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Automorphisms of Manifolds and Algebraic *K*-Theory: Part III

Michael S. Weiss Bruce E. Williams



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Abstract

The structure space $\mathcal{S}(M)$ of a closed topological *m*-manifold *M* classifies bundles whose fibers are closed *m*-manifolds equipped with a homotopy equivalence to M. We construct a highly connected map from $\mathcal{S}(M)$ to a concoction of algebraic L-theory and algebraic K-theory spaces associated with M. The construction refines the well-known surgery theoretic analysis of the block structure space of Min terms of *L*-theory.

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CHAPTER 1

Introduction

The structure space $\mathcal{S}(M)$ of a closed topological *m*-manifold *M* is the classifying space for bundles $E \to X$ with an arbitrary *CW*-space *X* as base, closed topological manifolds as fibers and with a fiber homotopy trivialization

$$E \to M \times X$$

(a homotopy equivalence and a map over X). The points of $\mathcal{S}(M)$ can loosely be imagined as pairs (N, f) where N is a closed m-manifold and $f: N \to M$ is a homotopy equivalence. To explain the relationship between $\mathcal{S}(M)$ and automorphisms of M, we invoke $\mathcal{H}om(M)$, the topological group of homeomorphisms from M to M, and G(M), the grouplike topological monoid of homotopy equivalences from M to M. In practice we work with simplicial models of $\mathcal{H}om(M)$ and G(M). The homotopy fiber of the inclusion $\mathcal{BH}om(M) \to \mathcal{BG}(M)$ is homotopy equivalent to a union of connected components of $\mathcal{S}(M)$.

The main result of this paper is a calculation of the homotopy type of $\mathcal{S}(M)$ in the so-called concordance stable range, in terms of L- and algebraic K-theory. With m fixed as above, we construct a homotopy invariant functor

$$(Y,\xi) \mapsto \mathbf{LA}_{\bullet\%}(Y,\xi,m)$$

from spaces Y with spherical fibrations ξ to spectra. The spectrum $\mathbf{LA}_{\bullet\%}(Y, \xi, m)$ is a concoction of the *L*-theory and the algebraic K-theory of spaces [27] associated with Y, compounded with an assembly construction [21]. (The subscript % is for homotopy fibers of assembly maps.) In the case where Y = M (nonempty and connected for simplicity) and ξ is ν , the normal fibration of M, there is a "local degree" map

$$\Omega^{\infty+m} \mathbf{LA}_{\bullet\%}(M,\nu,m) \longrightarrow 8\mathbb{Z} \subset \mathbb{Z}.$$

There is then a highly connected map

(1.1)
$$\mathcal{S}(M) \longrightarrow \operatorname{fiber}\left[\Omega^{\infty+m} \mathbf{LA}_{\bullet\%}(M,\nu,m) \xrightarrow{\operatorname{local deg.}} 8\mathbb{Z}\right]$$

where fiber in this case means the fiber over $0 \in 8\mathbb{Z}$, an infinite loop space. The connectivity estimate is given by the concordance stable range. In practice that translates into m/3 approximately, but in theory it is more convoluted and the reader is referred to definition 11.5. The result has a generalization to the case in which M is compact with nonempty boundary. It looks formally the same. Points of $\mathcal{S}(M)$ can be imagined as pairs (N, f) where N is a compact manifold with boundary and $f: (N, \partial N) \to (M, \partial M)$ is a homotopy equivalence of pairs restricting to a homeomorphism of ∂N with ∂M .

We now give a slightly more detailed, although still sketchy, definition of the spectrum $\Omega^m \mathbf{LA}_{\bullet\%}(Y,\xi,m)$. (Details can be found in chapter 9.) It is the total

homotopy fiber of a commutative square

The left-hand column is the quadratic L-theory assembly map. (This has a variety of equivalent descriptions and in the simplest of these it depends only on Y and the orientation double covering w_{ξ} of Y determined by ξ . We do not insist on such a simple description because that would make the horizontal maps in the diagram more obscure; therefore $\mathbf{L}_{\bullet}(Y,\xi)$ rather than $\mathbf{L}_{\bullet}(Y,w_{\xi})$ is the notation which we prefer.) The right-hand column is the Waldhausen A-theory assembly map with $S^1 \wedge$ and homotopy orbit construction inflicted. Both columns use a category of (finitely dominated) retractive spaces or spectra over Y, subject to finiteness conditions and equipped with a notion of Spanier-Whitehead duality which depends on ξ and m. The horizontal maps are variants of a natural transformation Ξ which was defined in [**37**]; the precise relationship will be clarified in a moment. By opting for finitely dominated retractive spaces in both columns we have implicitly selected the decoration p. (The infinite loop space $\Omega^{\infty+m} \mathbf{LA}_{\bullet\%}(Y, \xi, m)$ is decoration independent, i.e., any consistent choice of decoration from the list $h, p, \ldots, \langle -i \rangle, \ldots, \langle -\infty \rangle$ gives the same result up to a homotopy equivalence.)

This is a definition which relates $\Omega^m \mathbf{LA}_{\bullet\%}(Y,\xi,m)$ to known and trusted concepts in algebraic L- and K-theory. For our constructions we prefer another definition of $\Omega^m \mathbf{LA}_{\bullet\%}(Y,\xi,m)$ as the homotopy fiber of the map between homotopy pullbacks of the rows in the commutative diagram

where the left-hand column is the assembly map in a form of visible symmetric L-theory. The visible symmetric L-theory will be reviewed in chapter 3. There are forgetful natural transformations

$$\mathbf{L}_{\bullet}(Y,\xi) \longrightarrow \mathbf{VL}^{\bullet}(Y,\xi)$$

which fit into a homotopy cartesian square

$$\begin{array}{cccc} \mathbf{L}^{\%}_{\bullet}(Y,\xi) & \longrightarrow & \mathbf{VL}^{\bullet\%}(Y,\xi) \\ & & & \downarrow \\ & & & \downarrow \\ \mathbf{L}_{\bullet}(Y,\xi) & \longrightarrow & \mathbf{VL}^{\bullet}(Y,\xi). \end{array}$$

This will also be reviewed in chapter 3. Together with the norm fibration sequence

$$S^1 \wedge \mathbf{A}(Y,\xi,m)_{h\mathbb{Z}/2} \longleftarrow \mathbf{A}(Y,\xi,m)^{th\mathbb{Z}/2} \xleftarrow{\text{incl.}} \mathbf{A}(Y,\xi,m)^{h\mathbb{Z}/2}$$

(and a variant with $\mathbf{A}^{\%}$ instead of \mathbf{A}), this explains why the two competing definitions of $\mathbf{LA}_{\bullet\%}(Y,\xi,m)$, relying on diagrams (1.2) and (1.3) respectively, are consistent.

1. INTRODUCTION

Our reasons for preferring the second definition of $\mathbf{LA}_{\bullet\%}$ are strategic. Quadratic *L*-theory famously serves as a receptacle for relative invariants, such as surgery obstructions of degree one normal maps $X \to Y$. By contrast, visible symmetric *L*-theory is a mild refinement of symmetric *L*-theory and as such a good receptacle for absolute invariants: generalized signatures of Poincaré duality spaces, say. In particular a Poincaré duality space Y of formal dimension m, and with Spivak normal fibration ξ , determines a characteristic element

$$v_L(Y) \in \Omega^{\infty+m} \mathbf{VL}^{\bullet}(Y,\xi)$$

which can be viewed as a refined signature. (We think of it as a point in an infinite loop space, not a connected component of an infinite loop space.) Refining this some more to pick up algebraic K-theory information, we get

(1.4)
$$\sigma(Y) \in \Omega^{\infty+m} \mathbf{VLA}^{\bullet}(Y,\xi,m)$$

where

$$\Omega^{m} \mathbf{VLA}^{\bullet}(Y,\xi,m) := \operatorname{holim} \begin{pmatrix} \mathbf{A}(Y,\xi,m)^{h\mathbb{Z}/2} \\ & & \downarrow^{\operatorname{inclusion}} \\ \Omega^{m} \mathbf{VL}^{\bullet}(Y,\xi) \xrightarrow{\Xi} \mathbf{A}(Y,\xi,m)^{th\mathbb{Z}/2} \end{pmatrix}.$$

This refinement expresses a compatibility between $v_L(Y)$ and the self-dual Euler characteristic

$$v_K(Y) \in \Omega^{\infty}(\mathbf{A}(Y,\xi,m)^{h\mathbb{Z}/2})$$

We construct $\sigma(Y)$ in chapter 9. The construction enjoys continuity properties. It can be applied to the fibers of a fibration $E \to B$ whose fibers are Poincaré duality spaces E_b of formal dimension m, so that we obtain a section of a fibration on B whose fibers are certain infinite loop spaces.

Now suppose that the Poincaré duality space Y is a closed manifold of dimension m. Then the point $\sigma(Y)$ lifts across the visible L-theory and A-theory assembly maps to a point

(1.5)
$$\sigma^{\%}(Y) \in \Omega^{\infty+m} \mathbf{VLA}^{\bullet\%}(Y,\xi,m)$$

We construct $\sigma^{\%}(Y)$ in chapter 10. Again this construction enjoys continuity properties: it can be applied it to the fibers of a fibre bundle $E \to B$ whose fibers are closed manifolds E_b of dimension m, so that we obtain a section of a fibration on B whose fibers are certain infinite loop spaces. (This is very hard to establish, like the continuity property of excisive Euler characteristics in [10]. Relying on [16], we reduce to the case of fiber bundles with discrete structure group.)

In particular, the space $\mathcal{S}(M)$ carries a universal bundle $E \to \mathcal{S}(M)$ of closed manifolds with a fiber homotopy trivialization $E \simeq \mathcal{S}(M) \times M$. Therefore each point $(N, f) \in \mathcal{S}(M)$ determines an element $f_*\sigma^{\%}(N) \in \Omega^{\infty+m}\mathbf{VLA}^{\bullet\%}(M,\nu_M,m)$, whose image in $\Omega^{\infty+m}\mathbf{VLA}^{\bullet}(M,\nu_M,m)$ under assembly comes with a preferred path to $\sigma(M) \in \Omega^{\infty+m}\mathbf{VLA}^{\bullet}(M,\nu_M,m)$. This gives us the map (1.1): here it comes as a map from $\mathcal{S}(M)$ to the homotopy fiber, over the point $\sigma(M)$, of the assembly map

$$\Omega^{\infty+m}\mathbf{VLA}^{\bullet\%}(M,\nu_M,m)\longrightarrow \Omega^{\infty+m}\mathbf{VLA}^{\bullet}(M,\nu_M,m) .$$

Similar ideas, i.e., a firework of characteristics and signatures, can be used to show that the map (1.1) is highly connected; we give an overview in chapter 2 before developing the details.

This result has many precursors. The most fundamental and best known of these belong to surgery theory. From the surgery point of view it is very natural to introduce certain "block" structure spaces such as

$$\widetilde{\mathcal{S}}^s(M)$$
, $\widetilde{\mathcal{S}}^h(M)$.

These are designed in such a way that $\pi_0 \widetilde{\mathcal{S}}^s(M)$ and $\pi_0 \widetilde{\mathcal{S}}^h(M)$ are identifiable with, respectively, the subset of $\pi_0 \mathcal{S}(M)$ determined by the simple homotopy equivalences, and the quotient set of $\pi_0 \mathcal{S}(M)$ determined by the *h*-cobordism relation. In addition they have the property

$$\pi_i \widetilde{\mathcal{S}}^s(M) \cong \pi_0 \widetilde{\mathcal{S}}^s(M \times D^i) , \quad \pi_i \widetilde{\mathcal{S}}^h(M) \cong \pi_0 \widetilde{\mathcal{S}}^h(M \times D^i) .$$

This is obviously very useful in calculations. The surgery-theoretic calculations of these spaces are of the form

(1.6)
$$\begin{aligned} \mathcal{S}^{s}(M) \simeq & \text{fiber} \left[\Omega^{\infty+m} \mathbf{L}^{s}_{\bullet\%}(M, w) \longrightarrow 8\mathbb{Z} \right], \\ \widetilde{\mathcal{S}}^{h}(M) \simeq & \text{fiber} \left[\Omega^{\infty+m} \mathbf{L}^{h}_{\bullet\%}(M, w) \longrightarrow 8\mathbb{Z} \right], \end{aligned}$$

where $\mathbf{L}_{\bullet\%}^s$ and $\mathbf{L}_{\bullet\%}^h$ are homotopy invariant functors from spaces with double coverings to spectra. (In particular *w* denotes the orientation covering of *M*.) The functors $\mathbf{L}_{\bullet\%}^s$ and $\mathbf{L}_{\bullet\%}^h$ can be defined entirely in terms of algebraic *L*-theory, again compounded with assembly. They are therefore fully 4-periodic:

$$\Omega^{4}\mathbf{L}^{s}_{\bullet\%}(X,v) \simeq \mathbf{L}^{s}_{\bullet\%}(X,v) , \qquad \Omega^{4}\mathbf{L}^{h}_{\bullet\%}(X,v) \simeq \mathbf{L}^{h}_{\bullet\%}(X,v) .$$

This calculation of $\widetilde{\mathcal{S}}^{s}(M)$ and $\widetilde{\mathcal{S}}^{h}(M)$ is sometimes called the Casson-Sullivan-Wall-Quinn-Ranicki theorem. An earlier version of it, describing the homotopy groups of the block structure space(s), is known as the Casson-Sullivan-Wall long exact sequence. The space level formulation was championed by Quinn. The complete and final reduction to *L*-theory, at the space level, is mainly due to the untiring efforts of Ranicki. This took many years.

Our calculation of structure spaces $\mathcal{S}(M)$ in the concordance stable range is in agreement with the surgery theoretic calculation of block structure spaces. For example, there is a commutative diagram

where the upper horizontal arrow is the restriction of the map (1.1). Passing to vertical homotopy fibers, we obtain a highly connected map

 $\widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M) \longrightarrow \Omega^{\infty}(\mathbf{A}^{s}_{\%}(M,\nu,m)_{h\mathbb{Z}/2})$.

This is reminiscent of a highly connected map

$$\operatorname{TOP}(M)/\operatorname{TOP}(M) \longrightarrow \Omega^{\infty}(\mathbf{H}^{s}(M)_{h\mathbb{Z}/2})$$

constructed in [36]; see also [41] for notation. These two maps are intended to be the same, modulo Waldhausen's identification of the *h*-cobordism spectrum $\mathbf{H}^{s}(M)$ with $\mathbf{A}^{s}_{\%}(M)$. We do not quite prove that here, but we come close to it. It will be the theme of another paper in this series.

Meanwhile Burghelea and Lashof [6, cor. D] obtained results on the homotopy type of $\mathcal{S}(M)$. Localizing at odd primes, they were able to construct a highly connected map

$$\Omega \mathcal{S}(M) \longrightarrow \Omega \widetilde{\mathcal{S}}(M) \times \Omega^{\infty+1} \mathbf{A}_{\%}(M,\nu,m)^{h\mathbb{Z}/2}$$

(The localization is applied to $\mathcal{S}(M)$, $\widetilde{\mathcal{S}}(M)$ and $\mathbf{A}_{\%}(M, \nu, m)$ before other operations are carried out: Ω in both sides, $\Omega^{\infty+1}$ and the homotopy fixed point operation in the right-hand side.) After localization of $\mathbf{A}_{\%}(M, \nu, m)$ at odd primes, the homotopy fixed point spectrum $\mathbf{A}_{\%}(M, \nu, m)^{h\mathbb{Z}/2}$ is a wedge summand of $\mathbf{A}_{\%}(M)$ which depends only on ν and the parity of m.

With hindsight, the Burghelea-Lashof result can be explained in terms of our calculation of $\mathcal{S}(M)$ described above and the surgery-theoretic calculation of the block structure space. At odd primes, the six-term diagram (1.3) simplifies because the Tate constructions in the middle column (again to be applied after the localization of $\mathbf{A}_{\%}(Y, \xi, m)$) vanish. Therefore at odd primes

 $\Omega^m \mathbf{LA}_{\bullet\%}(Y,\xi,m) \simeq \Omega^m \mathbf{L}_{\bullet\%}(Y,\xi) \vee \mathbf{A}_{\%}(Y,\xi,m)^{h\mathbb{Z}/2} .$

This paper is a continuation of [36] and [37]. In another sense it is a continuation of [10]. For technical support, we use a fair amount of controlled topology as in [3], the Thurston-Mather-McDuff-Segal discrete approximation theory [16] for homeomorphism groups as in [10], and Spanier-Whitehead duality theory with its implications for algebraic K-theory as in [39].

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CHAPTER 2

Outline of proof

In the introduction, we gave a rough description of certain invariants of type signature and Euler characteristic for manifolds and Poincaré duality spaces. This led us to a map of the form (1.1). We wish to show that the map is highly connected. The main tools in the proof are

- (i) a controlled version of the Casson-Sullivan-Wall-Quinn-Ranicki (CSWQR) theorem in surgery theory;
- (ii) more invariants of type signature and Euler characteristic for manifolds and Poincaré duality spaces in a controlled setting;
- (iii) a simple downward induction, where the induction beginning relies on (i) while (ii) enables us to do the induction steps.

Let $\mathcal{S}(M \times \mathbb{R}^i; c)$ be the *controlled* structure space of $M \times \mathbb{R}^i$; here we view $M \times \mathbb{R}^i$ as an open dense subset of the join $M * S^{i-1}$. An element of $\mathcal{S}(M \times \mathbb{R}^i; c)$ should be thought of as a pair (N, f) where N is a manifold of dimension m + i, without boundary, and $f: N \to M \times \mathbb{R}^i$ is a *controlled* homotopy equivalence [3]. There is also a controlled block structure space

$$\widetilde{\mathcal{S}}^{cs}(M \times \mathbb{R}^i; c)$$

where the decoration cs (controlled simple) indicates that we allow only structures with vanishing controlled Whitehead torsion.

The homotopy type of $\widetilde{\mathcal{S}}^{cs}(M \times \mathbb{R}^i; c)$ can be described by a formula which combines the CSWQR ideas with controlled algebra [3]: namely,

(2.1)
$$\widetilde{\mathcal{S}}^{cs}(M \times \mathbb{R}^i; c) \simeq \text{ fiber } \left[\Omega^{\infty + m + i} \mathbf{L}^{cs}_{\bullet \%}(M \times \mathbb{R}^i, \nu; c) \longrightarrow 8\mathbb{Z} \right]$$

where $\mathbf{L}^{cs}_{\bullet}(M \times \mathbb{R}^{i}, \nu; c)$ is the controlled quadratic *L*-theory (with vanishing controlled Whitehead torsion) of the control space $(M * S^{i-1}, M \times \mathbb{R}^{i})$. Taking *i* to the limit we have

$$\operatorname{colim}_{i \ge 0} \ \widetilde{\mathcal{S}}^{cs}(M \times \mathbb{R}^i; c) \ \simeq \ \operatorname{fiber} \left[\begin{array}{c} \operatorname{colim}_{i \ge 0} \ \Omega^{\infty + m + i} \mathbf{L}^{cs}_{\bullet \, \%}(M \times \mathbb{R}^i, \nu; c) \longrightarrow 8\mathbb{Z} \end{array} \right]$$

where the colimits are formed using product with \mathbb{R} in various shapes. Moreover, it is well-known [36] that the inclusions

$$\underset{i \ge 0}{\operatorname{colim}} \ \mathcal{S}(M \times \mathbb{R}^i; c) \longleftarrow \ \underset{i \ge 0}{\operatorname{colim}} \ \mathcal{S}^{cs}(M \times \mathbb{R}^i; c) \longrightarrow \ \underset{i \ge 0}{\operatorname{colim}} \ \widetilde{\mathcal{S}}^{cs}(M \times \mathbb{R}^i; c)$$

are homotopy equivalences. Therefore we have (2.2)

$$\operatorname{colim}_{i \ge 0} \ \mathcal{S}(M \times \mathbb{R}^i; c) \ \simeq \ \operatorname{fiber} \left[\begin{array}{c} \operatorname{colim}_{i \ge 0} \ \Omega^{\infty + m + i} \mathbf{L}^{cs}_{\bullet \ \%}(M \times \mathbb{R}^i, \nu; c) \longrightarrow 8\mathbb{Z} \end{array} \right]$$

and this is the starting point for our downward induction.

2. OUTLINE OF PROOF

Next we discuss the induction steps. Let (\bar{Y}, Y) be a control space. For the present purposes we can take this to mean that \bar{Y} is compact metrizable, and Y is open dense in \bar{Y} . A choice of spherical fibration ξ on Y and integer m makes the Waldhausen category of locally finitely dominated retractive spaces over Y into a Waldhausen category with duality (see [10] for details). By forming *L*-theory, *K*-theory etc., we define spectrum-valued functors

$$(\bar{Y}, Y, \xi) \mapsto \begin{cases} \mathbf{L}_{\bullet}(Y, \xi; c), \\ \mathbf{VL}^{\bullet}(Y, \xi; c), \\ \mathbf{A}(Y; c), \\ \mathbf{LA}_{\bullet}(Y, \xi, m; c), \\ \mathbf{VLA}^{\bullet}(Y, \xi, m; c) \end{cases}$$

much as before. (Three of these can be viewed as functors of a general Waldhausen category with duality; the ones having a V in their name use more special features.) The symbol c is a shorthand for control conditions, allowing us to avoid direct reference to the inclusion $Y \to \overline{Y}$. There are natural assembly transformations

(2.3)

$$\begin{aligned}
\mathbf{L}_{\bullet}^{\%}(Y,\xi;c) &\longrightarrow \mathbf{L}_{\bullet}(Y,\xi;c), \\
\mathbf{VL}^{\bullet}^{\%}(Y,\xi;c) &\longrightarrow \mathbf{VL}^{\bullet}(Y,\xi;c), \\
\mathbf{A}^{\%}(Y;c) &\longrightarrow \mathbf{A}(Y;c), \\
\mathbf{LA}_{\bullet}^{\%}(Y,\xi,m;c) &\longrightarrow \mathbf{LA}_{\bullet}(Y,\xi,m;c), \\
\mathbf{VLA}^{\bullet}^{\%}(Y,\xi,m;c) &\longrightarrow \mathbf{VLA}^{\bullet}(Y,\xi,m;c).
\end{aligned}$$

where the domain is now designed so that its homotopy groups are the *locally finite* generalized homology groups of Y with (twisted where appropriate) coefficients in $\mathbf{L}_{\bullet}(\star,\xi), \mathbf{VL}^{\bullet\%}(\star,\xi), \mathbf{A}(\star), \mathbf{LA}_{\bullet}(\star,\xi,m)$ and $\mathbf{VLA}^{\bullet\%}(\star,\xi,m)$. Here \star should be thought of as a variable point in Y, and we restrict ξ from Y to that point where necessary. The homotopy fibers of the assembly maps (2.3) are denoted by

(2.4)
$$\begin{array}{rcl} \mathbf{L}_{\bullet\%}(Y,\xi;\,c) \\ \simeq & \mathbf{VL}^{\bullet}_{\%}(Y,\xi;\,c), \\ & \mathbf{A}_{\%}(Y;\,c), \\ & \mathbf{LA}_{\bullet\%}(Y,\xi,m;\,c) \\ \simeq & \mathbf{VLA}^{\bullet}_{\%}(Y,\xi,m;\,c), \end{array}$$

respectively. (The homotopy equivalences asserted here are nontrivial; they are established in chapter 6.) If (\bar{Y}, Y) happens to be a *controlled* Poincaré duality space of formal dimension m and with Spivak normal fibration ξ , then there is a signature invariant

(2.5)
$$\sigma(Y) \in \Omega^{\infty+m} \mathbf{VLA}^{\bullet}(Y, \xi, m; c)$$

which generalizes (1.4). This invariant has the expected naturality and continuity properties. It is constructed in chapter 9.

If Y happens to be a manifold of dimension m and $\xi = \nu$ is its normal bundle, then (\bar{Y}, Y) is automatically a controlled Poincaré duality space of formal dimension m and the signature invariant $\sigma(Y)$ lifts across the assembly map (2.3) to an element

(2.6)
$$\sigma^{\%}(Y) \in \Omega^{\infty+m} \mathbf{VLA}^{\bullet\%}(Y,\xi,m),$$

generalizing (1.5). This lift is constructed in chapter 10. In particular, the space $\mathcal{S}(M \times \mathbb{R}^i; c)$ carries a universal bundle where each fiber is an (m + i)-manifold N together with a controlled homotopy equivalence $f: N \to M \times \mathbb{R}^i$. We may compactify each fiber N to a control space $\overline{N} = N \cup S^{i-1}$ in such a way that N

is open dense in \overline{N} and f extends to a map from \overline{N} to $M * S^{i-1}$. Therefore each point $(N, f) \in \mathcal{S}(M \times \mathbb{R}^i; c)$ determines an element

$$f_*\sigma^{\%}(N) \in \Omega^{\infty+m+i} \mathbf{VLA}^{\bullet\%}(M \times \mathbb{R}^i, \nu, m+i; c)$$

whose image in $\Omega^{\infty+m+i}$ **VLA**• $(M \times \mathbb{R}^i, \nu, m+i; c)$ under assembly (2.3) comes with a preferred path to $\sigma(M \times \mathbb{R}^i)$. If this construction were to enjoy certain continuity properties, it would give us a map

$$\mathcal{S}(M \times \mathbb{R}^i; c) \longrightarrow \Omega^{\infty + m + i} \mathbf{LA}_{\bullet\%}(M \times \mathbb{R}^i, \nu, m + i; c)$$

generalizing (1.1), where we think of the target as the homotopy fiber over the point $\sigma(M \times \mathbb{R}^i)$ of the appropriate assembly map in controlled **VLA**[•] theory of $M \times \mathbb{R}^i$. Unfortunately we could not avoid some sacrifices in establishing the continuity properties, and so we only get a map

(2.7)
$$\mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^{i}; c) \longrightarrow \Omega^{\infty + m + i} \mathbf{LA}_{\bullet\%}(M \times \mathbb{R}^{i}, \nu, m + i; c)$$

where $\mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^i; c) \subset \mathcal{S}(M \times \mathbb{R}^i; c)$ is the union of the connected components of $\mathcal{S}(M \times \mathbb{R}^i; c)$ which are reducible in the sense that they come from $\pi_0 \mathcal{S}(M)$. Combining the maps (2.7) for all $i \geq 0$ results in a commutative ladder

where the vertical arrows are given by product with $\mathrm{id}_{\mathbb{R}}$ in the left-hand column, and product with $\sigma^{\%}(\mathbb{R})$ in the right-hand column. Each vertical arrow in the lefthand column induces a surjection on π_0 . At the bottom of the ladder we recognize the map (1.1) and at the top we recognize with a small effort (see chapter 13) the map of (2.2). In particular, all homotopy fibers of the horizontal map at the top of the ladder are either contractible or empty. We use downward induction to establish a similar property for *all* horizontal maps in the ladder:

(†) for each of these maps, all homotopy fibers are highly connected or empty.

It is enough show that in each square

(2.9)
$$\begin{array}{c} \mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^{i+1}; c) \longrightarrow \Omega^{\infty+m+i+1} \mathbf{LA}_{\bullet\%}(M \times \mathbb{R}^{i+1}, \nu, m+i+1; c) \\ \uparrow \\ \mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^{i}; c) \longrightarrow \Omega^{\infty+m+i} \mathbf{LA}_{\bullet\%}(M \times \mathbb{R}^{i}, \nu, m+i; c) \end{array}$$

of the ladder, all total homotopy fibers are highly connected or empty.

Each *vertical* homotopy fiber in the left-hand column can be identified with a union of connected components of a controlled *h*-cobordism space $\mathcal{H}(N \times \mathbb{R}^i; c)$, where N is some closed *m*-manifold homotopy equivalent to M. By an easy calculation carried out mainly in chapter 7, the vertical homotopy fibers in the right-hand column have the form

$$\Omega^{\infty} \mathbf{A}_{\%}(M \times \mathbb{R}^{i}; c) \simeq \Omega^{\infty} \mathbf{A}_{\%}(N \times \mathbb{R}^{i}; c)$$

With these descriptions, the map between matching vertical homotopy fibers in (2.9) extends to a controlled form

(2.10)
$$\mathcal{H}(N \times \mathbb{R}^i; c) \longrightarrow \Omega^{\infty} \mathbf{A}_{\%}(N \times \mathbb{R}^i; c)$$

of Waldhausen's map relating h-cobordism spaces to A-theory. This is verified in chapter 13. The map (2.10) is highly connected. So all its homotopy fibers are highly connected, and so our claim regarding (2.9) is proved, and claim (†) is also established. In particular, any homotopy fiber of our map

$$\mathcal{S}(M) \longrightarrow \Omega^{\infty+m} \mathbf{LA}_{\bullet\%}(M,\nu,m)$$

is highly connected of empty. It only remains to show that the nonempty homotopy fibers correspond to elements of $\Omega^{\infty+m} \mathbf{LA}_{\bullet\%}(M,\nu,m)$ whose connected component is in the kernel of the local degree homomorphism to $8\mathbb{Z}$.

For this we use the commutative diagram

$$\begin{array}{ccc} \pi_0 \mathcal{S}(M) \longrightarrow \pi_m \mathbf{LA}_{\bullet\%}(M,\nu,m) \longrightarrow 8\mathbb{Z} \\ \text{induced by incl.} & & & \text{forget} & = \\ & & & \\ \pi_0 \widetilde{\mathcal{S}}^h(M) \longrightarrow \pi_m \mathbf{L}_{\bullet\%}(M,\nu) \longrightarrow 8\mathbb{Z} \end{array}$$

where the lower row is short exact. The left-hand vertical arrow is onto by definition. Its fibers are the orbits of an action of $Wh(\pi_1 M)$ on $\pi_0 \mathcal{S}(M)$. By direct calculation, and almost by construction, the middle vertical arrow (which is a group homomorphism) is also onto and its kernel is the image of a homomorphism

(2.11)
$$\pi_0((\mathbf{A}^h_{\%}(M,\nu,m))_{h\mathbb{Z}/2}) \longrightarrow \pi_m \mathbf{L} \mathbf{A}_{\bullet\%}(M,\nu,m).$$

Here $\pi_0((\mathbf{A}^h_{\%}(M,\nu,m))_{h\mathbb{Z}/2})$ is a quotient of $Wh(\pi_1 M)$. Hence we need to show that the action of the Whitehead group in the upper left-hand term corresponds in the upper middle term to a translation action, using the homomorphism (2.11). This is done in chapter 13.

CHAPTER 3

Visible *L*-theory revisited

Mishchenko [17] and Ranicki [19], [20] introduced "symmetric structures" on certain chain complexes over rings with involution with a view to understanding signature invariants and product formulae in surgery theory. For a ring R with involution (=involutory antiautomorphism) $r \mapsto \bar{r}$ and a bounded chain complex Cof finitely generated projective left R-modules, a symmetric structure of dimension m on C is a chain map of $\mathbb{Z}[\mathbb{Z}/2]$ -module chain complexes

$$\varphi: \Sigma^m W \longrightarrow C^t \otimes_R C$$
.

Here C^t is C with the right R-module structure defined by $xr = \bar{r}x$ and W denotes the standard resolution of \mathbb{Z} by free $\mathbb{Z}[\mathbb{Z}/2]$ -modules:

$$\mathbb{Z}[\mathbb{Z}/2] \xleftarrow{1-T} \mathbb{Z}[\mathbb{Z}/2] \xleftarrow{1+T} \mathbb{Z}[\mathbb{Z}/2] \xleftarrow{1-T} \mathbb{Z}[\mathbb{Z}/2] \xleftarrow{1+T} \cdots$$

The value of φ on $1 \in W_0$ is an *m*-cycle in $C^t \otimes_R C$, corresponding to a degree *m* chain map from the dual C^{-*} to *C*. If this is a chain homotopy equivalence, φ is called *nondegenerate*. The bordism groups of objects (C, φ) as above, with nondegenerate φ of dimension *m*, are the symmetric *L*-groups $L^m(R)$. (This definition of $L^m(R)$ is in agreement with [**21**] but in slight disagreement with [**19**] because we do not require that $C_k = 0$ for $k \notin \{0, 1, \ldots, n\}$.) Ranicki's analogous description of the quadratic *L*-groups $L_n(R)$, in which the homotopy fixed point construction $\max_{\mathbb{Z}[\mathbb{Z}/2]}(W, -)$ is replaced by a homotopy orbit construction $W \otimes_{\mathbb{Z}[\mathbb{Z}/2]} -$, makes it easy to define multiplication operators

$$L^m(R_1) \otimes L_n(R_2) \longrightarrow L_{m+n}(R_1 \otimes R_2).$$

A (connected) Poincaré duality space X of formal dimension m with fundamental group π and orientation character $w: \pi \to \{\pm 1\}$ determines

- an involution on $R = \mathbb{Z}\pi$ given by $g \mapsto w(g) \cdot g^{-1}$ for $g \in \pi \subset \mathbb{Z}\pi$,
- a chain complex C over $R = \mathbb{Z}\pi$, the singular or cellular chain complex of the universal cover of X, and
- a nondegenerate *n*-dimensional structure φ on *C*, obtained by evaluating the Eilenberg–Zilber diagonal chain map on the fundamental class of *X*.

The corresponding element in $L^m(R)$ is the symmetric signature $\sigma^*(X)$ of X. If X is a closed manifold and $f: X' \to Y$ is a degree one normal map from a closed *n*-manifold to a Poincaré duality space of formal dimension *n*, then

$$(\mathrm{id}_X \times f) : X \times X' \to X \times Y$$

is also a degree one normal map. The surgery obstructions $\sigma_*(f)$ and $\sigma_*(\mathrm{id}_X \times f)$ are related by

$$\sigma_*(\mathrm{id}_X \times f) = \sigma^*(X) \cdot \sigma_*(f) \in L_{m+n}(\mathbb{Z}\pi \otimes \mathbb{Z}\pi'),$$

using the above product, with $\pi' = \pi_1(Y)$.

A few years later it was found [32] that the symmetric *L*-groups admit a homological description relative to the quadratic *L*-groups. That is, there is a long exact sequence

$$\cdots \to L_n(R) \to L^n(R) \to \widehat{L}^n(R) \to L_{n-1}(R) \to \cdots$$

and the calculation of the relative terms $\widehat{L}^n(R)$ is "only" a matter of homological algebra. Efforts to reduce the homological algebra to an absolute minimum eventually led to the visible symmetric *L*-groups $VL^n(R)$, defined for group rings $R = \mathbb{Z}\pi$. We now recall their definition, following [**33**].

Let $R = \mathbb{Z}\pi$ with involution $g \mapsto w(g) \cdot g^{-1}$ for some homomorphism $w: \pi \to \{\pm 1\}$. Let C be a chain complex of f.g. projective left R-modules, bounded as before. A symmetric structure of dimension n on C can be viewed as an n-cycle in

$$((C^t \otimes C)^{h\mathbb{Z}/2})_{\pi} \cong ((C^t \otimes C)_{\pi})^{h\mathbb{Z}/2}$$

where the various subscripts and superscripts indicate orbit constructions and homotopy fixed point constructions for the appropriate symmetry groups, here π and $\mathbb{Z}/2$. (Note that π acts diagonally on $C^t \otimes C$.) A visible symmetric structure of dimension m on C is an m-cycle in

$$((C^t \otimes C)^{h\mathbb{Z}/2})_{h\tau}$$

where $(-)_{h\pi}$ means $(-\otimes P)_{\pi}$ for a resolution P of the trivial module \mathbb{Z} by projective $\mathbb{Z}\pi$ -modules. In contrast to $C^t \otimes C$ and $(C^t \otimes C)_{h\mathbb{Z}/2}$, the chain complex $(C^t \otimes C)^{h\mathbb{Z}/2}$ is not bounded below if $C \neq 0$, so that there is no good reason to think that the augmentation-induced chain map

$$((C^t \otimes C)^{h\mathbb{Z}/2})_{h\pi} \longrightarrow ((C^t \otimes C)^{h\mathbb{Z}/2})_{\pi}$$

should induce an isomorphism in homology. In fact visible symmetric structures generally carry more information than symmetric structures. It is sometimes convenient to organise both types of structures into homotopy classes: the groups of such homotopy classes are denoted

$$Q^m(C) = H_m((C^t \otimes C)^{h\mathbb{Z}/2})_{\pi}), \qquad VQ^m(C) = H_m((C^t \otimes C)^{h\mathbb{Z}/2})_{h\pi}).$$

With a view to generalizations later on, we mention that there is a homotopy (co)cartesian square of chain complexes

where the ${}^{th\mathbb{Z}/2}$ superscript denotes a Tate construction hom_{$\mathbb{Z}[\mathbb{Z}/2]$}(\widehat{W} ,—), with

$$\begin{aligned} & \widehat{W}_k &= \mathbb{Z}[\mathbb{Z}/2] \quad \text{for all } k \in \mathbb{Z}, \\ & d_k \colon \widehat{W}_k \to \widehat{W}_{k-1} \quad ; \quad z \mapsto (1 + (-1)^k T) z \, . \end{aligned}$$

This is reflected in a long exact Mayer–Vietoris sequence

$$\dots \to \widehat{Q}^{n+1}(C) \to VQ^n(C) \to Q^n(C) \oplus V\widehat{Q}^n(C) \to \widehat{Q}^n(C) \to VQ^{n-1}(C) \to \dots$$

where $V\widehat{Q}^n(C) = H_n(((C^t \otimes C)^{th\mathbb{Z}/2})_{h\pi})$ and $\widehat{Q}^n(C) = H_n(((C^t \otimes C)^{th\mathbb{Z}/2})_{\pi}).$

The visible hyperquadratic theory has the property of being invariant under a "change of rings". That is, for a bounded chain complex C of f.g. left projective $\mathbb{Z}\pi$ -modules and a homomorphism $h: \pi \to \pi'$ we have

$$V\widehat{Q}^*(C) \cong V\widehat{Q}^*(C')$$

where $C' = \mathbb{Z}\pi' \otimes_{\mathbb{Z}\pi} C$. (It is understood that π and π' are equipped with homomorphisms to $\{\pm 1\}$ and that *h* respects these.) We also note that

$$V\widehat{Q}^* = \widehat{Q}^*$$

in the case $\pi = \{1\}$. These two properties, suitably sharpened, could be used to characterize the visible hyperquadratic theory in terms of the ordinary hyperquadratic one. But we need not go into that.

A visible symmetric structure on C is considered nondegenerate if the induced symmetric structure is nondegenerate. The bordism groups of chain complexes Cas above with a nondegenerate m-dimensional visible symmetric structure are the visible symmetric L-groups $VL^m(\mathbb{Z}\pi)$. A mild improvement on the Mishchenko-Ranicki construction of the symmetric signature of a Poincaré duality space X of formal dimension m gives the visible symmetric signature

$$\sigma^*(X) \in VL^m(\mathbb{Z}\pi).$$

Other useful features of the symmetric L-groups (such as the products and the product formula for surgery obstructions) can be transferred to the visible symmetric L-groups by means of the forgetful homomorphisms $VL^m(\mathbb{Z}\pi) \to L^m(\mathbb{Z}\pi)$.

The main result of [33] is a long exact sequence relating the quadratic *L*-groups of $\mathbb{Z}\pi$ to the visible symmetric *L*-groups, with "easy" relative terms:

$$\cdots \to L_n(\mathbb{Z}\pi) \to VL^n(\mathbb{Z}\pi) \to \bigoplus_{i+j=n} H_i(B\pi; \widehat{L}^j(\mathbb{Z})) \to L_{n-1}(\mathbb{Z}\pi) \to \cdots$$

Ranicki [21] found a generalization of this from the group ring case to the case of a simplicial group ring $\mathbb{Z}[\Omega X]$, and used it in a revised approach to his total surgery obstruction theory, a project going back to [18]. He defines a visible symmetric L-theory spectrum which we (not he) denote by $\mathbf{VL}^{\bullet}(\mathbb{Z}[\Omega X])$, with homotopy groups $VL^{n}(\mathbb{Z}[\Omega X])$. There is a long exact sequence

$$\cdots \to L_n(\mathbb{Z}\pi) \to VL^n(\mathbb{Z}[\Omega X]) \to \bigoplus_{i+j=n} H_i(X; \widehat{L}^j(\mathbb{Z})) \to L_{n-1}(\mathbb{Z}\pi) \to \cdots,$$

identical with the one above when $X = B\pi$. One of Ranicki's main results in [21] states roughly that the closed manifold structures on an oriented Poincaré duality space X of formal dimension n are in canonical bijection with the connected components of the homotopy fiber of the assembly map

$$\Omega^{\infty+m}(X_+ \wedge \mathbf{VL}^{\bullet}(\mathbb{Z})) \longrightarrow \Omega^{\infty+m}\mathbf{VL}^{\bullet}(\mathbb{Z}[\Omega X])$$

over the point determined by $\sigma^*(X)$, where $\mathbf{VL}^{\bullet}(\ldots)$ denotes visible *L*-theory spectra. (To be more precise, each of these connected components determines a class in $\pi_m(\Omega^{\infty}(X_+ \wedge \mathbf{VL}^{\bullet}(\mathbb{Z})))$ which has a *local signature* $d \in 8\mathbb{Z}+1$; the condition d = 1 must be added.) This is obviously relevant to our program.

We come to a definition of visible symmetric structures in the setting of retractive spaces and retractive spectra. Let Y_1 and Y_2 be finitely dominated retractive spaces over X, with retractions r_1 and r_2 . We recall first the definition of an "unstable" Spanier–Whitehead product $Y_1 \downarrow Y_2$ from [39, 1.A.3]. This is the based space obtained by first forming the external smash product

$$Y_1 \ _X \wedge_X \ Y_2 \quad = \quad Y_1 \times Y_2 / \sim$$

where ~ identifies (y_1, x) with $(r_1(y_1), x)$ and (x, y_2) with $(x, r_2(y_2))$; then taking the homotopy pullback of

$$X \xrightarrow{\text{diagonal}} X \times X \xleftarrow{\text{retraction}} Y_1 \ _X \wedge_X \ Y_2$$

and then dividing that by the homotopy pullback of

$$X \xrightarrow{\text{diagonal}} X \times X \xleftarrow{\text{id}} X \times X.$$

We make this unstable SW product "stable" essentially by applying $\Omega^{\infty}\Sigma^{\infty}$. More technically, however, we have to work in a stable category of retractive spaces with objects of the form (Y, k) where Y is a (finitely dominated) retractive space over X and $k \in \mathbb{Z}$. The set of morphisms from (Y, k) to (Y', ℓ) in the stable category is

$$\operatorname{colim}_i \operatorname{mor}_{\operatorname{uns}}(\Sigma^{i-k}Y, \Sigma^{i-\ell}Y')$$

where Σ is short for Σ_X and mor_{uns} refers to morphisms in the ordinary category of retractive spaces over X. It is worth noting that $(\Sigma^k Y, k)$ is isomorphic to (Y, 0) in the stable category, so that (Y, k) can be regarded as a formal k-fold desuspension of Y alias (Y, 0).

DEFINITION 3.1. We let

$$(Y_1, k) \odot (Y_2, \ell) = \operatorname{colim}_n \Omega^{2n} (\Sigma^{n-k} Y_1 \downarrow \Sigma^{n-\ell} Y_2).$$

More generally, we let

$$(Y_1,k) \odot_j (Y_2,\ell) = \operatorname{colim}_n \Omega^{2n} \Sigma^j (\Sigma^{n-k} Y_1 \land \Sigma^{n-\ell} Y_2)$$

so that $(Y_1, k) \odot (Y_2, \ell) = (Y_1, k) \odot_0 (Y_2, \ell)$, and denote the Ω -spectrum with *j*-th term $(Y_1, k) \odot_j (Y_2, \ell)$ by

$$(Y_1,k)\odot_{\bullet}(Y_2,\ell)$$
.

(By convention $\Omega^m Z$, for a based space Z and an integer $m \ge 0$, is the geometric realization of the simplicial set whose *n*-simplices are the based maps from the one-point compactification of $\Delta^n \times \mathbb{R}^m$ to Z. Hence all the spaces and spectra in definition 3.1 are CW spaces and CW spectra.)

Note that \odot_{\bullet} comes with a structural symmetry $(Y_1, k) \odot_{\bullet} (Y_2, \ell) \cong (Y_2, \ell) \odot_{\bullet} (Y_1, k)$ determined by the obvious symmetry of λ . For $Y_1 = Y_2 = Y$ and $k = \ell$ we obtain an Ω -spectrum $(Y, k) \odot_{\bullet} (Y, k)$ with an action of $\mathbb{Z}/2$.

DEFINITION 3.2. An *n*-dimensional symmetric structure on (Y, k) is an element of $\Omega^n((Y,k) \odot (Y,k))^{h\mathbb{Z}/2}$. An *n*-dimensional visible symmetric structure on (Y,k)is an element of $\Omega^n((Y,k) \odot (Y,k))^{\mathbb{Z}/2}$.

The first part of this definition comes from [39], but the second part is new. (We are extremely grateful to John Klein for suggesting it as an improvement on some earlier attempts of ours.) It is best understood from the point of view of equivariant homotopy theory. The Ω -spectrum $(Y, k) \odot_{\bullet} (Y, k)$, with the action of $\mathbb{Z}/2$, turns out to be the "underlying spectrum" of a $\mathbb{Z}/2$ -spectrum in the sense of the equivariant theory [8], [15]. (See also [2].) We shall explain this using the following (conservative) language. CONVENTIONS 3.3. Let G be a finite group, W the regular representation of G. Let nW be the direct sum of n copies of W, with one-point compactification S^{nW} . A G-spectrum C is a family of well-based G-spaces C_{nW} , defined for all sufficiently large positive integers n, together with based G-maps

$$S^W \wedge C_{nW} \to C_{(n+1)W},$$

with the diagonal action of G on $S^W \wedge C_{nW}$. The underlying spectrum $u\mathbf{C}$ of \mathbf{C} is the ordinary spectrum whose j-th space is

$$\operatorname{colim}_n \Omega^{nW} \Sigma^j C_{nW}.$$

It is an ordinary Ω -spectrum with a degreewise action of G (coming from the conjugation action of G on $\Omega^{nW} \Sigma^j C_{nW}$, for each n and j). The subspectrum of fixed points, $(u\mathbf{C})^G$, is often called the *fixed point spectrum of* \mathbf{C} . It is again an Ω -spectrum.

REMARK. The above definition of a G-spectrum is economical in that we only use the representations nW for bookkeeping. The price for that is a mildly undermotivated definition of the "underlying spectrum". As before, the loop spaces Ω^{nW} which appear in the definition of the underlying spectrum are to be constructed as geometric realizations of certain simplicial sets, so that the passage to the (co)limit is safe from the point of view of homotopy theory. Note that if G is trivial, $G = \{1\}$, then $u\mathbf{C}$ is simply a CW-substitute for \mathbf{C} .

Beware that the expressions *spectrum* and Ω -*spectrum*, as used here, correspond roughly to *prespectrum* and *spectrum*, respectively, in the language of [8] and [15] for example.

Returning to definition 3.2 now, we have that $(Y, k) \odot_{\bullet} (Y, k)$ is the underlying spectrum of the $\mathbb{Z}/2$ -spectrum given by $nW \mapsto S^{(n-k)W} \wedge Y^{\lambda 2}$ where $Y^{\lambda 2}$ is short for $Y \downarrow Y$. (For $G = \mathbb{Z}/2$ we like to identify the regular representation W with the permutation representation on \mathbb{R}^2 .) Clearly this $\mathbb{Z}/2$ -spectrum (not its underlying spectrum) can be described as

$$\mathbf{S}_{\mathbb{Z}/2}^{-kW} \wedge Y^{\lambda 2}$$

where $\mathbf{S}_{\mathbb{Z}/2}^{-kW}$ is a shifted $\mathbb{Z}/2$ -sphere spectrum, given by $nW \mapsto S^{(n-k)W}$.

PROPOSITION 3.4. For any finite group G, any well-based G-space Z which is free away from the base point and any G-spectrum \mathbf{C} , the fixed point spectrum $(u(\mathbf{C} \wedge Z))^G$ is homotopy equivalent to the homotopy orbit spectrum $(u(\mathbf{C} \wedge Z))_{hG}$. Under this identification, the inclusion of $(u(\mathbf{C} \wedge Z))^G$ in the homotopy fixed point spectrum $(u(\mathbf{C} \wedge Z))^{hG}$ corresponds to the norm map.

COROLLARY 3.5. There is a natural homotopy fiber sequence of spectra

$$((Y,k)\odot_{\bullet}(Y,k))_{h\mathbb{Z}/2} \longrightarrow ((Y,k)\odot_{\bullet}(Y,k))^{\mathbb{Z}/2} \longrightarrow \Sigma^{\infty-k}(Y/X)$$

where $\Sigma^{\infty-k}(Y|X)$ means a CW-substitute for the suspension spectrum of (Y|X).

PROOF OF THE COROLLARY. Let $T = Y^{\lambda^2}$ and $T' = (E\mathbb{Z}/2)_+ \wedge T$. Let $f: T' \to T$ be the projection. The homotopy cofiber sequence of $\mathbb{Z}/2$ -spaces $T' \longrightarrow T \longrightarrow \operatorname{cone}(f)$ determines a homotopy fiber sequence of spectra

$$(u(\mathbf{C}\wedge T'))^{\mathbb{Z}/2} \longrightarrow (u(\mathbf{C}\wedge T))^{\mathbb{Z}/2} \longrightarrow (u(\mathbf{C}\wedge \operatorname{cone}(f)))^{\mathbb{Z}/2}$$

with $\mathbf{C} = \mathbf{S}_{\mathbb{Z}/2}^{-kW}$. The middle term in the sequence is

$$((Y,k)\odot_{\bullet}(Y,k))^{\mathbb{Z}/2}$$

by construction. Proposition 3.4 identifies the left-hand term with

$$(u(\mathbf{C}\wedge T'))_{h\mathbb{Z}/2} \simeq (u(\mathbf{C}\wedge T))_{h\mathbb{Z}/2} = ((Y,k)\odot_{\bullet}(Y,k))_{h\mathbb{Z}/2}.$$

The expression $(u(\mathbf{C} \wedge \operatorname{cone}(f)))^{\mathbb{Z}/2}$ can be identified with $\Sigma^{\infty-k}(Y/X)$ as follows. It is an Ω -spectrum whose *j*-th space is

 $\operatorname{colim}_n \operatorname{map}_*^{\mathbb{Z}/2}(S^{nW}, S^{j\mathbb{R}\oplus (n-k)W} \wedge \operatorname{cone}(f))$

where $\operatorname{map}_{*}^{\mathbb{Z}/2}(...)$ denotes a space of equivariant based maps and $j\mathbb{R}$ denotes a j-dimensional trivial representation of $\mathbb{Z}/2$. Because $\operatorname{cone}(f)$ is non-equivariantly contractible, equivariant based maps from S^{nW} to $S^{j\mathbb{R}\oplus(n-k)W} \wedge \operatorname{cone}(f)$ are essentially determined by their restrictions to the fixed point sets. Hence the above expression for the j-th space of $(u(\mathbb{C} \wedge \operatorname{cone}(f)))^{\mathbb{Z}/2}$ simplifies to

 $\operatorname{colim}_n \operatorname{map}_*(S^n, S^{j+n-k} \wedge \operatorname{cone}(f)^{\mathbb{Z}/2}).$

Since $\operatorname{cone}(f)^{\mathbb{Z}/2}$ is $T^{\mathbb{Z}/2} \simeq Y/X$, this simplifies even more to $\Omega^{\infty} \Sigma^{\infty+j-k}(Y/X)$, which is the *j*-th space in the Ω -spectrification of $\Sigma^{\infty-k}(Y/X)$.

PROOF OF THE PROPOSITION. This is a standard fact from equivariant homotopy theory. We begin with a preliminary remark about pathologies. Choose a based G-CW-space Z' which is G-free away from the base point and a G-map $e: Z' \to Z$ which is a weak equivalence. Because Z is well-based, v induces weak equivalences

$$\mathrm{id} \wedge e : C \wedge Z' \to C \wedge Z$$

for any well-based G-space C, in particular for $C = C_{nW}$. It follows that e induces a homotopy equivalence of underlying spectra,

$$u(\mathbf{C} \wedge Z') \to u(\mathbf{C} \wedge Z),$$

and a homotopy equivalence of the fixed point spectra, $(u(\mathbf{C} \wedge Z'))^G \to (u(\mathbf{C} \wedge Z))^G$. Therefore, without loss of generality, Z is a based G-CW-space which is G-free away from the base point.

We shall prove the two parts of the proposition together using a characterization of the norm map as an "assembly" transformation. (For this idea we are again indebted to John Klein.) Let \mathcal{C} be the category of all based G-CW-spaces which are G-free away from the base point, with based G-maps as morphisms. Let \mathbf{F} be a functor from \mathcal{C} to spectra which takes homotopy equivalences to weak equivalences and takes \star to a weakly contractible spectrum. Then there exists a natural transformation $\alpha: \mathbf{F}^{\%} \to \mathbf{F}$ where $\mathbf{F}^{\%}$ is another functor from \mathcal{C} to spectra, and

- **F**[%] respects weak equivalences,
- $\mathbf{F}^{\%}$ respects (weak) homotopy pushout squares
- $\mathbf{F}^{\%}$ respects arbitrary wedges up to weak equivalence,
- $\alpha: \mathbf{F}^{\%}(Z) \to \mathbf{F}(Z)$ is a weak equivalence when $Z = G_+$.

The pair $(\mathbf{F}^{\%}, \alpha)$ is essentially determined by \mathbf{F} and α is called the assembly transformation for \mathbf{F} . For the case $G = \{1\}$, the proof (and a more detailed statement)

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can be found in [38] and the general case follows the same lines. (One possible definition of $\mathbf{F}^{\%}(Z)$ for arbitrary Z in C is as follows. Take the geometric realization of the simplicial spectrum

$$n \mapsto Z(n) \wedge \mathbf{F}(\Delta^n_+ \wedge G_+)$$

where Z(n) is the based set of singular *n*-simplices of Z; then divide out by the diagonal action of G.)

Now put $\mathbf{F}(Z) := (u(\mathbf{C} \wedge Z))^{hG}$. This \mathbf{F} clearly takes homotopy equivalences to weak equivalences and takes the trivial space \star to a trivial spectrum. It also respects (weak) homotopy pushout squares, but it does not satisfy the wedge axiom for infinite wedges. The norm transformation

$$(u(\mathbf{C} \wedge Z))_{hG} \to \mathbf{F}(Z)$$

satisfies all the properties which characterize the assembly for \mathbf{F} . Therefore it *is* the assembly for \mathbf{F} . We can now give our proof by showing that the natural transformation

$$(u(\mathbf{C} \wedge Z))^G \longrightarrow \mathbf{F}(Z)$$

given by the inclusion of fixed point spectra in homotopy fixed point spectra also satisfies all the properties which characterize the assembly for the functor \mathbf{F} .

Of the four properties listed, three hold by inspection. So it only remains to check that the inclusion of $(u(\mathbf{C} \wedge Z))^G$ in $(u(\mathbf{C} \wedge Z))_{hG}$ is a homotopy equivalence when $Z = G_+$. From the definitions, an element of $\pi_n((u(\mathbf{C} \wedge G_+))^G)$ is represented by a *G*-map

$$f: S^{(n+j)\mathbb{R}\oplus iW} \longrightarrow S^{j\mathbb{R}} \wedge C_{iW} \wedge G_+$$

with "large" i and j (where $j\mathbb{R}$ for example denotes a trivial j-dimensional representation). We may assume that f is transverse to $0 \times C_{iW} \times G$. The inverse image of $0 \times C_{iW} \times 1$ is then a framed smooth closed (n + i|G|)-dimensional submanifold M of $(n + j)\mathbb{R} \oplus iW$. Clearly $M \cap gM = \emptyset$ for $g \in G \setminus \{1\}$ and we have a map f|M from M to C_{iW} . Conversely, given any framed smooth closed (n+i|G|)-dimensional submanifold M of $(n + j)\mathbb{R} \oplus iW$ with $M \cap gM = \emptyset$ for $g \in G \setminus \{1\}$, and a map q from M to C_{iW} , the Pontryagin–Thom construction gives us an appropriate f for which $M = f^{-1}(0 \times C_{iW} \times 1)$ and f|M = q. In the limit, when j and i tend to infinity, the condition $M \cap gM = \emptyset$ for $G \setminus \{1\}$ becomes irrelevant and so the n-th homotopy group under consideration is identified with the n-dimensional framed bordism group of $u\mathbb{C}$, i.e., with $\pi_n(u\mathbb{C})$. Moreover, this identification clearly agrees with the homomomorphism induced by the composition

$$(u(\mathbf{C} \wedge G_+))^G \longrightarrow (u(\mathbf{C} \wedge G_+))^{hG} \longrightarrow u(\mathbf{C} \wedge G_+) \longrightarrow u(\mathbf{C})$$

where the second arrow is forgetful and the third is induced by the (non-equivariant) map $\mathbf{C} \wedge G_+ \rightarrow \mathbf{C}$ which isolates the summand $\mathbf{C} \wedge \{1\}_+$. Since the composition of the last two arrows is a homotopy equivalence, the first arrow is a homotopy equivalence.

The following continuation of definition 3.2 is suggested by corollary 3.5.

DEFINITION 3.6. An *n*-dimensional visible hyperquadratic structure on (Y, k) is an element in $\Omega^n \Omega^\infty \Sigma^{\infty-k}(Y/X)$. An *n*-dimensional quadratic structure on (Y, k)is an element of $\Omega^n \Omega^\infty(((Y, k) \odot_{\bullet} (Y, k))_{h\mathbb{Z}/2})$. Alternatively, an *n*-dimensional quadratic structure on (Y, k) can be defined as an element of $\Omega^n \Omega^\infty$ of the homotopy fiber of the natural map $J: ((Y, k) \odot_{\bullet} (Y, k))^{\mathbb{Z}/2} \to \Sigma^{\infty-k}(Y/X)$. An *n*-dimensional visible symmetric structure on (Y, k) is considered nondegenerate if the underlying *n*-dimensional symmetric structure is nondegenerate. Writing $s\mathcal{R}(X)$ for the stable category of finitely dominated retractive spaces over X, we obtain the definition of a visible symmetric *L*-theory spectrum

$$\mathbf{VL}^{\bullet}(s\mathcal{R}(X)) = \mathbf{VL}^{\bullet}(X)$$

by substituting nondegenerate visible symmetric structures for nondegenerate symmetric structures throughout in the construction of the symmetric *L*-theory spectrum $\mathbf{L}^{\bullet}(s\mathcal{R}(X)) = \mathbf{L}^{\bullet}(X)$. See [39]. The standard map

$$\mathbf{L}_{\bullet}(X) \to \mathbf{L}^{\bullet}(X)$$

can be factorized as $\mathbf{L}_{\bullet}(X) \to \mathbf{VL}^{\bullet}(X) \to \mathbf{L}^{\bullet}(X)$. This is clear from definition 3.6.

We write $\mathbf{V}\widehat{\mathbf{L}}^{\bullet}(X)$ for the mapping cone (in the category of spectra) of the above map $\mathbf{L}_{\bullet}(X) \to \mathbf{V}\mathbf{L}^{\bullet}(X)$.

THEOREM 3.7. The functor $X \mapsto \mathbf{V}\widehat{\mathbf{L}}^{\bullet}(X)$ is homotopy invariant and excisive.

"Homotopy invariance" is intended to mean that the functor takes weak equivalence to homotopy equivalences, and this is clear. (An equivalent formulation says that, for each X, the maps $\mathbf{V}\widehat{\mathbf{L}}^{\bullet}(X) \to \mathbf{V}\widehat{\mathbf{L}}^{\bullet}(X \times [0,1])$ induced by $x \mapsto (x,0)$ and $x \mapsto (x,1)$ are homotopic. They are indeed homotopic because the exact functors which they induce are related by a chain of natural weak equivalences.) The excision property means that the functor takes empty space to a contractible spectrum and takes weak homotopy pushout squares (also known as cocartesian squares) of spaces to homotopy pushout squares (equivalently, homotopy pullback squares) of spectra. Our proof of the excision property relies on three decomposition lemmas.

For the first of these, suppose that X is the union of two closed subspaces X_a and X_b with intersection X_{ab} , such that the inclusions $X_{ab} \to X_a$ and $X_{ab} \to X_b$ are cofibrations. Let $r: E \to X$ be a fibration with section s making E into a homotopy finite retractive space over X. Let Y be a finite retractive space over X with a morphism $f: Y \to E$ of retractive spaces over X. We assume that Y is decomposed as

$$Y := Y_a \cup Y_b$$

where Y_a , Y_b and $Y_{ab} = Y_a \cap Y_b$ are finite retractive spaces over X_a , X_b and X_{ab} , respectively, with cofibrations $Y_{ab} \cup X_a \to Y_a$ and $Y_{ab} \cup X_b \to Y_b$.

LEMMA 3.8. The morphism $f: Y \to E$ has a factorization of the form

$$Y \xrightarrow{f_1} Z \xrightarrow{g} E$$

where

- (i) Z is a finite retractive space over X
- (ii) f_1 is a cofibration
- (iii) g is a weak equivalence
- (iv) the decomposition of Y extends to a similar decomposition

$$Z := Z_a \cup Z_b$$

where Z_a , Z_b and $Z_{ab} = Z_a \cap Z_b$ are finite retractive spaces over X_a , X_b and X_{ab} , respectively, with cofibrations $Z_{ab} \cup X_a \to Z_a$ and $Z_{ab} \cup X_b \to Z_b$. PROOF. Since the inclusions $X_{ab} \to X_a$ and $X_{ab} \to X_b$ are cofibrations, we can easily reduce to the situation where X_{ab} has collar neighborhoods $X_{ab} \times [-1, 0]$ in X_a and $X_{ab} \times [0, 1]$ in X_b . Ignoring condition (iv), we can easily produce a factorization $f = gf_1$ with properties (i), (ii) and (iii); then Z has a filtration

$$X = Z^{-1} \subset Z^0 \subset Z^1 \subset \dots \subset Z^k = Z$$

where Z^i is the relative *i*-skeleton. Suppose now that Z^{i-1} is already decomposed as in (iv). We can assume that the attaching data for the cells we must attach to obtain Z^i have the form of a commutative diagram

such that the composition $\coprod D^i \to E \to X$ is transverse to the subspace $X_{ab} \times \{0\}$ of $X_{ab} \times [-1, +1] \subset X$. Triangulating the pair $(\coprod D^i, \coprod S^{i-1})$ in such a way that the inverse image of $X_{ab} \times \{0\}$ is a subcomplex, we can also arrange that the attaching map $S^{i-1} \to Z^{i-1}$ is cellular for the chosen triangulation. (Here we are using the assumption that E is fibered over X.) Using the triangulation cell structure on the attached $\coprod D^i$, we obtain a "new" relative CW structure on Z^i which extends the CW structure on Z^{i-1} and in which Z^i decomposes as in (iv). We continue inductively.

Keeping the notation of lemma 3.8, we define an *n*-dimensional quadratic structure on (Z_a, Z_b, k) , for $n \ge 0$, to be an element in $\Omega^n \Omega^\infty$ of the homotopy pushout of the diagram

$$((Z_{ab},k) \odot_{\bullet} (Z_{ab},k))_{h\mathbb{Z}/2} \longrightarrow ((Z_{b},k) \odot_{\bullet} (Z_{b},k))_{h\mathbb{Z}/2}$$

$$\downarrow$$

$$((Z_{a},k) \odot_{\bullet} (Z_{a},k))_{h\mathbb{Z}/2}.$$

Here the \odot_{\bullet} products are taken with respect to the base spaces X_a , X_{ab} and X_b , respectively. By a similar generalization process, we arrive at the notions of a visible symmetric structure on (Z_a, Z_b, k) , and the notion of a visible hyperquadratic structure on (Z_a, Z_b, k) . The corresponding abelian groups of path classes are denoted

$$Q_n(Z_a, Z_b, k), \quad VQ^n(Z_a, Z_b, k), \quad VQ^n(Z_a, Z_b, k)$$

respectively. They are actually defined for all $n \in \mathbb{Z}$ as n-th homotopy groups of the appropriate homotopy pushout spectra. Clearly the map $g: \mathbb{Z} \to E$ in lemma 3.8 induces homomorphisms from $Q_n(\mathbb{Z}_a, \mathbb{Z}_b, k)$ to $Q_n(\mathbb{E}, k)$ and from $VQ^n(\mathbb{Z}_a, \mathbb{Z}_b, k)$ to $VQ^n(\mathbb{E}, k)$. We ask whether these homomorphisms become isomorphisms "in the (co)limit". To speak of a colimit we need an indexing category \mathcal{C} , and in our case this should clearly have objects of the form

$$q: Z \to E$$

where Z is finite, g is a weak equivalence of retractive spaces over X and Z is decomposed into Z_a and Z_b with intersection Z_{ab} as before. Morphisms in the category, say from (g, Z, Z_a, Z_b) to (g', Z', Z'_a, Z'_b) , are cofibrations $u: Z \to Z'$ with g'u = g and $u(Z_a) \subset Z'_a$, $u(Z_b) \subset Z'_b$. Lemma 3.8 implies that C is *directed*. That is, for any two objects Z and Z' in C (in shorthand notation) there exists an object Z'' and morphisms $Z \to Z'', Z' \to Z''$; and given any two morphisms $u, v: Z \to Z'$ there exists a morphism $w: Z' \to Z'''$ such that wu = wv.

LEMMA 3.9. In the above notation, we have

$$\begin{array}{ccc} \operatorname{colim}_{(g,Z,Z_a,Z_b)} & Q_n(Z_a,Z_b,k) & \xrightarrow{\cong} & Q_n(E,k) \,, \\ \operatorname{colim}_{(g,Z,Z_a,Z_b)} & VQ^n(Z_a,Z_b,k) & \xrightarrow{\cong} & VQ^n(E,k) \\ \operatorname{colim}_{(g,Z,Z_a,Z_b)} & V\widehat{Q}^n(Z_a,Z_b,k) & \xrightarrow{\cong} & V\widehat{Q}^n(E,k) \end{array}$$

where the direct limits are taken over C.

PROOF. The second isomorphism is a consequence of the first and the third (and the "five lemma"), because by corollary 3.5 there are exact sequences of type

$$\cdots \to V\widehat{Q}^{n+1} \to Q_n \to VQ^n \to V\widehat{Q}^n \to Q_{n-1} \to \cdots$$

The third isomorphism is obvious from definition 3.6. It remains to establish the first isomorphism. Let $E_a := E|X_a, E_b := E|X_b$ and $E_{ab} = E|X_{ab}$. These are all fibered retractive spaces over the appropriate base spaces: X_a , X_a and X_{ab} , respectively. Therefore, in forming the λ product $E_a \lambda E_a$, for example, we can proceed more directly than otherwise by forming the fiberwise smash product of E_a with E_a over X_a , and then dividing out by the zero section X_a . This leads immediately to a homotopy pushout square consisting of the four spaces $E_a \lambda E_a$, $E_b \lambda E_{ab} \lambda E_{ab}$ and $E \lambda E$, and consequently a homotopy pushout square of spectra

where the \odot_{\bullet} products are taken with respect to the appropriate base spaces: X_{ab} , X_a , X_b and X. (Note that, strictly speaking, the \land product and the \odot_{\bullet} product have only been defined for finitely dominated retractive spaces. We have no reason to think that E_a , E_b and E_{ab} are all finitely dominated, but the definition of \land extends without difficulties.) We have therefore

$$Q_n(E_a, E_b, k) \xrightarrow{\cong} Q_n(E, k)$$

for all n, with the obvious interpretation of $Q_n(E_a, E_b, k)$. This reduces our task to showing that

$$\underset{(g,Z,Z_a,Z_b)}{\operatorname{colim}} Q_n(Z_a,Z_b,k) \xrightarrow{\cong} Q_n(E_a,E_b,k).$$

By a Mayer–Vietoris and five lemma argument, this reduces further to showing that the homomorphisms

$$\begin{array}{cc} \operatorname{colim}_{(g,Z,Z_a,Z_b)} & Q_n(Z_a,k) & \stackrel{\cong}{\longrightarrow} & Q_n(E_a,k) \\ \operatorname{colim}_{(g,Z,Z_a,Z_b)} & Q_n(Z_b,k) & \stackrel{\cong}{\longrightarrow} & Q_n(E_b,k) \\ \operatorname{colim}_{(g,Z,Z_a,Z_b)} & Q_n(Z_{ab},k) & \stackrel{\cong}{\longrightarrow} & Q_n(E_{ab},k) \end{array}$$

Licensed to University of Edinburgh. Prepared on Sun Nov 15 16:25:09 EST 2015for download from IP 129.215.149.96. License or copyright restrictions may apply to redistribution; see http://www.ams.org/publications/ebooks/terms are all isomorphisms. By lemma 3.8, we may now enlarge the indexing categories to allow objects (f, Y, Y_a, Y_b) where Y is a finite retractive space over X with a decomposition $Y = Y_a \cup Y_b$ etc., and where $f: Y \to E$ is any map (not necessarily a weak equivalence) of retractive spaces over X. Then E_a for example can easily be identified with the homotopy direct limit of the Y_a , etc., and Q_* takes the homotopy direct limits to direct limits, so that the isomorphisms become obvious.

LEMMA 3.10. Let $X = X_a \cup X_b$ as in lemma 3.8. Let Z be a finite retractive space over X with a decomposition $Z = Z_a \cup Z_b$ as in lemma 3.8, so that Z_a is retractive over X_a and Z_b is retractive over X_b . Let $k, n \in \mathbb{Z}$. Then there exist a finite retractive space V over X with a decomposition $V = V_a \cup V_b$ as in lemma 3.8, an integer ℓ and a nondegenerate element η in π_n of the homotopy pushout of

$$(V_a, \ell) \odot_{\bullet} (Z_a, k) \longleftarrow (V_{ab}, \ell) \odot_{\bullet} (Z_{ab}, k) \longrightarrow (V_b, \ell) \odot_{\bullet} (Z_b, k)$$

such that the images of η in

$$\pi_n((V,\ell)\odot_{\bullet}(Z,k)), \qquad \pi_n((V_a,\ell)\odot_{\bullet}(Z_a/Z_{ab},k)), \\\pi_n((V_b,\ell)\odot_{\bullet}(Z_b/Z_{ab},k)), \qquad \pi_{n-1}((V_{ab},\ell)\odot_{\bullet}(Z_{ab},k))$$

are all nondegenerate.

PROOF. The guiding principle here is the fact that passage from finite retractive spaces to cellular chain complexes over the appropriate group(oid) rings respects and detects nondegenerate pairings. This is due to [26]. The relevant group(oid) rings here are $\mathbb{Z}[\pi_1(X_a)], \mathbb{Z}[\pi_1(X_b)], \mathbb{Z}[\pi_1(X_{ab})]$ and $\mathbb{Z}[\pi_1(X)]$. Note also that a change of rings, such as the passage from chain complexes over $\mathbb{Z}[\pi_1(X_{ab})]$ to chain complexes over $\mathbb{Z}[\pi_1(X_a)]$ by means of

$$\mathbb{Z}[\pi_1(X_a)] \otimes_{\mathbb{Z}[\pi_1(X_{ab})]}$$

respects nondegenerate pairings. It follows that cobase change, such as the passage from retractive spaces over X_{ab} to retractive spaces over X_a by means of

 $X_a \sqcup_{X_{ab}}$

respects nondegenerate pairings. Consequently we can construct V and V_a , V_b in the following way. We first find an (n-1)-dual for (Z_{ab}, k) as a stable retractive space over X_{ab} . This amounts to finding a retractive space V_{ab} over X_{ab} , an integer ℓ and a nondegenerate element η_{ab} in $\pi_{n-1}((V_{ab}, \ell) \odot_{\bullet} (Z_{ab}, k))$. Next we find an n-dual for the pair $((Z_a, k), (Z_{ab}, k))$ which extends our chosen (n-1)-dual for (Z_{ab}, k) . This amounts to finding V_a , a cofibration V_{ab} and an element η_a in π_n of the mapping cone of

$$(V_{ab}, \ell) \odot_{\bullet} (Z_{ab}, k) \longrightarrow (V_a, \ell) \odot_{\bullet} (Z_a, k)$$

whose image in $\pi_n(((V_a, \ell) \odot_{\bullet} (Z_a/Z_{ab}, k)))$ is nondegenerate and whose image in $\pi_{n-1}((V_{ab}, \ell) \odot_{\bullet} (Z_{ab}, k))$ is η_{ab} . (It may be necessary to increase ℓ .) We proceed similarly with the pair $((Z_b, k), (Z_{ab}, k))$ to obtain V_b and η_b . Then we define

$$V := V_a \sqcup_{V_{ab}} V_b$$

and find η in π_n of the homotopy pushout of

$$(V_a, \ell) \odot_{\bullet} (Z_a, k) \longleftarrow (V_{ab}, \ell) \odot_{\bullet} (Z_{ab}, k) \longrightarrow (V_b, \ell) \odot_{\bullet} (Z_b, k)$$

mapping to η_a and $-\eta_b$ under the appropriate projections. The existence of such an η follows from a suitable Mayer–Vietoris sequence. The image of η in the homotopy

group $\pi_n((V, \ell) \odot_{\bullet} (Z, k))$ will automatically be nondegenerate, by Vogell's chain complex criterion.

As another preliminary for the proof of theorem 3.7, we offer a lengthy discussion of how elements in the homotopy group

$$V\widehat{L}^n(X) = \pi_n \mathbf{V}\widehat{\mathbf{L}}(X)$$

can be represented. For that discussion we return briefly to the case of the Ltheory of a group ring $\mathbb{Z}\pi$. Let \mathcal{D} be the category of bounded (above and below) chain complexes of f.g. left projective $\mathbb{Z}\pi$ -modules. We regard \mathcal{D} as a Waldhausen category in which the cofibrations are the chain maps which are split injective in each dimension. Cofibrations $C \to D$ in \mathcal{D} are often regarded as pairs (D, C).

Let E be an object of \mathcal{D} with an n-dimensional symmetric structure φ . The inclusion of E in the algebraic mapping cone of $\varphi_0: E^{n-*} \to E$ classifies an "extension" with base E in the shape of a short exact sequence

$$0 \to C \to D \to E \to 0$$

where $C \simeq \Sigma^{-1} \operatorname{cone}(\varphi_0)$ and $D \simeq E^{n-*}$. According to Ranicki [19], the symmetric structure φ on E has a preferred lift to an n-dimensional nondegenerate symmetric structure $(\bar{\varphi}, \partial \bar{\varphi})$ on the pair (D, C), so that $\varphi/\partial \bar{\varphi} = \varphi$ under the identification $D/C \cong E$. (Warning: $\bar{\varphi}$ is an n-chain in $\hom_{\mathbb{Z}[\mathbb{Z}/2]}(W, D^t \otimes_{\mathbb{Z}\pi} D)$ with boundary $\partial \bar{\varphi}$ in the image of $\hom_{\mathbb{Z}[\mathbb{Z}/2]}(W, C^t \otimes_{\mathbb{Z}\pi} C)$. Our notation for symmetric and quadratic structures on pairs deviates from Ranicki's.) This resolution procedure of Ranicki's leads to a bijective correspondence between homotopy types of chain complexes Ewith an n-dimensional symmetric structure, and homotopy types of chain complex pairs (D, C) with an n-dimensional nondegenerate symmetric structure. There is a similar correspondence for visible symmetric and quadratic structures.

Of particular interest to us is the mixed case, i.e. the case of nondegenerate symmetric pairs (or visible symmetric pairs) with quadratic boundary. For a pair (D, C) with an *n*-dimensional nondegenerate symmetric structure $(\bar{\varphi}, \partial \bar{\varphi})$, improving the (n-1)-dimensional symmetric structure $\partial \bar{\varphi}$ on *C* to a quadratic structure on *C* amounts to "trivializing" the induced (n-1)-dimensional hyperquadratic structure

$$J(\partial \bar{\varphi}) \in \hom_{\mathbb{Z}[\mathbb{Z}/2]}(\widehat{W}, C^t \otimes C)$$

on C, in other words, finding an *n*-chain with boundary $J(\partial \bar{\varphi})$. But since the functor

$$C \mapsto \hom_{\mathbb{Z}[\mathbb{Z}/2]}(\widehat{W}, C^t \otimes C)$$

respects homotopy cofiber sequences, finding such a trivialization is equivalent to finding a trivialization for the suspended hyperquadratic structure on ΣC . By Ranicki's correspondence, if we write E = D/C and $\varphi = \bar{\varphi}/\partial\bar{\varphi}$, the suspension of $J(\partial\bar{\varphi})$ can be identified with the image of $J\varphi$ under the inclusion

$$E \to \operatorname{cone}(\varphi_0)$$
.

Summarizing, there is a bijective correspondence between homotopy types of ndimensional nondegenerate symmetric pairs in \mathcal{D} with quadratic boundary, and homotopy types of single objects E in \mathcal{D} with an n-dimensional symmetric structure φ and a trivialization τ of the image of $J\varphi$ under the inclusion $E \to \operatorname{cone}(\varphi_0)$. This remains correct if symmetric and hyperquadratic structures are replaced with visible symmetric and visible hyperquadratic structures throughout. The correspondence extends to nullbordisms. For more precision we suppose again that $((D, C), (\bar{\varphi}, \partial \bar{\varphi}))$ is an *n*-dimensional nondegenerate symmetric pair with quadratic boundary. Let $(E, \varphi) = (D/C, \bar{\varphi}/\partial \bar{\varphi})$ be its Thom complex. Let τ be the trivialization of the image of $J\varphi$ under $E \to \operatorname{cone}(\xi_0)$ determined by the preferred trivialization of $J(\partial \bar{\varphi})$. Then a (nondegenerate) nullbordism of the nondegenerate pair $((D, C), (\bar{\varphi}, \partial \bar{\varphi}))$ determines a nullbordism of (E, φ, τ) , by which is meant:

- a chain complex pair $(F, \partial F)$ with (n+1)-dimensional symmetric structure $(\xi, \partial \xi)$, where $\partial F = E$ and $\partial \xi = \varphi$;
- a trivialization $(\kappa, \partial \kappa)$ with $\partial \kappa = \tau$ of the image of $J(\xi, \partial \xi)$ under the inclusion $(F, \partial F) \to (\operatorname{cone}(\xi_0), \operatorname{cone}(\partial \xi_0))$. Here ξ_0 is regarded as a chain map from $\operatorname{cone}(F^{n-*} \to E^{n-*})$ to F.

Conversely, a nullbordism of (E, φ, τ) determines a nondegenerate nullbordism of the nondegenerate pair $((D, C), (\bar{\varphi}, \partial \bar{\varphi}))$. Again, this remains correct if symmetric and hyperquadratic structures are replaced with visible symmetric and visible hyperquadratic structures throughout.

Returning now to the stabilization $s\mathcal{R}(X)$ of the category of finitely dominated retractive spaces over a fixed space X, we remark that these correspondences apply, mutatis mutandis, in $s\mathcal{R}(X)$. The fact that there are strictly speaking no "canonical" *n*-duals in $s\mathcal{R}(X)$ does complicate matters slightly (but only at first). For an object (Y, k) in $s\mathcal{R}(X)$ with an *n*-dimensional symmetric or visible symmetric structure φ , the correct way to determine an *n*-dual $(Y, k - n)^*$ with $n \ge 0$ is to find an object (Z, ℓ) in $s\mathcal{R}(X)$ and a nondegenerate element $\eta \in \Omega^n((Z, \ell) \odot (Y, k))$. Modulo a "trivial" enlargement of (Y, k), a morphism $f: (Z, \ell) \to (Y, k)$ and a path from $f_*(\eta)$ to $\varphi_0 \in \Omega^n((Y, k) \odot (Y, k))$ can then be found. (The trivial enlargement is an object of $s\mathcal{R}(X)$ related to (Y, k) by a morphism which is both a cofibration and a weak equivalence.) The morphism f then deserves to be regarded as the adjoint of φ_0 . Therefore, when we write

$$\varphi_0: (Y, k-n)^* \longrightarrow (Y, k),$$

we mean $f: (Z, \ell) \to (Y, k)$.

PROOF OF THEOREM 3.7. We assume that $X = X_a \cup X_b$ and $X_a \cap X_b = X_{ab}$ as in lemma 3.8. We need to show that the gluing homomorphism

$$\alpha_n : V\widehat{L}^n(X_a, X_b) \to V\widehat{L}^n(X)$$

is an isomorphism, where $V\hat{L}^n(X_a, X_b)$ is the *n*-th homotopy group of the homotopy pushout of

$$\mathbf{V}\widehat{\mathbf{L}}^{\bullet}(X_a) \leftarrow \mathbf{V}\widehat{\mathbf{L}}^{\bullet}(X_{ab}) \to \mathbf{V}\widehat{\mathbf{L}}^{\bullet}(X_b)$$

We establish this only when n > 0. The case n < 0 can be handled in the same way. (Replace \odot by \odot_j for some j with $j + n \ge 0$ in the argument below.)

Starting with the surjectivity part and assuming n > 0, we represent an element of $V\hat{L}^n(X)$ by an object (Z, k) in $s\mathcal{R}(X)$, an *n*-dimensional visible symmetric structure φ on (Z, k) and a "trivialization" τ of the *n*-dimensional visible hyperquadratic structure on cone (φ_0) obtained by pushing $J\varphi$ forward along the inclusion $(Z, k) \to \operatorname{cone}(\varphi_0)$. Here we view φ_0 as a morphism from an *n*-dual of (Z, k) to (Z, k). We may assume that Z is not only finitely dominated, but finite. (Otherwise replace Z by $Z \wedge \Sigma Z$, which has zero finiteness obstruction; also, replace φ and τ by their images under appropriate maps induced by the inclusion $Z \to Z \vee \Sigma Z$.) By lemma 3.8 and lemma 3.9 we may then assume that

$$Z = Z_a \cup Z_b$$
, $\varphi = \varphi' + \varphi''$

where Z_a and Z_b are as in lemma 3.8 and $(\varphi', \partial \varphi')$, $(\varphi'', \partial \varphi'')$ are visible symmetric structures on the pairs $((Z_a, k), (Z_{ab}, k))$ and $((Z_b, k), (Z_{ab}, k))$, respectively, with $\partial \varphi' = -\partial \varphi''$. (In more detail, if Z and φ do not come equipped with such a decomposition, then we first use the Serre construction to enlarge Z to a fibered retractive space E over X. The fibered retractive space E can in turn be approximated as in lemma 3.8 by another retractive space Z' over X which is decomposed into Z'_a and Z'_b . Then lemma 3.9 can be applied, etc.) The equations $\varphi = \varphi' + \varphi''$ and $\partial \varphi' = -\partial \varphi''$ can be more accurately expressed by saying that φ is parametrized by S^n and that its restrictions to the "upper" and "lower" hemispheres of S^n define n-dimensional visible symmetric structures on the pairs $((Z_a, k), (Z_{ab}, k))$ and $((Z_b, k), (Z_{ab}, k))$, respectively. Under these conditions, φ and φ' , φ'' represent an element of what we have called $VQ^n(Z_a, Z_b, k)$ in lemma 3.9.

Now by lemma 3.10 we may assume that we have an *n*-dual (V, ℓ) for (Z, k) which is also decomposed, $V = V_a \cup V_b$. Hence we have a decomposition in the shape of a pushout square

and the inclusion of Z in $\operatorname{cone}(\varphi_0)$ respects the decompositions. Hence, finally, the trivialization τ automatically decomposes in the same manner. This completes the solution of our decomposition problem and so establishes the surjectivity part of the proof.

A relative version (which we will not write out in detail) of the argument shows that, if the original representative $((Z, k), \varphi, \tau)$ is nullbordant, in the sense which we gave to the word "nullbordant" earlier, then the lift across

$$\alpha_n : V\widehat{L}^n(X_a, X_b) \to V\widehat{L}^n(X)$$

which we have constructed is also nullbordant. Hence our surjectivity proof amounts to a homomorphism of bordism groups which is right inverse to α_n . By a straightforward inspection, it is also left inverse to α_n .

CHAPTER 4

The hyperquadratic *L*-theory of a point

The *L*-theory of a point is, in our terminology, the *L*-theory of the (stabilization of) the category of finite based CW-spaces with the standard notion of Spanier–Whitehead duality. In this chapter we "calculate" the homotopy types of the spectra

$$\widehat{\mathbf{L}}^{\bullet}(\star), \mathbf{V}\widehat{\mathbf{L}}^{\bullet}(\star)$$
.

The calculations will not be used for anything else in this paper, but they are interesting for a number of reasons. In particular we shall see that the inclusion

$$V\widehat{L}^{ullet}(\star)
ightarrow \widehat{L}^{ullet}(\star)$$

is not a homotopy equivalence (which spoils the analogy with the linear version of visible symmetric L-theory, outlined above). But in fact it deviates very little from being a homotopy equivalence, and the source turns out to be nothing more or less than a cleaned-up version of the target.

Understanding $\mathbf{V}\widehat{\mathbf{L}}^{\bullet}(\star)$ and $\widehat{\mathbf{L}}^{\bullet}(\star)$ has a lot to do with understanding the "homology theories"

$$(Y,k) \mapsto \Sigma^{\infty-k}Y, \qquad (Y,k) \mapsto ((Y,k) \odot_{\bullet} (Y,k))^{th\mathbb{Z}/2}$$

and the natural transformation from the first to the second which is implicit in corollary 3.5. This natural transformation can be made explicit by (re)defining $\Sigma^{\infty-k}Y$ as the homotopy cofiber of the improved norm map

$$((Y,k)\odot_{\bullet}(Y,k))_{h\mathbb{Z}/2}\longrightarrow ((Y,k)\odot_{\bullet}(Y,k))^{\mathbb{Z}/2}$$

of proposition 3.4 and corollary 3.5, and (re)defining $((Y,k) \odot_{\bullet} (Y,k))^{th\mathbb{Z}/2}$ as the homotopy cofiber of the ordinary norm map

$$((Y,k)\odot_{\bullet}(Y,k))_{h\mathbb{Z}/2}\longrightarrow ((Y,k)\odot_{\bullet}(Y,k))^{h\mathbb{Z}/2}$$

Because we are dealing with homology theories, we can simplify through a chain of natural weak homotopy equivalences,

$$((Y,k) \odot_{\bullet} (Y,k))^{th\mathbb{Z}/2} \simeq \cdots \simeq (-k) \text{-fold shift of } Y \land (S^0 \odot_{\bullet} S^0)^{th\mathbb{Z}/2}.$$

and more obviously $\Sigma^{\infty-k}Y \simeq Y \wedge \Sigma^{\infty-k}S^0$. This is explained in [38]. Now it is easy to identify $S^0 \odot_{\bullet} S^0$ with the sphere spectrum **S** through a chain of equivariant homotopy equivalences (using the flip action of $\mathbb{Z}/2$ on $S^0 \odot_{\bullet} S^0$, and the trivial action of $\mathbb{Z}/2$ on **S**). Hence what we need to understand is $\mathbf{S}^{th\mathbb{Z}/2}$.

The Segal conjecture [7] for a single point with the (trivial) action of $\mathbb{Z}/2$ means that $\mathbf{S}^{h\mathbb{Z}/2}$ is homotopy equivalent to a certain completion of the fixed point spectrum of the equivariant sphere spectrum $\mathbf{S}_{\mathbb{Z}/2}$. The fixed point spectrum of the equivariant sphere spectrum can be identified with the *K*-theory of the symmetric monoidal category of finite $\mathbb{Z}/2$ -sets [8] and therefore breaks up as

$$\mathbf{S} \vee \mathbf{S}_{h\mathbb{Z}/2}$$

where the summands correspond to the isomorphism types of irreducible $\mathbb{Z}/2$ -sets. If we identify the Burnside ring $\pi_0(\mathbf{S} \vee \mathbf{S}_{h\mathbb{Z}/2})$ with $\mathbb{Z} \oplus \mathbb{Z}$, then the augmentation ideal \mathcal{I} consists of the elements of the form (2z, -z). We have to complete at \mathcal{I} . It is therefore to our advantage to reconsider the splitting of the equivariant fixed point spectrum: write

$$(u(\mathbf{S}_{\mathbb{Z}/2}))^{\mathbb{Z}/2} \simeq \mathbf{S} \vee \Gamma(-\text{transfer})$$

where $\Gamma(-\text{transfer})$ is the graph of the negative of the transfer from $\mathbf{S}_{h\mathbb{Z}/2}$ to \mathbf{S} . Then the \mathcal{I} is the π_0 of the second summand. Its powers are the ideals $2^n \mathcal{I}$. Therefore, indicating completion at 2 by a left-hand superscript c, we have

$$\mathbf{S}^{h\mathbb{Z}/2} \simeq \mathbf{S} \vee {}^{c}\Gamma(-\text{transfer})$$
.

In this decomposition, the norm map

$$\mathbf{S}_{h\mathbb{Z}/2} \to \mathbf{S}^{h\mathbb{Z}/2}$$

has first component equal to the transfer and second component equal to the identity (followed by completion). In calculating the homotopy cofiber, we may replace the source by its 2–completion and 2–complete the first summand of the target as well; the homotopy cofiber remains the same. We summarize the result in the following

LEMMA 4.1. The homology theory $(Y,k) \mapsto ((Y,k) \odot_{\bullet} (Y,k))^{th\mathbb{Z}/2}$ has coefficient spectrum ${}^{c}\mathbf{S}$.

Evaluating the natural transformation $\Sigma^{\infty-k}Y \to ((Y,k) \odot_{\bullet} (Y,k))^{th\mathbb{Z}/2}$ just constructed on the object $(Y,k) = (S^0,0)$, we have a map from $\Sigma^{\infty-k}Y = \mathbf{S}$ to $((Y,k) \odot_{\bullet} (Y,k))^{th\mathbb{Z}/2} \simeq {}^{c}\mathbf{S}$.

LEMMA 4.2. The map under consideration is the inclusion $\mathbf{S} \to {}^{c}\mathbf{S}$.

PROOF. The homotopy fiber sequence of corollary 3.5 splits when $(Y, k) = (S^0, 0)$. Our map can therefore be obtained from the composition

$$\mathbf{S} \vee \mathbf{S}_{h\mathbb{Z}/2} \simeq (u(\mathbf{S}_{\mathbb{Z}/2}))^{\mathbb{Z}/2} \to \mathbf{S}^{h\mathbb{Z}/2} \to \mathbf{S}^{th\mathbb{Z}/2},$$

which we analyzed earlier, by restricting to the summand S.

We now recall, following [32], how the chain bundle method for determining hyperquadratic L-theory (and certain variations on that) works in the linear case, and then transport the technology to the nonlinear situation.

Let R be a ring with involution R, let B be a bounded (below and above) chain complex of f.g. projective left R-modules and let γ be a 0-dimensional cycle in

$$\hom_{\mathbb{Z}[\mathbb{Z}/2]}(\hat{W}, (B^{-*})^t \otimes_R B^{-*}) = ((B^{-*})^t \otimes_R B^{-*})^{th\mathbb{Z}/2}$$

Such a thing is called a *chain bundle* on B and will be treated as a chain complex analogue of a spherical fibration.

In particular, let (C, φ) be a symmetric Poincaré chain complex over R, of formal dimension n. Then C has a *normal* chain bundle ν , which comes together with an (n + 1)-chain τ in

$$(C^t \otimes C)^{th\mathbb{Z}/2}$$

whose boundary is the difference between $J\varphi$ and $(\varphi_0)_*(\Sigma^n\nu)$. Here $\Sigma^n\nu$ is the *n*-fold homological suspension of ν , an *n*-cycle in $(C^{n-*})^t \otimes C^{n-*})^{th\mathbb{Z}/2}$. Because

 φ_0 is invertible up to chain homotopy, the pair consisting of ν and τ is sufficiently unique.

Given B and a chain bundle γ on B, a (B, γ) -structure on a symmetric Poincaré chain complex (C, φ) of formal dimension n consists of a chain map $f: C \to B$ and an (n+1)-chain τ in $(C^t \otimes_R C)^{th\mathbb{Z}/2}$ whose boundary is the difference of $J\varphi$ and $(\varphi_0)_*\Sigma^n(f^*\gamma)$. The chain τ gives an identification of $f^*\gamma$ with the normal chain bundle of (C, φ) .

Let $\mathbf{L}^{\bullet}(R; B, \gamma)$ be the algebraic bordism spectrum constructed from the bordism theory of symmetric algebraic Poincaré complexes (C, φ) over R with a (B, γ) structure. In the case where B = 0 this is the quadratic L-theory of R, and in the general case there is a comparison map

$$\mathbf{L}^{\bullet}(R) \to \mathbf{L}_{\bullet}(R; B, \gamma)$$

with homotopy cofiber $\widehat{\mathbf{L}}_{\bullet}(R; B, \gamma)$. The main result of [32] is a long exact sequence

 $\cdots \longrightarrow \widehat{L}^n(R; B, \gamma) \longrightarrow Q^n(B) \xrightarrow{J_{\gamma}} \widehat{Q}^n(B) \longrightarrow \widehat{L}^{n-1}(R; B, \gamma) \cdots$ where $J_{\gamma}[\varphi] := J[\varphi] - (\varphi_0)_*(\Sigma^n[\gamma])$ with $\Sigma^n[\gamma] \in \widehat{Q}(B^{n-*})$. There is also a stunted version of this. To get that we make the changes

$$W \rightsquigarrow W_{<0}, \qquad \widehat{W} \rightsquigarrow \widehat{W}_{<0}$$

in the above (passing to 0-skeletons). More practically we define

$$Q_n^-(B) := H_n(W^- \otimes_{\mathbb{Z}[\mathbb{Z}/2]} (B^t \otimes B))$$

where W^- is the dual of $\widehat{W}_{\leq 0}$, or alternatively, the mapping cone of the (chain) map $W \to \mathbb{Z}[\mathbb{Z}/2]$ which takes $1 \in W_0$ to 1 + T. This gives an obvious inclusioninduced map $\iota: H_n(B^t \otimes_R B) \to Q_n^-(\Sigma B)$. The stunted version of the above long exact sequence is another long exact sequence

$$\cdots \widehat{L}^n(R; B, \gamma) \longrightarrow H_n(B^t \otimes B) \xrightarrow{\iota_{\gamma}} Q_n^-(B) \longrightarrow \widehat{L}^{n-1}(R; B, \gamma) \cdots$$

in which $\iota_{\gamma}[f] := \iota[f] - f_*(\Sigma^n[\gamma])$ for a chain map $f: \Sigma^{n-*}B \to B$. The long exact sequences can be set up as the homotopy group sequences associated with certain homotopy fiber sequences of spectra. Beware that the crucial maps of spectra which induce J_{γ} and ι_{γ} are not **H**Z-module maps, although their sources and targets are **H**Z-module spectra.

Suppose now that B = B and $\gamma = \gamma(u)$ are universal; that is, the natural map $H_0(\hom_R(C, B)) \to Q^0(C^{-*})$ given by $f \mapsto f^*\gamma$ is an isomorphism for every chain complex C (bounded above and below, f.g. projective in each degree). Then it is not hard to identify the groups $L^n(R; B, \gamma)$ with the ordinary symmetric L-groups. The above long exact sequence therefore specializes to

$$\cdots \longrightarrow \widehat{L}^n(R) \longrightarrow H_n(B^t \otimes B) \xrightarrow{\iota_{\gamma}} Q_n^-(B(u)) \longrightarrow \widehat{L}^{n-1}(R) \cdots$$

It must be said that this is a little harder to justify and use, because in most cases B can no longer be chosen to be bounded above and below and f.g. in each degree. However, B can always be constructed as a direct limit of chain complexes B' satisfying these finiteness assumptions, and γ can be constructed as an element in the inverse limit of the chain bundle groups associated with these (sub)complexes. The above long exact sequence for the universal B and γ is then obtained as a direct limit for the long exact sequences associated with the subcomplexes B' and chain bundles $\gamma|B'$. In the case where B is not universal, we always have a comparison chain map to the universal specimen. We may think of B as a classifying object for another cohomology theory which comes with a natural transformation to ordinary chain bundle theory.

Returning to finite spectra, we see that to calculate $\hat{\mathbf{L}}^{\bullet}(\star)$ and $\mathbf{V}\hat{\mathbf{L}}^{\bullet}(\star)$, we must replace B by $\mathbf{S}^{th\mathbb{Z}/2} \simeq {}^{c}\mathbf{S}$ and by \mathbf{S} , respectively, in the above. Since \mathbf{S} has better finiteness properties than ${}^{c}\mathbf{S}$, the visible case is easier and we begin with that. By analogy with one of the long exact sequences just described (the "stunted version"), we obtain a homotopy fiber sequence of spectra

$$\mathbf{V}\widehat{\mathbf{L}}^{\bullet}(\star) \xrightarrow{} \mathbf{S} \wedge \mathbf{S} \xrightarrow{\iota_{\gamma}} (\mathbf{S} \wedge \mathbf{S})^{-}_{h\mathbb{Z}/2}$$

Here $(\mathbf{S} \wedge \mathbf{S})_{h\mathbb{Z}/2}^{-}$ is the homotopy cofiber of the transfer from $(\mathbf{S} \wedge \mathbf{S})_{h\mathbb{Z}/2}$ to $\mathbf{S} \wedge \mathbf{S}$. (Aside. For the present purposes, the "right" notion of smash product of two spectra \mathbf{E} and \mathbf{F} would be the spectrum with *i*-th term $\Omega^{i}(E_{i} \wedge F_{i})$, where the loop coordinates are associated with the antidiagonal of $\mathbb{R}^{i} \times \mathbb{R}^{i}$. This is "commutative" but neither associative nor unital, so it is one of many naive smash products.)

The spectrum $V\hat{\mathbf{L}}^{\bullet}(\star)$ is a ring spectrum. There is no need to make that very precise here, since all we need is the unit map for the ring structure $\mathbf{S} \longrightarrow V\hat{\mathbf{L}}^{\bullet}(\star)$. Evaluation on π_0 shows that this unit map is a homotopy right inverse for the map $V\hat{\mathbf{L}}^{\bullet}(\star) \rightarrow \mathbf{S} \wedge \mathbf{S}$ in the homotopy fiber sequence just above. Therefore ι_{γ} has a preferred nullhomotopy and we have

THEOREM 4.3.
$$\mathbf{V} \widehat{\mathbf{L}}^{\bullet}(\star) \simeq \mathbf{S} \vee \Omega(\mathbf{S}_{h\mathbb{Z}/2}) = \mathbf{S} \vee \mathbb{R} \mathbf{P}_{-1}^{\infty}$$

This is surprising. We have shown that the unit map for $\mathbf{V}\widehat{\mathbf{L}}^{\bullet}(\star)$ is the injection of a wedge summand \mathbf{S} , up to homotopy equivalence. In particular, multiplication by 8 does not annihilate its homotopy class. It follows immediately that the standard homomorphism $L_0(\star) \to VL^0(\star)$ does not send the (signature 8) generator to 8 times the unit of $VL^0(\star)$.

COROLLARY 4.4. For a space X with CW-approximation $X' \to X$, we have

 $\mathbf{V}\widehat{\mathbf{L}}^{\bullet}(X) \simeq X'_{+} \wedge \left(\mathbf{S} \vee \mathbb{R}\mathbf{P}_{-1}^{\infty} \right).$

COMMENT. This is a formal consequence of theorem 4.3 and the excision theorem 3.7. Beware that the definition of of $\mathbf{V}\widehat{\mathbf{L}}^{\bullet}(X)$ which we use here depends on a specific *SW* product in $s\mathcal{R}(X)$. There are "twisted" versions which will be considered later.

Next we calculate $\widehat{\mathbf{L}}^{\bullet}(\star)$. The only new aspect in this calculation is that our basic "homology theory" is now $(Y,k) \mapsto ((Y,k) \odot_{\bullet} (Y,k))^{h\mathbb{Z}/2}$ and the representing object is ${}^{c}\mathbf{S}$. From the point of view of chain bundle theory, generalized to the nonlinear setting, this means that our calculation of the hyperquadratic *L*-theory of a point is going to be almost identical with that of the visible hyperquadratic *L*-theory of a point. The difference is that \mathbf{S} has to be replaced by ${}^{c}\mathbf{S}$ where applicable. Noting that $({}^{c}\mathbf{S} \wedge {}^{c}\mathbf{S})^{-}_{h\mathbb{Z}/2} \simeq (\mathbf{S} \wedge \mathbf{S})^{-}_{h\mathbb{Z}/2}$, we obtain a homotopy fiber sequence of spectra

$$\widehat{\mathbf{L}}^{ullet}(\star) \longrightarrow {}^{c}\mathbf{S} \wedge {}^{c}\mathbf{S} \longrightarrow (\mathbf{S} \wedge \mathbf{S})^{-}_{h\mathbb{Z}/2}.$$

The map ι_{γ} in this case can immediately be understood by comparison with the visible case. It must be zero because its restriction to $\mathbf{S} \wedge \mathbf{S}$ is zero and the homotopy groups of the target are all 2-torsion. Therefore:

THEOREM 4.5.
$$\widehat{\mathbf{L}}^{\bullet}(\star) \simeq ({}^{c}\mathbf{S} \wedge {}^{c}\mathbf{S}) \vee \Omega(\mathbf{S}_{h\mathbb{Z}/2}^{-}) = ({}^{c}\mathbf{S} \wedge {}^{c}\mathbf{S}) \vee \mathbb{R}\mathbf{P}_{-1}^{\infty}.$$
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CHAPTER 5

Excision and restriction in controlled *L*-theory

We start with the Waldhausen category $\mathcal{R}^{\mathrm{ld}}(\bar{Q}, Q)$ of [10, dfn.7.1]. Here \bar{Q} is locally compact Hausdorff, Q is open in \bar{Q} and we add the assumption that \bar{Q} has a countable base. We recall that the objects of $\mathcal{R}^{\mathrm{ld}}(\bar{Q}, Q)$ are retractive spaces over Q which are dominated (in a controlled homotopy sense) by locally finite and finite dimensional retractive spaces with a controlled CW-structure.

Often we stabilize with respect to the suspension functor Σ and write the result as $s\mathcal{R}^{\mathrm{ld}}(\bar{Q},Q)$. Objects in the stable category can be written as (Y,k) for some Yin $\mathcal{R}^{\mathrm{ld}}(\bar{Q},Q)$ and $k \in \mathbb{Z}$. In the stabilized category, we want to introduce a Spanier– Whitehead (external) product in the sense of [**39**, dfn.1.1]. (This has been done in [**39**, 1.A.7], but it will not hurt to present it from a slightly different angle.)

DEFINITION 5.1. Let \overline{Q}^{\bullet} be the one-point compactification of \overline{Q} . Let Y be a retractive space over Q, with retraction $r: Y \to Q$. We write

 $Y \cup_{O} \bar{Q}^{\bullet}$

for the union of Y and \bar{Q}^{\bullet} along Q, equipped with the coarsest topology such that the inclusion $Y \to Y \cup_Q \bar{Q}^{\bullet}$ embeds Y as an open subset, and the retraction $r \cup \mathrm{id} \colon Y \cup_Q \bar{Q}^{\bullet} \to \bar{Q}^{\bullet}$ is continuous. (This means that a subset V of $Y \cup_Q \bar{Q}^{\bullet}$ is a neighborhood of some $z \in \bar{Q}^{\bullet} \smallsetminus Q$ in $Y \cup_Q \bar{Q}^{\bullet}$ iff V contains $(r \cup \mathrm{id})^{-1}(W)$ for some neighborhood W of z in \bar{Q}^{\bullet} .) Let $Y/\!\!/Q$ be the topological quotient of $Y \cup_Q \bar{Q}^{\bullet}$ by the subspace \bar{Q}^{\bullet} ,

$$Y/\!\!/Q = \frac{Y \cup_Q \bar{Q}^{\bullet}}{\bar{Q}^{\bullet}} \,.$$

REMARK. In the important special case where Y has a locally finite controlled CW-structure relative to Q, the special quotient $Y/\!\!/Q$ can be described directly in terms of the ordinary quotient Y/Q, which is a based CW-space. Namely, $Y/\!\!/Q$ is the topological inverse limit of the based CW-spaces Y/Y' where Y' runs through the cofinite based CW-subspaces of Y. (Here "cofinite" means that $Y \setminus Y'$ is a union of finitely many cells.) In general, the homotopy groups of $Y/\!\!/Q$.

Example. Let $\bar{Q} = [0, 1]$ and Q = [0, 1[. Let $T = \{1 - 2^{-i} | i = 0, 1, 2, 3, ...\}$ and $Y = Q \amalg_T Q$. With the inclusion of the first copy of Q as the zero section, Y becomes a retractive space over Q. It has an obvious locally finite controlled CW-structure relative to Q. The ordinary quotient Y/Q is a wedge of infinitely many circles. Its fundamental group is free on generators g_1, g_2, g_3, \ldots corresponding to the 1-cells of Y/Q. In particular it is countably infinite. But Y//Q is homeomorphic to the Hawaiian earring. Its fundamental group is an inverse limit of finitely generated free groups, and it is uncountable. Similarly, for the suspension $\Sigma_Q Y$ (taken in the

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category of retractive spaces over Q), we have

$$\pi_2(\Sigma_Q Y/Q) \cong \bigoplus_{i=1}^{\infty} \mathbb{Z}, \qquad \pi_2(\Sigma_Q Y/\!\!/Q) \cong \prod_{i=1}^{\infty} \mathbb{Z}.$$

REMARK. Because of [10] we are stuck with the notation (\bar{Q}, Q) for control spaces, even though we do not require that Q be dense in \bar{Q} . We will consequently try to avoid the overline notation for topological closures. (The overline notation is also used in chapter A for something completely unrelated.)

DEFINITION 5.2. Let Y and Z be objects of $\mathcal{R}^{\mathrm{ld}}(\bar{Q}, Q)$. To define their SW product $Y \odot Z$, we introduce first an unstable form $Y \downarrow Z$ of it. We define it as the geometric realization of a based simplicial set. An *n*-simplex of this simplicial set is a pair (f, γ) where

- (i) f is a continuous map from the standard n-simplex Δ^n to $Y/\!\!/Q \wedge Z/\!\!/Q$;
- (ii) γ is a continuous assignment $c \mapsto \gamma_c$ of paths in Q, defined for $c \in \Delta^n$ with f(c) not equal to the base point \star .

The paths γ_c are to be parametrized by [-1, +1] and must satisfy $\gamma_c(-1) = r_Y f_Y(c)$ and $\gamma_c(+1) = r_Z f_Z(c)$, where r_Y, r_Z are the retractions and $f_Y(c), f_Z(c)$ are the coordinates of f(c). Finally there is a control condition:

> For z in $\overline{Q}^{\bullet} \smallsetminus Q$ and any neighborhood V of z in \overline{Q}^{\bullet} , there exists a smaller neighborhood W of z in \overline{Q}^{\bullet} such that, for any $c \in \Delta^n$ with $f(c) \neq \star$, the path γ_c either avoids W or runs entirely in V.

DEFINITION 5.3. For Y and Z in $\mathcal{R}^{\mathrm{ld}}(\bar{Q}, Q)$ and integers $k, \ell \in \mathbb{Z}$, let

$$(Y,k) \odot (Z,\ell) = \operatorname{colim}_n \Omega^{2n} (\Sigma^{n-k}Y \downarrow \Sigma^{n-\ell}Z)$$

More generally let $(Y, k) \odot_{\bullet} (Z, \ell)$ be the Ω -spectrum with j-th space

 $(Y,k) \odot_i (Z,\ell) = \operatorname{colim}_n \Omega^{2n} \Sigma^j (\Sigma^{i-k} Y \downarrow \Sigma^{i-\ell} Z).$

REMARK. We have $\mathcal{R}^{\mathrm{ld}}(\bar{Q}, Q) = \mathcal{R}^{\mathrm{ld}}(\bar{Q}^{\bullet}, Q)$ (an equality of Waldhausen categories). The meaning of $Y \odot Z$ is the same in both categories. But there is a difference between passage to germs near $\bar{Q} \smallsetminus Q$ (which we consider next) and passage to germs near $\bar{Q}^{\bullet} \searrow Q$ (which we are not interested in).

Next we work in the germ category $\mathcal{RG}^{\mathrm{ld}}(\bar{Q}, Q)$ of [10, dfn.7.1] and its stable form, $s\mathcal{RG}^{\mathrm{ld}}(\bar{Q}, Q)$. Let Y and Z be objects of $\mathcal{RG}^{\mathrm{ld}}(\bar{Q}, Q)$. Note that Y and Z are honest retractive spaces over Q. Again, to define their SW product $Y \odot Z$ in the germwise setting (recycled notation), we begin with an unstable form $Y \downarrow Z$ (also recycled notation) which is the geometric realization of a simplicial set. An *n*-simplex in this simplicial set is a germ of triples (U, f, γ) where

- (i) $U = \overline{U} \cap Q$ for an open neighborhood \overline{U} of $\overline{Q} \smallsetminus Q$ in \overline{Q} ;
- (ii) f is a continuous map from Δ^n to $(Y_U / U) \wedge (Z_U / U)$, where $Y_U = r_Y^{-1}(U)$ and $Z_U = r_Z^{-1}(U)$;
- (ii) γ is a continuous assignment of paths ... (as before).

We impose the same control condition on γ as before. In (ii), we regard Y_U and $Z_U = r_Z^{-1}(U)$ as retractive spaces over U, and U is the nonsingular part of the control space (\bar{U}, U) . Note that \bar{U} is the union of U and the singular set $\bar{Q} \setminus Q$. (It is not defined as the closure of U in \bar{Q} .) Passage to germs is achieved by taking the direct limit over all possible U. (It is a direct limit but the indexing is contravariant, i.e., we approach it by making U smaller and smaller.)

DEFINITION 5.4. Put $(Y,k) \odot (Z,\ell) := \operatorname{colim}_n \Omega^{2n}(\Sigma^{n-k}Y \downarrow \Sigma^{j-\ell}Z)$. More generally let $(Y,k) \odot_{\bullet} (Z,\ell)$ be the Ω -spectrum with *j*-th space

$$(Y,k) \odot_j (Z,\ell) = \operatorname{colim}_n \Omega^{2n} \Sigma^j (\Sigma^{i-k} Y \downarrow \Sigma^{i-\ell} Z).$$

REMARK. Later we will have to consider twisted versions of the above, depending on a spherical fibration on Q.

It is straightforward to verify that the above definitions of \odot and \odot_{\bullet} in the stable categories $s\mathcal{R}^{\mathrm{ld}}(\bar{Q},Q)$ and $s\mathcal{RG}^{\mathrm{ld}}(\bar{Q},Q)$ satisfy the conditions of [**39**, §1] for *SW* products. It is less straightforward to verify that they also satisfy the axioms of [**39**, §2], which are about existence and uniqueness of "duals". But this has been verified in [**39**, §2.A]. Hence there are associated quadratic *L*-theory spectra [**39**] which we denote by $\mathbf{L}_{\bullet}((\bar{Q},Q))$ and $\mathbf{L}_{\bullet}((\bar{Q},Q)_{\infty})$, respectively. Also, visible symmetric structures on objects of $s\mathcal{R}^{\mathrm{ld}}(\bar{Q},Q)$ and $s\mathcal{RG}^{\mathrm{ld}}(\bar{Q},Q)$ can be defined by analogy with definition 3.2. Hence there are visible symmetric *L*-theory spectra denoted by $\mathbf{VL}^{\bullet}((\bar{Q},Q))$ and $\mathbf{VL}^{\bullet}((\bar{Q},Q)_{\infty})$, respectively.

We now specialize to the case $(\bar{Q}, Q) = \mathbb{J}X = (X \times [0, 1], X \times [0, 1[)$ where X is an ENR. In fact we think of $X \mapsto L^{\bullet}(\mathbb{J}X_{\infty})$ and $X \mapsto L_{\bullet}(\mathbb{J}X_{\infty})$ as covariant functors on the category \mathcal{E}^{\bullet} whose objects are the ENR's and where a morphism from X_1 to X_2 is a based map $X_1^{\bullet} \to X_2^{\bullet}$ of the one-point compactifications. (This is the same thing as a proper map from an open subset of X_1 to X_2 .)

THEOREM 5.5. The spectrum valued functor $X \mapsto \mathbf{E}(X)$, where $\mathbf{E}(X)$ means $\mathbf{L}_{\bullet}(\mathbb{J}X_{\infty})$, is homotopy invariant and excisive. In detail:

- The projection from $X \times [0,1]$ to X induces a homotopy equivalence of $\mathbf{E}(X \times [0,1])$ with $\mathbf{E}(X)$.
- For an open subset V of X, the collapse map j: X[•] → V[•] and the inclusion
 i: X \ V → X determine a homotopy fiber sequence of spectra

$$\mathbf{E}(X \smallsetminus V) \xrightarrow{i_*} \mathbf{E}(X) \xrightarrow{j_*} \mathbf{E}(V).$$

• For a disjoint union $X = \coprod_{i=1}^{\infty} X_i$ of ENR's, the projections $X^{\bullet} \to X_i^{\bullet}$ induce an isomorphism

$$\pi_* \mathbf{E}(X) \longrightarrow \prod_{i=1}^{\infty} \pi_* \mathbf{E}(X_i) \; .$$

The coefficient spectrum $\mathbf{E}(\star)$ is homotopy equivalent to $\Sigma \mathbf{L}_{\bullet}(\star)$.

REMARK. The spectrum $\mathbf{L}_{\bullet}(\star)$ can be viewed as the quadratic *L*-theory spectrum of the sphere spectrum, where the latter is regarded as a (brave new) ring with involution. By the π - π -theorem, $\mathbf{L}_{\bullet}(\star)$ is homotopy equivalent to the quadratic *L*-theory spectrum of the ring with involution \mathbb{Z} , also known as the quadratic *L*-theory spectrum of the trivial group. See [**37**].

Statements similar to theorem 5.5 have been proved in many places. See [9] and [3], for example. Our proof below imitates the proof of the analogous statement for A-theory (= algebraic K-theory of spaces) in [35, §§6–9]. This requires some preparations.

We will work with based CW-spaces, which we generally view as CW-spaces relative to \star . On the set of cells (not including \star) of such a Y, there is a partial ordering: $e \ge e'$ if the smallest based CW-subspace containing e also contains e'. We say that Y is dimensionwise locally finite [**35**, Dfn.6.1] if, for every cell e in Y (not allowing \star) and every $j \ge 0$ there are only finitely many *j*-cells in *Y* which are $\ge e$. For example, a wedge of infinitely many based compact *CW*-spaces is dimensionwise locally finite.

We replace the categories $\mathcal{RG}^{\mathrm{ld}}(\bar{Q}, Q)$ by more tractable ones, denoted $\mathcal{R}(\star;\bar{Q},Q)_{\infty}$ in [35, §6]. An object of $\mathcal{R}(\star;\bar{Q},Q)_{\infty}$ is a dimensionwise locally finite based CW-space Y where the set of cells (excluding \star) is equipped with a map to Q. This map must satisfy the usual control condition: given $n \ge 0$ and $z \in \overline{Q} \setminus Q$ and a neighborhood V of z in \overline{Q} , there exists a smaller neighborhood W of z in \overline{Q} such that the closure of any n-cell with label in W is contained in a compact based CW-subspace for which the cell labels are all in V. In addition, for any $n \ge 0$ and any compact region of Q, the set of n-cells of Y with labels in that compact region is required to be finite. (There is also a finite domination condition to which we return in a moment.) A morphism $Y \to Z$ is a sequence (f_n) of compatible cellular map germs between the skeletons, $f_n: Y_U^n \to Z^n$, where f_n need only be defined on the cells of Y^n with labels in some open $U \subset Q$, where $U = \overline{U} \cap Q$ for some open neighborhood \overline{U} of the singular set. The maps f_n are subject to a straightforward control condition formulated in terms the of cell labels. There is a good notion of "controlled homotopy" in the category $\mathcal{R}(\star; Q, Q)_{\infty}$, so that the weak equivalences in $\mathcal{R}(\star; Q, Q)_{\infty}$ can simply be defined as the morphisms which are invertible up to controlled homotopy. The cofibrations are, by definition, those morphisms $Y \to Z$ whose underlying CW map germ is a composition of CW isomorphisms and CW subspace inclusions. It remains to make the finite domination condition on objects Y explicit. This is automatically satisfied if $Y = Y^n$ for some n. In general it means that for some n and all $m \ge n$, the inclusion $Y^n \to Y^m$ admits a (controlled) homotopy right inverse, so that Y^m is a homotopy retract of Y^n (in the "germ" sense). See [35, §6] for more details.

We come to the definition of $Y \downarrow Z$ (again recycled notation) for objects Y and Z of $\mathcal{R}(\star; \bar{Q}, Q)_{\infty}$. Again this is defined as the geometric realization of a simplicial set. An *n*-simplex in this simplicial set corresponds to a germ of certain pairs (U, f). Here

- $U = \overline{U} \cap Q$ for an open neighborhood \overline{U} of the singular set in \overline{Q} ;
- f is a continuous map from Δ^n to $(Y_U/\!\!/U) \wedge (Z_U/\!\!/U)$.

Here Y_U and Z_U are the largest based CW-subspaces of Y and Z, respectively, containing only cells with labels in U. We impose the usual control condition:

For $z \in \overline{Q} \setminus Q$ and any neighborhood V of z in \overline{Q} , there exists a smaller neighborhood W of z in \overline{Q} such that, for any $c \in \Delta^n$ with $f(c) \neq \star$, either both $f_Y(c)$ and $f_Z(c)$ are in cells with labels in V, or both are in cells with labels outside W.

(Note the absence of "paths".) We pass to germs by taking the direct limit over all possible R. Let

$$(Y,k) \odot (Z,\ell) := \operatorname{colim}_{n} \Omega^{2n} (\Sigma^{n-k}Y \downarrow \Sigma^{n-\ell}Z), (Y,k) \odot_{i} (Z,\ell) := \operatorname{colim}_{n} \Omega^{2n} \Sigma^{j} (\Sigma^{n-k}Y \downarrow \Sigma^{n-\ell}Z).$$

Then \odot and \odot_{\bullet} satisfy the axioms for an *SW*-product listed in [39].

Now we specialize to the situation(s) where $(Q, Q) = \mathbb{J}X$ for some X in \mathcal{E}^{\bullet} . We abbreviate $\mathbf{E}'(X) = \mathbf{L}_{\bullet}(\mathcal{R}(\star; \bar{Q}, Q)_{\infty})$. Again we want to view the assignment

$$X \mapsto \mathbf{E}'(X)$$

as a covariant functor on \mathcal{E}^{\bullet} . Indeed, every morphism $X_1 \to X_2$ in \mathcal{E}^{\bullet} , alias based map $f: X_1^{\bullet} \to X_2^{\bullet}$, has a factorization

$$X_1^{ullet} \to V^{ullet} \to X_2^{ullet}$$

where $V = X_1 \setminus f^{-1}(\infty)$. In this factorization, the second morphism $V^{\bullet} \to X_2^{\bullet}$ is induced by a proper map $V \to X_2$ and this determines in a straightforward way a map $\mathbf{E}'(V) \to \mathbf{E}'(X_2)$. The first morphism $X_1^{\bullet} \to V^{\bullet}$ induces an exact functor from $\mathcal{R}(\star; \mathbb{J}X)_{\infty}$ to $\mathcal{R}(\star; \mathbb{J}V)_{\infty}$, hence a map $\mathbf{E}'(X_1) \to \mathbf{E}'(V)$, roughly as follows. For an object Y of $\mathcal{R}(\star; \mathbb{J}X)_{\infty}$, the largest based CW-subspace of Y having all its cell labels in $V \times [0, 1]$ is an object of $\mathcal{R}(\star; \mathbb{J}V)_{\infty}$.

Later we will show, following [35, §9], that $\mathbf{E}'(X)$ is related to $\mathbf{E}(X)$ in theorem 5.5 by a chain of natural weak equivalences. But first we will prove the analogue of theorem 5.5 for the functor \mathbf{E}' .

The main ingredients in this proof are certain approximation statements, related to Waldhausen's approximation theorem [27]. To state these we fix X (an ENR) and an open $V \subset X$. On the category $\mathcal{R}(\star; \mathbb{J}X)_{\infty}$ we have, in addition to the standard notion of weak equivalence, a coarser one denoted by ω . Namely, a morphism is regarded as a weak ω -equivalence if the induced morphism in $\mathcal{R}(\star; \mathbb{J}V)_{\infty}$ is a weak equivalence. We write $\mathcal{R}_{\omega}(\star; \mathbb{J}X)_{\infty}$ for $\mathcal{R}(\star; \mathbb{J}X)_{\infty}$ equipped with the coarse notion of weak equivalence. We write $\mathcal{R}^{\omega}(\star; \mathbb{J}X)_{\infty}$ for the full subcategory of $\mathcal{R}(\star; \mathbb{J}X)_{\infty}$ consisting of the objects which are weakly ω -equivalent to the zero object, and this is equipped with the standard notion of weak eqivalence inherited from $\mathcal{R}(\star; \mathbb{J}X)_{\infty}$.

LEMMA 5.6. The functors of stable categories determined by the inclusion functor from $\mathcal{R}(\star; \mathbb{J}(X \setminus V))_{\infty}$ to $\mathcal{R}^{\omega}(\star; \mathbb{J}X)_{\infty}$ and the restriction from $\mathcal{R}_{\omega}(\star; \mathbb{J}X)_{\infty}$ to $\mathcal{R}(\star; \mathbb{J}V)_{\infty}$ satisfy the hypotheses of Waldhausen's approximation theorem.

PROOF. In the first case, the hypotheses are verified in $[35, \S3, \S7]$, and this works even without stabilization. In the second case, a closely related statement is also proved in $[35, \S3, \$7]$, with more general assumptions. Specialized to our situation this says that the induced functor

$$\mathcal{R}_{\omega}(\star; \mathbb{J}X)^{\mathrm{lf}}_{\infty} \longrightarrow \mathcal{R}(\star; \mathbb{J}V)^{\mathrm{lf}}_{\infty}$$

between the full subcategories of finite dimensional objects satisfies the hypotheses of the approximation theorem. Given that all weak equivalences in the categories $\mathcal{R}(\star; \mathbb{J}X)_{\infty}$ and $\mathcal{R}(\star; \mathbb{J}V)_{\infty}$ are invertible up to homotopy, it is easy to extend this result from the full subcategories of finite dimensional objects to the ambient categories, at the price of stabilizing, by means of the next lemma.

LEMMA 5.7. Every object of $\mathcal{R}(\star; \mathbb{J}X)_{\infty}$ becomes weakly equivalent to a finite dimensional object after at most two suspensions.

PROOF. The excision and homotopy invariance theorem for the algebraic Ktheory functor $X \mapsto K(\mathcal{R}(\star; \mathbb{J}X)_{\infty})$ is proved in [**35**, 7.1, 7.2]. The coefficient spectrum is analyzed in [**35**, 8.2,8.3] and it is found to have a vanishing π_0 . In particular, the K_0 group of $\mathcal{R}(\star; \mathbb{J}X)_{\infty}$ is zero. Therefore, by standard finiteness obstruction theory, all objects of $\mathcal{R}(\star; \mathbb{J}X)_{\infty}$ are weakly equivalent to finite dimensional ones after two suspensions. (This is more fully explained in the proof of [**35**, 9.5], especially in the statement labelled (**). The point is that the general case can be reduced to the situation where an object is a homotopy retract of another object whose cells are all concentrated in one dimension. If that dimension is at least two, the homotopy retraction can be linearized without any loss of information.) $\hfill\square$

For an object Y of $\mathcal{R}(\star; \mathbb{J}X)_{\infty}$ let $\mu(Y)$ be the monoid of endomorphisms of Y which are mapped to the identity by the restriction functor from $\mathcal{R}(\star;\mathbb{J}X)_{\infty}$ to $\mathcal{R}(\star;\mathbb{J}V)_{\infty}$. We note that, for objects Y and Z of $\mathcal{R}(\star;\mathbb{J}X)_{\infty}$, the product monoid $\mu(Y) \times \mu(Z)$ acts on the SW-product $Y \odot Z$. We like to think of $\mu(Y) \times \mu(Z)$ as a category with one object. The action is a functor on that category. Hence there is a canonical map

$$\underset{\mu(Y)\times\mu(Z)}{\text{hocolim}} Y \odot Z \longrightarrow j_*(Y) \odot j_*(Z)$$

where $j_*: \mathcal{R}(\star; \mathbb{J}X)_{\infty} \to \mathcal{R}(\star; \mathbb{J}V)_{\infty}$ is the restriction functor.

The next approximation lemma about SW-products and its corollary (about quadratic structures) are adaptations of [22, 2.7, 14.1].

LEMMA 5.8. For a finite dimensional object Y of $\mathcal{R}(\star; \mathbb{J}X)_{\infty}$, the monoid $\mu(Y)$ is directed in the following sense: given $f_1, f_2 \in \mu(Y)$, there is $f_3 \in \mu(Y)$ such that $f_3f_1 = f_3 = f_3f_2$. For two finite dimensional objects Y and Z in $\mathcal{R}(\star; \mathbb{J}X)_{\infty}$, the canonical map of unstable SW products

$$\underset{\mu(Y)\times\mu(Z)}{\text{hocolim}} Y \land Z \longrightarrow j_*(Y) \land j_*(Z)$$

is a homotopy equivalence.

PROOF. The statement about directedness is a consequence of the following observations. For every $f \in \mu(Y)$, there exist a neighborhood U(f) of $X \times \{1\}$ in $X \times [0, 1]$ and a neighborhood W(f) of $V \times \{1\}$ in $V \times [0, 1]$, with $U(f) \supset W(f)$, such that

f has a representative which is defined on every cell with label in U(f) and which is the identity on any cell e whose label is in U(f) and whose image f(e) has nonempty intersection with some cell having label in W(f).

Conversely, given any neighborhood W of $V \times \{1\}$ in $V \times [0, 1]$, there exists $g \in \mu(Y)$ such that some representative of g is (undefined or) zero on all cells of Y whose labels are not in W. (This is best proved by downward induction on the dimension of Y. Assume $Y = Y^n$. Choose a representative of an endomorphism of Y^n/Y^{n-1} which is zero on cells with labels outside W, and which belongs to $\mu(Y^n/Y^{n-1})$. There is a smaller neighborhood W' of $V \times \{1\}$ in $V \times [0, 1]$ such that this representative is the identity on all cells with labels in W'. Next, choose a representative of an endomorphism of Y^{n-1} which belongs to $\mu(Y^{n-1})$ and is zero on cells with labels outside W'. The two representatives then combine to give an endomorphism of $Y^n = Y$ with the required property.) Combining these two observations, we can choose $f_3 \in \mu(Y)$ in such a way that it vanishes on all cells with labels outside $W(f_1) \cap W(f_2)$, and then clearly $f_3f_1 = f_3 = f_3f_2$.

Now for the statement about SW-products: it is already clear from the foregoing that we have an identification of (geometric realizations of) simplicial sets

$$\operatorname{colim}_{\mu(Y)\times\mu(Z)} Y \downarrow Z \cong j_*(Y) \downarrow j_*(Z)$$

where \land denotes the unstable form of the SW-product. As the colimit is a colimit of based CW-spaces and based cellular maps over a directed category, we may replace it by a homotopy colimit.

COROLLARY 5.9. For a finite dimensional object Y of $\mathcal{R}(\star; JX)_{\infty}$ and $k \in \mathbb{Z}$, there is a canonical homotopy equivalence of spectra

$$\underset{\mu(Y)}{\text{hocolim}} ((Y,k) \odot_{\bullet} (Y,k))_{h\mathbb{Z}/2} \longrightarrow ((j_*Y,k) \odot_{\bullet} (j_*Y,k))_{h\mathbb{Z}/2}.$$

PROOF OF THM. 5.5, EXCISION, WITH **E'** INSTEAD OF **E**. Writing i_* for the inclusion functor

$$\mathcal{R}(\star; \mathbb{J}(X \smallsetminus V))_{\infty} \longrightarrow \mathcal{R}^{\omega}(\star; \mathbb{J}X)_{\infty}$$

we have natural homotopy equivalences $Y \downarrow Z \longrightarrow i_*(Y) \downarrow i_*(Z)$. Consequently the homotopy classification of (nondegenerate) quadratic structures is the same for an object of $s\mathcal{R}(\star; \mathbb{J}(X \smallsetminus V))_{\infty}$ and its image in $s\mathcal{R}^{\omega}(\star; \mathbb{J}X)_{\infty}$. Therefore and by the first part of lemma 5.6, the map

$$i_*: \mathbf{L}_{\bullet}(\mathcal{R}(\star; \mathbb{J}(X \smallsetminus V))_{\infty}) \longrightarrow \mathbf{L}_{\bullet}(\mathcal{R}^{\omega}(\star; \mathbb{J}X)_{\infty})$$

is a homotopy equivalence. For the rest of the argument, we use an L-theoretic precursor, due to Ranicki, of Waldhausen's fibration theorem in algebraic K-theory [27]. Applied to our situation this gives a homotopy fiber sequence of spectra

$$\mathbf{L}_{\bullet}(\mathcal{R}^{\omega}(\star;\mathbb{J}X)_{\infty})\longrightarrow\mathbf{L}_{\bullet}(\mathcal{R}(\star;\mathbb{J}X)_{\infty})\longrightarrow\mathbf{L}_{\bullet}(\mathcal{R}(\star;\mathbb{J}X)_{\infty},\mathcal{R}^{\omega}(\star;\mathbb{J}X)_{\infty})$$

where $\mathbf{L}_{\bullet}(\mathcal{R}(\star; \mathbb{J}X)_{\infty}, \mathcal{R}^{\omega}(\star; \mathbb{J}X)_{\infty})$ denotes the bordism theory of objects in the (stable category of) $\mathcal{R}(\star; \mathbb{J}X)_{\infty}$ equipped with a quadratic structure which is non-degenerate modulo $\mathcal{R}^{\omega}(\star; \mathbb{J}X)_{\infty}$. See e.g. [21, §3] and [25]; see also remark D.2 below. Therefore it only remains to show that the map

$$\mathbf{L}_{\bullet}(\mathcal{R}(\star; \mathbb{J}X)_{\infty}, \mathcal{R}^{\omega}(\star; \mathbb{J}X)_{\infty}) \longrightarrow \mathbf{L}_{\bullet}(\mathcal{R}(\star; \mathbb{J}V)_{\infty})$$

induced by j is a homotopy equivalence. We verify that the induced maps of homotopy groups $L_n(\ldots)$ are isomorphisms for all $n \in \mathbb{Z}$. For the surjectivity part, fix an object (Y',k) in $s\mathcal{R}(\star; \mathbb{J}V)_{\infty}$ and an *n*-dimensional nondegenerate quadratic structure ψ' on it. By lemma 5.6, we may assume that Y' = j_*Y for some Y in $\mathcal{R}(\star; \mathbb{J}X)_{\infty}$. By lemma 5.8, there is an *n*-dimensional quadratic structure ψ on (Y,k) such that $j_*\psi$ is homotopic to ψ' . Then ψ is automatically nondegenerate modulo $s\mathcal{R}^{\omega}(\star; \mathbb{J}X)_{\infty}$, so that $((Y,k),\psi)$ represents a class in $L_n(\mathcal{R}(\star; \mathbb{J}X)_\infty, \mathcal{R}^\omega(\star; \mathbb{J}X)_\infty)$ which maps to the class of $((Y', k), \psi')$ in $L_n(\mathcal{R}(\star; \mathbb{J}V)_\infty)$. For the injectivity part, fix an object (Z, k) in $s\mathcal{R}(\star; \mathbb{J}X)_\infty$ with a quadratic structure φ which is nondegenerate modulo $s\mathcal{R}(\star; \mathbb{J}V)_{\infty}$, and assume that $((Z',k),\varphi') := ((j_*Z,k),j_*\varphi)$ is nullbordant. Then there exist a cofibration $u': (Z', k) \to (T', \ell)$ in $s\mathcal{R}(\star; \mathbb{J}V)_{\infty}$ and a nullhomotopy τ' of $u'_*\varphi'$ such that $((Z',k) \to (T',\ell), (\tau',\partial\tau'))$ with $\partial\tau' = \varphi'$ is a nondegenerate quadratic pair in $s\mathcal{R}(\star;\mathbb{J}V)_{\infty}$. By lemma 5.6, we may assume that u' is obtained from a cofibration $u: (Z,k) \to (T,\ell)$ in $s\mathcal{R}^{\omega}(\star; \mathbb{J}X)_{\infty}$ by applying j_* . By lemma 5.8, on composing u with an appropriate endomorphism of T (and restoring the cofibration property by means of a mapping cylinder construction), we may also assume that τ' is obtained from a nullhomotopy τ for $u_*\varphi$ by applying j_* . Then $((Z,k) \to (T,\ell), (\tau,\partial\tau))$ is a quadratic pair in $s\mathcal{R}(\star; \mathbb{J}X)_{\infty}$ which is nondegenerate modulo $s\mathcal{R}^{\omega}(\star; \mathbb{J}X)_{\infty}$. Hence $((Z, k), \varphi)$ represents the zero class. PROOF OF THM. 5.5, HOMOTOPY INVARIANCE, \mathbf{E}' INSTEAD OF \mathbf{E} . It suffices to show that the inclusion $i: X \times \{0\} \to X \times [0, 1]$ induces a homotopy equivalence $i_*: \mathbf{E}'(X) \to \mathbf{E}'(X \times [0, 1])$. By the excision property which we just established, it is also enough to show that $\mathbf{E}'(X \times [0, 1])$ is contractible. This uses an Eilenberg swindle. The details are as in [**35**, §4], except for a correction to [**35**, §4] in remark D.3 below.

PROOF OF THM. 5.5, DISJOINT UNION AXIOM, \mathbf{E}' INSTEAD OF \mathbf{E} . We leave it to the reader as a matter of inspection.

PROOF OF THM. 5.5, COEFFICIENT SPECTRUM, **E'** INSTEAD OF **E**. This is similar to the excision part. We take $X = \star$. We introduce a Waldhausen category $\mathcal{R}(\star; \mathbb{J}\star)$, defined like $\mathcal{R}(\star; \mathbb{J}\star)_{\infty}$ but without the germ relation. Thus an object of $\mathcal{R}(\star; \mathbb{J}\star)$ is a based CW-space with a map from the set of cells (excluding the base point) to [0, 1[. A morphism $Y \to Z$ in $\mathcal{R}(\star; \mathbb{J}\star)$ is a based cellular map (not a germ of such maps) from Y to Z, subject to the usual control condition. There is a finite domination condition on objects Y, which says that for some n, each Y^m with $m \geq n$ is a homotopy retract of Y^n in the appropriate controlled homotopy category.

In addition to the standard notion of weak equivalence in $\mathcal{R}(\star; \mathbb{J}\star)$, we have a coarse notion ω of weak equivalence. Namely, a morphism in $\mathcal{R}(\star; \mathbb{J}\star)$ is a weak ω -equivalence if the induced morphism in $\mathcal{R}(\star; \mathbb{J}\star)_{\infty}$ is a weak equivalence. As in the excision part, we obtain from general principles a homotopy fiber sequence of spectra

$$\mathbf{L}_{\bullet}(\mathcal{R}^{\omega}(\star;\mathbb{J}\star))\longrightarrow \mathbf{L}_{\bullet}(\mathcal{R}(\star;\mathbb{J}\star))\longrightarrow \mathbf{L}_{\bullet}(\mathcal{R}(\star;\mathbb{J}\star),\mathcal{R}^{\omega}(\star;\mathbb{J}\star)).$$

The Waldhausen category $\mathcal{R}^{\omega}(\star; \mathbb{J}\star)$ has an exact subcategory consisting of those objects Y which have only finitely many cells. This is equivalent to the category of based finite CW-spaces, so that its L-theory spectrum is $\mathbf{L}_{\bullet}(\star)$. The inclusion of this exact subcategory in $\mathcal{R}^{\omega}(\star;\mathbb{J}\star)$ satisfies the conditions of the approximation theorem; for the proof, see [35, 8.3]. The homotopy classification of (nondegenerate) quadratic structures on an object in the (stabilized) subcategory is the same whether we classify in the subcategory or in the ambient category. Consequently we have

$$\mathbf{L}_{\bullet}(\mathcal{R}^{\omega}(\star;\mathbb{J}\star)) \simeq \mathbf{L}_{\bullet}(\star).$$

It remains to show that the "passage to germs" functor from $\mathcal{R}(\star; \mathbb{J}_{\star})$ to $\mathcal{R}(\star; \mathbb{J}_{\star})_{\infty}$ induces a homotopy equivalence of spectra

$$\mathbf{L}_{\bullet}(\mathcal{R}(\star;\mathbb{J}\star),\mathcal{R}^{\omega}(\star;\mathbb{J}\star))\longrightarrow \mathbf{L}_{\bullet}(\mathcal{R}(\star;\mathbb{J}\star)_{\infty})$$

and this can be done by considering the homotopy groups. We need to know that the functor of stable categories determined by $\mathcal{R}_{\omega}(\star; \mathbb{J}_{\star}) \to \mathcal{R}(\star; \mathbb{J}_{\star})_{\infty}$ satisfies the conditions of the approximation theorem; for this, see again [**35**, 8.3] and make use of lemma 5.7 above. The other ingredient is an approximation lemma for quadratic structures analogous to lemma 5.8, but applicable to the "passage to germs" functor from $\mathcal{R}(\star; \mathbb{J}_{\star})$ to $\mathcal{R}(\star; \mathbb{J}_{\star})_{\infty}$. We leave the remaining details to the reader. \Box

PROOF OF THEOREM 5.5: COMPARING \mathbf{E}' and \mathbf{E} . Recall that

$$\mathbf{E}(X) = \mathbf{L}_{\bullet}(\mathcal{RG}^{\mathrm{Id}}(\bar{Q}, Q)), \qquad \mathbf{E}'(X) = \mathbf{L}_{\bullet}(\mathcal{R}(\star; \bar{Q}, Q)_{\infty})$$

where $(\bar{Q}, Q) = \mathbb{J}X$. The Waldhausen categories $\mathcal{RG}^{\mathrm{ld}}(\bar{Q}, Q)$ and $\mathcal{R}(\star; \bar{Q}, Q)_{\infty}$ are related, for a general control space (\bar{Q}, Q) , by exact functors

$$\mathcal{RG}^{\mathrm{ld}}(\bar{Q},Q) \xleftarrow{\mathrm{inclusion}} \mathcal{RG}^{\mathrm{lf}}(\bar{Q},Q) \xrightarrow{v} \mathcal{R}(\star;\bar{Q},Q)_{\infty}$$

Here $\mathcal{RG}^{\mathrm{lf}}(\bar{Q}, Q)$ is defined very much like $\mathcal{RG}^{\mathrm{ld}}(\bar{Q}, Q)$, but the objects Y come equipped with a finite dimensional controlled CW-structure relative to Q and morphisms are required to be cellular relative to Q. See the proof of [**35**, 9.5] for details. (Except for a homotopy finiteness condition, which is unimportant in our setting thanks to lemma 5.7, the category $\mathcal{RG}^{\mathrm{ld}}(\bar{Q}, Q)$ is identical with something denoted $t\mathcal{R}(\star; \bar{Q}, Q)_{\infty}$ in that proof, and $\mathcal{RG}^{\mathrm{lf}}(\bar{Q}, Q)$ is denoted \mathcal{B} there.) It is also proved in [**35**, §9] that the two exact functors in the chain, viewed as functors of the associated stable categories, satisfy the conditions of the approximation theorem (again modulo lemma 5.7). Finally v respects the SW-products, in the strong sense that we have a binatural homotopy equivalence $Y \odot Z \to v(Y) \odot v(Z)$, for Y and Z in $\mathcal{RG}^{\mathrm{lf}}(\bar{Q}, Q)$. It follows that u and v induce homotopy equivalences of the associated quadratic L-theory spectra.

Our next goal is to formulate an excision theorem similar to theorem 5.5 for the visible symmetric L-theory. (We do not have, and we do not need, an analogue of the excision theorem for a controlled version of ordinary symmetric L-theory.) This is straightforward modulo chapter 3.

DEFINITION 5.10. An *n*-dimensional visible symmetric structure on an object (Y,k) in $\mathcal{RG}^{\mathrm{ld}}(\bar{Q},Q)$ is an element of $\Omega^n((Y,k)\odot(Y,k))^{\mathbb{Z}/2}$, with $(Y,k)\odot(Y,k)$ as in definition 5.4. An *n*-dimensional visible symmetric structure on an object (Y,k) in $\mathcal{R}(\star;\bar{Q},Q)_{\infty}$ is an element of $\Omega^n((Y,k)\odot(Y,k))^{\mathbb{Z}/2}$, with the appropriate definition of \odot_{\bullet} for the category $\mathcal{R}(\star;\bar{Q},Q)_{\infty}$.

Again $(Y, k) \odot_{\bullet} (Y, k)$ turns out to be the underlying spectrum of a $\mathbb{Z}/2$ -spectrum which we can describe as a shifted suspension spectrum

$$\mathbf{S}_{\mathbb{Z}/2}^{-kW} \wedge Y^{\lambda 2}.$$

(The meaning of $Y^{\lambda 2} = Y \wedge Y$ depends on the category, which may be $\mathcal{RG}^{\mathrm{ld}}(\bar{Q}, Q)$ or $\mathcal{R}(\star; \bar{Q}, Q)_{\infty}$.) The analogues of corollary 3.5 hold in both categories, and they are still corollaries of proposition 3.4.

Let Y be an object of $\mathcal{RG}^{\mathrm{ld}}(\bar{Q},Q)$ or $\mathcal{R}(\star;\bar{Q},Q)_{\infty}$. Let \bar{U} be an open neighborhood of the singular set in \bar{Q} and put $U = \bar{U} \cap Q$. Recall that Y_U means $r^{-1}(U)$ for Y in $\mathcal{RG}^{\mathrm{ld}}(\bar{Q},Q)$; for Y in $\mathcal{R}(\star;\bar{Q},Q)_{\infty}$ it means the largest based CW-subspace of Y having all its cell labels in U. In the second case we also introduce the notation $Y_U/\!\!/\star$ for the topological inverse limit of the based spaces Y_U/Y'_U where Y'_U runs though all cofinite based CW-subspaces of Y_U .

COROLLARY 5.11. In the setting of definition 5.10, there is a natural homotopy fiber sequence of spectra

$$((Y,k)\odot_{\bullet}(Y,k))_{h\mathbb{Z}/2} \longrightarrow ((Y,k)\odot_{\bullet}(Y,k))^{\mathbb{Z}/2} \xrightarrow{J} \operatorname{hocolim} \Sigma^{\infty-k}(Y_U/\!\!/ U).$$

COROLLARY 5.12. For any object Y of $\mathcal{R}(\star; \bar{Q}, Q)_{\infty}$ and $k \in \mathbb{Z}$, there is a homotopy fiber sequence of spectra

$$((Y,k)\odot_{\bullet}(Y,k))_{h\mathbb{Z}/2} \longrightarrow ((Y,k)\odot_{\bullet}(Y,k))^{\mathbb{Z}/2} \xrightarrow{J} \operatorname{hocolim} \Sigma^{\infty-k}(Y_U/\!\!/\star).$$

In these two corollaries, U runs through the open subsets of Q of the form $U = \overline{U} \cap Q$ where \overline{U} is an open neighborhood of the singular set in \overline{Q} . The homotopy colimits are reduced (taken in the based category) and the suspension spectrum construction $\Sigma^{\infty-k}$ is meant to include a *CW*-approximation mechanism.

There is also an analogue of corollary 5.9. We keep the assumptions and notation of that corollary to state the analogue:

COROLLARY 5.13. For a finite dimensional object Y of $\mathcal{R}(\star; \mathbb{J}X)_{\infty}$ and $k \in \mathbb{Z}$, there is a canonical homotopy equivalence of spectra

$$\underset{\mu(Y)}{\operatorname{hocolim}} ((Y,k) \odot_{\bullet} (Y,k))^{\mathbb{Z}/2} \longrightarrow ((j_*Y,k) \odot_{\bullet} (j_*Y,k))^{\mathbb{Z}/2}$$

PROOF. By corollaries 5.9 and 5.12, and a five lemma argument, it is enough to verify that the canonical map

 $\underset{\mu(Y)}{\operatorname{hocolim}} \quad \underset{U}{\operatorname{hocolim}} \quad \Sigma^{\infty-k}(Y_U /\!\!/ U) \quad \longrightarrow \quad \underset{W}{\operatorname{hocolim}} \quad \Sigma^{\infty-k}((j_*Y)_W /\!\!/ \star)$

is a homotopy equivalence. But this is obvious.

With these tools available, the visible symmetric L-theory version of theorem 5.5, which we are about to state, can be proved in strict analogy with the original (quadratic L-theory) version.

THEOREM 5.14. The spectrum valued functor $X \mapsto \mathbf{E}(X)$, where $\mathbf{E}(X)$ means $\mathbf{VL}^{\bullet}(\mathbb{J}X_{\infty})$, is homotopy invariant and excisive. The coefficient spectrum $\mathbf{E}(\star)$ is homotopy equivalent to $\Sigma \mathbf{VL}^{\bullet}(\star)$.

CHAPTER 6

Control and visible *L*-theory

In this chapter our goal is to generalize theorem 3.7 to a controlled setting, as far as possible.

Fix a compact Hausdorff space S. For most of this chapter the only control spaces we shall be interested in are of the form (\bar{X}, X) , with compact \bar{X} and an identification $\bar{X} \setminus X \cong S$. The only morphisms $f:(\bar{X}_1, X_1) \to (\bar{X}_2, X_2)$ between such control spaces that we shall be interested in are those which are relative to S. These objects and morphisms form a category \mathcal{K}^S .

We can also speak of homotopies between morphisms in \mathcal{K}^S . These will also be relative to S, and they allow us to define a homotopy category \mathcal{HK}^S . A morphism in \mathcal{K}^S is a *cofibration* if it is an embedding which has the homotopy extension property, for such homotopies.

THEOREM 6.1. On \mathcal{K}^S , the functor $(\bar{X}, X) \mapsto \mathbf{V}\widehat{\mathbf{L}}^{\bullet}((\bar{X}, X))$ is homotopy invariant, excisive and satisfies a strong "wedge" axiom.

This needs a few explanations. The homotopy invariance property means that the functor takes homotopy equivalences in \mathcal{K}^S to homotopy equivalences of spectra. Here homotopy equivalences in \mathcal{K}^S refers to morphisms in \mathcal{K}^S which become invertible in $\mathcal{H}\mathcal{K}^S$.

For the excision property, suppose given a pushout diagram

$$\begin{array}{ccc} (\bar{X}_{ab}, X_{ab}) & \longrightarrow & (\bar{X}_{a}, X_{a}) \\ & & & \downarrow \\ & & & \downarrow \\ (\bar{X}_{b}, X_{b}) & \longrightarrow & (\bar{X}, X) \end{array}$$

in \mathcal{K}^S where all the arrows are cofibrations (and without loss of generality, all are inclusions). It is being claimed that in such a case

is homotopy cocartesian, and also that $V\widehat{\mathbf{L}}^{\bullet}$ applied to the initial object (S, \emptyset) of \mathcal{K}^{S} is a weakly contractible spectrum.

For the strong wedge axiom, suppose that (\bar{X}, X) is in \mathcal{K}^S and X is a topological disjoint union of subspaces X_{α} , where α runs through some (countable) set. Let \bar{X}_{α} be the union of X_{α} and the singular set $\bar{X} \setminus X$. Then the embedding

$$(\bar{X}_{\alpha}, X_{\alpha}) \to (\bar{X}, X)$$

is a morphism in \mathcal{K}^S , for every α . It follows from the ordinary excision property (just above) that the induced homomorphisms

$$V\widehat{L}^n((\bar{X}_\alpha, X_\alpha)) \longrightarrow V\widehat{L}^n((\bar{X}, X))$$

are split injective with a preferred splitting, since (\bar{X}, X) is the coproduct in \mathcal{K}^S of $(\bar{X}_{\alpha}, X_{\alpha})$ and $(\bar{X}, X \smallsetminus X_{\alpha})$. The decomposition of (\bar{X}, X) into the $(\bar{X}_{\alpha}, X_{\alpha})$ could be regarded as a generalized wedge decomposition (because it is when S is a point). It is not in general a coproduct decomposition. Nevertheless, it is being claimed that the projections $V\hat{L}^n((\bar{X}, X)) \to V\hat{L}^n((\bar{X}_{\alpha}, X_{\alpha}))$ induce an isomorphism

$$V\widehat{L}^n((\bar{X},X)) \longrightarrow \prod_{\alpha} V\widehat{L}^n((\bar{X}_{\alpha},X_{\alpha})).$$

We turn to the proofs. The homotopy invariance property in theorem 6.1 can be proved by the same argument as the homotopy invariance property in theorem 3.7. The excision property for the special case of a coproduct (that is, (\bar{X}, X) in \mathcal{K}^S with $X = X_1 \amalg X_2$) and the strong "wedge axiom" are valid by inspection. This leaves the general excision property. It is correct to say that the proof of the excision property in theorem 3.7 carries over, but some clarifications are nevertheless in order. The difficulty is that in lemma 3.8, lemma 3.9 etc., which were part of the proof of theorem 3.7, we made essential use of the concept of *fibration*. Here we will need a corresponding concept of *controlled fibration* and this is not completely obvious.

Fix a control space (\bar{X}, X) with compact \bar{X} for simplicity, but not necessarily in \mathcal{K}^S . Let $p: E \to X$ be any map. Suppose that a collection of open subspaces $E_{\lambda} \subset E$ has been specified, where λ runs though a directed set; suppose also that the indexing is monotone, so that $\lambda_1 < \lambda_2$ implies $E_{\lambda_1} \subset E_{\lambda_2}$. We assume that Eis the union of the E_{λ} .

DEFINITION 6.2. The map $p: E \to X$ together with the directed system of open subspaces $\{E_{\lambda}\}$ such that $E = \bigcup E_{\lambda}$ is a *controlled Serre fibration system* if the following holds. For every controlled finite dimensional locally finite CW-space Yover X, every controlled map $f: Y \to E_{\lambda}$ and controlled homotopy $h: Y \times [0, 1] \to X$ starting with $p \circ f$, there exists $\kappa > \lambda$ and a homotopy $Y \times [0, 1] \to E_{\kappa}$ which lifts h and starts with f.

REMARK. Controlled fibration systems can be pulled back along maps of control spaces. More precisely, if $f:(\bar{X}_1, X_1) \to (\bar{X}_2, X_2)$ is a map of control spaces (compact \bar{X}_1 and \bar{X}_2), and $(p: E \to X_2, \{E_\lambda\})$ is a controlled Serre fibration system over X_2 , then the pullback f^*E with the subspaces f^*E_λ is a controlled Serre fibration system over X_1 .

LEMMA 6.3. Let Z be a retractive space over X, with retraction $r: Z \to X$. Then there exists a controlled Serre fibration system $(p: E \to X, \{E_{\lambda}\})$ and an embedding $Z \to \bigcap E_{\lambda}$ over X, inducing controlled homotopy equivalences $Z \to E_{\lambda}$ for all λ .

PROOF. This is supposed to be a controlled version of the Serre construction in ordinary fibration theory. Let W be an open neighborhood of the diagonal in $X \times X$, invariant under permutation of the two factors X. We say that W is *controlled* if its closure in $\overline{X} \times \overline{X}$ is disjoint from $X \times (\overline{X} \setminus X)$. (Equivalently, Wis controlled if, for every $x \in \overline{X} \setminus X$ and every neighborhood V of x in \overline{X} , there exists a smaller neighborhood V' of z in \overline{X} such that for any $(y_1, y_2) \in W$ with $y_1 \in V'$ we have $y_2 \in V$.) Ordered by inclusion, these controlled neighborhoods form a directed system Λ . For W in Λ let E_W be the space of pairs (z, ω) where $z \in Z$ and $\omega : [0, 1] \to X$ is a path in X with $\omega(0) = r(z)$, subject to the control condition determined by W. (Namely, all points of the form $(\omega(s), \omega(t))$ are in W.) Let $p_W : E_W \to X$ be defined by $p_W(z, \omega) = \omega(1)$. The inclusion $Z \to E_W$ is clear. It is a map over X and as such it is a controlled homotopy equivalence. The inclusions $E_W \to E_{W'}$ for W' > W in Λ are also clear. We let $E = \bigcup E_W$. The remaining details are left to the reader.

Returning to \mathcal{K}^S and the proof of theorem 6.1, it only remains to say that lemma 3.10 carries over to the controlled situation without essential changes. In the controlled version of lemma 3.8, the fibration $E \to X$ should be replaced by a controlled Serre fibration system as in definition 6.2; the morphisms f should land in some E_{λ_1} by assumption and the morphism g should be constructed to land in some E_{λ_2} where $\lambda_2 \ge \lambda_1$. In the controlled version of lemma 3.9, the groups $Q_n(E;k), VQ^n(E;k)$ and $V\hat{Q}^n(E,k)$ should be replaced by the direct limits over λ of $Q_n(E_{\lambda};k), VQ^n(E_{\lambda};k)$ and $V\hat{Q}^n(E_{\lambda},k)$, respectively. Constructions like $E_a = E|X_a|$ should be read as restrictions or pullbacks of controlled fibration systems. These two lemmas (in the controlled version) feed into the proof of theorem 6.1 via the controlled Serre construction, lemma 6.3. The proof then proceeds as in the situation without control.

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CHAPTER 7

Control, stabilization and change of decoration

Our main point in this chapter is to make connections between the *L*-theory and algebraic *K*-theory of a control space (\bar{Q}, Q) on one side, and the *L*-theory and algebraic *K*-theory of $(\bar{Q} \times \bar{\mathbb{R}}, Q \times \mathbb{R})$ on the other side, where $\bar{\mathbb{R}} = [-\infty, +\infty]$. We are going to be more specific, as follows. Let *X* be a compact Hausdorff space. Fix an integer $i \ge 0$ and form the control space $(X * S^{i-1}, X * S^{i-1} \setminus S^{i-1})$. We usually identify $X * S^{i-1} \setminus S^{i-1}$ with $X \times \mathbb{R}^i$ and so write $(X * S^{i-1}, X \times \mathbb{R}^i)$. Less formally still, we like to refer to $X \times \mathbb{R}^i$ only and use a letter *c* as in $\mathbf{L}_{\bullet}(X \times \mathbb{R}^i; c)$ to indicate that $X \times \mathbb{R}^i$ is the nonsingular part of a control space $(X * S^{i-1}, X \times \mathbb{R}^i)$. Our goal is then to compare constructions like $\mathbf{L}_{\bullet}(X \times \mathbb{R}^i; c)$ and $\mathbf{A}(X \times \mathbb{R}^i; c)$ with $\mathbf{L}_{\bullet}(X \times \mathbb{R}^{i+1}; c)$ and $\mathbf{A}(X \times \mathbb{R}^{i+1}; c)$.

In particular, there is an exact functor $\times \mathbb{R}$ from retractive spaces over $X \times \mathbb{R}^i$ (subject to various conditions) to retractive spaces over $X \times \mathbb{R}^{i+1}$. Because of the geometric applications that we have in mind, it is important for us to come to grips with the induced maps in controlled *L*-theory and *A*-theory. From an algebraic viewpoint, this is not the best starting point. Instead we start by using fibration theorems from algebraic *K*-theory and *L*-theory to see connections between the controlled algebraic *K* or *L*-theory of $X \times \mathbb{R}^i$ and the controlled algebraic *K* or *L*-theory of $X \times \mathbb{R}^{i+1}$. These connections are of the well-known type. We have suppressed some of the more mechanical details in the proofs.

Let $\mathbf{L}^{h}_{\bullet}(X \times \mathbb{R}^{i}; c)$ be the controlled *L*-theory spectrum of $(X * S^{i-1}, X \times \mathbb{R}^{i})$, constructed using locally finite and finite dimensional retractive spaces (with a controlled relative *CW*-structure) over $X \times \mathbb{R}^{i}$ rather than locally finitely dominated ones. Similarly, $\mathbf{A}^{h}(X \times \mathbb{R}^{i}; c)$ is the controlled *A*-theory constructed using locally finite and finite dimensional retractive spaces (with a controlled relative *CW*-structure) over $X \times \mathbb{R}^{i}$.

THEOREM 7.1. We have

$$\mathbf{L}_{\bullet}(X \times \mathbb{R}^{i}; c) \simeq \Omega \mathbf{L}_{\bullet}^{h}(X \times \mathbb{R}^{i+1}; c), \\
 \mathbf{V}\mathbf{L}^{\bullet}(X \times \mathbb{R}^{i}; c) \simeq \Omega \mathbf{V}\mathbf{L}^{\bullet h}(X \times \mathbb{R}^{i+1}; c), \\
 \mathbf{A}(X \times \mathbb{R}^{i}; c) \simeq \Omega \mathbf{A}^{h}(X \times \mathbb{R}^{i+1}; c).$$

Indeed there is a homotopy cartesian square of inclusion-induced maps

with contractible off-diagonal terms; and there are analogous homotopy cartesian squares for VL^{\bullet} and A.

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PROOF. We concentrate on the quadratic L-theory case, the other two cases being very similar. The first step is to replace the control conditions by stronger ones. Let $p: X \times \mathbb{R}^i \to \mathbb{R}^i$ be the projection. Given retractive spaces Y_1 and Y_2 over $X \times \mathbb{R}^i$ and a controlled map $f: Y_1 \to Y_2$ (which is understood to be relative to $X \times \mathbb{R}^i$ but need not respect the retractions to $X \times \mathbb{R}^i$), we say that f is bounded if there exists a real number $a \ge 0$ such that $||p(y) - pf(y)|| \le a$ for all $y \in Y_1$. Similarly, there is a notion of bounded and controlled homotopy between bounded and controlled maps (between retractive spaces over $X \times \mathbb{R}^i$). A morphism $f: Y_1 \to Y_2$ between retractive spaces over $X \times \mathbb{R}^i$ (i.e., a map which respects both the zero sections and the retractions) is a bounded map for trivial reasons; we call it a *weak equivalence* (in the bounded sense) if it is invertible up to bounded homotopies relative to $X \times \mathbb{R}^i$. A finite dimensional controlled CW-structure on a retractive space Y over $X \times \mathbb{R}^i$, relative to $X \times \mathbb{R}^i$, is *bounded* if there exists a > 0 such that the image of each cell in \mathbb{R}^i has diameter $\leq a$. We use all that to introduce a Waldhausen category

$$\mathcal{R}^{\mathrm{ld}}(X \times \mathbb{R}^i; b)$$

similar to $\mathcal{R}^{\mathrm{ld}}(X \times \mathbb{R}^i; c)$, but with all control conditions replaced by the corresponding boundedness conditions. There is also a stable version

$$s\mathcal{R}^{\mathrm{ld}}(X \times \mathbb{R}^i; b)$$

obtained by adjoining formal desuspensions. There is also a preferred SW-product on $s\mathcal{R}^{\mathrm{ld}}(X \times \mathbb{R}^i; b)$, which can be constructed roughly as in definition 5.2. (Replace Q there by $X \times \mathbb{R}^i$. Replace the control condition on γ there by the condition that, for some a > 0, the images in \mathbb{R}^i of all paths γ_c have diameter $\leq a$.)

There is an inclusion of Waldhausen categories

$$s\mathcal{R}^{\mathrm{Id}}(X \times \mathbb{R}^{i}; b) \longrightarrow s\mathcal{R}^{\mathrm{Id}}(X \times \mathbb{R}^{i}; c).$$

This is compatible with the SW-products and preserves nondegenerate pairings. (The claim about nondegeneracy can be reduced to the case of a paring between two objects which have all their cells in a single dimension.) Hence we have an induced map in L-theory

$$\mathbf{L}_{\bullet}(s\mathcal{R}^{\mathrm{ld}}(X \times \mathbb{R}^{i}; b)) \longrightarrow \mathbf{L}_{\bullet}(s\mathcal{R}^{\mathrm{ld}}(X \times \mathbb{R}^{i}; c)).$$

It is a key fact, and one whose proof we want to skip, that this map is a homotopy equivalence. See [3]. Similarly there is an inclusion map

$$\mathbf{L}_{\bullet}(s\mathcal{R}^{\mathrm{lf}}(X \times \mathbb{R}^{i+1}; b)) \longrightarrow \mathbf{L}_{\bullet}(s\mathcal{R}^{\mathrm{lf}}(X \times \mathbb{R}^{i+1}; c))$$

of L-theory spectra, where the superscript lf refers to retractive spaces with a locally finite, finite dimensional and bounded CW-structure relative to $X \times \mathbb{R}^{i+1}$. This is again a homotopy equivalence.

For the remainder of this proof we work in the bounded setting. We use the Waldhausen fibration theorem [27] or rather its analogue in *L*-theory due to Ranicki/Vogel. The category to which we will apply it is

$$s\mathcal{R}^{\mathrm{lf}}(X \times \mathbb{R}^{i+1}; b)$$

and for the purposes of this proof we abbreviate this to $s\mathcal{R}$. In addition to the default notion of weak equivalence in $s\mathcal{R}$, we introduce two coarser notions involving passage to germs. Let Y_1 and Y_2 be retractive spaces over $X \times \mathbb{R}^{i+1}$. A *u*-germ of bounded and controlled maps from Y_1 to Y_2 is represented by a bounded and

controlled map from the portion of Y_1 lying over $X \times \mathbb{R}^i \times [a, \infty[$, for some $a \in \mathbb{R}$, to Y_2 . Two such representatives define the same *u*-germ if they are both defined on $X \times \mathbb{R}^i \times [a', \infty[$ for some $a' \in \mathbb{R}$ and agree there. Similarly, a *v*-germ of bounded and controlled maps from Y_1 to Y_2 is represented by a bounded and controlled map from the portion of Y_1 lying over $X \times \mathbb{R}^i \times [-\infty, a[$, for some $a \in \mathbb{R}$, to Y_2 . Call a morphism in $s\mathcal{R}$ a *u*-equivalence if its mapping cone is weakly equivalent to the zero object in the *u*-germ sense. Call it a *v*-equivalence if its mapping cone is weakly equivalent to zero in the *v*-germ sense. Using the notation which Waldhausen uses in his formulation of the fibration theorem, we obtain a commutative diagram of Waldhausen categories with SW-duality

In the resulting commutative diagram of *L*-theory spectra

$$(7.3) \qquad \begin{array}{ccc} \mathbf{L}_{\bullet}(s\mathcal{R}^{u} \cap s\mathcal{R}^{v}) & \longrightarrow & \mathbf{L}_{\bullet}(s\mathcal{R}^{u}) & \longrightarrow & \mathbf{L}_{\bullet}(s\mathcal{R}^{u}, s\mathcal{R}^{u} \cap s\mathcal{R}^{v}) \\ & \downarrow & & \downarrow & & \downarrow \\ & \mathbf{L}_{\bullet}(s\mathcal{R}^{v}) & \longrightarrow & \mathbf{L}_{\bullet}(s\mathcal{R}) & \longrightarrow & \mathbf{L}_{\bullet}(s\mathcal{R}, s\mathcal{R}^{v}) \end{array}$$

the rows are homotopy fibration sequences by the fibration theorem. The terms $\mathbf{L}_{\bullet}(s\mathcal{R}^u)$ and $\mathbf{L}_{\bullet}(s\mathcal{R}^v)$ are contractible because the Waldhausen categories involved are flasque. (More precisely there is an Eilenberg swindle argument for contractibility, as follows. The translation $x \mapsto x - n$ acting on \mathbb{R} induces an endofunctor κ_n of $s\mathcal{R}^u$. The sum

$$\tau = \bigvee_{n \ge 0} \kappa_n$$

is again an endofunctor of $s\mathcal{R}^u$. While it is not strictly true that $\tau \cong id \vee \tau$, it is easy to relate τ and $id \vee \tau$ by a chain of natural equivalences, preserving SWduality. Hence the identity map of $\mathbf{L}_{\bullet}(s\mathcal{R}^u)$ is contractible.) Next, the right-hand vertical arrow in diagram (7.3) is a homotopy equivalence by an easy application of the approximation theorem. (What we have in mind here is an *L*-theoretic cousin of Waldhausen's approximation theorem in algebraic *K*-theory. This is very neatly formulated in [25].) Hence the left-hand square in diagram (7.3) is homotopy cartesian and so

$$\mathbf{L}_{\bullet}(s\mathcal{R}^u \cap s\mathcal{R}^v) \simeq \Omega \mathbf{L}_{\bullet}(s\mathcal{R}).$$

Finally the projection $X \times \mathbb{R}^i \times \mathbb{R} \to X \times \mathbb{R}^i$ induces a map from $\mathbf{L}_{\bullet}(s\mathcal{R}^u \cap s\mathcal{R}^v)$ to $\mathbf{L}_{\bullet}((X * S^{i-1}, X \times \mathbb{R}^i))$ which is a homotopy equivalence by another (but easier) application of the approximation theorem. This leads to a situation where we can map the *b*-variant (*b* for bounded) of the commutative square (7.2) to the commutative square (7.3) by a map which is a termwise homotopy equivalence. (Here we adopt a generous interpretation of $s\mathcal{R}^{\mathrm{lf}}(X \times \mathbb{R}^{i+1}; b)$ as the full Waldhausen subcategory of $s\mathcal{R}^{\mathrm{ld}}(X \times \mathbb{R}^{i+1}; b)$ consisting of all objects which can be related to locally finite and finite dimensional objects by a chain of weak equivalences.) Indeed, on lower right-hand terms our map is an identity map; on upper left-hand terms it is a homotopy equivalence by the observation about $\mathbf{L}_{\bullet}(s\mathcal{R}^u \cap s\mathcal{R}^v)$ just made; and the remaining terms, in both squares, are contractible. Therefore the square (7.2) is also homotopy cartesian.

COROLLARY 7.4. The maps

$$\begin{array}{rccc} \times \mathbb{R} \colon \mathbf{L}_{\bullet}(X \times \mathbb{R}^{i}; c) & \longrightarrow & \Omega \mathbf{L}_{\bullet}^{h}(X \times \mathbb{R}^{i+1}; c) , \\ \times \mathbb{R} \colon \mathbf{V} \mathbf{L}^{\bullet}(X \times \mathbb{R}^{i}; c) & \longrightarrow & \Omega \mathbf{V} \mathbf{L}^{\bullet h}(X \times \mathbb{R}^{i+1}; c) \end{array}$$

induced by the exact functor $\times \mathbb{R}$ are homotopy equivalences.

REMARK 7.5. Before proving corollary 7.4 we need to clarify its meaning. The notation $\times \mathbb{R}$ is self-explanatory to the extent that it describes an exact functor from $s\mathcal{R}^{\mathrm{Id}}(X \times \mathbb{R}^i; c)$ to $s\mathcal{R}^{\mathrm{If}}(X \times \mathbb{R}^{i+1}; c)$, with the usual generous interpretation of the *lf* superscript. In addition, we need to know that it takes *n*-dualities to (n+1)-dualities. More precisely, there is a binatural transformation

$$Y \odot_n Z \longrightarrow (Y \times \mathbb{R}) \odot_{n+1} (Z \times \mathbb{R}),$$

for Y and Z in $s\mathcal{R}^{\mathrm{ld}}(X \times \mathbb{R}^i; c)$, which commutes with the symmetry actions of $\mathbb{Z}/2$ and preserves nondegenerate pairings. In this chapter we use only some formal properties of $\times \mathbb{R}$, which is why we defer a more detailed description to chapters 9 and 10. From the point of view of chapter 9, the two maps in corollary 7.4 are given by external product with $\sigma(\mathbb{R}) \in \Omega^{\infty+1} \mathbf{VL}^{\bullet h}(\mathbb{R}; c)$, the controlled (visible symmetric) signature of \mathbb{R} as a manifold with control map id : $\mathbb{R} \to \mathbb{R}$.

We also need to clarify the meaning of the Ω prefix in corollary 7.4. Let **E** be a spectrum, e.g. in the sense of [1], with *n*-th space E_n . It is generally a good idea to make a distinction between the spectrum $\Omega \mathbf{E}$ whose *n*-th space is ΩE_n and the spectrum $\mathbf{E}[-1]$ whose *n*-th space is E_{n-1} . The canonical maps $E_{n-1} \to \Omega E_n$ do not fit together to make a spectrum map $\mathbf{E}[-1] \to \Omega \mathbf{E}$. The two spectra can nevertheless be related by a chain of weak homotopy equivalences. Also, it is easy to identify $\Omega^{\infty+m}\mathbf{E}$ with $\Omega^{\infty}\mathbf{E}[-m]$ because it is easy to identify $\mathbf{S}^0[m]$ with $S^m \wedge \mathbf{S}^0$. In any case, we do in this paper sometimes write $\Omega \mathbf{E}$, or $\Omega^m \mathbf{E}$, when we ought to write $\mathbf{E}[-1]$ or $\mathbf{E}[-m]$. It was difficult to avoid. In the statement of theorem 7.1, the prefix Ω is an honest Ω . By contrast in the statement of corollary 7.4 it should be read as a shift operator.

PROOF OF COROLLARY 7.4. We concentrate on the quadratic *L*-theory case. Despite remark 7.5, it is meaningful to say that the map $\times \mathbb{R}$ which we are discussing induces the same homomorphism on homotopy groups as the map

$$\mathbf{L}_{\bullet}(X \times \mathbb{R}^{i}; c) \longrightarrow \Omega \mathbf{L}_{\bullet}^{h}(X \times \mathbb{R}^{i+1}; c)$$

resulting from the homotopy cartesian square (7.2). This is enough and it is what we shall prove. Start with a representative (Y, φ) of an element α_0 of $\pi_n \mathbf{L}_{\bullet}(X \times \mathbb{R}^i; c)$, so that Y is a retractive space over $X \times \mathbb{R}^i$. In order to see the corresponding element α_1 of $\pi_{n+1} \mathbf{L}_{\bullet}(X \times \mathbb{R}^{i+1}; c)$, according to (7.2), we proceed as follows. We choose an (n+1)-dimensional nullbordism for (Y, φ) as a quadratic Poincaré object in

$$\mathcal{R}^{\mathrm{Id}}(X \times \mathbb{R}^{i} \times] - \infty, 0]; c)$$

and another for (Y, φ) as a quadratic Poincaré object in

$$\mathcal{R}^{\mathrm{Id}}(X \times \mathbb{R}^i \times [0, \infty[; c)$$

We glue the two together along the common boundary (Y, φ) to obtain an (n+1)dimensional quadratic Poincaré object in $\mathcal{R}^{\mathrm{ld}}(X \times \mathbb{R}^{i+1}; c)$. This represents α_1 . Now the most obvious choices for the two (n+1)-dimensional nullbordisms are $(Y, \varphi) \times] - \infty, 0]$ and $(Y, \varphi) \times [0, \infty)$, where we are using informal notation. Then the representative for α_1 which we get is $(Y, \varphi) \times \mathbb{R}$. LEMMA 7.6. The map $\times \mathbb{R} : \mathbf{A}(X \times \mathbb{R}^i; c) \longrightarrow \mathbf{A}(X \times \mathbb{R}^{i+1}; c)$ is nullhomotopic.

PROOF. For each retractive space Y over $X \times \mathbb{R}^{i}$ we have a cofibration sequence of retractive spaces over $X \times \mathbb{R}^{i+1}$, as follows:

$$Y_0 \longrightarrow Y \times \mathbb{R} \longrightarrow Y_\lambda \amalg Y_\rho$$
.

Here Y_0 is the union of $Y \times 0$ and $X \times \mathbb{R}^i \times \mathbb{R} = X \times \mathbb{R}^{i+1}$; it is clear that the cofiber of the inclusion $Y_0 \to Y \times \mathbb{R}$, taken in the category of retractive spaces over $X \times \mathbb{R}^{i+1}$, breaks up into a coproduct of two retractive spaces Y_λ and Y_ρ over $X \times \mathbb{R}^{i+1}$, in such a way that Y_λ is trivial over $X \times \mathbb{R}^i \times [0, \infty[$ and Y_ρ is trivial over $X \times \mathbb{R}^i \times [-\infty, 0]$. By the additivity theorem, the map $\times \mathbb{R}$ is therefore homotopic (by a preferred homotopy) to the sum of three maps induced by the exact functors taking Y to Y_0 , Y_λ and Y_ρ , respectively. The functors taking Y to Y_0 and Y_λ induce maps from $\mathbf{A}(X \times \mathbb{R}^i; c)$ to $\mathbf{A}(X \times \mathbb{R}^{i+1}; c)$ which clearly factor through the contractible spectrum

$$\mathbf{A}(X \times \mathbb{R}^i \times] - \infty, 0]; c).$$

The functor taking Y to Y_{ρ} induces a map from $\mathbf{A}(X \times \mathbb{R}^{i}; c)$ to $\mathbf{A}(X \times \mathbb{R}^{i+1}; c)$ which factors through the contractible spectrum $\mathbf{A}(X \times \mathbb{R}^{i} \times [0, \infty[; c). \square$

COROLLARY 7.7. There is a commutative diagram of spectra with action of $\mathbb{Z}/2$,



where $S_!^1$ denotes the based space S^1 with the conjugation action of $\mathbb{Z}/2$ (fixed point set S^0) and $S_!^1 \wedge \mathbf{A}(X \times \mathbb{R}^i, n; c)$ has the diagonal action of $\mathbb{Z}/2$.

INTERPRETATION AND PROOF. The vertical arrow is induced by the inclusion $S^0 \to S_!^1$, which is a $\mathbb{Z}/2$ -map with the trivial action of $\mathbb{Z}/2$ on S^0 . We identify $\mathbf{A}(X \times \mathbb{R}^i, n; c)$ with $S^0 \wedge \mathbf{A}(X \times \mathbb{R}^i, n; c)$. For the horizontal arrow, we interpret $\mathbf{A}^h(X \times \mathbb{R}^{i+1}, n+1; c)$ as the 0-connected cover of $\mathbf{A}(X \times \mathbb{R}^{i+1}, n+1; c)$. Therefore the map $\times \mathbb{R}$ of lemma 7.6 factors through $\mathbf{A}^h(X \times \mathbb{R}^{i+1}, n+1; c)$, with $\mathbb{Z}/2$ -action. (Indeed, it induces the zero homomorphism on π_0 as it is nullhomotopic.) The dashed arrow remains to be constructed.

A statement equivalent to the above is that the map $\times \mathbb{R}$, top horizontal arrow, admits a (non-equivariant) nullhomotopy H such that the concatenation of H and the conjugate tHt (where $t \in \mathbb{Z}/2$ is the generator) produces a map

$$\mathbf{A}(X \times \mathbb{R}^{i}, n; c) \longrightarrow \Omega \mathbf{A}^{h}(X \times \mathbb{R}^{i+1}, n+1; c)$$

which is a homotopy equivalence. To establish that, it is enough to produce a nullhomotopy H such that the concatenation of H and tHt produces a map from $\mathbf{A}(X \times \mathbb{R}^i, n; c)$ to $\Omega \mathbf{A}^h(X \times \mathbb{R}^{i+1}, n+1; c)$ which is homotopic to the map of theorem 7.1 (up to sign). We are going to show that the nullhomotopy H constructed in lemma 7.6 has this property.

We are comparing two (non-equivariant) maps from $\mathbf{A}(X \times \mathbb{R}^i, n; c)$ to

$$\Omega \mathbf{A}^h(X \times \mathbb{R}^{i+1}, n+1; c) \simeq \Omega \mathbf{A}(X \times \mathbb{R}^{i+1}, n+1; c) .$$

By inspection, both maps can be described by means of external products. The first (concatenation of H and tHt) is given by external product with an element a of $\pi_1 \mathbf{A}(\mathbb{R}, 1; c)$ and the second (from theorem 7.1) is given by product with an element b of $\pi_1 \mathbf{A}(\mathbb{R}, 1; c)$. To describe a, we use the map

$$\mathbf{A}(\star,0) \xrightarrow{\times \mathbb{R}} \mathbf{A}(\mathbb{R},1\,;\,c)$$

of spectra with action of $\mathbb{Z}/2$. From lemma 7.6, we have a preferred nullhomotopy H for it. The concatenation of H and tHt induces a map from $\mathbf{A}(\star)$ to $\Omega\mathbf{A}(\mathbb{R}; c)$, and so a homomorphism from $\pi_0\mathbf{A}(\star)$ to $\pi_1\mathbf{A}(\mathbb{R}; c)$. Let a be the image of $1 \in \pi_0\mathbf{A}(\star)$ under that map. Define b as the image of $1 \in \pi_0\mathbf{A}(\star)$ under the isomorphism $\pi_0\mathbf{A}(\star, 0) \cong \pi_1\mathbf{A}(\mathbb{R}, 1; c)$ implied by theorem 7.1. Now it remains only to show that $a = \pm b$. This is an exercise in duality which we leave to the reader.

COROLLARY 7.8. The following is homotopy cartesian,

where lemma 7.6 is used to extend $\times \mathbb{R}$: $\mathbf{A}(X \times \mathbb{R}^i; c) \to \mathbf{A}(X \times \mathbb{R}^{i+1}; c)$ to the cone on $\mathbf{A}(X \times \mathbb{R}^i; c)$.

PROOF. By corollary 7.7, this amounts to saying that

$$\begin{array}{c} \mathbf{E}^{h\mathbb{Z}/2} \xrightarrow{\mathrm{incl}} (S^{1}_{!} \wedge \mathbf{E})^{h\mathbb{Z}/2} \\ \downarrow_{\mathrm{incl} \circ \mathrm{forget}} & \downarrow_{\mathrm{forget}} \\ S^{1}_{\lambda} \wedge \mathbf{E} \xrightarrow{\mathrm{incl}} S^{1} \wedge \mathbf{E} \end{array}$$

is homotopy cartesian, where **E** is $\mathbf{A}(X \times \mathbb{R}^i, n; c)$ and S^1_{λ} is the lower half of S^1 (a closed interval with boundary equal to S^0).

For the next lemma, let \mathcal{R} be any Waldhausen category with an SW-product \odot_{\bullet} satisfying the axioms of [**39**, §2]. We define $\mathcal{R}^{(1)}$ as above, so $\mathcal{R}^{(1)}$ is \mathcal{R} with a new SW-product which is a shift of the old one. To describe K-theoretic and L-theoretic relationships between \mathcal{R} and $\mathcal{R}^{(1)}$, we introduce the Waldhausen category \mathcal{PR} of pairs in \mathcal{R} : an object is a cofibration $C_0 \to C_1$ in \mathcal{R} . Compare [**39**, §1]. There is a canonical SW product on \mathcal{PR} making the inclusion and forgetful functors

$$\mathcal{R}^{(1)} \longrightarrow \mathcal{PR} \longrightarrow \mathcal{R}$$

given by $C \mapsto (* \to C)$ and $(C_0 \to C_1) \mapsto C_0$ duality-preserving. The resulting diagram of spectra with $\mathbb{Z}/2$ -action

(7.9)
$$\mathbf{K}(\mathcal{R}^{(1)}) \longrightarrow \mathbf{K}(\mathcal{PR}) \longrightarrow \mathbf{K}(\mathcal{R})$$

is a homotopy fiber sequence of spectra by the additivity theorem. It admits a nonequivariant homotopy splitting $u: \mathbf{K}(\mathcal{R}) \to \mathbf{K}(\mathcal{PR})$ induced by the exact functor $C \mapsto (C, C)$; by the aforementioned notorious fact from [39] and [40], the map $u \lor tu$ from $\mathbf{K}(\mathcal{R}) \lor \mathbf{K}(\mathcal{R})$ to $\mathbf{K}(\mathcal{PR})$ is a homotopy equivalence (and a $\mathbb{Z}/2$ -map). It follows that in the homotopy fiber sequence

(7.10)
$$\mathbf{K}(\mathcal{R}^{(1)})^{th\mathbb{Z}/2} \longrightarrow \mathbf{K}(\mathcal{P}\mathcal{R})^{th\mathbb{Z}/2} \longrightarrow \mathbf{K}(\mathcal{R})^{th\mathbb{Z}/2}$$

determined by (7.9), the middle term is contractible. For much more elementary reasons,

(7.11)
$$\mathbf{L}_{\bullet}(\mathcal{R}^{(1)}) \longrightarrow \mathbf{L}_{\bullet}(\mathcal{PR}) \longrightarrow \mathbf{L}_{\bullet}(\mathcal{R})$$

is also a homotopy fiber sequence with contractible middle term. Putting those observations together, we have a commutative diagram

(7.12)
$$\begin{array}{c} \mathbf{L}_{\bullet}(\mathcal{R}^{(1)}) \longrightarrow \mathbf{L}_{\bullet}(\mathcal{P}\mathcal{R}) \longrightarrow \mathbf{L}_{\bullet}(\mathcal{R}) \\ \downarrow \Xi & \downarrow \Xi & \downarrow \Xi \\ \mathbf{K}(\mathcal{R}^{(1)})^{th\mathbb{Z}/2} \longrightarrow \mathbf{K}(\mathcal{P}\mathcal{R})^{th\mathbb{Z}/2} \longrightarrow \mathbf{K}(\mathcal{R})^{th\mathbb{Z}/2} \end{array}$$

where the rows are homotopy fiber sequences and the terms in the middle column are contractible. This implies a relationship between left-hand column and right-hand column which may be loosely described as the *shift invariance* of Ξ .

For a basic application of this, let H be a subgroup of the group $K_0(\mathcal{R})$ which is invariant under the involution determined by the SW-product. Let $\mathcal{R}^H \subset \mathcal{R}$ be the full Waldhausen subcategory consisting of all objects C whose class $[C] \in K_0(\mathcal{R})$ belongs to H. The SW-product on \mathcal{R} restricts to one on \mathcal{R}^H which also satisfies the axioms of [**39**, §2].

LEMMA 7.13. The following square is homotopy cartesian:

$$\begin{split} \mathbf{L}_{\bullet}(\mathcal{R}^{H}) & \stackrel{\Xi}{\longrightarrow} & \mathbf{A}(\mathcal{R}^{H})^{th\mathbb{Z}/2} \\ & \downarrow \text{ inclusion } & \downarrow \text{ inclusion } \\ \mathbf{L}_{\bullet}(\mathcal{R}) & \stackrel{\Xi}{\longrightarrow} & \mathbf{A}(\mathcal{R})^{th\mathbb{Z}/2}. \end{split}$$

PROOF. It is well known [23] that the relative *n*-th homotopy group of the inclusion map $\mathbf{L}_{\bullet}(\mathcal{R}^H) \to \mathbf{L}_{\bullet}(\mathcal{R})$ is isomorphic to the Tate cohomology group

$$H^{-n}(\mathbb{Z}/2;K_0/H)$$

where $K_0 = K_0(\mathcal{R})$. More precisely, let an element in this relative homotopy group be represented by a pair $C \to D$ in \mathcal{R} with a nondegenerate quadratic structure of formal dimension n, and with C in \mathcal{R}^H . Poincaré duality implies that $[C] \in K_0(\mathcal{R})/H$ is in the $(-1)^n$ eigensubgroup of the standard involution. Hence [C] determines an element in the above-mentioned Tate cohomology group. This describes the isomorphism. — But now it is also clear that the relative n-th homotopy group of the inclusion map $\mathbf{A}(\mathcal{R}^H)^{th\mathbb{Z}/2} \to \mathbf{A}(\mathcal{R})^{th\mathbb{Z}/2}$ is isomorphic to

$$\hat{H}^{-n}(\mathbb{Z}/2;K_0/H).$$

Indeed this relative homotopy group can be identified with π_n of the spectrum $(\mathbf{A}(\mathcal{R})/\mathbf{A}(\mathcal{R}^H))^{th\mathbb{Z}/2}$, and here $\mathbf{A}(\mathcal{R})/\mathbf{A}(\mathcal{R}^H)$ is an Eilenberg-MacLane spectrum with (at most) one nonzero homotopy group,

$$\pi_0(\mathbf{A}(\mathcal{R})/\mathbf{A}(\mathcal{R}^H)) = K_0/H$$

Using these identifications, the relative n-th homotopy group for the two columns in the above commutative square are already abstractly identified and we want to show that Ξ induces that identification of the π_n groups. This is clear in the case n = 0 by inspection. The case of arbitrary n follows from the case n = 0 by the shift invariance of Ξ , described in diagram (7.12).

PROPOSITION 7.14. The following commutative squares are homotopy cartesian:

$$\begin{split} \Omega^{n}\mathbf{L}_{\bullet}(X \times \mathbb{R}^{i}; c) & \stackrel{\Xi}{\longrightarrow} & \mathbf{A}(X \times \mathbb{R}^{i}, n; c)^{th\mathbb{Z}/2} \\ & \downarrow \times \mathbb{R} & \downarrow \times \mathbb{R} \\ \Omega^{n+1}\mathbf{L}_{\bullet}(X \times \mathbb{R}^{i+1}; c) & \stackrel{\Xi}{\longrightarrow} & \mathbf{A}(X \times \mathbb{R}^{i+1}, n+1; c)^{th\mathbb{Z}/2} , \\ \Omega^{n}\mathbf{VL}^{\bullet}(X \times \mathbb{R}^{i}; c) & \stackrel{\Xi}{\longrightarrow} & \mathbf{A}(X \times \mathbb{R}^{i}, n; c)^{th\mathbb{Z}/2} \\ & \downarrow \times \mathbb{R} & \downarrow \times \mathbb{R} \\ \Omega^{n+1}\mathbf{VL}^{\bullet}(X \times \mathbb{R}^{i+1}, n+1; c) & \stackrel{\Xi}{\longrightarrow} & \mathbf{A}(X \times \mathbb{R}^{i+1}, n+1; c)^{th\mathbb{Z}/2} . \end{split}$$

INTERPRETATION AND PROOF. Let $\mathcal{R} = \mathcal{R}^{\mathrm{ld}}(X \times \mathbb{R}^i; c)$ and let $\mathcal{R}^{(n)}$ be the same with a shifted SW-product. Where we have $\Omega^n \mathbf{L}_{\bullet}(X \times \mathbb{R}^i; c) = \Omega^n \mathbf{L}^{\bullet}(\mathcal{R})$ etc. in the statement above, we really mean

 $\mathbf{L}_{\bullet}(\mathcal{R}^{(n)})$

and justify that as in remark 7.5.

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We prove the first statement, the other being similar. The diagram can be enlarged to

In diagram (7.15), the lower square is homotopy cartesian by lemma 7.13, and in the upper square the left-hand column is a homotopy equivalence by corollary 7.4. It is therefore enough to show that the right-hand column in the upper square of (7.15) is also a homotopy equivalence. This follows easily from corollary 7.7. \Box

We turn to a slightly different but related theme: the homotopy invariance properties of constructions such as \mathbf{L}_{\bullet} , \mathbf{VL}^{\bullet} and \mathbf{A} when applied to control spaces such as $(X * S^{i-1}, X \times \mathbb{R}^i)$ where X is compact Hausdorff. It is easy to show homotopy invariance in the variable X, but we are also interested in the other variable, the sphere.

LEMMA 7.16. For a compact Hausdorff space X, and I = [0, 1], the projection $(X * S^{i-1}) \times I \to X * S^{i-1}$ induces homotopy equivalences

COROLLARY 7.17. The constructions \mathbf{L}_{\bullet} , \mathbf{VL}^{\bullet} and \mathbf{A} applied to control spaces of the form $(X * S^{i-1}, X \times \mathbb{R}^i)$, with compact Hausdorff X, are homotopy invariant as functors of the sphere variable.

PROOF OF COROLLARY MODULO LEMMA. What we mean here by homotopy invariant is that homotopic maps f, g from S^{i-1} to S^{j-1} induce homotopic maps

$$f_*, g_*: \mathbf{A}((X * S^{i-1}, X \times \mathbb{R}^i)) \longrightarrow \mathbf{A}((X * S^{j-1}, X \times \mathbb{R}^j))$$

etc. To prove it, we use theorem 7.1; this increases i and j by 1, and replaces **A** by \mathbf{A}^{h} . Now we can apply the lemma.

PROOF OF LEMMA 7.16. We concentrate on the **A**-theory case (and return to the explicit notation where control spaces are shown with their singular parts).

Now we have several choices of Waldhausen category with an algebraic K-theory spectrum that deserves to be called $\mathbf{A}^h((X * S^{i-1}, X \times \mathbb{R}^i))$. Among these we choose one which is more "algebraic" than the one which we normally prefer. This is described in [**35**, §6]. The objects of this Waldhausen category, call it \mathcal{R} for now, are certain retractive CW-spaces Y over X with a finite dimensional relative CW-structure and a map from the set of cells to \mathbb{R}^i . This labelling map must satisfy certain local finiteness and control conditions. (The control conditions are expressed in terms of the compactification D^i of \mathbb{R}^i .) The morphisms in \mathcal{R} are retractive cellular maps (over and under X) which, again, satisfy certain control conditions (expressed in terms of the cell labels). The relationship between this Waldhausen category \mathcal{R} and the usual one, $\mathcal{R}^{\mathrm{lf}}((X * S^{i-1}, X \times \mathbb{R}^i))$, is essentially given by cobase change along the projection $X \times \mathbb{R}^i \to X$, regarded as a functor

$$\mathcal{R}^{\mathrm{lf}}((X * S^{i-1}, X \times \mathbb{R}^i)) \to \mathcal{R}.$$

Note that the objects in $\mathcal{R}^{lf}((X * S^{i-1}, X \times \mathbb{R}^i))$ are retractive spaces over $X \times \mathbb{R}^i$ with a relative *CW*-structure. See [**35**, §9] for related ideas.

We introduce a similar Waldhausen category \mathcal{Q} with an algebraic K-theory spectrum that deserves to be called $\mathbf{A}^h(((X * S^{i-1}) \times I, X \times \mathbb{R}^i \times I)))$. Its objects are retractive spaces over X with a relative CW-structure and a map from the set of cells to $\mathbb{R}^i \times I$, subject to certain local finiteness and control conditions (which are formulated using the compactification $D^i \times I$ of $\mathbb{R}^i \times I$). It is convenient to stabilize the two categories by introducing formal desuspensions; then we have $s\mathcal{Q}$ and $s\mathcal{R}$. Now our task is to show that the functor

$$s\mathcal{Q} \to s\mathcal{R}$$

induced by the projection $\mathbb{R}^i \times I \to \mathbb{R}^i$ induces a homotopy equivalence of the algebraic K-theory spectra. Note that this functor does not satisfy the first hypothesis of the approximation theorem (which is about "detection" of weak equivalences). We use the fibration theorem instead. In addition to the standard notion of weak equivalence in $s\mathcal{Q}$, we therefore introduce a coarser notion. Let I' =]0, 1] and call a morphism in $s\mathcal{Q}$ a v-equivalence if the germ near $S^{i-1} \times I'$ of its mapping cone is weakly equivalent to zero in the usual controlled sense. Note here that $S^{i-1} \times I'$ is

part of the singular set $S^{i-1} \times I$ of the control space $((X * S^{i-1}) \times I, X \times \mathbb{R}^i \times I))$. By the fibration theorem there is a homotopy fibration sequence of K-theory spectra,

$$\mathbf{K}(s\mathcal{Q}^v) \to \mathbf{K}(s\mathcal{Q}) \to \mathbf{K}(s\mathcal{Q}_v) \,.$$

Hence it is enough to verify that

- (i) sQ_v is "flasque" enough so that $\mathbf{K}(sQ_v)$ is contractible ;
- (ii) the composition of exact functors $sQ^v \to sQ \to sR$ satisfies the hypotheses of the approximation theorem, so that $\mathbf{K}(sQ^v) \simeq \mathbf{K}(sR)$.

As regards (i), the situation is really quite similar to the one which we have considered in the "homotopy invariance" part of the proof of theorem 5.5. To make this connection clearer we note that sQ_v can be replaced by a simpler category $sQ_{(v)}$ with the same objects, where the morphisms themselves are germs of retractive maps over $X \times \mathbb{R}^i \times I$ defined near (i.e., over an arbitrarily small neighborhood of) $S^{i-1} \times I'$. Indeed, another application of the approximation theorem shows that the forgetful functor $sQ_v \to sQ_{(v)}$ induces a homotopy equivalence of the K-theory spectra. Up to an equivalence of categories, $sQ_{(v)}$ however depends only on

$$(W, W \smallsetminus (S^{i-1} \times I'))$$

where W can be any neighborhood of $S^{i-1} \times I'$ in $(X * S^{i-1}) \times I$. We can take W to be homeomorphic to the mapping cylinder of the projection

$$X \times S^{i-1} \times I' \to S^{i-1} \times I'.$$

In particular when X is a point, the control pair $(W, W \setminus (S^{i-1} \times I'))$ is what we have called $\mathbb{J}(S^{i-1} \times I')$. The same arguments for "flasqueness" as in the homotopy invariance part of the proof of theorem 5.5 apply to the category $s\mathcal{Q}_{(v)}$; this includes remark D.3 below.

As regards (ii), it is hard to verify directly that the hypotheses of the approximation theorem are satisfied by the functor $s\mathcal{Q}^v \to s\mathcal{R}$. It seems wiser therefore to introduce a full Waldhausen subcategory $s\mathcal{Q}^{(v)}$ of $s\mathcal{Q}^v$ consisting of the objects Y whose set of cell labels avoids a neighborhood (depending on Y) of $S^{i-1} \times I'$ in $(X * S^{i-1}) \times I$. Then it is clear that the composition

$$s\mathcal{Q}^{(v)} \to s\mathcal{Q}^v \to s\mathcal{Q} \to s\mathcal{R}$$

is an equivalence of (Waldhausen) categories. Hence it remains only to verify that the inclusion $s\mathcal{Q}^{(v)} \to s\mathcal{Q}^v$ satisfies the hypotheses of the approximation theorem. The first condition (about detection of weak equivalences) is clearly satisfied. For the second condition, suppose given a cofibration

$$f: Y_1 \to Y_2$$

in $s\mathcal{Q}^v$, with Y_1 in $s\mathcal{Q}^{(v)}$. From the definitions, it is easy to construct a factorization

$$Y_1 \rightarrow Y_3 \rightarrow Y_2$$

of f, where $Y_1 \to Y_3$ is still a cofibration, Y_3 is also in $s\mathcal{Q}^{(v)}$ and the morphism from Y_3 to Y_2 is a "domination" (i.e. there exists a controlled map $Y_2 \to Y_3$, not necessarily a morphism, which is right inverse to $Y_3 \to Y_2$ up to a controlled homotopy). Attaching additional cells to Y_3 where necessary, one can then easily improve $Y_3 \to Y_2$ to a weak equivalence in $s\mathcal{Q}^v$. (Solve the corresponding problem in $s\mathcal{R}$ first and use that solution as a model.)

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CHAPTER 8

Spherical fibrations and twisted duality

It is well known [26] that a spherical fibration ξ on a space X determines a twisted SW product in the category of finitely dominated retractive spaces on X. This is compatible with the so-called w-twisted involution on $\mathbb{Z}[\pi_1 X]$ where $w:\pi_1(X) \to \mathbb{Z}$ is $w_1(\xi)$. We recall some of the details, following [39, 1.A.9] rather more than [26].

We assume that $\xi: E \to X$ is a fibration with fibers homotopy equivalent to S^d and with a preferred section which is a fiberwise cofibration [13]. (What we have in mind is, for example, the fiberwise one-point compactification of a vector bundle on X, with the preferred section which picks out the point at infinity in each fiber.) Let Y_1 and Y_2 be finitely dominated retractive spaces over X, with retractions r_1 and r_2 . Again we start by defining an unstable Spanier–Whitehead product $Y_1 \downarrow Y_2$. This is the based space obtained by first forming the external smash product

$$Y_1 \land A_X Y_2 = Y_1 \times Y_2 / \sim$$

where ~ identifies (y_1, x) with $(r_1(y_1), x)$ and (x, y_2) with $(x, r_2(y_2))$; then forming the homotopy pullback of

$$E \xrightarrow{\text{diagonal} \circ \text{proj.}} X \times X \xleftarrow{\text{retraction}} Y_1 X \wedge_X Y_2 ;$$

then collapsing the subspace consisting of all elements in the homotopy pullback which are mapped to the base point under the forgetful projection to

$$(E/X) \wedge (Y_1/X) \wedge (Y_2/X)$$
.

To make this unstable SW product into an SW product on the stable category $s\mathcal{R}(X)$ of finitely dominated retractive spaces over X, we proceed much as in Chapter 2.

DEFINITION 8.1. We let

$$(Y_1,k) \odot (Y_2,\ell) = \operatorname{colim}_n \Omega^{2n+d} (\Sigma^{n-k} Y_1 \land \Sigma^{n-\ell} Y_2).$$

More generally we let

$$(Y_1,k) \odot_j (Y_2,\ell) = \operatorname{colim}_n \Omega^{2n+d} \Sigma^j (\Sigma^{n-k} Y_1 \land \Sigma^{n-\ell} Y_2),$$

so that $(Y_1, k) \odot (Y_2, \ell) = (Y_1, k) \odot_0 (Y_2, \ell)$, and denote the Ω -spectrum with *j*-th term $(Y_1, k) \odot_j (Y_2, \ell)$ by $(Y_1, k) \odot_{\bullet} (Y_2, \ell)$.

This comes with a structural symmetry $(Y_1, k) \odot_{\bullet} (Y_2, \ell) \cong (Y_2, \ell) \odot_{\bullet} (Y_1, k)$ determined by the obvious symmetry of λ . For $Y_1 = Y_2 = Y$ and $k = \ell$ we obtain an Ω -spectrum $(Y, k) \odot_{\bullet} (Y, k)$ with an action of $\mathbb{Z}/2$.

Note the d in definition 8.1 which is the formal fiber dimension of ξ . This causes a slight disagreement with the conventions of [**39**, 1.A.9], but it is convenient here.

Definition 8.1 is essentially insensitive to a stabilization of ξ by fiberwise suspension. More precisely, a spherical fibration ξ on X as above determines an SW product \odot_{\bullet} on $s\mathcal{R}(X)$, and the fiberwise suspension $\Sigma\xi$ determines another, which we (temporarily) denote by \odot'_{\bullet} . There is a natural homotopy equivalence

$$(Y_1,k) \odot_{\bullet} (Y_2,\ell) \longrightarrow (Y_1,k) \odot'_{\bullet} (Y_2,\ell)$$

which respects the canonical involutions. Note also that the ξ -twisted version of $(Y_1, k) \odot_{\bullet} (Y_2, \ell)$ is naturally homeomorphic to the standard one (definition 3.1) if ξ is a trivial sphere bundle $S^0 \times X \to X$.

Definition 3.2 can be re-used with the ξ -twisted interpretation of the *SW* product. The ξ -twisted versions of corollary 3.5 and definition 3.6 take a slightly more complicated form. For a finitely dominated retractive space Y over X, let Y^{ξ} be the fiberwise smash product of Y and $E = E(\xi)$ over X. As before, suppose that the formal fiber dimension of ξ is d.

COROLLARY 8.2. There is a natural homotopy fiber sequence of spectra

$$((Y,k) \odot_{\bullet} (Y,k))_{h\mathbb{Z}/2} \longrightarrow ((Y,k) \odot_{\bullet} (Y,k))^{\mathbb{Z}/2} \longrightarrow \Sigma^{\infty-k-d}(Y^{\xi}/X).$$

DEFINITION 8.3. An *n*-dimensional visible hyperquadratic structure on (Y, k)is an element in $\Omega^n \Omega^{\infty} \Sigma^{\infty-k-d} (Y^{\xi}/X)$. An *n*-dimensional quadratic structure on (Y, k) is an element of $\Omega^n \Omega^{\infty} (((Y, k) \odot_{\bullet} (Y, k))_{h\mathbb{Z}/2})$. Alternatively, an *n*dimensional quadratic structure on (Y, k) can be defined as an element of $\Omega^n \Omega^{\infty}$ of the homotopy fiber of the natural map $J: ((Y, k) \odot_{\bullet} (Y, k))^{\mathbb{Z}/2} \to \Sigma^{\infty-k-d} (Y^{\xi}/X)$.

We write $\mathbf{L}_{\bullet}(X,\xi)$, $\mathbf{VL}^{\bullet}(X,\xi)$ and $\mathbf{L}^{\bullet}(X,\xi)$ for the *L*-theory spectra determined by the ξ -twisted *SW* product on the stable category of finitely dominated retractive spaces over *X*. Theorem 3.7 remains correct (with essentially the same proof) in this more general setting and can informally be regarded as a statement for spaces *X* over *BG*, the classifying space for stable spherical fibrations. A *weak homotopy equivalence*, in that setting, is a map of spaces over *BG* which is a weak homotopy equivalence of spaces. A *cocartesian square*, in that setting, is a commutative square of spaces over *BG* which is cocartesian as a square of spaces.

We come to an outline of a calculation of $\mathbf{VL}^{\bullet}(X,\xi)$ (which will not be used elsewhere in the paper). Let $\mathbf{Th}(X,\xi)$ be the Thom spectrum of X and ξ . For convenience or otherwise, we index that in such a way that the (d+k)-th space is $S^k \wedge (E/X)$. There is a natural map

$$\hat{\sigma}: \mathbf{Th}(X,\xi) \longrightarrow \mathbf{V}\widehat{\mathbf{L}}^{\bullet}(X,\xi)$$

the visible hyperquadratic signature map.

In more detail, $\pi_n \operatorname{Th}(X, \xi)$ can be identified with the group of bordism classes of normal spaces of formal dimension n over (X, ξ) . A normal space of formal dimension n over (X, ξ) consists of a finitely dominated space Y, a map $f: Y \to X$ and a stable (pointed) map $\eta: S^{n+d} \to E(f^*\xi)/Y$. (The image under the Thom isomorphism of the homology class carried by $[\eta]$ is a class in $H_n(Y; \mathbb{Z}^w)$, where \mathbb{Z}^w is the twisted integer coefficient system determined by $w_1(f^*\xi)$. It is called the fundamental class of the normal space, but it is not subject to any Poincaré duality condition. If it does satisfy the Poincaré duality condition as formulated by Wall [30], then that makes Y into a Poincaré duality space with Spivak normal bundle $f^*\xi$.) Therefore normal spaces generalize Poincaré duality spaces, and in fact the visible hyperquadratic signature map is a variant of a better known and easier– to–understand map σ from the Poincaré duality bordism spectrum $\mathbf{Bm}_{PD}(X,\xi)$ of (X,ξ) to the visible symmetric *L*-theory spectrum $\mathbf{VL}^{\bullet}(X,\xi)$. The two maps fit into a commutative square

In both the normal bordism theory and the visible hyperquadratic theory, there are external products. They have the form

$$\mathbf{Th}(X,\xi) \wedge \mathbf{Th}(X',\xi') \longrightarrow \mathbf{Th}(X \times X',\xi \times \xi') ,$$
$$\mathbf{V}\widehat{\mathbf{L}}^{\bullet}(X,\xi) \wedge \mathbf{V}\widehat{\mathbf{L}}^{\bullet}(X',\xi') \longrightarrow \mathbf{V}\widehat{\mathbf{L}}^{\bullet}(X \times X',\xi \times \xi')$$

where $\xi \times \xi'$ is the external fiberwise smash product of ξ and ξ' . The composition

$$\mathbf{Th}(X,\xi) \wedge \mathbf{V}\widehat{\mathbf{L}}^{\bullet}(\star) \xrightarrow{\sigma \wedge \mathrm{id}} \mathbf{V}\widehat{\mathbf{L}}^{\bullet}(X,\xi) \wedge \mathbf{V}\widehat{\mathbf{L}}^{\bullet}(\star) \xrightarrow{\operatorname{ext. prod.}} \mathbf{V}\widehat{\mathbf{L}}^{\bullet}(X,\xi)$$

is a natural transformation between excisive functors on the category of spaces over BG. It is an equivalence when X is a point (mapping to the base point of BG). Hence it is always an equivalence and we have

THEOREM 8.4. $\mathbf{V}\widehat{\mathbf{L}}^{\bullet}(X,\xi) \simeq \mathbf{Th}(X,\xi) \wedge (\mathbf{S} \vee \mathbb{R}\mathbf{P}_{-1}^{\infty}).$

Moreover, the fact that $\hat{\sigma}$ commutes with external products immediately leads to a "calculation" of $\hat{\sigma}$:

PROPOSITION 8.5. The hyperquadratic signature map

$$\hat{\sigma}$$
: $\mathbf{Th}(X,\xi) \longrightarrow \mathbf{VL}^{\bullet}(X,\xi) \simeq \mathbf{Th}(X,\xi) \land (\mathbf{S} \lor \mathbb{R}\mathbf{P}_{-1}^{\infty})$

is homotopic to the inclusion of the wedge summand $\mathbf{Th}(X,\xi) \wedge \mathbf{S} \simeq \mathbf{Th}(X,\xi)$.

Next we need a ξ -twisted version of chapter 5. We begin with a control space (\bar{Q}, Q) as in 5.3, and a spherical fibration $\xi: E \to Q$ of formal fiber dimension d, with a distinguished section.

Let Y and Z be objects of $\mathcal{R}^{\mathrm{ld}}(\bar{Q}, Q)$. To define their SW product $Y \odot Z$, we introduce first an unstable form $Y \downarrow Z$ of it. We define it as the geometric realization of a based simplicial set. An *n*-simplex of this simplicial set is a pair (f, γ) where

- (i) f is a continuous map from the standard *n*-simplex Δ^n to the topological inverse limit of the spaces $(Y/Q) \wedge (E^P/Q) \wedge (Z^P/Q)$, where P runs through the large closed subsets of Q;
- (ii) γ is a continuous assignment $c \mapsto \gamma_c$ of paths in Q, defined for $c \in \Delta^n$ with f(c) not equal to the base point \star .

The paths γ_c are to be parametrized by [-1, +1] and must satisfy

$$\gamma_c(-1) = r_Y f_Y(c) , \qquad \gamma_c(+1) = r_Z f_Z(c) , \qquad \gamma_c(0) = \xi f_E(c)$$

There is a control condition:

For $q \in \overline{Q} \setminus Q$ and any neighborhood V of q in \overline{Q} , there exists a smaller neighborhood W of q in \overline{Q} such that, for any $c \in \Delta^n$ with $f(c) \neq \star$, the path γ_c either avoids W or runs entirely in V. DEFINITION 8.6. For Y and Z in $\mathcal{R}^{\mathrm{ld}}(\bar{Q}, Q)$ and integers $k, \ell \in \mathbb{Z}$, let

$$(Y,k) \odot (Z,\ell) = \operatorname{colim}_n \Omega^{2n+d} (\Sigma^{n-k}Y \land \Sigma^{n-\ell}Z).$$

More generally let $(Y, k) \odot_{\bullet} (Z, \ell)$ be the spectrum with *j*-th space

 $(Y,k) \odot_i (Z,\ell) = \operatorname{colim}_n \Omega^{2n+d} \Sigma^j (\Sigma^{n-k} Y \downarrow \Sigma^{n-\ell} Z).$

This is the appropriate definition for the category $s\mathcal{R}^{\mathrm{ld}}(\bar{Q},Q)$. There is also a germ version, for the category $s\mathcal{RG}^{\mathrm{ld}}(\bar{Q},Q)$, which generalizes definition 5.4. With that generalized definition, theorem 5.5 generalizes as follows:

THEOREM 8.7. The spectrum valued functor $X \mapsto \mathbf{E}(X)$ is homotopy invariant and excisive. Here X can be a space over BG or a space with a spherical fibration ξ , and $\mathbf{E}(X)$ means $\mathbf{L}^{\bullet}(\mathbb{J}X_{\infty})$, defined using the ξ -twisted SW product.

We leave the detailed formulation to the reader. The proof is essentially identical with the proof of theorem 5.5.

Furthermore definition 5.10 can be re-used in the twisted setting and leads to a generalization of theorem 5.14:

THEOREM 8.8. The spectrum valued functor $X \mapsto \mathbf{E}(X)$ is homotopy invariant and excisive. Here X can be a space over BG or a space with a spherical fibration ξ , and $\mathbf{E}(X)$ means $\mathbf{VL}^{\bullet}(\mathbb{J}X_{\infty})$, defined using the ξ -twisted SW product.

The results of chapters 6 and 7 also have generalizations to the twisted setting. We do not formulate them here explicitly. They will however be used in chapters 12 and 13.

CHAPTER 9

Homotopy invariant characteristics and signatures

This chapter is formally analogous to parts of [10, §6]. We begin with a space X, a spherical fibration ξ on X (with a distinguished section which is a fiberwise cofibration), and an integer $n \ge 0$. From these data we produce a spectrum

$$VLA^{\bullet}(X, \xi, n).$$

In the case where X is a finitely dominated Poincaré duality space of formal dimension n and ξ is the Spivak normal fibration of X, we also construct a characteristic element

$$\sigma(X) \in F(X,\xi,n) := \Omega^{\infty+n} \mathbf{VLA}^{\bullet}(X,\xi,n)$$

This refines the Mishchenko–Ranicki (visible) symmetric signature of X which, in the nonlinear setting, is an element of $\Omega^{\infty+n}\mathbf{VL}^{\bullet}(X,\xi)$ or of $\Omega^{\infty+n}\mathbf{L}^{\bullet}(X,\xi)$. The construction has certain naturality properties. As in [10, §1], these properties imply that every *family* of finitely dominated formally *n*-dimensional Poincaré duality spaces X_b , depending on a parameter $b \in B$, determines a characteristic section $\sigma(p)$ of a fibration on B whose fibers are, essentially, the spaces $F(X_b, \xi_b, n)$.

Let \mathcal{R} be any Waldhausen category with stable SW-product satisfying the axioms of [**39**, §2]. We also need the shifted cousins of this SW-product and write $\mathcal{R}^{(n)}$ for \mathcal{R} with the shifted SW-product, as in chapter 7. The SW-product determines a duality involution on $\mathbf{K}(\mathcal{R}^{(n)})$ as explained in [**39**]. (See remark D.1 below for a correction.) Using that, we have the map

(9.1)
$$\Xi: \mathbf{L}^{\bullet}(\mathcal{R}^{(n)}) \longrightarrow \mathbf{K}(sR^{(n)})^{th\mathbb{Z}/2}$$

of [39]. We also write this in the form

(9.2)
$$\Xi: \Omega^n \mathbf{L}^{\bullet}(\mathcal{R}) \longrightarrow \mathbf{K}(\mathcal{R}^{(n)})^{th\mathbb{Z}/2}$$

noting that $\mathbf{L}^{\bullet}(\mathcal{R}^{(n)}) \simeq \Omega^{n} \mathbf{L}^{\bullet}(\mathcal{R})$, in the spirit of remark 7.5.

Now suppose that \mathcal{R} is $s\mathcal{R}(X)$ with the ξ -twisted stable SW-product introduced in chapter 8. Then we have good reasons to write

(9.3)
$$\Xi: \Omega^n \mathbf{L}^{\bullet}(X,\xi) \longrightarrow \mathbf{A}(X,\xi,n)^{th\mathbb{Z}/2}$$

for the map (9.2). We can also restrict to $\Omega^n \mathbf{VL}^{\bullet}(X,\xi)$ to get

(9.4)
$$\Xi: \Omega^n \mathbf{VL}^{\bullet}(X,\xi) \longrightarrow \mathbf{A}(X,\xi,n)^{th\mathbb{Z}/2}.$$

This brings us to the definition of VLA[•] anticipated in chapter 1:

DEFINITION 9.5. Let $\Omega^n \mathbf{VLA}^{\bullet}(X,\xi,n)$ be the homotopy pullback of

$$\Omega^{n}\mathbf{VL}^{\bullet}(X,\xi) \xrightarrow{\Xi} \mathbf{A}(X,\xi,n)^{th\mathbb{Z}/2} \xleftarrow{\text{incl.}} \mathbf{A}(X,\xi,n)^{h\mathbb{Z}/2} .$$

Licensed to University of Edinburgh. Prepared on Sun Nov 15 16:25:09 EST 2015for download from IP 129.215.149.96. License or copyright restrictions may apply to redistribution; see http://www.ams.org/publications/ebooks/terms Now let X be a finitely dominated Poincaré duality space of formal dimension n, with Spivak normal bundle $\xi: E \to X$ and a choice of a stable map $\eta: S^{n+d} \to E/X$ such that $[\eta]$ maps to a fundamental class for X under the Thom isomorphism. (We assume as usual that ξ has fibers $\simeq S^d$ and comes with a distinguished section. Together, ξ and η are determined by X up to contractible choice and ξ can therefore be regarded as the Spivak normal fibration of X.)

Let $s\mathcal{R} = s\mathcal{R}(X)$ with the ξ -twisted SW product. Let

$$F(X,\xi,n) = \Omega^{\infty+n} \mathbf{VLA}^{\bullet}(X,\xi,n)$$

which can also be described as the homotopy pullback of the diagram of infinite loop spaces

(9.6)
$$\Omega^{\infty} \mathbf{VL}^{\bullet}(s\mathcal{R}^{(n)}) \xrightarrow{\Xi} \Omega^{\infty}(\mathbf{K}(s\mathcal{R}^{(n)})^{th\mathbb{Z}/2}) \xleftarrow{\text{incl.}} \Omega^{\infty}(\mathbf{K}(s\mathcal{R}^{(n)})^{h\mathbb{Z}/2}).$$

To construct $\sigma(X) \in F(X,\xi,n)$, we begin as usual with $S^0 \times X$, viewed as a retractive space over X alias object of $s\mathcal{R}$. The composite stable map

$$S^{n+d} \longrightarrow E/X \longrightarrow X_+ \wedge (E/X)$$

defines an element φ in $(S^0 \times X) \odot_0 (S^0 \times X)$, for the *SW* product in $s\mathcal{R}^{(n)}$. This is fixed under the symmetry involution and nondegenerate, and so $S^0 \times X$ and φ together determine a vertex, say v_L , in the standard simplicial set model for $\Omega^{\infty} \mathbf{VL}^{\bullet}(s\mathcal{R}^{(n)})$. They also determine a (homotopy) fixed point for the duality involution on $\Omega^{\infty} \mathbf{K}(s\mathcal{R}^{(n)})$, which we may regard as a vertex v_K in $\Omega^{\infty}(\mathbf{K}(s\mathcal{R}^{(n)})^{h\mathbb{Z}/2})$. The images of v_L and v_K respectively under the two arrows in diagram (9.6) agree by the definition of Ξ , for which we refer again to [**39**, §9].

DEFINITION 9.7. Let $\sigma(X) \in F(X,\xi,n)$ be the element determined by v_L , v_K and the constant path connecting their images under the two arrows in diagram (9.6).

It remains to be said how the assignment σ is a *characteristic* [10, 1.1] on a suitable category of formally *n*-dimensional Poincaré duality spaces and homotopy equivalences. This forces on us another revision of $F(X, \xi, n)$, in fact an enlargement, which will as usual leave the homotopy type unchanged.

We have some freedom in interpreting the term $\Omega^{\infty} \mathbf{VL}^{\bullet}(s\mathcal{R}^{(n)})$ from the definition of $F(X,\xi,n)$. For a start we can regard it as the 0-th infinite loop space in the Ω -spectrum $\mathbf{VL}^{\bullet}(s\mathcal{R}^{(n)})$. This was defined, following [39, §9], as the geometric realization of a Δ -set

$$[m] \mapsto \operatorname{vsp}_0(s\mathcal{R}^{(n)}(m)).$$

Here $s\mathcal{R}^{(n)}(m)$ is a category of certain functors from the poset of faces of Δ^m to $s\mathcal{R}^{(n)}$. The notation $vsp_0(s\mathcal{R}^{(n)}(m))$ means: the set of visible symmetric Poincaré objects of formal dimension zero in $s\mathcal{R}^{(n)}(m)$, that is, objects of $s\mathcal{R}^{(n)}(m)$ with an appropriate nondegenerate visible symmetric structure. Let

$$\operatorname{vsp}_0'(s\mathcal{R}^{(n)}(m))$$

be the classifying space of the *category* of visible symmetric Poincaré objects of formal dimension zero in $s\mathcal{R}^{(n)}(m)$. A morphism between visible symmetric Poincaré objects of formal dimension zero in $s\mathcal{R}^{(n)}(m)$ is a morphism in $s\mathcal{R}^{(n)}(m)$ respecting the visible symmetric structures; such a morphism is automatically a weak equivalence in $s\mathcal{R}^{(n)}(m)$. Then

$$[m] \mapsto \operatorname{vsp}_0'(s\mathcal{R}^{(n)}(m))$$

is a Δ -space. Its geometric realization is an enlarged version of the above construction of the 0-th infinite loop space in the Ω -spectrum $\mathbf{VL}^{\bullet}(s\mathcal{R}^{(n)})$. It is homotopy equivalent to the original, by a standard argument which exploits the fact that several ways of realizing a bisimplicial set give the same result. The map Ξ extends easily to this enlarged version. Consequently we end up with an enlarged version of $F(X, \xi, n)$. Using this, it is easy to promote $\sigma(X) \in F(X, \xi, n)$ of definition 9.7 to a characteristic.

Namely, let X_i for i = 0, 1, ..., k be Poincaré duality spaces of formal dimension n, with Spivak normal fibrations $\xi_i : E_i \to X$ with fibers $\simeq S_i^d$ where

$$d_k \le d_{k-1} \le \dots \le d_0$$

and preferred zero sections which are fiberwise cofibrations. Let $\eta_i: S^{n+d_i} \to E_i/X_i$ be stable maps representing fundamental classes for the X_i . Let homotopy equivalences $u_i: X_i \to X_{i-1}$ be given for $i = 1, \ldots, k$, covered by maps

$$v_i \colon \Sigma_{X_i}^{d_{i-1}-d_i} E_i \longrightarrow E_{i-1}$$

respecting the zero sections, and such that $v_i\eta_i = \eta_{i-1}$. We can describe these data by a diagram

 $(X_0,\xi_0,\eta_0) \leftarrow (X_1,\xi_1,\eta_1) \leftarrow \cdots \leftarrow (X_{k-1},\xi_{k-1},\eta_{k-1}) \leftarrow (X_k,\xi_k,\eta_k) .$

With the new definition of F, the diagram determines a map from the standard k-simplex Δ^k to $F(X_0, \xi_0, n)$, which we could call the *characteristic of the diagram*. This assignment extends definition 9.7 and it has the naturality properties which make it into a characteristic (on a certain category \mathcal{P}_n) as defined in [10, 1.1]. The objects of \mathcal{P}_n are triples (X, ξ, η) as above. It is very fortunate that we can allow continuous variation of η ; that is to say, the characteristic σ depends continuously on the "reductions" η .

We spell out what [10, 1.6] means here. Let $p: Y \to B$ be a fibration whose fibers Y_b are finitely dominated Poincaré duality spaces of formal dimension n. We assume for simplicity that B is a simplicial complex. We also need a fiberwise Spivak normal fibration. Suppose that this comes in the shape of a fibration $\xi: E \to Y$ with preferred section, with fibers $\simeq S^d$, and a map

$$\eta: B_+ \wedge S^{n+d} \longrightarrow E / \sim$$

over B, where E/\sim is the pushout of $E \leftarrow Y \rightarrow B$. Every simplex $K \subset B$ determines a Poincaré duality space $Y_K = p^{-1}(K)$ with Spivak normal fibration $\xi_K : E_K \rightarrow Y_K$, where E_K is the portion of E above K, and a family of reductions $\eta_b : S^{n+d} \rightarrow E_K/Y_K$ where $b \in K$. These data determine a map

$$\sigma(Y_K): K \to F(Y_K, \xi_K, n)$$

using the continuity property of σ . As K varies, we have maps

 $\operatorname{hocolim}_{K} K \longrightarrow \operatorname{hocolim}_{K} F(Y_{K}, \xi_{K}, n) \longrightarrow \operatorname{hocolim}_{K} \star$.

The space on the right is the barycentric subdivision of B; the map on the right is a quasifibration which we can also describe as $F_B(Y,\xi,n) \to B$; the composite map from left to right is a homotopy equivalence. Hence the map on the left can be viewed as a "homotopy section" $\sigma(p)$ of $F_B(Y,\xi,n) \to B$, and this is what we wanted.

Next we need a generalization or adaptation of definitions 9.5 and 9.7 to the controlled setting. Let (\bar{X}, X) be a control space with compact \bar{X} , let n be an integer ≥ 0 and let ξ be a spherical fibration on X, with fibers $\simeq S^d$ and with a distinguished section which is a fiberwise cofibration. We have the ξ -twisted stable SW-product on $s\mathcal{R}^{\mathrm{ld}}(\bar{X}, X)$ from chapter 8.

DEFINITION 9.8. Let $\Omega^n \mathbf{VLA}^{\bullet}((\bar{X}, X), \xi, n)$ be the homotopy pullback of

$$\Omega^{n}\mathbf{VL}^{\bullet}((\bar{X},X),\xi) \xrightarrow{\Xi} \mathbf{A}((\bar{X},X),\xi,n)^{th\mathbb{Z}/2} \xleftarrow{\text{incl.}} \mathbf{A}((\bar{X},X),\xi,n)^{h\mathbb{Z}/2}$$

Let $F((\bar{X}, X), \xi, n) = \Omega^{\infty+n} \mathbf{VLA}^{\bullet}((\bar{X}, X), \xi, n)$, to be thought of as Ω^{∞} of the spectrum $\Omega^n \mathbf{VLA}^{\bullet}((\bar{X}, X), \xi, n)$ just defined. Now, in order to get a signature element $\sigma(\bar{X}, X) \in F((\bar{X}, X), \xi, n)$, we have to throw in a finite domination assumption and a Poincaré duality assumption. Suppose therefore that $S^0 \times X$, as a retractive space over X, is finitely dominated in the controlled sense. Suppose that in addition to the data (\bar{X}, X) and ξ , we are given a stable map

$$\eta: S^{n+d} \longrightarrow E /\!\!/ X.$$

There is a diagonal inclusion of $E/\!\!/ X$ in $(S^0 \times X) \odot_0 (S^0 \times X)$ for the SW product in $s\mathcal{R}^{(n)}$ which we are considering. The composition of η and the diagonal is then a 0-dimensional visible symmetric structure on $S^0 \times X$ as an object of $s\mathcal{R}^{(n)}$.

DEFINITION 9.9. If this is nondegenerate, we call (\bar{X}, X) together with ξ and η a controlled Poincaré duality space. In that case let $\sigma(\bar{X}, X) \in F((\bar{X}, X), \xi, n)$ be the element determined (as in definition 9.7) by $S^0 \times X$ with the above nondegenerate visible symmetric structure.

The naturality properties of $\sigma(\bar{X}, X)$ are similar to those of $\sigma(X)$ in definition 9.7. The morphisms we are most interested in have the form

$$((X_0, X_0), \xi_0, \eta_0) \leftarrow ((X_1, X_1), \xi_1, \eta_1)$$

with an underlying map $f: \overline{X}_1 \to \overline{X}_0$ which takes X_1 to X_0 and restricts to a homeomorphism from $\overline{X}_1 \smallsetminus X_1$ to $\overline{X}_0 \smallsetminus X_0$. Existence of an inverse up to homotopy $\overline{X}_0 \to \overline{X}_1$ is required; the inverse and the maps in the homotopy are subject to the same conditions as f.

REMARK 9.10. Definition 9.9 is good enough for our purposes, but it is not exactly a generalization of definition 9.7 because of the compactness condition on \bar{X} . In the case where $\bar{X} = X$, this amounts to compactness of X, which we did not assume in 9.7.

CHAPTER 10

Excisive characteristics and signatures

The goal here is very easy to formulate:

(i) For a closed *n*-manifold X with normal bundle ξ , we need to specify a preferred lift

$$\sigma^{\%}(X) \in F^{\%}(X,\xi,n)$$

of $\sigma(X) \in F(X, \xi, n)$, across a suitable assembly map.

(ii) For any compact control space (X, X) in which X is an *n*-manifold with normal bundle ξ (without boundary but not necessarily compact), we need to specify a preferred lift

$$\sigma^{\%}(\bar{X}, X) \in F^{\%}(X, \xi, n)$$

of $\sigma(\bar{X}, X) \in F((\bar{X}, X), \xi, n)$, across a suitable assembly map.

To clarify, $\sigma(X) \in F(X, \xi, n)$ and $\sigma(\overline{X}, X) \in F((\overline{X}, X), \xi, n)$ are the characteristic elements of definition 9.7 and definition 9.9. The space $F^{\%}(X, \xi, n)$ is designed in such a way that its homotopy groups are the homology groups of X with *locally finite* coefficients $\Omega^n \mathbf{VLA}^{\bullet}(x, \xi_x, n)$, depending on $x \in X$.

The details are slightly more complicated, but we follow [10, §7] closely. In the case (i) we define $F^{\%}(X,\xi,n)$ and the assembly map to $F(X,\xi,n)$ by means of a homotopy fiber sequence

$$F^{\%}(X,\xi,n) \longrightarrow F(X,\xi,n) \longrightarrow F(\mathbb{J}X,\xi,n)$$

where $\mathbb{J}X$ is the control space $(X \times [0,1], X \times [0,1[))$. The second map in the sequence is induced by the inclusion of $X \cong X \times \{0\}$ in $X \times [0,1[$; the spherical fibration ξ is extended to $X \times [0,1[$ in the obvious way. We have the definition of $F(\mathbb{J}X,\xi,n)$ from the end of the previous chapter. In the case (ii) we introduce

$$\mathbb{J}(\bar{X}, X) = \left(\frac{X \times [0, 1]}{\sim}, X \times [0, 1[\right)\right)$$

where ~ identifies points in $(\bar{X} \setminus X) \times [0,1]$ with the same coordinate in $\bar{X} \setminus X$. Then again $F^{\%}((\bar{X},X),\xi,n)$ and the assembly map to $F((\bar{X},X),\xi,n)$ are defined by means of a homotopy fiber sequence

$$F^{\%}((\bar{X},X),\xi,n) \longrightarrow F((\bar{X},X),\xi,n) \longrightarrow F(\mathbb{J}(\bar{X},X),\xi,n).$$

The second map in the sequence is induced by the inclusion of $(\bar{X} \times \{0\}, X \times \{0\})$ in $\mathbb{J}(\bar{X}, X)$.

In case (i), this leaves us with the task of trivializing the image of $\sigma(X)$ under $F(X,\xi,n) \to F(\mathbb{J}X,\xi,n)$. We also need to show that $F^{\%}(X,\xi,n)$ as defined above is excisive etc., so that the forgetful map from $F^{\%}(X,\xi,n)$ to $F(X,\xi,n)$ can be called an assembly map. In the more general case (ii), our task is to trivialize

the image of $\sigma(\bar{X}, X)$ under $F((\bar{X}, X), \xi, n) \to F(\mathbb{J}(\bar{X}, X), \xi, n)$, and to establish excision properties etc. for $F^{\%}((\bar{X}, X), \xi, n)$.

Trivializing $\sigma(X)$ in $F(\mathbb{J}X, \xi, n)$. We use $F(\mathbb{J}\star, \zeta, 0)$ where \star means "a point" and ζ is the trivial spherical fibration (with fiber S^0). We have $\sigma(\star) \in F(\mathbb{J}\star, \zeta, 0)$. Multiplication with $\sigma(X)$ can be regarded as a based map

$$F(\mathbb{J}\star,\zeta,0)\longrightarrow F(\mathbb{J}X,\xi,n)$$

which takes $\sigma(\star)$ to $\sigma(X)$. Hence it is enough to show that $F(\mathbb{J}\star, \zeta, 0)$ is contractible. This can be proved by a standard Eilenberg swindle argument. Think of $\mathbb{J}\star$ as $([0,\infty], [0,\infty[$). Translation by +1 is an endofunctor f of $s\mathcal{R}(\mathbb{J}\star)$. Let

$$g = \bigvee_{i=0}^{\infty} f^i$$

which is again an endofunctor of $s\mathcal{R}(\mathbb{J}_{\star})$. For the induced self-maps f_* and g_* of $F(\mathbb{J}_{\star}, \zeta, 0)$, we clearly have $f_* \simeq \mathrm{id}$, hence $g_*f_* \simeq g_*$. But since $g \cong \mathrm{id} \lor gf$, it is also true that $g_* \sim \mathrm{id} + g_*f_*$. Hence the identity of $F(\mathbb{J}_{\star}, \zeta, 0)$ is nullhomotopic. \Box

REMARK. This line of reasoning simplifies [10, 7.8]. Observe that it is precisely the manifold property of X which ensures that there is a map *multiplication* with $\sigma(X)$ from $F(\mathbb{J}\star, \zeta, 0)$ to $F(\mathbb{J}X, \xi, n)$. The construction does therefore not generalize to Poincaré duality spaces X.

REMARK. What the argument really gives us is a lift of $\sigma(X) \in F(X, \xi, n)$ to an element $\sigma^{\%}(X)$ in the homotopy pullback of the following diagram of pointed spaces:

$$F(X,\xi,n) \xrightarrow{\text{incl.}} F(\mathbb{J}X,\xi,n) \xleftarrow{\times \sigma(X)} F(\mathbb{J}\star,\zeta,0).$$

Since the right-hand term is contractible, this homotopy pullback is an acceptable substitute for the homotopy fiber of $F(X,\xi,n) \to F(\mathbb{J}X,\xi,n)$.

Trivializing $\sigma(\bar{X}, X)$ in $F(\mathbb{J}(\bar{X}, X), \xi, n)$. Again we use $F(\mathbb{J}\star, \zeta, 0)$ and the element $\sigma(\star) \in F(\mathbb{J}\star, \zeta, 0)$. Multiplication with $\sigma(\bar{X}, X)$ is a based map

$$F(\mathbb{J}\star,\zeta,0)\longrightarrow F(\mathbb{J}(X,X),\xi,n)$$

which takes $\sigma(\star)$ to $\sigma(X)$. Since $F(\mathbb{J}\star, \zeta, 0)$ is contractible, this achieves the trivialization.

In order to establish excision properties for $F^{\%}(X,\xi,n)$, we begin by clarifying how the categories

$$s\mathcal{R}(X), \quad s\mathcal{R}^{\mathrm{ld}}(\mathbb{J}X), \quad s\mathcal{R}\mathcal{G}^{\mathrm{ld}}(\mathbb{J}X)$$

fit together. (Recall that the decoration "ld" stands for *locally finitely dominated* and the \mathcal{G} in \mathcal{RG} stands for germs.) This will be done from the point of view of Waldhausen's fibration and approximation theorems [27, §1.6]. We assume for now that X is a compact space, not necessarily a manifold. In addition to the standard subcategory $w(s\mathcal{R}^{\mathrm{ld}}(\mathbb{J}X))$ of weak equivalences in $s\mathcal{R}^{\mathrm{ld}}(\mathbb{J}X)$, we introduce a coarser notion of weak equivalence κ , that is, a larger subcategory $\kappa(s\mathcal{R}^{\mathrm{ld}}(\mathbb{J}X))$. A morphism in $s\mathcal{R}^{\mathrm{ld}}(\mathbb{J}X)$ is a κ -equivalence if its mapping cone is equivalent to zero in $s\mathcal{RG}^{\mathrm{ld}}(\mathbb{J}X)$. Adopting Waldhausen's notation, we write

$$s\mathcal{R}^{\mathrm{Id}}(\mathbb{J}X)^{\kappa}$$

for the full subcategory of $s\mathcal{R}^{\mathrm{ld}}(\mathbb{J}X)$ consisting of all objects κ -equivalent to the zero object. This is a Waldhausen category with the usual *w*-equivalences as the weak equivalences. We also write

$$s\mathcal{R}^{\mathrm{Id}}(\mathbb{J}X)_{\kappa}$$

for $s\mathcal{R}^{\mathrm{ld}}(\mathbb{J}X)$ equipped with the coarser notion of weak equivalence, that is, κ -equivalence.

LEMMA 10.1. The forgetful functor $s\mathcal{R}^{\mathrm{ld}}(\mathbb{J}X)_{\kappa} \to s\mathcal{RG}^{\mathrm{ld}}(\mathbb{J}X)$ satisfies the hypotheses App1 and App2 of Waldhausen's approximation theorem.

PROOF. Property App1 means that morphisms which are taken to weak equivalences by the functor in question are already weak equivalences. This is trivially true in our case. To establish App2, we abbreviate $C_1 = s\mathcal{R}^{\mathrm{ld}}(\mathbb{J}X)_{\kappa}$ and $C_2 = s\mathcal{R}\mathcal{G}^{\mathrm{ld}}(\mathbb{J}X)$. Fix an object Y_1 in C_1 and a morphism $f: Y_1 \to Y_2$ in C_2 . We need to find an object \overline{Y}_2 in C_1 and morphisms $g: Y_1 \to \overline{Y}_2$ in C_1 as well as $h: \overline{Y}_2 \to Y_2$ in C_2 such that f = hg and h is a weak equivalence. This will be called an App2factorization. (Waldhausen's App2 also requires that g be a cofibration. But if g is not a cofibration, it can easily be converted into one by means of a mapping cylinder construction, so there is no need to pay any attention to that.)

Case 1. Here we assume in addition to all the above that Y_1 is the zero object of C_1 . It will be convenient to use the following terminology: an object Y of C_2 is clean if there exists a weak equivalence $\bar{Y} \to Y$ where \bar{Y} belongs to C_1 . Our task is then to show that all objects of C_2 are clean. This is not easy. It is however mostly axiomatic business and we start by summarizing related results of [34].

Let \mathcal{C} be any Waldhausen category satisfying the saturation axiom for weak equivalences [34, §0], and equipped with a cylinder functor satisfying the cylinder axiom. For objects C and D of \mathcal{C} let $\mathcal{M}(C, D)$ be the following category. The objects are diagrams in \mathcal{C} of the form $C \to D' \leftarrow D$ in \mathcal{C} , where the second arrow is both a cofibration and a weak equivalence; a morphism from $C \to D' \leftarrow D$ to $C \to D'' \leftarrow D$ is a morphism $D' \to D''$ in \mathcal{C} making the diagram



commutative. (Then $D' \to D''$ is a weak equivalence in C.) There is a composition law in the shape of a functor

$$\mathcal{M}(D, E) \times \mathcal{M}(C, D) \longrightarrow \mathcal{M}(C, E)$$
.

It follows that we can make a new category \mathcal{HC} with the same objects as \mathcal{C} , where $\operatorname{mor}_{\mathcal{HC}}(C, D) = \pi_0 B \mathcal{M}(C, D)$. (The \mathcal{H} is meant to be reminiscent of *homotopy*, but it does not mean that \mathcal{HC} is obtained from \mathcal{C} by organizing morphism sets in \mathcal{C} into equivalence classes for a relation called homotopy.) The functor $(C, D) \mapsto B \mathcal{M}(C, D)$ of two variables takes weak equivalences in either the left or right variable to homotopy equivalences of CW-spaces [34, §1]. Therefore:

All weak equivalences in C become invertible in \mathcal{HC} . (Consequently \mathcal{HC} can also be obtained from C by formally adding inverses for all morphisms in C which are both cofibrations and weak equivalences.)
The functor $C \mapsto B\mathcal{M}(C, D)$, for fixed D in \mathcal{C} , takes cofiber squares in \mathcal{C} to homotopy pullback squares of CW-spaces [34, §2]. Here a *cofiber square* in \mathcal{C} is a (commutative) pushout square in which either the vertical arrows or the horizontal arrows are cofibrations. This has the following consequence:

If the functor suspension from \mathcal{HC} to \mathcal{HC} is an equivalence of categories, then it makes \mathcal{HC} into a triangulated category where the distinguished triangles are obtained (up to isomorphism) from cofiber sequences in \mathcal{C} :

$$C \xrightarrow{f} D \longrightarrow \operatorname{cone}(f) \longrightarrow \Sigma C$$

(We use Verdier's axioms for a triangulated category as in [31]. The invertibility of the suspension functor is not used in the formulation of the remaining axioms.) Now we add two more conditions on C. For the first, suppose that



is a commutative diagram in \mathcal{C} whose rows are cofiber sequences. The condition is that if two of the vertical arrows are weak equivalences, then so is the third. The second condition is that if A, B are objects of \mathcal{C} such that the unique morphism $\star \to A \lor B$ is a weak equivalence, then $\star \to A$ and $\star \to B$ are weak equivalences. When these conditions hold, in addition to the above, then we have from [34, §3]:

A morphism $f: C \to D$ in \mathcal{C} is a weak equivalence if and only if it becomes invertible in \mathcal{HC} .

Now we specialize to $C := C_2$. Our task is (still) to show that all objects of C_2 are clean. We proceed by showing first that the class of clean objects in C_2 determines and is determined by a full triangulated subcategory of $\mathcal{H}C_2$. This breaks down into two statements:

- (i) If $Y \to Z$ is a weak equivalence in \mathcal{C}_2 and Z is clean, then Y is clean.
- (ii) If $Y \to Z$ is a cofibration where Y and Z are both clean, then the cofiber Z/Y is also clean.

Both of these are easy to verify and we only sketch the first. Suppose that $g: \overline{Z} \to Z$ is a weak equivalence, where \overline{Z} belongs to \mathcal{C}_1 . Since $Y \to Z$ is a weak equivalence in \mathcal{C}_2 , there exists a homotopy inverse $f: Z \to Y$ in the controlled setting, relative to the reference space $X \times [0, 1[$ but not necessarily *over* it. Then $fg: \overline{Z} \to Y$ is a controlled homotopy equivalence and we can view it as a morphism in \mathcal{C}_2 with source in \mathcal{C}_1 if we are willing to equip \overline{Z} with a new retraction map to $X \times [0, 1[$. Therefore Y is clean.

Now we know that the clean objects form a triangulated full subcategory of \mathcal{HC}_2 . Also, it follows easily from the definition of \mathcal{C}_2 that every object in \mathcal{HC}_2 is a direct summand of a clean object. Now general triangulated category principles imply that there is a subgroup of the Euler-Poincaré group of \mathcal{HC}_2 such that the clean objects are precisely those whose Euler-Poincaré characteristic is in that subgroup. (The Euler-Poincaré group of \mathcal{HC}_2 is an abelian group with one generator [Y] for every isomorphism class of objects Y of \mathcal{HC}_2 , and relations $[Y_1] = [Y_0] + [Y_2]$ for every distinguished triangle $Y_0 \to Y_1 \to Y_2 \to \Sigma Y_0$.) To finish the argument, it is therefore enough to show that the Euler-Poincaré group is trivial. But the Euler-Poincaré group is clearly a quotient of K_0 of \mathcal{C}_2 , which we know is zero.

Case 2. Here we assume that Y_1 is in $s\mathcal{R}^{\mathrm{lf}}(\mathbb{J}X)_{\kappa}$, so it has a locally finite and finite dimensional controlled CW structure. By case 1, we can enlarge $f: Y_1 \to Y_2$ to a diagram



in \mathcal{C}_2 , where \bar{Y}_2 belongs to \mathcal{C}_1 and e is a weak equivalence in \mathcal{C}_2 . We can find a CW subobject $Y'_1 \subset Y_1$ such that the inclusion $Y'_1 \to Y_1$ is a weak equivalence in $s\mathcal{R}^{\mathrm{ld}}(\mathbb{J}X)_{\kappa}$ and $f|Y'_1$ admits a factorization up to controlled homotopy (relative to zero sections) through e, say $f|Y'_1 \simeq e\varphi$. The factorization and the homotopy together allow us to think of the homotopy pushout (relative to zero sections) of

$$Y_1 \xleftarrow{\text{incl.}} Y'_1 \xrightarrow{\varphi} \bar{Y}_2$$

as an object P in $s\mathcal{R}^{\mathrm{ld}}(\mathbb{J}X)_{\kappa}$. Then $\overline{Y}_2 \subset P$ while $e: \overline{Y}_2 \to Y_2$ and $f: Y_1 \to Y_2$ extend simultaneously and canonically to a weak equivalence $P \to Y_2$ in \mathcal{C}_2 . This solves our factorization problem.

Case 3: the general case. Let $r: Y_1 \to X \times [0,1[$ be the retraction. We are assuming that Y_1 can be dominated by a locally finite and finite dimensional controlled CW-object W. The domination can be thought of in the following way. There are a morphism $W \to Y_1$ in \mathcal{C}_1 , a controlled map $Y_1 \to W$ relative to zero sections which need not be a morphism in \mathcal{C}_1 , and a controlled homotopy $(u_t: Y_1 \to Y_1)_{t \in [0,1]}$ relative to zero sections such that $u_0 = \text{id and } u_1$ is the composition $Y_1 \to W \to Y_1$. Write $Y_{1,t}$ for Y_1 with the retraction ru_t . The above description of the domination shows us that the morphism $fu_1: Y_{1,1} \to Y_2$ in \mathcal{C}_2 admits a factorization

$$Y_{1,1} \longrightarrow W \longrightarrow Y_2$$

where the first arrow is a morphism in C_1 and the second arrow is in C_2 . Applying case 2 above to the second arrow, $W \to Y_2$, and pre-composing with $Y_{1,1} \to W$ afterwards, we deduce that $fu_1: Y_{1,1} \longrightarrow Y_2$ has a factorization

$$Y_{1,1} \xrightarrow{g} \bar{Y}_2 \xrightarrow{h} Y_2$$

where g is in C_1 and h is a weak equivalence in C_2 . Enlarge \overline{Y}_2 to the mapping cylinder Z(g) of g (relative to zero sections), with a retraction to $X \times [0, 1]$ so that the slice $Y_1 \times t$ of the cylinder has the retraction ru_t . Then there is an inclusion

$$Y_{1,0} \to Z(g)$$

in \mathcal{C}_1 , and it only remains to extend $h: \overline{Y}_2 \to Y_2$ to a morphism $Z(g) \to Y_2$ in \mathcal{C}_2 . Define the extension so that it agrees with fu_t on the slice $Y_1 \times t$ of the cylinder. \Box

LEMMA 10.2. The maps

$$\mathbf{K}(s\mathcal{R}(X)) \longrightarrow \mathbf{K}(s\mathcal{R}^{\mathrm{ld}}(\mathbb{J}X)^{\kappa}) \ , \qquad \mathbf{VL}^{\bullet}(s\mathcal{R}(X)) \longrightarrow \mathbf{VL}^{\bullet}(s\mathcal{R}^{\mathrm{ld}}(\mathbb{J}X)^{\kappa})$$

induced by the inclusion $s\mathcal{R}(X) \to s\mathcal{R}^{\mathrm{ld}}(\mathbb{J}X)^{\kappa}$ are homotopy equivalences.

PROOF. For the K-theory case, apply Waldhausen's approximation theorem to the inclusion $s\mathcal{R}(X \times [0,1[) \to s\mathcal{R}^{\mathrm{ld}}(\mathbb{J}X)^{\kappa})$. The inclusion of $s\mathcal{R}(X)$ in $s\mathcal{R}(X \times [0,1[)$ clearly also induces a homotopy equivalence in K-theory. For the **VL**[•]-theory case, argue similarly, using a **VL**[•]-theory version of the approximation theorem. (This has essentially the same hypotheses App1 and App2. The conclusion, that the functor in question induces a homotopy equivalence of VL^{\bullet} -spectra, comes from a direct comparison of homotopy groups.)

COROLLARY 10.3. The inclusions $s\mathcal{R}(X) \to s\mathcal{R}^{\mathrm{ld}}(\mathbb{J}X)$ and the "passage to germ" functor $s\mathcal{R}^{\mathrm{ld}}(\mathbb{J}X) \to s\mathcal{R}\mathcal{G}^{\mathrm{ld}}(\mathbb{J}X)$ lead to homotopy fiber sequences of spectra,

$$\mathbf{A}(X) \longrightarrow \mathbf{A}(\mathbb{J}X) \longrightarrow \mathbf{A}(\mathbb{J}X_{\infty}) ,$$
$$\mathbf{VL}^{\bullet}(X,\xi,n) \longrightarrow \mathbf{VL}^{\bullet}(\mathbb{J}X,\xi,n) \longrightarrow \mathbf{VL}^{\bullet}(\mathbb{J}X_{\infty},\xi,n)$$

PROOF. By Waldhausen's fibration theorem there is a homotopy fiber sequence

$$\mathbf{K}(s\mathcal{R}^{\mathrm{ld}}(\mathbb{J}X)^{\kappa}) \longrightarrow \mathbf{K}(s\mathcal{R}^{\mathrm{ld}}(\mathbb{J}X) \longrightarrow \mathbf{K}(s\mathcal{R}^{\mathrm{ld}}(\mathbb{J}X)_{\kappa}).$$

With the definition $\mathbf{A}(\mathbb{J}X) = \mathbf{K}(s\mathcal{R}^{\mathrm{ld}}(\mathbb{J}X))$ and the identications of lemma 10.1 and lemma 10.2, this turns into a homotopy fiber sequence

$$\mathbf{A}(X) \longrightarrow \mathbf{A}(\mathbb{J}X) \longrightarrow \mathbf{A}(\mathbb{J}X_{\infty}).$$

The same reasoning applies in the VL^{\bullet} -theory case. Of course one needs to know that Waldhausen's fibration theorem has an analogue in *L*-theory. There is such an analogue, as follows.

Let \mathcal{C} be any Waldhausen category with weak equivalences $w\mathcal{C}$ and with an SW product \odot satisfying the axioms of [**39**, §2]. Suppose that another subcategory $\kappa \mathcal{C}$ of \mathcal{C} with $w\mathcal{C} \subset \kappa \mathcal{C} \subset \mathcal{C}$ is specified. Suppose that \mathcal{C} with weak equivalences $\kappa \mathcal{C}$ and the same SW product \odot also satisfy the axioms of [**39**, §2]. As before, let \mathcal{C}_{κ} stand for \mathcal{C} with the coarse notion κ of weak equivalences, and \mathcal{C}^{κ} for the full Waldhausen subcategory of \mathcal{C} consisting of the objects which are κ -equivalent to zero. Then there is a homotopy cartesian square of spectra



with contractible lower left-hand term. We shorten this as usual to a homotopy fiber sequence

$$\mathbf{L}^{\bullet}(\mathcal{C}^{\kappa}) \longrightarrow \mathbf{L}^{\bullet}(\mathcal{C}) \longrightarrow \mathbf{L}^{\bullet}(\mathcal{C}_{\kappa}).$$

This is the basic fibration theorem for symmetric L-theory. There is a version for quadratic L-theory and also one for visible symmetric L-theory, when that is defined.

The proof of the symmetric L-theory fibration theorem is as follows, in outline. Suppose that (C, D, φ) is a symmetric Poincaré pair in \mathcal{C} . In more detail, we assume that D and C are related by a cofibration $D \to C$ and that φ is a homotopy fixed point for the action of $\mathbb{Z}/2$ on $\Omega^n(C \odot C/D \odot D)$, satisfying the appropriate nondegeneracy condition. Then n is the formal dimension of the Poincaré pair. The symmetric structure φ on the pair (C, D) descends to a symmetric structure $\varphi/\partial\varphi$ on C/D, which may be degenerate. It is a basic fact of Ranicki's algebraic theory of surgery that the passage from the Poincaré pair (C, D, φ) to the (single) symmetric object $(C/D, \varphi/\partial\varphi)$ is reversible. We have already used it in the proof of theorem 5.5; see [19], [32] and [25] for more details on the inverse construction. These details on the inverse construction imply that C/D with the symmetric structure $\varphi/\partial\varphi$ is nondegenerate in \mathcal{C}_{κ} if and only if D belongs to \mathcal{C}^{κ} . Hence the bordism theory of Poincaré pairs (C, D, φ) with C in C and D in C^{κ} is "equivalent" to the bordism theory of Poincaré objects in \mathcal{C}_{κ} . This amounts to a homotopy equivalence of spectra, from the homotopy cofiber of

$$\mathbf{L}^{\bullet}(\mathcal{C}^{\kappa}) \to \mathbf{L}^{\bullet}(\mathcal{C})$$

to $\mathbf{L}^{\bullet}(\mathcal{C}_{\kappa})$. That in turn can be reformulated as a homotopy fiber sequence of spectra $\mathbf{L}^{\bullet}(\mathcal{C}^{\kappa}) \to \mathbf{L}^{\bullet}(\mathcal{C}) \to \mathbf{L}^{\bullet}(\mathcal{C}_{\kappa})$.

COROLLARY 10.4. The inclusions $s\mathcal{R}(X) \to s\mathcal{R}^{\mathrm{ld}}(\mathbb{J}X)$ and the passage to germs functor $s\mathcal{R}^{\mathrm{ld}}(\mathbb{J}X) \to s\mathcal{R}\mathcal{G}^{\mathrm{ld}}(\mathbb{J}X)$ lead to a homotopy fiber sequence of spectra

$$\mathbf{VLA}^{\bullet}(X,\xi,n) \longrightarrow \mathbf{VLA}^{\bullet}(\mathbb{J}X,\xi,n) \longrightarrow \mathbf{VLA}^{\bullet}(\mathbb{J}X_{\infty},\xi,n).$$

PROOF. What we are really saying is that \mathbf{VLK}^{\bullet} turns the square of Waldhausen categories with SW-duality

into a homotopy cartesian square of spectra, with contractible lower left-hand term. This follows directly from the analogous statements for \mathbf{K} -theory and \mathbf{VL}^{\bullet} -theory, which we have from the previous corollary, and the definition of \mathbf{VLK}^{\bullet} in terms of \mathbf{K} and \mathbf{VL}^{\bullet} .

Excision properties of $F^{\%}(X,\xi,n)$. We have defined $F^{\%}(X,\xi,n)$ as the homotopy fiber of a map

$$F(X,\xi,n) \longrightarrow F(\mathbb{J}X,\xi,n).$$

Unravelling that and using corollary 10.4, we obtain an identification of $F^{\%}(X,\xi,n)$ with $\Omega^{\infty+1}\mathbf{VLK}^{\bullet}(s\mathcal{R}\mathcal{G}^{\mathrm{ld}}(\mathbb{J}X))$. We know from theorem 5.14 that $\Omega^{\infty}\mathbf{VL}^{\bullet}$ applied to $s\mathcal{R}\mathcal{G}^{\mathrm{ld}}(\mathbb{J}X)$ gives a homotopy invariant and excisive functor of X. Also, $\Omega^{\infty}\mathbf{K}$ applied to $s\mathcal{R}\mathcal{G}^{\mathrm{ld}}(\mathbb{J}X)$ is an excisive and homotopy invariant functor of X by [**35**, §6-9]. It follows that $\Omega^{\infty}\mathbf{VLK}^{\bullet}$ applied to $s\mathcal{R}\mathcal{G}^{\mathrm{ld}}(\mathbb{J}X)$ is an excisive and homotopy invariant functor of X.

Excision properties of $F^{\%}((\bar{X}, X), \xi, n)$. This is similar to the case of $F^{\%}(X, \xi, n)$. We assume that \bar{X} is compact; we need not assume that X is a manifold. There is a commutative square of Waldhausen categories

which in the case $X = \overline{X}$ deviates very little from (10.5). Waldhausen's fibration theorem can be applied to this, after some minor re-definitions. Hence $F^{\%}((\overline{X}, X), \xi, n)$ is identified (via a chain of natural homotopy equivalences) with

$$\Omega^{\infty+1}\mathbf{VLK}^{\bullet}(s\mathcal{RG}^{\mathrm{Id}}(\mathbb{J}X)).$$

Therefore it is essentially a functor of the locally compact space X alone, and it is excisive in the locally finite sense of theorem 5.5 and theorem 5.14. Indeed this follows from theorem 5.14 and the analogous theorem for K-theory, [**35**, §6-9]. \Box

Naturality properties of $\sigma^{\%}(X)$. The naturality properties of $\sigma^{\%}$ are analogous to those of σ in the previous chapter. That is, a diagram of closed manifolds, homeomorphisms and stable normal bundle isomorphisms

$$(X_0,\xi_0) \leftarrow (X_1,\xi_1) \leftarrow \dots \leftarrow (X_{k-1},\xi_{k-1}) \leftarrow (X_k,\xi_k)$$

determines a map from Δ^k to $F^{\%}(X_0, \xi_0, n)$. This assignment extends $\sigma^{\%}$ and commutes with the usual face and degeneracy operators acting on such diagrams. (We omit the details, except for pointing out that each ξ_i can be viewed as a spherical fibration on X_i via fiberwise one-point compactification.)

Naturality properties of $\sigma^{\%}(\bar{X}, X)$. Let (\bar{X}_i, X_i) for $i = 0, 1, \ldots, k$ be compact control spaces where each X_i is an *n*-manifold. Suppose that they are arranged in a diagram of homeomorphisms of control spaces and stable normal bundle isomorphisms

$$((\bar{X}_0, X_0), \xi_0) \leftarrow ((\bar{X}_1, X_1), \xi_1) \leftarrow \dots \leftarrow ((\bar{X}_k, X_k), \xi_k)$$

The diagram then determines a map from Δ^k to $F^{\otimes}((\bar{X}_0, X_0), \xi_0)$. This assignment commutes with the usual face and degeneracy operators acting on such diagrams.

CHAPTER 11

Algebraic approximations to structure spaces: Set-up

NOTATION 11.1. We use the symbol % in the *subscript* position to describe homotopy fibers of assembly maps. For example: $\mathbf{L}^{\bullet}_{\%}(X)$ is the homotopy fiber of the assembly map $\mathbf{L}^{\bullet\%}(X) \to \mathbf{L}^{\bullet}(X)$, assuming X has the homotopy type of a CW-space. Similarly: $\mathbf{L}_{\bullet\%}(X)$ is the homotopy fiber of the assembly map in quadratic L-theory and $\mathbf{LA}_{\bullet\%}(X,\xi,n)$ is the homotopy fiber of the assembly map in quadratic LA-theory.

From now on we will often suppress the \bar{X} in a control space (\bar{X}, X) and make up for that with a semicolon followed by a c to indicate a controlled context. For example, instead of writing $F^{\%}((\bar{X}, X), \xi, n))$ and $F((\bar{X}, X), \xi, n))$ as in chapter 10, we may write

$$F^{\%}(X,\xi,n;c)$$
, $F(X,\xi,n;c)$.

In that spirit, $F_{\%}(X,\xi,n;c)$ is the homotopy fiber of the assembly map alias forgetful map $F^{\%}(X,\xi,n;c) \to F(X,\xi,n;c)$ which we have from chapter 10. This assumes that X is the nonsingular part of a control space (\bar{X},X) with compact \bar{X} .

CONSTRUCTION 11.2. Let M be a compact m-manifold with normal bundle ν . We introduce a map φ as in (1.1) from $\mathcal{S}(M)$ to

$$F_{\%}(M,\nu,m) = \Omega^{\infty+m} \mathbf{VLA}^{\bullet}_{\%}(M,\nu,m) \simeq \Omega^{\infty+m} \mathbf{LA}_{\bullet\%}(M,\nu,m)$$

REMARK. The setting is away from ∂M . Points of $\mathcal{S}(M)$ correspond to homotopy equivalences of pairs $f: (N, \partial N) \to (M, \partial M)$ where the induced map $\partial N \to \partial M$ is a homeomorphism. We tend to think of $F_{\%}(M, \nu, m)$ as the homotopy fiber of the assembly map

$$\Omega^{\infty+m}\mathbf{VLA}^{\bullet\%}(M,\nu,m)\longrightarrow \Omega^{\infty+m}\mathbf{VLA}^{\bullet}(M,\nu,m)$$

over the visible symmetric signature $\sigma(M)$. Because this homotopy fiber has a canonical base point, determined by $\sigma^{\%}(M)$ etc., it can easily be identified by a translation with the homotopy fiber of assembly over the base point, whenever that is convenient.

CONSTRUCTION 11.3. Notation being as in the previous construction, there is a "local degree" homomorphism from $\pi_m \mathbf{L}_{\bullet}^{\mathscr{H}}(M,\nu)$ to the group $L_0(\mathbb{Z})^{\pi_0 M}$. It is onto.

This is the composition of Poincaré duality, $\pi_m \mathbf{L}_{\bullet}^{\%}(M,\nu) \cong H^0(M; \mathbf{L}_{\bullet}(\star))$, with an evaluation map from $H^0(M; \mathbf{L}_{\bullet}(\star))$ to $L_0(\mathbb{Z})^{\pi_0 M}$. We admit that $L_0(\mathbb{Z})$ could also be described as $L_0(\star)$.

THEOREM 11.4. Assume $\dim(M) \ge 5$. The diagram

$$\mathcal{S}(M) \xrightarrow{\varphi} \Omega^{\infty+m} \mathbf{LA}_{\bullet\%}(M,\nu,m) \xrightarrow{\text{local degree}} L_0(\mathbb{Z})^{\pi_0 M}$$

is a homotopy fiber sequence in the concordance stable range.

In this theorem, $L_0(\mathbb{Z}) \cong 8\mathbb{Z}$ is viewed as a discrete space. The local degree is defined on $\pi_m \mathbf{LA}_{\bullet\%}(M,\nu,m)$ via the forgetful map to $\pi_m \mathbf{L}_{\bullet}^{\%}(M,\nu)$. Details on the meaning of "concordance stable range" are given in the definition which follows.

DEFINITION 11.5. For a compact manifold N let k_N be the minimum of all positive integers k such that the stabilization map of concordance spaces

$$\Omega \mathcal{H}(N \times D^i) \longrightarrow \Omega \mathcal{H}(N \times D^{i+1})$$

is k-connected for i = 0, 1, 2, ... The precise meaning of theorem 11.4 is that

- the map from $\pi_0 \mathcal{S}(M)$ to ker $[\pi_m \mathbf{LA}_{\bullet\%}(M,\nu,m) \to L_0(\mathbb{Z})^{\pi_0 M}]$ determined by φ is bijective;
- the restricted map, from a component of $\mathcal{S}(M)$ represented by a homotopy equivalence $(N, \partial N) \to (M, \partial M)$ to the corresponding component of $\Omega^{\infty+m} \mathbf{LA}_{\bullet\%}(M, \nu, m)$, is $(k_N + 1)$ -connected.

REMARK. It follows from [12] and smoothing theory that

$$k_N \ge \min\left\{\frac{m-7}{2}, \frac{m-4}{3}\right\}$$

if N admits a smooth structure, $m = \dim(N)$. Similar estimates for non-smooth manifolds were part of the topology folklore in the 1970's, but completely convincing proofs of these have apparently not been found. Unfortunately, even if Madmits a smooth structure, there may be some components of $\mathcal{S}(M)$, represented by $(N, \partial N) \to (M, \partial M)$ say, whose source manifold N does not admit a smooth structure.

Our only goal in the rest of the paper is to prove theorem 11.4. An overview of the proof has already been given in chapter 2. We are going to break this up into smaller pieces.

LEMMA 11.6. The map in construction 11.2 identifies $\pi_0 \mathcal{S}(M)$ with the kernel of the local degree homomorphism from $\pi_m \mathbf{LA}_{\bullet\%}(M,m)$ to $L_0(\mathbb{Z})^{\pi_0 M}$.

For $i \geq 0$ we introduce the controlled structure space $\mathcal{S}(M \times \mathbb{R}^i; c)$, using the compactification $M * S^{i-1}$ of $M \times \mathbb{R}^i$ to define the control criteria. Let

$$\mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^i; c) \subset \mathcal{S}(M \times \mathbb{R}^i; c)$$

be the union of the connected components which are reducible, i.e., in the image of $\pi_0 \mathcal{S}(M) \to \pi_0 \mathcal{S}(M \times \mathbb{R}^i; c)$.

CONSTRUCTION 11.7. For $i \ge 0$ we construct a map φ from $\mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^i; c)$ to

$$F_{\%}(M \times \mathbb{R}^{i}, \nu, m+i; c) \simeq \Omega^{\infty+m+i} \mathbf{LA}_{\bullet\%}(M \times \mathbb{R}^{i}, \nu, m+i; c)$$

This map agrees with construction 11.2 when i = 0. The resulting square (2.9) commutes for every $i \ge 0$. (We are actually slightly more general than in (2.9) because M can have a nonempty boundary.)

Let $f: N \to M$ be a homotopy equivalence of compact *m*-manifolds, relative to a homeomorphism $\partial N \to \partial M$. Thus $f: N \to M$ determines a point in $\mathcal{S}(M)$. LEMMA 11.8. The homotopy fiber of

 $\mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^{i}; c) \xrightarrow{\times \mathbb{R}} \mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^{i+1}; c)$

over the point determined by $f \times \mathbb{R}^{i+1} : N \times \mathbb{R}^{i+1} \to M \times \mathbb{R}^{i+1}$ is homotopy equivalent to a union of components of $\mathcal{H}(N \times \mathbb{R}^i; c)$. Indeed it is homotopy equivalent to the pullback of the following diagram.

$$\mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^{i}; c)$$

$$\downarrow^{\mathrm{inclusion}}$$

$$\mathcal{H}(N \times \mathbb{R}^{i}; c) \xrightarrow{\mathrm{upper \ bdry}} \mathcal{S}(N \times \mathbb{R}^{i}; c)$$

LEMMA 11.9. The homotopy fiber of

$$F_{\%}(M \times \mathbb{R}^{i}, \nu, m+i; c) \xrightarrow{\times \mathbb{R}} F_{\%}(M \times \mathbb{R}^{i+1}, \nu, m+i+1; c)$$

is homotopy equivalent to $\Omega^{\infty} \mathbf{A}_{\%}(M \times \mathbb{R}^{i}; c)$.

In combination with some results from [10], lemmas 11.8 and 11.9 lead to the following technical statement for the square (2.9). Again we allow nonempty ∂M . Notation: Select $x \in S^{\mathrm{rd}}(M \times \mathbb{R}^{i+1}; c)$ with image

$$y \in \Omega^{\infty+m+i+1} \mathbf{LA}_{\bullet\%}(M \times \mathbb{R}^{i+1}, \nu, m+i+1; c)$$
.

Let Φ_x and Ψ_y be the vertical homotopy fibers over x and y respectively in (2.9). Select $a \in \pi_0 \Phi_x$ with image $b \in \pi_0 \Psi_y$. Let $\Phi_{x,a}$ and $\Psi_{y,b}$ be the connected components determined by a and b, respectively. Select an element of $\pi_0 \mathcal{S}(M)$ whose image in $\pi_0 \mathcal{S}^{rd}(M \times \mathbb{R}^i; c)$ agrees with the image of a under the forgetful map. Represent that element of $\pi_0 \mathcal{S}(M)$ by a homotopy equivalence $N \to M$ (restricting to a homeomorphism $\partial N \to \partial M$).

PROPOSITION 11.10. The map $\Phi_x \to \Psi_y$ determined by (2.9) induces an injection on π_0 . The restricted map $\Phi_{x,a} \to \Psi_{y,b}$ is $(k_N + i + 1)$ -connected.

One more ingredient that we need for our downward induction procedure is the compatibility of the Casson-Sullivan-Wall-Quinn-Ranicki homotopy equivalence (2.2) and diagram (2.8). In the following formulation we make no distinction between (telescopic) homotopy colimits and ordinary colimits.

LEMMA 11.11. In the limit $i = \infty$, the horizontal maps of diagram (2.8) agree with the map (2.2): there is a homotopy commutative square

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Proof of thm. 11.4 modulo lemmas 11.6, 11.11 and prop. 11.10. The proof follows the outline given in chapter 2. Note the added precision coming from definition 11.5. $\hfill \Box$

CHAPTER 12

Algebraic approximations to structure spaces: Constructions

The constructions are based on one general principle which we recall from [10]. Let \mathcal{C} and \mathcal{D} be small categories, $\mathcal{C} \subset \mathcal{D}$. Let U_0 and U_1 be functors from \mathcal{D} to spaces, with a natural transformation $a: U_0 \to U_1$. Suppose that U_0 and U_1 take every morphism in \mathcal{D} to a homotopy equivalence of spaces. Then every choice of point $x = (x_0, x_1, \gamma)$ in the homotopy pullback of

holim
$$U_1$$

 \downarrow^{res}
holim $U_0|\mathcal{C} \xrightarrow{a_*}$ holim $U_1|\mathcal{C}$

and choice of object δ in \mathcal{D} determine a map

(12.1)
$$\operatorname{hofiber}_{\delta}[B\mathcal{C} \to B\mathcal{D}] \longrightarrow \operatorname{hofiber}_{x_1(\delta)}[U_0(\delta) \to U_1(\delta)].$$

Here $x_0 \in \text{holim } U_0|\mathcal{C} \text{ and } x_1 \in \text{holim } U_1$, while $\gamma: [0,1] \to \text{holim } U_1|\mathcal{C} \text{ is a path connecting the images of } x_0 \text{ and } x_1$. We have written $x_1(\delta) \in U_1(\delta)$ for the coordinate of x_1 in $U_1(\delta)$.

Sketch of construction of (12.1). Because U_0 and U_1 take every morphism in \mathcal{D} to a homotopy equivalence of spaces, the projections

 $\operatorname{hocolim} U_0 \longrightarrow B\mathcal{D}$, $\operatorname{hocolim} U_1 \longrightarrow B\mathcal{D}$

are quasi-fibrations. The element $x_1 \in \text{holim } U_1$ determines a section s_1 of the fibration associated to $\text{hocolim}U_1 \to B\mathcal{D}$, and x_0 determines a section s_0 of the fibration associated to $\text{hocolim}U_0|\mathcal{C} \to B\mathcal{C}$. Pulling these data back to

$$\mathcal{S} := \operatorname{hofiber}_{\delta}[B\mathcal{C} \to B\mathcal{D}]$$

yields two trivialized fibrations on S with fibers $U_0(\delta)$ and $U_1(\delta)$, respectively, and two sections t_0 and t_1 of these. We may view t_0 and t_1 as maps $S \to U_0(\delta)$ and $S \to U_1(\delta)$, respectively. The path γ determines a homotopy $a_*t_0 \simeq t_1$. The map t_1 is trivialized (equipped with a homotopy to a constant map) because, as a section, it is the pullback of a section defined on $B\mathcal{D}$.

Variants. We allow \mathcal{C} and \mathcal{D} to be categories *internal to the category of topological spaces*. This means that \mathcal{D} for example has a space of objects and a space of morphism, the maps *source* and *target* from morphism space to object space are continuous, and so on. In such a case we try our best to ensure that the functors U_0 and U_1 from \mathcal{D} to spaces factor through the discrete category $\pi_0 \mathcal{D}$ with object set $\pi_0 \text{ob}(\mathcal{D})$ and morphism set $\pi_0 \text{mor}(\mathcal{D})$.

We now specialize (12.1) to obtain construction 11.2 Suppose to begin that M has empty boundary.

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DEFINITION 12.2. Let \mathcal{D} be the following category. The objects are finitely dominated Poincaré duality spaces X of formal dimension m, together with a spherical fibration $\xi: E \to X$ with fiber $\simeq S^d$, a preferred section for that, and a stable map $\eta: S^{m+d} \to E/X$ which carries a fundamental class. Mostly for convenience, we require X to be compact Hausdorff and homotopy equivalent to M. A morphism from (X_1, ξ_1, η_1) to (X_0, ξ_0, η_0) is a pair (u, v) where u is a homotopy equivalence from X_1 to X_0 and

$$v \colon \Sigma_{X_1}^{d_1 - d_0} E_1 \longrightarrow E_0$$

is a homotopy equivalence which covers u, respects the zero sections and satisfies $v\eta_1 = \eta_0$. (It is understood that $d_1 \ge d_0$, where d_1 and d_0 are the formal fiber dimensions of $\xi_1: E_1 \to X_1$ and $\xi_0: E_0 \to X_0$.)

We allow continuous variation of η in objects (X, ξ, η) . This makes the set of objects of \mathcal{D} into a space. (Some conditions on underlying sets should be added to our definition of objects to ensure that the class of objects of \mathcal{D} is indeed a set.) We also allow continuous variation of v in morphisms (u, v) as above. This makes the set of morphisms of \mathcal{D} into a space, in such a way that "source" and "target" are continuous maps from the morphism space to the object space. Hence \mathcal{D} is a topological category. (By allowing continuous variation of the η 's and the v's , we achieve that $B\mathcal{D}$ is homotopy equivalent to BG(M), where G(M) is the topological monoid of homotopy automorphisms of M. This follows from the uniqueness of Spivak normal fibrations ; see chapter D.)

The category \mathcal{D} has a subcategory \mathcal{C} which is defined like this. An object (X, ξ, η) of \mathcal{D} , as above, belongs to \mathcal{C} precisely if X is a closed m-manifold, ξ is a sphere bundle and $\eta: S^{m+d} \to E/X$ restricts to a homeomorphism from $\eta^{-1}(E \smallsetminus X)$ to $E \smallsetminus X$. (The condition on η means that η is the "Thom collapse" associated with an embedding of N in some euclidean space.) A morphism (u, v) in \mathcal{D} , as above, belongs to \mathcal{C} if its source and target are in \mathcal{C} , and both u and v are homeomorphisms. Continuous variation of η in objects (X, ξ, η) and of v in morphisms (u, v) are allowed as before. The result is that $B\mathcal{C}$ is homotopy equivalent to a disjoint union of classifying spaces BHom(N), where N runs through a maximal set of pairwise non-homeomorphic closed m-manifolds which are homotopy equivalent to M. Recall from chapter A) that Hom(N) is the homeomorphism group of N with the discrete topology.

The manifold M itself, equipped with a euclidean normal bundle ν etc., can be viewed as an object of C. We note that

$$\operatorname{Str}(M) := \operatorname{hofiber}_M[B\mathcal{C} \hookrightarrow B\mathcal{D}]$$

is a good combinatorial model for $\mathcal{S}(M)$. More precisely, $\operatorname{Str}(M)$ comes with a forgetful map to $\mathcal{S}(M)$ which is a homology equivalence. We like to think of that map as an inclusion.

The homotopy invariant signature σ which we have constructed in chapter 9 determines or is a point in holim $F|\mathcal{D}$. To be more precise, $F|\mathcal{D}$ means the functor taking (X, ξ, η) in \mathcal{D} to $F(X, \xi, m)$ as defined in chapter 9; then we have $\sigma(X) \in F(X, \xi, m)$ with the naturality properties discussed at the end of that same chapter. Similarly, the excisive signature $\sigma^{\%}$ which we have constructed in chapter 10 is a point in holim $F^{\%}|\mathcal{C}$. Here $F^{\%}|\mathcal{C}$ means the functor taking (N, ξ, η) to $F^{\%}(N, \xi, m)$ as defined in chapter 10.

We apply (12.1) with this choice of \mathcal{D} and \mathcal{C} , with $U_0 = F^{\%}|\mathcal{D}$ and $U_1 = F|\mathcal{D}$, and with $x_1 = \sigma$, $x_0 = \sigma^{\%}$. The result is a map

(12.3)
$$\operatorname{Str}(M) \longrightarrow \operatorname{hofiber}_{\sigma(M)} [F^{\%}(M,\nu,m) \to F(M,\nu,m)].$$

We can write (12.3) in the form

$$\operatorname{Str}(M) \longrightarrow F_{\%}(M)$$
.

By obstruction theory, for which we can use that $F_{\%}(M)$ is an *H*-space, this map has an extension to the mapping cylinder of $Str(M) \to S(M)$. The extension is unique up to contractible choice. Restrict to S(M) to obtain

(12.4)
$$\varphi_M : \mathcal{S}(M) \longrightarrow F_{\%}(M, \nu, m)$$

To conceal the dependence on a contractible choice, and also to make the contractibility of the choice more obvious, we could replace the target $F_{\%}(M,\nu,m)$ by the homotopy equivalent

hocolim
$$[\mathcal{S}(M) \leftarrow \operatorname{Str}(M) \to F_{\%}(M,\nu,n)].$$

A systematic way to extend the construction to the case where M has a nonempty boundary is to use algebraic K-theory of pairs. This requires a few definitions which we have banished to appendix C. Meanwhile the notation that comes with these definitions is self-explanatory. The construction of φ_M in (12.4) generalizes mechanically to produce a map

$$\varphi_{(M,\partial M)}: \mathcal{S}(M,\partial M) \longrightarrow F_{\%}(\partial M \subset M, \nu, m)$$

where the definition of the target is based on K-theory and L-theory of pairs, such as $(M, \partial M)$. The maps $\varphi_{(M,\partial M)}$ and $\varphi_{\partial M}$ fit into a commutative square

$$(12.5) \qquad \begin{array}{c} \mathcal{S}(M,\partial M) & \xrightarrow{\varphi_{(M,\partial M)}} & F_{\%}(\partial M \subset M,\nu,m) \\ & & & & \downarrow^{\text{forget}} & & & \downarrow^{\text{forget}} \\ & & \mathcal{S}(\partial M) & \xrightarrow{\varphi_{\partial M}} & F_{\%}(\partial M,\nu,m-1). \end{array}$$

We pass to vertical homotopy fibers (over the base points in the lower row) and obtain a map of the form

(12.6)
$$\varphi_M : \mathcal{S}(M) \longrightarrow F_{\%}(M, \nu, m)$$
.

There is a slight generalization which we shall also need. Let M be a compact manifold such that ∂M is the union of two codimension zero submanifolds $\partial_0 M$ and $\partial_1 M$, with $\partial_0 M \cap \partial_1 M = \partial \partial_0 M = \partial \partial_1 M$. There is a structure space



Licensed to University of Edinburgh. Prepared on Sun Nov 15 16:25:09 EST 2015for download from IP 129.215.149.96. License or copyright restrictions may apply to redistribution; see http://www.ams.org/publications/ebooks/terms which fits into a homotopy fiber sequence

$$\mathcal{S}(M,\partial_1 M) \longrightarrow \mathcal{S}\begin{pmatrix} \partial(\partial_0 M) & \longrightarrow & \partial_0 M \\ \downarrow & & \downarrow \\ \partial_1 M & \longrightarrow & M \end{pmatrix} \longrightarrow \mathcal{S}(\partial_0 M, \partial\partial_0 M)$$

We can therefore make a map of the form

 $\varphi_{(M,\partial_1 M)}: \mathcal{S}(M,\partial_1 M) \longrightarrow F_{\%}(\partial M_1 \subset M,\nu,m)$ (12.7)

by passing to vertical homotopy fibers in the square

$$(12.8) \qquad \begin{array}{c} \mathcal{S}\begin{pmatrix} \partial\partial_{0}M \longrightarrow \partial_{0}M \\ \downarrow & \downarrow \\ \partial_{1}M \longrightarrow M \end{pmatrix} \longrightarrow F_{\%}\begin{pmatrix} \partial\partial_{0}M \longrightarrow \partial_{0}M \\ \downarrow & \downarrow \\ \partial_{1}M \longrightarrow M \end{pmatrix} \\ \downarrow & \downarrow \\ \mathcal{S}(\partial_{0}M, \partial\partial_{0}M) \longrightarrow F_{\%}(\partial\partial_{0}M \subset \partial_{0}M, \nu, m-1) \end{array}$$

This completes construction 11.2.

Construction 11.7 uses very similar ideas and recycled notation. There is a small complication due to the fact that we do not have a good discrete homology approximation theorem for the topological group of controlled automorphisms of $M \times \mathbb{R}^i$, assuming that M is compact. Instead we have such a theorem (in chapter A) for the topological group of homeomorphisms $M * S^{i-1} \to M * S^{i-1}$ taking S^{i-1} to S^{i-1} , and we have to make the best of that. Suppose to begin with that M is closed.

DEFINITION 12.9. Generalizing definition 12.2, we introduce a category \mathcal{D} whose objects are certain control spaces (\bar{X}, X) with compact Hausdorff \bar{X} , together with a spherical fibration $\xi: E \to X$ with fiber $\simeq S^d$, a preferred section for that, and a stable map $\eta: S^{m+d} \to E/\!\!/X$. Compare definition 5.1. These data are required to satisfy the conditions of definition 9.9 for a controlled Poincaré duality space. In addition we require that $\bar{X} \smallsetminus X = S^{i-1}$ and that (\bar{X}, X) be homotopy equivalent as a control space to $(M * S^{i-1}, M \times \mathbb{R}^i)$; and moreover we mean a homotopy equivalence such that the maps and homotopies involved restrict to the identity on the singular sets S^{i-1} . A morphism from $(\bar{X}_1, X_1, \xi_1, \eta_1)$ to $(X_0, X_0, \xi_0, \eta_0)$ is a pair (u, v) where

- u is a map of control spaces from (\bar{X}_1, X_1) to (\bar{X}_0, X_0) which restricts to the identity $S^{i-1} \to S^{i-1}$ on singular sets; • $v: \Sigma_{X_1}^{d_1-d_0} E_1 \longrightarrow E_0$ covers $u|X_1$, respects the zero sections and satisfies

Continuous variation of η in objects (\bar{X}, X, ξ, η) and continuous variation of v in morphisms v is allowed. The result is that

(12.10)
$$B\mathcal{D} \simeq BG(M \times \mathbb{R}^i; c) \simeq BG(M)$$

where $G(M \times \mathbb{R}^i; c)$ is the grouplike topological monoid of controlled homotopy automorphisms of $M \times \mathbb{R}^i$, and G(M) is the grouplike topological monoid of homotopy automorphisms of M.

The category \mathcal{D} has a subcategory \mathcal{C} which is defined like this. An object (\bar{X}, X, ξ, η) of \mathcal{D} belongs to \mathcal{C} precisely if (\bar{X}, X) is homeomorphic to $(N * S^{i-1}, N \times \mathbb{R}^i)$ for some closed *m*-manifold N, the spherical fibration ξ is a sphere bundle and $\eta: S^{m+d} \to E/\!\!/X$ restricts to a homeomorphism from $\eta^{-1}(E \smallsetminus X)$ to $E \smallsetminus X$. A morphism (u, v) in \mathcal{D} , as above, belongs to \mathcal{C} if its source and target are in \mathcal{C} , and both u and v are homeomorphisms. Continuous variation of η in objects (\bar{X}, X, ξ, η) and continuous variation of v in morphisms v is allowed. The result is that

$$B\mathcal{C} \simeq \prod_{\beta} B\text{Hom}(N_{\beta} \times \mathbb{R}^{i}; c).$$

Here each N_{β} is a closed *m*-manifold homotopy equivalent to *M*. We select a maximal set of such manifolds N_{β} such that the $N_{\beta} \times \mathbb{R}^{i}$ are pairwise controlled non-homeomorphic. Furthermore $\mathsf{Hom}(N_{\beta} \times \mathbb{R}^{i})$ is the discrete group of controlled homeomorphisms from $N_{\beta} \times \mathbb{R}^{i}$ to itself.

Generalizing from i = 0 to $i \ge 0$, we obtain functors $F|\mathcal{D}$ and $F^{\%}|\mathcal{D}$ and elements

$$\sigma \in \operatorname{holim} F | \mathcal{D} , \qquad \sigma^{\%} \in \operatorname{holim} F^{\%} | \mathcal{C} .$$

They lead as before to a map

(12.11)
$$\operatorname{Str}(M \times \mathbb{R}^{i}; c) \longrightarrow F_{\%}(M \times \mathbb{R}^{i}; c)$$

where

$$\operatorname{Str}(M \times \mathbb{R}^i; c) := \operatorname{hofiber}_{M \times \mathbb{R}^i}[B\mathcal{C} \hookrightarrow B\mathcal{D}]$$

There is an obvious inclusion map from $\operatorname{Str}(M \times \mathbb{R}^i; c)$ to $\mathcal{S}^{\operatorname{rd}}(M \times \mathbb{R}^i; c)$. It is not clear whether this is a homology equivalence. But there is a fix. Let

$$\Gamma := \operatorname{hofiber}[\operatorname{Hom}(S^{i-1}) \to \operatorname{Hom}(S^{i-1})]$$

where $\mathcal{H}om(S^{i-1})$ is the homeomorphism group of S^{i-1} and $\mathsf{Hom}(S^{i-1})$ is the underlying discrete group. Then Γ is a topological group whose classifying space $B\Gamma$ has the homology of a point. The group Γ acts, via projection to the discrete group $\mathsf{Hom}(S^{i-1})$, on both $\mathrm{Str}(M \times \mathbb{R}^i; c)$ and $F_{\mathcal{H}}(M \times \mathbb{R}^i; c)$, preserving base points. The map (12.11) is a Γ -map and we obtain therefore an induced map of reduced Borel constructions

(12.12)
$$\operatorname{Str}(M \times \mathbb{R}^{i}; c)_{rh\Gamma} \longrightarrow F_{\%}(M \times \mathbb{R}^{i}; c)_{rh\Gamma}$$

Now $\operatorname{Str}(M \times \mathbb{R}^i; c)_{rh\Gamma}$ is a good combinatorial model for $\mathcal{S}^{\operatorname{rd}}(M \times \mathbb{R}^i; c)$ in the sense that the forgetful map

(12.13)
$$\operatorname{Str}(M \times \mathbb{R}^{i}; c)_{h\Gamma} \longrightarrow \mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^{i}; c)_{rh\Gamma}$$

is a homology equivalence. See appendix B. Therefore the homotopy pushout of

$$\mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^{i}; c)_{rh\Gamma} \longleftrightarrow \mathrm{Str}(M \times \mathbb{R}^{i}; c)_{rh\Gamma} \longrightarrow F_{\%}(M \times \mathbb{R}^{i}; c)_{rh\Gamma}$$

is homotopy equivalent to $F_{\%}(M \times \mathbb{R}^i; c)$ by an obvious inclusion. Another obvious inclusion, that of $\mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^i; c)$ into the same homotopy pushout, can therefore be described loosely in the form

(12.14)
$$\mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^i; c) \longrightarrow F_{\%}(M \times \mathbb{R}^i; c)$$

As in the case i = 0, the construction can be extended to the case where M is compact with nonempty boundary, and some relative cases. The details of this are left to the reader.

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CHAPTER 13

Algebraic models for structure spaces: Proofs

PROOF OF LEMMA 11.8. There is the following commutative square of controlled structure spaces:

(13.1)
$$\begin{array}{c} \mathcal{S}(M \times \mathbb{R}^{i} \times] - \infty, 1], \ M \times \mathbb{R}^{i} \times 1 \ ; \ c) \longrightarrow \mathcal{S}(M \times \mathbb{R}^{i+1} \ ; \ c) \\ \uparrow \\ \mathcal{S}(M \times \mathbb{R}^{i} \times [0, 1], \ M \times \mathbb{R}^{i} \times 1 \ ; \ c) \longrightarrow \mathcal{S}(M \times \mathbb{R}^{i} \times [0, \infty[\ ; \ c). \end{array}$$

Details on the spaces involved: All manifolds which appear in the diagram, such as $M \times \mathbb{R}^i \times [0, 1]$ etc., are codimension zero submanifolds of $M \times \mathbb{R}^{i+1}$. We make them into control spaces by taking closures inside $M * S^i$. This amounts to adding the equator S^{i-1} of S^i to $M \times \mathbb{R}^i \times [0, 1]$, the closed upper hemisphere of S^i to $M \times \mathbb{R}^i \times [0, \infty[$ and the closed lower hemisphere of S^i to $M \times \mathbb{R}^i \times [-\infty, 1]$. Beware that structure space notation of the form $\mathcal{S}(C, D)$ usually assumes that Cis a manifold, D is a codimension zero closed submanifold of ∂C and the structures considered are trivial over the closure of the complement of D in ∂C . Hence there is a homotopy fiber sequence $\mathcal{S}(C) \to \mathcal{S}(C, D) \to \mathcal{S}(D)$.

Details on the maps: The maps in the square are given by obvious extension of structures in all cases. For the left-hand vertical map for example, we extend by gluing with the trivial structure (identity) on $M \times \mathbb{R}^i \times] - \infty, 0]$. For the lower horizontal map, we extend a structure f by gluing with $\partial f \times id$, where ∂f is the boundary structure (on $M \times \mathbb{R}^i \times 1$) determined by f and id means the identity map on $[1, \infty]$.

In the square (13.1), the lower left-hand term is clearly homotopy equivalent to the controlled *h*-cobordism space $\mathcal{H}(M \times \mathbb{R}^i; c)$. The upper left-hand term is homotopy equivalent to $\mathcal{S}(M \times \mathbb{R}^i; c)$, via restriction of structures to the boundary. With this identification, the upper horizontal arrow is just $\times \mathbb{R}$ up to homotopy. The lower right-hand term is contractible. The square is homotopy cartesian. (Modulo a replacement of controlled structure spaces by the homotopy equivalent bounded structure spaces, this goes back to Anderson and Hsiang [4]. See also [36] for some added details.) Therefore the square amounts to a homotopy fibration sequence

$$\mathcal{H}(M \times \mathbb{R}^i; c) \longrightarrow \mathcal{S}(M \times \mathbb{R}^i; c) \xrightarrow{\times \mathbb{R}} \mathcal{S}(M \times \mathbb{R}^{i+1}; c).$$

This completes the proof in the case where $f: N \to M$ is the identity of M. The general case follows by a translation argument: there is a commutative diagram

(13.2)
$$\begin{array}{c} \mathcal{S}(N \times \mathbb{R}^{i+1}; c) \xrightarrow{\simeq} \mathcal{S}(M \times \mathbb{R}^{i+1}; c) \\ \times \mathbb{R} \uparrow & \times \mathbb{R} \uparrow \\ \mathcal{S}(N \times \mathbb{R}^{i}; c) \xrightarrow{\simeq} \mathcal{S}(M \times \mathbb{R}^{i}; c) \end{array}$$

where the horizontal arrows are given by composition with $f \times \mathbb{R}^i$ and $f \times \mathbb{R}^{i+1}$, respectively.

PROOF OF LEMMA 11.9. This is easy from definitions 9.5 and 9.8, proposition 7.14 and corollary 7.7. We allow ourselves to use twisted versions of proposition 7.14 and corollary 7.7, in the spirit of chapter 8; the twist is determined by ν , the normal bundle of M (not shown in the notation by our current conventions). From the first homotopy cartesian square in 7.14, twisted version, we get that the square

$$\begin{split} \Omega^{m+i}\mathbf{LA}_{\bullet}(M\times\mathbb{R}^{i},m+i,\,;\,c) & \xrightarrow{\text{forget}} \mathbf{A}(M\times\mathbb{R}^{i},m+i\,;\,c)^{h\mathbb{Z}/2} \\ & \swarrow \\ \Omega^{m+i+1}\mathbf{LA}_{\bullet}(M\times\mathbb{R}^{i+1},m+i+1\,;\,c) \xrightarrow{\text{forget}} \mathbf{A}(M\times\mathbb{R}^{i+1},m+i+1\,;\,c)^{h\mathbb{Z}/2} \end{split}$$

is also homotopy cartesian. (Compare the horizontal homotopy fibers.) Hence the homotopy fiber of the left-hand column, which we are interested in, is identified with the homotopy fiber of the right-hand column. By corollary 7.7, twisted version, this can be identified with the homotopy fiber of

where $\mathbb{Z}/2$ acts by flip on the Ω and the map ev is given by evaluation at the center of a loop. Applying Ω^{∞} to that homotopy fiber, we get

$$\Omega^{\infty+1}\mathbf{A}^{h}(M\times\mathbb{R}^{i+1};c) \simeq \Omega^{\infty}\mathbf{A}(M\times\mathbb{R}^{i};c)$$

(using corollary 7.7 again). Summarizing, we have established a homotopy fiber sequence of infinite loop spaces

$$\Omega^{\infty} \mathbf{A}(M \times \mathbb{R}^{i}; c)$$

$$\downarrow$$

$$\Omega^{\infty+m+i} \mathbf{L} \mathbf{A}_{\bullet}(M \times \mathbb{R}^{i}, m+i, ; c)$$

$$\downarrow \times \mathbb{R}$$

$$\Omega^{\infty+m+i+1} \mathbf{L} \mathbf{A}_{\bullet}(M \times \mathbb{R}^{i+1}, m+i+1; c)$$

The entire argument is compatible with assembly; i.e., we could instead have started with a homotopy cartesian square like the first one in 7.14, but with $\mathbf{LA}^{\%}$ and $\mathbf{A}^{\%}$ or $\mathbf{LA}_{\%}$ and $\mathbf{A}_{\%}$ instead of \mathbf{LA} and \mathbf{A} . So lemma 11.9 is established.

We turn to proposition 11.10. A simplification which comes from the proof of lemma 11.9 is that instead of working with the commutative square (2.9), we may work with the simplified version

(13.3)
$$\begin{aligned} \mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^{i+1}; c) & \longrightarrow & \Omega^{\infty} \mathbf{A}_{\%}(M \times \mathbb{R}^{i+1}, m+i+1; c)^{h\mathbb{Z}/2} \\ & \uparrow & \uparrow & \uparrow \\ \mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^{i}; c) & \longrightarrow & \Omega^{\infty} \mathbf{A}_{\%}(M \times \mathbb{R}^{i}, m+i; c)^{h\mathbb{Z}/2}. \end{aligned}$$

The horizontal arrows in diagram (13.3) are obtained from those in diagram (2.9) by composing with forgetful maps. This can be simplified some more by concatenating with the homotopy cartesian square of corollary 7.8. We also use abbreviations A, $A^{\%}$ and $A_{\%}$ for $\Omega^{\infty} \mathbf{A}$, $\Omega^{\infty} \mathbf{A}^{\%}$ and $\Omega^{\infty} \mathbf{A}_{\%}$, respectively, where possible. The result is, after a rotation,

(13.4)
$$\begin{array}{c} \mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^{i}; c) & \xrightarrow{\times \mathbb{R}} & \mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^{i+1}; c) \\ \downarrow^{j\chi_{\%}} & & \downarrow^{\chi_{\%}} \\ \mathrm{cone} \ A_{\%}(M \times \mathbb{R}^{i}; c) & \xrightarrow{\times \mathbb{R}} & A_{\%}(M \times \mathbb{R}^{i+1}; c) \end{array}$$

The labels $\chi_{\%}$ refer to constructions which extract controlled A-theory characteristics (playing off the excisive type $\chi^{\%}$ against the controlled homotopy invariant type χ , and using discrete approximation technology for that). The left-hand column is a map of type $\chi_{\%}$ followed by the inclusion

$$j: A_{\%}(M \times \mathbb{R}^i; c) \longrightarrow \text{cone } A_{\%}(M \times \mathbb{R}^i; c).$$

Therefore Φ_x , Ψ_y , $\Phi_{x,a}$ and $\Psi_{y,b}$ in the statement of proposition 11.10 can be redefined in terms of diagram (13.4), now as horizontal homotopy fibers or components of such. Another simplification comes from diagram (13.2): without loss of generality, x is the base point of $\mathcal{S}(M \times \mathbb{R}^{i+1}; c)$, so that y is the base point of $A_{\%}(M \times \mathbb{R}^{i+1}; c)$. Yet another small simplification: without loss of generality, Mis connected and based.

PROOF OF PROPOSITION 11.10, FIRST PART. We pass to horizontal homotopy fibers Φ_{\star} and Ψ_{\star} in diagram (13.4). Commutativity of the diagram leads to a map

$$(13.5) \qquad \Phi_{\star} \to \Psi_{\star}$$

We have already identified Φ_{\star} with a union of connected components of the controlled *h*-cobordism space $\mathcal{H}(M \times \mathbb{R}^{i}; c)$, and Ψ_{\star} of (13.5) is clearly homotopy equivalent to $\Omega A_{\%}(M \times \mathbb{R}^{i+1}; c)$, and also to $A_{\%}(M \times \mathbb{R}^{i}; c)$. After deleting some more connected components of the source, if necessary, we can therefore write (13.5) in the form

(13.6)
$$\mathcal{H}^{\mathrm{rd}}(M \times \mathbb{R}^{i}; c) \longrightarrow A_{\%}(M \times \mathbb{R}^{i}; c) .$$

Here π_0 of $\mathcal{H}^{\mathrm{rd}}(M \times \mathbb{R}^i; c)$ can be identified with the image of π_0 of $\mathcal{H}(M)$ in π_0 of $\mathcal{H}(M \times \mathbb{R}^i; c)$. The aim is to show that the map (13.6), constructed as a restriction of (13.5), is identical with a map

(13.7)
$$\chi_{\%}: \mathcal{H}^{\mathrm{rd}}(M \times \mathbb{R}^{i}; c) \longrightarrow A_{\%}(M \times \mathbb{R}^{i}; c).$$

which merits the label $\chi_{\%}$ because it is obtained by playing off excisive characteristics against (controlled) homotopy invariant characteristics. We also want to show that (13.5) itself induces an injection on π_0 . To clarify this last statement, we note that π_0 of $\Psi_{\star} \simeq A_{\%}(M \times \mathbb{R}^i; c)$ is

$$\begin{aligned} & \text{Wh}(\pi_1 M) & \text{if } i = 0 \\ & \tilde{K}_0(\mathbb{Z}\pi_1 M) & \text{if } i = 1 \\ & K_{1-i}(\mathbb{Z}\pi_1 M) & \text{if } i > 1 \end{aligned}$$

We aim to show that the map $\pi_0 \Phi_* \to \pi_0 \Psi_*$ induced by (13.5) associates to a controlled *h*-cobordism its controlled Whitehead torsion. This implies that it is injective, by the classification of controlled *h*-cobordisms.

Here the proof proper begins. In order to make a closer connection with diagram (13.1) we enlarge the left-hand column in diagram (13.4), without changing homotopy types, to get

(13.8)

$$\begin{aligned}
\mathcal{S}^{\mathrm{rd}}\left(M \times \mathbb{R}^{i} \times] - \infty, 1], M \times \mathbb{R}^{i} \times 1; c\right) &\longrightarrow \mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^{i+1}; c) \\
\downarrow^{j_{\chi_{\%}}} & \downarrow^{\chi_{\%}} \\
\Omega^{\infty}\left(\mathbf{A}_{\%}\left(\begin{array}{c}M \times \mathbb{R}^{i} \times 1 \\ \downarrow \\ M \times \mathbb{R}^{i} \times] - \infty, 1]; c\right) \cup \mathbf{C}\right) &\longrightarrow A_{\%}(M \times \mathbb{R}^{i+1}; c)
\end{aligned}$$

Here $\mathbf{C} := \operatorname{cone} \mathbf{A}_{\%}(M \times \mathbb{R}^{i}; c)$ and the union $\cup \mathbf{C}$ is taken along the common subspectrum $\mathbf{A}_{\%}(M \times \mathbb{R}^{i}; c)$. We use the embedding

$$\mathbf{A}_{\%}(M \times \mathbb{R}^{i}; c) \longrightarrow \mathbf{A}_{\%} \begin{pmatrix} M \times \mathbb{R}^{i} \times 1 \\ \downarrow \\ M \times \mathbb{R}^{i} \times] - \infty, 1 \end{bmatrix}; c \end{pmatrix}$$

induced by exact functors of type product with the cofibration $\{1\} \rightarrow]-\infty, 1]$. The top row in (13.8) is already familiar as the top row of diagram (13.1) and the lower row is an A-theory counterpart. (A small complication here: the map

$$\mathcal{S}^{\mathrm{rd}}\left(M \times \mathbb{R}^{i} \times] - \infty, 1\right], M \times \mathbb{R}^{i} \times 1; c\right)$$

$$\downarrow^{\chi_{\%}}$$

$$A_{\%}\left(\begin{array}{c}M \times \mathbb{R}^{i} \times 1\\ \downarrow\\ M \times \mathbb{R}^{i} \times] - \infty, 1\right]; c\right)$$

implicit in the left-hand column of (13.8) needs to be defined in a roundabout way as in (12.6) or (12.7), using $A_{\%}$ instead of $F_{\%}$.) In order to make the connection with diagram (13.1) closer still, we think of (13.8) as one face of a cubical diagram whose opposite face is

The upper row in (13.9) is already familiar as the lower row of diagram (13.1) and the lower row in (13.9) is an A-theory counterpart of the upper row. The promised cube is a map from (13.9) to (13.8). The cube therefore has a geometric face

$$(13.10) \qquad \begin{array}{c} \mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^{i} \times] - \infty, 1], \ M \times \mathbb{R}^{i} \times 1; \ c) \longrightarrow \mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^{i+1}; \ c) \\ \uparrow \\ \mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^{i} \times [0, 1], \ M \times \mathbb{R}^{i} \times 1; \ c) \longrightarrow \mathcal{S}(M \times \mathbb{R}^{i} \times [0, \infty[; \ c)) \end{array}$$

which is a sub-square of (13.1), obtained by selecting some connected components in each term of (13.1). Therefore by lemma 11.8 it is almost a homotopy pullback square. More precisely, the map induced from the initial (lower left-hand) term to the homotopy pullback of the other three terms has all homotopy fibers empty or contractible. (It might not induce a surjection on π_0 .) The cube also has an *A*-theory face

and this is a homotopy pullback square by inspection (using the additivity theorem). The conclusion from this diagrammatic reasoning is that the map (13.5) agrees with

from the cube, after appropriate restriction. It is clear that

$$\mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^i \times [0,1], M \times \mathbb{R}^i \times 1; c) \simeq \mathcal{H}^{\mathrm{rd}}(M \times \mathbb{R}^i; c).$$

The canonical inclusion

 $\mathbf{A}_{\%}(M \times \mathbb{R}^{i} \times [0,1]; c) \longrightarrow \mathbf{A}_{\%}\left(M \times \mathbb{R}^{i} \times 1 \rightarrow M \times \mathbb{R}^{i} \times [0,1]; c\right) \cup \mathbf{C}$

is a homotopy equivalence, and $A_{\%}(M \times \mathbb{R}^i \times [0,1]; c) \simeq A_{\%}(M \times \mathbb{R}^i; c)$. With some elementary simplifications we are therefore allowed to write (13.12), alias restriction of (13.5), in the form (13.7). This was one of our declared goals. Our construction or description of (13.7) deviates slightly from the standard because we are in a controlled setting, and because we have used discrete approximations of homeomorphism groups of manifolds as in [16] instead of the technology of simplicial sets and simple maps [27], [28].

It remains to prove that the map $\pi_0 \Phi_* \to \pi_0 \Psi_*$ induced by (13.5) is the controlled Whitehead torsion map. This comes from a mild improvement on the diagrammatic reasoning just developed. Let us admit that by using

$$\mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^i \times [0, 1], M \times \mathbb{R}^i \times 1; c)$$

as the initial term in square (13.10), we did not make the best choice. Instead we could have used

$$T \cup \mathcal{S}^{\mathrm{rd}}(\mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^i \times [0, 1], M \times \mathbb{R}^i \times 1; c))$$

where T is any subset of $\mathcal{S}(M \times \mathbb{R}^i \times [0, 1], M \times \mathbb{R}^i \times 1; c)$ which has exactly one element in each connected component mapping to $\mathcal{S}^{rd}(M \times \mathbb{R}^i; c) \subset \mathcal{S}(M \times \mathbb{R}^i; c)$ under the forgetful map (and no other elements). If we enlarge (13.10) in this manner, we have a slightly better approximation to a homotopy pullback square: the canonical map from initial term to the homotopy pullback of the other three terms is still a homotopy equivalence on base point components, but it also induces a bijection on π_0 . It is straightforward to extend the map from (13.10) to (13.11) which we already have to this enlargement of (13.10), using $\chi_{\%}$ constructions as before. This allows us to see what (13.5) does on π_0 .

PROOF OF PROPOSITION 11.10, SECOND PART. Here the goal is to show that the map (13.5) restricts to a $(k_N + i + 1)$ -connected map

$$\Phi_{\star,a} \to \Psi_{\star,b}$$

in the notation of proposition 11.10. Note that N depends on a. There is an easy reduction to the case where a is the base point component, a = 0. This is again a translation argument. The component a of the homotopy fiber Φ_{\star} is homotopy equivalent to the base point component of

hofiber_{*}
$$[\mathcal{S}(N \times \mathbb{R}^i; c) \to \mathcal{S}(N \times \mathbb{R}^{i+1}; c)]$$

Therefore replacing M by N will do the trick. The details are left to the reader.

We are now looking at $\Phi_{\star,0} \to \Psi_{\star,0}$ and we may think of that as the restriction of (13.7) to the base point components. Therefore it only remains to show that the restriction of (13.7) to base point components is $(k_M + i + 1)$ -connected. We use induction on *i*. For i = 0 we know that (13.7) is Waldhausen's map and so is $(k_M + 1)$ -connected after restriction to base point components; see beginning of section 10 in [10]. For i > 0, there is a homotopy pullback square of controlled *h*-cobordism spaces and inclusion-induced maps,

$$(13.13) \qquad \begin{array}{c} \mathcal{H}(M \times \mathbb{R}^{i-1} \times] - \infty, 1]; c) \longrightarrow \mathcal{H}^{\mathrm{rd}}(M \times \mathbb{R}^{i}; c) \\ \uparrow \\ \mathcal{H}(M \times \mathbb{R}^{i-1} \times [-1, 1]; c) \longrightarrow \mathcal{H}(M \times \mathbb{R}^{i-1} \times [-1, \infty[; c]). \end{array}$$

This is closely related to the homotopy pullback square

which we have from theorem 7.1. In both squares, the off-diagonal terms are contractible. Using discrete approximation of homeomorphism groups as in chapters A and B, we can construct a map of type $\chi_{\%}$ from a mildly corrupted version of (13.13) to (13.14). For our purposes, a sufficiently mild form of corruption is to preserve all terms in (13.13) except the initial term, to preserve the base component in the initial term as well, and to replace each other component of the initial term by a selected point in it. The existence of such a map implies that if

$$\mathcal{H}^{\mathrm{rd}}(M \times [-1,1] \times \mathbb{R}^{i-1}; c) \xrightarrow{\chi_{\%}} A_{\%}(M \times [-1,1] \times \mathbb{R}^{i-1}; c)$$

is $(k_M + i)$ -connected on base point components, then the looping of (13.7) induces a bijection on π_0 and is also $(k_M + i)$ -connected on base points components. The looping of (13.7) is therefore unconditionally $(k_M + i)$ -connected, and so (13.7) is $(k_M + i + 1)$ -connected on base point components. This is our induction step: a reduction from i to i - 1 for the price of replacing M by $M \times [-1, 1]$.

REMARK 13.15. Although formula (2.1) is meant as a quotation, a historical essay around it is in order, with a view to the proof of lemma 11.11. Perhaps the best known, most polished and least complicated method for establishing a homotopy fiber sequence

$$\widetilde{\mathcal{S}}^{s}(M \times \mathbb{R}^{i}; c) \longrightarrow \Omega^{\infty + m + i} \mathbf{L}^{s}_{\bullet \%}(M \times \mathbb{R}^{i}; c) \longrightarrow L_{0}(\mathbb{Z})^{\pi_{0}M}$$

is the one developed by Ranicki [21, §18]. We summarize the key points, referring to [21] for definitions and clarifications.

- (1) The cases i > 0 can be reduced to the case i = 0 by a torus trick.
- (2) A compact polyhedron X and a homotopy equivalence $e: M \to X$ transverse to the triangulation of X are chosen. A degree one normal map $f: N \to M$ between closed manifolds determines (if $f \circ e$ is transverse to the triangulation of X) chain complexes C(N) and C(M) "dissected" over X, with dissected nondegenerate symmetric structures of formal dimension m. The map $f_*: C(N) \to C(M)$ respects the symmetric structures and so determines a splitting

$$C(N) \simeq D \oplus C(M)$$

where the "kernel" D is again dissected and comes with a nondegenerate symmetric structure. But D is globally (after assembly) contractible. Hence the dissected symmetric structure has an automatic refinement to a dissected quadratic structure.

(3) $\mathbf{L}_{\bullet}^{\%}(M)$ admits a description as the bordism theory of chain complexes dissected over X, with a dissected quadratic Poincaré structure. With that description, the assembly map is induced by the assembly ("universal" assembly) of dissected chain complexes.

(4) The local degree homomorphism from $\pi_m \mathbf{L}_{\bullet}^{\mathscr{H}}(M)$ to $L_0(\mathbb{Z})^{\pi_0 M}$ is zero on elements determined by degree one normal maps $f: N \to M$. (The reason is that an oriented map f between manifolds of the same dimension which has "global" degree one, in the sense that it respects fundamental classes, will also have local degree one, i.e., the generic cardinality of $f^{-1}(x)$ is 1 for any x in the target. It is this relationship between global and local degree which has been shown to fail spectacularly [5] in the world of ANR homology manifolds.)

These ideas, generalized to a parameterized setting (where the parameter space is Δ^k for k = 0, 1, 2, ...) lead to a map

$$\widetilde{\mathcal{S}}^{s}(M \times \mathbb{R}^{i}; c) \longrightarrow \Omega^{\infty + m + i} \mathbf{L}^{s}_{\bullet\%}(M \times \mathbb{R}^{i}; c).$$

Proving that the map is a homotopy equivalence except for a deviation in π_0 is another matter and there is no need to go into that here. What we need to do is this: unravel each of the items (1)–(3) and relate it to the methods (e.g. the *characteristic element* method and the *control method*) which we have favored in this paper.

(1a). The torus trick consists in using that $\mathcal{S}^{s}(M \times \mathbb{R}^{i}; c)$ and $\mathbf{L}^{s}_{\mathfrak{H}}(M \times \mathbb{R}^{i}; c)$ are homotopy retracts of $\mathcal{S}^{s}(M \times (S^{1})^{i})$ and $\mathbf{L}^{s}_{\mathfrak{H}}(M \times (S^{1})^{i})$, respectively. The retracting maps

$$\begin{aligned} \mathcal{S}^s(M\times (S^1)^i) &\longrightarrow \mathcal{S}^s(M\times \mathbb{R}^i\,;\,c), \\ \mathbf{L}^s_{\bullet\%}(M\times (S^1)^i) &\longrightarrow \mathbf{L}^s_{\bullet\%}(M\times \mathbb{R}^i\,;\,c) \end{aligned}$$

are obvious transfer maps in both cases.

(2a). Dissection theory works with retractive spectra over X and visible symmetric structures just as well as with chain complexes and symmetric structures. It remains true that a dissected visible symmetric structure on a dissected retractive spectrum Y over X (subject to some finiteness conditions) automatically lifts to a dissected quadratic structure if Y is globally weakly equivalent to zero.

(3a). The interpretation of the assembly map in terms of assembly of dissected chain complexes works just as well with dissected retractive spaces and spectra. Given a degree one normal map $f: N \to M$ which is a homotopy equivalence, there are two slightly different ways of using the dissection argument to extract *L*-theoretic information.

• One way is to form the visible symmetric kernel (mapping cone of the stable Umkehr map $M \amalg X \to N \amalg X$, with a nondegenerate visible symmetric structure). It is dissected over X along with its nondegenerate visible symmetric structure. Since f is a homotopy equivalence, the kernel is globally contractible and the dissected nondegenerate visible symmetric structure on it automatically lifts to a dissected nondegenerate quadratic structure. The global contractibility of the kernel implies a preferred global nullbordism of the kernel (i.e., a nullbordism of the assembled kernel with the assembled quadratic structure). The dissected kernel and the global nullbordism then determine a point in the homotopy fiber (over the base point, zero) of the assembly map

$$\Omega^{\infty+m} \mathbf{L}_{\bullet}^{\%}(X) \longrightarrow \Omega^{\infty+m} \mathbf{L}_{\bullet}(X) \,.$$

• Another way is to note that $M \amalg X$ and $N \amalg X$ can themselves be dissected over X, and come with dissected nondegenerate visible symmetric

structures. Before assembly, these may not be equivalent; after assembly they are certainly equivalent via $f \amalg \text{id} : N \amalg X \to M \amalg X$. Hence these data determine a point in a homotopy fiber of the assembly map

$$\Omega^{\infty+m}\mathbf{VL}^{\bullet\%}(X)\longrightarrow \Omega^{\infty+m}\mathbf{VL}^{\bullet}(X).$$

This time we take the homotopy fiber over the point in $\Omega^{\infty+m}\mathbf{VL}^{\bullet}(X)$ determined by $M \amalg X$ with its (assembled) nondegenerate visible symmetric structure.

There is an obvious compatibility between the two methods. By theorem 3.7, we lose no *L*-theoretic information by relying on the second method.

But now another little problem remains. We have two descriptions of the assembly map in *L*-theory, one in terms of dissections and another one using control. How are they related? An easy way to make a connection is to use both approaches simultaneously. Very schematically, we have one description of $VL^{\bullet\%}$ -theory as

 $\mathbf{VL}^{\bullet}(X, \text{dissected})$

and another as the homotopy fiber of an inclusion

 $\mathbf{VL}^{\bullet}(X) \longrightarrow \mathbf{VL}^{\bullet}(X \times [0, 1[, \text{controlled}),$

as in chapter 10. There is a third description of $\mathbf{VL}^{\bullet\%}(X)$ as the homotopy fiber of an inclusion

 $\mathbf{VL}^{\bullet}(X, \text{dissected}) \longrightarrow \mathbf{VL}^{\bullet}(X \times [0, 1], \text{ controlled and dissected}).$

(The dissections are always over X, even where we are dealing with retractive spaces over $X \times [0, 1[$. *Control* refers to the control space $\mathbb{J}X$.) Validating this third formula amounts to showing that

 $\mathbf{VL}^{\bullet}(X \times [0, 1], \text{controlled and dissected})$

is contractible. (This is left to the reader.) This makes the connection which we were after. Moreover it does that in such a way that the two standard methods (by dissection and by control) of lifting manifold signatures in VL-spaces across the assembly map are seen to agree.

PROOF OF LEMMA 11.11. The homotopy colimit of

$$\mathbf{A}_{\%}(M) \xrightarrow{\times \mathbb{R}} \mathbf{A}_{\%}(M \times \mathbb{R}; c) \xrightarrow{\times \mathbb{R}} \mathbf{A}_{\%}(M \times \mathbb{R}^{2}; c) \to \cdots$$

is contractible by lemma 7.6. It follows that

hocolim_i ($\mathbf{A}_{\%}(M \times \mathbb{R}^{i}, m+i; c)$)_{hZ/2}

is also contractible. (The dimension indicator m + i here specifies the involution or the *SW*-product, which is obtained from the standard one by (m+i)-fold looping). Therefore the inclusion

$$\operatorname{hocolim}_{i}(\mathbf{A}_{\%}(M \times \mathbb{R}^{i}, m+i; c))^{h\mathbb{Z}/2} \longrightarrow \operatorname{hocolim}_{i}(\mathbf{A}_{\%}(M \times \mathbb{R}^{i}, m+i; c))^{th\mathbb{Z}/2}$$

is a homotopy equivalence. Therefore the forgetful map

hocolim_i $F_{\%}(M \times \mathbb{R}^{i}; c) \longrightarrow \text{hocolim}_{i} \Omega^{\infty + m + i} \mathbf{L}_{\bullet\%}(M \times \mathbb{R}^{i}; c)$

is also a homotopy equivalence. Hence in the limit $i \to \infty$, it does not matter whether we use the map φ in construction 11.7 or a simplified version

(13.16)
$$\mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^{i}; c) \longrightarrow \Omega^{\infty + m + i} \mathbf{L}_{\bullet\%}(M \times \mathbb{R}^{i}; c)$$

constructed purely in terms of VL-theoretic signatures (as in chapters 9 and 10, but without any algebraic K-theory). When i > 0, this map has an automatic lift

(13.17)
$$\mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^{i}; c) \longrightarrow \Omega^{\infty + m + i} \mathbf{L}^{cs}_{\bullet \%}(M \times \mathbb{R}^{i}; c)$$

because all components of $S^{\mathrm{rd}}(M \times \mathbb{R}^i; c)$ are represented by (controlled) simple structures. Furthermore, it is easy to extend (13.17) to the block structure space

$$\widetilde{\mathcal{S}}^{\mathrm{rd}}(M \times \mathbb{R}^i; c)$$
.

The reason is that we can extend (13.17) by adding on another simplicial direction. In other words, there are maps

(13.18)
$$\mathcal{S}^{\mathrm{rd}}(M \times \Delta^k_! \times \mathbb{R}^i; c) \longrightarrow \Omega^{\infty + m + i} \mathbf{L}^{cs}_{\bullet \%}(M \times \Delta^k_! \times \mathbb{R}^i; c)$$

for every $k \ge 0$, generalizing (13.17); the notation $\Delta_!^k$ means that we think of Δ^k as a diagram of manifolds (the faces of any codimension), not as a single manifold with boundary. By taking geometric realizations over k (and noting, as we have done before, that on the target side all the face operators are homotopy equivalences), we obtain a single map

$$\widetilde{\mathcal{S}}^{\mathrm{rd}}(M \times \mathbb{R}^{i}; c) \longrightarrow \Omega^{\infty + m + i} \mathbf{L}_{\bullet \%}(M \times \Delta^{k}_{\mathsf{I}} \times \mathbb{R}^{i}; c).$$

Now we let the new simplicial direction take over by restricting (13.18) to appropriate 0-skeletons for each k, and noting that that does not affect the homotopy type of the geometric realization over k. Then what we have is just the standard map of (2.1), restricted to a union of connected components, namely $\mathcal{S}^{rd} \subset \mathcal{S}^{cs}$. This identification relies on remark 13.15.

PROOF OF LEMMA 11.6. Without loss of generality, ${\cal M}$ is connected. There is a commutative square

(13.19)

$$\pi_{0}(\mathbf{A}^{h}_{\%}(M,\nu,m)_{h\mathbb{Z}/2})$$

$$\downarrow$$

$$\pi_{0}\mathcal{S}(M) \longrightarrow \pi_{m}\mathbf{LA}_{\bullet\%}(M,\nu,m)$$

$$\downarrow$$

$$\pi_{0}\widetilde{\mathcal{S}}^{h}(M) \longrightarrow \pi_{m}\mathbf{L}_{\bullet\%}^{h}(M,w_{\nu}) \xrightarrow{\text{loc. deg.}} \mathbb{Z}$$

with exact lower row and exact middle column. The middle column is part of the long exact sequence in homotopy groups of a homotopy fiber sequence

$$\Omega^m \mathbf{LA}_{\bullet\%}(M,\nu,m) \longrightarrow \Omega^m \mathbf{L}^h_{\bullet\%}(M,w_\nu) \xrightarrow{(1.2)} S^1 \wedge \mathbf{A}^h_{\%}(M,\nu,m)_{h\mathbb{Z}/2} .$$

We need to understand the composition

$$\widetilde{\mathcal{S}}^{h}(M) \xrightarrow{(1.6)} \Omega^{\infty+m} \mathbf{L}^{h}_{\bullet\%}(M, w_{\nu}) \xrightarrow{(1.2)} \Omega^{\infty-1}(\mathbf{A}^{h}_{\%}(M, \nu, m)_{h\mathbb{Z}/2})$$

on the 1-skeleton, as a map taking the subspace $\mathcal{S}(M)$ to the base point. For this purpose we may replace the right-hand term by the space $B(\mathbb{Z} \otimes_{\mathbb{Z}} Wh)$, where Wh

is the Whitehead group of $\pi_1 M$ with $\mathbb{Z}/2$ acting on Wh by the standard involution, and on \mathbb{Z} by $(-1)^m$. Then we have

$$(\mathcal{S}^h(M), \mathcal{S}(M)) \longrightarrow (B(\mathbb{Z} \otimes_{\mathbb{Z}} \mathrm{Wh}), \star)$$

and this is easy to understand. (See remark 13.20 below.) It takes all 0-simplices in the source to the base point of $B(\mathbb{Z} \otimes_{\mathbb{Z}} Wh)$. It takes a 1-simplex corresponding to an *h*-cobordism (over *M*) of torsion *t* to the loop in $B(\mathbb{Z} \otimes_{\mathbb{Z}/2} Wh)$ determined by $1 \otimes t$. Therefore the left-hand square in (13.19) is a pullback square. \Box

REMARK 13.20. It cannot hurt to say once again that we like to construct $\widetilde{S}^h(M)$ as the geometric realization of a simplicial *space*. When M is closed, the space of k-simplices can be described (for example) as

hofiber
$$_{M \times \Delta^k}[B\mathcal{C}_k \to B\mathcal{D}_k]$$

where

- C_k is the discrete groupoid whose objects are compact manifolds modelled on $\mathbb{R}^m \times \Delta^k$ (that is, compact manifolds with an atlas having charts in $\mathbb{R}^m \times \Delta^k$ and changes of charts respecting the (m + k - 1)-dimensional faces), and whose morphisms are homeomorphisms respecting the face structure;
- \mathcal{D}_k is a Poincaré duality space analogue of \mathcal{C}_k (still a discrete category, but not strictly a groupoid).

This description is also useful for us because the space of 0-simplices is homology equivalent to $\mathcal{S}(M)$, and so can be used as a good substitute for $\mathcal{S}(M)$. In the proof above, where 0-simplices and 1-simplices are mentioned, think space of 0-simplices and space of 1-simplices.

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APPENDIX A

Homeomorphism groups of some stratified spaces

When we evaluate characteristic invariants of manifolds on a manifold M, we expect and normally have enough invariance under the discrete group of automorphisms of M. It can be very hard to establish invariance under the topological group of automorphisms of M. In [10] we relied on the Mather-McDuff-Segal-Thurston theory [16] which gives homology isomorphisms between the classifying spaces of the discrete and topological automorphism groups of a compact topological manifold M. We need a similar tool to prove invariance of various controlled characteristics of $M \times \mathbb{R}^i$ (for compact M) under the topological group of *controlled* homeomorphisms $M \times \mathbb{R}^i \to M \times \mathbb{R}^i$. Recall that $M \times \mathbb{R}^i$ is identified with the nonsingular part of the control space $(M * S^{i-1}, S^{i-1})$. Three different options come to mind:

- (i) Look for an extension of the Mather-McDuff-Segal-Thurston theory to controlled automorphisms, hoping that the classifying spaces of the discrete and topological groups of controlled automorphisms of $M \times \mathbb{R}^i$ have the same homology.
- (ii) Try to make the Mather-McDuff-Segal-Thurston theory work for automorphisms of the control space $(M * S^{i-1}, S^{i-1})$, and compare with the existing theory for automorphisms of the singular part S^{i-1} .
- (iii) Attempt a reduction to the control-free setting by using the belt buckle trick, which implies that the classifying space of the topological controlled automorphism group of $M \times \mathbb{R}^i$ is a homotopy retract of the classifying space of the topological automorphism group of $M \times \mathbb{R}^i/\mathbb{Z}^i$.

In the end we decided for option (ii). We could not handle (i). The relationship between (i) and (ii) is nevertheless simple. In (i), the focus is on automorphisms of the control space $(M * S^{i-1}, S^{i-1})$ which extend the identity on the singular part S^{i-1} ; in (ii), all automorphisms of $(M * S^{i-1}, S^{i-1})$ as a control space are allowed. Option (iii) looks like an attractive shortcut. We decided against it mainly because it does not respect symmetries of \mathbb{R}^i such as linear automorphisms. These symmetries are important to us in view of [**36**]. Another argument against (iii) is that it generates a demand for belt buckle machinery on the algebraic side, so that it is not really a shortcut.

The present chapter is therefore a short review of [16], the main point being that everything carries over from a manifold setting to a more general setting where the geometric objects are well-behaved stratified spaces with manifold strata. We wonder whether a more abstract formulation can be given, perhaps in sheaf-theoretic language. In any case we assume some familiarity with [16].

The main result of the chapter is theorem A.15 with $A = \emptyset$. In chapter B we spell out what it means for controlled automorphism spaces and structure spaces.

DEFINITION A.1. A stratified space is a space X together with a locally finite partition into locally closed subsets X_i , called the *strata*, such that the closure of each stratum X_i in X is a union of strata. By an *automorphism* of a stratified space X, we understand a homeomorphism $X \to X$ mapping each stratum X_i to itself.

DEFINITION A.2. A stratified space is a TOP stratified space if it is paracompact Hausdorff and each stratum X_i is a manifold (of some dimension n_i , with empty boundary).

For us, the most important example of a TOP stratified space is the join

$$X = M * S^{i-1}$$

where M is a closed manifold. We partition this into two strata. One of these is the embedded copy of S^{i-1} . The other stratum, i.e. the complement of S^{i-1} in $M * S^{i-1}$, can be identified with $M \times \mathbb{R}^i$. Automorphisms of $M * S^{i-1}$, with this stratification, must map the sphere S^{i-1} to itself. (This is not a vacuous condition since, for example, M could also be a sphere in which case $M * S^{i-1}$ is homeomorphic to a sphere.)

Another important type of stratified space with two strata is furnished by manifolds with boundary. The boundary can be regarded as one stratum, the complement of the boundary as the other stratum.

Combining these two examples, one has a canonical stratification of $M * S^{i-1}$ into three strata when M is a manifold with nonempty boundary.

DEFINITION A.3. The open cone cL on a stratified space L is defined as the quotient of $L \times [0, 1[$ by $L \times \{0\}$. It is canonically stratified with strata $cL_i \setminus \star$ and \star , where L_i denotes a stratum of L and \star is the base of the cone.

The next definition is due to Siebenmann [24]:

DEFINITION A.4. A stratified space X is *locally conelike* if, for each stratum X_i and each $x \in X_i$, there exist an open neighborhood U of x in X_i , a compact stratified space L and a stratification-preserving homeomorphism (relative to U) of $cL \times U$ with an open neighborhood of x in X.

For a topological manifold M and i > 0, the stratified space $X = M * S^{i-1}$ with two strata (as explained above) is locally conelike.

NOTATION A.5. A locally conclike TOP stratified space will be called a CS space. (This is slightly different from Siebenmann's definition of a CS space, in which there can be only one stratum of dimension n for each $n \ge 0$, but the partition into strata is not required to be locally finite.)

Generalizing some of the conventions of [16], we denote by $\mathcal{H}om(X)$ and $\mathsf{Hom}(X)$ the topological group of automorphisms of a CS space X (with the compact-open topology), and the underlying discrete group, respectively. More generally, for a closed subset A of X let $\mathcal{H}om(X, \operatorname{rel} A)$ and $\mathsf{Hom}(X, \operatorname{rel} A)$ be the topological group and the underlying discrete group of automorphisms of X which agree with the identity in some *neighborhood* of A. Following [16], we topologize $\mathcal{H}om(X, \operatorname{rel} A)$ as a direct limit

$$\operatorname{colim}_U \left\{ h \in \mathcal{H}om(X) \middle| h(x) = x \text{ for } x \in U \right\}$$

where $\{h \in \mathcal{H}om(X) | h(x) = x \text{ for } x \in U\}$ has the subspace topology inherited from $\mathcal{H}om(X)$ and U runs over the set of all open neighborhoods of A in X. (Hence the inclusion map from $\mathcal{H}om(X, \text{rel } A)$ to $\mathcal{H}om(X)$ is continuous, but it need not be an embedding.) Let

$$\bar{B}\mathcal{H}om(X, \mathrm{rel}\ A)$$

be the homotopy fiber of the inclusion $BHom(X, rel A) \to BHom(X, rel A)$.

LEMMA A.6. For any compact $K \subset \mathcal{H}om(X, \operatorname{rel} A)$, there exists a neighborhood U of A in X such that $f|U = \operatorname{id}$ for all $f \in K$.

PROOF. Suppose that for some compact $K \subset \mathcal{H}om(X, \operatorname{rel} A)$ there is no such U. Then there exists $x \in A$ and a sequence $(y_n)_{n \in \mathbb{N}}$ in X converging to x, and a sequence $(f_n)_{n \in \mathbb{N}}$ in K such that $f_n(y_n) \neq y_n$ for all n. Choose a metric d on X inducing the given topology. Let $V_n \subset \mathcal{H}om(X, \operatorname{rel} A)$ consist of all g such that $d(g(y_k), y_k) < d(f_k(y_k), y_k)$ for all $k \geq n$. Then V_n is open in $\mathcal{H}om(X, \operatorname{rel} A)$ by definition of the topology on $\mathcal{H}om(X, \operatorname{rel} A)$. Clearly $V_n \subset V_{n+1}$ and

$$\bigcup_{n} V_n = \mathcal{H}om(X, \text{rel } A)$$

so that the V_n for $n \in \mathbb{N}$ constitute an open covering of $\mathcal{H}om(X, \operatorname{rel} A)$. But $f_n \notin V_n$, so that none of the V_n contains K. Therefore K is not compact. \Box

DEFINITION A.7. A clean subspace of a stratified space X is a closed subspace Y of X whose frontier $\operatorname{Fr}(Y)$ in X admits a stratification and a bicollar neighborhood $V \cong \operatorname{Fr}(Y) \times \mathbb{R}$ in X (where the homeomorphism $V \cong \operatorname{Fr}(Y) \times \mathbb{R}$ respects the stratifications.)

One example of a clean subspace of a stratified space which we need is as follows. Let X be a CS space. Let z be a point in a stratum X_i , let U be a neighborhood of x in X_i and let $e: cL \times U \to X$ be an open embedding as in definition A.4. Let $U' \subset U$ be an open ball containing z whose closure in U is a (compact) disk and let $c'L \subset cL$ be an open subcone $L \times [0, r[$ for some r with $0 < r < \infty$. We call the image of $c'L \times U'$ under the embedding $e: cL \times U \to X$ a quasi-ball (about z) in X.

LEMMA A.8. The complement of a quasi-ball in a CS space X is a clean subset of X.

PROOF. The quasi-ball is contained in an open subset of X which is identified with $cL \times U$. (We may as well assume $X = cL \times U$.) Without loss of generality, U can be identified with a euclidean open ball of radius 1. Now the spaces cL and U come with evident actions of the topological monoid [0, 1[, a submonoid of (\mathbb{R}, \cdot) , and so we get a diagonal action of [0, 1[on $cL \times U$. Without loss of generality the quasi-ball is $\frac{1}{2} \cdot (cL \times U)$. A bicollar for its frontier is then defined by the embedding

$$(x,t) \mapsto \frac{e^t}{e^t + 1} \cdot x \in cL \times U$$

for $t \in \mathbb{R}$ and x in the frontier.

LEMMA A.9. Any point in a CS space has an open neighborhood which is a quasi-ball. $\hfill \Box$

The following theorem generalizes a result of Mather in [14] which may have been the starting point of the theory developed in [16]. Mather's result serves in [16] as an induction beginning in a handle induction, and our generalization has a similar purpose here.

THEOREM A.10. Let A be closed in a CS space X. Assume that the complement of A is a quasi-ball. Then BHom(X, rel A) is acyclic, BHom(X, rel A) is contractible and hence $\overline{B}Hom(X, rel A)$ is acyclic.

PROOF. The contractibility of $B\mathcal{H}om(X, \text{rel } A)$ is a consequence of an Alexander trick showing that in fact $\mathcal{H}om(X, \text{rel } A)$ is contractible.

Our proof that BHom(X, rel A) is acyclic is a straightforward generalization of Mather's proof in the unstratified case [14]. As in the proof of lemma A.8 we can assume that $X = cL \times U$ and that the quasi-ball is $E = \frac{1}{2} \cdot (cL \times U)$. We are looking at automorphisms $h: cL \times U \to cL \times U$ which are the identity outside $r_h \cdot (cL \times U)$ for some positive $r_h < 1/2$ depending on h. Copying Mather's argument in the unstratified case, we begin by choosing a sequence of disjointly embedded codimension zero disks $D_i \subset \frac{1}{2}U$, for $i = 1, 2, 3, \ldots$ Let U_i be the relative interior of D_i in U. Then $E_i = (2^{-i} \cdot cL) \times U_i$ is a quasi-ball in E for each i, and $E_1, E_2,$ E_3, \ldots have disjoint compact closures in E. These choices can easily be made in such a way that there exists $h_0 \in \text{Hom}(X, \text{rel } X \setminus E) = \text{Hom}(X, \text{rel } A)$ which maps E_i onto E_{i+1} for $i = 1, 2, 3, \ldots$.

Let $G_i = \text{Hom}(X, \text{rel } X \setminus E_i)$. The inclusion

$$BG_i \to BHom(X, rel X \smallsetminus E)$$

induces a surjection in integer homology. The reasoning is that any finitely generated subgroup of $\operatorname{Hom}(X, \operatorname{rel} X \smallsetminus E)$ is conjugate in $\operatorname{Hom}(X, \operatorname{rel} X \smallsetminus E)$ to a subgroup of G_i . (Namely, for any g_1, g_2, \ldots, g_k in $\operatorname{Hom}(X, \operatorname{rel} X \smallsetminus E)$ there is r < 1/2 such that g_1, \ldots, g_k agree with the identity outside $r \cdot (cL \times U)$. Then there exists $g \in \operatorname{Hom}(X, \operatorname{rel} X \smallsetminus E)$ mapping the closure of $r \cdot (cL \times U)$ to E_i , so that conjugation with g takes g_1, \ldots, g_k to the subgroup $\operatorname{Hom}(X, \operatorname{rel} X \smallsetminus E_i)$.)

The next step is to introduce the subgroup G of $\mathsf{Hom}(X, \mathrm{rel}\ X \setminus E)$ consisting of all automorphisms h which have h(x) = x for all $x \notin \bigcup_i E_i$. Then the restriction homomorphisms $h \mapsto h|E_i$ lead to isomorphisms

$$G \cong \prod_{i=1}^{\infty} G_i \cong \prod_{i=1}^{\infty} G_1.$$

(The first isomorphism uses the fact that a bijective continuous map between compact Hausdorff spaces is a homeomorphism. The second isomorphism uses conjugation with appropriate powers of $h_{0.}$) Let $\sigma: G \to \operatorname{Hom}(X, \operatorname{rel} X \setminus E)$ be the inclusion. Define homomorphisms

$$u, v, w: G_1 \longrightarrow G \cong \prod_{i=1}^{\infty} G_1$$

by u(g) = (g, 1, 1, ...), v(g) = (1, g, g, ...) and w(g) = (g, g, g, ...). Using conjugation with the element h_0 , we see that σv and σw induce the same homomorphism in integral homology,

$$\sigma_* v_* = \sigma_* w_* \colon H_*(BG_1) \longrightarrow H_*(B\operatorname{Hom}(X, \operatorname{rel} X \setminus E)).$$

Now assume inductively that our vanishing statement has been established in degrees s for 0 < s < k. In particular $H_s(BHom(X, rel X \setminus E)) = 0$ and $H_s(BG_1) = 0$ for 0 < s < k. Writing w as a composition

$$G_1 \xrightarrow{\text{diagonal}} G_1 \times G_1 \xrightarrow{\text{id} \times v'} G_1 \times G'$$

where $G' = \prod_{i=2}^{\infty}$, and using the Künneth formula for the homology of $BG_1 \times BG_1$, we get for any $z \in H_k(BG_1)$ that

$$w_*(z) = (z \times 1) + (1 \times v'_*(z)) \in H_k(B(G_1 \times G')) = H_k(BG).$$

Hence $\sigma_* w_*(z) = \sigma_* u_*(z) + \sigma_* v_*(z)$ in $H_k(B\text{Hom}(X, \text{rel } X \setminus E))$. Since $\sigma_* v_* = \sigma_* w_*$, this means $\sigma_* u_*(z) = 0$ and since $\sigma_* u_*$ is surjective this implies

$$H_k(B\operatorname{\mathsf{Hom}}(X,\operatorname{rel} X\smallsetminus E)) = 0.$$

NOTATION A.11. Let Y be a clean subspace of a CS space X and let $A \subset X$ be closed. We assume that $Fr(Y) \setminus A$ has compact closure in X. Generalizing some of the notation in [16] again, we write

$$\mathcal{H}om(X, Y, \mathrm{rel} A)$$

for the topological submonoid of $\mathcal{H}om(X, \operatorname{rel} A)$ consisting of the automorphisms $X \to X$ in $\mathcal{H}om(X, \operatorname{rel} A)$ which embed Y in itself. Let $\mathcal{H}om_0(X, Y, \operatorname{rel} A)$ be the identity component of $\mathcal{H}om(X, Y, \operatorname{rel} A)$. The underlying discrete monoid is $\operatorname{Hom}_0(X, Y, \operatorname{rel} A)$. We write

$$B\mathcal{H}om_0(X, Y, \mathrm{rel}\ A) := \mathrm{hofiber}[B\mathrm{Hom}_0(X, Y, \mathrm{rel}\ A) \longrightarrow B\mathcal{H}om_0(X, Y, \mathrm{rel}\ A)]$$

The set-theoretic quotient of $Hom_0(X, Y, rel A)$ by the equivalence relation

 $h \sim h' \iff h = h'$ on some neighborhood of Y

will be written as $\mathsf{Emb}_0^X(Y, \mathrm{rel}\ A)$. There is no preferred topology on this monoid. But let

 $\mathcal{E}mb(Y, \operatorname{rel} A)$

be the space of all embeddings $Y \to Y$ which are the identity on some neighborhood of $Y \cap A$, with the direct limit topology as before; and let $\mathcal{E}mb_0^X(Y, \text{rel } A)$ be the image of the restriction map

$$\mathcal{H}om_0(X, Y, \mathrm{rel}\ A) \longrightarrow \mathcal{E}mb(Y, \mathrm{rel}\ A)$$

with the subspace topology induced from $\mathcal{E}mb(Y, \operatorname{rel} A)$. As noted in [16, §2], this topology agrees with the quotient topology obtained by viewing $\mathcal{E}mb_0^X(Y, \operatorname{rel} A)$ as a quotient of $\mathcal{H}om_0(X, Y, \operatorname{rel} A)$. For this McDuff uses an isotopy extension theorem as in [11]; we need a variant for CS spaces [24, 6.5]. Finally let

$$\overline{B}\mathcal{E}mb_0^X(Y, \operatorname{rel} A) := \operatorname{hofiber}[B\mathsf{Emb}_0^X(Y, \operatorname{rel} A) \longrightarrow B\mathcal{E}mb_0^X(Y, \operatorname{rel} A)].$$

The following propositions A.12 and A.13 as well as corollary A.14 generalize propositions 2.1, 2.2 and corollary 2.4 in [16], respectively. For the proofs, see [16] and the remark following corollary A.14 below.

PROPOSITION A.12. Let X be a CS space, Y a clean subset of X and A a closed subset of X such that $Fr(Y) \setminus A$ has compact closure. Then the inclusion

$$B\mathcal{H}om_0(X, Y, \mathrm{rel}\ A) \longrightarrow B\mathcal{H}om(X, \mathrm{rel}\ A)$$

is a weak homotopy equivalence.

PROPOSITION A.13. If X, Y and A are as in proposition A.12, then the sequence

$$\bar{B}\mathcal{H}om_0(X, \operatorname{rel} Y \cup A) \longrightarrow \bar{B}\mathcal{H}om_0(X, Y, \operatorname{rel} A) \longrightarrow \bar{B}\mathcal{E}mb_0^X(Y, \operatorname{rel} A)$$

is an integer homology fibration sequence. In other words, the inclusion of the space $\bar{B}\mathcal{H}om_0(X, \operatorname{rel} Y \cup A)$ into the homotopy fiber of the restriction map $\bar{\rho}$ induces an isomorphism of (untwisted) integer homology groups.

COROLLARY A.14. Let X, Y and A be as in proposition A.12. Assume that $\overline{B}\mathcal{H}om(X, \operatorname{rel} Z)$ is acyclic for Z = A, Y and $Y \cup A$. Then it is acyclic for $Z = Y \cap A$ also.

REMARK. Corollary A.14 is a formal consequence of propositions A.12 and A.13. See §2 of [16] for the deduction.

The proof of McDuff's proposition 2.1 in [16], which is the unstratified case of proposition A.12 above, occupies §4 of [16]. Its backbone is a lemma about topological monoids, lemma 4.1 in [16]. The deduction of McDuff's proposition 2.1 from that lemma occupies only half a page (right after the statement of the lemma) and carries over to the stratified case with only trivial changes.

Similarly, the proof of McDuff's proposition 2.2 in [16], which is the unstratified case of proposition A.13 above, occupies §3 of [16]. It relies on a string of lemmas about topological groups and monoids. The deduction of McDuff's proposition 2.2 from these lemmas occupies only half a page (at the end of §3 in [16]) and carries over to the stratified case with only trivial changes.

THEOREM A.15. The space $B\mathcal{H}om(X, \operatorname{rel} A)$ is acyclic if A is closed in X and $X \smallsetminus A$ has compact closure in X.

PROOF. Imitating the strategy of [16], we deduce this directly from theorem A.10 and corollary A.14. Let Z be a stratum of X which is not contained in A and which is minimal among the strata of X with that property (i.e., all strata of X which belong to the closure of Z in X, except Z itself, are contained in A). By induction, we may assume:

(i) $\overline{B}\mathcal{H}om(X, \operatorname{rel} N)$ is acyclic for each closed $N \subset X$ which is a neighborhood of $A \cup Z$.

In addition, we will assume to begin with that

(ii) there exists $z \in Z \smallsetminus A$ and a quasi-ball neighborhood V about z which contains all of $Z \smallsetminus A$.

(This assumption will be removed at a later stage.) Because of (ii), the open subset $Z \setminus A$ of Z is identified with an open subset of a standard euclidean space, and can be triangulated. We fix a triangulation.

For every closed A' which is a neighborhood of A in X, we can choose a triangulation of $Z \setminus A$ such that $Z \setminus A'$ is covered by finitely many open stars $\operatorname{st}(z_1), \ldots, \operatorname{st}(z_r)$ where z_1, \ldots, z_r are vertices of the triangulation. Making V sufficiently slim, we can arrange that the portion V_i of V lying over $\operatorname{st}(z_i)$ has empty intersection with A. Let $A_i = X \setminus V_i$. For $S \subset \{1, \ldots, r\}$ let A_S be the union of the A_i with $i \in S$. Then

(iii) each A_S is a clean subspace of X and its complement is a quasi-ball, or the empty set. Hence $\bar{B}\mathcal{H}om(X, \mathrm{rel} A_S)$ is acyclic by theorem A.10. Now $B\mathcal{H}om(X, \operatorname{rel} A_1 \cap A_2 \cap \cdots \cap A_i)$ is acyclic for $i = 1, \ldots, r$. This can easily be shown by induction on i, using corollary A.14 and the last part of (iii) just above.

Finally choose a closed neighborhood N of $A\cup Z$ such that

$$N \cap A_1 \cap A_2 \cap \dots \cap A_r \quad \subset \quad A'.$$

By assumption (i), we also know that $B\mathcal{H}om(X, \text{rel } N)$ is acyclic. Hence, by corollary A.14 again,

$$B\mathcal{H}om(X, \mathrm{rel}\ N \cap A_1 \cap A_2 \cap \cdots \cap A_r)$$

is acyclic. We have now shown that, for any closed neighborhood A' of A, there exists another closed neighborhood A'' of A with $A'' \subset A'$ such that $\overline{B}\mathcal{H}om(X, \operatorname{rel} A'')$ is acyclic. By lemma A.6, this implies that $\overline{B}\mathcal{H}om(X, \operatorname{rel} A)$ is acyclic.

Now we repeat the argument, but without hypothesis (ii). As before, we fix a closed neighborhood A' of A in X and we choose finitely many quasi-balls V_1, \ldots, V_r about points in $Z \setminus A$ such that the union of the V_i contains $Z \setminus A'$ and is contained in $X \setminus A$. We define A_i and A_S as before, for $S \subset \{1, \ldots, r\}$. We have less information about the A_S this time, but at least we know that condition (ii) is satisfied with A_S in place of A. Therefore

(iv) $\overline{B}\mathcal{H}om(X, \operatorname{rel} A_S)$ is acyclic for each $S \subset \{1, \ldots, r\}$,

by the first part of this proof, which relied on (ii). We can now finish the argument as in the first part of the proof, using (ivh instead of (iii). \Box

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APPENDIX B

Controlled homeomorphism groups

For a closed manifold M, view $M \times \mathbb{R}^i$ as part of a control space

$$(M * S^{i-1}, M \times \mathbb{R}^i).$$

We do not know whether the inclusion

$$B\text{Hom}(M \times \mathbb{R}^i; c) \to B\mathcal{H}om(M \times \mathbb{R}^i; c)$$

is a homology equivalence and we have given up on that. Let Γ be the homotopy fiber of $\operatorname{Hom}(S^{i-1}) \to \operatorname{Hom}(S^{i-1})$, a topological group acting on $B\operatorname{Hom}(M \times \mathbb{R}^i; c)$ and on $B\operatorname{Hom}(M \times \mathbb{R}^i; c)$ via the forgetful homomorphism to $\operatorname{Hom}(S^{i-1})$. The Borel construction $B\operatorname{Hom}(M \times \mathbb{R}^i; c)_{h\Gamma}$ comes with a projection

 $B\text{Hom}(M \times \mathbb{R}^i; c)_{h\Gamma} \longrightarrow B\Gamma$

which has a zero section $B\Gamma \to B\text{Hom}(M \times \mathbb{R}^i; c)_{h\Gamma}$ since the action of Γ respects the base point of $B\text{Hom}(M \times \mathbb{R}^i; c)$. By the *reduced* Borel construction

BHom $(M \times \mathbb{R}^i; c)_{rh\Gamma}$

we mean the quotient of $B\text{Hom}(M \times \mathbb{R}^i; c)_{h\Gamma}$ by the image of the zero section. The quotient map

$$B\text{Hom}(M \times \mathbb{R}^i; c)_{h\Gamma} \longrightarrow B\text{Hom}(M \times \mathbb{R}^i; c)_{rh\Gamma}$$

induces an isomorphism in homology (for any local coefficient system on the target) because $B\Gamma$ has the homology of a point. Similarly, there is a reduced Borel construction $B\mathcal{H}om(M \times \mathbb{R}^i; c)_{rh\Gamma}$.

LEMMA B.1. The inclusion $B\mathcal{H}om(M \times \mathbb{R}^i; c) \to B\mathcal{H}om(M \times \mathbb{R}^i; c)_{rh\Gamma}$ is a homotopy equivalence.

PROOF. The action of Γ on $B\mathcal{H}om(M \times \mathbb{R}^i; c)$ extends to an action of the contractible topological group hofiber $[\mathcal{H}om(S^{i-1}) \to \mathcal{H}om(S^{i-1})]$. Therefore

 $B\text{Hom}(M \times \mathbb{R}^i; c)_{h\Gamma} \simeq B\text{Hom}(M \times \mathbb{R}^i; c) \times B\Gamma$

and so the reduced Borel construction $B\text{Hom}(M \times \mathbb{R}^i; c)_{rh\Gamma}$ is homotopy equivalent to the pushout or homotopy pushout of

$$B\operatorname{Hom}(M \times \mathbb{R}^i; c) \times B\Gamma \longleftarrow B\Gamma \longrightarrow \star$$
.

Hence the inclusion $B\mathcal{H}om(M \times \mathbb{R}^i; c) \to (B\mathcal{H}om(M \times \mathbb{R}^i; c))_{rh\Gamma}$ induces an isomorphism in homology for any local coefficient system on the target. By the Seifertvan Kampen theorem, it also induces an isomorphism on fundamental groups. \Box

LEMMA B.2. The inclusion of reduced Borel constructions

 $B\text{Hom}(M \times \mathbb{R}^i; c)_{rh\Gamma} \longrightarrow B\mathcal{H}om(M \times \mathbb{R}^i; c)_{rh\Gamma}$

is a homology equivalence.
PROOF. It is enough to show that the inclusion of ordinary Borel constructions

$$B\text{Hom}(M \times \mathbb{R}^i; c)_{h\Gamma} \longrightarrow B\mathcal{H}om(M \times \mathbb{R}^i; c)_{h\Gamma}$$

is a homology equivalence. And it is also enough to show that the inclusion

$$B\operatorname{Hom}(M \times \mathbb{R}^i; c)_{h\Gamma} \longrightarrow B\mathcal{H}om(M \times \mathbb{R}^i; c)_{h\Gamma'}$$

is a homology equivalence, where $\Gamma' = \text{hofiber}[\mathcal{H}om(S^{i-1}) \to \mathcal{H}om(S^{i-1})]$. For that we identify $B\text{Hom}(M \times \mathbb{R}^i; c)_{h\Gamma}$ with the homotopy fiber of the composition

$$B\mathrm{Hom}((M * S^{i-1}, M \times \mathbb{R}^i)) \xrightarrow{\mathrm{forget}} B\mathrm{Hom}(S^{i-1}) \xrightarrow{\mathrm{incl.}} B\mathcal{H}om(S^{i-1})$$

That composition fits into a commutative square

$$\begin{array}{c} B\mathrm{Hom}((M*S^{i-1},M\times\mathbb{R}^i)) \longrightarrow B\mathcal{H}om(S^{i-1}) \\ & \downarrow \\ B\mathcal{H}om((M*S^{i-1},M\times\mathbb{R}^i)) \longrightarrow B\mathcal{H}om(S^{i-1}) \end{array}$$

whose left-hand column is a homology isomorphism by chapter A. Given that

$$\pi_1 B \mathcal{H}om(S^{i-1}) = \pi_0 \mathcal{H}om(S^{i-1}) \cong \{\pm 1\}$$

is as uncomplicated as it is, it is not hard to deduce a homology isomorphism between the homotopy fibers of the rows. This is exactly what we need since the homotopy fiber in the lower row is $B\mathcal{H}om(M \times \mathbb{R}^i; c)_{h\Gamma'}$.

EXAMPLE B.3. Let M be a compact manifold such that ∂M is the union of two codimension zero submanifolds $\partial_0 M$ and $\partial_1 M$ with intersection $\partial_0 M \cap \partial_1 M =$ $\partial \partial_0 M = \partial \partial_1 M$. What is the recommended approximation (by the classifying space of a discrete group etc.) to $B\mathcal{H}om(M \times \mathbb{R}^i, \partial_1 M \times \mathbb{R}^i; c)$? We also write $B\mathcal{H}om((M, \partial_1 M) \times \mathbb{R}^i; c)$ for $B\mathcal{H}om(M \times \mathbb{R}^i, \partial_1 M \times \mathbb{R}^i; c)$.

- (i) In the case where $\partial M = \emptyset$, we can use $B\text{Hom}(M \times \mathbb{R}^i; c)_{rh\Gamma}$.
- (ii) In the case where i = 0, we can use $B\text{Hom}(M\text{rel }\partial_0 M)$. This is covered by theorem A.15, or the unstratified version in [16]. Note however that $\text{Hom}(M\text{rel }\partial_0 M)$ consists of homeomorphisms $M \to M$ which agree with the identity on some neighborhood of $\partial_0 M$.
- (iii) In the general case, we resort to approximating the forgetful map

 $B\mathcal{H}om((M,\partial_0 M,\partial_1 M)\times\mathbb{R}^i;c)\longrightarrow B\mathcal{H}om((\partial_0 M,\partial_0 M)\times\mathbb{R}^i;c)$

by the forgetful map

 $B\text{Hom}((M,\partial_0 M,\partial_1 M) \times \mathbb{R}^i; c)_{rh\Gamma} \longrightarrow B\text{Hom}((\partial_0 M,\partial_0 M) \times \mathbb{R}^i; c)_{rh\Gamma}.$

APPENDIX C

K-theory of pairs and diagrams

Let $f: \mathcal{W}_s \to \mathcal{W}_t$ be an exact functor between Waldhausen categories. Make a new Waldhausen category $\mathcal{P} = \mathcal{P}(\mathcal{W}_s \to \mathcal{W}_t)$ whose objects are triples (a_s, a_t, g) where a_s and a_t are objects of \mathcal{W}_s and \mathcal{W}_t , respectively, and $g: f(a_s) \to a_t$ is a cofibration in \mathcal{W}_t . A morphism in \mathcal{P} from (a_s, a_t, g) to (b_s, b_t, h) is a pair of morphisms $(q_s: a_s \to b_s, q_t: a_t \to b_t)$ such that the square

in \mathcal{W}_1 commutes. Call (q_s, q_t) a weak equivalence if q_s and q_t are weak equivalences; call it a cofibration if q_s and q_t and the induced map from the pushout of

to b_t are cofibrations. The additivity theorem implies immediately that

$$\mathbf{K}(\mathcal{P}) \simeq \mathbf{K}(\mathcal{W}_s) imes \mathbf{K}(\mathcal{W}_t) \simeq \mathbf{K}(\mathcal{W}_s) \lor \mathbf{K}(\mathcal{W}_t)$$

by means of the coordinate functors $(a_s, a_t, g) \mapsto a_s$ and $(a_s, a_t, g) \mapsto a_t$.

If both \mathcal{W}_s and \mathcal{W}_t are equipped with SW products, denoted \odot , and the functor f respects these, then \mathcal{W}_f has a preferred SW product given by

$$(a_s, a_t, g) \odot (b_s, b_t, h) := \text{hofiber}[g \odot h : a_s \odot b_s \longrightarrow a_t \odot b_t]$$

The splitting $\mathbf{K}(\mathcal{P}) \simeq \mathbf{K}(\mathcal{W}_s) \times \mathbf{K}(\mathcal{W}_t)$ may not respect the involution on $\mathbf{K}(\mathcal{P})$ determined by the SW product on \mathcal{P} . But there is still a homotopy fiber sequence of spectra with involution,

(C.1)
$$\mathbf{K}(\mathcal{W}_t^{(1)}) \longrightarrow \mathbf{K}(\mathcal{P}) \xrightarrow{\text{forget}} \mathbf{K}(\mathcal{W}_s)$$

where $\mathcal{W}_t^{(1)}$ is \mathcal{W}_t with a new SW product, defined in terms of the original one by $(a_t, b_t) \mapsto \Omega(a_t \odot b_t)$.

In the case where (X, Y) is a pair of spaces, \mathcal{W}_s and \mathcal{W}_t are certain categories of retractive spaces over Y and X which we use to define the spectra $\mathbf{A}(Y)$ and $\mathbf{A}(X)$, respectively, and $f: \mathcal{W}_s \to \mathcal{W}_t$ is the functor induced by the inclusion $Y \to X$, we may write $\mathbf{A}(Y \subset X)$ or $\mathbf{A}(Y \to X)$ instead of $\mathbf{K}(\mathcal{P})$. Furthermore, if X comes with a spherical fibration ν , then we have n-duality SW products on \mathcal{W}_s and \mathcal{W}_t determined by ν and n, and we may use notation such as

(C.2)
$$\mathbf{A}(X,\nu,n+1) \longrightarrow \mathbf{A}(Y \subset X,\nu,n+1) \xrightarrow{\text{forget}} \mathbf{A}(Y,\nu,n)$$

as an alternative to the notation in (C.1).

These definitions generalize mechanically to the situation where we have a commutative square of spaces and inclusion maps,

$$\begin{array}{ccc} X_{01} \rightarrow X_0 \\ \downarrow & \downarrow \\ X_1 \rightarrow X \end{array}$$

giving rise to a square of exact functors between categories of retractive spaces,

$$\begin{array}{ccc} \mathcal{W}_{01} \longrightarrow \mathcal{W}_{0} \\ \downarrow & \downarrow \\ \mathcal{W}_{1} \longrightarrow \mathcal{W} \end{array} .$$

Namely, we let

(C.3)

$$\mathbf{A} \begin{pmatrix} X_{01} \rightarrow X_0 \\ \psi & \psi \\ X_1 \rightarrow X \end{pmatrix} := \mathbf{K}(\mathcal{P}(\mathcal{P}_u \rightarrow \mathcal{P}_\ell))$$

where \mathcal{P}_u and \mathcal{P}_ℓ are the Waldhausen category of pairs determined by $\mathcal{W}_{01} \to \mathcal{W}_0$ and $\mathcal{W}_1 \to \mathcal{W}$, respectively. If X comes equipped with a spherical fibration ν , then there is a homotopy fiber sequence of spectra with involution

$$\mathbf{A}(X_1 \to X, \nu, n+1)$$

$$\downarrow$$

$$\mathbf{A}\begin{pmatrix} X_{01} \longrightarrow X_0 \\ \downarrow & \downarrow, \nu, n+1 \\ X_1 \longrightarrow X \end{pmatrix}$$

$$\downarrow$$

$$\mathbf{A}(X_{01} \to X_0, \nu, n).$$

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APPENDIX D

Corrections and Elaborations

REMARK D.1. There is an unfortunate oversight in [39], as follows. In [39, 7.1] we have an enlarged model $xK(\mathcal{C})$ of the K-theory space $K(\mathcal{C})$ of a Waldhausen category \mathcal{C} with Spanier–Whitehead product,

$$xK(\mathcal{C}) = \Omega |xw\mathcal{S}_{\bullet}\mathcal{C}|.$$

Here $xwS_{\bullet}C$ is an enlarged model of Waldhausen's $wS_{\bullet}C$. It is a simplicial category with a degreewise involution which *anticommutes* with the simplicial operators. The involution at the category level induces an involution on $|xwS_{\bullet}C|$. It should have been pointed out just before [**39**, 7.2] that this must be combined with the "reverse loops" operation to give the preferred involution on $\Omega|xwS_{\bullet}C| = xK(C)$. With that convention, the standard inclusion of |xwC| in $\Omega|xwS_{\bullet}C|$ respects the preferred involutions. This is used in [**39**, §9].

REMARK D.2. We have frequently encountered the following constellation in this paper: a Waldhausen category \mathcal{D} with SW-product satisfying all the usual axioms and a Waldhausen subcategory $\mathcal{C} \subset \mathcal{D}$ closed under weak equivalences and "duals". Then it is often useful to have something like a "quotient" of \mathcal{D} by \mathcal{C} , designed in such a way that the algebraic K-theory spectrum of the quotient is homotopy equivalent to the mapping cone of $\mathbf{K}(\mathcal{C}) \to \mathbf{K}(\mathcal{D})$, and similarly for the various L-theory spectra. From the point of view of algebraic K-theory the easiest and best approach is to continue using \mathcal{D} , but with a new notion of weak equivalence where all morphisms in \mathcal{D} whose mapping cones are in \mathcal{C} qualify as weak equivalences. From an L-theory point of view, this seems (at first) less fortunate because the old SW product in \mathcal{D} , with the new notion of weak equivalence, will normally violate one of the basic conditions for an SW product [39, 1.1]. It is probably possible to repair this by introducing a new SW product to go along with the new notion of weak equivalence in \mathcal{D} . But this may not always be worthwhile. What we have tended to do instead is to continue working with the old SW product in \mathcal{D} , and to look for pairings in \mathcal{D} which are nondegenerate for the new notion of weak equivalence (i.e., nondegenerate modulo \mathcal{C}). This is unproblematic, but it is slightly embarrassing that the situation has not been looked at in detail in [39]. It is all the more reassuring that Ranicki in [21] has got it right; but Ranicki has it only in the setting of chain complex categories.

REMARK D.3. The proof of proposition 4.3 in [35] is slightly wrong. (The "functor" τ as defined there is not a functor; its "induced morphisms" do not satisfy the required control condition.) Here is a corrected proof. For each integer $i \ge 0$, define $\psi_i: [0,1] \rightarrow]0,1]$ so that $\psi_i(t) = (1 - \log_2(t))/(1+i)$ if $t \ge 2^{-i}$ and $\psi_i(t) = 1$ otherwise. There is an endomorphism of the control space $(Z \times [0,1], Z \times [0,1[))$ given by the formula

$$((x,s),t) \mapsto ((x,s \cdot \psi_i(t)),t)$$
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for $(x,s) \in Z = X \times [0,1]$. Denote the induced endofunctor of $\mathcal{A}^{\rho}(\mathbb{J}Z)^{\wedge}_{\infty}$ by σ_i . Note that $\sigma_0 = \mathrm{id}$, and all the σ_i are related by invertible natural transformations (so that σ_i is isomorphic to σ_j for any $i, j \geq 0$). We re-define τ by the formula

$$\tau(A) = \bigoplus_{i \ge 0} \sigma_i$$

and this time it is an endofunctor of $\mathcal{A}^{\rho}(\mathbb{J}Z)_{\infty}^{\wedge}$. There is an Eilenberg swindle in the shape of a natural isomorphism of functors $\tau \cong \mathrm{id} \oplus \tau$. This uses the natural isomorphisms of functors $\sigma_i \to \sigma_{i+1}$ mentioned earlier. Hence, for the self-map τ_* of the infinite loop space $K(\mathcal{A}^{\rho}(\mathbb{J}Z)_{\infty}^{\wedge})$ we have $\tau_* + \mathrm{id} \simeq \tau_*$.

REMARK D.4. The existence and uniqueness of Spivak normal fibrations for a Poincaré duality space X of formal dimension n has been repeatedly used in this paper. What does it mean ? Let G_k be the space of based homotopy automorphisms of S^{k-1} . Let ξ_k be the canonical quasi-fibration on BG_k with fibers $\simeq S^{k-1}$. Form the space $U_k(X)$ of pairs (g, η) where $g: X \to BF_k$ and η is a based map from S^{n+k} to the Thom space of $g^*\xi_k$. (Here, *Thom space* means the mapping cone of the projection.) The existence and uniqueness claim is that

$$U_{\infty}(X) = \operatorname{colim}_{k} U_{k}(X)$$

is contractible. Wall [29] shows that $U_{\infty}(X)$ is connected, and it seems likely that similar (Spanier-Whitehead duality) arguments could be employed to show that $U_{\infty}(X)$ is contractible. A different argument is as follows. Suppose first that Xhas the homotopy type of a compact CW-complex. For $k \gg 0$ we consider pairs (N, e) where N is a compact smooth codimension zero submanifold of \mathbb{R}^{n+k} , the inclusion $\partial N \to N$ induces in isomorphism on π_1 , and $e: N \to X$ is a homotopy equivalence. We can call such a thing a regular neighborhood of X in \mathbb{R}^{n+k} . These regular neighborhoods can be regarded as 0-simplices of a suitable simplicial set where the j-simplices are certain regular neighborhoods of $X \times \Delta^j$ in $\mathbb{R}^{n+k} \times \Delta^j$. Let $V_k(X)$ be its geometric realization. It is relatively easy to verify that

$$V_{\infty}(X) = \operatorname{colim}_k V_k(X)$$

is contractible; this amounts to an existence and uniqueness statement for regular neighborhoods of X in euclidean space. Hence it is enough to show that there exist compatible homotopy equivalences

$$V_k(X) \to U_k(X)$$

for large enough k. Indeed, for any (N, e) in $V_k(X)$, the homotopy fibers of $e_{|\partial N}: \partial N \to X$ are (k-1)-spheres [19]. Then $N/\partial N$ can be regarded as the Thom space of a spherical fibration on X and the Pontryagin-Thom collapse from $\mathbb{R}^{n+k} \cup \infty$ to $N/\partial N$ is a Spivak reduction. This gives maps $V_k(X) \to U_k(X)$. To understand why these maps $V_k(X) \to U_k(X)$ should be highly connected, fix some (g, η) in $U_k(X)$. The mapping cone of $g^*\xi_k$ is a quotient of the mapping cylinder. The mapping cylinder comes with a map δ to [0, 1] measuring the "distance" to the zero section X. If η is transverse to $\delta^{-1}(1/2)$, then η^{-1} of δ^{-1} of [0, 1/2] is a codimension zero smooth compact submanifold N of S^{n+k} avoiding the base point of S^{n+k} . This comes with an obvious map $e: N \to X$, and more importantly, with a map of pairs \bar{e} from $(N, \partial N)$ to the (disk bundle, sphere bundle) pair associated with $g^*\xi_k$. Now e may not be a homotopy equivalence. But embedded surgery on $(N, \partial N)$, with the goal of making \bar{e} embedded bordant to a homotopy equivalence

of pairs, will repair that. Hence the map $V_k(X) \to U_k(X)$ is 0-connected. A parameterized version of this argument shows that $V_k(X) \to U_k(X)$ is *j*-connected for any *j*.

If X is finitely dominated with nonzero finiteness obstruction, then $X \times S^1$ has zero finiteness obstruction, thanks to M. Mather; see also [29]. But $R_{\infty}(X)$ is a homotopy retract of $R_{\infty}(X \times S^1)$; hence $R_{\infty}(X)$ is again contractible.

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