The word problem in fundamental groups of sufficiently large irreducible 3-manifolds

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We treat the word problem in its topological version, i.e., given a space (more precisely: given a space in terms of some finite notation), to decide whether or not a given closed curve in this space is contractible. We describe a solution of this problem for manifolds which (are known to) belong to the class mentioned in the title.

The restriction to irreducible manifolds is, of course, not really necessary. It only is more convenient than the requirement that every irreducible summand of the manifold be either (known to be) sufficiently large or (known to be) simply connected—which would do just as well. This remark applies to compact submanifolds of the 3-sphere.

The algorithm described is essentially a corollary to work by Haken, namely the method of (algorithmically) splitting a manifold in a certain way and to the extent that finally only balls are left [2]. The surfaces we use for splitting are required to be "good" in the sense described in (1.1)(this property is slightly more restrictive than the one Haken uses). By definition, a good surface has certain properties with respect to non-singular discs. We then show it has the analogous properties with respect to singular discs, (1.6), (1.7). While this is pretty obvious (with the loop theorem lurking in the background), to prove it is the hard part of the paper.

Now, if we split a manifold we simultaneously cut into pieces any curve in that manifold. Similarly, if that curve bounded a singular disc, there will be left pieces of that singular disc. The idea of the algorithm may now be phrased as follows. Each time we are going to split the manifold (which already is part of the algorithm), we first normalize the singular disc (or what is left from it) while the original curve is kept fixed, taking advantage of the fact that all splitting is done at good surfaces. Here the existence of the singular disc, of course, is merely hypothetical. In the end we check algorithmically whether or not there is a singular disc which is a candidate for a piece of the original singular disc. If so, we simplify the original curve, and start all over again. If not, that curve is proved to be non-contractible. Since the method just indicated is not too suitable for communication we have to generalize it a bit. This is done in § 2.

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1. Preliminaries

We are in the piecewise linear category throughout.

A *manifold* is 3-dimensional, compact, and orientable; we call it *sufficiently large* if it contains an incompressible surface.

In the manifold M_1 let F_1 be a (compact, connected, orientable) surface (such that $F_1 \cap \partial M_1 = \partial F_1$). Then, the *manifold M_2 obtained by splitting M_1 at F_1*, has by definition the property: ∂M_2 contains surfaces F' and F'' which are copies of F_1, and identifying F' and F'' gives a projection

$$p_1\colon (M_2, F' \cup F'') \longrightarrow (M_1, F_1) .$$

(1.1) Let M be a manifold which is irreducible (i.e., any 2-sphere in M bounds a ball in M). In ∂M (which may be empty) let J be a (finite) graph any point of which has order at most 3. Let F be a (compact, connected, orientable) surface in M, ($F \cap \partial M = \partial F$), such that ∂F is in general position to J. F will be called *good* (*with respect to J*) if and only if the following hold:

(1) F is incompressible (i.e., F is not a 2-sphere; and if D is a disc in M, $D \cap F = \partial D$, then $\partial D = \partial \tilde{D}$ for some disc $\tilde{D}$ in F).

(2) F is boundary-incompressible (i.e., no component of ∂F bounds a disc in ∂M; and if D is a disc in M such that $D \cap (F \cup \partial M) = \partial D$, and $D \cap F$ is an arc k in ∂D, $k \cap \partial F = \partial k$, then there exists a (non-singular) disc $\tilde{D}$ in F, such that $\partial \tilde{D} \subset k \cup \partial M$).

(3) There is no surface in M which has properties (1) and (2), and which has bigger characteristic than F has.

(4) Let D be a disc in M such that $D \cap (F \cup \partial M) = \partial D$, and $D \cap F$ is an arc k in ∂D, $k \cap \partial F = \partial k$. If $D \cap J$ consists of at most one point, then there exists a disc $\tilde{D}$ in F such that $\partial \tilde{D} \subset k \cup \partial M$, and that $\tilde{D} \cap J$ consists of not more points than $D \cap J$.

(1.2) LEMMA. *Let M be a connected irreducible manifold. If $\partial M \neq \emptyset$, and if M is not a ball, then there exists a good surface in M.*

PROOF. It is known, e.g. [7, (1.4)], that there exists in M an incompressible surface the boundary of which is a non-bounding cycle in ∂M; from this the lemma follows.

(1.3) PROPOSITION. *Let M be a connected irreducible manifold. If (it is* *known that) there exists in M a good surface, then a good surface in M can be constructed.*

This is a corollary of results of Haken which we state in the following lemmas.

(1.4) LEMMA. *For any value of χ, there is an algorithm for finding out if there exists in M an incompressible and boundary-incompressible surface F with characteristic $\chi(F) \geqq \chi$. If such a surface exists, the algorithm will construct one*; [3].

(1.5) LEMMA. *For any value of η, there is an algorithm for finding out if there exists in M a boundary-incompressible disc F with $\eta(F) \leqq \eta$, where $\eta(F)$ denotes the number of intersection points $F \cap J$. If such a disc exists, the algorithm will construct one*; [3].

PROOF OF (1.3). By (1.4), we may construct an incompressible and boundary-incompressible surface F which has maximal characteristic. By (1.5), we may construct it in such a way that $\eta(F)$ is minimal if F is a disc. Since any point of J has order $\leqq 3$, we simultaneously may achieve general position of ∂F with respect to J.

Case 1. *Assume F is a disc, and* (1.1.4) *does not hold.* There are two discs which can be obtained by composing D, cf. (1.1.4), with part of F. At least one of them, say D^*, must be boundary-incompressible. Since $\eta(D^*) < \eta(F)$, we have a contradiction.

Case 2. *Assume F is not a disc, and* (1.1.4) *does not hold.* Since F is boundary-incompressible, there exists a disc $\tilde{D}$ in F such that $\partial\tilde{D} = (\tilde{D} \cap D) \cup (\tilde{D} \cap \partial M)$. Since F has maximal characteristic, the disc $D \cup \tilde{D}$ is boundary-compressible, i.e., $\partial(D \cup \tilde{D})$ bounds a disc D' in ∂M. Our assumption (1.1.4) is wrong, means $\overline{(\partial D' - \partial F)} \cap J$ consists of at most one point (which we may assume to be a general point of J), and $\partial D' \cap \partial F \cap J$ has at least one more point than $\overline{(\partial D' - \partial F)} \cap J$. If $D' \cap J$ is not connected, we may pass to a smaller disc with similar properties; so we assume $D' \cap J$ is connected. We now take away from D' those components of $D' - (J \cup \partial F)$ which contain $\partial D' - \partial F$, and, in addition, that edge of the graph $J \cup \partial F$ which pierces $\partial D' - \partial F$, if it exists. This way we make D' collapse to a complex which consists of arcs in $J \cup \partial F$ and discs which have their boundary in $J \cup \partial F$. Thus we see $\partial D'$ corresponds to a circuit in $J \cup \partial F$ which runs at most twice through each edge. Since there are only finitely many circuits like this, and since it can be checked whether a simple closed curve in ∂M bounds a disc, we can find out by trial whether a disc like the above D' actually exists. If we hit upon D', we can make $\eta(F)$ smaller by an obvious isotopy of F.

(1.6) LEMMA. *Let F be a good surface in the manifold M. Let $f: D \rightarrow M$ be a singular disc such that $f^{-1}(F) = \partial D$. Then there exists $f': D \rightarrow M$, $f' \mid \partial D = f \mid \partial D, f'(D) \subset F$.*

PROOF. If this were wrong, the loop theorem would show that F is compressible.

(1.7) LEMMA. *In the manifold M let F be a surface which is good with respect to the graph J. Let $f: D \rightarrow M$ be a singular disc such that $f^{-1}(F)$ is an arc k in $\partial D, f^{-1}(\partial M) = \partial D - \mathring{k}$, and that $f^{-1}(J)$ is at most one point, disjoint to ∂k.*

(1) *Then there exists $f': D \rightarrow M$, such that $f' \mid k = f \mid k, f'(D) \subset F$, $f'(\partial D - k) \subset \partial M$,*

(2) *and that $f'^{-1}(J)$ consists of not more points than $f^{-1}(J)$.*

PROOF OF (1). Construct N by splitting M at F. There is a lifting $\tilde{f}: D \rightarrow N$ of f. Assuming assertion (1) is wrong, and denoting by G that component of ∂N which contains $\tilde{f}(\partial D)$, we find that the loop $\tilde{f} \mid \partial D$ is not contained in the normal subgroup of $\pi_1(G)$ which is generated by $p^{-1}(\partial M)$, (here, $p: N \rightarrow M$ is the natural projection). Hence the loop theorem gives us a non-singular $\tilde{D}$ in $N, \tilde{D} \cap \partial N = \tilde{D} \cap G = \partial \tilde{D}$ such that $\partial \tilde{D}$ is not contained in that same normal subgroup. An analysis of the proof of the loop theorem shows we may assume $\partial \tilde{D} \cap p^{-1}(\partial F)$ consists of at most two points: If it is empty, we conclude F is compressible, if it consists of two points, we conclude F is boundary-compressible; but F is neither of these.

PROOF OF (2). We may assume $f(\partial D - \mathring{k}) \cap J$, if not empty, is a general point of J; for otherwise, since any point of J has order $\leqq 3$, we can achieve that f, in addition, has this property.

Case 1. *F is a disc.* (Notation as in (1).) If f maps ∂k to one point, the assertion is trivial; so we assume it does not.

Let k_1 and k_2 be the arcs in $p^{-1}(\partial F)$ which are bounded by $\tilde{f}(\partial k)$. We construct G' from G by removing a point from each $\mathring{k}_1$ and $\mathring{k}_2$. Then $\tilde{f} \mid \partial D$ is not contained in the normal subgroup of $\pi_1(G')$ which is generated by $\mathring{(p^{-1}(\partial M))}$. So, using the loop theorem, we conclude there exists a disc D' in N, $D' \cap \partial N = \partial D'$, $D' \cap p^{-1}(\partial F) = \tilde{f}(D) \cap p^{-1}(\partial F)$, such that $D' \cap p^{-1}(J)$ consists of not more points than $\tilde{f}^{-1}(p^{-1}(J))$. Considering the disc $p(D')$, we see at least one of the arcs $p(k_1)$ and $p(k_2)$ must intersect J in not more points than $p(D')$ does, since F is good.

Case 2. *F is not a disc.* By (1), there is a deformation (of pairs), constant on $\partial D - \mathring{k}$, from $f: (D, k) \rightarrow (M, F)$ to g' such that $g'(\partial D) \subset \partial M$; g' may be chosen so that $g'|k$ is locally homeomorphic. Since F has maximal characteristic and is not a disc, it follows from the loop theorem that $g' | \partial D$ can be extended to a map $g: D \to \partial M$. Since no component of ∂F is contractible, g may be chosen so that $g^{-1}(F) = k$.

We construct the graph J^* as follows: If $g(\partial D - k) \cap J \neq \emptyset$, let I be a neighborhood in J of this point, and define $J' = J - \mathring{I}$. If $g(\partial D - k) \cap J = \emptyset$, let $J' = J$. Call the "positive side" of $g(k)$ that one which is approached by $g(\mathring{D})$. Then, J^* is to be the union of all those points which in J' can be reached from the positive side of $g(k)$, without crossing $g(k)$.

By a small deformation of g which is constant on ∂D and does not alter $g^{-1}(F)$, we achieve general position of g with respect to J^*. If now l is any simple closed curve or arc in J^*, then $g^{-1}(l)$ is a system of simple closed curves and arcs, and if $l \cap g(k) = \partial l$, then for any of them we have $\tilde{l} \cap \partial D = \partial \tilde{l}$. Furthermore, if $\tilde{l}$ is any such arc and if $\partial \tilde{l} \subset \partial D$, then $g(\partial \tilde{l})$ consists of two different points, because otherwise we would conclude that $g | k$ were not locally homeomorphic. In particular we see there is no deformation of g, constant on ∂D, by which any point of J^* could be uncovered.

Let l be an arc in J^* such that $l \cap g(k) = \partial l$. Then in $g^{-1}(l)$ there is an arc $\tilde{l}$ such that $\tilde{l} \cap k = \partial \tilde{l}$. There is a disc D_1 in D which is bounded by $\tilde{l}$ and an arc in k; $\tilde{l}$ may be chosen so that $D_1 \cap g^{-1}(l)$ does not contain an arc other than $\tilde{l}$. Since $g|k$ is locally homeomorphic, the map $g|\partial D_1: \partial D_1 \to g(\partial D_1)$ has non-zero degree. Hence there must exist a disc in ∂M which is bounded by l and an arc in $g(k)$.

Let l be a simple closed curve in $J^* - g(k)$. Then $g^{-1}(l)$ consists of mutually disjoint simple closed curves which bound discs $D_1, D_2, \cdots$ in $\mathring{D}$. At least one of the maps $g | \partial D_j: \partial D_j \to l$ must have non-zero degree, for otherwise we could uncover l by a deformation of g which is constant on ∂D, in contradiction to what we found above. Hence l bounds a disc in ∂M.

By the preceding arguments we see we may engulf J^* by a system of discs, D', with the following properties:

(a) $D' \cap \partial F = D' \cap g(k) = \partial D' \cap g(k)$, and every component of D' contributes exactly one arc to this intersection;

(b) $J^* - g(k)$ is contained in the interior of D';

(c) D' collapses to a complex which contains $J^* \cup (D' \cap \partial F)$, and which consists of arcs in $J^* \cup g(k)$ and discs which have their boundary in $J^* \cup g(k)$;

(d) $(\partial D' - \partial F) \cap g(\partial D - k) = \emptyset$;

(e) if $g(\partial D - k) \cap J = \emptyset$, then $(\partial D' - \partial F) \cap J = \emptyset$; in the other case $(\partial D' - \partial F) \cap I$ consists of either one or two points (where I is the neighborhood of $g(\partial D - k) \cap J$ in J chosen above); in the last case, these points are separated by $g(\partial D) \cap I$, and are contained in different components of D'.

Since F is good, we see that $J^* \cap g(k) = \emptyset$ in the first case of (e), so in this case we are finished.

In the first one of the two remaining cases, $J^* \cap g(k)$ is at most one point since F is good (and, in particular, we may assume D' has only one component). But we still have to consider the possibility that $g^{-1}(J) \cap k$ consists of more than one point. Having normalized g, by a deformation which is constant on ∂D and leaves unaltered $g^{-1}(F)$, first to achieve general position with respect to $\overline{(\partial D' - \partial F)}$, and secondly to remove closed curves from $g^{-1}(\partial D' - \partial F)$, we find $g^{-1}(D')$ consists of disjoint discs $D'_1, \cdots, D'_m$ such that $D'_j \cap \partial D$ is an arc in k; we have $m > 1$. By another deformation of the same type, we achieve that the $g \mid D'_j$ are homeomorphisms onto D', and that g is in general position with respect to J. If now I is that neighborhood of $g(\partial D - k) \cap J$ in J, as chosen above, we see $g^{-1}(I)$ contains arcs $l_1, \cdots, l_m$, with l_j escaping from D'_j. None of the l_j reenters any D'_j, and only one of them goes to $\partial D - k$.

Hence, if instead of J^* we now define $\tilde{J}$ to be the union of those points which in J can be reached from the positive side of $g(k)$ (cf. the definition of J^*), we see the above engulfing arguments go through for $\tilde{J}$. So, there is a disc D'' in ∂M such that $D'' \cap \partial F$ is an arc in $\partial D''$, and that $\tilde{J} - g(k)$ is contained in the interior of D''. Hence $\tilde{J} \cap g(k)$ ought to be empty, since F is good.

In the last one of our cases, we may assume at once that D' has only two components. We observe that $I \cap J^*$ consists of two points. On the other hand, $g^{-1}(I) \cap \partial D$ is only one point. Therefore one of our above finding-a-disc-arguments shows that any arc in $\overline{(\partial D' - \partial F)} \cup I$ which has both its end points in ∂F, is parallel in ∂M to an arc in ∂F. So there is a disc D^* in ∂M such that $D^* \cap \partial F$ is an arc in ∂D^* and that $J^* \cup I - g(k)$ is contained in the interior of D^*. Hence again $J^* \cap g(k)$ ought to be empty.

2. The algorithm

Let M_1 be a connected manifold (given as a simplicial complex) which is (known to be) irreducible and sufficiently large. In ∂M_1 (which may be empty) let there be given a graph J_1 any point of which has order $\leqq 3$, e.g., the empty graph. In M_1 there exists by assumption an incompressible surface. If M_1 is not a ball (i.e., if and only if no component of ∂M_1 is a 2-sphere), there exists therefore a surface in M_1 which is good with respect to J_1, and so, by Haken's algorithm, cf. (1.3), a good surface F_1 in M_1 can be constructed. Construct M_2 by splitting M_1 at F_1; denote by $p_1\colon M_2 \longrightarrow M_1$ the natural projection, and define $J_2 = p_1^{-1}(J_1 \cup \partial F_1)$; because of the general position of $J_1 \cap \partial F_1$, any point of J_2 has order $\leqq 3$.

M_2 is irreducible, and every component of M_2 has non-empty boundary; so, if there is a component of M_2 which is not a ball, there exists a surface in M_2 which is good (with respect to the graph J_2), (1.2). We repeat the above construction to get $F_2 \subset M_2$, $p_2: M_3 \longrightarrow M_2$, and $J_3 = p_2^{-1}(J_2 \cup \partial F_2)$. And so on.

By a result of Haken, this procedure will stop after a finite number of steps ([2, if, p. 101], details will be given in [3]). Let

$$F_r \subset M_r, \quad p_r: M_{r+1} \longrightarrow M_r, \quad J_{r+1} = p_r^{-1}(J_r \cup \partial F_r), \; 1 \leqq r \leqq m \,,$$

be the sequence of data finally obtained.

We are going to describe an algorithm $\mathfrak{A}_r$ to answer the following questions (α), (β), (γ):

(α) Let l be the circle, and $f: l \longrightarrow M_r$ a map. Is f contractible?

(β, γ) Let l be the interval, and $f: l \longrightarrow M_r$ a map such that $f(\partial l) \subset \partial M_r - J_r$. Is there a homotopy, constant on ∂l, from f to $\tilde{f}$ such that $\tilde{f}(l) \subset \partial M_r$, and

(β) $\tilde{f}(l) \cap J_r = \varnothing$?

(γ) $\tilde{f}^{-1}(J_r)$ is at most one point?

Since every component of M_{m+1} is a ball, we can take for $\mathfrak{A}_{m+1}$ the obvious algorithm. So, it will suffice to describe $\mathfrak{A}_r$ under the assumption that $\mathfrak{A}_s$, for $r < s \leqq m + 1$, has been described already.

After a small deformation of $f: (l, \partial l) \longrightarrow (M_r, \partial M_r - J_r)$, we will have $f^{-1}(\partial M_r) = \partial l$, and f will be in general position with respect to F_r; in particular, $f^{-1}(F_r)$ will be disjoint to ∂l and will consist of finitely many points; denote by $j(f)$ the number of these points.

Instead of the algorithm $\mathfrak{A}_r$, we shall describe a series of algorithms, $\mathfrak{A}_r^j$, $j = 0, 1, \cdots$, where $\mathfrak{A}_r^j$ is to answer the above questions (α), (β), (γ) for those f, for which $j(f) = j$. In describing $\mathfrak{A}_r^j$, we may then assume $\mathfrak{A}_r^i$ has been described for $i < j$.

(2.1) *Description of* $\mathfrak{A}_r^0$. Since $f(l) \cap F_r = \varnothing$, a lifting of $f: l \longrightarrow M_r$ to $f^*: l \longrightarrow M_{r+1}$ exists (and can be constructed). To f^* we apply the algorithm $\mathfrak{A}_{r+1}$.

PROPOSITION. *The answer to question* (α)*(resp.* (β) *or* (γ)*) on* f *is "yes" if and only if* $\mathfrak{A}_{r+1}$ *gives the answer "yes" to question* (α) *(resp.* (β) *or* (γ)*) on* f^*.

(2.2) *Description of* $\mathfrak{A}_r^j$, $j \geqq 1$. *First reduction.* Let l' be one of those arcs in l for which $l' \cap f^{-1}(F_r) = \partial l'$. There exists a lifting of $f \mid l'$ to

$$f^*: (l', \partial l') \longrightarrow (M_{r+1}, \partial M_{r+1} - J_{r+1}) \,.$$

Using $\mathfrak{A}_{r+1}$, we check question (β) for f^*. If the answer turns out to be "yes", we have deformed f into a map f' such that after a general position deformation we will have $j(f') = j(f) - 2$. Thus we may apply $\mathfrak{A}_r^{j-2}$.

Second reduction. Let l' be one of those arcs in l for which $\partial l' \cap \partial l \neq \varnothing$ and $l' \cap (f^{-1}(F_r) \cup \partial l) = \partial l'$. There exists a lifting of $f \mid l'$ to

$$f^*: (l', \partial l') \longrightarrow (M_{r+1}, \partial M_{r+1} - J_{r+1}) .$$

Using $\mathfrak{A}_{r+1}$, we check question (γ) for f^*. If the answer turns out to be "yes", we have deformed f, by a deformation which is constant on $l - \mathring{l}'$, to f' such that $f'(l') \subset F_r \cup \partial M_r$, and $l' \cap f'^{-1}(J_r \cup \partial F_r)$ is at most one point. Since $f'(l') \cap \partial F_r \neq \varnothing$, we have $f'(l') \cap J_r = \varnothing$. Thus, if we denote the arc

$$l - l' \cup (l' \cap f'^{-1}(F_r))$$

by l'', and $f' \mid l''$ by f'', we see the answer to our question on f is "yes", if and only if the corresponding question on f'' has the answer "yes". After a general position deformation, we have $j(l'') = j(l) - 1$, and so we may apply $\mathfrak{A}_r^{j-1}$.

PROPOSITION. *If both reductions fail to apply for every choice of l', then the answer to question (α) (resp. (β) or (γ)) is "no".*

PROOF OF PROPOSITIONS (2.1) AND (2.2). Assume the answer to question (α) (resp. (β) or (γ)) is "yes", i.e., if D is the disc, and if we identify l with ∂D (resp. with an arc in ∂D), there exists an extension of $f: l$ to $g: D$ such that $g(\partial D - \mathring{l})$ is contained in ∂M_r, and $g^{-1}(J_r) \cap (\partial D - \mathring{l})$ is empty or is at most one point, respectively.

By small deformations which are constant on l and which do not alter $g^{-1}(J_r) \cap (\partial D - \mathring{l})$, we achieve $g^{-1}(\partial M_r) = \partial D - \mathring{l}$, $g(\partial D - \mathring{l}) \cap J_r \cap F_r = \varnothing$, and g is in general position with respect to F_r. Then $g^{-1}(F_r)$ consists of mutually disjoint simple closed curves and arcs, and for any component k of $g^{-1}(F_r)$, we have $k \cap \partial D = \partial k$, and $\partial k \cap g^{-1}(J_r) \cap (\partial D - \mathring{l}) = \varnothing$.

We assume g has been so chosen that it has the above properties and that the number of components of $g^{-1}(F_r)$ is as small as possible. By lemmas (1.6) and (1.7) this implies that every component of $g^{-1}(F_r)$ is an arc, and that at least one endpoint of any such arc lies in l.

(1) Let k be a component of $g^{-1}(F_r)$ such that $\partial k \subset l$. Then for one of the discs split off D by k, we have $\partial D' \subset k \cup l$. Passing to another k, if necessary, we may assume $D' \cap g^{-1}(F_r) = k$. Thus we see the first reduction applies.

(2) Assume the above case does not occur for any component of $g^{-1}(F_r)$; let k be a component. $g^{-1}(J_r) \cap (\partial D - \mathring{l})$ is at most one point and is disjoint

to k. Therefore one of the discs split off D by k, say D', does not contain this point. Passing to another k, if necessary, we may assume $D' \cap g^{-1}(F_r) = k$. Hence the second reduction applies.

(3) $g^{-1}(F_r) = \varnothing$. This case will occur in the situation of (2.1); which proves that proposition.

3. Remark

There exist irreducible manifolds with infinite fundamental group which are not sufficiently large. The ones I know, [7, § 2], have quite pleasant fundamental groups in which the word problem is definitely solvable. It is conceivable that other ones might be obtained from a manifold which is a regular neighborhood of a singular surface, by attaching handle bodies to kill the boundary (as indicated in the remark in [7, § 2]). Yet, these have a good chance to be "almost sufficiently large" in the sense that there exists a finite cover which is sufficiently large. If this should indeed happen, we can again solve the word problem, as follows:

Let M be a manifold which is (known to be) almost sufficiently large. By a well-known procedure, [6, § 58], we construct all coverings $M_j \to M$, $j_1(n) \leqq j < j_2(n)$, the index of which is a given number n; and we let $j_1(n+1) = j_2(n)$. Let $\mathfrak{H}_j$ be Haken's algorithm for constructing an incompressible surface in M_j, ($\mathfrak{H}_j$ is a finite produre only if such a surface exists at all); let $\mathfrak{H}_j^i$, $i = 1, 2, \cdots$, be the steps of $\mathfrak{H}_j$. Running along the short diagonals $i + j =$ const., we eventually will hit upon a covering $M_m \to M$ such that M_m is sufficiently large. Using the covering projection and the solution of the word problem in $\pi_1(M_m)$, we solve the word problem in $\pi_1(M)$.

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