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FRAMING THE EXCEPTIONAL LIE GROUP G_2

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§1. INTRODUCTION

RECENTLY, attention has been given to framings on Lie groups with a view to producing non-trivial elements in the homotopy groups of spheres and finding some kind of pattern in their construction. We refer to Atiyah, Smith[3], [13] and Gershenson[7] for work in this direction and to Steer[14] for a more comprehensive treatment of the subject of framed Lie groups and their Hopf invariants to which this paper is a contribution.

Our principal aim is to show that Toda's element κ in the 14-stem of the homotopy groups of spheres can be represented by a framing on G_2. During the course of the work we are led to consider framings on $SU(3)$. It turns out that all elements in the stable 8-stem can be represented by such framings.

The general plan is to start with a representation of a compact Lie group G of order k and embed G in codimension k with a framing which represents the zero element of $\pi_{n+k}(S^k)$, where n is the dimension of G. This trivial framing is twisted by the representation to provide a new element of $\pi_{n+k}(S^k)$, which is hopefully non-trivial. To test this we apply the generalized Hopf invariant of the classical EHP sequence. Background information on the homotopy theory involved here can be found in Toda's book[15]. The success of the method depends of course on constructing elements in $\pi_{n+k}S^k$ which are not desuspensions, otherwise the Hopf invariant vanishes. It turns out that there is a correlation between the least orders of faithful representations of certain Lie groups of low rank and the dimensions of spheres S^k on which interesting elements of $\pi_{n+k}S^k$ first arise. Furthermore there is a pattern of Hopf invariants corresponding to classical fibrations of these groups. We shall illustrate these assertions in the case of G_2. As a focal point of this paper let us state the following result.

1.1. THEOREM. *There is a natural framed embedding of G_2 in $\mathbf{R}^{21}$ arising from a maximal orbit of three copies of the fundamental representation ρ: $G_2 \subset SO(7)$. This framing represents* 0 *in $\pi_{21}S^7$. If we twist this trivial framing by the representation ρ we obtain a non-zero element of $\pi_{21}S^7$ which stabilizes to Toda's element κ in the* 14-*stem.*

We may also twist the trivial framing of G_2 by the inverse map ρ^{-1}: $G_2 \to SO(7)$. This also gives a non-trivial element of $\pi_{21}S^7$ but it stabilizes to zero in the 14-stem. Hopf invariant calculations and Toda's tables are sufficient to check the non-triviality of the twisted framings unstably. The Hopf invariants of framings on G_2 are identified with framings on $SU(3)$ whose Hopf invariants in turn are identified with framings on $SU(2)$. The pattern of Hopf invariants referred to above corresponds in these cases to the fibrations

$$SU(2) \to SU(3) \to S^5, \qquad SU(3) \to G_2 \to S^6.$$

The problem of the stable identification of framings on G_2 and $SU(3)$ is rather subtle. To solve the problem completely we need the e-invariants of framings on $SU(3)$ in conjunction with their Hopf invariants and Toda's tables[15]. In §8 the e-invariant calculations are carried out, but for a comprehensive treatment of this matter we refer to Ray[12] where a formula is developed in general homology theory for the change of e-invariant of a twisted framing. In §3 we give a way of viewing the Hopf invariant in terms of a certain "intermediate bordism group". The intermediate bordism groups are defined in §2. They lie between the homotopy groups of the orthogonal group and the homotopy groups of spheres and give rise to a certain filtration of the latter which may be of some independent interest. More information relevant to these intermediate bordism groups and Hopf invariants is given in §6 and in §7 where we have also collected a few problems together which arise quite naturally in the course of our work. For convenience we have provided in §4 a brief description of the exceptional Lie group G_2 and its main properties, which are used in §5 to set up the required embeddings of G_2 in $\mathbf{R}^{21}$ and to reduce the calculation of Hopf invariants to elementary matrix theory.

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§2. FRAMED MANIFOLDS

Background information on framed manifolds and the Thom–Pontrjagin construction can be found in Milnor's book[11]. We just recall some of the basic ideas which we shall be using subsequently. A framing on a manifold M^n smoothly embedded in euclidean space $\mathbf{R}^{n+k}$ consists of an ordered set of vectors $v_1(x), \ldots, v_k(x)$ varying smoothly with $x \in M^n$ and providing a basis for the normal space at x of M^n in $\mathbf{R}^{n+k}$. There is an equivalence relation of bordism between such manifolds. The set of equivalence classes of closed framed n-manifolds in $\mathbf{R}^{n+k}$ forms a group under disjoint union. The Thom–Pontrjagin construction sets up an isomorphism between this bordism group and the homotopy group $\pi_{n+k}S^k$.

Let M^n be framed in $\mathbf{R}^{n+k}$ by vectors $v_1, \ldots, v_k$ which we consider as columns of an $(n+k)\times k$ matrix V. Let $g: M^n \to GL(k)$ be a smooth map assigning to a point x in M the non-singular matrix A. We can form the matrix product $W = VA$ whose columns give a new basis of the normal space at $x \in M$. The new framed manifold (M, W) obtained from (M, V) by *twisting* the framing V with the map g is not in general bordant to (M, V). If however g is nullhomotopic then a smooth nullhomotopy gives rise to a cylindrical bordism between (M, V) and (M, W). This situation arises, for example, when we apply the Gram–Schmidt process for transforming a basis into an orthogonal one. We shall assume in practice therefore that our framing vectors are orthogonal.

The suspension homomorphism of homotopy theory

$$E: \pi_{n+k}(S^k) \to \pi_{n+k+1}(S^{k+1}) \tag{2.1}$$

has the following geometric interpretation. Let α in $\pi_{n+k}(S^k)$ be represented by a manifold M^n in $\mathbf{R}^{n+k}$ framed by vectors $v_1, \ldots, v_k$. Let

$$t = (0, \ldots, 0, 1) \tag{2.2}$$

be the unit vector along the last axis of $\mathbf{R}^{n+k+1}$. Then $E(\alpha)$ is represented by the manifold M^n framed in $\mathbf{R}^{n+k+1}$ by the vectors $v_1, \ldots, v_k, t$.

The problem of desuspending an element in $\pi_{n+k+1}S^{k+1}$ can be thought of in two stages, firstly lowering the embedding dimension of the manifold and secondly deforming the framing so that the last framing vector always points in the direction t. With a view to distinguishing these problems we now introduce some intermediate bordism groups. Consider closed n-manifolds embedded in $\mathbf{R}^{n+r}$ but framed in $\mathbf{R}^{n+k}$ where $1 \leq r \leq k$. For example, if $r = 1$, we are dealing with framed hypersurfaces whereas, if $r = k$, there is no extra constraint on the embedding. A bordism group of such manifolds is constructed in the usual way. We just remark that a bording manifold is constrained to lie in $\mathbf{R}^{n+r} \times I$ but its framing is allowed to take place in $\mathbf{R}^{n+k} \times I$. Let $\pi^{k,r}_{n+k}$ denote this bordism group of manifolds framed in codimension k but embedded in codimension r. We can admit the case $r = 0$ and define $\pi^{k,0}_{n+k}$ as $\pi_n SO(k)$, on the grounds that the only manifold which embeds in S^k with codimension 0 is S^k itself and the only bording manifold allowed is the cylinder. Of course, for $r > 0$, we can work with n-manifolds embedded in $\mathbf{R}^{n+r}$ or S^{n+r}; the two are equivalent by stereographic projection. The following sequence arises naturally by regarding a manifold embedded in codimension r as embedded in codimension $r+1$.

$$\pi^{k,0}_{n+k} \to \pi^{k,1}_{n+k} \to \cdots \to \pi^{k,k-1}_{n+k} \to \pi^{k,k}_{n+k}. \tag{2.3}$$

We note that $\pi^{k,k}_{n+k}$ is the usual bordism group isomorphic to $\pi_{n+k}S^k$ and the composite map of the sequence is the J-homomorphism

$$\pi_n SO(k) \to \pi_{n+k}S^k \tag{2.4}$$

in geometric guise. The images of $\pi^{k,r}_{n+k}$ in $\pi_{n+k}S^k$ provide a filtration starting with the image of J. We shall say a little more about these intermediate bordism groups in §7 but we wish now to concentrate attention on the penultimate group $\pi^{k+1,k}_{n+k+1}$.

§3. THE HOPF INVARIANT

The suspension map (2.1) factors through the group $\pi^{k+1,k}_{n+k+1}$ because we add on a new framing vector but leave the manifold embedded in the same dimension. It is a sensible question therefore to ask for an obstruction to desuspending elements in $\pi^{k+1,k}_{n+k+1}$ with reference to the sequence

$$E: \pi_{n+k}S^k \to \pi^{k+1,k}_{n+k+1} \to \pi_{n+k+1}S^{k+1}. \tag{3.1}$$

So consider M^n embedded in $\mathbf{R}^{n+k}$ and framed in $\mathbf{R}^{n+k+1}$ by orthogonal vectors $v_1, \ldots, v_{k+1}$. Using the notation (2.2) we define a map

$$f: M^n \to S^k \tag{3.2}$$

by the vector of inner products

$$(\langle t, v_1(x)\rangle, \ldots, \langle t, v_{k+1}(x)\rangle) \tag{3.3}$$

normalized to have unit length. We should note of course that the vector (3.3) is never zero because t lies in the space spanned by $v_1, \ldots, v_{k+1}$. The map f is nullhomotopic if the framing on M can be deformed so that v_{k+1} always points along t and then $v_1, \ldots, v_k$ lie in $\mathbf{R}^{n+k}$. To obtain an obstruction to desuspending the bordism class of M we go one step further. Let $N^{n-k} = f^{-1}(p)$ denote the inverse image of a regular value of $p \in S^k$. By a suitable adjustment of the framed manifold M, if necessary, we may assume that

$$p = (0, \ldots, 0, 1) \tag{3.4}$$

so that N is described as the set of points $x \in M$ at which v_{k+1} points in the t-direction. The manifold N is framed in M by vectors $w_1, \ldots, w_k$ which project under the differential of f onto the standard framing vectors at $p \in S^k$. The bordism class of (N, W) in M corresponds to the homotopy class of f under the Thom–Pontrjagin construction[11]. Now we adjoin the framing vectors $v_1, \ldots, v_{k+1}$ of M to obtain a framing of N in $\mathbf{R}^{n+k+1}$. But, by definition, v_{k+1} always point in the direction t on N so, effectively, N is framed in $\mathbf{R}^{n+k}$ by the vectors

$$w_1, \ldots, w_k, \quad v_1, \ldots, v_k. \tag{3.5}$$

This construction is well defined on bordism classes and gives rise to a homomorphism

$$h: \pi^{k+1,k}_{n+k+1} \to \pi^{2k,2k}_{n+k}, \tag{3.6}$$

which we call a Hopf invariant. This terminology is justified by the commutativity of the square

$$\begin{array}{ccc} \pi^{k+1,k}_{n+k+1} & \xrightarrow{h} & \pi_{n+k}S^{2k} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle E} \\ \pi_{n+k+1}S^{k+1} & \xrightarrow[H]{} & \pi_{n+k+1}S^{2k+1}, \end{array} \tag{3.7}$$

where F is the natural map described in (2.3), E is suspension and H is a version of the classical Hopf invariant. A proof of commutativity will be supplied in §6 where we give a homotopy interpretation of $\pi^{k+1,k}_{n+k+1}$.

A geometric description of G. Whitehead's generalization of Hopf's original invariant was first given by Kervaire[10] and a comprehensive treatment of Hopf invariants appears in Boardman–Steer[4]. Our constructions are modifications of Kervaire's ideas as he applied them to spheres and our results can be deduced from [4]. We give an independent proof because the intermediate bordism groups may be of some independent interest. Our framed manifold N is the "self-linking" class of [4].

To summarize this section we state a working definition of the Hopf invariant by paraphrasing the above discussion. Suppose $\alpha \in \pi_{n+k+1}S^{k+1}$ can be represented by a manifold M^n framed in $\mathbf{R}^{n+k+1}$ but embedded in $\mathbf{R}^{n+k}$. Then its Hopf invariant is represented by the submanifold $N \subset M$ on which the last framing vector points along the last coordinate axis of $\mathbf{R}^{n+k+1}$. The framing on N arises from the obstruction $f: M \to S^k$ to desuspending the given framing on M together with this framing restricted to N. Our Hopf invariant applies to the subclass of elements in the penultimate filtration of $\pi_{n+k+1}S^{k+1}$. This includes elements in the metastable range as one can see by a forward reference to Proposition 6.3.

In §5 we compute the Hopf invariants of certain framings on G_2. Frequently a framed embedding of an n-dimensional Lie group G in R^{n+k} arises in a natural way from a representation ρ of G on R^{n+k}. This framed embedding we denote by (G, ρ). If $\alpha: G \to GL(k)$ is a map, frequently another representation, then $(G, \rho)^\alpha$ denotes the framed manifold (G, ρ) twisted by α in the manner described at the beginning of §2. In [14] Steer shows how to associate with a representation ϕ of G a well defined element $[G, \phi]$ in the stable stem $\pi_n{}^S$. This element is represented stably by the twisted framing $(G, \theta \oplus \phi)^{\theta^{-1}}$ where θ is a representation admitting an orbit with trivial isotropy group. The construction is independent of the choice of θ.

§4. THE EXCEPTIONAL LIE GROUP G_2

In this section we gather together some standard facts about G_2 which are needed to define the framings in $\mathbf{R}^{21}$. Information about G_2 can be found, for example, in Borel [5]. We define G_2 as the automorphism group of the Cayley number system. A Cayley number is a pair (a, b) of quaternions, and numbers $u = (a, b)$, $v = (c, d)$ are multiplied according to the formula

$$uv = (ac - \bar{d}b, da + b\bar{c}). \tag{4.1}$$

This defines a real algebra structure on the set $\mathbf{K}$ of Cayley numbers which we identify with $\mathbf{R}^8$. Numbers of the form $(a, 0)$ form a subalgebra of $\mathbf{K}$ isomorphic to the quaternions $\mathbf{H}$ and we have inclusions $\mathbf{R} \subset \mathbf{C} \subset \mathbf{H} \subset \mathbf{K}$. The number $1 \in \mathbf{R}$ acts as the identity of $\mathbf{K}$. The real multiples of 1, which we identify with $\mathbf{R}$, constitute the centre of $\mathbf{K}$. Cayley multiplication satisfies $\|uv\|^2 = \|u\|^2\|v\|^2$ where $\|u\|$ is the usual norm of $u \in \mathbf{R}^8$. We have a conjugation operator

$$\bar{u} = (\bar{a}, -b) \tag{4.2}$$

which extends the usual quaternionic conjugation. The inner product of vectors u, $v \in \mathbf{R}^8$ is given in terms of conjugation by the formula

$$2\langle u, v\rangle = u\bar{v} + v\bar{u}. \tag{4.3}$$

The self-conjugative Cayley numbers are precisely the reals $\mathbf{R}$. The skew-conjugate numbers satisfying $\bar{u} = -u$ form the 7-dimensional orthogonal complement of $\mathbf{R}$, which we identify with $\mathbf{R}^7$ and call its elements pure Cayley numbers. The pure Cayley numbers are characterized by the condition

$$p^2 = -r^2, \tag{4.4}$$

where r is real. Two pure Cayley numbers u, v are orthogonal if and only if

$$uv = -vu \tag{4.5}$$

and in this case uv is again a pure Cayley number.

Let $\{e_1, \ldots, e_7\}$ denote the standard basis of $\mathbf{R}^7$. The multiplication table of basis elements can be drawn up from (4.1). We note in particular the following defining relations

$$e_3 = e_1e_2, \quad e_5 = e_1e_4, \quad e_6 = e_2e_4, \quad e_7 = e_3e_4. \tag{4.6}$$

An automorphism of $\mathbf{K}$ is a non-singular linear transformation $T: \mathbf{R}^8 \to \mathbf{R}^8$ satisfying $T(uv) = T(u)T(v)$ for all u, $v \in \mathbf{K}$. It is easy to verify, using (4.3), (4.4) that T leaves the centre $\mathbf{R}$ pointwise fixed, commutes with conjugation, stabilizes the pure Cayley numbers and is automatically an orthogonal transformation. The automorphism group G_2 may therefore be considered as a subgroup of $O(7)$ acting on the pure Cayley numbers $\mathbf{R}^7$. It is easy to see that G_2 is a closed subgroup of $O(7)$ and is therefore a compact Lie group.

Let $\{f_1, \ldots, f_7\}$ be any orthonormal basis of $\mathbf{R}^7$ satisfying condition (4.6). Then the multiplication table of $\{f_i\}$ is the same as that of $\{e_i\}$ so that the association $T(e_i) = f_i$ defines an automorphism of $\mathbf{K}$ and every automorphism of $\mathbf{K}$ arises this way. In other words an element of $G_2 \subset O(7)$ is determined by its effect on the basic elements e_1, e_2, e_4 which may be transformed into any orthonormal system f_1, f_2, f_4 subject only to the condition that f_4 is orthogonal to f_1f_2. From this it readily follows that G_2 acts transitively on the unit sphere $S^6 \subset \mathbf{R}^7$ and on the Stiefel manifold $V_{7,2}$ of orthonormal 2-frames in 7-space. The stabilizers of e_1 and (e_1, e_2) are respectively $SU(3)$ and $SU(2)$, hence we obtain fibrations

$$SU(3)\to G_2\to S^6, \qquad SU(2)\to G_2\to V_{7,2}.$$

Clearly G_2 is connected so that $G_2\subset SO(7)$. We also see that the dimension of G_2 is 14.

§5. FRAMING G_2 IN $\mathbf{R}^{21}$

Consider three vectors u, v, $w\in\mathbf{R}^7$ satisfying the following system of equations

$$\begin{gathered}\|u\|^2=\|v\|^2=\|w\|^2=1,\\ \langle v,w\rangle=\langle w,u\rangle=\langle u,v\rangle=0,\\ \langle uv,w\rangle=0.\end{gathered}\tag{5.1}$$

The first six equations define the Stiefel manifold of 3-frames in 7-space. In the seventh equation uv is the product of pure Cayley numbers and is again pure Cayley by 4.5. If we write

$$u=ge_1,\qquad v=ge_2,\qquad w=ge_4,\tag{5.2}$$

where $g\in SO(7)$ and e_i are the basic vectors used in the previous section, we see that the equations (5.1) are precisely the conditions for g to lie in G_2 where it is uniquely determined. The equations (5.1) define therefore an embedding of G_2 in $\mathbf{R}^{21}$ as a real algebraic variety. This embedding can also be described as a maximal orbit arising from three copies of the fundamental representation $\rho: G_2\subset SO(7)$. The gradient vectors associated with the equations are listed below in partitioned columns (ignoring the factor 2 in the first three).

$$\begin{matrix} u & 0 & 0 & 0 & w & v & vw\\ 0 & v & 0 & w & 0 & u & wu\\ 0 & 0 & w & v & u & 0 & uv\end{matrix}\tag{5.3}$$

These seven vectors in $\mathbf{R}^{21}$ are linearly independent at each point of G_2 where they form a basis of the normal space. This framing represents the zero element in $\pi_{21}S^7$. To see why this is true we replace the equation $\|u\|^2=1$ by the inequality $\|u\|^2\le 1$ to provide a manifold with boundary, framed by the gradients of the other six equations and giving a bordism to zero of the gradient framing of G_2.

The value of the gradient framing at the identity of G_2 is

$$\begin{matrix} e_1 & 0 & 0 & 0 & e_4 & e_2 & e_6\\ 0 & e_2 & 0 & e_4 & 0 & e_1 & -e_5\\ 0 & 0 & e_4 & e_2 & e_1 & 0 & e_3\end{matrix}\tag{5.4}$$

and we observe that 5.3 is the left translate of 5.4 by the matrix

$$\begin{pmatrix} g & 0 & 0\\ 0 & g & 0\\ 0 & 0 & g\end{pmatrix}$$

where $g\in G_2$.

Following the notation of Steer[14] for stable elements let

$$[G_2,3\rho]^{\rho^n}=[G_2,(3+n)\rho]\tag{5.5}$$

denote the trivial framing $[G_2,3\rho]$ twisted by the nth power of the fundamental representation $\rho: G_2\to SO(7)$. We shall consider the two cases $n=\pm 1$. Explicitly, the vectors of the twisted framings are the columns of the 21×7 matrix

$$\begin{pmatrix} g & 0 & 0\\ 0 & g & 0\\ 0 & 0 & g\end{pmatrix}\begin{pmatrix} e_1 & 0 & 0 & 0 & e_4 & e_2 & e_6\\ 0 & e_2 & 0 & e_4 & 0 & e_1 & -e_5\\ 0 & 0 & e_4 & e_2 & e_1 & 0 & e_3\end{pmatrix}g^n,\tag{5.6}$$

where g is thought of as a 7×7 matrix.

To calculate the Hopf invariants of $(G_2,3\rho)^{\rho^{-1}}$ and $(G_2,3\rho)^{\rho}$ we first observe that equations (5.1) effectively embed G_2 in the sphere S^{20} given by the equation

$$\|u\|^2+\|v\|^2+\|w\|^2=3.$$

Under stereographic projection the radial direction on the sphere corresponds to the direction t in the earlier notation (2.2). We may therefore regard $(G_2, 3\rho)^{\rho^*}$ as an element in the intermediate bordism group $\pi_{21}^{7,6}$ and calculate its Hopf invariant according to the rule laid down in §3. The obstruction map (3.3) is formed in this case by taking the inner products of the framing vectors (5.6) and the radial direction on S^{20}, which works out to be

$$(1, 1, 1, 0, 0, 0)g^n.$$

For $n = -1$ this obstruction map is homotopic to the projection map of the fibration

$$SU(3) \to G_2 \xrightarrow{f_1} S^6, \tag{5.7}$$

where f_1 assigns to a 7×7 matrix its first column. For $n = 1$ the obstruction map is homotopic to the projection map of the fibration

$$SU(3) \to G_2 \xrightarrow{f_2} S^6, \tag{5.8}$$

where f_2 assigns to a 7×7 matrix its first row. Although the fibrations (5.7) and (5.8) are equivalent under matrix transposition in $SO(7)$, the maps f_1 and f_2 are not homotopic and this makes a difference in the computation of Hopf invariants. In both cases the obstruction manifold N of §2 is $SU(3)$ because we may take any point, in particular $e_1 \in S^6$, as the regular value of the projection map and $SU(3)$ is the stabilizer of e_1.

To continue the computation of the Hopf invariant, according to §3, we must now find vectors $w_1, \ldots, w_6$ which frame $SU(3)$ in G_2 and project under the differential of f_i onto the standard framing vectors $e_2, \ldots, e_7$ at the point e_1 in S^6. For this purpose it is helpful if we first consider the fibration

$$SO(n) \to SO(n+1) \xrightarrow{f} S^n, \tag{5.9}$$

where f is either projection onto the first row or projection onto the first column and $SO(n)$ is embedded in $SO(n+1)$ as the set of matrices with 1 in the top left corner, i.e. $SO(n)$ is the stabilizer of the standard basic vector e_1 in R^{n+1}. We may identify the tangent space at the identity of $SO(n+1)$ as the set of $(n+1) \times (n+1)$ skew-symmetric matrices. If $K_1, \ldots, K_n$ are skew-symmetric matrices forming a basis of the normal space of $SO(n)$ in $SO(n+1)$ then the matrices $AK_1, \ldots, AK_n$ form such a basis at the point $A \in SO(n)$. Clearly, the first rows of these matrices are independent of $A \in SO(n)$ and can be chosen as $e_2, \ldots, e_{n+1}$. This explains how to frame $SO(n)$ in $SO(n+1)$ if f is projection onto the first row. If we project the normal vectors onto their first columns we get the negatives of $Ae_2, \ldots, Ae_{n+1}$. Apart from the sign problem this indicates that we must twist the framing $AK_1, \ldots, AK_n$ by A^{-1} in the normal space in order to get the correct framing of $SO(n)$ in $SO(n+1)$ when f is projection onto the first column.

What we have said about the fibration (5.9) applies equally well to a subfibration, in particular to (5.7) and (5.8) in the case $n = 6$. The correct framing of $SU(3)$ in G_2 for case (5.8) is simply the left translate of a normal basis at the identity by $g \in SU(3)$, whereas in (5.7) this framing must be twisted by g^{-1}. Our model for $SU(3)$ in $\mathbf{R}^{21}$ is given by the equations (5.1) together with the seven equations defined by $u = e_1$. This characterizes $SU(3) \subset G_2$ as the stabilizer of e_1. By evaluating gradients we find that the normal space at the identity of $SU(3)$ in $\mathbf{R}^{21}$ is spanned by the thirteen vectors

$$\begin{matrix} e_i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e_2 & 0 & e_4 & 0 & e_1 & -e_5 \\ 0 & 0 & e_4 & e_2 & e_1 & 0 & e_3 \end{matrix}, \tag{5.10}$$

where $1 \leq i \leq 7$. The following six vectors lie in this normal space and are tangential to G_2

$$\begin{matrix} e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ -e_1 & 0 & 0 & 0 & \frac{1}{2}e_5 & 0 \\ 0 & 0 & -e_1 & 0 & -\frac{1}{2}e_3 & 0 \end{matrix}. \tag{5.11}$$

To verify this we note that each vector of (5.11) is orthogonal to the vectors of (5.4), which form a basis of the normal space of G_2 at the identity. From the above discussion we are now ready to write down explicitly the framing vectors $w_1, \dots, w_6$ of $SU(3)$ in G_2. In case (5.8) they are given by the columns of the matrix

$$\begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g \end{pmatrix} \begin{pmatrix} e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ -e_1 & 0 & 0 & 0 & \frac{1}{2}e_5 & 0 \\ 0 & 0 & -e_1 & 0 & -\frac{1}{2}e_3 & 0 \end{pmatrix} \tag{5.12}$$

and in case (5.7) they are given by the columns of the matrix

$$\begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g \end{pmatrix} \begin{pmatrix} e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ -e_1 & 0 & 0 & 0 & \frac{1}{2}e_5 & 0 \\ 0 & 0 & -e_1 & 0 & -\frac{1}{2}e_3 & 0 \end{pmatrix} g^{-1}, \tag{5.13}$$

where $g \in SU(3)$ is regarded as a matrix in $SO(6)$ for the purpose of twisting on the right.

To complete our computation of the Hopf invariant we must juxtapose the framings $w_1, \dots, w_6$ and $v_1, \dots, v_7$ according to the rule in §3. Applying this to (5.12) and (5.6) when $n = 1$ we obtain the following framing of $SU(3)$ in $\mathbf{R}^{21}$:

$$\begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g \end{pmatrix} \begin{pmatrix} e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_1 & 0 & 0 & 0 & e_4 & e_2 & e_6 \\ -e_1 & 0 & 0 & 0 & \frac{1}{2}e_5 & 0 & 0 & e_2 & 0 & e_4 & 0 & e_1 & -e_5 \\ 0 & 0 & -e_1 & 0 & -\frac{1}{2}e_3 & 0 & 0 & 0 & e_4 & e_2 & e_1 & 0 & e_3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}$$

With the aid of a few column operations this framing can be deformed into

$$\begin{pmatrix} g & 0 & 0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_2 & 0 & e_4 & 0 & e_1 & -e_5 \\ 0 & 0 & g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_4 & e_2 & e_1 & 0 & e_3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} \tag{5.15}$$

'We should recall from §2 that a deformation of a twisting does not alter the bordism class of the framed manifold. Now the middle matrix has the 7×7 identity matrix in its top left position. We may therefore replace the first matrix of 5.15 by

$$\begin{matrix} 1 & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g \end{matrix}$$

and the third by $\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$ which can be deformed into $\begin{pmatrix} 1 & 0 \\ 0 & g^2 \end{pmatrix}$. Consequently 5.15 is the 7-fold suspension of the following framing of $SU(3)$ in $\mathbf{R}^{14}$

$$\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} e_2 & 0 & e_4 & 0 & e_1 & -e_5 \\ 0 & e_4 & e_2 & e_1 & 0 & e_3 \end{pmatrix} g^2. \tag{5.16}$$

Actually the equations (5.1) with $u = e_1$ define an embedding of $SU(3)$ in $\mathbf{R}^{12}$ which is just a maximal orbit of two copies of the fundamental representation λ of $SU(3)$ on $\mathbf{C}^3$. This again is given by equations whose gradient vectors at the identity may be taken as

$$\begin{matrix} e_2 & 0 & e_4 & -e_5 \\ 0 & e_4 & e_2 & e_3 \end{matrix}.$$

The gradient framing is trivial for the same reason as we gave for G_2. The framing (5.16) is the double suspension of the gradient framing of $SU(3)$ twisted by λ^2. Again following the notation of Steer[14], let

$$[SU(3), 2\lambda]^{\lambda^n} = [SU(3), (2+n)\lambda] \tag{5.17}$$

stand for the trivial framing of $SU(3)$ in $\mathbf{R}^{14}$ twisted by the nth power of the fundamental representation $\lambda: SU(3) \to SO(6)$. We have shown that the Hopf invariant of $(G_2, 3\rho)^\rho$ is the 7-fold suspension of $(SU(3, 2\lambda)^{\lambda^2}$. Carrying out an analogous procedure in case (5.7) on the vectors (5.13) juxtaposed with the original framing vectors (5.6) when $n = -1$, we arrive at the 7-fold suspension of the following framing on $SU(3)$ in $\mathbf{R}^{14}$

$$\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} e_2 & 0 & e_4 & 0 & e_1 & -e_5 \\ 0 & e_4 & e_2 & e_1 & 0 & e_3 \end{pmatrix} g^{-1} \tag{5.18}$$

and this is the framing $[SU(3), \lambda]$. To summarize the work so far, we have the following result.

5.19. Proposition. *Hopf invariants of twisted framings on G_2 are given in terms of twisted framings on $SU(3)$ by*

$$H(G_2, 3\rho)^{\rho} = E^7(SU(3), 2\lambda)^{\lambda^2}, \qquad H(G_2, 3\rho)^{\rho^{-1}} = E^7(SU(3), 2\lambda)^{\lambda^{-1}},$$

where E denotes suspension, $(G_2, 3\rho)^{\rho^n}$ is an element of $\pi_{21}S^7$ and $(SU(3), 2\lambda)^{\lambda^n}$ is an element of $\pi_{14}S^6$.

We could proceed now to calculate the Hopf invariants of twisted framings on $SU(3)$ by a process similar to the one employed above for G_2, but based this time on the fibration $SU(2) \to SU(3) \to S^5$. In this way one obtains, for example,

$$H(SU(3), 2\lambda)^{\lambda^{-1}} = E^7(SU(2), \mu)^{\mu^{-1}},$$

where $(SU(2), \mu)^{\mu^{-1}}$ is the element in $\pi_7 S^4$ obtained by twisting the trivial framing on $SU(2) = S^3$ by the inverse of the fundamental representation $\mu: SU(2) \subset SO(4)$. This element is well known to be ν, in Toda's notation, represented by the Hopf map $S^7 \to S^4$.

For further information about Hopf invariants of twisted framings on G_2 and $SU(3)$ we refer to [14].

Consulting Toda's tables in [15] we learn that any element in $\pi_{14}S^6$ whose Hopf invariant is stably equal to ν is itself stably non-trivial. This implies therefore that $(G_2, 3\rho)^{\rho^{-1}}$ is a nontrivial element of $\pi_{21}S^7$. Hopf invariant calculations alone in $\pi_{14}S^6$ are not sufficient to identify the elements $[SU(3), n\lambda]$. For example, $\bar{\nu}$ and $\bar{\nu} + \epsilon$ both have Hopf invariant ν. In §8 we shall show that $[SU(3), 4\lambda] = \bar{\nu}$, $[SU(3), \lambda] = \bar{\nu} + \epsilon$. It then follows from Toda's tables that stably $[G_2, 4\rho] = \kappa$, $[G_2, 2\rho] = 0$.

§6. THE INTERMEDIATE BORDISM GROUPS

In this section we first give a homotopy interpretation of the group $\pi^{k+1,k}_{n+k+1}$ introduced in §2. Let τ denote the tangent vector bundle of the sphere S^k and $M(\tau)$ its Thom complex.

6.1. Proposition. *There is an isomorphism $r: \pi^{k+1,k}_{n+k+1} \to \pi_{n+k}M(\tau)$.*

Proof. To describe the map r, consider a manifold M^n embedded in $\mathbf{R}^{n+k}$. Let t be the unit vector along the last axis of $\mathbf{R}^{n+k+1}$. As we move round the manifold M, the vector t, relative to the moving frame of reference, appears as a point of S^k and the hyperplane $\mathbf{R}^{n+k}$, in which M is embedded, appears to move normal to t. In this way we can assign to each point in a suitable tubular neighbourhood of M in $\mathbf{R}^{n+k}$ a tangent vector to S^k and by compactification we obtain a map $f: S^{n+k} \to M(\tau)$ whose homotopy class depends only on the bordism class of M. This describes r. An inverse homomorphism

$$S: \pi_{n+k}M(\tau) \to \pi^{k+1,k}_{n+k+1}$$

is constructed in the usual way by starting with a map $f: S^{n+k} \to M(\tau)$, making it transversal to the zero-section of $S^k \subset M(\tau)$, and then taking $M^n = f^{-1}(S^k)$ embedded in S^{n+k} or equivalently $\mathbf{R}^{n+k}$. We frame M in $\mathbf{R}^{n+k+1}$ by pulling back the cononical trivialisation of the Whitney sum $\tau \oplus \epsilon$ of the tangent bundle τ and the trivial line bundle ϵ of S^k in $\mathbf{R}^{k+1}$. This canonical trivialisation assigns to a point $x \in S^k$ the standard basic vectors $e_1, \ldots, e_{k+1}$ of $\mathbf{R}^{k+1}$. The proof that r, s are mutual inverses goes through as for the Thom–Pontrjagin construction. For future reference we note that the subset of points $x \in M = f^{-1}(S^k)$ at which the last framing vector v_{k+1} points in the direction t is $N = f^{-1}(e_{k+1})$, where e_{k+1} is here regarded as a point of S^k.

6.2. Proposition. *Under the identification of $\pi^{k+1,k}_{n+k+1}$ with $\pi_{n+k}M(\tau)$ and $\pi^{2k,2k}_{n+k}$ with $\pi_{n+k}S^{2k}$ the Hopf invariant $h: \pi^{k+1,k}_{n+k+1} \to \pi^{2k,2k}_{n+k}$ is induced by the map $p: M(\tau) \to S^{2k}$ which projects the 2-cell complex $M(\tau)$ onto its top cell.*

Proof. We recall that $M(\tau)$ is the 1-point compactification of τ and can be given the structure of a 2-cell complex $S^k \cup e^{2k}$, where the k-skeleton is the compactification of the fibre of τ over the vertex of the base sphere, and the $2k$-cell comes from the product of the fibre and the k-cell of the base. The projection map p collapses the k-skeleton to the point at infinity in S^{2k}.

Let $f: S^{n+k} \to M(\tau)$ be a map which we assume transversal to the zero section of $M(\tau)$. Let

$M = f^{-1}(S^k)$ be the inverse image of the zero-section of τ, framed in $\mathbf{R}^{n+k+1}$ by vectors $v_1, \ldots, v_{k+1}$ which are the pull-backs of the canonical vectors $e_1, \ldots, e_{k+1}$ as explained in Proposition (6.1). We may assume that the point $e_{k+1} \in S^k$ is away from the fibre collapsed by p to the point at infinity in S^{2k}. Let $z = p(e_{k+1}) \in S^{2k}$. Then z is a regular value of the composite map pf. According to the Thom–Pontrjagin construction the proof of Proposition (6.2) is complete if we can show that the manifold

$$N = (pf)^{-1}(z) = f^{-1}(e_{k+1}),$$

framed in S^{n+k} by the pull-back of the standard framing of $z \in S^{2k}$, agrees with the description of the Hopf invariant of §3. Certainly the description of N is correct as noted at the end of the proof of Proposition (6.1). We can frame the point e_{k+1} by the vectors $f_1, \ldots, f_k$ which are tangential to the zero-section $S^k \subset \tau$ together with the canonical vectors $e_1, \ldots, e_k$ along the fibre at e_{k+1}. The images of these vectors under p may be taken as the standard framing of z in S^{2k}. Then N is framed by vectors $w_1, \ldots, w_k, v_1, \ldots, v_k$ where $w_1, \ldots, w_k$ are tangential to M and project onto the framing vectors $f_1, \ldots, f_k$ of e_{k+1} in S^k under the map $f\colon M \to S^k$, and $v_1, \ldots, v_k$ are the restriction to N of the original framing of M in $\mathbf{R}^{n+k+1}$ which, by definition, project onto $e_1, \ldots, e_k$. This description of the framed manifold N is now in accordance with §3 and the proof of Proposition (6.2) is complete.

6.3. Proposition. *Under the identification of $\pi^{k+1,k}_{n+k+1}$ with $\pi_{n+k}M(\tau)$ and $\pi^{k+1,k+1}_{n+k+1}$ with $\pi_{n+k+1}S^{k+1}$, the natural map $F\colon \pi^{k+1,k}_{n+k+1} \to \pi^{k+1,k+1}_{n+k+1}$ is induced by a map $q\colon M(\tau) \to \Omega S^{k+1}$ which is an equivalence of $2k$-skeleta.*

Proof. We recall that the loop space ΩS^{k+1} has the homotopy type of a cell complex of the form $S^k \cup e^{2k} \cup \cdots \cup e^{mk} \cup \cdots$ and the attaching map of the $2k$-skeleton $S^k \cup e^{2k}$ is the Whitehead product $[\iota, \iota]$. This is also the attaching map of the Thom complex $M(\tau)$. We describe q in terms of its adjoint $q'\colon S^1 \wedge M(\tau) \to S^{k+1}$. Now $S^1 \wedge M(\tau)$ can be identified with the Thom complex $M(\tau \oplus \epsilon)$ and we define q' to be the compactification of the projection $\tau \oplus \epsilon \to R^{k+1}$ arising from the canonical trivialisation of $\tau \oplus \epsilon$. We note that q' collapses the zero-section of $\tau \oplus \epsilon$ to a point and maps a fibre in a degree one fashion. It follows that q has degree one on bottom cells. If k is even, a cohomology ring argument shows that q is an equivalence of $2k$-skeleta. This argument does not work for k odd, but we can use the Pontrjagin ring in all cases.

To verify that q induces the natural map F we start with a map $f\colon S^{n+k} \to M(\tau)$ corresponding to the manifold $M = f^{-1}(S^k)$ framed in S^{n+k+1} by the method described in the proof of Proposition (6.1). The adjoint of the composite map qf is $q'Sf\colon S^{n+k+1} \to S^1 \wedge M(\tau) \to S^{k+1}$, where Sf denotes the suspension of f. Now q' collapses the zero-section of $\tau \oplus \epsilon$ to a point $z \in S^{k+1}$. It follows that $M = (q'Sf)^{-1}z$. The framing on M obtained by pulling back the standard framing of z in S^{2k} agrees with the original framing on M. This completes the proof of Proposition (6.3).

We are now in a position to show that our Hopf invariant h agrees with the classical Hopf invariant as defined, for example, by James [9]. With the aid of Propositions (6.1), (6.2), (6.3) the proof that diagram (3.7) commutes is reduced to the homotopy commutativity of the following square

$$\begin{array}{ccc} M(\tau) & \xrightarrow{p} & S^{2k} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle j} \\ \Omega S^{k+1} & \xrightarrow[H]{} & \Omega S^{2k+1}, \end{array} \tag{6.4}$$

where H is the classical Hopf invariant of James, p and q are the maps defined in Propositions (6.2), (6.3) and j is the inclusion of the bottom cell. Of course, j induces the suspension map E in diagram 3.7.

The commutativity of (6.4) follows quickly from James' definition of H in terms of the reduced product construction [9].

§7. REMARKS

By analogy with Proposition (6.1) and its proof one can find an interpretation of all the intermediate bordism groups $\pi^{k,r}_{n+k}$ in terms of Thom complexes of certain vector bundles over

Stiefel manifolds. The vector bundles in question are those associated with the standard fibrations $0(r)\to 0(k)\to V_{k,k-r}$. Again, by analogy with the arguments of §3, one can define intermediate Hopf invariants $\pi_{n+k}^{k,k-r}\to\pi_{n+k-r}S^{k-r+s}$, where S is the dimension of the Stiefel manifold $V_{k,r}$. We start with a manifold M^n embedded in $\mathbf{R}^{n+k-r}$ and framed by vectors $v_1,\ldots,v_k$ in $\mathbf{R}^{n+k}$. Let $t_1,\ldots,t_r$ be the unit vectors along the last r coordinate axes of $\mathbf{R}^{n+k}$. The matrix of inner products $(\langle t_i, v_j\rangle)$ can be regarded as an obstruction map $M^n\to V_{k,r}$ to r-fold desuspension of the framed manifold M. Lifting back a regular value as in §3 we obtain a framed manifold N^{n-s} in $\mathbf{R}^{n+k-r}$ which represents the intermediate Hopf invariant. For example, we located the twisted framing $(G_2, 3\rho)^{\rho^{-1}}$ in $\pi_{21}^{7,6}$ in order to calculate its Hopf invariant. Actually $(G_2, 3\rho)^{\rho^{-1}}$ pulls back to $\pi_{21}^{7,5}$ because the equations (5.1) embed G_2 in $S^6\times S^6\times S^6$, which itself embeds in $\mathbf{R}^{19}$. We may therefore calculate its intermediate Hopf invariant $\pi_{21}^{7,5}\to\pi_{19}S^{16}$ which turns out in this case to be associated with the fibration $S^3\to G_2\to V_{7,2}$ and is in fact $\nu\in\pi_3^{S}$. This identifies a certain element in $\pi_{21}^{7,5}$ whose order is at least 24.

Unfortunately these intermediate Hopf invariants do not in general extend to the whole of the homotopy groups of spheres. Of course, the first of these intermediate invariants does always extend and is the classical Hopf invariant, as we have demonstrated. There is an explanation for this behaviour but we shall not pursue the matter here. It is, however, an interesting problem to find the filtrations of well known elements in the homotopy groups of spheres and work out their intermediate Hopf invariants. By using the classical higher Hopf invariants (not to be confused with our intermediate invariants), P. J. Eccles has been able to locate framed manifolds which, unstably, have top filtration. But it remains a problem to decide whether the filtration introduced in §2 has any significance for the stable homotopy groups of spheres. For example, which elements in the stable homotopy groups of spheres have filtration one? In other words, which elements can be represented by framed hypersurfaces? If we think of elements in the image of J as having filtration 0 then our filtration measures to what extent an element fails to be in the image of J. What is the filtration of $\bar\nu$, for example? We shall show in §8 that $\bar\nu$ can be represented by a framing on $SU(3)$ and $SU(3)$ can be embedded in $\mathbf{R}^{11}$, but it is conceivable that the filtration of $\bar\nu$ is less than 3. We know that $\bar\nu$ is not in the image of J so its filtration is certainly positive.

§8. THE e-INVARIANT OF FRAMINGS ON $SU(3)$

Let $\lambda: SU(3)\to SU(3)$ denote the identity map regarded as the fundamental representation of $SU(3)$ on $\mathbf{C}^3$. We embed $SU(3)$ in $\mathbf{C}^6$ as the set of complex 3×2 matrices

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}$$

satisfying the equations

$$\sum a_i\bar a_i = 1,\qquad \sum b_i\bar b_i = 1,\qquad \sum a_i\bar b_i = 0. \tag{8.1}$$

This locus is just a maximal orbit of the representation 2λ. The gradients of (8.1), regarded as four real polynomials, provide a normal framing of the 8-dimensional manifold $SU(3)$ in $\mathbf{R}^{12}$. As an element in $\pi_{12}S^4$ this gradient framing $(SU(3), 2\lambda)$ is zero. In $\mathbf{R}^{14}$ we can twist the double suspension of the trivial framing by powers λ^n of the fundamental representation λ, regarded now as a map $SU(3)\to SO(6)$. In terms of Toda's generators $\bar\nu$, ϵ, for the stable 8-stem $\pi_8^{S}\cong \mathbf{Z}_2\oplus\mathbf{Z}_2$ we can identify the twisted framings $[SU(3), 2\lambda]^{\lambda^n} = [SU(3), (2+n)\lambda]$ as follows.

8.2. THEOREM. $[SU(3), k\lambda] = 0$, $\bar\nu$, ϵ, $\bar\nu+\epsilon$ *according as* $k = 2, 0, 3, 1 \bmod 4$.

The proof of this theorem, which occupies the rest of this section, is based on a computation of the stable cohomotopy ring of $SU(3)$ and a computation of e-invariants. Our result provides a useful cross-check on some of the delicate calculations in [14] where Theorem (8.2) was first put forward. We refer also to [13].

Let $\bar\pi^0(X)$ denote the reduced stable cohomotopy group in dimension 0 of the finite cell complex X. Since cohomotopy theory is a generalized cohomology theory with products, $\bar\pi^0(X)$ is a commutative ring. Of course $\bar\pi^0(S^n) = \pi_n^{S}$, the stable n-stem.

8.3. PROPOSITION. *As a group*

$$\tilde{\pi}^0(SU(3)) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_{24},$$

where the cyclic summands are generated respectively by a, b, c, defined as follows: under the map

$$\tilde{\pi}^0(SU(3)) \leftarrow \tilde{\pi}^0(S^8),$$

induced by projection of SU(3) onto the top cell, the elements a, b are the images of $\bar{\nu}$, ϵ. Under the map

$$\tilde{\pi}^0(S^1 \wedge CP^2) \leftarrow \tilde{\pi}^0(SU(3))$$

induced by the inclusion of the suspension of CP^2 in SU(3), the element c maps onto a generator of

$$\tilde{\pi}^0(S^1 \wedge CP^2) \cong \mathbf{Z}_{24}.$$

The ring structure of $\tilde{\pi}^0(SU(3))$ is given by $c^2 = a$. All other products vanish.

Proof. The additive structure of $\tilde{\pi}^0(SU(3))$ follows immediately from the cofibrations

$$S^1 \wedge CP^2 \to SU(3) \to S^8, \qquad S^3 \to S^1 \wedge CP^2 \to S^5$$

and the fact that the top cell of $SU(3)$ splits off stably because $SU(3)$ is parallelisable. We are using the well known fact that $SU(3)$ has a cell structure of the form $S^3 \cup_\eta e^5 \cup e^8$, where the attaching map of the 5-cell is the suspension of the Hopf map. This can be checked by examining the sphere bundle $S^3 \to SU(3) \to S^5$. We have also used the following values of stable stems

$$\pi_3^S \cong \mathbf{Z}_{24}, \qquad \pi_4^S = 0, \qquad \pi_5^S = 0, \qquad \pi_8^S \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2.$$

It should be noted that c is not uniquely defined, but can be altered by any combination of a and b. However, c^2 is unique because a, b come from a suspension so that $a^2 = b^2 = 0$ and they have order 2. The ring structure on $\tilde{\pi}^0(X)$ is induced from the smash product of stable maps $X \to S^0$ and the diagonal map $\Delta: X \to X \wedge X$. It follows quickly, for dimensional and connectivity reasons, that $ca = cb = 0$. The only non-trivial computation to be made is the value of c^2. I am indebted to Grant Walker for showing me how to do this in terms of Toda brackets. We wish to investigate the stable composite map

$$SU(3) \xrightarrow{\Delta} SU(3) \wedge SU(3) \xrightarrow{c \wedge c} S^0 \wedge S^0.$$

Since $SU(3)$ is 8-dimensional and $S' \wedge CP^2$ is 5-dimensional we have the following factorization through the 10-skeleton of $SU(3) \wedge SU(3)$

$$SU(3) \xrightarrow{p} S^8 \xrightarrow{s} S^1 \wedge CP^2 \wedge S^1 \wedge CP^2 \xrightarrow{h \wedge h} S^0 \wedge S^0, \tag{8.4}$$

where p denotes projection onto the top cell, $sp = \Delta$, and h is the restriction of c to $S^1 \wedge CP^2$. Our problem is reduced to identifying $(h \wedge h)s$. Now the restriction of c to $S^3 \subset S^1 \wedge CP^2$ is the generator ν of the 3-stem. It follows that the restriction h of c to $S^1 \wedge CP^2$ may be regarded as an extension of ν corresponding to a null homotopy of $\nu\eta$ where η generates the 1-stem. Now we consider the factorization

$$S^8 \xrightarrow{s} S^1 \wedge CP^2 \wedge S^1 \wedge CP^2 \xrightarrow{1 \wedge h} S^1 \wedge CP^2 \wedge S^0 \xrightarrow{h \wedge 1} S^0 \wedge S^0.$$

We shall show that the composite $(1 \wedge h)s$ may be regarded as a co-extension of ν, corresponding to a null homotopy of $\eta\nu$. To verify this, consider the diagram

$$\begin{array}{ccc} S^8 \xrightarrow{s} S^1 \wedge CP^2 \wedge S^1 \wedge CP^2 & \xrightarrow{1 \wedge h} & S^1 \wedge CP^2 \wedge S^0 \\ \downarrow{\scriptstyle q \wedge 1} & & \downarrow{\scriptstyle q \wedge 1} \\ S^5 \wedge S^1 \wedge CP^2 & \xrightarrow[1 \wedge h]{} & S^5 \wedge S^0, \end{array}$$

where q is projection of $S^1 \wedge CP^2$ onto the top cell. Now the map s has degree one on the 8-skeleton because it arose from the diagonal map Δ of $SU(3)$ into $SU(3) \wedge SU(3)$, and in the cohomology ring of $SU(3)$ the product of the 3-dimensional and 5-dimensional generators is the

8-dimensional generator. It follows that the composite $(q \wedge 1)s$ also has degree one on the 8-skeleton and we deduce

$$(q \wedge 1)(1 \wedge h)s = (1 \wedge h)(q \wedge 1)s = \nu.$$

This demonstrates that $(1 \wedge h)s$ is a co-extension of ν, and putting the extension with the co-extension we obtain, by definition of Toda brackets,

$$(h \wedge h)s = (h \wedge 1)(1 \wedge h)s = \langle \eta, \nu, \eta \rangle.$$

Now $\langle \eta, \nu, \eta \rangle$ is Toda's definition of $\bar{\nu}$. Referring to (8.4) it follows that c^2 in $\tilde{\pi}^0(SU(3))$ is the projection of $\bar{\nu}$ in $\tilde{\pi}^0(S^8)$ and this concludes the proof that $c^2 = a$.

To proceed further with the proof of Theorem (8.2) we collect together some standard facts which relate various constructions like the Hopf construction, the J-map and framed embeddings. We refer to [12] for a full account of the ideas involved here. Consider a map $\alpha : X \to SO(N)$, where N is sufficiently large compared to the dimension of the finite cell complex X. The Hopf construction assigns to α the composite

$$X \xrightarrow{\alpha} SO(N) \xrightarrow{u} \Omega_1{}^N S^N \xrightarrow{v} \Omega_0{}^N S^N, \tag{8.5}$$

where u is the inclusion of the orthogonal transformations of $\mathbf{R}^N$ into the space of all base point preserving maps of S^N with degree 1. The map v translates $\Omega_1{}^N S^N$ onto $\Omega_0{}^N S^N$, the maps of S^N with degree 0, by subtracting a fixed map of degree one with respect to the H-space structure of $\Omega^N S^N$ given by loop addition. Regarding α as an element of $\tilde{K}_R{}^{-1}(X)$ and taking the adjoint of $v\,u\,\alpha$ in (8.5), we obtain a natural map

$$J : \tilde{K}_R{}^{-1}(X) \to \tilde{\pi}^0(X). \tag{8.6}$$

The J-map is not a homomorphism in general but satisfies a quadratic formula

$$J(\alpha + \beta) = J(\alpha) + J(\beta) + J(\alpha)J(\beta), \tag{8.7}$$

which is easily checked from the definition of J and the fact that loop composition in $\Omega^N S^N$ induces the cohomotopy ring structure. If X is a suspension, then products vanish and J is a homomorphism. In particular, if X is a sphere, we have the classical J homomorphism.

Now consider a framed embedding f of a manifold M^n in $\mathbf{R}^{n+k}$ together with a twisting t of the framing

$$M \times \mathbf{R}^k \xrightarrow{t} M \times \mathbf{R}^k \xrightarrow{f} \mathbf{R}^{n+k}. \tag{8.8}$$

The twisting t is given by a formula

$$t(x, v) = (x, \alpha(x)v), \qquad x \in M, \qquad v \in R^k, \tag{8.9}$$

where $\alpha : X \to SO(k)$. The composite ft is again a framed embedding. Associated with f we have the collapsing map of one-point compactifications

$$F : (\mathbf{R}^{n+k})^* \to (M \times \mathbf{R}^k)^*. \tag{8.10}$$

Let P denote the compactification of the projection $p : M \times \mathbf{R}^k \to \mathbf{R}^k$. The composite map

$$PF : (\mathbf{R}^{n+k})^* \to (\mathbf{R}^k)^* \tag{8.11}$$

is precisely the Thom–Pontrjagin construction on the framed manifold (M, f). Now let T denote the compactification of the twisting map t. Then $PT^{-1}F$ is the Thom–Pontrjagin construction on the framed manifold (M, ft). Consider the difference element

$$u = PT^{-1}F - PF : S^{n+k} \to S^k. \tag{8.12}$$

We explain in terms of the following diagram how the difference element u can be factored through the Hopf construction.

$$\begin{array}{ccccc} (\mathbf{R}^{n+k})^* & \xrightarrow{F} & (M \times \mathbf{R}^k)^* & \xrightarrow{G} & S^k \wedge M \\ & & & \searrow^{PT^{-1}-P} & \downarrow H \\ & & & & (\mathbf{R}^k)^*. \end{array} \tag{8.13}$$

The map G collapses an axis $(\mathbf{R}^k)^*$ of $(M \times \mathbf{R}^k)^*$ to a point and we identify the quotient space

$(M\times \mathbf{R}^k)^*/(\mathbf{R}^k)^*$ with the k-fold suspension of M. The maps PT^{-1} and P agree up to homotopy on the axis $(\mathbf{R}^k)^*$, both being maps of degree one. Hence their difference $PT^{-1}-P$ factors through $S^k\wedge M$. The factor map H is just another way of describing the Hopf construction (8.5) applied to the inverse of the map $\alpha: M\to SO(k)$ associated with the twisting t as in (8.9). The composite map $GF=\Phi$ is an element of $\tilde{\pi}_n^S(M^n)$, and may be regarded as the reduced fundamental class of the framed manifold (M,f) with respect to stable homotopy theory. Let us summarise the result as follows.

8.14. PROPOSITION. *Let (M^n,f) be a framed n-manifold representing the element $[M^n,f]$ in the n-stem. Let t be a twisting of the framing associated with an element $\alpha\in\tilde{K}_R^{-1}(M)$. Then the difference element is given by the formula*

$$[M,ft]-[M,f]=H\Phi,$$

where H is the Hopf construction applied to the inverse of α, so $H=J(-\alpha)$, and Φ is the reduced fundamental class of (M,f) in stable homotopy theory.

In particular, if (M,f) is a boundary framing then $[M,ft]=H\Phi$ provides a factorization of the twisted framing through the cohomotopy group $\tilde{\pi}^0(M)$. We should also note at this point that the composite map

$$S^{n+k}\xrightarrow{\Phi}S^k\wedge M\xrightarrow{Q}S^{n+k} \tag{8.15}$$

has degree one, where Q is induced by projection of M onto the top cell. It follows that in stable cohomotopy Q^* embeds π_n^S as a direct summand in $\tilde{\pi}^0(M)$ with Φ^* as a left inverse.

Let us now apply the above work to the case $M=SU(3)$ and the trivial framing $(SU(3),2\lambda)$. The problem of identifying the twisted framings $[SU(3),k\lambda]$ can be analysed in terms of the following sequence

$$\mathbf{Z}\xrightarrow{q}\tilde{K}_C^{-1}(SU(3))\xrightarrow{r}\tilde{K}_R^{-1}(SU(3))\xrightarrow{J}\tilde{\pi}^0(SU(3))\xrightarrow{\Phi}\pi_8^S. \tag{8.16}$$

The map q assigns to the integer n the nth power λ^{-n} of the inverse of λ thought of now as a map of $SU(3)$ into the infinite unitary group. This is just $-n\lambda$ as an element of $\tilde{K}^{-1}(SU(3))$. The map r is realification from complex K-theory, and the maps J,Φ are explained above. In this case Φ is the reduced fundamental class associated with the framed embedding $(SU(3),2\lambda)$. Let

$$\Psi=Jrq \tag{8.17}$$

denote the composite map, so that $\Phi\Psi$ evaluated on an integer n tells us which element of the 8-stem we get by twisting the trivial framing $[SU(3),2\lambda]$ by λ^{-n}.

In Proposition (8.3) let us now choose the generator c to satisfy $\Phi(c)=0$. This is possible because Φ is an inverse for projection onto the top cell as noted in (8.15). Only the sign of c now remains ambiguous. In terms of the generators a, b, c of $\tilde{\pi}^0(SU(3))$ now chosen, we can state the crucial result needed in the proof of Theorem (8.2).

8.18. PROPOSITION. $\Psi(1)=a+b+c$.

The proof of Theorem 8.2 follows quickly from this proposition and (8.7). As a consequence of this proposition we have for example

$$\begin{aligned}\Psi(2)&=\Psi(1)+\Psi(1)+\Psi(1)^2 \text{ by } 8.7\\ &=2(a+b+c)+(a+b+c)^2 \quad \text{by}\\ &=2c+a \quad \text{by } 8.3\end{aligned} \tag{8.18}$$

Hence $\Phi\Psi(2)=\Phi(a)=\bar{\nu}$ and so

$$[SU(3),0]=\bar{\nu}. \tag{8.19}$$

As another example consider $\Psi(-1)$. Although the J-map is not a homomorphism we have $J(0)=0$. Hence

$$0=\Psi(-1+1)=\Psi(-1)+\Psi(1)+\Psi(-1)\Psi(1).$$

Hence $\Psi(-1)=b-c$, hence $\Phi\Psi(-1)=\epsilon$, and so $[SU(3),3\lambda]=\epsilon$. In general $\Psi(n)=n\Psi(1)-(1/2)n(n-1)\Psi(1)^2$, which gives the result stated in (8.2).

The proof of Proposition 8.18 is in two stages. Let

$$\Psi(1) = \lambda a + \mu b + \omega c, \tag{8.20}$$

where $\lambda, \mu \in \mathbf{Z}_2$ and $\omega \in \mathbf{Z}_{24}$. In the first stage we determine the coefficient ω by considering the following diagram.

$$\begin{array}{ccccc}
\tilde{K}_C^{-1}(SU(3)) & \xrightarrow{r} & \tilde{K}_R^{-1}(SU(3)) & \xrightarrow{J} & \tilde{\pi}^0(SU(3)) \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{K}_C^{-1}(S^1 \wedge CP^2) & \xrightarrow{r} & \tilde{K}_R^{-1}(S^1 \wedge CP^2) & \xrightarrow{J} & \tilde{\pi}^0(S^1 \wedge CP^2) \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{K}_C^{-1}(S^3) & \xrightarrow{r} & \tilde{K}_R^{-1}(S^3) & \xrightarrow{J} & \tilde{\pi}^0(S^3).
\end{array} \tag{8.21}$$

The vertical maps are induced by inclusions. The diagram commutes by naturality of r, J. Now $\tilde{K}_C^{-1}(SU(3)) \cong \mathbf{Z} \oplus \mathbf{Z}$, generated by the fundamental representation λ and its conjugate $\bar{\lambda}$ [8]. Moreover, λ restricts to a generator of $\tilde{K}_C^{-1}(S^3)$ and its realification generates $\tilde{K}_R^{-1}(S^3)$. Along the bottom row of diagram (8.21) the J-map is the classical J-homomorphism, which is known to send the generator of $\tilde{K}_R^{-1}(S^3)$ onto ν in $\pi^0(S^3)$. From the definition of c in Proposition (8.3) it follows that the coefficient ω is ± 1 and we can now fix the sign of c so that $\omega = 1$.

It is interesting to note at this point that we can already prove (8.19). The computation does not depend on the coefficients λ, μ in (8.20). In [14] Steer shows that $[SU(3), 0]$ is the canonical tangential framing which features in the work of Smith and Atiyah [3]. We have confirmed that the tangential framing on $SU(3)$ gives $\bar{\nu}$. A similar calculation shows that $[SU(3), 4\lambda] = \bar{\nu}$, and this is the result needed in §5 to show that $[G_2, 4\rho] = \kappa$. We refer to [14] for a proof that $[G_2, 0]$, the tangential framing on G_2, is also κ.

The second stage in the proof of Proposition (8.18) is the difficult one. We want to show that $\lambda = \mu = 1$ in (8.20), which is equivalent to showing that $[SU(3), \lambda] = \bar{\nu} + \epsilon$. From Hopf invariant calculations we know that $[SU(3), \lambda]$ is either $\bar{\nu}$ or $\bar{\nu} + \epsilon$. The problem is settled by the following result.

8.22. Proposition. *The e-invariant of* $[SU(3), \lambda]$ *is* 1.

We refer to [12] for a comprehensive treatment of e-invariants in general homology theory. In the present paper we shall just indicate the main ideas as they apply to our particular problem.

Let n and k be divisible by 8. From the work of Adams [1], the e-invariant of a map $u: S^{n+k} \to S^k$ can be calculated from the short exact sequence

$$0 \leftarrow \tilde{K}_R(S^k) \leftarrow \tilde{K}_R(C) \leftarrow K_R(S^{n+k+1}) \leftarrow 0, \tag{8.23}$$

where C is the mapping cone of u. Let $\sigma \in \tilde{K}_R(C)$ map onto a generator of $\tilde{K}_R(S^k) \cong \mathbf{Z}$ and let $\tau \in \tilde{K}_R(C)$ be the image of the generator of $\tilde{K}_R(S^{n+k+1}) \cong \mathbf{Z}_2$. Evaluating the Adams' operation ψ_3 on σ we get

$$\psi_3(\sigma) = a\sigma + e\tau. \tag{8.24}$$

The coefficient e is the required e-invariant of u.

To find the e-invariant of a framed manifold we transport the calculations of ψ_3 onto the manifold, as Adams does in the case of a sphere when calculating the e-invariant on the image of the J-homomorphism.

Let us start therefore with the data of (8.8), namely a framed embedding of M^n in $\mathbf{R}^{n+k}$ and a twisting of the framing. Consider the following diagram:

$$\begin{array}{ccccccc}
S^k \wedge M & \xrightarrow{H} & S^k & \xrightarrow{i_1} & D & \xrightarrow{j_1} & S^{k+1} \wedge M \\
\Phi\uparrow & & I\uparrow & & \Phi_1\uparrow & & \Phi_2\uparrow \\
S^{n+k} & \xrightarrow[u]{} & S^k & \xrightarrow[i_2]{} & C & \xrightarrow[j_2]{} & S^{k+1}.
\end{array} \tag{8.25}$$

The map Φ is the reduced fundamental class as explained in (8.13), H is the Hopf construction associated with the twisting and u is the difference map (8.12). The space D is the mapping cone of H and Φ induces a map of cofibrations if we insert the identity map I of S^k. The first square then commutes by (8.14). The rest of the maps in (8.25) are defined in the usual way.

At this point we need the following result.

8.26 LEMMA. *Let B denote the k-plane bundle over the suspension $S^1 \wedge M$ associated with a map $\alpha: M \to SO(k)$. Let $H: S^k \wedge M \to S^k$ be the Hopf construction of α. The mapping cone D of H has the same homotopy type as the Thom complex of B, and in the cofibration*

$$S^k \wedge M \xrightarrow{H} S^k \xrightarrow{i} D \xrightarrow{j} S^{k+1} \wedge M \tag{8.27}$$

the map i may be taken as the inclusion of a fibre in the Thom complex.

Adams proves this fact when M is a sphere in [1] and the general case is similar.

If the map α lifts to Spin(k) then the bundle B is orientable for real K-theory, and in that case we have a Thom isomorphism

$$\Theta: K_R(S^1 \wedge M) \to \tilde{K}_R(D). \tag{8.28}$$

Apply real K-theory to diagram (8.25), on the assumption that we are now working with an orientable twisting, we obtain a diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \tilde{K}_R(S^k) & \xleftarrow{i_1^*} & \tilde{K}_R(D) & \xleftarrow{j_1^*} & \tilde{K}_R(S^{k+1} \wedge M) & \longleftarrow & 0 \\
 & & \downarrow{\scriptstyle I^*} & & \downarrow{\scriptstyle \Phi_1^*} & & \downarrow{\scriptstyle \Phi_2^*} & & \\
0 & \longleftarrow & \tilde{K}_R(S^k) & \xleftarrow[i_2^*]{} & \tilde{K}_R(C) & \xleftarrow[j_2^*]{} & \tilde{K}_R(S^{n+k+1}) & \longleftarrow & 0.
\end{array}$$

In order to evaluate ψ_3 in (8.24) we may choose

$$\sigma = \Phi_1^* \Theta(1), \tag{8.30}$$

where $\Theta(1)$ is the Thom class in $\tilde{K}_R(D)$. Again from the work of Adams [1] we have the following formula in the real K-theory of D

$$\psi_3 \Theta(1) = \Theta \rho_3(B), \tag{8.31}$$

where ρ_3 is the cannibalistic class of Adams and Bott. The virtual bundle $\rho_3(B)$ over $S^1 \wedge M$ is induced from B by a virtual representation of Spin(k) whose character is given explicitly in [1]. In the case when $M = SU(3)$ and B is the bundle over $S^1 \wedge SU(3)$ induced by the fundamental representation $\lambda: SU(3) \to SU(3)$, the virtual bundle $\rho_3(B)$ is induced by the representation $SU(3) \to SO$ whose character is

$$(z_1 + 1 + z_1^{-1})(z_2 + 1 + z_2^{-1})(z_3 + 1 + z_3^{-1}). \tag{8.32}$$

One easily checks that the representation is in fact

$$\lambda^2 + \bar{\lambda}^2 + \lambda\bar{\lambda}. \tag{8.33}$$

Now $\lambda^2 + \bar{\lambda}^2$ is the underlying real representation of a complex representation, namely λ^2. However $\lambda\bar{\lambda}$ is not in the image of realification. In fact a simple computation with characters shows that

$$\lambda\bar{\lambda} = 1 + \text{Ad}, \tag{8.34}$$

where Ad: $SU(3) \to SO(8)$ is the adjoint representation. At this point we need to know something about the K-theory of $SU(3)$

8.35. LEMMA. *As groups*

$$\tilde{K}_C^{-1}(SU(3)) \cong \mathbf{Z} \oplus \mathbf{Z}, \qquad \tilde{K}_R^{-1}(SU(3)) \cong \mathbf{Z} \oplus \mathbf{Z}_2,$$

where generators of the cyclic summands are respectively λ, $\bar{\lambda}$, $r(\lambda)$, μ. Here $r(\lambda)$ denotes the realification of λ and μ is the image of the generator of $\tilde{K}_R^{-1}(S^8) \cong \mathbf{Z}_2$ under the projection of $SU(3)$ onto its top cell.

The proof of this lemma follows quickly from the cofibration used in the proof of (8.3), and a comparison of the complex and real K-theories via realification. We note that $r(\lambda) = r(\bar{\lambda})$ so that μ is not in the image of realification.

8.36. LEMMA. *The adjoint representation of* $SU(3)$, *regarded as an element of* $\tilde{K}_R^{-1}(SU(3))$, *is given by* $\mathrm{Ad} = 3r(\lambda) + \mu$ *in terms of the generators of Lemma* (8.35).

The presence of $3r(\lambda)$ in the formula for Ad is easy to verify by looking at the complexification $c: \tilde{K}_R^{-1}(SU(3)) \to \tilde{K}_C^{-1}(SU(3))$ and using the fact that the tensor product $\lambda\bar{\lambda}$ of representations is homotopic to $3(\lambda + \bar{\lambda})$ as maps into the infinite unitary group. But $\lambda + \bar{\lambda} * cr(\lambda)$, and from (8.34) we deduce that either $\mathrm{Ad} = 3r(\lambda)$ or $3r(\lambda) + \mu$. The difficulty is to show that the latter holds. We delay the proof of this to the end of the section and proceed now to complete our calculation of the e-invariant of $[SU(3), \lambda]$.

8.37. COROLLARY. *In* $K_R^{-1}(SU(3))$ *we have* $\rho_3(B) = ml + nr(\lambda) + \mu$ where m and n are integers.

This follows from (8.36) and the fact that $\rho_3(B)$ is induced by the map (8.33) of $SU(3)$ into the infinite unitary group.

Now the Thom class in complex K-theory may be taken as the complexification of the real Thom class, because complexification $c: \tilde{K}_R(S^k) \to \tilde{K}_C(S^k)$ is an isomorphism when k is a multiple of eight. The results below now follow quickly by a comparison of complex K-theory and real K-theory under the realification map.

8.38. COROLLARY. *With reference to* 8.28 *and diagram* 8.29 *we have*

$$\Theta\rho_3(B) = p(1) + qj^*r(\lambda) + j^*\mu,$$

where p and q are integers.

This follows from (8.37) and the above remarks.

With reference to (8.24), (8.30) and diagram (8.29) we have

$$\begin{aligned}\psi_3\sigma &= \psi_3\Phi_1^*\Theta(1)\\ &= \Phi_1^*\psi_3\Theta(1)\\ &= \Phi_1^*\Theta(\rho_3(B))\\ &= p\Phi_1^*\Theta(1) + q\Phi_1^*j_1^*r(\lambda) + \Phi_1^*j_1^*\mu.\end{aligned}$$

Now $\Phi_1^*j_1^*r\lambda = r\Phi_1^*j_1^*\lambda = rj_2^*\Phi_2^*\lambda = 0$, because $\tilde{K}_c(S^{k+1}) = 0$. Moreover Φ_2 has degree one on the top cell because it is the suspension of Φ in diagram (8.25). Hence

$$\Phi_1^*j_1^*\mu = j_2^*\Phi_2^*\mu = \tau,$$

which leads finally to the conclusion that $\psi_3\sigma = p\sigma + \tau$ and the proof that the e-invariant of $[SU(3), \lambda]$ is 1.

We return now to the proof of Lemma (8.36). Let T denote the maximal torus of $SU(3)$. The adjoint representation of $SU(3)$ restricted to T is a real representation of T whose character is

$$k = 2 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_1\bar{z}_3 + z_3\bar{z}_1 + z_2\bar{z}_3 + z_3\bar{z}_2. \tag{8.39}$$

Clearly k has the form

$$k = cr(1 + z_1\bar{z}_2 + z_2\bar{z}_3 + z_3\bar{z}_1), \tag{8.40}$$

where as usual r denotes realification and c denotes complexification. In other words, the restriction of the adjoint representation of $SU(3)$ to the maximal torus T is the realification of the complex representation

$$D = 1 + z_1\bar{z}_2 + z_2\bar{z}_3 + z_3\bar{z}_1. \tag{8.41}$$

This allows us to construct the following ladder of fibrations

$$\begin{array}{ccccccc} U & \xrightarrow{r_1} & SO & \xrightarrow{q_1} & SO/U & \xrightarrow{c_1} & BU\\ {\scriptstyle D}\uparrow & & {\scriptstyle \mathrm{Ad}}\uparrow & & {\scriptstyle D_1}\uparrow & & {\scriptstyle D_2}\uparrow\\ T & \xrightarrow[r_2]{} & SU(3) & \xrightarrow[q_2]{} & SU(3)/T & \xrightarrow[c_2]{} & BT \end{array} \tag{8.42}$$

where U, SO are the infinite unitary and special orthogonal groups; BU, BT are classifying spaces, Ad is the adjoint representation regarded now as a map of $SU(3)$ into SO, and D is the map of T into U arising from Ad in the manner described above. The top sequence of (8.42) is the well known Bott sequence relating real and complex K-theory which can be formulated in the following way

$$\tilde{K}_C^{-1} \xrightarrow{r} \tilde{K}_R^{-1} \xrightarrow{\xi} \tilde{K}_R^{-2} \xrightarrow{c} \tilde{K}_C^{-2}, \tag{8.43}$$

where r_1 in (8.42) corresponds to realification in (8.43), the quotient map q_1 in (8.42) corresponds to the module action of the generator ξ of $K_R^{-1}(pt)$ and c_1 in (8.42) corresponds to complexification in (8.43). We are using here the periodicity theorems of Bott to identify SO/U with $\Omega^2 BO$ and BU with $\Omega^2 BU$. The rest of the maps in (8.42) are defined in the usual way.

We know from (8.35) that $\mu \in \tilde{K}_R^{-1}(SU(3))$ is not in the image of realification. In fact, one easily verifies that $\tilde{K}_R^{-2}(SU(3)) \cong \mathbf{Z}_2$ with $\xi\mu$ as the generator. Consequently, in order to verify the presence of μ in the formula for Ad in (8.36), it is necessary and sufficient to show that the composite map q_1Ad in diagram (8.42) is non-trivial. This is equivalent to showing that $D_1 q_2$ is non-trivial. Let δ in $\tilde{K}_R^{-2}(SU(3)/T)$ denote the element represented by the map (8.44) D_1. Then our problem is to show that $q_2^*(\delta)$ is the non-zero element of $\tilde{K}_R^{-2}(SU(3))$. This we do in several stages. First of all we need to know something about the space $SU(3)/T$. We refer to [6] for general information about homogeneous spaces and just recall here that for any compact Lie group G with maximal torus T, the space G/T has a cell structure with cells only in even dimensions and the number of cells (counting the 0-cell) is the order of the Weyl group of G. In particular

$$SU(3)/T = S^2 \vee S^2 \cup e^4 \cup e^4 \cup e^6 \tag{8.45}$$

from which we deduce immediately that $\tilde{K}_R^{-1}(SU(3)/T) = 0$ and from the Bott sequence (8.43) we see that the complexification map

$$c: \tilde{K}_R^{-2}(SU(3)/T) \to \tilde{K}_C^{-2}(SU(3)/T) \tag{8.46}$$

is injective.

Again from (8.45) we deduce that $\tilde{K}_R^{-2}(SU(3)/T)$ is a free group of rank 3. Let ω be the generator of $\tilde{K}_R^{-2}(SU(3)/T)$ which comes from the top cell in (8.45) under the projection map $SU(3)/T \to S^6$. With reference to (8.44) we can state the following.

8.47. Lemma. $\delta = 3\omega$.

By (8.46) and the fact that $\tilde{K}_R^{-2}(S^6) \xrightarrow{c} \tilde{K}_C^{-2}(S^6)$ is an isomorphism, we can verify (8.47) by working in complex K-theory. With reference to diagram (8.42) and the definition of δ in (8.44), we see that $c(\delta)$ is represented by the map $C_1 D_1$. But $C_1 D_1 = D_2 C_2$ and, since D_2 is the classifying map of D, we are now in a position to formulate our problem in terms of the natural map $\alpha: R(T) \to K(SU(3)/T)$ from the representation ring of T to the K-theory of $SU(3)/T$, which appears in the work of Atiyah and Hirzebruch[2]. In fact $D_2 C_2$ is essentially $\alpha(D)$. The character of D was given in (8.41) and we can now quote the value of the Chern character of $c(\delta)$, namely

$$e^{z_1} e^{\bar{z}_2} + e^{z_2} e^{\bar{z}_3} + e^{z_3} e^{\bar{z}_1}. \tag{8.48}$$

In this context z_1, z_2, z_3 are interpreted as elements of the second cohomology group of $SU(3)/T$, which generate $\tilde{H}^*(SU(3)/T)$ subject to the single relation

$$(1 + z_1)(1 + z_2)(1 + z_3) = 1. \tag{8.49}$$

The contribution of (8.48) to the reduced cohomology of $SU(3)/T$ works out to be $3z_1 z_2^2$ and $z_1 z_2^2$ generates $H^6(SU(3)/T)$. Since the Chern character maps $\tilde{K}_C(S^6)$ isomorphically onto $H^6(S^6)$ and is monomorphic from $K^*_C(SU(3)/T)$ to $H^*(SU(3)/T; Q)$ we deduce the result stated in (8.47).

Referring back to (8.44), we continue the proof that $q_2^*(\delta)$ is the non-trivial element of $\tilde{K}_R^{-2}(SU(3))$ by looking at the following map of cofibrations

$$\begin{array}{ccccc} S^1 \wedge CP^2 & \longrightarrow & SU(3) & \xrightarrow{p_1} & S^8 \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle q_3} \\ S^2 \vee S^2 \cup e^4 \cup e^4 & \longrightarrow & SU(3)/T & \xrightarrow[p_2]{} & S^6. \end{array} \tag{8.50}$$

The maps p_1 and p_2 project onto top cells, q_2 is the quotient map which induces the map q of 4-skeleta and the map q_3 of top cells.

8.51. LEMMA. *The map q_3 is the non-trivial element in the 2-stem.*

If we assume this result for the moment, then we know that the induced map q_3^*: $\tilde{K}_R^{-2}(S^6) \to \tilde{K}_R^{-2}(S^8)$ is non-trivial. it now follows easily from (8.47), (8.50) and the definition of ω that $q_3^* \delta$ is non-trivial.

The proof of Lemma (8.51) can be done in two ways. We can factor the quotient map q_2 into a composite of quotient maps

$$SU(3) \xrightarrow{r_1} SU(3)/S^1 \xrightarrow{r_2} SU(3)/T$$

and verify that r_1 and r_2 both carry the Hopf map on the top cells. The non-trivial element in the 2-stem is of course the square of the Hopf map in stable homotopy. There is a standard argument using Steenrod squares in the mapping cones of r_1 and r_2 to detect the Hopf maps. The details are straightforward but long and tedious. We shall not write them out here.

In the alternative argument for proving (8.51) we first observe that there are only two homotopy classes of maps from $SU(3)$ to S^6, represented by the trivial map and the composite

$$SU(3) \xrightarrow{p_1} S^8 \xrightarrow{\eta^2} S^6,$$

where p_1 is projection onto the top cell and η^2 is the square of the Hopf map. This is easily checked from the cell structure of $SU(3)$. The inverse image of a regular value of the composite

$$SU(3) \xrightarrow{q_2} SU(3)/T \xrightarrow{p_2} S^6$$

is a torus in $SU(3)$ with non-trivial framing and this detects the non-trivial element $SU(3) \to S^6$.

This completes our proof that the e-invariant of $[SU(3), \lambda]$ is 1. We refer to [12] for more details of many of the arguments given above and applications to other examples like framings on Sp(2).

REFERENCES

1. J. F. ADAMS: On the groups $J(X)$—II, *Topology* **3** (1965), 137–173; J. F. ADAMS: On the groups $J(X)$—IV, *Topology* **5** (1966), 21–74.
2. M. F. ATIYAH and F. HIRZEBRUCH: Vector bundles and homogeneous spaces, *Proc. Symp. Pure Math.* Vol. 3, *Am. Math. Soc.* (1961), 7–38.
3. M. F. ATIYAH and L. SMITH: Compact Lie groups and the stable homotopy of spheres, *Topology* **13** (1974), 135–142.
4. J. M. BOARDMAN and B. STEER: On Hopf invariants, *Comment. Math. Helv.* **42** (1967), 180–221.
5. A. BOREL: Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compact, *Ann. Math.* **57** (1953), 115–207.
6. A. BOREL and F. HIRZEBRUCH: Characteristic classes and homogeneous spaces—I, II, *Am. J. Math.* **80** (1958), 458–538; **81** (1959), 315–382.
7. H. H. GERSHENSON: A problem in compact Lie groups and cobordism, *Pacific J. Math.* **51** (1974), 189–202.
8. L. HODGKIN: On the K-theory of Lie groups, *Topology* **6** (1967), 1–36.
9. I. M. JAMES: On the suspension triad, *Ann. Math.* **63** (1956), 191–247.
10. M. KERVAIRE: An interpretation of G. Whitehead's generalization of the Hopf invariant, *Ann. Math.* **69** (1959), 345–365.
11. J. MILNOR: *Topology from the differential viewpoint.* The University Press of Virginia (1965).
12. N. RAY: Invariants of reframed manifolds (in preparation).
13. L. SMITH: Framings of sphere bundles over spheres, the plumbing pairing, and the framed bordism class of rank 2 simple Lie groups, *Topology* **13** (1974), 401–415.
14. B. STEER: Orbits and the homotopy class of a compactification of a classical map (to appear).
15. H. TODA: Composition methods in the homotopy of spheres, *Ann. Math. Stud.* **49**, Princeton (1962).

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