K-Theory **1** (1988), 575–626.

Automorphisms of Manifolds and Algebraic K-Theory: I

MICHAEL WEISS
SFB 170, Math. Institut, Bunsenstrasse 3–5, 3400 Göttingen, West Germany.

and

BRUCE WILLIAMS
Mathematics Department, University of Notre Dame, Notre Dame, IN 46556, U.S.A.

(Received: 3 September 1987)

Abstract. We investigate the homotopy type of $\widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M)$, where M is a compact manifold, $\mathrm{TOP}(M)$ is the simplicial group of homeomorphisms of M which restrict to the identity on ∂M, and $\widetilde{\mathrm{TOP}}(M)$ is the simplicial group of block homeomorphisms of M which restrict to the identity on ∂M. In the so-called topological concordance stable range of M, we obtain an expression in terms of the topological Whitehead spectrum of M. If M is smooth, we also investigate the homotopy type of $\widetilde{\mathrm{DIFF}}(M)/\mathrm{DIFF}(M)$; in the smooth concordance stable range of M, it has an expression in terms of the smooth Whitehead spectrum of M.

Key words. Diffeomorphism, homeomorphism, concordance, bounded homeomorphism, coordinate free spectrum.

Introduction

Let M be a compact topological manifold. Denote by $G(M)$ and $\mathrm{TOP}(M)$ the spaces of self-homotopy equivalences and self-homeomorphisms of M which are the identity on ∂M. We want to investigate the difference between $G(M)$ and $\mathrm{TOP}(M)$, or $G(M)/\mathrm{TOP}(M)$.

Recall that surgery theory, notably the Sullivan–Wall long exact sequence, analyses $G(M)/\widetilde{\mathrm{TOP}}(M)$. (Here $\widetilde{\mathrm{TOP}}(M)$ is the simplicial set of block homeomorphisms of M; its k-simplices are the self-homeomorphisms of $\Delta^k \times M$ which are the identity on $\Delta^k \times \partial M$ and which preserve the faces $d_i\Delta^k \times M$ for $0 \leqslant i \leqslant k$.) It remains to understand $\widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M)$.

Let $\mathscr{C}^{\mathrm{TOP}}(M)$ be the space of topological concordances of M (see Hatcher [18] or Waldhausen [41]). If the stabilization maps

$$\mathscr{C}^{\mathrm{TOP}}(M) \to \mathscr{C}^{\mathrm{TOP}}(M \times D^1) \to \mathscr{C}^{\mathrm{TOP}}(M \times D^2) \longrightarrow \cdots$$

are all k-connected, then we say that k is in the topological concordance stable range for M. The direct limit $\mathscr{C}^{\mathrm{TOP}}(M \times D^\infty)$ of the spaces $\mathscr{C}^{\mathrm{TOP}}(M \times D^j)$ is an infinite loop space. It determines a spectrum whose suspension (!) we denote, for one reason or another, by $\Omega\underline{\underline{\mathrm{Wh}}}s^{\mathrm{TOP}}(M)$. We construct an action of Z_2 on $\Omega\underline{\underline{\mathrm{Wh}}}s^{\mathrm{TOP}}(M)$. We are particularly interested in the homotopy orbit spectrum $S^\infty_+ \wedge_{Z_2} \Omega \underline{\underline{\mathrm{Wh}}}s^{\mathrm{TOP}}(M)$ and its zeroth infinite loop space, written $Q(S^\infty_+ \wedge_{Z_2} \Omega \underline{\underline{\mathrm{Wh}}}s^{\mathrm{TOP}}(M))$. Here S^∞ plays the role of EZ_2, and the subscript $+$ marks an added base point.

THEOREM A (topological version). *There exists a map*

$$\Phi^s\colon \widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M) \to Q(S^\infty_+ \wedge_{Z_2} \Omega \underline{\underline{\mathrm{Wh}}}s^{\mathrm{TOP}}(M)),$$

which is $(k+1)$*-connected if* k *is in the topological concordance stable range for* M.

Remark. Using Theorem A and the filtration of S^∞ by skeletons S^i, one obtains a spectral sequence for the analysis of $\pi_*(\widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M))$ in the concordance stable range. This is known and due to Hatcher [19]. If we localize at odd primes, then Theorem A is a result of Burghelea and Lashof [9]; see also Burghelea and Fiedorowicz [8] and Hsiang and Jahren [21].

THEOREM A (smooth version). *If* M *is smooth, then there is a map*

$$\Phi^s\colon \widetilde{\mathrm{DIFF}}(M)/\mathrm{DIFF}(M) \to Q(S^\infty_+ \wedge_{Z_2} \Omega \underline{\underline{\mathrm{Wh}}}s^{\mathrm{DIFF}}(M))$$

which is $(k+1)$*-connected if* k *is in the smooth concordance stable range for* M.

We hope the notation in the smooth version is self-explanatory. The smooth version can be used to analyse $G(M)/\mathrm{DIFF}(M)$, just as the topological version can be used to analyse $G(M)/\mathrm{TOP}(M)$. The proofs of the topological and smooth versions are identical, and we will concentrate mostly on the topological case in this introduction and throughout the paper. Note, however, that concordance stability is better understood in the smooth case. Kiyoshi Igusa has shown that if M is smooth and $k < \dim(M)/3$ approximately, then k is in the smooth and in the topological concordance stable range for M. See Igusa [23, 24].

Our proof of Theorem A proceeds by separating the combinatorial aspects of $\widetilde{\mathrm{TOP}}(M)$ from its geometrical aspects. The method is:

Euclidean Stabilization. Let $\mathrm{TOP}^b(M \times \mathbb{R}^i)$ be the topological or simplicial group of homeomorphisms $f\colon M \times \mathbb{R}^i \to M \times \mathbb{R}^i$ such that f is the identity on $\partial M \times \mathbb{R}^i$, and

> there exists an $\varepsilon(f) > 0$ with $\|\mathrm{pr}_2 f(m,z) - z\| < \varepsilon(f)$ for all $m \in M$, $z \in \mathbb{R}^i$, where $\mathrm{pr}_2\colon M \times \mathbb{R}^i \to \mathbb{R}^i$ is the projection.

We call f a bounded homeomorphism. The bounded theory was introduced and first exploited by Anderson and Hsiang [2].

Of course, there is also a block version $\widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^i)$ and we get a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{TOP}(M) & \to & \mathrm{TOP}^b(M \times \mathbb{R}^1) & \to & \mathrm{TOP}^b(M \times \mathbb{R}^2) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\widetilde{\mathrm{TOP}}(M) & \to & \widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^1) & \to & \widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^2) & \longrightarrow & \cdots
\end{array}$$

where the horizontal arrows are given by crossing with the identity on $\mathbb{R}^1$, or by Euclidean stabilization. Write

$$\mathrm{TOP}^b(M \times \mathbb{R}^\infty) = \bigcup_i \mathrm{TOP}^b(M \times \mathbb{R}^i) \quad \text{and}$$

$$\widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^\infty) = \bigcup_i \widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^i).$$

The next result implies that Euclidean stabilization kills the difference between 'honest' and blocked.

THEOREM B. *The inclusion* $\mathrm{TOP}^b(M \times \mathbb{R}^\infty) \hookrightarrow \widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^\infty)$ *is a homotopy equivalence.*

The stabilization map $\widetilde{\mathrm{TOP}}(M) \to \widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^\infty)$ is also close to being a homotopy equivalence; for example it is so if M is simply connected and, therefore, $\mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M)$ is approximately the same as $\widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M)$, and is much easier to handle. Using the Anderson–Hsiang theory, we construct a spectrum $\Omega\underline{\underline{\mathrm{Wh}}}^{\mathrm{TOP}}(M)$ with Z_2-action whose 0-connected cover is $\Omega\underline{\underline{\mathrm{Wh}}}s^{\mathrm{TOP}}(M)$ and whose homotopy groups in negative dimensions are the negative algebraic K-groups of $\mathbb{Z}\pi_1(M)$. We then analyse $\mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M)$ in terms of $\Omega\underline{\underline{\mathrm{Wh}}}^{\mathrm{TOP}}(M)$ and use combinatorial methods to pick up the trifles lost through Euclidean stabilization. This is summarized in the next result.

THEOREM C. *There exists a map*

$$\Phi\colon \mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M) \to Q(S^\infty_+ \wedge_{Z_2} \Omega\underline{\underline{\mathrm{Wh}}}^{\mathrm{TOP}}(M))$$

which fits into a commutative square

$$\begin{array}{ccc}
\widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M) & \xrightarrow{\quad\Phi^s\quad} & Q(S^\infty_+ \wedge_{Z_2} \Omega\underline{\underline{\mathrm{Wh}}}s^{\mathrm{TOP}}(M)) \\
\big\downarrow & & \big\downarrow \\
\widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M) & & \\
\simeq \big\uparrow & & \\
\mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M) & \xrightarrow{\quad\Phi\quad} & Q(S^\infty_+ \wedge_{Z_2} \Omega\underline{\underline{\mathrm{Wh}}}^{\mathrm{TOP}}(M)).
\end{array}$$

The square is a homotopy pullback square if $\dim(M) \geqslant 5$.

In future papers on this subject, we want to use the known relationship between concordance theory and algebraic K-theory to obtain numerical results.

Leitfaden: Sections 2 and 3 contain the geometric part of the proof of Theorem A, and Section 4 contains the necessary combinatorics. Sections 1 and 5 contain introductory and supplementary material about bounded homeomorphisms, for which we claim no originality. Sections 0 and 6 are about language.

0. Preliminaries

Simplicial sets are popular in homotopy theory for two different reasons. Firstly, many important spaces, such as Eilenberg–MacLane spaces or classifying spaces in K-theory, can be conveniently defined in simplicial language. Secondly, certain necessary constructions (of mapping objects, say) can be performed easily in the category of simplicial sets when they are painful in the category of topological spaces.

We are mostly interested in the second aspect, and we have found it necessary to introduce yet another substitute for the notion of space which does not suffer from the combinatorial rigidity that simplicial sets inevitably have. Our reason for avoiding rigidity is that we wish to use the language of coordinate free spectra in Sections 2 and 3; in particular, some of our 'spaces' will come equipped with an action of the orthogonal group $O(n)$, and the action should be continuous. The use of simplicial sets in this situation would obscure even the simplest arguments.

0.1. DEFINITION. A virtual space is a contravariant set-valued functor Y on the category of topological spaces and continuous maps, satisfying the sheaf condition:

> If X is a topological space with an open covering $\{U_i \mid i \in J\}$, and if for each $i \in J$ an element s_i in $Y(U_i)$ is given such that
>
> $$s_{i\,|\,U_i \cap U_j} = s_{j\,|\,U_i \cap U_j} \quad \text{in} \quad Y(U_i \cap U_j)$$
>
> for all $(i, j) \in J \times J$, then there exists a unique $s \in Y(X)$ such that $s_{|U_i} = s_i$ for all $i \in J$.

A continuous map between virtual spaces Y_1, Y_2 is a natural transformation $Y_1 \to Y_2$. A pointed virtual space is a virtual space Y together with a continuous map $* \to Y$, where $*$ is the constant one-point functor.

0.2. REMARKS. (i) The notion of quasi-space in Kirby and Siebenmann [27] is very similar in character. In Siebenmann's words 'a quasi-space is a "space" of which we want to know only the sets of maps to it of certain specified pleasant spaces'. The same could be said of virtual spaces; see 0.3 below.

(ii) The category of virtual spaces is a topos, by definition of that word. See the introductions to Johnstone [25], Barr and Wells [4], and Wraith [45].

0.3 EXAMPLE. Every topological space Y can be regarded as a virtual space in the obvious way: Let $Y(X)$ be the set of continuous maps from X to Y, if X is another topological space. The category of topological spaces and continuous maps is therefore contained in the category of virtual spaces and continuous maps, as a full subcategory (by the Yoneda lemma). If U is a virtual space and X is a topological space, then $U(X)$ can be identified with the set of continuous maps from X to U.

0.4. EXAMPLES. (i) Let M be a compact topological manifold as in the introduction, and let V be a finite-dimensional real Hilbert space. Let $\mathrm{TOP}^b(M \times V)$ be the virtual space which to each topological space X associates the set of locally bounded homeomorphisms

$$f: X \times M \times V \to X \times M \times V$$

preserving the projection to X, and restricting to the identity on $X \times \partial M \times V$. ('Locally bounded' means that any $x \in X$ has a neighbourhood $U \subset X$ such that the set of real numbers $\{d(z, f(z)) \mid z \in U \times M \times V\}$ is bounded. Here d is the distance measured in the V-direction only.)

(ii) Suppose now that M is smooth. An element $f: X \times M \times V \to X \times M \times V$ of $\mathrm{TOP}^b(M \times V)(X)$ will be called smooth if, for each point $x \in X$, the restriction

$$f_x: \{x\} \times M \times V \to \{x\} \times M \times V$$

is smooth, and if the higher derivatives $D(f_x), D^2(f_x), \ldots$ vary continuously in x. (Each derivative $D^n(f_x)$ is a continuous section of some vector bundle over $\{x\} \times M \times V$; letting x vary, one obtains a section of some vector bundle over $X \times M \times V$, and this is still required to be continuous, for all $n > 0$. We do not put any bounds on the higher derivatives.)

The smooth elements of $\mathrm{TOP}^b(M \times V)(-)$ define a virtual subspace $\mathrm{DIFF}^b(M \times V)$ of $\mathrm{TOP}^b(M \times V)$.

0.5. CONSTRUCTIONS with virtual spaces. Since virtual spaces form a topos, practically all categorical constructions can be performed with them. We mention a few explicitly.

(i) The product of an arbitrary family of virtual spaces is again a virtual space.

(ii) Let Y be a virtual space. A virtual subspace $A \subset Y$ is a subfunctor which is a virtual space in its own right.

(iii) Take $A \subset Y$ as in (ii). The diagram $* \leftarrow A \rightarrow Y$, where $*$ is the one-point functor, has a pushout in the category of virtual spaces: Take the contravariant functor which to a topological space X associates the pointed set $Y(X) \amalg_{A(X)} *$, and subject it to the standard construction for converting presheaves into sheaves. The resulting virtual space $Y \amalg_A *$ has the required universal property.

The reader is warned that if A and Y happen to be genuine topological spaces, then the pushout $Y \amalg_A *$ in the category of virtal spaces will not usually agree with the pushout $Y \amalg_A *$ in the category of topological spaces. However, the virtual version behaves much better than the topological version, and it would be unwise not to use it (see Section 6). There is a risk of confusion here, but the consequences of such a confusion would be quite harmless.

(iv) As an application of (iii), define the wedge $Y_1 \vee Y_2$ of two pointed virtual spaces Y_1, Y_2 by taking $Y = Y_1 \amalg Y_2$ and $A = * \amalg *$ in (iii).

(v) Define the smash product $Y_1 \wedge Y_2$ of two pointed virtual spaces to be $(Y_1 \times Y_2) \amalg_{(Y_1 \vee Y_2)} *$.

(vi) To define the direct limit of a direct system

$$\cdots \to Y_{n-1} \to Y_n \to Y_{n+1} \text{---} \cdots \quad (n \in \mathbb{Z})$$

of virtual spaces, takes the contravariant functor $X \mapsto \lim_n Y_n(X)$ and subject it to the standard construction for turning presheaves into sheaves. (Again, direct limits in this sense should not be confused with direct limits in the category of topological spaces.)

(vii) If Y_1 and Y_2 are virtual spaces, then the rule which to every topological space X associates the set of continuous maps $X \times Y_1 \to Y_2$ is a contravariant functor with the sheaf property, or a virtual space. It is called the virtual space of continuous maps $Y_1 \to Y_2$, written $\mathrm{map}(Y_1, Y_2)$.

The definition is a little sloppy because the 'set' of continuous maps $X \times Y_1 \to Y_2$ need not be a set. But if Y_1 is a genuine topological space, then it can be identified with $Y_2(X \times Y_1)$ and is therefore a set.

If Y_1 and Y_2 are both pointed, then we can similarly define the virtual space of all pointed continuous maps from Y_1 to Y_2, written $\mathrm{map}_*(Y_1, Y_2)$.

(viii) If Y is a pointed virtual space, let $\Omega Y = \mathrm{map}_*(S^1, Y)$, using (vii). Note that

$$\Omega^n Y = \mathrm{map}_*(S^1 \wedge S^1 \wedge \cdots \wedge S^1, Y),$$

which is not quite the same as $\mathrm{map}_*(S^n, Y)$ because smash products are to be taken in the category of virtual spaces. But the difference is quite inessential.

(ix) Suppose that J is a virtual space with group structure. (This means that the sets $J(X)$ are groups, and the maps $J(X) \to J(X')$ induced by continuous maps $X \to X'$ are group homomorphisms.) Suppose further that $H \subset J$ is a virtual subspace which is also a subgroup. The rule which to every topological space X associates the set of left cosets $J(X)/H(X)$ is then a contravariant set-valued functor, but it need not have the sheaf property. Now apply the standard construction for converting presheaves into sheaves. The result is a virtual space J/H. For example, we could take $J = \mathrm{TOP}^b(M \times (V \oplus \mathbb{R}))$ and $H = \mathrm{TOP}^b(M \times V)$, or $J = \mathrm{DIFF}^b(M \times (V \oplus \mathbb{R}))$ and $H = \mathrm{DIFF}^b(M \times V)$. In fact, we will do so in Section 1.

More generally, suppose that H is a virtual space with group structure acting on a virtual space Y. Then it is possible to define a virtual orbit space Y/H in the same way.

0.6. REMARK. Let Y be a virtual space and let X be a topological space. To every $f \in Y(X)$ we can associate a map of sets

$$f^{\bullet}: X \to Y(*); \qquad x \mapsto f_{|\{x\}} \in Y(\{x\}) \cong Y(*).$$

Most of the virtual spaces that we will encounter are such that $f^{\bullet}$ determines f, for arbitrary X and $f \in Y(X)$. To specify a continuous map between virtual spaces with this property, say Y_1 and Y_2, it is sufficient to specify the map of sets $Y_1(*) \to Y_2(*)$.

0.7. DEFINITION. Two continuous maps $f_0, f_1: U \to Y$ between virtual spaces are homotopic if there exists a continuous map $f: U \times I \to Y$ such that $f_{|U \times \{0\}} = f_0$ and $f_{|U \times \{1\}} = f_1$.

Homotopy is an equivalence relation. To check for transitivity, suppose there are given two homotopies

$$h_\alpha: U \times [0, 1] \to Y, \qquad h_\omega: U \times [2, 3] \to Y$$

such that h_α connects f_0 with f_1 and h_ω connects f_1 with f_2. Let p_α: $U \times [0, 2[\to U \times [0, 1]$ be given by $p_\alpha(u, t) = (u, \min\{t, 1\})$ and let p_ω: $U \times]1, 3] \to U \times [2, 3]$ be given by $p_\omega(u, t) = (u, \max\{t, 2\})$. Let h: $U \times [0, 3] \to Y$ be the unique continuous map which equals $h_\alpha p_\alpha$ on $U \times [0, 2[$ and $h_\omega p_\omega$ on $U \times]1, 3]$. Then h is a homotopy connecting f_0 with f_2.

We can now say that a continuous map f: $U \to Y$ between virtual spaces is a weak homotopy equivalence if f_*: $U(X)/\sim \; \to Y(X)/\sim$ is an isomorphisms for all CW-spaces X, where $\sim$ denotes the homotopy relation.

0.8. DEFINITION. The materialization of a virtual space Y is the simplicial set Y^{mat} whose k-simplices are the continuous maps $\Delta^k \to Y$, for all $k \geqslant 0$.

It will be shown in a separate appendix (Section 6) that there is a sufficiently well defined continuous map from the geometric realization of Y^{mat} to Y which is a weak homotopy equivalence. Moreover, all the constructions in 0.5 behave well under materialization, in the sense that they yield easily predictable homotopy types. The moral is that we can pass freely from the world of virtual spaces to that of simplicial sets. We will in fact use virtual spaces when rigidity would be a hindrance, and simplicial sets when combinatorial arguments are needed.

1. Bounded Homeomorphisms and Diffeomorphisms

This section is a survey of results due in their final form mostly to Anderson and Hsiang [2], with ideas from Hsiang and Sharpe [22], Hatcher [18], Siebenmann [38], Edwards and Kirby [17], and M. Brown (unpublished). See Madsen and Rothenberg [28, 29] and Anderson and Pedersen [3] for recent applications of the bounded theory. The controlled theory of Chapman [15] and Quinn [35, 36] is also closely related.

For simplicity of notation, we present results and proofs in the topological category first, but it should always be kept in mind that we make similar claims for the smooth category. We comment on the similarities and minor differences in Remark 1.17.

We begin by stating two instrumental theorems: an isotopy extension theorem, and a wrapping theorem known under the name 'belt buckle trick'.

1.1. ISOTOPY EXTENSION THEOREM. *Let X be a topological manifold, $V \subset X$ an open subset, C a compact subset of V. Suppose there is give a continuous family of embeddings*

$$j_t: V \to X \quad \text{for } t \in \Delta^n$$

such that j_b is the inclusion for some $b \in \Delta^n$. Then there exists a continuous family of homeomorphisms

$$J_t: X \to X, \quad \text{with } t \in \Delta^n,$$

such that J_t agrees with j_t on C for all t, and $J_b = \mathrm{id}_X$. Further, if $j_{t|\partial V}$ is the inclusion $\partial V \subset \partial X \subset X$ for all t, then $J_{t|\partial X}$ can be the inclusion $\partial X \subset X$ for all t. (Continuity of the *families* $\{j_t\}$ *and* $\{J_t\}$ *refers to the compact-open topology.)*

Proof. See Siebenmann [39], 6.5.III, 6.6, 2.3, 1.3.(0) or §5 of Cernavskii [14].

It would be useful to have an isotopy extension theorem in the bounded case. Suppose, for instance, that X in Theorem 1.1 is equipped with a proper map $p\colon X \to \mathbb{R}^k$, that C is closed instead of compact, and that the family of embeddings $\{j_t\}$ satisfies a boundedness condition (which means that $\|p(j_t(x)) - p(x)\| < \varepsilon$ for some $\varepsilon > 0$ and all $x \in V$ and $t \in \Delta^n$). Does a (bounded) extension $\{J_t\}$ as in 1.1 exist? The answer is no; see Hirsch [20], ch. 8 ex. 9. We will use the belt buckle trick as a substitute for the missing isotopy extension theorems.

Define $\mathrm{TOP}^b(M \times \mathbb{R}^n)$ as in 0.4. Suppose that H is a finitely generated subgroup of the additive group $\mathbb{R}^n$, and let $\mathrm{TOP}^b(M \times \mathbb{R}^n; H) \subset \mathrm{TOP}^b(M \times \mathbb{R}^n)$ be the virtual subspace consisting of all bounded homeomorphisms which commute with the translations

$$M \times \mathbb{R}^n \to M \times \mathbb{R}^n; \quad (m, z) \mapsto (m, z + h)$$

for arbitrary $h \in H$.

1.2. BELT BUCKLE THEOREM. *Choose integers* $j, k, m \geqslant 0$. *Write* $\mathbb{R}^{j+k+m} = \mathbb{R}^j \times \mathbb{R}^k \times \mathbb{R}^m$. *The forgetful map*

$$u\colon \mathrm{TOP}^b(M \times \mathbb{R}^{j+k+m}; \mathbb{Z}^{k+m}) \mapsto \mathrm{TOP}^b(M \times \mathbb{R}^{j+k+m}; \mathbb{Z}^k)$$

has a homotopy splitting

$$w\colon \mathrm{TOP}^b(M \times \mathbb{R}^{j+k+m}; \mathbb{Z}^k) \to \mathrm{TOP}^b(M \times \mathbb{R}^{j+k+m}; \mathbb{Z}^{k+m}),$$

so that $uw \simeq$ *identity.*

1.3. LEMMA (for the proof of 1.2). *Let* X *be a topological space. Let* $\alpha_-, \beta_-, \alpha_+, \beta_+$ *be open embeddings* $X \times \mathbb{R} \to X \times \mathbb{R}$ *such that*

$$\alpha_-(X \times \mathbb{R}), \quad \beta_-(X \times \mathbb{R}) \subset X \times]-\infty, 0],$$

$$\alpha_+(X \times \mathbb{R}), \quad \beta_+(X \times \mathbb{R}) \subset X \times [0, +\infty[,$$

$$\left.\begin{array}{l} \alpha_- = \beta_- = \text{identity on } X \times]-\infty, -c] \\ \alpha_+ = \beta_+ = \text{identity on } X \times [c, +\infty] \end{array}\right\} \textit{ for some } c > 0.$$

Let $X \times \mathbb{R} \times \mathbb{Z}/(\alpha_-, \alpha_+)$ *be the quotient space obtained from* $(X \times \mathbb{R}) \times \mathbb{Z}$ *by identifying* $(\alpha_+(x, r), z)$ *with* $(\alpha_-(x, r), z + 1)$ *for all* $z \in \mathbb{Z}$. *Define* $X \times \mathbb{R} \times \mathbb{Z}/(\beta_-, \beta_+)$ *similarly. Then there is a canonical homeomorphism*

$$X \times \mathbb{R} \times \mathbb{Z}/(\alpha_-, \alpha_+) \to X \times \mathbb{R} \times \mathbb{Z}/(\beta_-, \beta_+)$$

which commutes with the translation $(x, r, z) \mapsto (x, r, z + 1)$.

Proof. The canonical homeomorphism is the composition of homeomorphisms

$$X \times \mathbb{R} \times \mathbb{Z}/(\alpha_-, \alpha_+) \cong X \times \mathbb{R} \times \mathbb{Z}/(\beta_-, \alpha_+) \cong X \times \mathbb{R} \times \mathbb{Z}/(\beta_-, \beta_+).$$

To see for example that

$$X \times \mathbb{R} \times \mathbb{Z}/(\alpha_-, \alpha_+) \cong X \times \mathbb{R} \times \mathbb{Z}/(\beta_-, \alpha_+),$$

note that the underlying sets can both be identified with $((X \times \mathbb{R}) - \mathrm{im}(\alpha_+)) \times \mathbb{Z}$. The identity is then a homeomorphism.

Some choices will be needed in the proof of Theorem 1.2. Choose homeomorphisms $e_+: \mathbb{R} \to]+2, +\infty[$ and $e_-: \mathbb{R} \to]-\infty, -2[$ which are the identity on $[+3, +\infty[$ and on $]-\infty, -3]$, respectively. Choose also a homeomorphism $\lambda: \mathbb{R} \times \mathbb{Z}/(e_-, e_+) \to \mathbb{R}$ commuting with the actions of $\mathbb{Z}$. (Here we use notation as in 1.3; the generator of $\mathbb{Z}$ acts on $\mathbb{R} \times \mathbb{Z}/(e_-, e_+)$ by $(r, z) \mapsto (r, z+1)$ and on $\mathbb{R}$ by $r \mapsto r+1$.) Choose λ so that the composition

$$\mathbb{R} \cong \mathbb{R} \times \{0\} \hookrightarrow \mathbb{R} \times \mathbb{Z}/(e_-, e_+) \xrightarrow{\lambda} \mathbb{R}$$

agrees with the identity in a neighbourhood of $0 \in \mathbb{R}$.

Proof of 1.2. (This is also given, in a slightly different setting, in Madsen and Rothenberg [29], Part III.) We can assume that $m = 1$. Let g be a point in $\mathrm{TOP}^b(M \times \mathbb{R}^{j+k+1}; \mathbb{Z}^k)$, and assume that g has bound $\leqslant 1$ with regard to the last coordinate. (This means that $\|pg(x) - p(x)\| \leqslant 1$ for all $x \in M \times \mathbb{R}^{j+k+1}$, where $p: M \times \mathbb{R}^{j+k+1} \to \mathbb{R}$ is the projection to the last coordinate.) Put $X = M \times \mathbb{R}^{j+k}$ in 1.3, and

$$\alpha_- = \mathrm{id}_X \times e_-, \quad \alpha_+ = \mathrm{id}_X \times e_+, \quad \beta_- = g\alpha_- g^{-1}, \quad \beta_+ = g\alpha_+ g^{-1}.$$

Then

$$X \times \mathbb{R} \times \mathbb{Z}/(\beta_-, \beta_+) \cong X \times \mathbb{R} \times \mathbb{Z}/(\alpha_-, \alpha_+)$$
$$\cong X \times (\mathbb{R} \times \mathbb{Z}/(e_-, e_+)) \cong X \times \mathbb{R}$$

by 1.3. and $X \times \mathbb{R} = M \times \mathbb{R}^{j+k+1}$. Therefore $g \times \mathrm{id}_{\mathbb{Z}}: X \times \mathbb{R} \times \mathbb{Z}/(\alpha_-, \alpha_+) \to X \times \mathbb{R} \times \mathbb{Z}/(\beta_-, \beta_+)$ can also be regarded as a homeomorphism $w(g)$ from $M \times \mathbb{R}^{j+k+1}$ to itself. This defines the map w on the virtual subspace of $\mathrm{TOP}^b(M \times \mathbb{R}^{j+k+1}; \mathbb{Z}^k)$ consisting of all g having bound $\leqslant 1$ with regard to the last coordinate. But the inclusion of this subspace is clearly a weak homotopy equivalence, or a homotopy equivalence after materialization.

Showing that uw is homotopic to the identity amounts to showing that the map $g \mapsto uw(g) \cdot g^{-1}$ is nullhomotopic. This is an easy consequence of the fact that $uw(g) \cdot g^{-1}$ agrees with the identity in a neighbourhood of $M \times \mathbb{R}^{j+k} \times \{0\}$, by construction. (Use an Alexander trick, which means pushing the two halves of $uw(g) \cdot g^{-1}$ towards $M \times \mathbb{R}^{j+k} \times \{+\infty\}$ and $M \times \mathbb{R}^{j+k} \times \{-\infty\}$ respectively, by conjugating with suitable translations.)

1.4. NOTATION. We define $\mathrm{TOP}^b(M \times \mathbb{R}^n)$ as in 0.4 and regard it either as a virtual space or as a simplicial set, using the materialization functor. If $n = 0$, we simply write

$\mathrm{TOP}(M)$. Note that homeomorphisms in $\mathrm{TOP}(M)$ are the identity on ∂M. Accordingly, $\mathrm{TOP}(M \times D^k)$ is the space of homeomorphisms of $M \times D^k$ which are the identity on $\partial(M \times D^k)$. Relative versions will be marked as such; for instance, if $\partial_0 M$ is a codimension zero submanifold of ∂M, we write $\mathrm{TOP}(M, \partial_0 M)$ for the virtual space (or simplicial set) of homeomorphisms of M which are the identity on $\partial M - \partial_0 M$.

In the sequel, it is sometimes helpful to think of certain homeomorphisms of certain manifolds as perturbations of the identity homeomorphism. For example, let $f: D^n \to D^n$ be a homeomorphism keeping S^{n-1} pointwise fixed; regard it as a perturbation, and remove it by radially shrinking the domain of perturbation to the centre of D^n. This shows that $\mathrm{TOP}(D^n)$ is contractible; of course, the trick is due to Alexander [1]. Anderson and Hsiang [2] employ a different Alexander trick which consists in pushing the domain of perturbation towards ∞. We often use the label 'alex' for constructions involving an Alexander trick of this type.

A typical example is the map from $\mathrm{TOP}^b(M \times D^k \times \mathbb{R}^n)$ to $\Omega^k \mathrm{TOP}^b(M \times \mathbb{R}^{k+n})$ defined as follows. Take a bounded homeomorphism $f: M \times D^k \times \mathbb{R}^n \to M \times D^k \times \mathbb{R}^n$, and regard it as a bounded homeomorphism $\hat{f}: M \times \mathbb{R}^k \times \mathbb{R}^n \to M \times \mathbb{R}^k \times \mathbb{R}^n$ by extending trivially outside $M \times D^k \times \mathbb{R}^n \subset M \times \mathbb{R}^k \times \mathbb{R}^n$. For $z \in \mathbb{R}^k$, let $\mathrm{tr}_z: M \times \mathbb{R}^k \times \mathbb{R}^n \to M \times \mathbb{R}^k \times \mathbb{R}^n$ be the translation by z. The map

$$z \mapsto \begin{cases} \mathrm{tr}_{-z} \cdot \hat{f} \cdot \mathrm{tr}_z & \text{if } z \in \mathbb{R}^k \\ \text{identity} & \text{if } z = \infty \end{cases}$$

is then a continuous map from $\mathbb{R}^k \cup \{\infty\}$ to $\mathrm{TOP}^b(M \times \mathbb{R}^k \times \mathbb{R}^n)$. Identifying $\mathbb{R}^k \cup \{\infty\}$ with S^k, regard it as a k-fold loop $\mathrm{alex}(f)$ in $\mathrm{TOP}^b(M \times \mathbb{R}^{k+n})$.

1.5. PROPOSITION. *The map*

$$\mathrm{alex}: \mathrm{TOP}^b(M \times D^k \times \mathbb{R}^n) \to \Omega^k \mathrm{TOP}^b(M \times \mathbb{R}^{k+n})$$

is a weak homotopy equivalence.

Proof. We may assume that $n = 0$, because otherwise we know from 1.2 that

$$\mathrm{alex}: \mathrm{TOP}^b(M \times D^k \times \mathbb{R}^n) \to \Omega^k \mathrm{TOP}^b(M \times \mathbb{R}^{k+n})$$

is a homotopy retract of another map

$$\mathrm{alex}: \mathrm{TOP}^b(M \times D^k \times \mathbb{R}^n; \mathbb{Z}^n) \to \Omega^k \mathrm{TOP}^b(M \times \mathbb{R}^{k+n}; \mathbb{Z}^n)$$

defined by the same method. The latter will be a weak homotopy equivalence if

$$\mathrm{alex}: \mathrm{TOP}^b(M \times (S^1)^n \times D^k) \to \Omega^k \mathrm{TOP}^b(M \times (S^1)^n \times \mathbb{R}^k)$$

is (use covering space arguments). The factor $(S^1)^n$ can be absorbed in the symbol M.

We may also assume that $k = 1$, because otherwise $D^k \cong D^1 \times D^1 \times D^1 \times \cdots \times D^1$, and the map can then be written as a k-fold iteration.

When $k = 1$ and $n = 0$, proceed as follows. Let E be the space of all pairs (f, g) where $f, g: M \times \mathbb{R} \to M \times \mathbb{R}$ are bounded homeomorphisms (equal to the identity on

$\partial M \times \mathbb{R}$) such that

$$g_{|M \times]-\infty, -1]} = \text{identity}, \qquad g_{|M \times [z, +\infty[} = f_{|M \times [z, +\infty[}$$

for some $z \geqslant 0$. (This is a virtual space, of course; the bounds on f and g are required to exist locally, as in 0.4, and z is also required to exist locally.) Let $E_0 \subset E$ be the subspace consisting of the pairs (f, g) with $f = \text{id}$. Clearly $E_0 \simeq \text{TOP}(M \times D^1)$. We will prove:

1.6. LEMMA. (i) *E is contractible.*
(ii) *The diagram*

$$E_0 \to E \xrightarrow[(f,g)\mapsto f]{} \text{TOP}^b(M \times \mathbb{R})$$

is a fibration (*after materialization*, cf. *end of Section* 0).

Proof of (i). If $z \in \mathbb{R}$, let $\text{tr}_z\colon M \times \mathbb{R} \to M \times \mathbb{R}$ be the translation by z. The map

$$E \to E; \quad (f, g) \mapsto ((\text{tr}_{-z} \cdot fg^{-1} \cdot \text{tr}_z) \cdot g, g)$$

is the identity if $z = 0$ and becomes $(f, g) \mapsto (g, g)$ as z tends to $+\infty$, since f and g agree on $M \times \{z\}$ for large z. Therefore, E can be deformed into the subspace E' consisting of all (f, g) with $f = g$. But E' is contractible, as is shown by the deformation

$$E' \times [0, +\infty] \to E'; \quad ((g, g), z) \mapsto (\text{tr}_z \cdot g \cdot \text{tr}_{-z}, \text{tr}_z \cdot g \cdot \text{tr}_{-z}).$$

(Remember that $g_{|M \times]-\infty, 0]} = \text{identity}$.)

Proof of (ii). Using the materialization functor, we regard the map $E \to \text{TOP}^b(M \times \mathbb{R})$ as one of simplicial groups. Our task is to show that it is a Kan fibration of simplicial groups, which amounts to saying that it maps onto the identity component of $\text{TOP}^b(M \times \mathbb{R})$. But the identity component of any simplicial group is generated by the simplices whose zeroth vertex is the base point.

Suppose then that $\{f_t\colon M \times \mathbb{R} \to M \times \mathbb{R} \mid t \in \Delta^n\}$ is a typical n-simplex in $\text{TOP}^b(M \times \mathbb{R})$; let $b \in \Delta^n$ be the zeroth vertex, and assume that $f_b = \text{id}$. Apply the isotopy extension theorem 1.1 with $X = M \times \mathbb{R}$, $C = C_{-1} \cup C_z$ the union of two small closed tubular neighbourhoods about $M \times \{-1\}$ and $M \times \{z\}$, for some (large) real number z; and $V = V_{-1} \cup V_z$ the union of two slightly larger open tubular neighbourhoods about $M \times \{-1\}$ and $M \times \{z\}$. Specify the embeddings j_t by

$$j_{t|V_{-1}} = \text{inclusion}, \qquad j_{t|V_z} = f_{t|V_z}.$$

(They are indeed embeddings because z is considerably larger than the uniform bound on $\{f_t\}$.) Now 1.1 yields a family of (possibly unbounded) homeomorphisms

$$J_t\colon M \times \mathbb{R} \to M \times \mathbb{R} \quad (t \in \Delta^n;\ J_b = \text{identity})$$

restricting to the identity on $\partial M \times \mathbb{R}$ and equal to j_t on C, and we let

$$g_t = \left\{\begin{array}{ll} J_t & \text{on } M \times [-1, z] \\ f_t & \text{on } M \times [z, +\infty[\\ \text{identity} & \text{on } M \times]-\infty, -1] \end{array}\right\} \text{ for } t \in \Delta^n.$$

Then $\{(f_t, g_t) \mid t \in \Delta^n\}$ is an n-simplex in E which lifts $\{f_t \mid t \in \Delta^n\}$. This proves 1.6.

Completion of the proof of 1.5: From 1.6 we get that

$$\mathrm{TOP}(M \times D^1) \xhookrightarrow{\simeq} E_0 \simeq \Omega\mathrm{TOP}^b(M \times \mathbb{R}),$$

but it is not clear that this homotopy equivalence agrees with the map alex of 1.5. To prove this we need the missing arrow in a commutative diagram

$$\begin{tikzcd}
\mathrm{TOP}(M \times D^1) \arrow[r, hook] \arrow[d, hook] & E_0 \arrow[d, hook] \\
\text{Cone on } \mathrm{TOP}(M \times D^1) \arrow[r, dashed] \arrow[d] & E \arrow[d] \\
\Sigma\mathrm{TOP}(M \times D^1) \arrow[r, "\mathrm{alex}"] & \mathrm{TOP}^b(M \times \mathbb{R}).
\end{tikzcd}$$

Write the cone on $\mathrm{TOP}(M \times D^1)$ as $\mathrm{TOP}(M \times D^1) \wedge [-\infty, +\infty]$, where $-\infty$ serves as the base point of $[-\infty, +\infty]$. Recall the definition of E as a space of certain pairs, and define the missing arrow by

$$f \wedge z \mapsto \begin{cases} (\mathrm{tr}_{-z} \cdot \hat{f} \cdot \mathrm{tr}_z, \hat{f}) & \text{if } z \geqslant 0, \\ (\mathrm{tr}_{-z} \cdot \hat{f} \cdot \mathrm{tr}_z, \mathrm{tr}_{-z} \cdot \hat{f} \cdot \mathrm{tr}_z) & \text{if } z \leqslant 0. \end{cases}$$

Here $f\colon M \times D^1 \to M \times D^1$ is a homeomorphism, z is a real number (or $+\infty$, $-\infty$), and $\hat{f}$ is obtained from f by extending trivially outside $M \times D^1 \subset M \times \mathbb{R}$. The proof of 1.5 is finished.

There is a slight refinement of 1.5, as follows. For simplicity take $n = 0$ in 1.5. Observe that $\Omega^k\mathrm{TOP}(M)$ is contained in $\mathrm{TOP}(M \times D^k)$ as the subgroup consisting of all homeomorphisms $M \times D^k \to M \times D^k$ preserving the projection to D^k. Also $\Omega^k\mathrm{TOP}(M) \subset \Omega^k\mathrm{TOP}^b(M \times \mathbb{R}^k)$ because $\mathrm{TOP}(M) \subset \mathrm{TOP}^b(M \times \mathbb{R}^k)$.

1.7. PROPOSITION. *There is a weak homotopy equivalence*

$$\mathrm{alex}\colon \mathrm{TOP}(M \times D^k)/\Omega^k\mathrm{TOP}(M) \to \Omega^k\mathrm{TOP}^b(M \times \mathbb{R}^k)/\Omega^k\mathrm{TOP}(M)_{\mathrm{fat}},$$

where $\mathrm{TOP}(M)_{\mathrm{fat}} \subset \mathrm{TOP}^b(M \times \mathbb{R}^k)$ *is a subgroup containing* $\mathrm{TOP}(M)$ *as a deformation retract.*

Proof. We work with virtual spaces again. Note that the map in Proposition 1.5 is a group homomorphism (always use the multiplication on $\Omega^k\mathrm{TOP}^b(M \times \mathbb{R}^k)$ induced from the multiplication on $\mathrm{TOP}^b(M \times \mathbb{R}^k)$). It sends $\Omega^k\mathrm{TOP}(M) \subset \mathrm{TOP}(M \times D^k)$ to the subgroup $\Omega^k\mathrm{TOP}(M)_{\mathrm{fat}} \subset \Omega^k\mathrm{TOP}^b(M \times \mathbb{R}^k)$, where $\mathrm{TOP}(M)_{\mathrm{fat}}$ consists of all homeomorphisms in $\mathrm{TOP}^b(M \times \mathbb{R}^k)$ preserving the projection to $\mathbb{R}^k$. The deformation retraction of $\mathrm{TOP}(M)_{\mathrm{fat}}$ into $\mathrm{TOP}(M)$ is clear (use an Alexander trick), and the composition

$$\Omega^k\mathrm{TOP}(M) \xrightarrow[\text{map of 1.5}]{} \Omega^k\mathrm{TOP}(M)_{\mathrm{fat}} \simeq \Omega^k\mathrm{TOP}(M)$$

is homotopic to the identity.

Define the bounded concordance space $\mathscr{C}^b(M \times \mathbb{R}^n)$ to be the virtual space of all bounded homeomorphisms $f: M \times \mathbb{R}^n \times D^1 \to M \times \mathbb{R}^n \times D^1$ which are the identity on $M \times \mathbb{R}^n \times \{-1\} \cup \partial(M \times \mathbb{R}^n) \times D^1$. If f is such a homeomorphism, i.e. a bounded concordance, let $\partial f: M \times \mathbb{R}^n \to M \times \mathbb{R}^n$ be the restriction of f to $M \times \mathbb{R}^n \times \{+1\} \cong M \times \mathbb{R}^n$.

1.8. PROPOSITION. *There is a weak homotopy equivalence*

$$\text{alex}: \mathscr{C}^b(M \times \mathbb{R}^n) \to \Omega(\mathrm{TOP}^b(M \times \mathbb{R}^{n+1})/\mathrm{TOP}^b(M \times \mathbb{R}^n)).$$

Proof. Given a bounded concordance $f: (M \times \mathbb{R}^n) \times D^1 \to (M \times \mathbb{R}^n) \times D^1$, define a bounded homeomorphism $\hat{f}: (M \times \mathbb{R}^n) \times \mathbb{R} \to (M \times \mathbb{R}^n) \times \mathbb{R}$ by the rule

$$\begin{aligned} \hat{f} &= f && \text{on } (M \times \mathbb{R}^n) \times D^1 \\ \hat{f} &= \mathrm{id} && \text{on } (M \times \mathbb{R}^n) \times]-\infty, -1] \\ \hat{f} &= \partial f \times \mathrm{id} && \text{on } (M \times \mathbb{R}^n) \times [+1, +\infty[. \end{aligned}$$

Then the formula $z \mapsto \mathrm{tr}_{-z} \cdot \hat{f} \cdot \mathrm{tr}_z$ defines a map from $\mathbb{R} \cup \{-\infty, +\infty\}$ to $\mathrm{TOP}^b(M \times \mathbb{R}^{n+1})$. Here tr_z is translation by z, acting on the last factor of $(M \times \mathbb{R}^n) \times \mathbb{R}$. If $z = -\infty$, then $\mathrm{tr}_{-z} \cdot \hat{f} \cdot \mathrm{tr}_z$ is the identity; if $z = +\infty$, then

$$\mathrm{tr}_{-z} \cdot \hat{f} \cdot \mathrm{tr}_z = \partial f \times \mathrm{id}: (M \times \mathbb{R}^n) \times \mathbb{R} \to (M \times \mathbb{R}^n) \times \mathbb{R}.$$

Therefore $\{\mathrm{tr}_{-z} \cdot \hat{f} \cdot \mathrm{tr}_z | z \in [-\infty, +\infty]\}$ defines a loop in

$$\mathrm{TOP}^b(M \times \mathbb{R}^{n+1})/\mathrm{TOP}^b(M \times \mathbb{R}^n),$$

which we call alex(f). This defines the map.

To prove that it is a homotopy equivalence, let Y be the homotopy fibre of the inclusion $\mathrm{TOP}^b(M \times \mathbb{R}^n) \hookrightarrow \mathrm{TOP}^b(M \times \mathbb{R}^{n+1})$. This is conveniently defined as a virtual space. The projection

$$Y \to \Omega(\mathrm{TOP}^b(M \times \mathbb{R}^{n+1})/\mathrm{TOP}^b(M \times \mathbb{R}^n))$$

is a weak homotopy equivalence (see Section 6). Also, the map which we just defined factors as

$$\mathscr{C}^b(M \times \mathbb{R}^n) \to Y \xrightarrow{\simeq} \Omega(\mathrm{TOP}^b(M \times \mathbb{R}^{n+1})/\mathrm{TOP}^b(M \times \mathbb{R}^n))$$

because for any bounded concordance f we can regard $\{\mathrm{tr}_{-z} \cdot \hat{f} \cdot \mathrm{tr}_z \mid z \in [-\infty, +\infty]\}$ as a typical point in Y. There is a strictly commutative diagram

$$\begin{tikzcd} \mathrm{TOP}^b(M \times \mathbb{R}^n \times D^1) \arrow[r, hook] \arrow[d, "\simeq"', "\text{alex}"] & \mathscr{C}^b(M \times \mathbb{R}^n) \arrow[r, "\partial"] \arrow[d] & \mathrm{TOP}^b(M \times \mathbb{R}^n) \arrow[d, "="] \\ \Omega\mathrm{TOP}^b(M \times \mathbb{R}^{n+1}) \arrow[r, hook] & Y \arrow[r] & \mathrm{TOP}^b(M \times \mathbb{R}^n) \end{tikzcd}$$

in which the rows are fibrations up to homotopy (by 1.2). Therefore, the arrow in the middle is a homotopy equivalence.

1.9. REMARK. There is a standard involution on $\mathscr{C}^b(M \times \mathbb{R}^n)$ which consists in turning a concordance f upside down and composing with $(\partial f \times D^1)^{-1}$. Define an involution on $\Omega(\mathrm{TOP}^b(M \times \mathbb{R}^n \times \mathbb{R})/\mathrm{TOP}^b(M \times \mathbb{R}^n))$ by conjugating with the flip -id: $\mathbb{R} \to \mathbb{R}$ on the last factor of $M \times \mathbb{R}^n \times \mathbb{R}$, and reversing loops. Then the map in Proposition 1.8 commutes with involutions.

1.10. PROPOSITION. *There is a weak homotopy equivalence*

$$\mathrm{alex}\colon \mathscr{C}^b(M \times D^k \times \mathbb{R}^n) \to \Omega^k \mathscr{C}^b(M \times \mathbb{R}^{k+n}).$$

Proof. Let $X = M \times D^1$, and $\partial_0 X = M \times \{+1\} \subset \partial X$. Then $\mathscr{C}^b(M \times D^k \times \mathbb{R}^n) = \mathrm{TOP}^b(X \times D^k \times \mathbb{R}^n, \partial_0 X \times D^k \times \mathbb{R}^n)$ and $\mathscr{C}^b(M \times \mathbb{R}^{k+n}) = \mathrm{TOP}^b(X \times \mathbb{R}^{k+n}, \partial_0 X \times \mathbb{R}^{k+n})$; see 1.4 for notation in relative cases. Therefore 1.10 is a special case of a relative version of 1.5, whose proof is similar to that of the absolute version. Note: the map in 1.10 is obtained in the usual way, by embedding D^k in $\mathbb{R}^k$ and pushing off towards infinity in all possible directions.

So far we have not discussed stabilization maps between concordance spaces. The stabilization map $\mathscr{C}(M) \to \mathscr{C}(M \times D^k)$ is defined so as to fit into a homotopy commutative diagram

$$\begin{tikzcd}
\mathscr{C}(M) \arrow[r, "\simeq"] \arrow[dd, dashed] & \mathrm{TOP}(M \times D^1, M \times \{+1\}) \arrow[d, "\times D^k"] \\
& \mathrm{TOP}(M \times D^k \times D^1, M \times D^k \times \{+1\} \bigcup M \times \partial D^k \times D^1) \arrow[d, "\simeq"] \\
\mathscr{C}(M \times D^k) \arrow[r, "\simeq"] & \mathrm{TOP}(M \times D^k \times D^1, M \times D^k \times \{+1\}).
\end{tikzcd}$$

(Note that $M \times \partial D^k \times D^1$ is just a collar attached to $M \times D^k \times \{+1\}$.)

An explicit description is as follows. Choose an embedding $e\colon D^k \times D^1 \to D^k \times D^1$ as in Figure 1. Given a concordance of M, say $f\colon M \times D^1 \to M \times D^1$, take products with

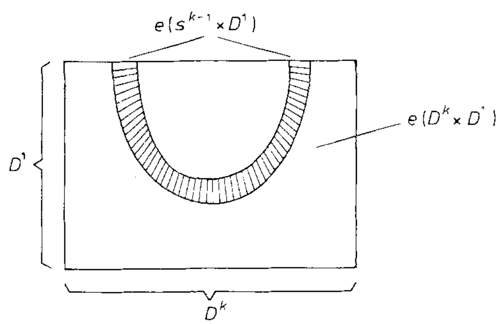

Fig. 1.

D^k to obtain a homeomorphism $M \times e(D^k \times D^1) \to M \times e(D^k \times D^1)$. Extend this over all of $M \times D^k \times D^1$ in the evident way to obtain a concordance of $M \times D^k$.

There are similar stabilization maps $\mathscr{C}^b(M \times \mathbb{R}^n) \to \mathscr{C}^b(M \times D^k \times \mathbb{R}^n)$. In 1.11 and 1.12 below, we combine these with the deloopings given by 1.10 and 1.8 to construct a spectrum $\Omega\underline{\underline{\mathrm{Wh}}}(M)$.

1.11. DEFINITIONS. Let $\mathscr{J}$ be the category of finite-dimensional real Hilbert spaces; a morphism from V to W will be a linear map $V \to W$ preserving the scalar product. If V is in $\mathscr{J}$, we let V^c be the one-point compactification of V; it is a pointed space with base point ∞.

Write $F(V) = \mathrm{TOP}^b(M \times (V \oplus \mathbb{R}))/\mathrm{TOP}^b(M \times V)$. If $V_1 \to V_2$ is a morphism in the category $\mathscr{J}$, write $V_2 = V_1 \oplus V_1^\perp$ and define an induced map $F(V_1) \to F(V_2)$ by taking the product with the identity on V_1. This makes F into a functor.

Suppose that V and W are objects of $\mathscr{J}$. For any $z \in V$ let $r_z\colon V \oplus W \oplus \mathbb{R} \to V \oplus W \oplus \mathbb{R}$ be the unique rotation which sends $(0, 0, 1) \in V \oplus W \oplus \mathbb{R}$ to a positive scalar multiple of $(z, 0, 1)$ and which restricts to the identity on the orthogonal complement of

$$\{(az, 0, b) \mid a, b \in \mathbb{R}\} \subset V \oplus W \oplus \mathbb{R}.$$

Define a continuous map

$$\sigma\colon V^c \wedge F(W) \to F(V \oplus W)$$

by

$$\sigma(z, f) = r_{-z}\cdot(\mathrm{id}_V \times f)\cdot r_z,$$

$$\sigma(\infty, f) = \text{base point},$$

where f is a point in $\mathrm{TOP}^b(M \times (W \oplus \mathbb{R}))$ (see 0.6.). We regard σ as a natural transformation between functors in two variables V and W.

(*Proof of continuity of* σ: Any doubts about continuity must be due to the exceptional role played by the point ∞ in V^c. There is another formula for σ in which $\infty \in V^c$ no longer appears exceptional, but $0 \in V^c$ does; the formula is

$$\sigma(z, f) = (r_{-z}\cdot(1_V \times f)\cdot r_z)(r_{-\infty z}\cdot(1_V \times f)\cdot r_{\infty z})^{-1},$$

with $r_{\infty z} = \lim_{a \to \infty} r_{az}$ $(a > 0)$ and f as before.)

The functor F and the binatural transformation σ form what is called a coordinate free spectrum; see Section 2. For the moment it is sufficient to observe that the spaces $F(\mathbb{R}^0)$, $F(\mathbb{R}^1)$, $F(\mathbb{R}^2), \ldots$ and the maps

$$\sigma\colon \Sigma F(\mathbb{R}^n) \cong (\mathbb{R}^1)^c \wedge F(\mathbb{R}^n) \to F(\mathbb{R}^{n+1})$$

constitute a spectrum in the usual sense. Call it $\Omega\underline{\underline{\mathrm{Wh}}}(M)$.

1.12. LEMMA. *The diagram*

$$\begin{tikzcd}
\mathscr{C}^b(M \times \mathbb{R}^n) \arrow[r, "\text{stabilization}"] \arrow[dd, "\simeq"', "1.8"] & \mathscr{C}^b(M \times D^k \times \mathbb{R}^n) \arrow[d, "\simeq"', "1.10"] \\
 & \Omega^k\mathscr{C}^b(M \times \mathbb{R}^{n+k}) \arrow[d, "\simeq"', "1.8"] \\
\Omega F(\mathbb{R}^n) \arrow[r, "\sigma"] & \Omega(\Omega^k F(\mathbb{R}^{n+k}))
\end{tikzcd}$$

commutes up to a preferred homotopy.

Proof. This is a consequence of two observations. For the first, choose $\varepsilon > 0$ and let $\mathrm{TOP}^{b=\varepsilon}(M \times \mathbb{R}^{k+1})$ consist of all bounded homeomorphisms in $\mathrm{TOP}^b(M \times \mathbb{R}^{k+1})$ with bound $\leqslant \varepsilon$ (see the proof of 1.2). Let $10\varepsilon \cdot D^{k+1} \subset \mathbb{R}^{k+1}$ be the disk of radius 10ε about the origin, and let $\mathrm{EMB}^{b=\varepsilon}(M \times 10\varepsilon \cdot D^{k+1}, M \times \mathbb{R}^{k+1})$ be the space of embeddings $j: M \times 10\varepsilon \cdot D^{k+1} \to M \times \mathbb{R}^{k+1}$ with bound $\leqslant \varepsilon$ (meaning that j is ε-close to the inclusion, the distance being measured in the $\mathbb{R}^{k+1}$-direction only).

First observation: The restriction map

$$\mathrm{res}: \mathrm{TOP}^{b=\varepsilon}(M \times \mathbb{R}^{k+1}) \to \mathrm{EMB}^{b=\varepsilon}(M \times 10\varepsilon \cdot D^{k+1}, M \times \mathbb{R}^{k+1})$$

has a homotopy left inverse q, so that $q \cdot \mathrm{res} \simeq \mathrm{id}$.

(*Proof.* Assume $\varepsilon = 1$. Inspection shows that the map $w: \mathrm{TOP}^{b=1}(M \times \mathbb{R}^{k+1}) \to \mathrm{TOP}^b(M \times \mathbb{R}^{k+1}; \mathbb{Z}^{k+1})$ from the proof of 1.2, factors through the space $\mathrm{EMB}^{b=1}(M \times 10 \cdot D^{k+1}, M \times \mathbb{R}^{k+1})$. But w has a homotopy left inverse by 1.2.)

For the second observation, let $K \subset \mathbb{R}^{k+1} = \mathbb{R}^k \oplus \mathbb{R}$ be a closed smooth connected codimension one submanifold without boundary. Suppose that there exists a compact set $C \subset K$ such that for all $x \in K - C$ the tangent space $\tau(x)$ of K at x contains the vertical axis $0 \oplus \mathbb{R} \subset \mathbb{R}^k \oplus \mathbb{R}$. (Always regard $\tau(x)$ as a linear subspace of $\mathbb{R}^{k+1}$.) Then $\mathbb{R}^{k+1} - K$ has two components, one of which has bounded image under the projection $\mathbb{R}^k \oplus \mathbb{R} \to \mathbb{R}^k$; call this the interior component.

Such a K gives rise to two maps g_1, g_2 from $\mathscr{C}(M)$ to the (virtual) space of maps of pairs

$$\mathrm{map}((D^k \times D^1, \partial(D^k \times D^1)), (\mathrm{TOP}^b(M \times \mathbb{R}^{k+1}), \mathrm{TOP}^b(M \times \mathbb{R}^k))),$$

as follows. For $x \in K$, let $n(x)$ be the inward normal vector of K at x, of length $\varepsilon/2$, where ε is very small. Identity $K \times D^1$ with a subset of $\mathbb{R}^{k+1}$ by the rule $(x, v) \mapsto x + v \cdot n(x)$ for $x \in K$ and $v \in D^1$. Given a point in $\mathscr{C}(M)$, say $f: M \times D^1 \to M \times D^1$, we now define a bounded homeomorphism $\bar{f}: M \times \mathbb{R}^{k+1} \to M \times \mathbb{R}^{k+1}$ in the expected way. Namely, $\bar{f}$ agrees with $\mathrm{id}_K \times f$ on $K \times (M \times D^1) \cong M \times (K \times D^1) \subset M \times \mathbb{R}^{k+1}$; it agrees with the identity on $M \times (\text{ext. comp. of } \mathbb{R}^{k+1} - (K + D^1))$, and with $\partial f \times \mathrm{id}$ on $M \times (\text{int. comp. of } \mathbb{R}^{k+1} - (K \times D^1))$. Then the map

$$\mathbb{R}^{k+1} \to \mathrm{TOP}^b(M \times \mathbb{R}^{k+1}); \qquad z \to \mathrm{tr}_{-z} \cdot \bar{f} \cdot \mathrm{tr}_z,$$

where tr denotes translations, extends to a map of pairs

$$(D^k \times D^1, \partial(D^k \times D^1)) \to (\mathrm{TOP}^b(M \times \mathbb{R}^{k+1}), \mathrm{TOP}^b(M \times \mathbb{R}^k))$$

provided we regard $D^k \times D^1$ as a compactification of $\mathbb{R}^k \times \mathbb{R}^1 \cong \mathbb{R}^{k+1}$ in the evident way. Call this extension $g_1(f)$.

Continuing with the same f, let $\hat{f}$: $M \times \mathbb{R} \to M \times \mathbb{R}$ be equal to f on $M \times D^1$, equal to the identity on $M \times]-\infty, -1]$ and equal to $\partial f \times \mathrm{id}$ on $M \times [+1, +\infty[$. As in the proof of 1.8, define a continuous map

$$\alpha_f: \mathbb{R} \cup \{-\infty, +\infty\} \to \mathrm{TOP}^b(M \times \mathbb{R})$$

by $z \mapsto \mathrm{tr}_{-z} \cdot \hat{f} \cdot \mathrm{tr}_z$. Let h: $D^1 \to \mathbb{R} \cup \{-\infty, +\infty\}$ be an orientation-preserving homeomorphism. There is a unique continuous map from $\mathbb{R}^{k+1}$ to $\mathrm{TOP}^b(M \times \mathbb{R}^{k+1})$ which is constant on the exterior component of $\mathbb{R}^{k+1} - (K \times D^1)$, constant on the interior component of $\mathbb{R}^{k+1} - (K \times D^1)$, and which sends $(x, v) \in K \times D^1 \subset \mathbb{R}^{k+1}$ to the bounded automorphism $\alpha_f(h(v)) \times \mathrm{id}_{\tau(x)}$ of

$$(M \times \mathbb{R}) \times \tau(x) \cong M \times (\mathbb{R} \times \tau(x)) \cong M \times \mathbb{R}^{k+1}.$$

(For $x \in K$, identify $\mathbb{R} \times \tau(x)$ with $\mathbb{R}^{k+1}$ by the rule $(r, t) \mapsto t + r \cdot n(x)$, where $r \in \mathbb{R}$ and $t \in \tau(x)$.) Again, this map from $\mathbb{R}^{k+1}$ to $\mathrm{TOP}^b(M \times \mathbb{R}^{k+1})$ extends to a map of pairs

$$(D^k \times D^1, \partial(D^k \times D^1)) \to (\mathrm{TOP}^b(M \times \mathbb{R}^{k+1}), \mathrm{TOP}^b(M \times \mathbb{R}^k))$$

where we regard $D^k \times D^1$ as a compactification of $\mathbb{R}^{k+1}$. Call it $g_2(f)$.

Second observation: The maps g_1 and g_2 are homotopic.

(*Proof.* Note that all bounded homeomorphisms in sight have bound $\leqslant \varepsilon$. Use the first observation to replace spaces of ε-bounded homeomorphisms by spaces of ε-bounded embeddings throughout. Since ε is the width of a tubular neighbourhood of K, it can be taken arbitrarily small. Letting it tend to zero, the homotopy from g_1 to g_2, or rather from $\mathrm{res} \cdot g_1$ to $\mathrm{res} \cdot g_2$, becomes obvious.)

In the application to 1.12, let K be the boundary of a smooth contractible codimension zero submanifold $W \subset \mathbb{R}^{k+1}$ such that

$$D^k \times [+1, +\infty[\subset W \subset D^k \times [-1, +\infty[\subset \mathbb{R}^k \times \mathbb{R} \cong \mathbb{R}^{k+1}.$$

Interpret g_1 and g_2 as maps with target $\Omega^{k+1}(\mathrm{TOP}^b(M \times \mathbb{R}^{k+1})/\mathrm{TOP}^b(M \times \mathbb{R}^k))$. Then g_1, g_2 are essentially the maps which 1.12 asserts to be homotopic, so long as $n = 0$ in 1.12. For $n > 0$ the proof is similar; the idea is to absorb the factor $\mathbb{R}^n$ in the symbol M.

In the corollary below, $Q(E)$ denotes the zeroth infinite loop space associated to a spectrum E, and $\Sigma^n E$ is the n-fold suspension of E.

1.13. COROLLARY. *There are homotopy equivalences*

$$Q(\Sigma^n \Omega \underline{\underline{\mathrm{Wh}}}(M)) \simeq \lim_{k \to \infty} \mathscr{C}^b(M \times D^k \times \mathbb{R}^{n+1})$$

for $n \geqslant -1$, the limit being taken with respect to stabilization. In particular, the loop *space of* $Q(\Omega\underline{\underline{\mathrm{Wh}}}(M))$ *is homotopy equivalent to* $\lim_{k\to\infty}\mathscr{C}(M\times D^k)$. (The limit should be interpreted as one of virtual spaces, cf. 0.5. (vi), or as one of simplicial sets; it is also the homotopy limit.)

The next topic to be discussed is Theorem B. Recall from the introduction that $\widetilde{\mathrm{TOP}}^b(M\times\mathbb{R}^n)$ is the simplicial set whose k-simplices are the bounded homeomorphisms $\Delta^k\times M\times\mathbb{R}^n\to\Delta^k\times M\times\mathbb{R}^n$ which preserve the blocks $d_i\Delta^k\times M\times\mathbb{R}^n$ for $0\leqslant i\leqslant k$. This is truly a simplicial set and not a virtual space; but even the fact that it is a simplicial set requires proof, because the degeneracy operators are not obvious.

Let Δ be the category with objects $[n]=\{0,1,\ldots,n\}$ for $n\geqslant 0$, and with monotone maps as morphisms, so that simplicial sets are contravariant functors from Δ to the category of sets. Suppose that $p\colon[k]\to[j]$ is an epimorphism in Δ. This induces a linear surjection $p_*\colon\Delta^k\to\Delta^j$ sending vertices to vertices. Let $V(p)$ be the space of linear maps $i\colon\Delta^j\to\Delta^k$ such that $p_*\cdot i=\mathrm{id}\colon\Delta^j\to\Delta^k$. These maps i are not required to send vertices to vertices, but they are determined by their effect on the vertices of Δ^j; therefore

$$V(p)\cong\prod_{s\in[j]}(p_*)^{-1}(\{s\}).$$

The evaluation $V(p)\times\Delta^j\to\Delta^k$; $(i,z)\to i(z)$ is onto. Now if y is a j-simplex in $\widetilde{\mathrm{TOP}}^b(M\times\mathbb{R}^n)$, then there is a unique k-simplex $p^*(y)$ in $\widetilde{\mathrm{TOP}}^b(M\times\mathbb{R}^n)$ making the following square commutative:

$$\begin{array}{ccc}
V(p)\times(\Delta^j\times M\times\mathbb{R}^n) & \twoheadrightarrow & \Delta^k\times M\times\mathbb{R}^n.\\
\Big\downarrow{\scriptstyle \mathrm{id}\times y} & & \Big\downarrow{\scriptstyle p^*(y)}\\
V(p)\times(\Delta^j\times M\times\mathbb{R}^n) & \twoheadrightarrow & \Delta^k\times M\times\mathbb{R}^n
\end{array}$$

This defines the degeneracy operators in $\widetilde{\mathrm{TOP}}^b(M\times\mathbb{R}^n)$.

Interpret $\mathrm{TOP}^b(M\times\mathbb{R}^n)$ as a simplicial set using the materialization functor; then there is an inclusion

$$\mathrm{TOP}^b(M\times\mathbb{R}^n)\hookrightarrow\widetilde{\mathrm{TOP}}^b(M\times\mathbb{R}^n).$$

Write $\mathrm{TOP}^b(M\times\mathbb{R}^\infty)$ for the simplicial set $\bigcup\mathrm{TOP}^b(M\times\mathbb{R}^n)$; similarly $\widetilde{\mathrm{TOP}}^b(M\times\mathbb{R}^\infty)=\bigcup\widetilde{\mathrm{TOP}}^b(M\times\mathbb{R}^n)$ (see the introduction).

1.14. 'THEOREM B'. *The inclusion of simplicial sets*

$$\mathrm{TOP}^b(M\times\mathbb{R}^\infty)\hookrightarrow\widetilde{\mathrm{TOP}}^b(M\times\mathbb{R}^\infty)$$

is a homotopy equivalence. (*Therefore so is the inclusion*

$$\mathrm{TOP}^b(M\times\mathbb{R}^\infty)/\mathrm{TOP}(M)\hookrightarrow\widetilde{\mathrm{TOP}}^b(M\times\mathbb{R}^\infty)/\mathrm{TOP}(M).)$$

Proof. We will show that the inclusion induces an isomorphism on π_k for all $k\geqslant 0$. Fix k. Write $\tilde{X}=\mathrm{TOP}^b(D^k\times M\times\mathbb{R}^\infty)$; regard this as a virtual space, preferably. Let $X\subset\tilde{X}$ consist of all bounded homeomorphisms in $\tilde{X}$ preserving the projection to D^k.

Clearly $X \simeq \Omega^k\mathrm{TOP}^b(M \times \mathbb{R}^\infty)$. There is a commutative square

$$\begin{tikzcd}
\pi_0(X) \arrow[r, "(3)"] \arrow[d, "(1)"'] & \pi_0(\tilde{X}) \arrow[d, "(2)"] \\
\pi_k(\mathrm{TOP}^b(M \times \mathbb{R}^\infty)) \arrow[r, "(4)"] & \pi_k(\widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^\infty))
\end{tikzcd}$$

with horizontal arrows induced by inclusion and vertical arrows defined ad hoc, but still obvious. Clearly, (1) is an isomorphism, since $X \simeq \Omega^k\mathrm{TOP}^b(M \times \mathbb{R}^\infty)$; clearly, (2) is onto. We will see in a moment that (3) is an isomorphism which forces (4) to be onto.

By 1.5, there is a homotopy equivalence

$$\tilde{X} = \mathrm{TOP}^b(D^k \times M \times \mathbb{R}^\infty) \xrightarrow{\simeq} \Omega^k\mathrm{TOP}^b(M \times \mathbb{R}^{\infty+k}),$$

so that the inclusion $X \hookrightarrow \tilde{X}$ corresponds to the inclusion

$$\Omega^k\mathrm{TOP}^b(M \times \mathbb{R}^\infty) \hookrightarrow \Omega^k\mathrm{TOP}^b(M \times \mathbb{R}^{\infty+k})$$

(see also 1.7). Therefore (3) is an isomorphism and (4) is onto. Injectivity of the homomorphism (4) can be proved by a relative version of the argument which proves surjectivity. We leave this to the reader.

1.15. REMARK. The homomorphism

$$\pi_k(\widetilde{\mathrm{TOP}}(M)) \to \pi_k(\widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^\infty)) \cong \pi_k(\mathrm{TOP}^b(M \times \mathbb{R}^\infty))$$

induced by the inclusion can be factorized as follows:

$$\begin{tikzcd}
 & \mathrm{im}[\pi_k(\mathrm{TOP}^b(M \times \mathbb{R}^k)) \to \pi_k(\mathrm{TOP}^b(M \times \mathbb{R}^{k+1}))] \arrow[d] \\
\pi_k(\widetilde{\mathrm{TOP}}(M)) \arrow[ur, dashed, "\cong"] \arrow[r] & \pi_k(\mathrm{TOP}^b(M \times \mathbb{R}^\infty)).
\end{tikzcd}$$

To define the lift, represent an element in $\pi_k(\widetilde{\mathrm{TOP}}(M))$ by a homeomorphism $\Delta^k \times M \to \Delta^k \times M$ which is the identity on $\partial(\Delta^k \times M)$. This determines an element in $\pi_0\mathrm{TOP}(\Delta^k \times M)) \cong \pi_0(\mathrm{TOP}(D^k \times M))$. Now use 1.5 to go from $\pi_0(\mathrm{TOP}(D^k \times M))$ to $\pi_k(\mathrm{TOP}^b(M \times \mathbb{R}^k))$. Checking that the dotted arrow is an isomorphism is straightforward.

Using 1.7 instead of 1.5, one obtains a relative version in which all simplicial groups in the diagram are divided by their common subgroup $\mathrm{TOP}(M)$.

1.16. REMARK. There is a well known relationship between bounded homeomorphisms/concordances and lower algebraic K-theory which is described in an appendix (Section 5). It will be used in proving Theorem C, but not in proving Theorem A. It can also be used in giving a quick proof of Theorem A when M is simply connected and $\dim(M) \geqslant 5$.

1.17. REMARK. To do justice to the title of this section we now discuss the smooth versions of 1.1 – 1.16.

The isotopy extension theorem 1.1 has a smooth version in which X is smooth, $\{j_t\}$ is a continuous family of smooth embeddings, and $\{J_t\}$ is a continuous family of diffeomorphisms. ('Continuity' refers to the compact-open or weak topology on the space of smooth maps; see Hirsch [20]. Note that a continuous family of smooth embeddings is not the same as a smooth family of smooth embeddings.) Replacing C in 1.1 by a compact smooth codimension zero submanifold containing C and contained in V, we see that we can take C itself to be a compact smooth codimension zero submanifold of the smooth manifold X. In this case the theorem can be deduced from chap. II, 2.2.1 of Cerf [13]. Cerf's result states, roughly, that the restriction map from the space of suitable diffeomorphisms $X \to X$ to the space of suitable embeddings $C \to X$ is a fibre bundle.

The belt buckle trick 1.2 works in the smooth case just as it does in the topological case. Therefore $\mathrm{DIFF}^b(M \times \mathbb{R}^{j+k+m}; \mathbb{Z}^m)$ is a retract of $\mathrm{DIFF}^b(M \times \mathbb{R}^{j+k+m}; \mathbb{Z}^{k+m})$, up to homotopy, for arbitrary j, k, $m \geqslant 0$. The proof of 1.2 used certain choices of homeomorphisms e_+, e_-, λ; in the smooth version these have to be diffeomorphisms.

The smooth version of 1.4 needs a detailed comment. We defined $\mathrm{DIFF}^b(M \times \mathbb{R}^n)$ in 0.4 as the virtual space of all bounded diffeomorphisms $M \times \mathbb{R}^n \to M \times \mathbb{R}^n$ which agree with the identity on $\partial M \times \mathbb{R}^n$. However, it is often more convenient to let $\mathrm{DIFF}^b(M \times \mathbb{R}^n)$ consist of all bounded diffeomorphisms $M \times \mathbb{R}^n \to M \times \mathbb{R}^n$ which agree with the identity on an infinitesimal neighbourhood of $\partial M \times \mathbb{R}^n$ (which means that the higher derivatives also agree on $\partial M \times \mathbb{R}^n$). This does not affect the homotopy type of $\mathrm{DIFF}^b(M \times \mathbb{R}^n)$, but it ensures, for example, that the proof of 1.5 works as it stands in the smooth case. Similarly, it is often convenient to define the smooth concordance space $\mathscr{C}^b(M \times \mathbb{R}^n)$ to consist of all bounded diffeomorphisms $f\colon (M \times \mathbb{R}^n) \times D^1 \to (M \times \mathbb{R}^n) \times D^1$ which agree with the identity on an infinitesimal neighbourhood of $\partial(M \times \mathbb{R}^n) \times D^1 \cup (M \times \mathbb{R}^n) \times \{-1\}$, and which agree with $\partial f \times \mathrm{id}\colon (M \times \mathbb{R}^n) \times D^1 \to (M \times \mathbb{R}^n) \times D^1$ on an infinitesimal neighbourhood of $(M \times \mathbb{R}^n) \times \{+1\}$. (Recall that ∂f is the restriction of f to $(M \times \mathbb{R}^n) \times \{+1\}$.) Again, it makes no difference to the homotopy type of the smooth concordance space $\mathscr{C}^b(M \times \mathbb{R}^n)$ whether or not we include these technical conditions, but we should in view of 1.8 and 1.10.

With these precautions, statements 1.5 – 1.13 and their proofs are valid in the smooth case. (It is a curious but undeniable fact that the Alexander trick of Anderson and Hsiang works in the topological category and in the smooth category, whereas the original Alexander trick of [1] only works in the topological category, or at best in the PL category. In the perturbation language introduced just before 1.5, we can explain this by saying that shrinking a smooth perturbation fails to shrink its derivatives, but pushing a smooth perturbation away to ∞ certainly takes care of the derivatives also.) In 1.13, note that $\Omega\underline{\underline{\mathrm{Wh}}}(M)$ is an abbreviation for $\Omega\underline{\underline{\mathrm{Wh}}}^{\mathrm{TOP}}(M)$, which must be replaced by $\Omega\underline{\underline{\mathrm{Wh}}}^{\mathrm{DIFF}}(M)$ in the smooth version.

In 1.14, define $\widetilde{\mathrm{DIFF}}^b(M \times \mathbb{R}^\infty)$ as the union of the simplicial sets $\widetilde{\mathrm{DIFF}}^b(M \times \mathbb{R}^n)$; a k-simplex in $\widetilde{\mathrm{DIFF}}^b(M \times \mathbb{R}^n)$ is a diffeomorphism $f\colon \Delta^k \times M \times \mathbb{R}^n \to \Delta^k \times M \times \mathbb{R}^n$

such that

$$f(d_i\Delta^k \times M \times \mathbb{R}^n) \subset d_i\Delta^k \times M \times \mathbb{R}^n$$

whenever $0 \leqslant i \leqslant k$. (The degeneracy operators in $\widetilde{\mathrm{DIFF}}^b(M \times \mathbb{R}^n)$ agree with those in $\widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^n) \supset \widetilde{\mathrm{DIFF}}^b(M \times \mathbb{R}^n)$.) Then $\widetilde{\mathrm{DIFF}}^b(M \times \mathbb{R}^n)$ is a simplicial group and, therefore, it has the Kan extension property. The smooth versions of 1.14 and 1.15 are valid, with the same proofs.

Finally, Remark 1.16 also makes sense in the smooth setting; see Section 5.

We return to the topological category for Sections 2, 3, and 4. Smooth versions exist.

2. Coordinate Free Spectra

In constructing the map

$$\mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M) \to Q(S^\infty_+ \wedge_{Z_2} \Omega\underline{\underline{\mathrm{Wh}}}(M))$$

promised in the introduction, we shall make essential use of the fact that the spectrum $\Omega\underline{\underline{\mathrm{Wh}}}(M)$ of 1.11 has the structure of a coordinate free spectrum in the sense of May [30]. In this section, we give a definition of coordinate free spectra, geared to our needs, and derive a few basic consequences.

We investigate covariant functors F from the category $\mathscr{I}$ defined in 1.11 to a suitable category of spaces – this could be the category of all topological spaces, or (preferably) the category of virtual spaces. To avoid distraction, let us be naive and work with ordinary topological spaces in this section.

A functor F from $\mathscr{I}$ to the category of topological spaces is continuous if, for arbitrary V, W in $\mathscr{I}$, the map

$$\mathrm{Mor}(V, W) \times F(V) \to F(W); \qquad (g, x) \mapsto g_*(x)$$

is continuous. Here $\mathrm{Mor}(V, W)$ is the space of morphisms with the usual topology.

2.1. DEFINITION. A coordinate free spectrum consists of a continuous functor

$$F: \mathscr{I} \to \text{category of pointed topological spaces}$$

and a map

$$\sigma: V^c \wedge F(W) \to F(V \oplus W)$$

natural in both variables V and W, such that the composition

$$F(W) \cong \{0\}^c \wedge F(W) \xrightarrow{\sigma} F(\{0\} \oplus W) \cong F(W)$$

is the identity for all W in $\mathscr{I}$.

We often write F instead of (F, σ). Note that the space $F(\mathbb{R}^0)$, $F(\mathbb{R}^1)$, $F(\mathbb{R}^2), \ldots$ and the suspension maps

$$\Sigma F(\mathbb{R}^n) \cong (\mathbb{R}^1)^c \wedge F(\mathbb{R}^n) \xrightarrow{\sigma} F(\mathbb{R}^{n+1})$$

form a spectrum in the usual sense, with a generous definition of that word. This will also be written F.

Examples of coordinate free spectra are:

$$F(V) = V^c = V^c \wedge S^0 \quad \text{(the sphere spectrum)}$$

or more generally

$$\mathrm{D}(V) = V^c \wedge Y,$$

where Y is a pointed CW-spaces. The maps σ are obvious in both cases.

2.2. PROPOSITION. *Let (F, σ) be a coordinate-free spectrum and let V, W, X be objects of $\mathscr{J}$. Then the following diagram is commutative up to a canonical homotopy:*

$$\begin{tikzcd}
V^c \wedge W^c \wedge F(X) \arrow[r, "\cong"] \arrow[d, "V^c \wedge \sigma"] & (V \oplus W)^c \wedge F(X) \arrow[d, "\sigma"] \\
V^c \wedge F(W \oplus X) \arrow[r, "\sigma"] & F(V \oplus W \oplus X).
\end{tikzcd}$$

Proof. Fix $v \in V, w \in W$. For $t \in [0, 1]$, let $v_t, w_t \in V \oplus W$ be defined by the equations

$$v_t + w_t = v + w, \qquad \langle v_t, w_t \rangle = 0,$$

where $\langle\ ,\ \rangle$ is the inner product;

$$w_t = c_t(w + tv),$$

for suitable c_t in $\mathbb{R}$. So $v_0 = v, w_0 = w$, but $v_1 = 0, w_1 = v + w$. See Figure 2, with $t = \frac{1}{3}$.

Define $f_{v,w,t}$: $F(X) \to F(V \oplus W \oplus X)$ to be the composition

$$\begin{tikzcd}
F(X) \arrow[d, "\cong"] & & F(V \oplus W \oplus X) \\
\{v_t\} \times \{w_t\} \times F(X) \arrow[d, phantom, "\cap"] & & \\
\langle v_t \rangle^c \wedge \langle w_t \rangle^c \wedge F(X) \arrow[d, "\langle v_t \rangle^c \wedge \sigma"] & & \\
\langle v_t \rangle^c \wedge F(\langle w_t \rangle \oplus X) \arrow[rr, "\sigma"] & & F(\langle v_t \rangle \oplus \langle w_t \rangle \oplus X). \arrow[uuu, "\text{inclusion}_*"']
\end{tikzcd}$$

Define f_t: $V^c \wedge W^c \wedge F(X) \to F(V \oplus W \oplus X)$ by

$$f_t(v, w, x) = f_{v,w,t}(x)$$

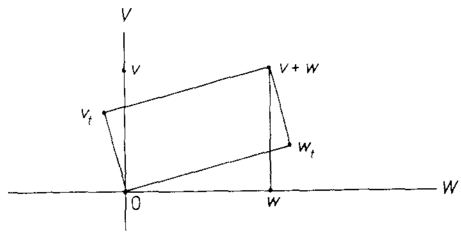

Fig. 2.

in case $v, w \neq \infty$. Then f_0 is equal to the composition

$$V^c \wedge W^c \wedge F(X) \to V^c \wedge F(W \oplus X) \to F(V \oplus W \oplus X)$$

and f_1 is equal to the composition

$$V^c \wedge W^c \wedge F(X) \xrightarrow{\cong} (V \oplus W)^c \wedge F(X) \xrightarrow{\sigma} F(V \oplus W \oplus X)$$

using the last clause of 2.1. So $\{f_t \mid 0 \leqslant t \leqslant 1\}$ is the required homotopy. Continuity is easily established by observing that if one of v, w is large, then one of v_t, w_t must be large for arbitrary $t \in [0, 1]$.

2.3 DEFINITION. An involution on a coordinate free spectrum (F, σ) is a natural transformation $tw: F \to F$ such that $tw \cdot tw =$ identity, and such that the following diagram is commutative for all V, W in $\mathscr{J}$:

$$\begin{array}{ccc} V^c \wedge F(W) & \xrightarrow{\sigma} & F(V \oplus W) \\ \Big\downarrow{\scriptstyle V^c \wedge tw} & & \Big\downarrow{\scriptstyle tw} \\ V^c \wedge F(W) & \xrightarrow{\sigma} & F(V \oplus W). \end{array}$$

For example, if F is the suspension spectrum associated with a CW-space Y, so that $F(V) = V^c \wedge Y$, then any involution on Y determines an involution on F. A more interesting example can be found in the next section.

Now suppose that P^n is a smooth compact manifold with boundary, smoothly embedded in a Euclidean space $\mathbb{R}^N$ for some large N. (Later we shall specialize by letting $P = \mathbb{R}P^n$.) Write τ^P or just τ for its tangent bundle. Note that the tangent space τ_x of P at $x \in P$ inherits an inner product from $\mathbb{R}^N$. If F is a coordinate free spectrum, we can therefore form a fibre bundle $F(\tau)$ over P whose fibre over $x \in P$ is $F(\tau_x)$.

Write P^{col} for the quotient P modulo ∂P. Let $Q(P^{col} \wedge F)$ be the zeroth infinite loop space associated with the spectrum $P^{col} \wedge F$; that is, $Q(P^{col} \wedge F)$ is the homotopy direct limit (= telescope) obtained from the spaces $\Omega^m(P^{col} \wedge F(\mathbb{R}^m))$ by letting m tend to ∞. (We use the compact open topology for loop spaces, and also for the space of continuous sections of $F(\tau)$ which occurs in the next proposition.) From now on the notation $\Gamma(\ldots)$ will be used for the space of sections of the fibre bundle '...'.

2.4. PROPOSITION. *There is a Poincaré duality cum stabilization map* st: $\Gamma(F(\tau)) \to Q(P^{\mathrm{col}} \wedge F)$.

Proof. This is obtained by composing two rather obvious maps. To describe the first, let ν be the normal bundle of P^n in $\mathbb{R}^N$, with Thom space $T(\nu)$. Again, each fibre ν_x of ν is a Hilbert space. Note that $T(\nu)$ is the union, but not the disjoint union, of the one-point compactifications $\nu_x^c = \nu_x \cup \{\infty\}$. Any section s of $F(\tau)$ determines a pointed map $T(\nu) \to F(\mathbb{R}^N)$;

$$y \in \nu_x^c \mapsto y \wedge s(x) \in \nu_x^c \wedge F(\tau_x) \mapsto \sigma(y \wedge s(x)) \in F(\nu_x \oplus \tau_x) \cong F(\mathbb{R}^N).$$

We have therefore constructed a map

(1): $\Gamma(F(\tau)) \to$ space of pointed maps from $T(\nu)$ to $F(\mathbb{R}^N)$.

The other map is a familiar Poincaré duality map. Take a pointed map f: $T(\nu) \to F(\mathbb{R}^N)$. Then the composition

$$S^N \cong \mathbb{R}^N \cup \{\infty\} \xrightarrow{\text{collapse}} T(\nu)^{\mathrm{col}} \xrightarrow{\text{projection} \wedge f} P^{\mathrm{col}} \wedge F(\mathbb{R}^N)$$

is an element in $\Omega^N(P^{\mathrm{col}} \wedge F(\mathbb{R}^N)) \subset Q(P^{\mathrm{col}} \wedge F)$. (We hope the notation $T(\nu)^{\mathrm{col}}$ is self-explanatory.) Therefore, we have constructed a map

(2): (space of pointed maps from $T(\nu)$ to $F(\mathbb{R}^N)) \to Q(P^{\mathrm{col}} \wedge F)$.

Combining (1) and (2) gives the map in 2.4. By 2.2, it is essentially independent of the integer N and the embedding $P \hookrightarrow \mathbb{R}^N$. Note that $P^{\mathrm{col}} = P_+$ if $\partial P = \emptyset$.

Suppose next that $P^n \subset U^m$ are closed smooth manifolds, with U^m embedded in $\mathbb{R}^N$. Then it is reasonable to search for a map $\Gamma(F(\tau^P)) \to \Gamma(F(\tau^U))$ to fit into a commutative diagram

$$\begin{array}{ccc} \Gamma(F(\tau^P)) & \longrightarrow & \Gamma(F(\tau^U)) \\ \downarrow{\scriptstyle st} & & \downarrow{\scriptstyle st} \\ Q(P_+ \wedge F) & \xrightarrow{\text{inclusion}} & Q(U_+ \wedge F). \end{array}$$

Such a map exists, but it requires some preparation. Choose a tubular neighbourhood of P in U, with fibres orthogonal to P.

2.5. NOTATION. Let the orthogonal tubular neighbourhood be given by a vector bundle r: $E \to P$ with zero section i: $P \to E$, and a smooth codimension zero embedding f: $E \to U$ such that $f \cdot i =$ inclusion: $P \to U$.

We will also need an isometric isomorphism α: $f^*(\tau^U) \to (ir)^* f^*(\tau^U)$ of vector bundles over E, restricting to the identity over $i(P) \subset E$. This can be chosen at random, or it can be manufactured using parallel transport in the Riemannian manifold U. In more detail, any point $x \in E$ can be connected with $ir(x)$ by a straight line segment; the image of the segment under f is a path in U along which tangent spaces can be transported.

2.6. PROPOSITION. *Any orthogonal tubular neighbourhood of P in U gives rise to a map $j: \Gamma(F(\tau^P)) \to \Gamma(F(\tau^U))$ making the square*

$$\begin{tikzcd} \Gamma(F(\tau^P)) \arrow[r] \arrow[d] & \Gamma(F(\tau^U)) \arrow[d] \\ Q(P_+ \wedge F) \arrow[r] & Q(U_+ \wedge F) \end{tikzcd}$$

commutative up to a preferred homotopy.

Proof. Let s be a section of $F(\tau^P)$. For $x \in P$ and $z \in E_x$, let $f(z) \in U$ be the image of z under f in 2.5. Define $j(s)$ by

$$j(s)(f(z)) = \text{image of } z \wedge s(x) \text{ under the composition}$$

$$E_x^c \wedge F(\tau_x^P) \xrightarrow{\sigma} F(E_x \oplus \tau_x^P) \cong F(\tau_x^U) \xrightarrow{F(\alpha)^{-1}} F(\tau_{f(z)}^U),$$

where α is the bundle isomorphism in 2.5. If $y \in U$ is not of the form $f(z)$ as above, put $j(s)(y) =$ base point. This defines the map j.

(*Digression*: If F is a coordinate free spectrum of fantasy spaces, then the formula for $j(s)$ does not give a continuous section unless we insist that $f: E \to U$ in 2.5 extend to an embedding $\bar{f}: \bar{E} \to U$ of the fibrewise disk compactification $\bar{E}$ of E, and that α be defined over all of $\bar{E}$. We call such a tubular neighbourhood regular.)

Commutativity of the square in 2.6 is proved by dividing the square into two, as suggested by the proof of 2.4. (Write $\mathrm{map}_*(\ldots)$ for spaces of pointed maps).

$$\begin{tikzcd} \Gamma(F(\tau^P)) \arrow[r, "j"] \arrow[d] & \Gamma(F(\tau^U)) \arrow[d] \\ \mathrm{map}_*(T(\nu^P), F(\mathbb{R}^N)) \arrow[r] \arrow[d] & \mathrm{map}_*(T(\nu^U), F(\mathbb{R}^N)) \arrow[d] \\ Q(P_+ \wedge F) \arrow[r] & Q(U_+ \wedge F). \end{tikzcd}$$

The vertical arrows in this diagram are defined in the proof of 2.4, and the horizontal arrow in the middle is composition with the collapsing map $T(\nu^U) \to T(\nu^P)$. Commutativity is now easy to check.

Now let $T \subset U$ be the compact codimension zero submanifold obtained by deleting the interior of a regular tubular neighbourhood of P in U.

2.7. PROPOSITION. *The diagram*

$$\begin{tikzcd} \Gamma(F(\tau^P)) \arrow[r, "j"] \arrow[d, "\mathrm{st}"] & \Gamma(F(\tau^U)) \arrow[r, "\text{restriction}"] \arrow[d, "\mathrm{st}"] & \Gamma(F(\tau^T)) \arrow[d, "\mathrm{st}"] \\ Q(P_+ \wedge F) \arrow[r, "\text{inclusion}"] & Q(U_+ \wedge F) \arrow[r, "\text{collapse}"] & Q(T^{\mathrm{col}} \wedge F) \end{tikzcd}$$

is commutative up to preferred homotopies.

Comment: Suppose given a diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p"] & B \arrow[r, "g"] \arrow[d, "q"] & C \arrow[d, "r"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \end{tikzcd}$$

of pointed spaces and continuous maps such that $gf = *$ and $g'f' = *$. Suppose we wish to show that it is sufficiently commutative for all practical purposes. Then we need three homotopies. The two obvious ones are

$$\{x_t\}: f'p \simeq qf \quad \text{and} \quad \{y_t\}: g'q \simeq rg, \quad \text{with } 0 \leqslant t \leqslant 1.$$

These two give rise to a homotopy between maps from A to C',

$$* = g'f'p \simeq g'qf \simeq rgf = *,$$

or a map from ΣA to C'. Clearly this map should be equipped with a nullhomotopy $\{z_t\}$.

In proving 2.7, construct the homotopies $\{x_t\}$ and $\{y_t\}$ in such a way that $\{g'x_t\}$ and $\{y_t f\}$ are strictly zero. Then take $\{z_t\}$ to be zero also.

We shall need twisted versions of 2.4, 2.6, and 2.7 which are a little harder to state. In the situation of 2.4, suppose that the smooth manifold $P^n \subset \mathbb{R}^N$ comes equipped with a double covering $g: \tilde{P} \to P$, and suppose that the coordinate-free spectrum comes equipped with an involution $tw: F \to F$. Write τ for the tangent bundle of P, and let $F^{tw}(\tau)$ be the fibre bundle over P whose fibre over $x \in P$ is

$$F^{tw}(\tau_x) = F(\tau_x) \times_{Z_2} g^{-1}(x),$$

where Z_2 acts on $F(\tau_x)$ by tw, and on $g^{-1}(x)$ by permutation.

2.8. PROPOSITION. *There is a stabilization map*

$$\Gamma(F^{tw}(\tau)) \to Q(\tilde{P}^{\mathrm{col}} \wedge_{Z_2} F),$$

with Z_2 acting on $\tilde{P}^{\mathrm{col}}$ by covering translations and on F by tw.

The proof resembles that of 2.4 and is left to the reader. Next, let P, U, T and F be as in 2.7, but suppose that U is equipped with a double covering $\tilde{U} \to U$ and that F is equipped with an involution tw.

2.9. PROPOSITION. *There is a diagram, commutative up to preferred homotopies,*

$$\begin{tikzcd} \Gamma(F^{tw}(\tau^P)) \arrow[r, "j"] \arrow[d, "\mathrm{st}"] & \Gamma(F^{tw}(\tau^U)) \arrow[r, "\mathrm{restriction}"] \arrow[d, "\mathrm{st}"] & \Gamma(F^{tw}(\tau^T)) \arrow[d, "\mathrm{st}"] \\ Q(\tilde{P}_+ \wedge_{Z_2} F) \arrow[r, "\mathrm{inclusion}"] & Q(\tilde{U}_+ \wedge_{Z_2} F) \arrow[r, "\mathrm{collapse}"] & Q(\tilde{T}^{\mathrm{col}} \wedge_{Z_2} F). \end{tikzcd}$$

We conclude this section with a few historical remarks. Coordinate-free spectra were introduced by May [30] and Puppe [34], perhaps in reaction to Boardman's work on smash products of spectra (see Vogt [40]). Our definition is slightly different from May's; it is more functorial, but does not incude the strict associativity that May requires. However, the proof of 2.2 shows that associativity of the suspension σ up to all higher coherences is automatic in our version. We are content with that, especially since our main example (in 1.11) does not satisfy strict associativity.

The result in 2.4 is a reformulation of Poincaré duality in the language of coordinate free spectra; in particular, the map st defined there is a homotopy equivalence if F is a coordinate free Ω-spectrum. This means that the adjoints $F(W) \to \Omega^V F(V \oplus W)$ of the suspension maps $\sigma\colon V^c \wedge F(W) \to F(V \oplus W)$ are homotopy equivalences for arbitrary V, W in $\mathscr{I}$. We do not claim any originality here: the same point of view is used, e.g. in Bödigheimer's work on configuration spaces [6]. A section space of the type discussed in 2.4 occurs in Theorem 1 of Anderson and Hsiang [2]; it is a very close relative of the section spaces we are going to use.

3. The Hyperplane Test

Let F be the coordinate free spectrum defined in 1.11. Its values $F(V)$, for V in $\mathscr{I}$, are virtual spaces. As we have indicated the results of Section 2 can be applied to F. They will be so applied; when all the work has been done the reader may want to use the materialization functor in order to see genuine maps between genuine spaces.

For V in $\mathscr{I}$, we let $-1\colon M \times (V \oplus \mathbb{R}) \to M \times (V \oplus \mathbb{R})$ be the homeomorphism sending (m, v, r) to $(m, -v, -r)$. Define $tw\colon F(V) \to F(V)$ by $tw(f) = (-1)\cdot f \cdot (-1)$, where f is a point in $\mathrm{TOP}^b(M \times (V \oplus \mathbb{R}))$ and represents a point in $F(V)$. Then tw is an involution as in 2.3.

Let τ be the tangent bundle of $\mathbb{R}P^n$; let $\mathbb{R}\tilde{P}^n = S^n$ and assume that $\mathbb{R}P^n$ is embedded in some $\mathbb{R}^N$. By 2.8, there is a Poincaré duality cum stabilization map

$$\Gamma(F^{tw}(\tau)) \to Q(S^n_+ \wedge_{Z_2} F)$$

with Z_2 acting on S^n by the antipodal map and on $F = \Omega\underline{\underline{\mathrm{Wh}}}(M)$ by tw. This is of interest to us because we want to compose it with the map in the next proposition.

3.1. PROPOSITION (Hyperplane test). *There is a continuous map*

$$\mathrm{TOP}^b(M \times \mathbb{R}^{n+1})/\mathrm{TOP}(M) \to \Gamma(F^{tw}(\tau)),$$

where τ is the tangent bundle of $\mathbb{R}P^n$.

Proof. Let $\tilde{\tau}$ be the tangent bundle of S^n. We regard S^n as a subset of $\mathbb{R}^{n+1}$, regardless of where $\mathbb{R}P^n$ lives; so $\tilde{\tau}_x \oplus \mathbb{R}$ is canonically and linearly identified with $\mathbb{R}^{n+1}$, for each $x \in S^n$.

To each f in $\mathrm{TOP}^b(M \times \mathbb{R}^{n+1})$ we must associate a section of $F^{tw}(\tau)$, or equivalently, an equivariant section of $F(\tilde{\tau})$. For any $x \in S^n$, we can regard f as an element of

$\mathrm{TOP}^b(M \times (\tilde{\tau}_x \oplus \mathbb{R}))$ since $\tilde{\tau}_x \oplus \mathbb{R} = \mathbb{R}^{n+1}$; therefore we can regard f as an element of

$$F(\tilde{\tau}_x) = \mathrm{TOP}^b(M \times (\tilde{\tau}_x \oplus \mathbb{R}))/\mathrm{TOP}^b(M \times \tilde{\tau}_x).$$

So f does give rise to a section of $F(\tilde{\tau})$; it is equivariant. It depends only on the class of f modulo $\mathrm{TOP}(M)$.

Now compose 3.1 with 2.8 to get a continuous map

$$\mathrm{TOP}^b(M \times \mathbb{R}^{n+1})/\mathrm{TOP}(M) \to Q(S^n_+ \wedge_{Z_2} F).$$

3.2. PROPOSITION. *The square*

$$\begin{tikzcd}
\mathrm{TOP}^b(M \times \mathbb{R}^n)/\mathrm{TOP}(M) \arrow[r] \arrow[d, "\text{inclusion}"] & Q(S^{n-1}_+ \wedge_{Z_2} F) \arrow[d, "\text{inclusion}"] \\
\mathrm{TOP}^b(M \times \mathbb{R}^{n+1})/\mathrm{TOP}(M) \arrow[r] & Q(S^n_+ \wedge_{Z_2} F)
\end{tikzcd}$$

is commutative up to a preferred homotopy.

Proof. This follows from 2.6, or rather its twisted version. By inspection, the square

$$\begin{tikzcd}
\mathrm{TOP}^b(M \times \mathbb{R}^n)/\mathrm{TOP}(M) \arrow[r, "\text{hyperplane test}"] \arrow[d, "\text{inclusion}"] & \Gamma(F^{tw}(\tau^{n-1})) \arrow[d, "j \text{ of } 2.6"] \\
\mathrm{TOP}^b(M \times \mathbb{R}^{n+1})/\mathrm{TOP}(M) \arrow[r, "\text{hyperplane test}"] & \Gamma(F^{tw}(\tau^n))
\end{tikzcd} \qquad (*)$$

is commutative up to a preferred homotopy, where τ^{n-1} and τ^n are the tangent bundles of $\mathbb{R}P^{n-1}$ and $\mathbb{R}P^n$, respectively.

3.3. COROLLARY. *The maps in* 3.2 *stabilize to yield a map*

$$\Phi\colon \mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M) \to Q(S^\infty_+ \wedge_{Z_2} F) = Q(S^\infty_+ \wedge_{Z_2} \Omega\underline{\underline{\mathrm{Wh}}}(M)).$$

This is the map promised in the introduction. It is most suggestive to think of Φ as a map between towers of fibrations whose effect on each stage is, in some sense, stabilization. This is the content of the next proposition, which is obtained by plugging together two diagrams. The first is the one in 2.9 with $P = \mathbb{R}P^{n-1}$ and $U = \mathbb{R}P^n$, so that T is contractible, and $\tilde{T}^{\mathrm{col}} = S^n \vee S^n$, where Z_2 acts by interchanging the wedge summands. The second is the diagram (*) from the proof of 3.2. There is only one reasonable way to plug these together. Note that the composition

$$\begin{tikzcd}
\mathrm{TOP}^b(M \times \mathbb{R}^{n+1})/\mathrm{TOP}(M) \arrow[r] & \Gamma(F^{tw}(\tau^U)) \arrow[d, "\text{restriction}"] \\
& \Gamma(F^{tw}(\tau^T)) \simeq F(\mathbb{R}^n)
\end{tikzcd}$$

agrees with the projection

$$\mathrm{TOP}^b(M \times \mathbb{R}^{n+1})/\mathrm{TOP}(M) \to \mathrm{TOP}^b(M \times \mathbb{R}^{n+1})/\mathrm{TOP}^b(M \times \mathbb{R}^n) = F(\mathbb{R}^n).$$

This proves what we want:

3.4. PROPOSITION. *The diagram*

$$\begin{array}{ccc}
\mathrm{TOP}^b(M \times \mathbb{R}^n)/\mathrm{TOP}(M) & \longrightarrow & Q(S^{n-1}_+ \wedge_{Z_2} F) \\
\downarrow & & \downarrow \\
\mathrm{TOP}^b(M \times \mathbb{R}^{n+1})/\mathrm{TOP}(M) & \longrightarrow & Q(S^n_+ \wedge_{Z_2} F) \\
\downarrow & & \downarrow \\
\mathrm{TOP}^b(M \times \mathbb{R}^{n+1})/\mathrm{TOP}^b(M \times \mathbb{R}^n) = F(\mathbb{R}^n) & \longrightarrow & Q(\Sigma^n F)
\end{array}$$

is commutative up to preferred homotopies. (The bottom horizontal arrow is the inclusion

$$F(\mathbb{R}^n) \subset \varinjlim \Omega^k F(\mathbb{R}^{n+k}) = Q(\Sigma^n F),$$

which may also be called stabilization.) Recall that three homotopies are needed, as in 2.7. Both columns are fibrations up to homotopy after materialization. (A diagram of pointed spaces and maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, with $gf = *$, is a fibration up to homotopy if the inclusion of X into the homotopy fibre of g is a homotopy equivalence.)

3.5. REMARK. Suppose that M is simply connected, $\dim(M) \geqslant 5$, and that k is in the topological concordance stable range for M. Then from 5.7 we know that $F(\mathbb{R}^n)$ is an n-connected $(n + 1)$-fold delooping of the concordance space $\mathscr{C}(M \times D^n)$. (See also 1.10 and 1.8.) We also know from 5.7 that $Q(\Sigma^n F)$ is an n-connected $(n + 1)$-fold delooping of the stabilized concordance space $\mathscr{C}(M \times D^\infty)$. By 1.12, the map $F(\mathbb{R}^n) \to Q(\Sigma^n F)$ in 3.4 is just an $(n + 1)$-fold delooping of the usual stabilization map $\mathscr{C}(M \times D^n) \to \mathscr{C}(M \times D^\infty)$, and is, therefore, $(k + n + 1)$-connected by assumption on k, and *a fortiori* $(k + 1)$-connected. An easy induction using 3.4 now shows that Φ in 3.3 is $(k + 1)$-connected. Therefore, Theorem A is proved for simply connected M with $\dim(M) \geqslant 5$, since then $\widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M) \simeq \mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M)$ by 1.14 and 5.7.

3.6. PHILOSOPHY. Here is some additional evidence for Theorem A in the nonsimply connected case. In 1.15 we identified $\pi_n(\widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M))$ with

$$\mathrm{im}\,[\pi_n(\mathrm{TOP}^b(M \times \mathbb{R}^n)/\mathrm{TOP}(M)) \to \pi_n(\mathrm{TOP}^b(M \times \mathbb{R}^{n+1})/\mathrm{TOP}(M))].$$

Go from there to

$$\mathrm{im}\,[\pi_n(Q(S^{n-1}_+ \wedge_{Z_2} \Omega \underline{\underline{\mathrm{Wh}}}(M)) \to \pi_n(Q(S^n_+ \wedge_{Z_2} \Omega \underline{\underline{\mathrm{Wh}}}(M)))]$$
$$\overset{!}{\cong} \pi_n(Q(S^\infty_+ \wedge_{Z_2} \Omega \underline{\underline{\mathrm{Wh}}}s(M)))$$

by the hyperplane test. (The isomorphism labelled !, can be deduced from a suitable definition of Postnikov covers, such as in Dold [16]; recall that $\Omega\underline{\underline{\mathrm{Wh}}}s(M)$ is the 0-connected Postnikov cover of $\Omega\underline{\underline{\mathrm{Wh}}}(M)$.) The result is a factorization

$$\begin{array}{ccc}
\pi_*(\widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M)) & \dashrightarrow & \pi_*(Q(S^\infty_+ \wedge_{Z_2} \Omega \underline{\underline{\mathrm{Wh}}}s(M))) \\
\downarrow & & \\
\pi_*(\widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M)) & & \downarrow \\
\downarrow \cong & & \\
\pi_*(\mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M)) & \longrightarrow & \pi_*(Q(S^\infty_+ \wedge_{Z_2} \Omega \underline{\underline{\mathrm{Wh}}}(M)))
\end{array}$$

which one would like to see induced by a map

$$\Phi^s\colon \widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M) \to Q(S^\infty_+ \wedge_{Z_2} \Omega \underline{\underline{\mathrm{Wh}}}s(M)).$$

3.7. DIGRESSION. There is a slightly different way of describing the connection between $\widetilde{\mathrm{TOP}}(M)/(\mathrm{TOP}(M)$ and concordance theory, in the spirit of Weiss [44]. To keep the discussion simple, let us concentrate on $\widetilde{\mathrm{TOP}}(M)$ rather than $\widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M)$. Fix an integer $n \geqslant 0$, and regard $\pi_n(\widetilde{\mathrm{TOP}}(M))$ as a factor group of $\pi_0(\mathrm{TOP}(M \times D^n))$, as in 1.15. We will construct

(i) a fibration $p\colon E \to S^{n-1}$ whose fibres are homotopy equivalent to the topological concordance space $\mathscr{C}(M \times D^{n-1})$;
(ii) an involution on the total space E, covering the antipodal involution on S^{n-1};
(iii) a map ψ from $\mathrm{TOP}(M \times D^n)$ to the space of equivariant sections of p.

Write $\hat{p}\colon \hat{E} \to \mathbb{R}P^{n-1}$ for the quotient of $p\colon E \to S^{n-1}$ by Z_2; accordingly write $\Gamma(\hat{p})$ for the space of equivariant sections of p, which is also the space of sections of $\hat{p}$.

In order to explain the connection with the approach used so far, we also construct the missing homotopy equivalence e in a commutative diagram

$$\text{(iv)}\qquad \begin{array}{ccc}
\mathrm{TOP}(M \times D^n) & \xrightarrow{\simeq} & \Omega^n \mathrm{TOP}^b(M \times \mathbb{R}^n) \\
\downarrow \psi & & \downarrow \text{hyperplane test of 3.1} \\
\Gamma(\hat{p}) & \xrightarrow[\simeq]{e} & \Omega^n(\Gamma(F^{tw}(\tau))).
\end{array}$$

Here are the details.

(i) For each $s \in S^{n-1} \subset \mathbb{R}^n$, let $\langle s \rangle \subset \mathbb{R}^n$ be the subspace generated by s, let $\langle s \rangle^\perp$ be the orthogonal complement, and let $D\langle s \rangle$, $D\langle s \rangle^\perp$ be the unit disks in $\langle s \rangle$ and $\langle s \rangle^\perp$, respectively. We identify $D\langle s \rangle^\perp \times D\langle s \rangle$ with $D\langle s \rangle^\perp + D\langle s \rangle \subset \mathbb{R}^n$.

Let E_s be the (virtual) space of self-homeomorphisms of $M \times D\langle s \rangle^\perp \times D\langle s \rangle$ which are the identity on

$$M \times D\langle s \rangle^\perp \times \{-s\} \bigcup \partial(M \times D\langle s \rangle^\perp) \times D\langle s \rangle.$$

Clearly,

$$E_s \cong \mathscr{C}(M \times D\langle s \rangle^\perp) \cong \mathscr{C}(M \times D^{n-1}).$$

Define $p\colon E \to S^{n-1}$ to be the fibre bundle such that $p^{-1}(s) = E_s$ for all $s \in S^{n-1}$. (This must be interpreted as a fibre bundle with virtual spaces as fibres, say.)

(ii) For $f \in E_s$, let $\partial f: M \times D\langle s\rangle^{\perp} \to M \times D\langle s\rangle^{\perp}$ be the restriction of f to $M \times D\langle s\rangle^{\perp} \times \{s\} \cong M \times D\langle s\rangle^{\perp}$. The map

$$E_s \to E_{-s}; \qquad f \mapsto (\partial f \times \mathrm{id}_{D\langle s\rangle})^{-1} \cdot f$$

is a homeomorphism (of virtual spaces); letting s range over S^{n-1} gives an involution on E which covers the antipodal involution on S^{n-1}.

(iii) Take an element f in $\mathrm{TOP}(M \times D^n)$, meaning a self-homeomorphism of $M \times D^n$ which is the identity on $\partial(M \times D^n)$. For any $s \in S^{n-1}$, regard f as an element of E_s by extending f trivially outside $M \times D^n \subset M \times (D\langle s\rangle^{\perp} + D\langle s\rangle)$. This gives an equivariant section $\psi(f)$ of $p: E \to S^{n-1}$.

(iv) Recall how the homotopy equivalence

$$\text{alex}: \mathrm{TOP}(M \times D^n) \xrightarrow{\simeq} \Omega^n \mathrm{TOP}^b(M \times \mathbb{R}^n)$$

was defined: Given $f \in \mathrm{TOP}(M \times D^n)$, define $\hat{f}: M \times \mathbb{R}^n \to M \times \mathbb{R}^n$ by extending f trivially outside $M \times D^n$. Then the rule $z \mapsto \mathrm{tr}_{-z} \cdot \hat{f} \cdot \mathrm{tr}_z$, where $z \in \mathbb{R}^n$ and tr_z denotes translation by z, defines a map from $\mathbb{R}^n \cup \{\infty\}$ to $\mathrm{TOP}^b(M \times \mathbb{R}^n)$, or an n-fold loop in $\mathrm{TOP}^b(M \times \mathbb{R}^n)$. This defines alex, as a map between virtual spaces.

Much the same method works if we pick f in the space E_s defined above, for fixed $s \in S^{n-1}$. Let $\hat{f}$ be equal to f on $M \times D\langle s\rangle^{\perp} \times D\langle s\rangle$; let it be equal to $\partial f \times \mathrm{id}$ on $(M \times D\langle s\rangle^{\perp}) \times \{ts \mid t \geqslant 1\}$, and let it be equal to the identity outside $M \times D\langle s\rangle^{\perp} \times \{ts \mid t \geqslant -1\}$. Picture (Figure 3): again, the rule $z \mapsto \mathrm{tr}_{-z} \cdot \hat{f} \cdot \mathrm{tr}_z$ defines a map from $\mathbb{R}^n \cup \{\infty\}$ to $\mathrm{TOP}^b(M \times \mathbb{R}^n)/\mathrm{TOP}^b(M \times \langle s\rangle^{\perp})$.

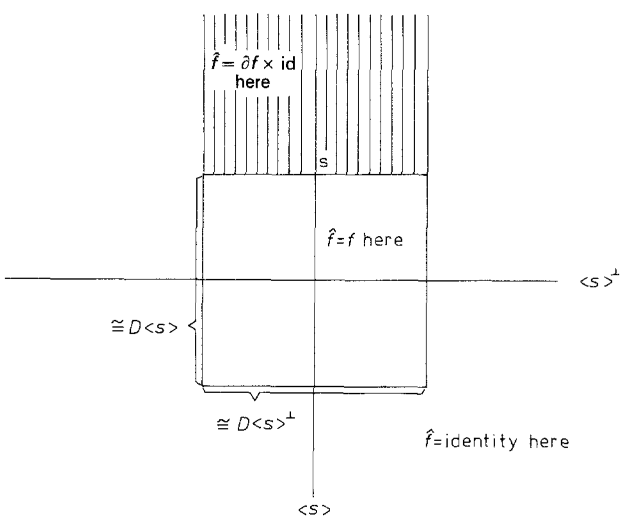

Fig. 3.

Letting s range over S^{n-1}, or rather over $\mathbb{R}P^{n-1}$, we obtain in this way a map of fibre bundles

$$\begin{array}{ccc} \hat{E} & \longrightarrow & `\ldots' \\ \big\downarrow{\scriptstyle \hat{p}} & & \big\downarrow{\scriptstyle \Omega^n F^{tw}(\tau)} \\ \mathbb{R}P^{n-1} & \xrightarrow{=} & \mathbb{R}P^{n-1} \end{array}$$

which is a homotopy equivalence on the fibres. It induces a homotopy equivalence

$$e\colon \Gamma(\hat{p}) \xrightarrow{\simeq} \Gamma(\Omega^n F^{tw}(\tau)) \cong \Omega^n \Gamma(F^{tw}(\tau)).$$

To see what the digression is good for, suppose that M is a point. Then M is a smooth manifold of dimension zero. The smooth version of Remark 3.5 shows that we have established Theorem A (smooth version) in this case, since a one-point space is simply connected. That is, we have constructed a map

$$\Phi^s\colon \widetilde{\mathrm{DIFF}}(*)/\mathrm{DIFF}(*) \to Q(S^\infty_+ \wedge_{Z_2} \Omega \underline{\underline{\mathrm{Wh}}}s^{\mathrm{DIFF}}(*)).$$

Note that DIFF$(*)$ is contractible, but $\widetilde{\mathrm{DIFF}}(*)$ is not; instead $\pi_j(\widetilde{\mathrm{DIFF}}(*))$ is (obviously) isomorphic to the group of pseudo-isotopy classes of oriented diffeomorphisms of S^j for all $j \geqslant 0$, which is, in turn, isomorphic to the group Θ_{j+1} of oriented smooth $(j+1)$-dimensional homotopy spheres if $j \geqslant 5$. Evaluating Φ^s on homotopy groups therefore gives invariants for exotic spheres. Moreover, the smooth version of Digression 3.7, with M a point, gives a more direct description for these invariants, namely, one which does not mention bounded diffeomorphisms. This is also the description used in [44], where exotic spheres are investigated by this method.

3.8. DIGRESSION. Here is another interesting point of view: the map Φ in 3.3 is a kind of Kahn–Priddy map. (See Kahn and Priddy [26] or Segal [37].) To explain why, we shall reformulate the results of Sections 2 and 3 in abstract (and sloppy) terms.

Let E be a continuous functor from the category $\mathscr{J}$ of 1.11 to the category of associative topological monoids. We assume that $\pi_0(E(V))$ is a group for each V in $\mathscr{J}$.

Examples:

(i) $E(V) = \mathrm{TOP}^b(M \times V)$,

(ii) $E(V) = \mathrm{DIFF}^b(M \times V)$ if M is smooth,

(iii) $E(V) = O(V) =$ orthogonal group of V,

(iv) $E(V) = G(V) =$ monoid of self-homotopy equivalences of the unit sphere $S(V) \subset V$.

There are many others. We associate with E a coordinate free spectrum F with involution by letting

$$F(V) = E(V \oplus \mathbb{R})/E(V);$$

the involution tw and the suspension maps have been defined explicitly in the special case when $E(V) = \mathrm{TOP}^b(M \times V)$, but the definitions generally make sense. The hyperplane test and 2.8 give a map

$$\Phi\colon E(\mathbb{R}^\infty) := \underrightarrow{\mathrm{holim}}\, E(\mathbb{R}^n) \to Q(S^\infty_+ \wedge_{Z_2} F)$$

of which 3.3 is a special case.

Now concentrate on examples (iii) and (iv). Clearly, the spectrum F in example (iii) is the sphere spectrum $\mathbf{S}^0$, with trivial involution. But the maps

$$O(V \oplus \mathbb{R})/O(V) \to G(V \oplus \mathbb{R})/G(V)$$

are approximatey (2dim(V))-connected for any V in $\mathscr{J}$. (See, e.g., Wall [43], Cor. 11.3.2.) It follows that the spectrum F in example (iv), stripped of its coordinate free structure, is also a sphere spectrum $\mathbf{S}^0$ with trivial involution. Therefore in example (iv) we obtain

$$\Phi\colon G \to Q(\mathbb{R}P^\infty_+),$$

where $G \subset Q(S^0)$ consists of the components of degree ± 1. It is not difficult to see that composing Φ with the transfer from $Q(\mathbb{R}P^\infty_+)$ to $Q(S^\infty_+) \simeq Q(S^0)$ results in

$$\text{inclusion} - c_1\colon G \to Q(S^0),$$

where c_1 is the constant map with value 1. So Φ is a Kahn–Priddy map.

4. Proof of Theorems A and C

In this section, we work with simplicial sets (rather than virtual spaces); the word space will often be used to mean simplicial set.

Let X be a pointed simplicial set with a filtration $\mathrm{Filt}_0(X) \subset \mathrm{Filt}_1(X) \subset \mathrm{Filt}_2(X) \subset \cdots \subset X$, so that

$$X = \bigcup_{i=0}^{\infty} \mathrm{Filt}_i(X).$$

Assume that $\mathrm{Filt}_i(X)$ contains the base point and has the Kan property for all i (then so does X). Call an n-simplex y in X positive if the corresponding simplicial map $f_y\colon \Delta^n \to X$ is filtration-preserving, which means that

$$f_y(i\text{-skeleton of } \Delta^n) \subset \mathrm{Filt}_i(X), \quad \text{for all } i.$$

The positive simplices form a simplicial subset

$$^{\mathrm{pos}}X \subset X$$

which is still filtered if we let $\mathrm{Filt}_i(^{\mathrm{pos}}X) = {}^{\mathrm{pos}}X \cap \mathrm{Filt}_i(X)$. Then $\mathrm{Filt}_i(^{\mathrm{pos}}X)$ has the Kan property for all i, and

$$i\text{-skeleton of } {}^{\mathrm{pos}}X \subset \mathrm{Filt}_i(^{\mathrm{pos}}X), \quad \text{for all } i.$$

Now assume additionally that X is a simplicial group and that $\mathrm{Filt}_i(X)$ is a simplicial subgroup for each i. Then ${}^{\mathrm{pos}}X$ is also a simplicial subgroup of X, and

$$({}^{\mathrm{pos}}X)/\mathrm{Filt}_0(X) \cong {}^{\mathrm{pos}}(X/\mathrm{Filt}_0(X)).$$

The isomorphism makes sense if we regard the simplicial set $X/\mathrm{Filt}_0(X)$ as filtered by simplicial subsets $\mathrm{Filt}_i(X)/\mathrm{Filt}_0(X)$.

We can interpret X as a tower of fibrations with stages $\mathrm{Filt}_{i+1}(X)/\mathrm{Filt}_i(X)$, and we can interpret ${}^{\mathrm{pos}}X$ as a tower of fibrations with stages $\mathrm{Filt}_{i+1}({}^{\mathrm{pos}}X)/\mathrm{Filt}_i({}^{\mathrm{pos}}X)$. The inclusion map

$$\mathrm{Filt}_{i+1}({}^{\mathrm{pos}}X)/\mathrm{Filt}_i({}^{\mathrm{pos}}X) \hookrightarrow \mathrm{Filt}_{i+1}(X)/\mathrm{Filt}_i(X)$$

induces an isomorphism in π_j for $j > i \geqslant 0$, whereas

$$\pi_j(\mathrm{Filt}_{i+1}({}^{\mathrm{pos}}X)/\mathrm{Filt}_i({}^{\mathrm{pos}}X)) = 0, \quad \text{for } j \leqslant i \geqslant 0.$$

This is clear from the definitions if the homotopy groups in question are interpreted as relative homotopy groups (of the inclusion maps $\mathrm{Filt}_i({}^{\mathrm{pos}}X) \hookrightarrow \mathrm{Filt}_{i+1}({}^{\mathrm{pos}}X)$ and $\mathrm{Filt}_i(X) \hookrightarrow \mathrm{Filt}_{i+1}(X)$). So the stages of the tower ${}^{\mathrm{pos}}X$ are Postnikov covers of the stages of X.

We now specialize by letting $X = \mathrm{TOP}^b(M \times \mathbb{R}^\infty)$, with filtration given by $\mathrm{Filt}_i(X) = \mathrm{TOP}^b(M \times \mathbb{R}^i)$, for $i \geqslant 0$.

4.1. PROPOSITION. *There is a map ${}^{\mathrm{pos}}\Phi$ making the following square commutative (up to a preferred homotopy):*

$$\begin{tikzcd}
{}^{\mathrm{pos}}\mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M) \arrow[r, dashed, "{}^{\mathrm{pos}}\Phi"] \arrow[d, hook] & Q(S^\infty_+ \wedge_{Z_2} \Omega\underline{\underline{\mathrm{Wh}}}s(M)) \arrow[d] \\
\mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M) \arrow[r, "\Phi"] & Q(S^\infty_+ \wedge_{Z_2} \Omega\underline{\underline{\mathrm{Wh}}}(M)).
\end{tikzcd}$$

Proof. The spaces $Q(S^\infty_+ \wedge_{Z_2} \Omega\underline{\underline{\mathrm{Wh}}}(M))$ and $Q(S^\infty_+ \wedge_{Z_2} \Omega\underline{\underline{\mathrm{Wh}}}s(M))$ have filtrations given by

$$\mathrm{Filt}_i(Q(S^\infty_+ \wedge_{Z_2} \Omega\underline{\underline{\mathrm{Wh}}}(M)) = Q(S^{i-1}_+ \wedge_{Z_2} \Omega\underline{\underline{\mathrm{Wh}}}(M)),$$
$$\mathrm{Filt}_i(Q(S^\infty_+ \wedge_{Z_2} \Omega\underline{\underline{\mathrm{Wh}}}s(M)) = Q(S^{i-1}_+ \wedge_{Z_2} \Omega\underline{\underline{\mathrm{Wh}}}s(M)).$$

The map Φ preserves filtrations; if we make the same requirement for ${}^{\mathrm{pos}}\Phi$, then existence and essential uniqueness of ${}^{\mathrm{pos}}\Phi$ is a straightforward consequence of obstruction theory. Suppose, namely, that we have already constructed a lift

$$\begin{tikzcd}
\mathrm{Filt}_i({}^{\mathrm{pos}}\mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M)) \arrow[r, dashed, "{}^{\mathrm{pos}}\Phi"] \arrow[d, hook] & Q(S^{i-1}_+ \wedge_{Z_2} \Omega\underline{\underline{\mathrm{Wh}}}s(M)) \arrow[d] \\
\mathrm{TOP}^b(M \times \mathbb{R}^i)/\mathrm{TOP}(M) \arrow[r, "\Phi"] & Q(S^{i-1}_+ \wedge_{Z_2} \Omega\underline{\underline{\mathrm{Wh}}}(M)).
\end{tikzcd}$$

In trying to extend this to a lift

$$\mathrm{Filt}_{i+1}({}^{\mathrm{pos}}\mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M)) \xrightarrow{{}^{\mathrm{pos}}\Phi} Q(S^i_+ \wedge_{Z_2} \Omega\underline{\underline{\mathrm{Wh}}}s(M))$$

we encounter obstructions in the relative homotopy groups

$$\pi_j(Q(S^i_+ \wedge_{Z_2} \Omega\underline{\underline{\mathrm{Wh}}}s(M)) \to Q(S^i_+ \wedge_{Z_2} \Omega\underline{\underline{\mathrm{Wh}}}(M)))$$

for $j > i$. (We can say $j > i$ because the i-skeleton of ${}^{\mathrm{pos}}\mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M)$ is contained in $\mathrm{Filt}_i(\ldots)$, where the lift is already defined.) But these relative homotopy groups are zero (for $j > i$). Therefore, obstructions vanish and choices are unique up to contractible indeterminacy.

4.2. PROPOSITION. *Write B^i for $(i-1)$-connected i-fold deloopings. For any $i \geqslant 0$, there is a diagram*

$$\begin{tikzcd}
\mathrm{Filt}_i({}^{\mathrm{pos}}\mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M)) \arrow[r, "{}^{\mathrm{pos}}\Phi"] \arrow[d, hook] & Q(S^{i-1}_+ \wedge_{Z_2} \Omega\underline{\underline{\mathrm{Wh}}}s(M)) \arrow[d] \\
\mathrm{Filt}_{i+1}({}^{\mathrm{pos}}\mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M)) \arrow[r, "{}^{\mathrm{pos}}\Phi"] \arrow[d] & Q(S^i_+ \wedge_{Z_2} \Omega\underline{\underline{\mathrm{Wh}}}s(M)) \arrow[d] \\
B^{i+1}\mathscr{C}(M \times D^i) \arrow[r] & B^{i+1}\mathscr{C}(M \times D^\infty)
\end{tikzcd}$$

(with $\mathscr{C}(M \times D^\infty) = \varinjlim_k \mathscr{C}(M \times D^k)$) whose columns are fibrations up to homotopy, and which is commutative up to preferred homotopies. (Three homotopies are required; we label them $\{x_t\}$, $\{y_t\}$ and $\{z_t\}$ as in the comment after 2.7.)

Proof. Note that $B^{i+1}\mathscr{C}(M \times D^\infty)$ is the i-connected Postnikov cover of $Q(\Sigma^i\Omega\underline{\underline{\mathrm{Wh}}}(M))$. If we replace $B^{i+1}\mathscr{C}(M \times D^\infty)$ by $Q(\Sigma^i\Omega\underline{\underline{\mathrm{Wh}}}(M))$ in the diagram, then its existence and commutativity up to three homotopies $\{x_t\}$, $\{y_t\}$ and $\{z_t\}$ are obvious from the proof of 4.1 and from 3.4. It is not difficult to lift the two maps with target $Q(\Sigma^i\Omega\underline{\underline{\mathrm{Wh}}}(M))$ to the Postnikov cover $B^{i+1}\mathscr{C}(M \times D^\infty)$. The difficult thing is to lift $\{y_t\}$ and $\{z_t\}$ to $B^{i+1}\mathscr{C}(M \times D^\infty)$. *Solution*: Requiring the existence of a lift of $\{z_t\}$ is tantamount to prescribing the lift of $\{y_t\}$ over the subspace $\mathrm{Filt}_i({}^{\mathrm{pos}}\mathrm{TOP}\ldots) \subset \mathrm{Filt}_{i+1}({}^{\mathrm{pos}}\mathrm{TOP}\ldots)$. The partial lift of $\{y_t\}$ can then be extended over all of $\mathrm{Filt}_{i+1}({}^{\mathrm{pos}}\mathrm{TOP}\ldots)$ because the inclusion $\mathrm{Filt}_i({}^{\mathrm{pos}}\mathrm{TOP}\ldots) \hookrightarrow \mathrm{Filt}_{i+1}({}^{\mathrm{pos}}\mathrm{TOP}\ldots)$ is i-connected.

4.3. PROPOSITION. *If k is in the topological concordance stable range for M, then the map ${}^{\mathrm{pos}}\Phi$ in 4.1 is $(k+1)$-connected. If $\dim(M) \geqslant 5$, the square in 4.1 is a homotopy pullback square.*

Proof. It k is in the topological concordance stable range for M, then the bottom horizontal arrow in diagram 4.2 is $(k+i+1)$-connected and therefore $(k+1)$-connected. Suppose for induction purposes that the top horizontal arrow in the same diagram is $(k+1)$-connected; then so is the middle horizontal arrow, which gives the induction step. Letting i tend to infinity we obtain the connectivity claim in 4.3.

For the proof of the last sentence of 4.3, we make the following observation. Suppose that W, X and Y are commutative squares of pointed spaces and maps of the form

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} ;$$

interpret W, X and Y as covariant functors from a category χ with four objects (the corners) to the category of pointed spaces. Suppose also that a natural fibration up to homotopy

$$W \xrightarrow{f} X \xrightarrow{g} Y$$

is given; this means that f and g are natural transformations such that

$$W(c) \xrightarrow{f} X(c) \xrightarrow{g} Y(c)$$

is a fibration up to homotopy (see 3.4) for each object c in χ. Suppose, finally, that W and Y are homotopy pullback squares. Is it true that X is a homotopy pullback square? The answer is yes if the upper left corners in W and Y are connected.

Use this as follows: Assume that $\dim(M) \geqslant 5$. Let

$$\square_i = \begin{tikzcd} \mathrm{Filt}_i({}^{\mathrm{pos}}\mathrm{TOP}^b(M \times \mathbb{R}^\infty))/\mathrm{TOP}(M) \arrow[r] & Q(S^{i-1}_+ \wedge_{Z_2} \Omega\underline{\underline{\mathrm{Wh}}}s(M)) \arrow[d] \\ \mathrm{TOP}^b(M \times \mathbb{R}^i)/\mathrm{TOP}(M) \arrow[r] \arrow[u, hook] & Q(S^{i-1}_+ \wedge_{Z_2} \Omega\underline{\underline{\mathrm{Wh}}}(M)) \end{tikzcd}$$

be the square from the proof of 4.1, and let

$$\square_{i+1}/\square_i = \begin{tikzcd} B^{i+1}\mathscr{C}(M \times D^i) \arrow[r] \arrow[d] & B^{i+1}\mathscr{C}(M \times D^\infty) \arrow[d] \\ \mathrm{TOP}^b(M \times \mathbb{R}^{i+1})/\mathrm{TOP}^b(M \times \mathbb{R}^i) \arrow[r] & Q(\Sigma^i\Omega\underline{\underline{\mathrm{Wh}}}(M)) \end{tikzcd}$$

where the horizontal arrows are stabilization maps and the vertical arrows are Postnikov covers. By 5.8, the square $\square_{i+1}/\square_i$ is a homotopy pullback square. By inductive assumption, so is $\square_i$. Therefore, so is $\square_{i+1}$, by the observation just made, since 4.2 gives a natural fibration up to homotopy $\square_i \to \square_{i+1} \to \square_{i+1}/\square_i$. Letting i tend to infinity completes the proof.

We see from Propositions 4.1 and 4.3 that all the things we ever wanted to know about $\widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M)$ are true for ${}^{\mathrm{pos}}\mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M)$. The moral is that we have to produce the missing homotopy equivalence in a commutative diagram

$$\begin{tikzcd} {}^{\mathrm{pos}}\mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M) \arrow[r, dashed, "\simeq"] & \widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M) \\ \mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M) \arrow[r, hook, "\simeq"] \arrow[u, hook] & \widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M). \arrow[u, hook] \end{tikzcd}$$

This looks like a combinatorial problem. We will solve it by constructing a bisimplicial set which contains ${}^{\mathrm{pos}}\mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M)$ and $\widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M)$ as its vertical and horizontal 0-skeleton, respectively, and which is homotopy equivalent to both. We begin with a few elementary facts about bisimplicial sets (see Waldhausen [42]).

4.4. DEFINITION. As usual, we let Δ be the category with objects $[n] = \{0, 1, \ldots, n\}$ for $n \geqslant 0$, and with monotone maps as morphisms. A bisimplicial set $\mathfrak{X}$ is a contravariant functor from $\Delta \times \Delta$ to the category of sets; we write $\mathfrak{X}[k, j]$ for the value of $\mathfrak{X}$ on $([k], [j])$. We can interpret $\mathfrak{X}$ as a contravariant functor

$$[k] \mapsto \mathfrak{X}[k, -]$$

from Δ to simplicial sets; in this case the simplicial maps

$$(f \times \mathrm{id})^*\colon \mathfrak{X}[k, -] \to \mathfrak{X}[m, -]$$

(induced by a monotone map $f\colon [m] \to [k]$) are called horizontal operators. See Figure 4. We can also regard $\mathfrak{X}$ as a contravariant functor

$$[j] \mapsto \mathfrak{X}[-, j]$$

from Δ to simplicial sets; then the simplicial maps

$$(\mathrm{id} \times f)^*\colon \mathfrak{X}[-, j] \to \mathfrak{X}[-, i]$$

(with $f\colon [i] \to [j]$ a monotone map) are called vertical operators. Finally we call $\mathfrak{X}[0, -]$ and $\mathfrak{X}[-, 0]$ the vertical and horizontal 0-skeleton, respectively.

The geometric realization of $\mathfrak{X}$ is

$$\coprod_{k, j \geqslant 0} \Delta^k \times \Delta^j \times \mathfrak{X}[k, j] \,/ \sim,$$

where $\sim$ denotes the usual relations.

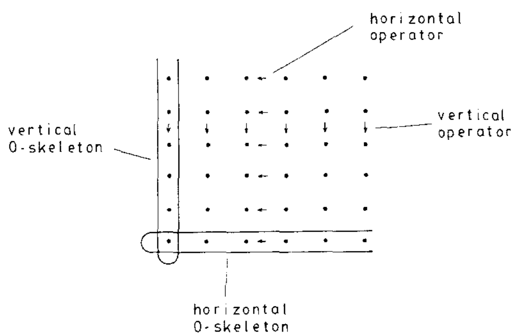

Fig. 4.

The next two lemmas are standard knowledge; formally, 4.6 is a consequence of 4.5.

4.5. LEMMA. (i) *Let $g\colon \mathfrak{U} \to \mathfrak{X}$ be a map of bisimplicial sets such that $g[k, -]\colon \mathfrak{U}[k, -] \to \mathfrak{X}[k, -]$ is a homotopy equivalence for each k (on geometric realizations).*

Then g itself is a homotopy equivalence (on geometric realizations).
(ii) *Ditto, but with vertical and horizontal interchanged.*

4.6. LEMMA. (i) *Let $\mathfrak{X}$ be a bisimplicial set in which all horizontal operators $\mathfrak{X}[k, -] \to \mathfrak{X}[n, -]$ are homotopy equivalences. Then $\mathfrak{X}$ is homotopy equivalent to its vertical 0-skeleton $\mathfrak{X}[0, -]$, i.e. the inclusion is a homotopy equivalence.*
(ii) *Ditto, but with vertical and horizontal interchanged.*

4.7. EXAMPLE. Let $\mathfrak{S}(n)$ be the bisimplicial group whose (k, j)-bisimplices are the bounded homeomorphisms

$$f\colon \Delta^k \times \Delta^j \times M \times \mathbb{R}^n \to \Delta^k \times \Delta^j \times M \times \mathbb{R}^n$$

such that

(i) f restricts to the identity on $\Delta^k \times \Delta^j \times \partial M \times \mathbb{R}^n$

(ii) $\mathrm{pr} \cdot f = \mathrm{pr}$, where pr is the projection to $\Delta^k \times \Delta^j$.

One can check by hand that the conditions 4.6(i), (ii) are satisfied by $\mathfrak{S}(n)$, and also by $\mathfrak{S} = \bigcup_n \mathfrak{S}(n)$.

The composite homotopy equivalence e given by

$$\mathrm{TOP}^b(M \times \mathbb{R}^n) \cong \mathfrak{S}(n)[0, -] \xrightarrow[\simeq]{\subset} \mathfrak{S}(n) \xleftarrow[\simeq]{\supset} \mathfrak{S}(n)[-, 0] \cong \mathrm{TOP}^b(M \times \mathbb{R}^n)$$

is homotopic to the identity on $\mathrm{TOP}^b(M \times \mathbb{R}^n)$; to put it differently, the two evident inclusions of $\mathrm{TOP}^b(M \times \mathbb{R}^n)$ into the geometric realization of $\mathfrak{S}(n)$ are canonically homotopic (and they are both homotopy equivalences by 4.6).

Sketch proof: Clearly $e^2 \simeq \mathrm{id}$. Construct a trisimplicial group whose (k, j, i)-trisimplices are the bounded self-homeomorphisms of $\Delta^k \times \Delta^j \times \Delta^i \times M \times \mathbb{R}^n$ preserving the projection to $\Delta^k \times \Delta^j \times \Delta^i$. Find that $e^3 \simeq \mathrm{id}$ also; therefore $e \simeq \mathrm{id}$.

4.8. EXAMPLE. Let $\mathfrak{T}(n)$ be the bisimplicial group whose (k, j)-bisimplices are the bounded homeomorphisms

$$f\colon \Delta^k \times \Delta^j \times M \times \mathbb{R}^n \to \Delta^k \times \Delta^j \times M \times \mathbb{R}^n$$

such that

(i) f restricts to the identity on $\Delta^k \times \Delta^j \times \partial M \times \mathbb{R}^n$
(ii) $\mathrm{pr}_2 \cdot f = \mathrm{pr}_2$, where pr_2 is the projection to Δ^j
(iii) $f(d_i\Delta^k \times \Delta^j \times M \times \mathbb{R}^n) = d_i\Delta^k \times \Delta^j \times M \times \mathbb{R}^n$ for $0 \leqslant i \leqslant k$, where d_i is the ith face.

In other words, f is fibre preserving in the vertical direction, but only block preserving in the horizontal direction.

Again one can check by hand that $\mathfrak{T}(n)$ satisfies condition 4.6.(ii), meaning that all vertical operators are homotopy equivalences. (Compare the homotopy groups.) Now let $\mathfrak{T}$ be the union of the $\mathfrak{T}(n)$. Then all maps in the commutative diagram

$$\begin{tikzcd} \mathrm{TOP}^b(M \times \mathbb{R}^\infty) \arrow[r, hook, "\text{vert.}", "\simeq"'] \arrow[d, hook, "="'] & \mathfrak{S} \arrow[d, hook] & \mathrm{TOP}^b(M \times \mathbb{R}^\infty) \arrow[l, hook', "\text{horiz.}"', "\simeq"] \arrow[d, hook, "\simeq"] \\ \mathrm{TOP}^b(M \times \mathbb{R}^\infty) \arrow[r, hook, "\text{vert.}"'] & \mathfrak{T} & \widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^\infty) \arrow[l, hook', "\simeq", "\text{horiz.}"'] \end{tikzcd}$$

must be homotopy equivalences (on geometric realizations) by Theorem B. (Arrows labelled vert. or horiz. are inclusions of vertical or horizontal 0-skeletons.)

4.9. EXAMPLE. By construction, $\mathfrak{T}$ in the preceding example is a filtered bisimplicial group; in particular, each simplicial set $\mathfrak{T}[k, -]$ is filtered. Define a bisimplicial group $\mathfrak{Z}$ in such a way that

$$\mathfrak{Z}[k, -] = {}^{\mathrm{pos}}(\mathfrak{T}[k, -]), \quad \text{for all } k \geqslant 0.$$

In some sense $\mathfrak{Z}$ is the ideal compromise between $\widetilde{\mathrm{TOP}}(M)$ and ${}^{\mathrm{pos}}\mathrm{TOP}^b(M \times \mathbb{R}^\infty)$, because

$$\mathfrak{Z}[0, -] \cong {}^{\mathrm{pos}}\mathrm{TOP}^b(M \times \mathbb{R}^\infty) \quad \text{and} \quad \mathfrak{Z}[-, 0] \cong \widetilde{\mathrm{TOP}}(M).$$

4.10. PROPOSITION. *The inclusions of the vertical and horizontal 0-skeletons,*

$${}^{\mathrm{pos}}\mathrm{TOP}^b(M \times \mathbb{R}^\infty) \hookrightarrow \mathfrak{Z} \quad \textit{and} \quad \widetilde{\mathrm{TOP}}(M) \hookrightarrow \mathfrak{Z},$$

are both homotopy equivalences (on geometric realizations).

We postpone the proof because it requires more bisimplicial machinery. Instead, here is the proof of Theorems A and C, modulo 4.10. We look at the bisimplicial set $\mathfrak{Z}/\mathfrak{S}(0)$ of 4.9 and 4.7. The inclusions of the horizontal and vertical 0-skeletons,

$$\widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M) \hookrightarrow \mathfrak{Z}/\mathfrak{S}(0)$$

and

$${}^{\mathrm{pos}}\mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M) \hookrightarrow \mathfrak{Z}/\mathfrak{S}(0),$$

are homotopy equivalences by 4.7 and 4.10. Therefore

$$\widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M) \simeq {}^{\mathrm{pos}}\mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M).$$

This is essentially what we had to prove, but we also wanted the homotopy equivalence to fit into a homotopy commutative diagram

$$\begin{tikzcd} {}^{\mathrm{pos}}\mathrm{TOP}^b(M \times \mathbb{R}^\infty)/(\mathrm{TOP}(M) \arrow[r, dashed, leftrightarrow, "\simeq"] \arrow[d, hook] & \widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M) \arrow[d, hook] \\ \mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M) \arrow[r, "\simeq"] & \widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M). \end{tikzcd}$$

Consider then the larger diagram

$$\begin{tikzcd}
{}^{\mathrm{pos}}\mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M) \arrow[r, hook, "\mathrm{ver.}", "\simeq"'] \arrow[d, hook] & \mathfrak{Z}/\mathfrak{S}(0) \arrow[d, phantom, "(*)"] & \widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M) \arrow[l, hook', "\mathrm{hor.}"', "\simeq"] \arrow[d, hook] \\
\mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M) \arrow[rr, hook, "i", "\simeq"'] \arrow[d, hook, "\mathrm{ver.}", "\simeq"'] & {} \arrow[d, phantom, "(**)"] & \widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M) \arrow[d, hook, "\mathrm{hor.}", "\simeq"'] \\
\mathfrak{S}/\mathfrak{S}(0) \arrow[rr, hook, "\simeq"'] & {} & \mathfrak{T}/\mathfrak{S}(0).
\end{tikzcd}$$

Deleting the arrow labelled i and inserting the inclusion $\mathfrak{Z}/\mathfrak{S}(0) \hookrightarrow \mathfrak{T}/\mathfrak{S}(0)$ instead, we obtain a strictly commutative diagram. Therefore, commutativity of square (*) up to a preferred homotopy is equivalent to commutativity of square (**) up to a preferred homotopy. But we know from 4.7 that the vertical and horizontal inclusions $\mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M) \hookrightarrow \mathfrak{S}/\mathfrak{S}(0)$ are canonically homotopic; therefore (**) is indeed commutative up to a preferred homotopy.

The machinery needed in proving 4.10 consists of a lemma and two remarks. The lemma is a refinement of 4.6 for bisimplicial groups $\mathfrak{X}$. Define

$$N\mathfrak{X}[k, -] = \bigcap_{i \neq 0} \ker(d_i \colon \mathfrak{X}[k, -] \to \mathfrak{X}[k-1, -])$$

where the d_i are the horizontal elementary face operators. Then $N\mathfrak{X}[k, -]$ is a simplicial subgroup of $\mathfrak{X}[k, -]$ for each $k \geqslant 0$. Define similarly $N\mathfrak{X}[-, j] \subset \mathfrak{X}[-, j]$ for all $j \geqslant 0$.

4.11. LEMMA. (i) *If $N\mathfrak{X}[k, -]$ is contractible for all $k > 0$, then the condition in* 4.6.(i) *is satisfied.*

(ii) *If $N\mathfrak{X}[-, j]$ is contractible for all $j > 0$, then the condition in* 4.6.(ii) *is satisfied.*

Proof (of (i)). Fix $n \geqslant 0$. The zeroth vertex map $\mathfrak{X}[n, -] \to \mathfrak{X}[0, -]$ is a split surjection; we must prove that its kernel W is a contractible simplicial group (because then the degeneracy map $\mathfrak{X}[0, -] \to \mathfrak{X}[n, -]$ will be a homotopy equivalence, and since n was arbitrary all horizontal operators will be homotopy equivalences). Filter W as follows: For each j between 0 and n, let $I(j)$ be the set of injective morphisms $[j] \to [n]$ in Δ which map $0 \in [j]$ to $0 \in [n]$. Let

$$W_j = \bigcap_{f \in I(j)} \ker(f^* \colon W \subset \mathfrak{X}[n, -] \to \mathfrak{X}[j, -]).$$

Then $W_0 = W$ and $W_n = \{1\}$. There is a restriction map

$$\prod_{f \in I(j)} f^* \colon W_{j-1}/W_j \to \prod_{f \in I(j)} N\mathfrak{X}[j, -]$$

which is clearly injective. Using degeneracy operators and the group structure in W_{j-1}, one can easily show it to be surjective. Therefore, the assumption in 4.11.(i) implies contractibility of W.

4.12. REMARK. Suppose that $\mathfrak{X}$ is a bisimplicial group such that $N\mathfrak{X}[k, -]$ is contractible for all $k > 0$. Then $\pi_*(\mathfrak{X}[0, -]) \cong \pi_*(\mathfrak{X})$ by 4.11 and 4.6. The homomorphism

$$\pi_*(\mathfrak{X}[-, 0]) \to \pi_*(\mathfrak{X}) \cong \pi_*(\mathfrak{X}[0, -])$$

has the following explicit description by transgression. Write

$$\operatorname{int} \mathfrak{X}[k, -] = \bigcap_{\text{all } i} \ker(d_i\colon \mathfrak{X}[k, -] \to \mathfrak{X}[k-1, -]).$$

Then $\operatorname{int} \mathfrak{X}[k, -] \subset N\mathfrak{X}[k, -]$, and $N\mathfrak{X}[k, -]$ is contractible if $k > 0$, so that

(a) $\Omega(N\mathfrak{X}[k, -]/\operatorname{int} \mathfrak{X}[k, -]) \simeq \operatorname{int} \mathfrak{X}[k, -]$

if $k > 0$. But the face operator d_0 gives an injection

(b) $N\mathfrak{X}[k, -]/\operatorname{int} \mathfrak{X}[k, -] \to \operatorname{int} \mathfrak{X}[k-1, -]$.

Putting (a) and (b) together, we get transgression maps

$$\operatorname{int} \mathfrak{X}[k, -] \to \Omega(\operatorname{int} \mathfrak{X}[k-1, -]), \qquad \text{for } k > 0.$$

Now represent an element in $\pi_k(\mathfrak{X}[-, 0])$ by a k-simplex in $\mathfrak{X}[-, 0]$ with all faces at the base point. This, then, is also a 0-simplex in $\operatorname{int} \mathfrak{X}[k, -]$ and represents an element in $\pi_0(\operatorname{int} \mathfrak{X}[k, -])$. Pass from there to $\pi_k(\operatorname{int} \mathfrak{X}[0, -]) = \pi_k(\mathfrak{X}[0, -])$ by iterated transgression. It is not difficult to see that the two homotopy classes under consideration, in $\pi_k(\mathfrak{X}[-, 0])$ and in $\pi_k(\mathfrak{X}[0, -])$, have the same image in $\pi_k(\mathfrak{X})$.

4.13. REMARK. For a generalization of 4.12, suppose that $\mathfrak{U} \subset \mathfrak{X}$ is an inclusion of bisimplicial groups such that $N\mathfrak{U}[k, -]$ and $N\mathfrak{X}[k, -]$ are both contractible for all $k > 0$. Then we know that $\mathfrak{U}$ and $\mathfrak{X}$ satisfy condition 4.6.(i) and, therefore, so does $\mathfrak{Y} = \mathfrak{X}/\mathfrak{U}$. Again, the homomorphism

$$\pi_*(\mathfrak{Y}[-, 0]) \to \pi_*(\mathfrak{Y}) \cong \pi_*(\mathfrak{Y}[0, -])$$

can be described by transgression: For $k \geqslant 0$, let

$$N\mathfrak{Y}[k, -] = \bigcap_{i \neq 0} d_i^{-1}(\text{base point}),$$

$$\operatorname{int} \mathfrak{Y}[k, -] = \bigcap_{\text{all } i} d_i^{-1}(\text{base point}),$$

where the d_i are the horizontal elementary face operators (going from $\mathfrak{Y}[k, -]$ to $\mathfrak{Y}[k-1, -]$). Inspection of 4.11 shows that the inclusion $N\mathfrak{X}[k, -]/N\mathfrak{U}[k, -] \to N\mathfrak{Y}[k, -]$ is an isomorphism of simplicial sets. Therefore, $N\mathfrak{Y}[k, -]$ is contractible if $k > 0$; therefore also the map

$$d_0\colon N\mathfrak{Y}[k, -] \to N\mathfrak{Y}[k-1, -]$$

is a Kan fibration onto its image β_{k-1}. We get transgression maps

$$\operatorname{int} \mathfrak{Y}[k,-] \simeq \Omega(\beta_{k-1}) \subset \Omega(\operatorname{int} \mathfrak{Y}[k-1,-])$$

as before.

Proof of 4.10: We will first show that $N\mathfrak{Z}[k,-]$ is contractible for all $k > 0$. Note that

$$N\mathfrak{Z}[k,-] = {}^{\mathrm{pos}}\mathrm{TOP}^b(M \times \Delta^k \times \mathbb{R}^\infty, M \times d_0\Delta^k \times \mathbb{R}^\infty),$$

where we use the filtration of $\mathbb{R}^\infty$ by subspaces $\mathbb{R}^i$ to make sense of the superscript 'pos'. See 1.4 for relative notation. There is an obvious identification of simplicial sets

$$^{\mathrm{pos}}\mathrm{TOP}^b(M \times \Delta^k \times \mathbb{R}^\infty, M \times d_0\Delta^k \times \mathbb{R}^\infty) \cong {}^{\mathrm{pos}}\mathscr{C}^b(M \times D^{k-1} \times \mathbb{R}^\infty);$$

also, $\pi_j({}^{\mathrm{pos}}\mathscr{C}^b(M \times D^{k-1} \times \mathbb{R}^\infty))$ is isomorphic to

$$\operatorname{im}[\pi_j(\mathscr{C}^b(M \times D^{k-1} \times \mathbb{R}^j)) \to \pi_j(\mathscr{C}^b(M \times D^{k-1} \times \mathbb{R}^{j+1}))]$$

for any $j \geqslant 0$, almost by definition. But the inclusion map $\mathscr{C}^b(M \times D^{k-1} \times \mathbb{R}^j) \to \mathscr{C}^b(M \times D^{k-1} \times \mathbb{R}^{j+1})$ is nullhomotopic. (By 1.8, it can be delooped to an inclusion map $F(\mathbb{R}^j) \to F(\mathbb{R}^{j+1})$, where

$$F(V) = \mathrm{TOP}^b(M \times D^{k-1} \times (V \oplus \mathbb{R}))/\mathrm{TOP}^b(M \times D^{k-1} \times V)$$

for any finite-dimensional real Hilbert space V. Replacing M by $M \times D^{k-1}$ in 1.11, we see that the inclusion $F(\mathbb{R}^j) \to F(\mathbb{R}^{j+1})$ is nullhomotopic; in fact there are two essentially different nullhomotopies, giving rise to a map $F(\mathbb{R}^j) \to \Omega F(\mathbb{R}^{j+1})$.) The conclusion is that $N\mathfrak{Z}[k,-]$ has trivial homotopy groups. This proves one half of 4.10, namely, that the inclusion $\mathfrak{Z}[0,-] \to \mathfrak{Z}$ is a homotopy equivalence.

We now use 4.12 to check that the homomorphism

$$\pi_*(\mathfrak{Z}[-,0] \to \pi_*(\mathfrak{Z}) \cong \pi_*(\mathfrak{Z}[0,-])$$

is an isomorphism. Note that

$$\operatorname{int} \mathfrak{Z}[k,-] = {}^{\mathrm{pos}}\mathrm{TOP}^b(M \times \Delta^k \times \mathbb{R}^\infty)$$

so that

$$\begin{aligned} &\pi_j(\operatorname{int} \mathfrak{Z}[k,-] \\ &\quad = \operatorname{im}[\pi_j(\mathrm{TOP}^b(M \times \Delta^k \times \mathbb{R}^j) \to \pi_j(\mathrm{TOP}^b(M \times \Delta^k \times \mathbb{R}^{j+1}))] \end{aligned}$$

whereas

$$\begin{aligned} &\pi_{j+1}(\operatorname{int} \mathfrak{Z}[k-1,-]) \\ &\quad = \operatorname{im}[\pi_{j+1}(\mathrm{TOP}^b(M \times \Delta^{k-1} \times \mathbb{R}^{j+1}) \to \pi_{j+1}(\mathrm{TOP}^b(M \times \Delta^{k-1} \times \mathbb{R}^{j+2}))]. \end{aligned}$$

The transgression

$$\pi_j(\operatorname{int} \mathfrak{Z}[k,-]) \to \pi_{j+1}(\operatorname{int} \mathfrak{Z}[k-1,-])$$

is then simply obtained from the Anderson–Hsiang isomorphism

$$\pi_j(\mathrm{TOP}^b(M \times \Delta^k \times \mathbb{R}^j)) \cong \pi_{j+1}(\mathrm{TOP}^b(M \times \Delta^{k-1} \times \mathbb{R}^{j+1}))$$

by passing to factor groups. Using 4.12, we then find that the homomorphisms

$$\pi_j(\widetilde{\mathrm{TOP}}(M)) = \pi_j(\mathfrak{Z}[-,0]) \to \pi_j(\mathfrak{Z}[0,-] = \pi_j({}^{\mathrm{pos}}\mathrm{TOP}^b(M \times \mathbb{R}^\infty))$$

have the following unsurprising description. Represent an element in $\pi_j(\widetilde{\mathrm{TOP}}(M))$ by a j-simplex with all faces at the base point. This represents an element in $\pi_0(\mathrm{TOP}(M \times \Delta^j)) \cong \pi_0(\mathrm{TOP}(M \times D^j)) \cong \pi_j(\mathrm{TOP}^b(M \times \mathbb{R}^j))$ and, therefore, an element in $\pi_j({}^{\mathrm{pos}}\mathrm{TOP}^b(M \times \mathbb{R}^\infty))$. It is quite easy to check that this homomorphism from $\pi_*(\widetilde{\mathrm{TOP}}(M))$ to $\pi_*({}^{\mathrm{pos}}\mathrm{TOP}^b(M \times \mathbb{R}^\infty))$ is an isomorphism. This proves the second half of 4.10.

4.14. REMARK. The last sentences of the proof give an explicit description of the isomorphism

$$\pi_*(\widetilde{\mathrm{TOP}}(M)) \cong \pi_*({}^{\mathrm{pos}}\mathrm{TOP}^b(M \times \mathbb{R}^\infty)).$$

Using 4.13 instead of 4.12, one obtains an equally explicit description of the isomorphism

$$\pi_*(\widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M)) \cong \pi_*({}^{\mathrm{pos}}\mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M)).$$

Since

$$\Phi^s\colon \widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M) \to Q(S^\infty_+ \wedge_{Z_2} \Omega \underline{\underline{\mathrm{Wh}}}{}^s(M))$$

is defined to be the composition of the homotopy equivalence

$$\widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M) \simeq {}^{\mathrm{pos}}\mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M)$$

with

$${}^{\mathrm{pos}}\Phi\colon {}^{\mathrm{pos}}\mathrm{TOP}^b(M \times \mathbb{R}^\infty)/\mathrm{TOP}(M) \to Q(S^\infty_+ \wedge_{Z_2} \Omega \underline{\underline{\mathrm{Wh}}}{}^s(M)),$$

this shows that the effect of Φ^s on homotopy groups is what it was supposed to be (return to 3.6).

5. Appendix: Geometry and Lower *K*-Theory

We need to recall the connection between bounded or controlled geometry and lower algebraic K-theory, as developed by Anderson and Hsiang [2], Quinn [35, 36], Chapman [15], and Pedersen [31].

Let N be a manifold (with $\partial N = \emptyset$) equipped with a proper map $p\colon N \to \mathbb{R}^j$. Assume that N has a bounded fundamental group (oid); see Pedersen [31]. Pedersen investigates equivalence classes of bounded h-cobordisms $(W; N, N')$ over N, under the equivalence relation given by bounded homeomorphism relative to N.

5.1. BOUNDED h-COBORDISM THEOREM. *Suppose that* $\dim(N) \geqslant 5$. *Equivalence classes of bounded h-cobordisms over N are in one-one correspondence with the elements of an algebraically defined group*

$$\kappa_{1-j}(\pi) = \begin{cases} \mathrm{Wh}(\pi), & \text{if } j = 0, \\ \tilde{K}_0(\mathbb{Z}\pi), & \text{if } j = 1, \\ K_{1-j}(\mathbb{Z}\pi), & \text{otherwise,} \end{cases}$$

which only depends on the fundamental group (oid) $\pi = \pi_1(N)$. *The product h-cobordism corresponds to the neutral element.*

See Pedersen [31] for details. Note that π must be finitely presented since we assume it is bounded. For the definition of the negative K-groups, see Pedersen [33]. In Pedersen's formulation, it is such that the proof of 5.1 can be quite analogous to that of the ordinary h-cobordism or s-cobordism theorem, which is contained in 5.1 as a special case ($j = 0$).

5.2. REMARKS. (i) Theorem 5.1 is valid in the smooth and in the topological category.

(ii) There is a mild generalization to the case where $\partial N \neq \emptyset$; in this case one classifies bounded h-cobordisms over N, equipped with a bounded product structure over ∂N. The obstruction groups (or classification groups) are the same.

5.3. COROLLARY. *Let M be a compact manifold as in section* 1. *If* $\dim(M) + n \geqslant 5$, *then*

$$\pi_j(\mathscr{C}^b(M \times \mathbb{R}^n)) \cong \kappa_{2+j-n}(\pi), \quad \textit{for } 0 \leqslant j < n,$$

where $\pi = \pi_1(M)$.

Proof. Write $M \times D^j \times \mathbb{R}^{n-j-1} = N$, keeping j fixed; then

$$\pi_j(\mathscr{C}^b(M \times \mathbb{R}^n)) \cong \pi_0(\mathscr{C}^b(M \times D^j \times \mathbb{R}^{n-j})) = \pi_0(\mathscr{C}^b(N \times \mathbb{R})),$$

by 1.10. Here we regard N as a manifold with control map equal to the projection $p: N \to \mathbb{R}^{n-j-1}$. We will describe an isomorphism

$$\beta: \pi_0(\mathscr{C}^b(N \times \mathbb{R})) \to h\mathrm{cob}(N \times [0, 1]),$$

where $h\mathrm{cob}(N \times [0, 1])$ is the group of equivalence classes of bounded h-cobordisms over $N \times [0, 1]$ trivialized over $\partial(N \times [0, 1])$. This reduces 5.3. to 5.1. (The group structure in $h\mathrm{cob}(N \times [0, 1])$ can be defined by juxtaposition, since $N \times [0, 1] \cup N \times [1, 2] = N \times [0, 2] \cong N \times [0, 1]$.)

For the definition of β, let $f: N \times D^1 \times \mathbb{R} \to N \times D^1 \times \mathbb{R}$ be a bounded concordance. Choose $z > 0$ so large that $N \times D^1 \times \{0\}$ and $f(N \times D^1 \times \{z\})$ are disjoint. Then the region enclosed by $N \times D^1 \times \{0\}$ and $f(N \times D^1 \times \{z\})$ is a bounded h-cobordisms over $N \times [0, z] \cong N \times [0, 1]$. See Figure 5. It is trivialized over

$\partial(N \times [0, 1])$ in the sense that there is an identification

$$\begin{array}{c} N \times D^1 \times \{0\} \cup \partial N \times D^1 \times [0, z] \cup f(N \times D^1 \times \{z\}) \\ \cong \big\downarrow \mathrm{id} \cup \mathrm{id} \cup f^{-1} \\ N \times D^1 \times \{0\} \cup \partial N \times D^1 \times [0, z] \cup N \times D^1 \times \{z\} = D^1 \times \partial(N \times [0, z]). \end{array}$$

This is a provisional definition of the map β. We will see below that $\beta(f)$ depends only on the component of f. It is clear that $\beta(fg) = \beta(f) + \beta(g)$ for arbitrary f, g.

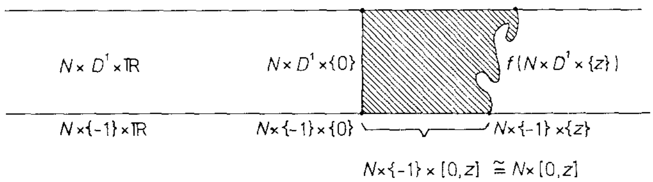

Fig. 5.

Suppose now that $\beta(f) = 0$. We must show that f belongs to the identity component of $\mathscr{C}^b(N \times \mathbb{R})$. By assumption, the bounded h-cobordism over $N \times [0, 1] \cong N \times [0, z]$ which we associated with f can be equipped with a bounded product structure extending the given product structure over $\partial(N \times [0, z])$. With this information, it is easy to deform f into a bounded concordance g such that g is the identity on $N \times D^1 \times \{0\}$. The usual Alexander trick then deforms g into the identity concordance.

The surjectivity of β can be proved by an Eilenberg swindle. Take any bounded h-cobordism μ over $N \times [0, 1]$, trivialized over $\partial(N \times [0, 1])$; and take another one which is inverse to μ, say $-\mu$. Let μ_i be the bounded h-cobordism over $N \times [i, i + 1]$ given by

$$\mu_i = \begin{cases} \mu, & \text{if } i \text{ is even,} \\ -\mu, & \text{if } i \text{ is odd.} \end{cases}$$

Let $X = \bigcup \mu_i$, so that X is a bounded h-cobordism over

$$N \times (\bigcup [i, i + 1]) = N \times \mathbb{R}.$$

Writing

$$X = \bigcup_{i \text{ even}} (\mu_i \cup \mu_{i+1})$$

and using a fixed bounded product structure on $\mu_i \cup \mu_{i+1} = \mu \cup -\mu$ for all even i, one obtains a bounded product structure

$$j_1 : X \xrightarrow{\cong} (N \times \mathbb{R}) \times D^1.$$

Writing

$$X = \bigcup_{i \text{ odd}} (\mu_i \cup \mu_{i+1})$$

one obtains another bounded product structure

$$j_2 \colon X \xrightarrow{\cong} (N \times \mathbb{R}) \times D^1.$$

Then $f = j_2(j_1)^{-1}$ is a bounded concordance of $N \times \mathbb{R}$ such that $\beta(f) = \mu$, as required.

To show that $\beta(f)$ only depends on the component of f, we invoke a continuity principle which is implicit in Pedersen [31]. It states the following: Suppose that a bounded h-cobordism (over a manifold L with control map $p \colon L \to \mathbb{R}^k$, for some $k > 0$) has a bounded product structure over some open subset $U \subset L$; suppose also that U contains the inverse image under p of a large disk about the origin in $\mathbb{R}^k$. (Here 'large' means large in comparison with the various bounds satisfied by the bounded h-cobordism and by the product structure over U.) Then the algebraic invariant $y \in \kappa_{1-k}(\pi_1(L))$ determined by the bounded h-cobordism (see 5.1) is zero.

Proof. Inspection shows that $\mathrm{BHS}(y) = 0$, where BHS is the Bass–Heller–Swan monomorphism from $\kappa_{1-k}(\pi_1(L))$ to $\mathrm{Wh}(\pi_1(L) \times \mathbb{Z}^k)$. See the definitions in Pedersen [32, 33].

For a continuous path

$$[0, 1] \to \mathscr{C}^b(N \times \mathbb{R}); \qquad t \mapsto f_t,$$

we now compare $\beta(f_t)$ and

$$\beta(f_{t+\varepsilon}) = \beta(f_t) + \beta(f_{t+\varepsilon} \cdot f_t^{-1}).$$

An application of 1.1 and the continuity principle just formulated, shows that $\beta(f_{t+\varepsilon} \cdot f_t^{-1}) = 0$ for arbitrary t and sufficiently small ε. Therefore, $\beta(f_t)$ is the same for all $t \in [0, 1]$.

5.4. COROLLARY. *If* $\dim(M) + n \geqslant 5$, *then* $\pi_0(\mathrm{TOP}^b(M \times \mathbb{R}^{n+1})/\mathrm{TOP}^b(M \times \mathbb{R}^n))$ *maps injectively to* $\kappa_{1-n}(\pi)$, *where* $\pi = \pi_1(M)$. (*The homotopy groups* π_j *for* $0 < j \leqslant n$ *are covered by* 1.8 *and* 5.3.)

Proof. Represent an element in $\pi_0(\mathrm{TOP}^b(M \times \mathbb{R}^{n+1})/\mathrm{TOP}^b(M \times \mathbb{R}^n))$ by some bounded homeomorphism $f \colon M \times \mathbb{R}^{n+1} \to M \times \mathbb{R}^{n+1}$. For sufficiently large $z > 0$, the region enclosed by $M \times \mathbb{R}^n \times \{-z\}$ and $f(M \times \mathbb{R}^n \times \{0\})$ is a bounded h-cobordism over $M \times \mathbb{R}^n \times \{-z\}$. Together with 5.1, this defines the map. Injectivity is obvious.

Let N be the manifold in 5.1 again. If $(W; N, N')$ is a bounded h-cobordism over N with torsion invariant $x \in \kappa_{1-j}(\pi)$, then it is also a bounded h-cobordism over N' with torsion invariant $y \in \kappa_{1-j}(\pi)$, say. Then $y = (-1)^n T(x)$, where $n = \dim(N)$ and T is the transposition or conjugation involution on $\kappa_{1-j}(\pi)$. It depends only on the first Stiefel–Whitney class $w_1 \colon \pi \to Z_2$ of N or of N'.

5.5. COROLLARY. *Let M be a compact manifold as in Section* 1. *If* $\dim(M) + n \geqslant 5$ *and* $j \geqslant 0$, *then there is a homomorphism*

$$\pi_j(\widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^{n+1})/\widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^n)) \to H_j(Z_2; \kappa_{1-n}(\pi))$$

which is an isomorphism if $j > 0$ *and a monomorphism if* $j = 0$. *Here,* Z_2 *acts on* $\kappa_{1-n}(\pi)$ *by* $(-1)^{m+n-1}T$, *where* $m = \dim(M)$ *and* $\pi = \pi_1(M)$.

This can also be written in the shape of a long exact 'Rothenberg' sequence relating the homotopy groups of $\widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^n)$ and $\widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^{n+1})$. See 1.14 for notation.

Indication of proof. Suppose for notational simplicity that $n = 0$. Represent an element of $\pi_j(\widetilde{\mathrm{TOP}}^b(M \times \mathbb{R})/\widetilde{\mathrm{TOP}}(M))$ by a j-simplex having all faces at the base point. This can be represented in turn by a bounded homeomorphism $f: \Delta^j \times M \times \mathbb{R} \to \Delta^j \times M \times \mathbb{R}$. Then the region enclosed by $f(\Delta^j \times M \times \{0\})$ and $\Delta^j \times M \times \{-z\}$ is an h-cobordism over $\Delta^j \times M \times \{-z\}$, trivialized over $\partial(\Delta^j \times M \times \{-z\})$, for large $z > 0$. It determines an element x in $\mathrm{Wh}(\pi) = \kappa_1(\pi)$. If $j > 0$, we have to show that $x + (-1)^{j+m-1}T(x) = 0$, because only then does x represent an element in $H_j(Z_2; \mathrm{Wh}(\pi))$. To this end observe that $x + (-1)^{j+m-1}T(x)$ is the Whitehead torsion of the inclusion $d_0\Delta^j \times M \times \{0\} \hookrightarrow X$, where $X = f(\Delta^j \times M \times \{0\})$ is the top of the h-cobordism under consideration. But this Whitehead torsion is clearly zero, as can be seen by applying f^{-1} to X. This completes the description of the homomorphism in 5.5 if $n = 0$; the arguments for $n > 0$ are analogous. Surjectivity (for $j > 0$) can be proved by a suitable Eilenberg swindle, and injectivity can be proved by a relative version of the argument which proves surjectivity.

The corollaries above are by no means new: 5.3 is due to Anderson and Hsiang [2], and 5.5 is implicit in Anderson and Pedersen [3]. They are equally valid in the smooth category (although we have only stated the topological versions) because of 5.2.(i). We now state secondary corollaries; again, it is understood that these are also valid in the smooth category.

5.6. COROLLARY. *Let M be a compact topological manifold. Then*

$$\pi_{-j}(\Omega\underline{\underline{\mathrm{Wh}}}(M)) \cong \kappa_{1-j}(\pi_1(M)), \quad \textit{for } j \geqslant 0.$$

Proof. By 1.13 we have

$$\pi_{-j}(\Omega\underline{\underline{\mathrm{Wh}}}(M)) \cong \varinjlim \pi_0(\mathscr{C}^b(M \times D^k \times \mathbb{R}^{j+1})),$$

where the limit runs over k and is taken with respect to stabilization. So 5.6 follows from 5.3 (and its proof).

5.7. COROLLARY. *Suppose that M is simply connected. Then $\Omega\underline{\underline{\mathrm{Wh}}}(M)$ is* 0-*connected. If also* $\dim(M) \geqslant 5$, *then the inclusion* $\widetilde{\mathrm{TOP}}(M) \hookrightarrow \widetilde{\mathrm{TOP}}^b(M \times \mathbb{R}^\infty)$ *is a homotopy equivalence; therefore* $\widetilde{\mathrm{TOP}}(M) \simeq \mathrm{TOP}^b(M \times \mathbb{R}^\infty)$ *by* 1.14. *If* $\dim(M) + n \geqslant 5$, *then* $\mathrm{TOP}^b(M \times \mathbb{R}^{n+1})/\mathrm{TOP}^b(M \times \mathbb{R}^n)$ *is n-connected.*

Proof. For any $j \geqslant 0$, the group $\kappa_{1-j}(\{1\})$ vanishes because it injects into $\kappa_1(\mathbb{Z}^j)$, e.g. by Pedersen [32], and because $\kappa_1(\mathbb{Z}^j)$ is zero according to Bass, Heller and Swan [5]. See also Carter [10–12].

5.8. COROLLARY. *With M as in* 5.6, *write*

$$F(\mathbb{R}^n) = \mathrm{TOP}^b(M \times \mathbb{R}^{n+1})/\mathrm{TOP}^b(M \times \mathbb{R}^n)$$

as in 1.11. *Assume that* $\dim(M) + n \geqslant 5$. *Then the inclusion*

$$F(\mathbb{R}^n) \hookrightarrow \lim_{k \to \infty} \Omega^k F(\mathbb{R}^{n+k}) = Q(\Sigma^n \Omega \underline{\underline{\mathrm{Wh}}}(M))$$

induces an isomorphism on π_j, for $0 < j \leqslant n$, and an injection on π_0. (The direct limit is one of virtual spaces, and is taken with respect to the maps σ defined in 1.11.)

Proof. Recall that 1.12 gives us a good understanding of the maps $\sigma\colon F(\mathbb{R}^n) \to \Omega F(\mathbb{R}^{n+1})$ once the functor Ω has been inflicted on them. It follows together with 5.3 that the induced map $\pi_j(F(\mathbb{R}^n)) \to \pi_j(\Omega F(\mathbb{R}^{n+1}))$ is an isomorphism for $0 < j \leqslant n$. Injectivity of the map $\pi_0(F(\mathbb{R}^n)) \to \pi_0(\Omega F(\mathbb{R}^{n+1}))$ is harder to prove, although 5.3 and 5.4 identify its source with a subset of its target. Concepts seem to fail at this point, so we use a trick.

Write $F(\mathbb{R}^n; M)$ instead of $F(\mathbb{R}^n)$, for greater precision. Feel free to define and use relative versions, such as $F(\mathbb{R}^n; M, \partial_0 M)$, where $\partial_0 M$ is a codimension zero submanifold of M. See 1.4 for relative notation.

Step 1: The map $\sigma\colon F(\mathbb{R}^n; M \times D^k) \to \Omega F(\mathbb{R}^{n+1}; M \times D^k)$ is an injection on π_0, if $k > 0$. (*Proof*: $F(\mathbb{R}^n; M \times D^k)$ is homotopy equivalent to a union of components of $\Omega^k F(\mathbb{R}^{n+k}; M)$ by 1.5. Again by 1.5, it is sufficient to know that $\sigma\colon F(\mathbb{R}^{n+k}; M) \to \Omega F(\mathbb{R}^{n+1+k}; M)$ is an injection on π_k, which we do know.)

Step 2: The inclusion of $\Omega F(\mathbb{R}^{n+1}; M \times D^k)$ in $\Omega F(\mathbb{R}^{n+1}; M \times D^k, M \times S^{k-1})$ is an injection on π_0. (*Proof*: Using 1.8 identify it with an inclusion map between concordance spaces, say i. This has an obvious left homotopy inverse r, so that $ri \simeq$ identity.)

Step 3: There is a commutative square

$$\begin{array}{ccc}
\pi_0(F(\mathbb{R}^n; M)) & \xrightarrow{\quad\alpha\quad} & \pi_1(F(\mathbb{R}^{n+1}; M)) \\
\downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\
\pi_0(F(\mathbb{R}^n; M \times D^k, M \times S^{k-1})) & \xrightarrow{\quad\beta\quad} & \pi_1(F(\mathbb{R}^{n+1}; M \times D^k, M \times S^{k-1}))
\end{array}$$

where the vertical arrows are obtained by taking products with D^k, and the horizontal arrows are induced by σ. Now suppose, for example, that $k = 4$. Then by 5.4 and a suitable relative version of 5.4, the map γ is injective and its image is contained in

$$\mathrm{im}[\pi_0(F(\mathbb{R}^n; M \times D^4) \to \pi_0(F(\mathbb{R}^n; M \times D^4, M \times S^3))]$$

because taking products with S^3 annihilates the algebraic torsion invariant of any bounded h-cobordism. Using steps 1 and 2, conclude that $\beta \cdot \gamma$ is injective. Therefore, α is injective.

6. Appendix: Materialization

Let Y be a virtual space. If $Y \neq \emptyset$, choose a base point in Y. Denote by $[X, Y]_{pt}$ the set of homotopy classes of pointed maps from X to Y, where X is any pointed connected CW-space. The contravariant functor $[-, Y]_{pt}$ satisfies the conditions in Brown's representation theorem [7]; the conclusion is that there exist a pointed connected CW-space X^u and a pointed continuous map f^u: $X^u \to Y$ inducing an isomorphism on homotopy groups. An obstruction theory argument then shows that

$$f^u_*: [-, X^u]_{pt} \to [-, Y]_{pt}$$

is an isomorphism of functors on the category of all pointed connected CW-spaces. (The same argument is normally used in proving Whiteheads theorem in homotopy theory.)

Arguing for each path component of Y separately, one can easily deduce that there exist a CW-space W^u and a continuous map g^u: $W^u \to Y$ which is a weak homotopy equivalence. See the definition preceding 0.8. Call g^u a CW-approximation of Y.

A more careful look at Brown's representation theorem gives the following result. If g: $W \to Y$ is any continuous map from a CW-space W to Y, then there exist a CW-space W^u containing W, and a continuous map g^u: $W^u \to Y$ extending g which is a weak homotopy equivalence. This can be used to show that CW-approximations of Y are sufficiently unique for all homotopy theoretic purposes. (Given two approximations, construct a third containing both of them, etc.)

Now let Y^{mat} be the simplicial set defined in 0.8, with geometric realization $|Y^{mat}|$. Let g^u: $W^u \to Y$ be a CW-approximation, and arrange W^u to be the geometric realization of a simplicial set. Then g^u determines a map $W^u \to |Y^{mat}|$ which is simplicial. For if f: $\Delta^k \to W^u$ is a k-simplex in W^u, then $g^u f$: $\Delta^k \to Y$ is a k-simplex in Y^{mat}. A brutal check on homotopy groups, which we leave to the reader, shows that this map $W^u \to |Y^{mat}|$ is a homotopy equivalence. Choosing a homotopy inverse $|Y^{mat}| \to W^u$, which is unique up to contractible choice, and composing with g^u we obtain a continuous map $|Y^{mat}| \to Y$ which is a weak homotopy equivalence.

6.1. OBSERVATION. Suppose that Y is a virtual space and W, X are CW-spaces. Then

$$[W, |(\mathrm{map}(X, Y))^{mat}|] \cong [W, \mathrm{map}(X, Y)] \cong [W \times X, Y] \cong [W \times X, |Y^{mat}|].$$

This shows that the virtual mapping spaces of 0.5. (vii) have the right homotopy type. Square brackets denote homotopy classes of maps.

6.2. OBSERVATION. If

$$\cdots \to Y_{n-1} \to Y_n \to Y_{n+1} \to \cdots$$

is a direct system of virtual spaces, with $n \in \mathbb{Z}$, then

$$\left(\lim_{n\to\infty} Y_n\right)^{\mathrm{mat}} \cong \lim_{n\to\infty} Y_n^{\mathrm{mat}}$$

(See 0.5.(vi); the limit on the right is one of simplicial sets.)

6.3. PROPOSITION. *Let $H \hookrightarrow J$ be an inclusion map of virtual spaces with group structure. Define J/H as in 0.5.(ix). Then the map $J^{\mathrm{mat}} \to (J/H)^{\mathrm{mat}}$ is onto with kernel H^{mat}, so that $(J/H)^{\mathrm{mat}} \cong J^{\mathrm{mat}}/H^{\mathrm{mat}}$.*

Proof. Inspection.

6.4. PROPOSITION. *Let $A \hookrightarrow Y$ be an inclusion map of virtual spaces. Write $Y_{\sim}$ for the virtual space quotient of Y by A (see 0.5.(iii)), and write $(Y^{\mathrm{mat}})_{\sim}$ for the simplicial set quotient of Y^{mat} by A^{mat}. Then the evident map from $(Y^{\mathrm{mat}})_{\sim}$ to $(Y_{\sim})^{\mathrm{mat}}$ is a homotopy equivalence of simplicial sets.*

Proof. Compose the evident map $|(Y^{\mathrm{mat}})_{\sim}| \to |(Y_{\sim})^{\mathrm{mat}}|$ with the canonical weak homotopy equivalence $|(Y_{\sim})^{\mathrm{mat}}| \to Y_{\sim}$. Our task is then to show that the resulting map $f\colon |(Y^{\mathrm{mat}})_{\sim}| \to Y_{\sim}$ is a weak homotopy equivalence.

Suppose then that $g\colon S^k \to Y_{\sim}$ is a continuous map, for some $k \geqslant 0$. We must factorize this through f, up to homotopy. By definition of $Y_{\sim}$ there exists an open covering $\{V_i\}$ of S^k and continuous maps $g_i\colon V_i \to Y$ such that the square

$$\begin{tikzcd}
V_i \arrow[r, "g_i"'] \arrow[d, hook] & Y \arrow[d] \\
S^k \arrow[r, "g"'] & Y_{\sim}
\end{tikzcd}$$

commutes for each i, and such that for arbitrary i, j we have either

$$g_{i|V_i \cap V_j} = g_{j|V_i \cap V_j}$$

or both $g_{i|V_i \cap V_j}$ and $g_{j|V_i \cap V_j}$ factor through $A \subset Y$. Now choose a triangulation of S^k such that each simplex is contained in one of the V_i. Choose an ordering on the set of vertices. This identifies S^k with the geometric realization of a simplicial set. Using this simplicial set structure on S^k, we see that the g_i define a simplicial map $\hat{g}$ from S^k to $(Y^{\mathrm{mat}})_{\sim}$. Namely, the restriction of g_i to any j-simplex in V_i gives a j-simplex in Y^{mat}. The image j-simplex in $(Y^{\mathrm{mat}})_{\sim}$ is well defined.

It is not difficult to see that $\hat{g}$ is the map we are looking for. Therefore

$$f_*\colon [S^k, (Y^{\mathrm{mat}})_{\sim}] \to [S^k, Y_{\sim}]$$

is onto for every $k \geqslant 0$. A relative version of the same argument shows that it is also injective. The usual obstruction theory argument then shows that

$$f_*: [X, (Y^{\mathrm{mat}})_\sim] \to [X, Y_\sim]$$

is a bijection for any CW-space X. This means that f is a weak homotopy equivalence.

Acknowledgement

It is a pleasure to thank Steve Ferry, Erik Pedersen, Bob Rigdon, Mark Steinberger, Bill Dwyer and Marcel Bökstedt for conversations, and Ib Madsen for drawing our attention to the theory of bounded homeomorphisms.

References

1. Alexander, J. W.: On the deformation of an n-cell, *Proc. Nat. Acad. Sci. U.S.A.* **9** (1923), 406–407.
2. Anderson, D. R. and Hsiang, W.-C.: The functors K_{-i} and pseudo-isotopies of polyhedra, *Ann. Math.* **105** (1977), 201–223.
3. Anderson, D. R. and Pedersen, E. K.: Semifree topological actions of finite groups on spheres, *Math. Annal.* **265** (1983), 23–44.
4. Barr, M. and Wells, C.: *Toposes, Triples and Theories*, Grundlehren series No. 278, Springer-Verlag, New York, Berlin, 1985.
5. Bass, H., Heller, A. and Swan, R.: The Whitehead group of a polynomial extension, *Publ. IHES.* **22** (1964).
6. Bödigheimer, C.-F.: Stable splittings of mapping spaces, Preprint Math. Gotting. series 42 (1985), Göttingen, F.R.G.
7. Brown, E. H. jr.: Cohomology theories, *Ann. Math.* **75** (1962), 467–484.
8. Burghelea, D. and Fiedorowicz, Z.: Hermitian algebraic K-theory of topological spaces, *Proc. 1982 Bielefeld Conf.*, Springer Lecture Notes in Math. 1046 (1984), 32–47.
9. Burghelea, D. and Lashof, R.: Geometric transfer and the homotopy type of the automorphism group of manifolds, *AMS Trans.* **269** (1982), 1–39.
10. Carter, D. W.: Localization in lower algebraic K-theory, *Comm. Algebra* **8** (1980), 603–622.
11. Carter, D. W.: Lower K-theory of finite groups, *Comm. Algebra* **8** (1980), 1927–1937.
12. Carter, D. W.: Lower K-theory of nilpotent groups, *J. Algebra* **66** (1980), 134–146.
13. Cerf, J.: Topologie de certains espaces de plongements, *Bull. Soc. Math. France* **89** (1961), 227–380.
14. Černavskii, A. V.: Local contractibility of the homeomorphism group of a manifold (Russian) *Math. Sbornik*, **79** (121) (1969), 307–356; (English) *Math. U.S.S.R. Sbornik* **8** (1969), No. 3, 287–333.
15. Chapman, T. A.: *Controlled Simple Homotopy Theory and Applications*, Springer Lecture Notes in Math. 1009 (1983).
16. Dold, A.: *Halbexakte Homotopiefunktoren*, Springer Lecture Notes in Math. 12 (1966).
17. Edwards, R. D. and Kirby, R. C.: Deformations of spaces of imbeddings, *Ann. Math.* **93** (1971), 63–88.
18. Hatcher, A. E.: Higher simple homotopy theory, *Ann. Math.* **102** (1975), 101–137.
19. Hatcher, A. E.: Concordance spaces, higher simple homotopy theory, and applications, *Proc. Symp. Pure Math.* **32**, part I, (1978), 3–21.
20. Hirsch, M. W.: *Differential Topology*, Graduate texts in Math., Springer-Verlag, New York, Heidelberg, Berlin, 1976.
21. Hsiang, W.-C. and Jahren, B.: A remark on the isotopy classes of diffeomorphisms of lens spaces, *Pacific J. Math.* **109** (1983), 411–423.
22. Hsiang, W.-C. and Sharpe, R.: Parametrized surgery and isotopy, *Pacific J. Math.* **67** (1976), 401–459.

23. Igusa, K.: Higher singularities of smooth functions are unnecessary, *Ann. Math.* **119** (1984), 1–58.
24. Igusa, K.: The stability theorem for smooth pseudo-isotopies, to appear in *K-Theory* **2** (1988).
25. Johnstone, P. T.: *Topos Theory*, Academic Press, London, New York, San Francisco (1977).
26. Kahn, D. S. and Priddy, S. B.: Applications of the transfer to stable homotopy theory, *AMS Bull.* **78** (1972), 981–987.
27. Kirby, R. C. and Siebenmann, L. C.: *Foundational Essays on Topological Manifolds, Smoothings and Triangulations*, Annals of Mathematics Studies 88, Princeton Univ. Press (1977).
28. Madsen, I. and Rothenberg, M.: Classifying G spheres, *AMS Bull.* **7**(1) (1982), 223–226.
29. Madsen, I. and Rothenberg, M.: On the classification of G spheres (Parts I, II, III), Aarhus Univ. Preprints, August/Sept. 1985.
30. May, J. P.: E_∞ *Ring Spaces* and E_∞ *Ring Spectra*, Springer Lecture Notes in Math. 577 (1977).
31. Pedersen, E. K.: On the bounded and thin h-cobordism theorem parametrized by $\mathbb{R}^k$, *Proc. 1985 Poznan Conference*, Springer Lecture Notes in Math. 1217 (1986), pp. 306–320.
32. Pedersen, E. K.: On the K_{-i} functors *J. Algebra* **90** (1984) 461–475.
33. Pedersen, E. K.: K_{-i} invariants of chain complexes, *Proc. 1982 Leningrad Conference*, Springer Lecture Notes in Math. 1060 (1984), 174–186.
34. Puppe, D.: On the stable homotopy category, Topology and its applications, *Budva 1972* (Beograd 1973), pp. 200–212.
35. Quinn, F.: Ends of maps I, *Ann. Math.* **110** (1979), 275–331.
36. Quinn, F.: Ends of maps II, *Invent. Math.* **68** (1982), 353–424.
37. Segal, G. B.: Operations in stable homotopy theory, *Proc. 1972 Oxford Algebraic Topology Symposium*, LMS Lecture Notes Series No. 11 (1974), pp. 105–110.
38. Siebenmann, L. C.: A total Whitehead torsion obstruction, *Comment. Mat. Helv.* **45** (1970), 1–48.
39. Siebenmann, L. C.: Deformations of homeomorphisms on stratified sets, *Comment. Math. Helv.* **47** (1972), 123–163.
40. Vogt, R.: Boardmans stable homotopy category, Aarhus Univ. Lecture Notes series No. 21, January 1970.
41. Waldhausen, F.: Algebraic K-theory of spaces, a manifold approach, *Canadian Mat. Soc. Conf. Proceedings*, Vol. 2, Part 1 (1982), pp. 141–184.
42. Waldhausen, F.: Algebraic K-theory of generalized free products, *Ann. Math.* **108** (1978), 135–256.
43. Wall, C. T. C.: *Surgery on Compact Manifolds*, Academic Press 1971.
44. Weiss, M.: Sphères exotiques et l'espace de Whitehead, *C.R. Acad. Sc. Paris*, **303**, Serie I, No. 17 (1986), pp. 885–888. (Note the following errata: in the diagram on p. 888, replace q by 2 and f' by f; in the diagram on p. 887 replace q by k and f' by f.)
45. Wraith, G. C.: *Lectures on Elementary Topoi*, Springer Lecture Notes in Math. 445 (1975), pp. 114–206.