, u) be a ring with anti-structure with $u = \pm 1$. Thus u is central and α is an anti-involution.

4.1 Theorem. *The map*

$$\Upsilon\colon \mathrm{Arf}^h(R, \alpha, u) \to \mathrm{Coker}(1 + \vartheta_R)$$

determined by

$$(a, b) \mapsto [a \otimes b, ab]$$

is a well-defined homomorphism.

We are a bit sloppy here in denoting the projection $HQ_1(R) \to \mathrm{Coker}(\mu_{R/2R})$ by 1.

Proof. Recall the presentation of $\mathrm{Arf}^h(R, \alpha, u)$ from theorem 2.4.
For all $a, b \in \Lambda_1(R)$ the element $(a \otimes b, ab)$ is a cycle in $HQ_1(R/2R)$:

$$(b \;\; 1 - y)(a \otimes b, ab) = ab + ba + ab + ba = 0.$$

Next we will check that Υ respects all the relations of the aforementioned presentation.

1. obvious
2. obvious
3. $[a \otimes b + b \otimes a, ab + ba] = 0$
4. $[a \otimes (x + \alpha(x)), a(x + \alpha(x)] = [a \otimes x + \alpha(x) \otimes a, ax + xa] = 0$
5. $[a \otimes \alpha(x)bx + xa\alpha(x) \otimes b, a\alpha(x)bx + xa\alpha(x)b] =$
   $[a\alpha(x)b \otimes x + xa \otimes \alpha(x)b, a\alpha(x)bx + xa\alpha(x)b] =$
   $[a\alpha(x)b \otimes x + bxa \otimes \alpha(x), a\alpha(x)bx + xa\alpha(x)b] = 0$
6. $\vartheta([a \otimes b, ab]) = [aba \otimes b + ab \otimes ba + ab \otimes ba, abab] = [aba \otimes b, abab]$
7. Suppose

   $$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix} \in \mathrm{GL}_{2n}(R) \;\text{ satisfies }\; t_{\alpha,u}\left(\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}\right) = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{-1}.$$

   Then using the relations for X, Y, Z and T, we equate

   $$\begin{aligned}
   &(1 + \vartheta)[X^\alpha \otimes T + Z \otimes Y^\alpha, X^\alpha T] \\
   &\quad = [X^\alpha \otimes T + Z \otimes Y^\alpha + \\
   &\qquad X^\alpha T X^\alpha \otimes T + Z Y^\alpha Z \otimes Y^\alpha + X^\alpha T \otimes T X^\alpha + Z Y^\alpha \otimes Y^\alpha Z + \\
   &\qquad (X^\alpha T + T X^\alpha) \otimes (Z Y^\alpha + Y^\alpha Z) + X^\alpha T \otimes T^\alpha X, X^\alpha T + (X^\alpha T)^2] \\
   &\quad = [X^\alpha Z Y^\alpha \otimes T + T X^\alpha Z \otimes Y^\alpha, X^\alpha Z Y^\alpha T] \\
   &\quad = [X^\alpha Z \otimes Y^\alpha T, X^\alpha Z Y^\alpha T].
   \end{aligned}$$

   Now theorem 3.3 finishes the job.

This finishes the proof. •

4.2 Remark. In the case that R is commutative and α is the identity we have

$$\begin{aligned}\operatorname{Coker}(1+\vartheta_R) &= \frac{R}{\operatorname{Span}\{x+x^2 \mid x \in R\}} \oplus \\ &\quad \frac{\Omega_R}{2\Omega_R + \delta R + \{(r+r^2\delta s)\delta s \mid r,s \in R\}} \\ &= \operatorname{Coker}(1+\theta_2\colon H_0(R) \to H_0(R/2R)) \oplus \\ &\quad \operatorname{Coker}(1+\theta_2\colon H_1(R) \to HC_1(R/2R)).\end{aligned}$$

This can be verified by a little examination of 2.13 and the definitions of θ_2 and ϑ in proposition 2.1, theorem 2.2 and theorem 2.16. The projection of

$$\Upsilon\colon \operatorname{Arf}^h(R,1,-1) \to \operatorname{Coker}(1+\theta_2) \oplus \operatorname{Coker}(1+\theta_2)$$

on the first summand is just the old primary Arf invariant. The secondary Arf invariant

$$\operatorname{Arf}^s(R,1,-1) \longrightarrow \frac{\Omega_R}{2\Omega_R + \delta R + \{(r+r^2\delta s)\delta s \mid r,s \in R\}}$$

factors through the projection of Υ on the second summand.

Chapter IV

Applications to group rings.

1 Quaternionic homology of group rings.

The following exposition is based upon the work of J.-L. Loday in [16].
Let k be a commutative ring with identity, G a group and $k[G]$ the group algebra of G over k. By providing $k[G]$ with the anti-involution $\bar{\ }$ determined by $\overline{g} = g^{-1}$ for all $g \in G$, $k[G] \otimes_k k[G]^n$ becomes a quaternionic module by means of example 1.5 of chapter III.

Notation. Denote by Γ the set of conjugacy classes of G and by $C: G \to \Gamma$ the map which assigns to $g \in G$ its conjugacy class $C(g)$. Further we choose a section $S: \Gamma \to G$ of C such that $S(C(g^{-1})) = (S(C(g))^{-1}$ for every $g \in G$ with $C(g) \neq C(g^{-1})$. Finally, for every set V endowed with a right G-action we supply the free k-module $k[V]$ with a $k[G]$-bimodule structure by letting G act trivially from the left-hand side on V.

1.1. For every $z \in G$, the right action $C(z) \times G \to C(z)$ of G on $C(z)$ defined by $(x, g) \mapsto g^{-1}xg$ for all $x \in C(z)$ and $g \in G$, makes $k[C(z)]$ into a $k[G]$-bimodule.

1.2 Lemma. The map

$$\phi: k[G] \otimes_k k[G]^n \to \bigoplus_{z \in \operatorname{Im} S} k[C(z)] \otimes_k k[G]^n$$

determined by

$$\phi(g \otimes g_1 \otimes \cdots \otimes g_n) := \begin{cases} g_1 \cdots g_n g \otimes g_1 \otimes \cdots \otimes g_n & \text{if } g g_1 \cdots g_n \in C(z) \\ 0 & \text{otherwise,} \end{cases}$$

is an isomorphism of simplicial modules with inverse determined by

$$h \otimes g_1 \otimes \cdots \otimes g_n \mapsto (g_1 \cdots g_n)^{-1} h \otimes g_1 \otimes \cdots \otimes g_n \text{ for all } h \in C(z).$$

Proof. See [16]. •

1.3 Definition. We say that $z \in G$ is of type
1 if $z = z^{-1}$,
2 if $z^{-1} \in C(z)$ and $z \neq z^{-1}$,
3 if $z^{-1} \notin C(z)$.

For each $i \in \{1, 2, 3\}$ let S_i denote the subset of $\operatorname{Im} S$ consisting of all elements of type i. Notice that $\operatorname{Im} S$ is the disjoint union of the S_i. Now S was chosen in such a way that $z \in S_3 \Leftrightarrow z^{-1} \in S_3$, This allows us to write S_3 as a disjoint union of sets S_3^+ and S_3^- such that $z \in S_3^+ \Leftrightarrow z^{-1} \in S_3^-$.

1.4 Definition. The simplicial module $k[C(z) \cup C(z^{-1})] \otimes_k k[G]^n$ becomes a quaternionic module by defining

$$x(g \otimes g_1 \otimes \cdots \otimes g_n) := (-1)^n (g_1 \cdots g_n)^{-1} g g_1 \cdots g_n \otimes (g_1 \cdots g_n)^{-1} g \otimes g_1 \otimes \cdots \otimes g_{n-1}$$

and

$$y(g \otimes g_1 \otimes \cdots \otimes g_n) := (-1)^{\frac{n(n+1)}{2}} (g_1 \cdots g_n)^{-1} g^{-1} g_1 \cdots g_n \otimes g_n^{-1} \otimes \cdots \otimes g_1^{-1}$$

1.5 Theorem.

$$\phi \colon k[G] \otimes_k k[G]^n \to \bigoplus_{z \in S_1 \cup S_2 \cup S_3^+} k[C(z) \cup C(z^{-1})] \otimes_k k[G]^n$$

is an isomorphism of quaternionic modules.

Proof. This is easy to check. The maps x and y were defined so as to make ϕ respect the quaternionic structure. •

1.6. For every group G we define

$$d_i \colon k[G]^{n+1} \to k[G]^n$$

by

$$\begin{array}{rcll} d_i(g_0 \otimes g_1 \otimes \cdots \otimes g_n) & := & g_0 \otimes \cdots \otimes g_i g_{i+1} \otimes \cdots \otimes g_n & \text{if} \quad 0 \le i < n \\ d_n(g_0 \otimes g_1 \otimes \cdots \otimes g_n) & := & g_0 \otimes \cdots \otimes g_{n-1}, & \end{array}$$

$$d \colon k[G]^{n+1} \to k[G]^n \qquad \text{by} \qquad d := \sum_{i=0}^{n} (-1)^i d_i$$

and

$$d' \colon k[G]^{n+1} \to k[G]^n \qquad \text{by} \qquad d' := \sum_{i=0}^{n-1} (-1)^i d_i.$$

Now the map

$$s \colon k[G]^{n+1} \to k[G]^{n+2} \text{ determined by } s(g_0 \otimes \cdots \otimes g_n) := 1 \otimes g_0 \otimes \cdots \otimes g_n$$

satisfies $sd + ds = sd' + d's = 1$ and therefore provides for a chain contraction of both the chain complexes

$$(k[G]^{*+1}, d) \qquad \text{and} \qquad (k[G]^{*+1}, d').$$

Now let G be a group and H a subgroup of G. Choose a set-theoretic section $\beta \colon H\backslash G \to G$, of the canonical projection $\pi \colon G \to H\backslash G$, satisfying $\beta(H) = 1$ and define $\gamma := \beta \circ \pi$.
In what follows we will give homotopy-inverse maps of the inclusion-induced maps

$$j_* \colon (k[H]^{*+1}, d) \to (k[G]^{*+1}, d)$$

$$j'_*\colon (k[H]^{*+1}, d') \to (k[G]^{*+1}, d')$$

and appropriate chain homotopies.
The chain map p_* determined by

$$p_n\colon k[G]^{n+1} \to k[H]^{n+1}$$

$$p_n(g_0 \otimes \cdots \otimes g_n) :=$$
$$g_0\gamma(g_0)^{-1} \otimes \gamma(g_0)g_1\gamma(g_0g_1)^{-1} \otimes \cdots \otimes \gamma(g_0g_1\cdots g_{n-1})g_n\gamma(g_0\cdots g_n)^{-1}$$

is a chain inverse to j_*, through the homotopies

$$h_n\colon k[H]^{n+1} \to k[H]^{n+2} \quad \text{defined by} \quad h_n := 0$$

and

$$\overline{h}_n\colon k[G]^{n+1} \to k[G]^{n+2} \quad \text{defined by} \quad \overline{h}_n := s(j_np_n - 1).$$

Thus

$$\begin{aligned} p_nj_n - 1 &= dh_n + h_{n-1}d \\ j_np_n - 1 &= d\overline{h}_n + \overline{h}_{n-1}d. \end{aligned}$$

Analogously we define

$$p'_n\colon k[G]^{n+1} \to k[H]^{n+1} \quad \text{by} \quad p'_n := 0,$$

$$h'_n\colon k[H]^{n+1} \to k[H]^{n+2} \quad \text{by} \quad h'_n := -s$$

and

$$\overline{h'}_n\colon k[G]^{n+1} \to k[G]^{n+2} \quad \text{by} \quad \overline{h'}_n := -s.$$

Then again p'_n determines a chain map and

$$\begin{aligned} p'_nj'_n - 1 &= d'h'_n + h'_{n-1}d' \\ j'_np'_n - 1 &= d'\overline{h'}_n + \overline{h'}_{n-1}d'. \end{aligned}$$

1.7 Definition. For all $z \in G$ one defines the subgroups G_z and $\overline{G_z}$ of G by

$$G_z := \{g \in G \mid gz = zg\}$$

and

$$\overline{G_z} := \left\{g \in G \mid g^{-1}zg \in \{z, z^{-1}\}\right\}.$$

Notice that

- $G_z = G_{z^{-1}}$.
- the correspondences $G_z\backslash G \to C(z)$ and $G_z\backslash\overline{G_z} \to \{z, z^{-1}\}$ determined by $G_za \mapsto a^{-1}za$ are bijective.
- $\overline{G_z}$ acts from the right on $\{z, z^{-1}\}$ by conjugation and this makes $k[z, z^{-1}]$ into a $k[\overline{G_z}]$-bimodule.

1.8 Theorem. *For all $z \in G$ the inclusion $\overline{G_z} \subseteq G$ induces a morphism*

$$k[z, z^{-1}] \otimes_k k[\overline{G_z}]^n \longrightarrow k[C(z) \cup C(z^{-1})] \otimes_k k[G]^n$$

$$a \otimes g_1 \otimes \cdots \otimes g_n \mapsto a \otimes g_1 \otimes \cdots \otimes g_n$$

of quaternionic modules.

Proof. We distinguish between three cases and keep 1.6 and definition 1.7 in mind.

1. For all z of type 1 we have $G_z = \overline{G_z}$ and the inclusion $G_z \subseteq G$ induces a morphism of quaternionic modules

$$\begin{array}{c} k[z] \otimes_k k[G_z]^n \\ \cong \big\downarrow \\ k \otimes_{k[G_z]} k[G_z]^{n+1} \\ \big\downarrow \\ k \otimes_{k[G_z]} k[G]^{n+1} \\ \cong \big\downarrow \\ k[C(z)] \otimes_k k[G]^n \end{array}$$

   mapping

$$z \otimes g_1 \otimes \cdots \otimes g_n \text{ to } z \otimes g_1 \otimes \cdots \otimes g_n.$$

   Formulas for x and y can be found in definition 1.4.

2. For all z of type 2 we have $C(z) = C(z^{-1})$ and the inclusion $\overline{G_z} \subseteq G$ induces a morphism of quaternionic modules

$$\begin{array}{c} k[z, z^{-1}] \otimes_k k[\overline{G_z}]^n \\ \cong \big\downarrow \\ k \otimes_{k[G_z]} k[\overline{G_z}]^{n+1} \\ \big\downarrow \\ k \otimes_{k[G_z]} k[G]^{n+1} \\ \cong \big\downarrow \\ k[C(z)] \otimes_k k[G]^n \end{array}$$

   mapping

$$a \otimes g_1 \otimes \cdots \otimes g_n \text{ to } a \otimes g_1 \otimes \cdots \otimes g_n.$$

   Formulas for x and y can be found in definition 1.4.

3. For all z of type 3 we have $G_z = G_{z^{-1}} = \overline{G_z}$ and the inclusion $G_z \subseteq G$ induces a morphism of quaternionic modules

$$\begin{array}{c} k[z,z^{-1}] \otimes_k k[G_z]^n \\ \cong\big\downarrow \\ (k \otimes_{k[G_z]} k[G_z]^{n+1}) \oplus (k \otimes_{k[G_{z^{-1}}]} k[G_{z^{-1}}]^{n+1}) \\ \big\downarrow \\ (k \otimes_{k[G_z]} k[G]^{n+1}) \oplus (k \otimes_{k[G_{z^{-1}}]} k[G]^{n+1}) \\ \cong\big\downarrow \\ k[C(z) \cup C(z^{-1})] \otimes_k k[G]^n \end{array}$$

mapping

$$a \otimes g_1 \otimes \cdots \otimes g_n \quad \text{to} \quad a \otimes g_1 \otimes \cdots \otimes g_n.$$

Formulas for x and y can be derived from definition 1.4.

In all cases this morphism induces a chain map of the associated quaternionic double complexes. •

By applying $k \otimes_{k[G_z]} -$ in the various situations of 1.6 that occur here, we see that these maps are chain equivalences on the columns by Shapiro's lemma. Further 1.6 enables us to compute explicit chain inverses and chain homotopies. To obtain the inverse homomorphism on the level of quaternionic homology we use the following lemma.

1.9 Lemma. Suppose $j\colon \mathcal{C} \to \overline{\mathcal{C}}$ is a chain map of double complexes

$$\begin{array}{ccccc} C_{20} & & & & \\ \big\downarrow d^v_{20} & & \vdots & & \\ C_{10} & \xleftarrow{d^h_{11}} & C_{11} & \cdots & \\ \big\downarrow d^v_{10} & & \big\downarrow d^v_{11} & & \\ C_{00} & \xleftarrow{d^h_{01}} & C_{01} & \xleftarrow{d^h_{02}} & C_{02} \end{array} \quad \xrightarrow{j} \quad \begin{array}{ccccc} \overline{C}_{20} & & & & \\ \big\downarrow \overline{d}^v_{20} & & \vdots & & \\ \overline{C}_{10} & \xleftarrow{\overline{d}^h_{11}} & \overline{C}_{11} & \cdots & \\ \big\downarrow \overline{d}^v_{10} & & \big\downarrow \overline{d}^v_{11} & & \\ \overline{C}_{00} & \xleftarrow{\overline{d}^h_{01}} & \overline{C}_{01} & \xleftarrow{\overline{d}^h_{02}} & \overline{C}_{02} \end{array}$$

which is a chain equivalence on the columns. Let $p_{*\,k}$ be a chain inverse of $j_{*\,k}$ and

$$\begin{aligned} p_{m\,k} j_{m\,k} - 1 &= d^v_{m+1\,k} h_{m\,k} + h_{m-1\,k} d^v_{m\,k} \\ j_{m\,k} p_{m\,k} - 1 &= \overline{d}^v_{m+1\,k} \overline{h}_{m\,k} + \overline{h}_{m-1\,k} \overline{d}^v_{m\,k}. \end{aligned}$$

Then

$$\tau\colon H_1(\operatorname{Tot} \overline{\mathcal{C}}) \to H_1(\operatorname{Tot} \mathcal{C})$$

defined by

$$[a,b] \mapsto [p_{10}a + p_{10}\overline{d}^h_{11}\overline{h}_{01}b + h_{00}d^h_{01}p_{01}b, p_{01}b]$$

for all $(a,b) \in \mathrm{Ker}(\overline{d}^v_{10}\ \ \overline{d}^h_{01})$, is the inverse of

$$j_*\colon H_1(\mathrm{Tot}\,\mathcal{C}) \to H_1(\mathrm{Tot}\,\overline{\mathcal{C}}).$$

Proof. The map j_* is an isomorphism since j is an equivalence on the columns. By definition of double complex:

$$\begin{aligned} d^h_{m-1\,k}d^v_{m\,k} + d^v_{m\,k-1}d^h_{m\,k} &= 0 \qquad \text{for all} \quad m,k \in \mathbb{N} \\ \overline{d}^h_{m-1\,k}\overline{d}^v_{m\,k} + \overline{d}^v_{m\,k-1}\overline{d}^h_{m\,k} &= 0 \qquad \text{for all} \quad m,k \in \mathbb{N} \end{aligned}$$

Now suppose $a \in \overline{C}_{10}$ and $b \in \overline{C}_{01}$ satisfy

$$\overline{d}^v_{10}a + \overline{d}^h_{01}b = 0.$$

Then

$$\begin{aligned} & d^v_{10}(p_{10}a + p_{10}\overline{d}^h_{11}\overline{h}_{01}b + h_{00}d^h_{01}p_{01}b) + d^h_{01}p_{01}b \\ &\quad = -p_{00}\overline{d}^h_{01}b - p_{00}\overline{d}^h_{01}\overline{d}^v_{11}\overline{h}_{01}b + p_{00}j_{00}d^h_{01}p_{01}b \\ &\quad = -p_{00}\overline{d}^h_{01}j_{01}p_{01}b + p_{00}\overline{d}^h_{01}j_{01}p_{01}b \\ &\quad = 0 \end{aligned}$$

proves that $\tau([a,b]) \in H_1(\mathrm{Tot}\,\mathcal{C})$. Further we equate

$$\begin{aligned} & j_*\tau([a,b]) - [a,b] \\ &\quad = [j_{10}p_{10}(a + \overline{d}^h_{11}\overline{h}_{01}b) + j_{10}h_{00}d^h_{01}p_{01}b - a, (j_{01}p_{01} - 1)b] \\ &\quad = [(j_{10}p_{10} - 1)(a + \overline{d}^h_{11}\overline{h}_{01}b) + j_{10}h_{00}d^h_{01}p_{01}b, 0] \\ &\quad = [j_{10}p_{10}(j_{10}p_{10} - 1)(a + \overline{d}^h_{11}\overline{h}_{01}b) + j_{10}p_{10}j_{10}h_{00}d^h_{01}p_{01}b, 0]. \end{aligned}$$

To obtain this last identity we used the fact that

$$(j_{10}p_{10} - 1)(a + \overline{d}^h_{11}\overline{h}_{01}b) + j_{10}h_{00}d^h_{01}p_{01}b \in \mathrm{Ker}(\overline{d}^v_{10})$$

and

$$(j_{10}p_{10} - 1)\,\mathrm{Ker}(\overline{d}^v_{10}) \subseteq \mathrm{Im}(\overline{d}^v_{20}).$$

To continue the computation we define $c := j_{10}p_{10}(j_{10}p_{10} - 1)(a + \overline{d}^h_{11}\overline{h}_{01}b)$.

$$\begin{aligned} & [c + j_{10}p_{10}j_{10}h_{00}d^h_{01}p_{01}b, 0] \\ &\quad = [c + j_{10}h_{00}p_{00}j_{00}d^h_{01}p_{01}b, 0] \\ &\quad = [c + j_{10}h_{00}p_{00}\overline{d}^h_{01}j_{01}p_{01}b, 0] \\ &\quad = [c + j_{10}h_{00}p_{00}\overline{d}^h_{01}(\overline{d}^v_{11}\overline{h}_{01} + 1)b, 0] \end{aligned}$$

$$
\begin{aligned}
&= [c - j_{10}h_{00}p_{00}\overline{d}^v_{10}a - j_{10}h_{00}p_{00}\overline{d}^v_{10}\overline{d}^h_{11}\overline{h}_{01}b, 0] \\
&= [c - j_{10}h_{00}d^v_{10}p_{10}(a + \overline{d}^h_{11}\overline{h}_{01}b), 0] \\
&= [c - j_{10}p_{10}(j_{10}p_{10} - 1)(a + \overline{d}^h_{11}\overline{h}_{01}b), 0] \\
&= 0.
\end{aligned}
$$

Thus we find $j_*\tau = 1$ and since j_* is already an isomorphism, this proves the lemma. •

1.10. We apply lemma 1.9 to the situation of theorem 1.8: Write $\mathcal{D}_1$ for the double complex

$$\mathcal{D}\left(k[z, z^{-1}] \otimes k[\overline{G_z}]^*\right),$$

$\mathcal{D}_2$ for the double complex

$$\mathcal{D}\left(k[C(z) \cup C(z^{-1})] \otimes k[G]^*\right)$$

and

$$j\colon \mathcal{D}_1 \to \mathcal{D}_2$$

for the chain map induced by the morphism of theorem 1.8. See definition 1.9 of chapter III for the definition of $\mathcal{D}$. Picture $\mathcal{D}_1$:

$$
\begin{array}{ccccc}
k[z, z^{-1}] \otimes k[\overline{G_z}]^2 & & & & \\
\Big\downarrow d^v_{20} & & & & \\
k[z, z^{-1}] \otimes k[\overline{G_z}] & \xleftarrow{d^h_{11}} & (k[z, z^{-1}] \otimes k[\overline{G_z}]) \oplus (k[z, z^{-1}] \otimes k[\overline{G_z}]) & & \\
\Big\downarrow d^v_{10} & & \Big\downarrow d^v_{11} & & \\
k[z, z^{-1}] & \xleftarrow{d^h_{01}} & k[z, z^{-1}] \oplus k[z, z^{-1}] & \xleftarrow{d^h_{02}} & k[z, z^{-1}] \oplus k[z, z^{-1}]
\end{array}
$$

and $\mathcal{D}_2$:

$$
\begin{array}{ccccc}
k[C(z) \cup C(z^{-1})] \otimes k[G]^2 & & & & \\
\Big\downarrow \overline{d}^v_{20} & & & & \\
k[C(z) \cup C(z^{-1})] \otimes k[G] & \xleftarrow{\overline{d}^h_{11}} & (k[C(z) \cup C(z^{-1})] \otimes k[G]) \oplus (k[C(z) \cup C(z^{-1})] \otimes k[G]) & & \\
\Big\downarrow \overline{d}^v_{10} & & \Big\downarrow \overline{d}^v_{11} & & \\
k[C(z) \cup C(z^{-1})] & \xleftarrow{\overline{d}^h_{01}} & k[C(z) \cup C(z^{-1})] \oplus k[C(z) \cup C(z^{-1})] & \xleftarrow{\overline{d}^h_{02}} &
\end{array}
$$

We use 1.6, definition 1.4, theorem 1.8 and lemma 1.9 to obtain the following formulas.

$$d^v_{10}, \overline{d}^v_{10} \quad : \quad a \otimes g \mapsto g^{-1}ag - a$$

$$
\begin{array}{rcl}
d^v_{20}, \overline{d}^v_{20} & : & a \otimes g_1 \otimes g_2 \mapsto g_1^{-1} a g_1 \otimes g_2 - a \otimes g_1 g_2 + a \otimes g_1 \\
d^v_{11}, \overline{d}^v_{11} & : & (a \otimes g_1, 0) \mapsto (-g_1^{-1} a g_1, 0) \\
 & & (0, b \otimes g_2) \mapsto (0, b - g_2^{-1} b g_2) \\
d^h_{01}, \overline{d}^h_{01} & : & (a, 0) \mapsto 0 \\
 & & (0, b) \mapsto b - b^{-1} \\
d^h_{11}, \overline{d}^h_{11} & : & (a \otimes g_1, 0) \mapsto a \otimes g_1 + g_1^{-1} a g_1 \otimes g_1^{-1} a \\
 & & (0, b \otimes g_2) \mapsto b \otimes g_2 + g_2^{-1} b^{-1} g_2 \otimes g_2^{-1} \\
d^h_{02}, \overline{d}^h_{02} & : & (a, 0) \mapsto (a, -a - a^{-1}) \\
 & & (0, b) \mapsto (b + b^{-1}, 0) \\
p_{10} & : & g_1^{-1} a g_1 \otimes g_2 \mapsto \gamma(g_1) g_1^{-1} a g_1 \gamma(g_1)^{-1} \otimes \gamma(g_1) g_2 \gamma(g_1 g_2)^{-1} \\
p_{01} & : & (g_1^{-1} a g_1, 0) \mapsto 0 \\
 & & (0, g_2^{-1} b g_2) \mapsto (0, \gamma(g_2) g_2^{-1} b g_2 \gamma(g_2)^{-1}) \\
h_{00} = 0 & & \\
\overline{h}_{01} & : & (g_1^{-1} a g_1, 0) \mapsto (a \otimes g_1, 0) \\
 & & (0, g_2^{-1} b g_2) \mapsto (0, b \otimes g_2 - b \otimes g_2 \gamma(g_2)^{-1})
\end{array}
$$

1.11 Theorem. *The inverse*

$$\tau\colon H_1(\mathrm{Tot}(\mathcal{D}_2)) \longrightarrow H_1(\mathrm{Tot}(\mathcal{D}_1))$$

of j_* is <u>determined</u> by $(x, y) \longmapsto$

$$(\gamma(g_1) g_1^{-1} a g_1 \gamma(g_1)^{-1} \otimes \gamma(g_1) g_2 \gamma(g_1 g_2)^{-1} + b \otimes b, (0, \gamma(g_4) g_4^{-1} c g_4 \gamma(g_4)^{-1})),$$

where

$$
\begin{array}{rcl}
x & = & g_1^{-1} a g_1 \otimes g_2 \in k[C(z) \cup C(z^{-1})] \otimes k[G] \\
y & = & (g_3^{-1} b g_3, g_4^{-1} c g_4) \in k[C(z) \cup C(z^{-1})] \oplus k[C(z) \cup C(z^{-1})].
\end{array}
$$

Proof. Under the given conditions we have

$$
\begin{array}{l}
p_{10} \overline{d}^h_{11} \overline{h}_{01} (g_3^{-1} b g_3, g_4^{-1} c g_4) \\
\quad = \quad p_{10} \overline{d}^h_{11} (b \otimes g_3, c \otimes g_4 - c \otimes g_4 \gamma(g_4)^{-1}) \\
\quad = \quad p_{10} (b \otimes g_3 + g_3^{-1} b g_3 \otimes g_3^{-1} b + c \otimes g_4 + g_4^{-1} c^{-1} g_4 \otimes g_4^{-1} \\
\qquad\qquad - c \otimes g_4 \gamma(g_4)^{-1} - \gamma(g_4) g_4^{-1} c^{-1} g_4 \gamma(g_4)^{-1} \otimes \gamma(g_4) g_4^{-1}) \\
\quad = \quad b \otimes g_3 \gamma(g_3)^{-1} + \gamma(g_3) g_3^{-1} b g_3 \gamma(g_3)^{-1} \otimes \gamma(g_3) g_3^{-1} b.
\end{array}
$$

Applying the first relation of the list of theorem 1.12 yields

$$[b \otimes g_3 \gamma(g_3)^{-1} + \gamma(g_3) g_3^{-1} b g_3 \gamma(g_3)^{-1} \otimes \gamma(g_3) g_3^{-1} b] = [b \otimes b].$$

Using the formula for τ in lemma 1.9 yields the desired result. •

1.12 Theorem. *For every* $g, g_1, g_2 \in \overline{G_z}$ *and* $a \in \{z, z^{-1}\}$, *the following relations are valid in* $H_1(\mathrm{Tot}(\mathcal{D}_1))$.

1. $[g_1^{-1} a g_1 \otimes g_2 + a \otimes (g_1 - g_1 g_2), 0, 0] = 0$,
2. $[0, z + z^{-1}, 0] = 0$,
3. $[0, a, a + a^{-1}] = 0$ *and* $[0, 0, 2(z + z^{-1})] = 0$,
4. $[z \otimes z, 0, z + z^{-1}] = 0$,
5. $[z \otimes g - z^{-1} \otimes g, 0, z - g^{-1} z g] = 0$,
6. $[z \otimes (g_1 + g_2 - g_1 g_2), 0, \epsilon(g_1, g_2)] = 0$, *where*

$$\epsilon(g_1, g_2) := \begin{cases} z - z^{-1} & \text{if } g_1, g_2 \notin G_z \\ 0 & \text{otherwise} \end{cases}.$$

Proof.

1 follows immediately from the definition of d_{20}^v.

2 is clear since $d_{02}^h(0, z) = (z + z^{-1}, 0)$.

3 $d_{02}^h(a, 0) = (a, -a - a^{-1})$ and *2* imply that $[0, 0, 2(z + z^{-1})] = 0$. The rest is obvious.

4 Using the definitions of d_{11}^h and d_{11}^v we find

$$\begin{aligned} 0 &= [z \otimes g + g^{-1} z g \otimes g^{-1} z, -g^{-1} z g, 0] \\ &= [z \otimes g + z \otimes (z - g), 0, z + z^{-1}] \quad \text{by } 1 \text{ and } 3 \\ &= [z \otimes z, 0, z + z^{-1}]. \end{aligned}$$

5 Using the definitions of d_{11}^h and d_{11}^v we equate

$$\begin{aligned} 0 &= [z \otimes g + g^{-1} z^{-1} g \otimes g^{-1}, 0, z - g^{-1} z g] \\ &= [z \otimes g + z^{-1} \otimes (1 - g), 0, z - g^{-1} z g] \quad \text{by } 1 \\ &= [z \otimes g - z^{-1} \otimes g, 0, z - g^{-1} z g]. \end{aligned}$$

Note that $[z \otimes 1, 0, 0] = 0$ by taking $g_1 = g_2 = 1$ in *1*.

6 If $g_1 \in G_z$, then *6* follows from *1*.
If $g_1 \notin G_z$, then *6* follows from *1* and *5*.

This completes the list of relations. •

Notation. For every group J we denote by J_{ab} the commutator quotient of J, i.e. $J_{\mathrm{ab}} = J/[J, J]$, and by $J_{\#}$ the quotient group $J_{\mathrm{ab}}/(J_{\mathrm{ab}})^2$.

1.13 Theorem. *Let* $k = \mathbb{F}_2$.

1. *For every z of type 1 the map*

$$\eta\colon H_1(\mathrm{Tot}(\mathcal{D}_1)) \longrightarrow ((G_z)_\#/ < z >) \times C_2$$

determined by

$$\left[\sum_i z \otimes g_i, n_1 z, n_2 z\right] \mapsto \left(\left[\prod_i g_i\right], t^{n_2}\right)$$

for all $n_1, n_2 \in k$ and $g_i \in G_z$, where C_2 denotes the cyclic group of order two generated by t, is an isomorphism.

2. *For every z of type 2 the map*

$$\eta\colon H_1(\mathrm{Tot}(\mathcal{D}_1)) \longrightarrow \frac{(\overline{G_z} \times_{C_2} C_4)_\#}{< [z, t^2] >}$$

determined by

$$\left[\sum_i a_i \otimes g_i, \rho(n_1)z + \rho(n_2)z^{-1}, \rho(n_3)z + \rho(n_4)z^{-1}\right] \mapsto \left[\prod_i g_i, t^n\right]$$

for all $n_1, n_2, n_3, n_4 \in \mathbf{Z}$, $a_i \in \{z, z^{-1}\}$ and $g_i \in \overline{G_z}$, satisfying the cycle condition $\sum \rho(w(g_i)) = \rho(n_3 - n_4)$, is an isomorphism. Here

$$\rho \text{ is the canonical map } \mathbf{Z} \to k,$$

$$n := \sum w(g_i) + 2\left(n_1 + n_2 + n_4 + \sum w'(a_i)w(g_i)\right),$$

$$w'(z) := 0,\ w'(z^{-1}) := 1,$$

$$w(g) := w'(g^{-1}zg) \text{ for all } g \in \overline{G_z}.$$

And $\overline{G_z} \times_{C_2} C_4$ is the pull-back of the diagram

$$\begin{array}{ccc} & & C_4 \\ & & \downarrow{\scriptstyle \pi_1} \\ \overline{G_z} & \xrightarrow{\pi_2} & C_2 \end{array}$$

Here C_4 denotes the cyclic group of order four generated by t, π_1 is the non-trivial map and $\pi_2(g) := t^{w(g)}$ for every $g \in \overline{G_z}$.

3. *For every z of type 3 the map*

$$\eta\colon H_1(\mathrm{Tot}(\mathcal{D}_1)) \longrightarrow (G_z)_\#$$

determined by

$$\left[\sum_i a_i \otimes g_i, n_1 z + n_2 z^{-1}, n_3 z + n_4 z^{-1}\right] \mapsto \left[\prod_i g_i z^{n_1+n_2+n_3}\right]$$

for all $n_1, n_2, n_3, n_4 \in k$, $a_i \in \{z, z^{-1}\}$ and $g_i \in G_z$, satisfying the cycle condition $n_3 = n_4$, is an isomorphism.

Proof. We will not enter into all the details of the proof; it is not difficult but rather tedious.

1 The data in 1.10 make it is easy to verify that the map on

$$\mathrm{Ker}(d^v_{10} \;\; d^h_{01}) = (k[z] \otimes k[G_z]) \oplus k[z] \oplus k[z]$$

determined by the expression in the definition of η is a homomorphism which vanishes on $\mathrm{Im}(d^h_{02})$, $\mathrm{Im}(d^v_{20})$ and $\mathrm{Im}(d^h_{11} \;\; d^v_{11})$.
Theorem 1.12 enables us to check that the inverse of η is determined by

$$([g], t^n) \mapsto [z \otimes g, 0, \overline{n} z]$$

for every $g \in G_z$, $n \in \mathbf{Z}$.

2 Again η is a well-defined homomorphism. The inverse homomorphism is determined by

$$[g, t^n] \mapsto [z \otimes g, 0, \rho(\mathrm{ent}((n+1)/2))z + \rho(\mathrm{ent}(n/2))z^{-1}]$$

for all $g \in \overline{G_z}$ and $n \in \mathbf{Z}$ satisfying $\rho(w(g)) = \rho(n)$.

3 The homomorphism η^{-1} maps $[g]$ to $[z \otimes g, 0, 0]$ for all $g \in G_z$.

Here ent denotes the entier function. •

Notation. Write

$$\Sigma(G) = \bigoplus_{z \in S_1} (((G_z)_\#/ < z >) \times C_2) \oplus \bigoplus_{z \in S_2} \frac{(\overline{G_z} \times_{C_2} C_4)_\#}{< [z, t^2] >} \oplus \bigoplus_{z \in S_3^+} (G_z)_\#$$

1.14 Theorem. *We have an isomorphism* $\Psi \colon HQ_1(\mathbb{F}_2[G]) \longrightarrow \Sigma(G)$.

Proof. By theorem 1.5, theorem 1.11 and theorem 1.13. •

2 Managing Coker$(1+\vartheta)$.

Before we start with our reflections on Coker$(1+\vartheta_{\mathbb{F}_2[G]})$ recall the theorems 1.5, 1.11, 1.13 and 1.14 of the previous section.

2.1 Lemma. The isomorphism Ψ induces an isomorphism

$$\Psi_1\colon \mathrm{Coker}(\nu) \hookrightarrow\!\!\!\!\rightarrow \mathrm{Coker}(\Psi\circ\nu)$$

and

$$\begin{aligned}\mathrm{Coker}(\Psi\circ\nu) &= \bigoplus\nolimits_{z\in S_1} (((G_z)_\#/<z>)\times C_2)\oplus \\ & \bigoplus\nolimits_{z\in S_2} \left(\overline{G_z}\right)_\# / <z> \oplus \\ & \bigoplus\nolimits_{z\in S_3^+} (G_z)_\#/<z>\end{aligned}$$

Proof. To determine Coker$(\Psi\circ\nu)$ we compute

$$\Psi(\nu(x)) = \Psi([1\otimes x, 0, 0])$$

for $x\in\mathbb{F}_2[G]$. We may assume that $x\in G$. There exist $z\in \mathrm{Im}\ S$ and $g\in G$ such that $x = g^{-1}zg$. Now

$$\phi(1\otimes g^{-1}zg) = g^{-1}zg\otimes g^{-1}zg,$$

$$\begin{aligned}\tau([g^{-1}zg\otimes g^{-1}zg,0,0]) &= [\gamma(g)g^{-1}zg\gamma(g)^{-1}\otimes\gamma(g)g^{-1}zg\gamma(zg)^{-1},0,0] \\ &= [\gamma(g)g^{-1}zg\gamma(g)^{-1}\otimes\gamma(g)g^{-1}zg\gamma(g)^{-1},0,0] \\ &= \begin{cases}[z\otimes z,0,0] & \text{if } z \text{ is of type 1}\\ [z^{\pm1}\otimes z^{\pm1},0,0] & \text{if } z \text{ is of type 2}\\ [z\otimes z,0,0] & \text{if } z \text{ is of type 3}\end{cases}\end{aligned}$$

applying the isomorphism η we find

$$\Psi([1\otimes g^{-1}zg,0,0]) = \begin{cases}([z],1)=1\in((G_z)_\#/<z>)\times C_2 & \text{if } z \text{ is of type 1}\\ [z,1]\in(\overline{G_z}\times_{C_2}C_4)_\#/<[z,t^2]> & \text{if } z \text{ is of type 2}\\ [z]\in(G_z)_\# & \text{if } z \text{ is of type 3}\end{cases}$$

The rest is clear now. •

2.2 Definition. Let G be a group. Define

$$\tilde F(z) := \begin{cases}\dfrac{(G_z)_\#}{<\{x\in G\mid x=z\ \vee\ x^2=z\}>}\times C_2, & \text{if } z \text{ is of type 1}\\ \dfrac{(\overline{G_z})_\#}{<\{x\in G\mid x=z\ \vee\ x^2=z\}>}, & \text{if } z \text{ is of type 2}\\ \dfrac{(G_z)_\#}{<\{x\in G\mid x=z\ \vee\ x^2=z\}>}, & \text{if } z \text{ is of type 3.}\end{cases}$$

2.3 Lemma. The isomorphism Ψ_1 induces an isomorphism

$$\Psi_2\colon \mathrm{Coker}(\mu) \hookrightarrow\!\!\!\!\rightarrow \mathrm{Coker}(\Psi_1 \circ \mu)$$

and

$$\mathrm{Coker}(\Psi_1 \circ \mu) = \bigoplus_{z \in S_1 \cup S_2 \cup S_3^+} \tilde{F}(z)$$

Proof. To determine $\mathrm{Coker}(\Psi_1 \circ \mu)$ we compute $\Psi_1([x \otimes x, 0, 0])$. Again we may assume that $x \in G$. There exist $z \in \mathrm{Im}\ S$ and $g \in G$ such that $x^2 = g^{-1}zg$. Observe that $(gxg^{-1})^2 = z$. Now

$$\phi(x \otimes x) = x^2 \otimes x = g^{-1}zg \otimes x.$$

Notice that $\gamma(gx) = \gamma(g)$ since $gxg^{-1} \in G_z$.

$$\begin{aligned}
\tau([g^{-1}zg \otimes x, 0, 0]) &= [\gamma(g)g^{-1}zg\gamma(g)^{-1} \otimes \gamma(g)x\gamma(gx)^{-1}, 0, 0] \\
&= [\gamma(g)g^{-1}zg\gamma(g)^{-1} \otimes \gamma(g)x\gamma(g)^{-1}, 0, 0] \\
&= \begin{cases} [z \otimes \gamma(g)g^{-1}gxg^{-1}g\gamma(g)^{-1}, 0, 0] & \text{if } z \text{ is of type 1} \\ [z^{\pm 1} \otimes \gamma(g)g^{-1}gxg^{-1}g\gamma(g)^{-1}, 0, 0] & \text{if } z \text{ is of type 2} \\ [z \otimes \gamma(g)g^{-1}gxg^{-1}g\gamma(g)^{-1}, 0, 0] & \text{if } z \text{ is of type 3} \end{cases}
\end{aligned}$$

applying the isomorphism η we find

$$\Psi_1([x \otimes x, 0, 0]) = \begin{cases} ([gxg^{-1}], 1) \in ((G_z)_\# / < [z] >) \times C_2 & \text{if } z \text{ is of type 1} \\ [gxg^{-1}] \in (\overline{G_z})_\# / < [z] > & \text{if } z \text{ is of type 2} \\ [gxg^{-1}] \in (G_z)_\# / < [z] > & \text{if } z \text{ is of type 3} \end{cases}$$

This proves the claim. •

The isomorphism Ψ_2 induces an isomorphism

$$\Psi_3\colon \mathrm{Coker}(1 + \vartheta) \hookrightarrow\!\!\!\!\rightarrow \mathrm{Coker}(\Psi_2(1 + \vartheta)\Psi^{-1}) :$$

$$\begin{array}{ccccc}
HQ_1(\mathbb{F}_2[G]) & \xrightarrow{1+\vartheta} & \mathrm{Coker}(\mu) & \longrightarrow & \mathrm{Coker}(1+\vartheta) \\
\downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \Psi_2} & & \downarrow{\scriptstyle \Psi_3} \\
\mathcal{S} & \longrightarrow & \mathrm{Coker}(\Psi_1 \circ \mu) & \longrightarrow & \mathrm{Coker}(\Psi_2(1+\vartheta)\Psi^{-1})
\end{array}$$

2.4 Lemma. $\mathrm{Coker}(\Psi_2(1+\vartheta)\Psi^{-1})$ arises from $\mathrm{Coker}(\Psi_1 \circ \mu)$ by imposing the following identifications. For all z of type

1 identify

$$([g], t^i) \in \tilde{F}(z) \quad \text{and} \quad ([g], t^i) \in \tilde{F}(1)$$

2 identify

$$[g] \in \tilde{F}(z) \quad \text{and} \quad \begin{cases} ([g], t^{w(g)}) \in \tilde{F}(z^2) & \text{if } z^2 \text{ is of type 1} \\ [g] \in \tilde{F}(z^2) & \text{if } z^2 \text{ is of type 2} \end{cases}$$

3 identify

$$[g] \in \tilde{F}(z) \quad \text{and} \quad \begin{cases} ([g],1) \in \tilde{F}(z^2) & \text{if } z^2 \text{ is of type 1} \\ [g] \in \tilde{F}(z^2) & \text{if } z^2 \text{ is of type 2} \\ [g] \in \tilde{F}(z^2) & \text{if } z^2 \text{ is of type 3.} \end{cases}$$

Proof.

1. Let $([g], t^i) \in ((G_z)_\#/ < z >) \times C_2$.

$$\Psi^{-1}([g], t^i) = [g^{-1}z \otimes g, 0, iz]$$

since

$$\begin{aligned} \phi(g^{-1}z \otimes g) &= (z \otimes g) \quad \text{and} \quad \phi(iz) = iz, \\ \tau([z \otimes g, 0, iz]) &= [z \otimes g\gamma(g)^{-1}, 0, iz] \;=\; [z \otimes g, 0, iz], \\ \eta([z \otimes g, 0, iz]) &= ([g], t^i). \end{aligned}$$

$\vartheta([g^{-1}z \otimes g, 0, iz]) = [g^{-1}z^2 \otimes g + z \otimes z + iz \otimes z, 0, iz^2] = [g^{-1} \otimes g, 0, i]$.

$$\Psi_2(g^{-1} \otimes g, 0, i]) = ([g], t^i) \in \tilde{F}(z^2) = \tilde{F}(1)$$

since

$$\begin{aligned} \phi(g^{-1} \otimes g) &= (1 \otimes g) \quad \text{and} \quad \phi(i) \;=\; i, \\ \tau([1 \otimes g, 0, iz]) &= [1 \otimes g\gamma(g)^{-1}, 0, i] \;=\; [1 \otimes g, 0, i], \\ \eta([1 \otimes g, 0, i]) &= ([g], t^i). \end{aligned}$$

2. Let $[g, t^i] \in (\overline{G_z} \times_{C_2} C_4)_\#/ < [z, t^2] >$.

$$\Psi^{-1}([g, t^i]) = [g^{-1}z \otimes g, 0, y],$$

where $y = \text{ent}((i+1)/2)z + \text{ent}(i/2)z^{-1}$, since

$$\begin{aligned} \phi(g^{-1}z \otimes g) &= (z \otimes g) \quad \text{and} \quad \phi(y) \;=\; y, \\ \tau([z \otimes g, 0, y]) &= [z \otimes g\gamma(g)^{-1}, 0, y] \;=\; [z \otimes g, 0, y], \\ \eta([z \otimes g, 0, y]) &= [g, t^{w(g)+2\,\text{ent}(i/2)}] \;=\; [g, t^i]. \end{aligned}$$

$\vartheta([g^{-1}z \otimes g, 0, y]) = [g^{-1}z^2 \otimes g, 0, \text{ent}((i+1)/2)z^2 + \text{ent}(i/2)z^{-2}]$.
Note that $[z \otimes z^{\pm 1}, 0, 0] = [z^{\pm 1} \otimes z, 0, 0] = 0$ in $\text{Coker}(\mu)$.
Define $y' := \text{ent}((i+1)/2)z^2 + \text{ent}(i/2)z^{-2}$.

$$\Psi_2([g^{-1}z^2 \otimes g, 0, y']) = \begin{cases} ([g], t^i) \in \tilde{F}(z^2) & \text{if } z^2 \text{ is of type 1} \\ [g] \in \tilde{F}(z^2) & \text{if } z^2 \text{ is of type 2} \end{cases}$$

since

$$\begin{aligned} \phi(g^{-1}z^2 \otimes g) &= (z^2 \otimes g) \quad \text{and} \quad \phi(y') \;=\; y', \\ \tau([z^2 \otimes g, 0, y']) &= [z^2 \otimes g\gamma(g)^{-1}, 0, y'] \;=\; [z^2 \otimes g, 0, y'], \\ \eta([z^2 \otimes g, 0, y']) &= \begin{cases} ([g], t^i) \in \tilde{F}(z^2) & \text{if } z^2 \text{ is of type 1} \\ [g] \in \tilde{F}(z^2) & \text{if } z^2 \text{ is of type 2} \end{cases} \end{aligned}$$

3. Let $[g] \in (G_z)_\#$.

$$\Psi^{-1}([g]) = [g^{-1}z \otimes g, 0, 0]$$

since

$$\begin{aligned} \phi(g^{-1}z \otimes g) &= (z \otimes g), \\ \tau([z \otimes g, 0, 0]) &= [z \otimes g\gamma(g)^{-1}, 0, 0] \;=\; [z \otimes g, 0, 0], \\ \eta([z \otimes g, 0, 0]) &= [g]. \end{aligned}$$

$\vartheta([g^{-1}z \otimes g, 0, 0]) = [g^{-1}z^2 \otimes g, 0, 0]$.

$$\Psi_2([g^{-1}z^2 \otimes g, 0, 0]) = \begin{cases} ([g], 1) \in \tilde{F}(z^2) & \text{if } z^2 \text{ is of type 1} \\ [g] \in \tilde{F}(z^2) & \text{if } z^2 \text{ is of type 2} \\ [g] \in \tilde{F}(z^2) & \text{if } z^2 \text{ is of type 3} \end{cases}$$

since

$$\begin{aligned} \phi(g^{-1}z^2 \otimes g) &= (z^2 \otimes g), \\ \tau([z^2 \otimes g, 0, 0]) &= [z^2 \otimes g\gamma(g)^{-1}, 0, 0] \;=\; [z^2 \otimes g, 0, 0], \\ \eta([z^2 \otimes g, 0, 0]) &= \begin{cases} ([g], 1) \in \tilde{F}(z^2) & \text{if } z^2 \text{ is of type 1} \\ [g] \in \tilde{F}(z^2) & \text{if } z^2 \text{ is of type 2} \\ [g] \in \tilde{F}(z^2) & \text{if } z^2 \text{ is of type 3} \end{cases} \end{aligned}$$

This completes the proof. •

2.5 Definition. Let G be a group. For every $z \in G$ we define $\sqrt{z}$ as the subgroup of $(G_z)_\#$ resp. $(\overline{G_z})_\#$ generated by the set

$$\{g \in G \mid g^{2^k} = z \text{ for some } k \in \mathbb{N}\}.$$

2.6 Definition. Define

$$\mathcal{J}(G) := \varinjlim_{z} F(z),$$

where

$$F(z) := \begin{cases} \dfrac{(G_z)_\#}{\sqrt{z}} \times C_2 & \text{if } z \text{ is of type 1} \\ \dfrac{(\overline{G_z})_\#}{\sqrt{z}} & \text{if } z \text{ is of type 2} \\ \dfrac{(G_z)_\#}{\sqrt{z}} & \text{if } z \text{ is of type 3} \end{cases}$$

and the limit is taken with respect to the homomorphisms

· $F(z) \longrightarrow F(x^{-1}zx)$ for every $x \in G$ defined by

$$\begin{cases} ([g], t^i) \mapsto ([x^{-1}gx], t^i) & \text{for all } z \text{ of type 1} \\ [g] \mapsto [x^{-1}gx] & \text{for all } z \text{ of type 2 and 3} \end{cases}$$

· $F(z) \to F(z^2)$ defined by

$$\begin{cases} ([g], t^i) \mapsto ([g], t^i) & \text{for all } z \text{ of type } 1 \\ [g] \mapsto \begin{cases} ([g], t^{w(g)}) & \text{if } z^2 \text{ is of type } 1 \\ [g] & \text{if } z^2 \text{ is of type } 2 \end{cases} & \text{for all } z \text{ of type } 2 \\ [g] \mapsto \begin{cases} ([g], 1) & \text{if } z^2 \text{ is of type } 1 \\ [g] & \text{if } z^2 \text{ is of type } 2 \\ [g] & \text{if } z^2 \text{ is of type } 3 \end{cases} & \text{for all } z \text{ of type } 3 \end{cases}$$

· $F(z) \to F(z^{-1})$ defined by $[g] \mapsto [g]$ for all z of type 3.

2.7 Remark.

$$\mathcal{J}(G) \cong \bigoplus_{c \in \mathcal{C}\ell(G)} Ł(c),$$

where

$$Ł(c) := \varinjlim_{z \in c} F(z).$$

2.8 Theorem.

$$\mathrm{Coker}(1 + \vartheta) \cong \mathcal{J}(G)$$

Proof. Obvious in view of lemma 2.4. •

2.9 Proposition. *Suppose (g, h) is an element of* $\mathrm{Arf}^h(G)$*. The invariant* Υ *of* chapter III *maps (g, h) to*

$$\begin{cases} [1, t] \in Ł([1]) & \textit{if } gh \textit{ is of type } 1 \\ [h] \in Ł([gh]) & \textit{if } gh \textit{ is of type } 2 \end{cases}$$

Note that gh is never of type 3*.*

Proof. Define $z := gh$. Note that $g^2 = h^2 = 1$ and $hzh = z^{-1}$. By definition $\Upsilon((g, h)) = [g \otimes h, gh] \in \mathrm{Coker}(1 + \vartheta)$. By the definitions of ϕ, τ and η:

$$\begin{aligned} \phi(gh) &= gh, \\ \phi(g \otimes h) &= hg \otimes h, \\ \tau([hg \otimes h, 0, gh]) &= \tau([hzh \otimes h, 0, z]) \\ &= [\gamma(h)hzh\gamma(h)^{-1} \otimes \gamma(h)h, 0, z] \\ &= [z^{-1} \otimes h, 0, z] \\ \eta([z^{-1} \otimes h, 0, z]) &= \begin{cases} ([h], t) \in \tilde{F}(z) & \text{if } z \text{ is of type } 1 \\ [h] \in \tilde{F}(z) & \text{if } z \text{ is of type } 2 \end{cases} \end{aligned}$$

Hence

$$\Psi_3([g \otimes h, gh]) = \begin{cases} ([h], t) = ([1], t) & \text{if } gh \text{ is of type } 1 \\ [h] & \text{if } gh \text{ is of type } 2 \end{cases}$$

•

2.10 Lemma. For all $z \in G$

$$\mathrm{Ker}(\overline{G_z} \to (\overline{G_z})_{\#}/\sqrt{z}) \subset G_z.$$

Proof. Every commutator is a product of squares: $xyx^{-1}y^{-1} = x^2(x^{-1}y)^2y^{-2}$. Every square of an element in $\overline{G_z}$ belongs to G_z. If $y^{2^k} = z$, then $y \in G_z$. •

2.11 Lemma. $\Upsilon((g,h))$ is never trivial in $\mathcal{J}(G)$.

Proof. Define $z := (gh)^{2^k}$, with k large. If z is of type 1, the statement is true by proposition 2.9. If z is of type 2, then $g \in \overline{G_z} \setminus G_z$. Therefore $[g] \in Ł([z])$ cannot be trivial. •

Now we review one of the examples we encountered in section 5 of chapter II.

Example. Let G be the group with presentation

$$G := \langle X, Y, S \mid S^2 = (XS)^2 = (YS)^2 = 1, \quad XY = YX\rangle.$$

To compute $\mathrm{Arf}^h(G)$ we determine

$$\mathcal{J}(G) = \bigoplus_{c \in \mathcal{C}\!\ell(G)} Ł(c).$$

It is immediately clear from the presentation of G that $\sqrt{1} = G$ and for all $z \in H$ we have $\overline{G_z} = G$. (Recall that H is the subgroup generated by X and Y.) Therefore

$$\left(\overline{G_z}\right)_{\#} = G/G^2 = G/\langle X^2, Y^2\rangle.$$

A little examination shows that

$$\mathcal{C}\!\ell(G) = \{[1]\} \cup \left\{[X^{2i}Y^{2j+1}],\ [X^{2k+1}Y^{2l}],\ [X^{2m+1}Y^{2n+1}] \quad \mid j,k,m \geq 0\right\}$$

and

$$Ł(c) = \begin{cases} C_2 & \text{if } c = [1] \\ G/\langle X^2, Y\rangle \cong C_2 \times C_2 & \text{if } c = [X^{2i}Y^{2j+1}] \\ G/\langle X, Y^2\rangle \cong C_2 \times C_2 & \text{if } c = [X^{2k+1}Y^{2l}] \\ G/\langle X^2, XY\rangle \cong C_2 \times C_2 & \text{if } c = [X^{2m+1}Y^{2n+1}]. \end{cases}$$

Proposition 5.2 of chapter II says that the elements

$$\begin{cases} (1,1) & \\ (X^{2i}Y^{2j+1}S, S) & \text{for } j \geq 0 \\ (X^{2i+1}Y^{2j}S, S) & \text{for } i \geq 0 \\ (X^{2i+1}Y^{2j+1}S, S) & \text{for } i \geq 0 \\ (X^{2i+1}Y^{2j+1}S, XS) & \text{for } j \geq 0 \\ (X^{2i+1}Y^{2j+1}S, YS) & \text{for } i \geq 0. \\ (X^{2i}Y^{2j+1}S, XS) & \text{for } j \geq 0 \end{cases}$$

generate $\mathrm{Arf}^h(G)$. But since

$$\begin{aligned}
\Upsilon((1,1)) &= t \in C_2 \\
\Upsilon((X^{2i}Y^{2j+1}S,S)) &= [S] \in Ł([X^{2i}Y^{2j+1}]) \\
\Upsilon((X^{2i+1}Y^{2j}S,S)) &= [S] \in Ł([X^{2i+1}Y^{2j}]) \\
\Upsilon((X^{2i+1}Y^{2j+1}S,S)) &= [S] \in Ł([X^{2i+1}Y^{2j+1}]) \\
\Upsilon((X^{2i+1}Y^{2j+1}S,XS)) &= [XS] \in Ł([X^{2i}Y^{2j+1}]) \\
\Upsilon((X^{2i+1}Y^{2j+1}S,YS)) &= [YS] \in Ł([X^{2i+1}Y^{2j}]) \\
\Upsilon((X^{2i}Y^{2j+1}S,XS)) &= [XS] \in Ł([X^{2i-1}Y^{2j+1}])
\end{aligned}$$

we may conclude that these elements constitute a basis for $\mathrm{Arf}^h(G)$.

We revert to one of the examples of chapter I.

Example. Let G be the group with presentation

$$G := \langle X, Y, S \mid S^2 = (XS)^2 = (YS)^4 = (Y^2S)^2 = 1, \quad XY = YX \rangle.$$

2.12 Proposition.

$$\begin{aligned}
\{(1,1)\} &\cup \left\{(X^{2i+1}Y^{2j}S,S) \mid i \geq 0\right\} \\
&\cup \left\{(X^{2i}Y^{4j+2}S,S) \mid j \geq 0\right\} \\
&\cup \left\{(X^{2i+1}Y^{4j+2}S,XS) \mid j \geq 0\right\}
\end{aligned}$$

is a basis for $\mathrm{Arf}^{s,h}(G)$.

Proof. We know already that these elements generate $\mathrm{Arf}^h(G)$. To prove independence we use our invariant Υ. We proceed to compute the summands $Ł(c)$ of value group $\mathcal{J}(G)$. It is not hard to verify that

$$\mathcal{C}\ell(G) = \{[1]\} \cup \left\{[X^{2i+1}Y^{2j}] \mid i \geq 0\right\} \cup \left\{[X^iY^{2j+1}] \mid j \geq 0\right\}.$$

We omit the proof.

$$Ł(c) = \begin{cases} C_2 & \text{if } c = [1] \\ G/\langle X, Y^2, (YS)^2 \rangle \cong C_2 \times C_2 & \text{if } c = [X^{2i+1}Y^{2j}] \\ G/\langle X^2, Y, (YS)^2 \rangle \cong C_2 \times C_2 & \text{if } c = [X^{2i}Y^{2j+1}] \\ G/\langle X^2, XY, (YS)^2 \rangle \cong C_2 \times C_2 & \text{if } c = [X^{2i+1}Y^{2j+1}]. \end{cases}$$

Note that the class of S is non-trivial in any $Ł(c)$. Further, the classes of X, S and XS in $Ł([X^{2i}Y^{2j+1}])$ as well as in $Ł([X^{2i+1}Y^{2j+1}])$ are distinct. Now we can use the list of images

$$\begin{aligned}
\Upsilon((1,1)) &= t \in C_2 \\
\Upsilon((X^{2i+1}Y^{2j}S,S)) &= [S] \in Ł([X^{2i+1}Y^{2j}]) \\
\Upsilon((X^{4i}Y^{4j+2}S,S)) &= [S] \in Ł([X^{2i}Y^{2j+1}]) \\
\Upsilon((X^{4i+2}Y^{4j+2}S,S)) &= [S] \in Ł([X^{2i+1}Y^{2j+1}]) \\
\Upsilon((X^{4i+1}Y^{4j+2}S,XS)) &= [XS] \in Ł([X^{2i}Y^{2j+1}]) \\
\Upsilon((X^{4i+3}Y^{4j+2}S,XS)) &= [XS] \in Ł([X^{2i+1}Y^{2j+1}])
\end{aligned}$$

to see that the assertion is true. •

Example. Let G be the group with presentation

$$\langle X, Y, Z, S \mid X, Y, Z \text{ commute }, S^2 = (XS)^2 = (YS)^2 = (ZS)^2 = 1\rangle.$$

Let $c \in \mathcal{C}\ell(G)$ be the class of XYZ. The invariant Υ maps

$$\xi := (XYS, SZ) + (XZS, SY) + (YZS, SX) + (XYZS, S) \in \mathrm{Arf}^h(G)$$

to the class $[SZSYSXS] = [1] \in Ł(c) = G/\langle X^2, Y^2, Z^2, XYZ\rangle$. But it is not clear at all whether ξ is trivial in $\mathrm{Arf}^h(G)$.

3 Groups with two ends.

We wish to prove that our invariant Υ is injective for all groups having two ends. For that purpose theorem 3.3 gives a suitable characterization of these groups.

Notation. Throughout this section

G	denotes a group,
E	denotes a finite group,
C	denotes the infinite cyclic group,
C_m	denotes the cyclic group of order m,
D	denotes the infinite dihedral group with presentation $< S, T \mid S^2 = (ST)^2 = 1 >$,
D_m	denotes the dihedral group of order $2m$ with presentation $< \sigma, \tau \mid \sigma^2 = (\sigma\tau)^2 = \tau^m = 1 >$.

3.1 Theorem. [20] *The following statements are equivalent;*

1. *G has two ends.*
2. *G has an infinite cyclic subgroup of finite index.*
3. *G has an infinite cyclic normal subgroup of finite index.*

Proof. We refer to *loc. cit.* for a proof. •

3.2 Definition. A group extension of C by E is a short exact sequence of groups and homomorphisms

$$1 \to C \to G \to E \to 1$$

The extension is called central if the image of C is central in G.

3.3 Theorem.

1. *$1 \to C \to G \to E \to 1$ is a central extension if and only if G fits into a pull-back diagram*

$$\begin{array}{ccc} G & \to & E \\ \downarrow & & \downarrow \\ C & \to & C_m \end{array}$$

2. *$1 \to C \to G \to E \to 1$ is a non-central extension if and only if G fits into a pull-back diagram*

$$\begin{array}{ccc} G & \to & E \\ \downarrow & & \downarrow \\ D & \to & D_m \end{array}$$

Proof. In the sequel we will regard C as a subgroup of G.

1. "$\Rightarrow$" Suppose $1 \to C \to G \xrightarrow{\pi} E \to 1$ is a central extension. Define the so-called transfer homomorphism $\phi\colon G \to C$ as follows: choose a set theoretic section $\alpha\colon E \to G$ of the projection $\pi\colon G \to E$ such that $\alpha(1) = 1$ and define

$$\phi(g) := \prod_{e \in E} \alpha(e) g \alpha(e\pi(g))^{-1} \quad \text{for all} \quad g \in G.$$

Note that

- $\alpha(e)g\alpha(e\pi(g))^{-1} \in \operatorname{Ker}\pi = C$ for all $e \in E$ and $g \in G$.
- ϕ does not depend on the choice of α:
  If α' is another section of π we have $\alpha(e)\alpha'(e)^{-1} \in C$ for every $e \in E$. Hence

$$\begin{aligned}\prod_{e \in E} \alpha(e) g \alpha(e\pi(g))^{-1} &= \prod_{e \in E} \alpha'(e) g \alpha'(e\pi(g))^{-1} \cdot \\ &\quad \prod_{e \in E} \alpha(e)\alpha'(e)^{-1} \cdot \\ &\quad \prod_{e \in E} \alpha'(e\pi(g))\alpha(e\pi(g))^{-1} \\ &= \prod_{e \in E} \alpha'(e) g \alpha'(e\pi(g))^{-1}\end{aligned}$$

- ϕ is a homomorphism:

$$\begin{aligned}\phi(g_1 g_2) &= \prod_{e \in E} \alpha(e) g_1 g_2 \alpha(e\pi(g_1 g_2))^{-1} \\ &= \prod_{e \in E} \alpha(e) g_1 \alpha(e\pi(g_1))^{-1} \cdot \alpha(e\pi(g_1)) g_2 \alpha(e\pi(g_1)\pi(g_2))^{-1} \\ &= \phi(g_1)\phi(g_2)\end{aligned}$$

- $\phi(c) = c^{|E|}$ for every $c \in C$. Here $|E|$ denotes the cardinality of E.

Now it is easy to verify that G fits into the pull-back diagram

$$\begin{array}{ccc} G & \xrightarrow{\pi} & E \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle p\phi'\alpha} \\ C & \xrightarrow{p} & C_m \end{array}$$

where $m := |E|/[C : \operatorname{Im}\phi]$,
$[C : \operatorname{Im}\phi]$ is the index of $\operatorname{Im}\phi$ in C,
$\phi' := \epsilon \circ \phi$,
ϵ is an isomorphism $\operatorname{Im}\phi \to C$ and
$p\colon C \to C_m$ is the canonical projection.
Note that $p\phi'\alpha$ does not depend on α.

"⇐" Suppose

$$\begin{array}{ccc} G & \xrightarrow{\pi} & E \\ \downarrow & & \downarrow \\ C & \xrightarrow{p} & C_m \end{array}$$

is a pull-back diagram, then

$$\begin{array}{ccccc} 1 \longrightarrow & C \longrightarrow & G & \xrightarrow{\pi} E \longrightarrow 1 \\ & c \mapsto & (c^m, 1) & \\ & & (c, e) & \mapsto e \end{array}$$

is a central extension.

2. "⇒" Suppose $1 \to C \to G \xrightarrow{\pi} E \to 1$ is a non-central extension. Choose a set theoretic section $\alpha\colon E \to G$ as before. The homomorphism

$$w\colon E \to \mathrm{Aut}(C) \cong C_2$$

defined by $w(e)(c) := \alpha(e)c\alpha(e)^{-1}$ for all $c \in C$ and $e \in E$, does not depend on the choice of α. Let $\phi\colon \mathrm{Ker}(w\pi) \to C$ be the transfer homomorphism associated to the central extension

$$1 \to C \to \mathrm{Ker}(w\pi) \xrightarrow{\pi} \mathrm{Ker}(w) \to 1.$$

Choose an element $u \in G \setminus \mathrm{Ker}(w\pi)$ and define

$$\psi\colon G \to D$$

$$\psi(g) := \begin{cases} \phi(g) & \text{if } g \in \mathrm{Ker}(w\pi) \\ \phi(gu^{-1})S & \text{if } g \in G \setminus \mathrm{Ker}(w\pi) \end{cases}.$$

For every $g \in \mathrm{Ker}(w\pi)$ we equate

$$\begin{aligned} \phi(ugu^{-1})^{-1} &= u^{-1}\phi(ugu^{-1})u \\ &= \prod_{e \in \mathrm{Ker}(w)} u^{-1}\alpha(e)ugu^{-1}\alpha(e\pi(ugu^{-1}))^{-1}u \\ &= \prod_{e \in \mathrm{Ker}(w)} u^{-1}\alpha(e)u\alpha(\pi(u)^{-1}e\pi(u))^{-1} \cdot \\ &\quad \prod_{e \in \mathrm{Ker}(w)} \alpha(\pi(u)^{-1}e\pi(u))g\alpha(\pi(u)^{-1}e\pi(u)\pi(g))^{-1} \cdot \\ &\quad \prod_{e \in \mathrm{Ker}(w)} \alpha(\pi(u)^{-1}e\pi(u)\pi(g))u^{-1}\alpha(e\pi(ugu^{-1}))^{-1}u \\ &= \phi(g). \end{aligned}$$

In particular $\phi(u^2) = 1$ and ψ is a homomorphism.
Again it is easy to verify that G fits into the pull-back diagram

$$\begin{array}{ccc} G & \xrightarrow{\pi} & E \\ \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle p\psi'\alpha} \\ D & \xrightarrow{p} & D_m \end{array}$$

where $2m := |E|/[D : \operatorname{Im}\psi]$.
Note that $m \cdot [D : \operatorname{Im}\psi] = |\operatorname{Ker}(w)|$ and $|E| = 2|\operatorname{Ker}(w)|$.
$\psi' := \epsilon \circ \psi$,
ϵ is an isomorphism $\operatorname{Im}\psi \to D$ and
$p \colon D \to D_m$ is the canonical projection.
Note that $p\psi'\alpha$ does not depend on α.

"$\Leftarrow$" If

$$\begin{array}{ccc} G & \xrightarrow{\pi} & E \\ \downarrow & & \downarrow \\ D & \xrightarrow{p} & D_m \end{array}$$

is a pull-back diagram, then

$$\begin{array}{ccccc} 1 \longrightarrow & C \longrightarrow & G & \xrightarrow{\pi} E \longrightarrow 1 \\ & c \mapsto & (c^m, 1) & \\ & & (d, e) & \mapsto e \end{array}$$

is obviously a non-central extension.

This completes the proof. •

4 Υ for groups with two ends.

This section is devoted to the following theorem.

4.1 Theorem. *The invariant* $\Upsilon\colon \mathrm{Arf}^h(G) \to \mathcal{J}(G)$ *is injective for all groups* G *having two ends.*

4.2 Lemma. For all $k \in \mathbb{N}$ the relation

$$(a, b) = (a, a(ab)^{2^k}) = (b, b(ab)^{2^k})$$

holds in $\mathrm{Arf}^h(G)$.

Proof. By the relations mentioned in remark 2.7 of chapter I, we have $(a, b) = (a, bab) = (a, a(ab)^2)$ and $(a, b) = (b, a)$. The rest is obvious. •

4.3 Lemma. If (a, az) and (b, bz) are elements of $\mathrm{Arf}^h(G)$ and abz^i has finite order, for some $i \in \mathbf{Z}$, then $(a, az) = (b, bz)$.

Proof. First we consider the case that $i = 0$. Let us write $x = ab$ and let's say that the order of x equals $2^k m$ with m odd.
Note that $a^2 = b^2 = (az)^2 = 1$ and $xz = zx$. Using the relations of remark 2.7 of chapter I we equate

$$\begin{aligned}
(b, bz) &= (ax, axz) \\
&= (ax^m, ax^m z) \\
&= (ax^m, ax^m z^{2^k}) \\
&= (ax^m, ax^m (ax^m az)^{2^k}) \\
&= (ax^m, az) \\
&= (az, ax^m) \\
&= (az, az(azax^m)^{2^k}) \\
&= (az, az(aza)^{2^k}) \\
&= (az, a) \\
&= (a, az)
\end{aligned}$$

The case $i \neq 0$ can be reduced to the previous case:

$$(b, bz) = (z^{-j}bz^j, z^{-j}bzz^j) = (bz^{2j}, bz^{2j+1})$$

$$(b, bz) = (b, bbzb) = (b, bz^{-1}) = (bz^{2j}, bz^{2j-1}) = (bz^{2j-1}, bz^{2j}),$$

thus $(b, bz) = (bz^i, bz^{i+1})$. •

4.4 Proposition. *In the case where* G *fits into a pull-back diagram*

$$\begin{array}{ccc}
G & \longrightarrow & E \\
\big\downarrow & & \big\downarrow \\
C & \longrightarrow & C_m
\end{array}$$

Υ *is injective.*

Proof. Let $x \in \mathrm{Ker}(\Upsilon)$. The relations in $\mathrm{Arf}^h(G)$ listed in remark 2.7 of chapter I and remark 2.7 on the structure of $\mathcal{J}(G)$ allow us to assume, without loss of generality, that

$$x = \sum (a_i, a_i z).$$

Every product of two elements of order two, is of finite order, since all elements of order two in G take the form $(1, e)$. So we may use lemma 4.3 to see that $x = 0$ or $x = (a, az)$. But according to lemma 2.11 $\Upsilon((a, az))$ is non-trivial, so the second case does not occur. •

It remains to show that Υ is injective for groups G which fit into a pull-back diagram

$$\begin{array}{ccc} G & \xrightarrow{\pi} & E \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\hat{p}} \\ D & \xrightarrow{p} & D_{2m} \end{array}$$

Intermezzo.

4.5 Definition. Let G be a group. Suppose we have 2-primary elements $a, b \in G$ which satisfy $[a, b^2] = [a^2, b] = 1$. Here $[x, y]$ denotes the commutator $xyx^{-1}y^{-1}$. Denote by H the subgroup of G generated by a and b. The matrix

$$\begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix} + \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}^{\alpha} = \begin{pmatrix} a + a^{-1} & 1 \\ 1 & b + b^{-1} \end{pmatrix} = \begin{pmatrix} a(1 + a^{-2}) & 1 \\ 1 & b(1 + b^{-2}) \end{pmatrix}$$

is invertible, since $1 + a^{-2}$ and $1 + b^{-2}$ are nilpotent and central in $\mathbb{F}_2[H]$. It is therefore legitimate to define

$$\wp(a, b) := \left[\begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] \in L^h(H).$$

We call such elements of $L^h(H)$ as well as their images in $L^h(G)$ pseudo-arfian.

Notice that $\wp(a, b)$ is not necessarily an element of $\mathrm{Arf}^h(H)$ or $\mathrm{Arf}^h(G)$. However, applying theorem 1.34 of chapter I to the ring $\mathbb{F}_2[H]$ and its nilpotent ideal $(a^2 + 1, b^2 + 1)$ yields an isomorphism

$$L^h(H) \hookrightarrow\!\!\!\!\!\rightarrow L^h(H/ < a^2, b^2 >),$$

which maps $\wp(a, b) \in L^h(H)$ to the Arf-element $(a, b) \in \mathrm{Arf}^h(H/ < a^2, b^2 >)$:

$$\begin{array}{ccccc} & & \wp(a,b) & \longmapsto & (a,b) \\ \mathrm{Arf}^h(G) & \longleftarrow & \mathrm{Arf}^h(H) & \hookrightarrow & \mathrm{Arf}^h(H/ < a^2, b^2 >) \\ \bigcap & & \bigcap & & \bigcap \\ L^h(G) & \longleftarrow & L^h(H) & \hookrightarrow\!\!\!\!\!\rightarrow & L^h(H/ < a^2, b^2 >) \end{array}$$

4.6. Let G be a group and $g, z \in G$. Assume $g^{-1}zg = z^{-1}$ and g is of finite order, say $2^r r_0$ with r_0 odd. Define H as the subgroup of G generated by z and $h := g^{r_0}$.
Since $h^{-1}zh = z^{-1}$, i.e. $h^2 = (hz)^2$ we obtain a pseudo-arf element $\wp(h, hz) \in L^h(H)$. The question is whether this element depends on h.

4.7 Theorem. *Let E be a finite group. The invariant ω_1^h of chapter II induces an isomorphism*

$$L^h(E) \longrightarrow \bigoplus k/\{x + x^2 \mid x \in k\}$$

Here the summation runs through all representations $\rho{:}\mathbb{F}_2[E] \to M_n(k)$ which take the form $\rho(e)^{-1} = P^{-1}\rho(e)^t P$ for all $e \in E$, for some invertible matrix P and t means matrix transpose. What's more, the image of $\wp(h, hz)$ under this isomorphism is the element which has $[\mathrm{Tr}(\rho(z))]$ at the place with index ρ. In particular $\wp(h, hz)$ does not depend on h.

Proof. Define $R := \mathbb{F}_2[E]/\operatorname{rad}(\mathbb{F}_2[E])$ where rad means Jacobson radical. For every ring A, we denote by $\widetilde{A}$ the truncated polynomial ring $A[T]/(T^3)$ and in this context (T) is the two-sided ideal of $\widetilde{A}$ generated by T. Consider the following diagram.

$$\begin{array}{ccc}
L^h(E) & & \\
\cong\big\downarrow 1 & & \\
L^h(R) & \xrightarrow{\quad\omega_1^h\quad} & H^0(K_1(\widetilde{R})) \\
\cong\big\downarrow 2 & & \cong\big\downarrow 6 \\
L^h\left(\prod D_i\right) & & H^0\left(\bigoplus K_1(\widetilde{D_i})\right) \\
\cong\big\downarrow 3 & & \cong\big\downarrow 7 \\
\bigoplus_j L^h(D_j) & & \bigoplus_j H^0\left(K_1(\widetilde{D_j})\right) \\
\cong\big\downarrow 4 & & \cong\big\downarrow \\
\bigoplus_j L^h(k_j) & & H^0\left(\bigoplus(k_j^* \oplus 1 + T\widetilde{k_j})\right) \\
\cong\big\downarrow 5 & & \cong\big\downarrow \\
\bigoplus_j \mathrm{Arf}^h(k_j) & & H^0\left(\bigoplus_j 1 + T\widetilde{k_j}\right) \\
\searrow & & \swarrow \\
 & \bigoplus_j \mathrm{Coker}(1 + \sigma_j) & \\
 & \big\downarrow\cong & \\
 & \bigoplus_j \mathbf{Z}/2 &
\end{array}$$

We elucidate the diagram.

1. It follows from theorem 1.34 of chapter I that $L^h(E)$ and $L^h(R)$ are isomorphic, because $\mathrm{rad}(\mathbb{F}_2[E])$ is a nilpotent ideal.

2. The ring R is artinian and $\mathrm{rad}(R) = 0$, so we can apply the Wedderburn-Artin theorem. In our case this reads: R is isomorphic to a direct product of full matrix rings over finite fields of characteristic two. Explicitly,

$$R \cong \prod D_i;$$

here $D_i := M_{n_i}(k_i)$ is the ring of $(n_i \times n_i)$-matrices over the finite field k_i and $\mathrm{char}(k_i) = 2$.
Denote by ρ_i the composition $\mathbb{F}_2[E] \to R \to \bigoplus D_i \to D_i$.

3. Let $^-$ denote the (anti-) involutions on R and $\prod D_i$ induced by the involution on $\mathbb{F}_2[E]$. Before we decompose $\prod D_i$ as a product of rings with involution, we fix some notations concerning finite fields of characteristic two. If k is such a field, the group of automorphisms of k is cyclic and generated by the Frobenius automorphism $\sigma\colon k \to k$ which assigns to an element x of k its square. The field trace $\mathrm{Tr}\colon k \twoheadrightarrow \mathbb{F}_2$ induces an isomorphism $\mathrm{Coker}(1+\sigma) \to \mathbb{F}_2$. If the degree of k over $\mathbb{F}_2$ is even there exists a unique automorphism of order two, which we denote by $\hat{\sigma}$.
Now, for a factor $D = M_n(k)$ of $\prod D_i$ we have three possible cases:

   - D is invariant under the involution, i.e. $D = \overline{D}$, and the restriction of $^-$ to k is $\hat{\sigma}$.
   - $D = \overline{D}$ and the restriction of $^-$ to k is the identity. Since the composition of the (anti-) involution $^-$ with matrix transpose is a k-linear automorphism of D, this composition takes the form $X \mapsto PXP^{-1}$, for some invertible matrix P. Further we may assume that P is symmetric, since this automorphism is of order two. Thus for all $X \in D$ we have $\overline{X} = P^{-1}X^tP$.
   - $\overline{D} \neq D$. So $D \times \overline{D}$ is a factor of $\prod D_i$. If $D \times D^\circ$ is endowed with the involution $(x, y) \mapsto (y, x)$, the map $D \times \overline{D} \to D \times D^\circ$ defined by $(x, y) \mapsto (x, \overline{y})$ is an isomorphism of rings with involution. Recall that $\circ$ means opposite multiplication.

   Thus we obtain a decomposition of $\prod D_i$ in which three different types of factors occur. The L-groups split accordingly. See e.g. [26]. We assert that only the groups $L^h(D)$, where D is of the second type, survive.
   In the first case we have

$$L^h(D, ^-, 1) \cong L^h(k, \hat{\sigma}, 1)$$

   by Morita invariance. But $L^h(k, \hat{\sigma}, 1) = 0$ by [24, §6].
   For the third case we will show that quadratic modules (M, θ) over the ring $D \times D^\circ$, with involution $\alpha(x, y) = (y, x)$, are in fact hyperbolic. Note that there is no need to worry about bases, because we are working in L^h.

Define $\lambda := (1,0)$, $M_1 := \lambda M$ and $M_2 := (1+\lambda)M$. So $M = M_1 \oplus M_2$. Since $b_\theta\colon M \to M^\alpha$ is an isomorphism and for all $m, n \in M$

$$b_\theta((1+\lambda)m)((1+\lambda)n) = \lambda(1+\lambda)b_\theta(m)(n) = 0,$$

the restriction of b_θ to M_2 yields an isomorphism $M_2 \to M_1^\alpha$. Now it is easy to verify that the map

$$(M, \theta) \longrightarrow H(M_1) = (M_1 \oplus M_1^\alpha, \upsilon)$$

defined by

$$m \mapsto (\lambda m, b_\theta((1+\lambda)m))$$

is an isometry. In Walls terminology [24] this says that $(D \times \{0\})M$ is a subkernel of M. This proves our assertion.
We use the index j to refer to summands of the second type.

4. $L^h(D_j, {}^-, 1)$ is isomorphic to $L^h(k_j, 1, 1)$ by Morita invariance.

5. Since the field trace $\mathrm{Tr}\colon k \longrightarrow \mathbb{F}_2$ is surjective we can choose an element a in k such that $\mathrm{Tr}(a) = 1$. The Arf invariant

$$\omega_1^h\colon \mathrm{Arf}^h(k, 1, 1) \to \mathrm{Coker}(1+\sigma) \cong \mathbf{Z}/2$$

maps $(a, 1)$ to the non-trivial element in $\mathbf{Z}/2$. Combining this with the fact that $L^h(k, 1, 1) \cong \mathbf{Z}/2$, see [24, §6], we find

$$L^h(k, 1, 1) = \mathrm{Arf}^h(k, 1, 1) \cong \mathbf{Z}/2.$$

6. It is almost immediately clear from the definition of K_1, that for all rings A, B one has $K_1(A \times B) \cong K_1(A) \oplus K_1(B)$.

7. The argument here is roughly the same as the one on the 'L-side' of the diagram (item 3). By Morita theory we have $K_1(\widetilde{D}) \cong K_1(\widetilde{k})$. Alternatively one can see this directly by looking at the definition of K_1. Since the projection $\widetilde{k} \to k$ splits we have

$$K_1(\widetilde{k}) \cong K_1(k) \oplus K_1(\widetilde{k}, (T)).$$

It is well-known that $K_1(k) = k^*$, the group of units in k and we already saw that $K_1(\widetilde{k}, (T)) = 1 + T\widetilde{k}$. To decompose $\bigoplus_i K_1(\widetilde{D_i})$ into invariant parts, we consider the same three possibilities:

- D is invariant under the involution and the restriction of ${}^-$ to k is $\hat{\sigma}$. In this case

$$H^0(K_1(\widetilde{D})) = H^0(k^* \oplus (1 + T\widetilde{k})) = H^0(k^*) \oplus H^0(1 + T\widetilde{k}).$$

But $H^0(k^*)$ vanishes, because k^* has odd order. And in the third section of chapter II we computed $H^0(1 + T\widetilde{k}) = C(k)$, but this also disappears since $H^0(k; \hat{\sigma}) = 0$.

- $D = \overline{D}$ and the restriction of $^{-}$ to k is the identity. By the same arguments as in the previous case we obtain $H^0(K_1(\widetilde{D})) = C(k)$, but now $C(k)$ is precisely $\mathrm{Coker}(1+\sigma)$.
- $D \neq \overline{D}$. Here the involution interchanges the summands $K_1(\widetilde{D})$ and $K_1(\widetilde{\overline{D}})$, so H^0 clearly dies.

Thus only the summands of the second type survive.

This completes the proof of the first part of what the theorem asserts. To prove the second part let $\rho: \mathbb{F}_2[E] \to D = M_n(k)$ be a representation of the special kind, i.e. D is of the second type, and assume that $\rho(h) = H$ and $\rho(z) = Z$. Then

$$\omega_1^h(\wp(h, hz)) = \omega_1^h\left(\left[\begin{pmatrix} h & 1 \\ 0 & hz \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]\right).$$

Now

$$\begin{pmatrix} h & 1 \\ 0 & hz \end{pmatrix} + (1+T)\begin{pmatrix} h & 1 \\ 0 & hz \end{pmatrix}^{\alpha} = \begin{pmatrix} h + (1+T)h^{-1} & 1 \\ (1+T) & hz + (1+T)(hz)^{-1} \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + (1+T)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^{\alpha} = \begin{pmatrix} 0 & 1 \\ (1+T) & 0 \end{pmatrix},$$

so

$$\begin{aligned}
&\omega_1^h(\wp(h, hz)) = \\
&\quad \left[\begin{pmatrix} h + (1+T)h^{-1} & 1 \\ (1+T) & hz + (1+T)(hz)^{-1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ (1+T) & 0 \end{pmatrix}^{-1}\right] = \\
&\quad \left[\begin{pmatrix} 1 & h(1+T)^{-1} + h^{-1} \\ hz + (1+T)(hz)^{-1} & 1 \end{pmatrix}\right] = \\
&\quad \left[\begin{pmatrix} 1 + h^2 z(1+T)^{-1} + h^{-2} z(1+T) & h(1+T)^{-1} + h^{-1} \\ 0 & 1 \end{pmatrix}\right] = \\
&\quad \left[\begin{pmatrix} 1 + h^2 z(1+T)^{-1} + h^{-2} z(1+T) & 0 \\ 0 & 1 \end{pmatrix}\right] = \\
&\quad \left[\left(1 + h^2 z(1+T)^{-1} + h^{-2} z(1+T)\right)\right] \in H^0(K_1(\widetilde{R})).
\end{aligned}$$

The image of this element in $H^0(1 + T\widetilde{k})$ equals

$$\begin{aligned}
&\left[\det\left(1 + H^2 Z(1+T+T^2) + H^{-2} Z(1+T)\right)\right] \\
&\quad = \left[\det\left(1 + H^2 Z T^2\right)\right]\left[\det\left(1 + (H^2 + H^{-2})Z(1+T)\right)\right] \\
&\quad = \left[\det\left(1 + H^2 Z T^2\right)\right] \\
&\quad = [1 + \mathrm{Tr}(H^2 Z) T^2] \\
&\quad = [1 + \mathrm{Tr}(Z) T^2]
\end{aligned}$$

Here

$$\det\big(1+(H^2+H^{-2})Z(1+T)\big)=1 \quad \text{and} \quad \mathrm{Tr}(H^2Z)=\mathrm{Tr}(Z)$$

by lemma 4.8, because $(H^2+H^{-2})Z$ and $1+H^2$ are nilpotent. Finally, the image of $[1+\mathrm{Tr}(Z)T^2]$ in $\mathrm{Coker}(1+\sigma)$ equals $[\mathrm{Tr}(Z)]=[\mathrm{Tr}(\rho(z))]$ according to the computations in chapter II. •

4.8 Lemma. If V is a finite dimensional k-vectorspace, $N\colon V\to V$ is a nilpotent linear map and s is an indeterminate, then

$$\mathrm{Tr}(N)=0 \;\text{ and }\; \det(1+sN)=1.$$

Proof. Suppose $N^n=0$. We apply induction on n.
If $n=1$ the matter is clear.
If $n>1$ consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & NV & \longrightarrow & V & \longrightarrow & V/NV & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle N} & & \big\downarrow{\scriptstyle N} & & \big\downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & NV & \longrightarrow & V & \longrightarrow & V/NV & \longrightarrow & 0 \end{array}$$

The first vertical map in this diagram has nilpotency degree $n-1$ and $N\colon V\to V$ takes the form $\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}$. This proves the assertions. •

<u>End intermezzo</u>.

4.9 Theorem. *If* $(a,b)+(c,d)\in\mathrm{Ker}(\Upsilon)$, *then* $(a,b)=(c,d)$ *in* $\mathrm{Arf}^h(G)$.

Proof. Note that $(a,b)+(c,d)\in\mathrm{Ker}(\Upsilon)$ if and only if

$$[ab]=[cd]\in\mathcal{C}\ell(G) \quad \text{and} \quad [bd]=1\in Ł([ab]).$$

Again the relations in $\mathrm{Arf}^h(G)$ and the structure of $\mathcal{J}(G)$ allow us to assume that $ab=cd$. Elements of order two in G either have the form $(1,e)$ with $e^2=1$ and $\hat{p}(e)=1$ or the form (ST^i,e) with $e^2=1$ and $\hat{p}(e)=p(ST^i)$. Thus we may assume that

$$(a,b)+(c,d)=((\Delta,e),(\Delta T^i,ez))+((\Delta T^j,ex),(\Delta T^{i+j},exz)),$$

where

- $\Delta=ST^\nu$: if $\Delta=1$ we are through by lemma 4.3
- $e,x,z\in E$ satisfy $e^2=(ex)^2=(ez)^2=1$ and $xz=zx$
- $[(T^j,x)]=1\in Ł([(T^i,z)])$.

Lemma 4.2 permits us to replace (T^i,z) by any power-of-two power of (T^i,z). Hence we may assume that z has odd order, let's say order $l_0=2l-1$.

Case 1: $i \neq 0$.
Write $m = 2^\mu m_0$ and $i = 2^\tau i_0$ with m_0 and i_0 odd. Since $(T^i, z) \in G$ and z has order l_0 in E, we have $m | i l_0$, i.e. $\mu \leq \tau$ and $m_0 | i_0 l_0$. If $\mu < \tau$, then

$$((\Delta, e), (\Delta T^i, ez)) = ((\Delta, e), (\Delta T^{i/2}, ez^l)),$$

by lemma 4.2, where

- $\cdot$ $(T^{i/2}, z^l) \in G$, because $il \equiv i/2 \pmod m$
- $\cdot$ $(T^{i/2}, z^l)^2 = (T^i, z)$.

So we may assume that $\mu = \tau$.
Further, conjugation by a suitable power of (T^j, x) allows us to replace (T^j, x) by any odd power of (T^j, x). Thus we may assume that x has order a power of 2, let's say 2^k.

- ⋄ If necessary we conjugate by $(T^m, 1)$ to achieve that $0 \leq j < 2m$.
- ⋄ If $m < j < 2m$, then conjugation by (T^{j-m}, x) yields

  $$((\Delta T^j, ex), (\Delta T^{i+j}, exz)) = ((\Delta T^{2m-j}, ex^{-1}), (\Delta T^{i+2m-j}, ex^{-1}z)),$$

  so we may assume that $0 \leq j \leq m$.
  It is important to note that these changes do not affect the order of x.
- ⋄ If $j = 0$, then lemma 4.3 gives the desired result.
- ⋄ If $j = m$, then conjugation by $(T^{(m+i)/2}, z^l)$ yields

  $$\begin{aligned} ((\Delta T^m, ex), (\Delta T^{i+m}, exz)) &= ((\Delta T^{-i}, exz^{-2l}), (\Delta, exz^{1-2l})) \\ &= ((\Delta, e), (\Delta T^i, ez)) \end{aligned}$$

  The second identity follows from lemma 4.3. Note that $(T^{(m+i)/2}, z^l) \in G$ if and only if $(i+m)/2 \equiv il \pmod m$. But this condition is satisfied because

  $$(i+m)/2 = 2^\mu (i_0 + m_0)/2 \equiv 2^\mu i_0 l = il \pmod{2^\mu m_0}.$$

  This finishes the proof in the case that $j = m$.
- ⋄ If $0 < j < m$, write $j = 2^\nu j_0$ with j_0 odd.
  We know that $(T^j, x) \in G$ and x has order 2^k in E, hence $m | j 2^k$, i.e. $\mu \leq k + \nu$ and $m_0 | j_0$. Taking the fact that $j < m$ into account this implies $\nu < \mu$.
  - $\cdot$ Choose $r \in \mathbb{N}$ such that $r > k + \nu$ and $l_0 | 2^r - 1$.
  - $\cdot$ Define $w := z^{l^{\mu-\nu}}$.

- Choose an ϵ which satisfies the congruence

$$j\epsilon + il^{\mu-\nu} \equiv j + i_0 2^\nu \pmod{m}.$$

This is possible:
mod m_0 it reads

$$\begin{aligned} i_0 2^\nu 2^{\mu-\nu} l^{\mu-\nu} &\equiv i_0 2^\nu \\ i_0 2^\nu((2l)^{\mu-\nu} - 1) &\equiv 0 \\ i_0 2^\nu((l_0+1)^{\mu-\nu} - 1) &\equiv 0, \end{aligned}$$

but since $m_0 | i_0 l_0$, this is automatically true;
mod 2^μ it reads

$$2^\nu j_0(\epsilon - 1) \equiv 2^\nu i_0 \pmod{2^\mu}$$

which is equivalent to

$$j_0(\epsilon - 1) \equiv i_0 \pmod{2^{\mu-\nu}},$$

but since j_0 is odd this is solvable.
- $ex(exex^\epsilon w)^{2^{r-\nu}} = ex(x^\epsilon w)^{2^{r-\nu}} = exz^{2^{r-\mu}}$.
- Define $\tilde{j} := j - (2^r - 1)2^\nu i_0$ and $\tilde{x} := x^\epsilon$.

These definitions and facts support the following computation.

$$\begin{aligned} ((\Delta T^j, ex), (\Delta T^{i+j}, exz)) &= \\ (r-\mu \text{ times lemma 4.2}) &= ((\Delta T^j, ex), (\Delta T^{i2^{r-\mu}+j}, exz^{2^{r-\mu}})) \\ (r-\nu \text{ times lemma 4.2}) &= ((\Delta T^j, ex), (\Delta T^{i_0 2^\nu + j}, e\tilde{x}w)) \\ (r \text{ times lemma 4.2}) &= ((\Delta T^{j+2^\nu i_0 - 2^{r+\nu} i_0}, e\tilde{x}), (\Delta T^{i_0 2^\nu + j}, e\tilde{x}w)) \\ &= ((\Delta T^{\tilde{j}}, e\tilde{x}), (\Delta T^{i_0 2^{r+\nu} + \tilde{j}}, e\tilde{x}w)) \\ (\mu-\nu \text{ times lemma 4.2}) &= ((\Delta T^{\tilde{j}}, e\tilde{x}), (\Delta T^{i2^r + \tilde{j}}, e\tilde{x}w)) \\ (r \text{ times lemma 4.2}) &= ((\Delta T^{\tilde{j}}, e\tilde{x}), (\Delta T^{i+\tilde{j}}, e\tilde{x}z)) \end{aligned}$$

Observe that $\tilde{j}$ is a multiple of $2^{\nu+1} m_0$. Thus we replace the old (T^j, x) by a new one. Apply one of the preceding steps if $j \geq m$ or $j \leq 0$.

Repeat this process until $\mu = \nu$, which implies $m|j$.

This completes the proof in this first case. We did not need the fact that $[(T^j, x)] = 1 \in Ł([(T^i, z)])$! This means that the primary Arf invariant is already good enough to detect the Arf-elements in this case.

Case 2: $i = 0$**.**
Our purpose is to show that

$$((\Delta, e), \Delta, ez)) = ((\Delta T^j, ex), (\Delta T^j, exz)) \in \mathrm{Arf}^h(G).$$

We apply induction on j, as follows.

◇ If $j < 0$ or $j > 2m$,
we conjugate by a suitable power of $(T^m, 1)$, to attain $0 \leq j \leq 2m$.

◇ If $m < j \leq 2m$,
we conjugate by $(\Delta T^m, e)$, to achieve $0 \leq j \leq m$.

◇ If $j = 0$,
lemma 4.3 does the job.

◇ If $j \not| m$,
we define $d := \gcd(j, m)$. Obviously $(T^d, x^{n_0}) \in \overline{G_{(1,z)}}$ for some $n_0 \in \mathbf{Z}$. Now conjugating by (T^d, x^{n_0}) allows us to replace j by $j - 2d$. Notice that $j - 2d > 0$.

◇ If $j|m$,
there are two possibilities. If there exists $(T^c, y) \in \overline{G_{(1,z)}}$ with $0 < c < j$, then we conjugate by (T^c, y) to replace j by $j - 2c$. We have

$$-j + 2 \leq j - 2c \leq j - 2.$$

Conjugating by (Δ, e), if necessary, yields

$$0 \leq j - 2c \leq j - 2.$$

If there is not such a c, then the elements of $\overline{G_{(1,z)}}$ either have the form $(T^{jv}, \cdot\cdot)$ or $(\Delta T^{jv}, \cdot\cdot)$. Since any element of

$$\mathrm{Ker}(\overline{G_{(1,z)}} \to F(1, z))$$

is a product of squares and 2-power roots of $(1, z)$, so is (T^j, x). A little examination reveals that this can only happen when there exist 2-power roots y_1 and y_2 of z such that $(\Delta, y_1), (\Delta T^j, y_2) \in G$
Consider the pull-back diagram

$$\begin{array}{ccc} G & \longrightarrow & E \\ \big\downarrow & & \big\downarrow \hat{p} \\ D & \longrightarrow & D_m \end{array}$$

and define

$$F_1 := \hat{p}^{-1}(< \sigma\tau^\nu >)$$

$$j_1 \colon F_1 \to G \text{ by} \qquad j_1(f) := \begin{cases} (1, f) & \text{if } \hat{p}(f) = 1 \\ (\Delta, f) & \text{otherwise} \end{cases}$$

$$F_2 := \hat{p}^{-1}(< \sigma\tau^{\nu+j} >)$$

$$j_2 \colon F_2 \to G \text{ by} \qquad j_2(f) := \begin{cases} (1, f) & \text{if } \hat{p}(f) = 1 \\ (\Delta T^j, f) & \text{otherwise} \end{cases}$$

$$F_0 := F_1 \cap F_2 = \mathrm{Ker}(\hat{p})$$

$$E_0 := \mathrm{Ker}(w \colon E \to \mathrm{Aut}(C))$$

Now $z \in F_0$, $e \in F_1$, $ex \in F_2$ and in the diagram

$$\begin{array}{ccc} F_0 & \subset & E_0 \\ \bigcap & & \bigcap \\ F_1 & \subset & E \end{array}$$

$[F_1 : F_0] = [E : E_0] = 2$ and $[E : F_1] = [E_0 : F_0] = m$.
We know there exist $y_1 \in F_1 \setminus F_0$ such that $y_1 z = z y_1$. Then we have $ey_1 \in F_0$ and $ey_1 z(ey_1)^{-1} = z^{-1}$, so 4.6 guarantees the existence of a pseudo-arf element $\wp(f_1, f_1 z) \in L^h(F_0)$. Analogous, the existence $y_2 \in F_2 \setminus F_0$, satisfying $y_2 z = z y_2$, yields a pseudo-arf element $\wp(f_2, f_2 z) \in L^h(F_0)$. through the element $exy_2 \in F_0$.
But these pseudo-arf elements must coincide by theorem 4.7.

$$\begin{array}{ccccc} & \wp(f_2, f_2 z) & & (ex, exz) \\ \wp(f_1, f_1 z) & L^h(F_0) & \longrightarrow & L^h(F_2) \\ & \downarrow & \searrow & \downarrow{\scriptstyle j_{2*}} \\ (e, ez) & L^h(F_1) & \xrightarrow{j_{1*}} & L^h(G) \end{array}$$

Therefore we may conclude that

$$j_{1*}((e, ez)) = j_{2*}((ex, exz)) \in L^h(G).$$

This completes the proof of theorem 4.9. •

proof of theorem 4.1. Suppose we have an element of $\mathrm{Arf}^h(G)$ which is killed by Υ. As before we may assume that it has the form $\sum (a_i, a_i z)$. We apply induction on the number of terms occuring in the expression for our element in $\mathrm{Arf}^h(G)$. Recall that we are dealing with terms like $((\Delta, e), (\Delta T^i, ez))$ and $((\Delta T^j, ex), (\Delta T^{i+j}, exz))$. If there are less than three terms theorem 4.9 does the job. Thus assume that the number of terms exceeds two. If a term $((1, \cdot\cdot), \cdots)$ appears, lemma 4.3 enables us to cancel two terms. Otherwise there are two cases:

- $i \neq 0$
  We can cancel terms by the first case of theorem 4.9 without having any information on (T^j, x).

- $i = 0$
  The following terms occur:

$$\begin{array}{l} ((\Delta, e), (\Delta T^i, ez)) \\ ((\Delta T^{j_1}, ex_1), (\Delta T^{i+j_1}, ex_1 z)) \\ ((\Delta T^{j_2}, ex_2), (\Delta T^{i+j_2}, ex_2 z)) \end{array}$$

Now define $j = \gcd(j_1, j_2)$, say $j = a_1 j_1 + a_2 j_2$. We conjugate the second and third term by a suitable power of

$$(T^j, x_1^{a_1} x_2^{a_2})$$

to obtain

$$((\Delta, e\tilde{x}), (\Delta, e\tilde{x}z))$$

or

$$((\Delta T^j, e\tilde{x}), (\Delta T^{i+j}, e\tilde{x}z)).$$

Applying lemma 4.3 once more, we can cancel terms.

We see that two terms cancel in all cases.

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Samenvatting

In dit proefschrift worden nieuwe invarianten in de algebraïsche L-theorie bestudeerd. Een interessant bijprodukt van deze studie is de ontwikkeling van gereduceerde machtsoperaties in cyclische homologie.

L-groepen van een ring met anti-structuur zijn in essentie Grothendieck en Whitehead groepen van de categorie van kwadratische modulen over die ring. Om meetkundige redenen is men vooral geïnteresseerd in L-groepen van groepenringen.

Hoofdstuk I behelst een eigenzinnige poging de relevante K- en L-theorie te etaleren en daarmee een geschikte setting voor de rest van het proefschrift te creëren. In hoofdstuk II worden K-theoretische invarianten geconstrueerd door de anti-structuur op de gegeven ring, op een nogal exotische manier, uit te breiden tot de ring van formele machtreeksen over die ring. Deze invarianten generaliseren klassieke invarianten zoals de Arf invariant. Uitvoerige berekeningen aan de waardengroepen van deze nieuwe invarianten leiden vervolgens tot de bestudering van homologische invarianten in hoofdstuk III. Hiertoe worden gereduceerde machtsoperaties in Hochschild, cyclische en quaternionische homologie gefabriceerd. Hoofdstuk IV tenslotte, is gewijd aan de toepassing van het voorafgaande op groepenringen. Het belangrijkste resultaat is hier dat de homologische invariant 'goed genoeg' is voor elke groep met twee einden (Stelling 4.1).

Curriculum Vitae

De schrijver van dit proefschrift werd geboren op 1 juli 1961 te Heerlen. In 1979 behaalde hij het VWO diploma aan het Eykhagencollege te Schaesberg. Daarna studeerde hij wiskunde aan de Katholieke Universiteit te Nijmegen. In 1985 slaagde hij cum laude voor zijn doctoraalexamen voorzien van een pedagogisch-didactische aantekening. In oktober 1985 trad hij als wetenschappelijk assistent in dienst van de Katholieke Universiteit. Tot maart 1990 verrichtte hij promotieonderzoek onder leiding van Dr. F.J.-B.J. Clauwens. De meeste resultaten van dat onderzoek zijn te vinden in dit proefschrift.