V. A. Yakubovich and V. M. Starzhinskii

LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

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PREFACE

Many problems in physics and engineering ultimately involve systems of linear differential equations with periodic coefficients. Till quite recently, research engineers, frequently employing nonrigorous methods, merely reduced these problems to Hill's equation or even to Mathieu's equation. Modern engineering problems often demand the investigation of systems with several degrees of freedom (dynamic stability of elastic systems, parametric resonance in high-power transmission lines and particle accelerators, numerous problems of celestial mechanics). In this connection one is often most interested in a quantitative description of phenomena for which there are no analogs in the simple system with one degree of freedom described by Hill's equation. The old techniques prove inadequate.

It is also noteworthy that, since Lyapunov and Poincaré, practical methods for investigating the stability of periodic motions described by nonlinear differential equations have centered increasingly around systems of linear differential equations with periodic coefficients.

Thanks to the efforts of many authors, the past two decades have seen significant progress made in the mathematical theory of such systems (particularly Hamiltonian systems). Practical methods are now available which frequently furnish solutions, quite satisfactory from the engineering standpoint, to many problems which previously seemed quite forbidding. It is characteristic of these methods that the computational difficulties increase only slowly with increase in the order of the system, so that they are particularly fruitful in regard to problems with many degrees of freedom. Many new and profound results have been obtained for Hill's equation as well.

This book aims at a systematic exposition of these results, previously available only in the periodical literature in a brief form, not readily understood by the nonspecialist. Most of the book is intended for the mathematically mature engineer, equipped with the basic mathematical tools of linear algebra and Lyapunov stability theory at the level of a technical college education. The exceptions are Chapters III and VIII, which demand more sophisticated mathematical tools and require an acquaintance with advanced university courses in mechanics, physics and mathematics.

The authors have not presented methods previously discussed in the monograph literature, such as the methods of Krylov-Bogolyubov-Mitropol'skii /10, 11, 66a, b, c/, treatment of Hill's equation through infinite determinants and continued fractions (see Strutt /89/, McLachlan /61/). The reader interested in the method of Lappo-Danilevskii is referred to the monographs /58a, b, 40b/.

The fundamental results of Lyapunov /60/ and Poincaré /111a, b/, closely bound up with the subject of this book, may be found in the well-known monographs of Krasovskii /54/, Malkin /62c/, Chetaev /102/. Many related questions are considered in Erugin /40a, c/, Malkin /62a, b/, Rozenvasser /79/, Shtokalo /104/, Cesari /100/, Hale /98/.

To facilitate the use of the book, the authors have striven, without fear of repetition, to highlight the final conclusions and to state them in a form comprehensible with minimum mathematical knowledge and minimum acquaintance with previous parts of the book.

The first two chapters are introductory. The content of each of the succeeding chapters is outlined in a brief introduction. In view of the frequent appearance in applications of problems involving parameters which are combinations of real physical quantities, the authors have paid special attention to effective techniques for constructing stability and instability domains in parameter spaces for systems with many degrees of freedom. These constructions, and the various computational procedures and stability tests provided in the text, are illustrated by many examples from mechanics, physics, and engineering, concentrated mainly in Chapters IV and VI.

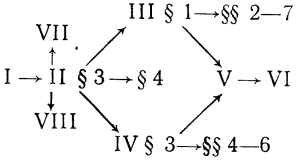
The authors lay no claims to the book's being a complete presentation of all current methods for investigation of differential equations with periodic coefficients. The material is of course a reflection of their own scientific interests, based on lectures read at the universities of Moscow and Leningrad. References to original papers related to the content of the book, as well as a list of relevant literature, may be found in the bibliographical notes at the end of the book.

Throughout the text we denote scalars (real or complex) by Greek or Roman italics (e. g., α, x, a, T, A), vectors by bold-type upright lower case letters ($\mathbf{x}$, $\mathbf{a}$, etc.), matrices by bold-type capitals ($\mathbf{X}$, $\mathbf{A}$, etc.), sets and subspaces by Gothic letters ($\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{L}$, etc.), operators by Gothic or Roman bold-type capitals.

In the numbering of formulas or subsections in the same chapter, the first numeral signifies the section number, the second the ordinal number of the formula or subsection in the section. In references to formulas, sections and subsections of other chapters a Roman numeral is added on the left to identify the chapter. Lemmas and theorems are numbered within subsections.

Chapters III, V and VIII were written by Yakubovich, the remaining chapters by both authors jointly.

The logical interdependence of the chapters is as follows:



The authors are indebted to V. B. Lidskii for many valuable remarks, which were taken into account in the final version.

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Chapter I

PREREQUISITES FROM LINEAR ALGEBRA AND MATRIX THEORY

This introductory chapter presents the algebraic prerequisites for the rest of the book. Proofs are given only when they are so simple and brief that it is pointless to refer the reader to specialized texts, and also when they do not appear in textbooks of linear algebra and matrix theory. The reader familiar with these subjects may skip this chapter, referring to it only as the need arises. For the benefit of those doing so, we mention the following nonstandard notation employed throughout the book: A block-diagonal matrix $\mathbf{A}=\mathbf{diag}\,(\mathbf{A}_1,\ldots,\mathbf{A}_k)$ will also be denoted by $\mathbf{A}=\mathbf{A}_1\dot{+}\mathbf{A}_2\dot{+}\ldots\dot{+}\mathbf{A}_k$. If

$\dot{+}$ = direct sum

$$x=\begin{pmatrix}\xi_1\\ \ldots\\ \xi_k\end{pmatrix},\quad y=\begin{pmatrix}\eta_1\\ \ldots\\ \eta_l\end{pmatrix},\quad z=\begin{pmatrix}\xi_1\\ \ldots\\ \xi_k\\ \eta_1\\ \ldots\\ \eta_l\end{pmatrix}$$

are column vectors, we write $z=x\dot{+}y$.

The main part of the book uses only the simplest material, presented in §§1.1 through 1.15 and 2.1, 2.3. The remainder of Chapter I, including §3, will be used only in Chapters III, IV and VIII.

§1. FINITE-DIMENSIONAL COMPLEX LINEAR SPACES AND MATRICES

1.1. Complex linear n-space

We define an n-vector $\mathbf{x}$ to be an ordered n-tuple of complex numbers $\xi_1, \xi_2, \ldots, \xi_n$, which we write as a column:

$$\mathbf{x}=\begin{pmatrix}\xi_1\\ \ldots\\ \xi_n\end{pmatrix}.$$

The set of all such vectors is denoted by $\mathfrak{R}_n$. We denote complex conjugation by a bar; thus

$$\bar{\mathbf{x}}=\begin{pmatrix}\bar{\xi}_1\\ \cdots\\ \bar{\xi}_n\end{pmatrix}$$

is the complex conjugate of the above vector $\mathbf{x}$. The vector with zero components will be denoted by $\mathbf{0}$. The transpose of an n-vector is an ordered n-tuple of numbers, written as a row:

$$\mathbf{x}^{\tau}=(\xi_1, \ldots, \xi_n).$$

The hermitian conjugate of a vector, $\mathbf{x}^*$, is defined by

$$\mathbf{x}^*=\bar{\mathbf{x}}^{\tau}=(\bar{\xi}_1, \ldots, \bar{\xi}_n).$$

The vectors $\mathbf{x}+\mathbf{y}$ and $\lambda\mathbf{x}$, where λ is a complex number and

$$\mathbf{y}=\begin{pmatrix}\eta_1\\ \cdots\\ \eta_n\end{pmatrix},$$

are defined by

$$\begin{pmatrix}\xi_1+\eta_1\\ \cdots\\ \xi_n+\eta_n\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\lambda\xi_1\\ \cdots\\ \lambda\xi_n\end{pmatrix}.$$

Addition of vectors and multiplication of vectors by complex numbers satisfy the usual rules:

$$\mathbf{x}+\mathbf{0}=\mathbf{x}, \quad \mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}, \quad \mathbf{x}+(\mathbf{y}+\mathbf{z})=(\mathbf{x}+\mathbf{y})+\mathbf{z}, \quad 1\cdot\mathbf{x}=\mathbf{x},$$
$$\alpha(\mathbf{x}+\mathbf{y})=\alpha\mathbf{x}+\alpha\mathbf{y}, \quad (\alpha+\beta)\mathbf{x}=\alpha\mathbf{x}+\beta\mathbf{x}, \quad \alpha(\beta\mathbf{x})=(\alpha\beta)\mathbf{x}.$$

Thus $\mathfrak{R}_n$ is a linear space (see, e.g., /21/, Chapter 1, §1; /103a/, Chapter 2, §11).

As usual, $(\mathbf{x}, \mathbf{y})=\mathbf{y}^*\mathbf{x}$ is the scalar product of vectors $\mathbf{x}$ and $\mathbf{y}$:

$$(\mathbf{x}, \mathbf{y})=\mathbf{y}^*\mathbf{x}=\xi_1\bar{\eta}_1+\ldots+\xi_n\bar{\eta}_n. \tag{1.1}$$

It follows from the definition (1.1) that

$$(\mathbf{y}, \mathbf{x})=\overline{(\mathbf{x}, \mathbf{y})}, \quad (\alpha\mathbf{x}, \beta\mathbf{y})=\alpha\bar{\beta}(\mathbf{x}, \mathbf{y}),$$
$$(\mathbf{x}+\mathbf{y}, \mathbf{z})=(\mathbf{x}, \mathbf{z})+(\mathbf{y}, \mathbf{z}), \quad (\mathbf{x}, \mathbf{y}+\mathbf{z})=(\mathbf{x}, \mathbf{y})+(\mathbf{x}, \mathbf{z})$$

and

$$(\mathbf{x}, \mathbf{x})=|\xi_1|^2+\ldots+|\xi_n|^2>0 \quad \text{if} \quad \mathbf{x}\neq\mathbf{0}.$$

The norm (modulus, length) of a vector $\mathbf{x}\in\mathfrak{R}_n$ is denoted by $|\mathbf{x}|$:

$$|\mathbf{x}|=\sqrt{(\mathbf{x}, \mathbf{x})}=\sqrt{|\xi_1|^2+\ldots+|\xi_n|^2}. \tag{1.2}$$

The norm has the following properties:

1) $|\mathbf{x}|>0$ if $\mathbf{x}\neq\mathbf{0}$,

2) $|\bar{\mathbf{x}}|=|\mathbf{x}|$,

3) $|\lambda\mathbf{x}|=|\lambda||\mathbf{x}|$ (where λ is a scalar),

4) $|\mathbf{x}+\mathbf{y}|\leqslant|\mathbf{x}|+|\mathbf{y}|$.

The proof of the first three properties is obvious; a proof of the fourth (the triangle inequality) may be found, e. g., in Shilov /103a/, Chapter 7, §50.5.

The Cauchy-Bunyakovskii inequality is

$$|(\mathbf{x}, \mathbf{y})|\leqslant|\mathbf{x}|\cdot|\mathbf{y}|$$

(for a proof, see /103a/, Chapter 7, §50.2). In scalar notation, this inequality is

$$|\sum\xi_\nu\bar{\eta}_\nu|^2\leqslant\sum|\xi_\nu|^2\cdot\sum|\eta_\nu|^2.$$

1.2. Matrices and operations on matrices

An $n\times m$ matrix is a tabular arrangement of nm complex numbers in n rows and m columns:

$$\mathbf{A}=\begin{Vmatrix}\alpha_{11} & \dots & \alpha_{1m}\\ \dots & \dots & \dots\\ \alpha_{n1} & \dots & \alpha_{nm}\end{Vmatrix}.$$

For example, a vector $\mathbf{x}$ and its transpose $\mathbf{x}^\tau$ are $n\times1$ and $1\times n$ matrices, respectively.* The word "matrix" alone, without indication of its order, will always mean a square matrix, i. e., an $n\times n$ matrix, which we write as

$$\mathbf{A}=\begin{Vmatrix}\alpha_{11} & \dots & \alpha_{1n}\\ \dots & \dots & \dots\\ \alpha_{n1} & \dots & \alpha_{nn}\end{Vmatrix}=\|\alpha_{ij}\|_1^n.$$

The elements $\alpha_{11}, \dots, \alpha_{nn}$ of a square matrix $\mathbf{A}$ will be called its diagonal elements.

Let

$$\mathbf{a}_j=\begin{pmatrix}\alpha_{j1}\\ \dots\\ \alpha_{jn}\end{pmatrix}$$

be the j-th column of the matrix $\mathbf{A}$. We shall also write the matrix as

$$\mathbf{A}=\|\mathbf{a}_1, \dots, \mathbf{a}_m\|.$$

* And therefore, apart from the notation of § 1.1, we shall also write

$$\mathbf{x}=\begin{Vmatrix}\xi_1\\ \dots\\ \xi_n\end{Vmatrix},\quad \mathbf{x}^\tau=\|\xi_1, \dots, \xi_n\|,\quad \mathbf{x}^*=\|\bar{\xi}_1, \dots, \bar{\xi}_n\|.$$

Let

$$\mathbf{A}=\|\alpha_{ij}\|, \quad \mathbf{B}=\|\beta_{ij}\|$$

be rectangular matrices of the same order and λ a complex number. The matrices $\mathbf{A}+\mathbf{B}$ and $\lambda\mathbf{A}$ are defined respectively as

$$\|\alpha_{ij}+\beta_{ij}\| \quad \text{and} \quad \|\lambda\alpha_{ij}\|.$$

The product of two square matrices $\mathbf{A}$ and $\mathbf{B}$ is the matrix $\mathbf{C}=\|\gamma_{jh}\|_1^n$ defined by

$$\gamma_{jh}=\sum_{\nu=1}^{n}\alpha_{j\nu}\beta_{\nu h} \tag{1.3}$$

(multiply the elements of the j-th column of $\mathbf{A}$ by the corresponding elements of the h-th column of $\mathbf{B}$ and add the results; the sum is the element at the intersection of the j-th row and h-th column of $\mathbf{C}$).

The product $\mathbf{C}=\mathbf{AB}$ may also be defined by (1.3) for rectangular matrices $\mathbf{A}$ and $\mathbf{B}$ of orders $m_1\times n$ and $n\times m_2$, respectively (the first factor has the same number of columns as the second has rows). In particular, the notation $\mathbf{y}=\mathbf{Ax}$, where

$$\mathbf{x}=\begin{pmatrix}\xi_1\\ \cdots\\ \xi_n\end{pmatrix}=\|\xi_i\|_1^n, \quad \mathbf{y}=\begin{pmatrix}\eta_1\\ \cdots\\ \eta_n\end{pmatrix}=\|\eta_i\|_1^n, \quad \mathbf{A}=\|\alpha_{ij}\|_1^n,$$

stands for the linear transformation

$$\eta_i=\alpha_{i1}\xi_1+\ldots+\alpha_{in}\xi_n \qquad (i=1, \ldots, n).$$

Multiplication of matrices is not commutative, i. e., in general $\mathbf{BA}\neq\mathbf{AB}$. If $\mathbf{BA}=\mathbf{AB}$, we shall say that $\mathbf{A}$ and $\mathbf{B}$ commute (or are permutable). Matrix multiplication is associative:

$$(\mathbf{AB})\mathbf{C}=\mathbf{A}(\mathbf{BC}),$$

and distributive:

$$(\mathbf{A}+\mathbf{B})\mathbf{C}=\mathbf{AC}+\mathbf{BC}, \quad \mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{AB}+\mathbf{AC}.$$

The identity matrix of order $n\times n$ is the matrix

$$\mathbf{I}_n=\|\delta_{ij}\|_1^n=\begin{Vmatrix}1 & 0 & \ldots & 0\\ 0 & 1 & \ldots & 0\\ \cdot & \cdot & \cdot & \cdot\\ 0 & 0 & \ldots & 1\end{Vmatrix}.$$

Here δ_{ij} is the Kronecker symbol: $\delta_{ij}=0$ for $j\neq i$, $\delta_{jj}=1$ $(i, j=1, \ldots, n)$.

It is easy to see that for any $n\times n$ matrix $\mathbf{A}$ and any n-vector $\mathbf{x}$

$$\mathbf{A}\mathbf{I}_n=\mathbf{I}_n\mathbf{A}=\mathbf{A},\quad \mathbf{I}_n\mathbf{x}=\mathbf{x}.$$

If each of the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}=\mathbf{AB}$ is partitioned into k^2 blocks of equal respective orders,

$$\mathbf{A}=\begin{pmatrix}\mathbf{A}_{11} & \dots & \mathbf{A}_{1k}\\ \dots & \dots & \dots\\ \mathbf{A}_{k1} & \dots & \mathbf{A}_{kk}\end{pmatrix},\quad \mathbf{B}=\begin{pmatrix}\mathbf{B}_{11} & \dots & \mathbf{B}_{1k}\\ \dots & \dots & \dots\\ \mathbf{B}_{k1} & \dots & \mathbf{B}_{kk}\end{pmatrix},\quad \mathbf{C}=\begin{pmatrix}\mathbf{C}_{11} & \dots & \mathbf{C}_{1k}\\ \dots & \dots & \dots\\ \mathbf{C}_{k1} & \dots & \mathbf{C}_{kk}\end{pmatrix}$$

(the matrices $\mathbf{A}_{jh}$, $\mathbf{B}_{jh}$, $\mathbf{C}_{jh}$ have the same orders), then the corresponding blocks $\mathbf{C}_{jh}$ may be determined by a formula analogous to (1.3):

$$\mathbf{C}_{jh}=\sum_{x=1}^{k}\mathbf{A}_{jx}\mathbf{B}_{xh}$$

(block multiplication rule). For example, if

$$\mathbf{A}=\begin{pmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{pmatrix},\quad \mathbf{B}=\begin{pmatrix}\mathbf{B}_{11} & \mathbf{B}_{12}\\ \mathbf{B}_{21} & \mathbf{B}_{22}\end{pmatrix},\quad \mathbf{C}=\begin{pmatrix}\mathbf{C}_{11} & \mathbf{C}_{12}\\ \mathbf{C}_{21} & \mathbf{C}_{22}\end{pmatrix}$$

then

$$\mathbf{C}_{11}=\mathbf{A}_{11}\mathbf{B}_{11}+\mathbf{A}_{12}\mathbf{B}_{21},\quad \mathbf{C}_{12}=\mathbf{A}_{11}\mathbf{B}_{12}+\mathbf{A}_{12}\mathbf{B}_{22},$$
$$\mathbf{C}_{21}=\mathbf{A}_{21}\mathbf{B}_{11}+\mathbf{A}_{22}\mathbf{B}_{21},\quad \mathbf{C}_{22}=\mathbf{A}_{21}\mathbf{B}_{12}+\mathbf{A}_{22}\mathbf{B}_{22}.$$

Consider a matrix $\mathbf{A}$ of the form

$$\begin{pmatrix}\mathbf{A}_1 & 0\\ 0 & \mathbf{A}_2\end{pmatrix},$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are square matrices of orders $n_1\times n_1$ and $n_2\times n_2$, respectively $(n_1+n_2=n)$, and $\mathbf{0}$ stands for $n_1\times n_2$ and $n_2\times n_1$ matrices all of whose elements are zero. We shall call $\mathbf{A}$ the direct sum of $\mathbf{A}_1$ and $\mathbf{A}_2$, writing

$$\mathbf{A}=\mathbf{A}_1\dot{+}\mathbf{A}_2.$$

(It is obvious that in general $\mathbf{A}_2\dot{+}\mathbf{A}_1\neq\mathbf{A}_1\dot{+}\mathbf{A}_2$.)

This definition may be extended to any finite number of summands: $\mathbf{A}=\mathbf{A}_1\dot{+}\mathbf{A}_2\dot{+}\dots\dot{+}\mathbf{A}_k$ will denote the matrix having the matrices $\mathbf{A}_1$, $\mathbf{A}_2$, ..., $\mathbf{A}_k$ along its principal diagonal and zeros elsewhere. The matrix $\mathbf{A}_1\dot{+}\mathbf{A}_2\dot{+}\dot{+}\dots\dot{+}\mathbf{A}_k$ will also be denoted by

$$\mathbf{A}=\operatorname{diag}(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_k).$$

In particular, any matrix $\mathbf{A}=\|\alpha_{ij}\|_1^n$ with zero off-diagonal elements $(\alpha_{ij}=0$ for $i\neq j)$ may be written $\mathbf{A}=\operatorname{diag}(\alpha_{11}, \dots, \alpha_{nn})$, and we shall call it a diagonal matrix. Let

$$\mathbf{A}=\mathbf{A}_1\dot{+}\dots\dot{+}\mathbf{A}_k,\quad \mathbf{B}=\mathbf{B}_1\dot{+}\dots\dot{+}\mathbf{B}_k,$$

where $\mathbf{A}_s, \mathbf{B}_s$ are matrices of the same orders. It follows from the block multiplication rule that

$$\mathbf{AB} = \mathbf{A}_1\mathbf{B}_1 \dotplus \ldots \dotplus \mathbf{A}_k\mathbf{B}_k.$$

Let $\mathbf{x}$ be a k-vector and $\mathbf{y}$ an l-vector. The column vector $\mathbf{z}$ of order $k+l$ with elements (from top to bottom) $\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_l$ will be called the direct sum of $\mathbf{x}$ and $\mathbf{y}$ and be denoted by

$$\mathbf{z} = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \mathbf{x} \dotplus \mathbf{y}$$

(in general, $\mathbf{y} \dotplus \mathbf{x} \neq \mathbf{x} \dotplus \mathbf{y}$).

Given a matrix

$$\mathbf{A} = \left\| \begin{matrix} \alpha_{11} & \ldots & \alpha_{1n} \\ \cdot & \cdot & \cdot \\ \alpha_{n1} & \ldots & \alpha_{nn} \end{matrix} \right\|,$$

we define the complex conjugate matrix $\bar{\mathbf{A}}$, the transpose $\mathbf{A}^\tau$ and the adjoint (or hermitian conjugate) matrix $\mathbf{A}^*$:

$$\bar{\mathbf{A}} = \left\| \begin{matrix} \bar{\alpha}_{11} & \ldots & \bar{\alpha}_{1n} \\ \cdot & \cdot & \cdot \\ \bar{\alpha}_{n1} & \ldots & \bar{\alpha}_{nn} \end{matrix} \right\|, \quad \mathbf{A}^\tau = \left\| \begin{matrix} \alpha_{11} & \ldots & \alpha_{n1} \\ \cdot & \cdot & \cdot \\ \alpha_{1n} & \ldots & \alpha_{nn} \end{matrix} \right\|, \quad \mathbf{A}^* = \bar{\mathbf{A}}^\tau = \left\| \begin{matrix} \bar{\alpha}_{11} & \ldots & \bar{\alpha}_{n1} \\ \cdot & \cdot & \cdot \\ \bar{\alpha}_{1n} & \ldots & \bar{\alpha}_{nn} \end{matrix} \right\|.$$

The elements of $\mathbf{A}^\tau$ are obtained from those of $\mathbf{A}$ by reflection in the principal diagonal, along which we have the numbers $\alpha_{11}, \ldots, \alpha_{nn}$, i. e., we interchange α_{ij} and α_{ji}; to obtain $\mathbf{A}^*$, we replace α_{ij} by $\bar{\alpha}_{ji}$.

For any $\mathbf{x}, \mathbf{y} \in \mathfrak{R}_n$,

$$(\mathbf{Ax}, \mathbf{y}) = (\mathbf{x}, \mathbf{A}^*\mathbf{y}), \tag{1.4}$$

as may be verified directly.

The norm of a matrix $\mathbf{A}$ is defined by

$$|\mathbf{A}| = \sqrt{\sum_{i,j=1}^{n} |\alpha_{ij}|^2}. \tag{1.5}$$

For example, the norm of the identity matrix $\mathbf{I}_n$ is

$$|\mathbf{I}_n| = \sqrt{n}.$$

Thus defined, the norm has the usual properties:

1) $|\mathbf{A}| > 0$ if $\mathbf{A} \neq 0$,

2) $|\mathbf{A}+\mathbf{B}| \leqslant |\mathbf{A}| + |\mathbf{B}|$,

3) $|\lambda\mathbf{A}| = |\lambda||\mathbf{A}|$ (where λ is a scalar),

and in addition:

4) $|\alpha_{ij}| \leqslant |\mathbf{A}| \quad (i, j = 1, \ldots, n)$,
5) $|\mathbf{A}^{\tau}| = |\mathbf{A}| = |\mathbf{A}^{*}| = |\mathbf{A}|$,
6) $|\mathbf{AB}| \leqslant |\mathbf{A}||\mathbf{B}|$,
7) $|\mathbf{Ax}| \leqslant |\mathbf{A}||\mathbf{x}| \quad (\mathbf{x} \in \mathfrak{R}_n)$.

The proof of properties 1, 3, 4, 5 is obvious. Property 2 is proved in the same way as property 4 of the norm of a vector. We prove property 6. Let $\mathbf{AB} = \|\gamma_{ij}\|_1^n$, where γ_{ij} are as in (1.3) and note that by the Cauchy-Bunyakovskii inequality for vectors (see § 1.1)

$$|\gamma_{ij}|^2 \leqslant \Big(\sum_{\nu} |\alpha_{i\nu}\beta_{\nu j}|\Big)^2 \leqslant \sum_{\nu} |\alpha_{i\nu}|^2 \sum_{\nu} |\beta_{\nu j}|^2.$$

Therefore

$$|\mathbf{AB}|^2 = \sum_{i,j} |\gamma_{ij}|^2 \leqslant \sum_i \sum_j \Big(\sum_{\nu} |\alpha_{i\nu}|^2 \sum_{\nu} |\beta_{\nu j}|^2\Big) = \Big(\sum_i \sum_{\nu} |\alpha_{i\nu}|^2\Big)\Big(\sum_j \sum_{\nu} |\beta_{\nu j}|^2\Big),$$

or, finally,

$$|\mathbf{AB}| \leqslant |\mathbf{A}||\mathbf{B}|.$$

The proof of property 7 is analogous.

There are various alternative definitions of the norm of a matrix, among them:

$$|\mathbf{A}| = \max_{\mathbf{x} \in \mathfrak{R}_n} \frac{|\mathbf{Ax}|}{|\mathbf{x}|}, \quad |\mathbf{A}| = \max_{i,j} |\alpha_{ij}|,$$
$$|\mathbf{A}| = \max_i \sqrt{\sum_j |\alpha_{ij}|^2}, \quad |\mathbf{A}| = \max_i \sum_j |\alpha_{ij}|,$$

and the definitions obtained by interchanging i and j in the last two definitions. All these definitions are equivalent, in the following sense (see /103a/, Chapter 12, §89). For any two of the above norms $|\mathbf{A}|_1$, $|\mathbf{A}|_2$, there exists a constant c_{12} such that for any $n \times n$ matrix

$$|\mathbf{A}|_1 \leqslant c_{12} |\mathbf{A}|_2.$$

We shall say that a sequence of matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots$ converges to a matrix $\mathbf{A}$ as $k \to \infty$, writing $\mathbf{A}_k \to \mathbf{A}$ $(k \to \infty)$, if $|\mathbf{A}_k - \mathbf{A}| \to 0$ as $k \to \infty$. This is equivalent to elementwise convergence of the sequence $\mathbf{A}_1, \mathbf{A}_2, \ldots$ to $\mathbf{A}$; i.e.,

$$\mathbf{A}_k = \|\alpha_{ij}^{(k)}\|_1^n \to \mathbf{A} = \|\alpha_{ij}\|_1^n$$

if and only if

$$\alpha_{ij}^{(k)} \to \alpha_{ij} \quad \text{as} \quad k \to \infty \qquad (i, j = 1, \ldots, n).$$

Let

$$\mathbf{A}(t) = \|\alpha_{ij}(t)\|_1^n$$

be a matrix-function defined in an interval (t_1, t_2), with integrable piecewise-continuous (see § II 1.1) elements $\alpha_{ij}(t)$ $(i, j = 1, \ldots, n)$. Integration and differentiation of matrix-functions are defined in elementwise fashion; for example,

$$\mathbf{B} = \int_{t_1}^{t_2} \mathbf{A}(t)\, dt$$

means that $\mathbf{B} = \|\beta_{ij}\|_1^n$, where

$$\beta_{ij} = \int_{t_1}^{t_2} \alpha_{ij}(t)\, dt.$$

The following property of the norm relates to integrals of a matrix-function:

8) $$\left| \int_{t_1}^{t_2} \mathbf{A}(t)\, dt \right| \leqslant \int_{t_1}^{t_2} |\mathbf{A}(t)|\, dt \qquad (t_2 > t_1). \tag{1.6}$$

This follows from the obvious inequalities

$$\left| \lim_{\max \Delta t_k \to 0} \sum_k \mathbf{A}(t_k)\, \Delta t_k \right| = \lim_{\max \Delta t_k \to 0} \left| \sum_k \mathbf{A}(t_k)\, \Delta t_k \right| \leqslant \lim_{\max \Delta t_k \to 0} \sum_k |\mathbf{A}(t_k)|\, \Delta t_k.$$

1.3. Determinant and inverse of a matrix

We shall denote the determinant of a matrix $\mathbf{B}$ by $\det \mathbf{B}$. We recall the main points of the theory of determinants (see, e. g., /103a/, Chapter I).

1) Transposition of a matrix leaves the determinant unchanged:

$$\det \mathbf{B}^{\tau} = \det \mathbf{B}.$$

Since $\det \overline{\mathbf{B}} = \overline{\det \mathbf{B}}$, we have the following relation for the adjoint:

$$\det \mathbf{B}^{*} = \overline{\det}\, \overline{\mathbf{B}}.$$

2) A homogeneous vector equation

$$\mathbf{B}\mathbf{x} = \mathbf{0} \qquad \left(\mathbf{B} = \|\beta_{ij}\|_1^n, \quad \mathbf{x} = \begin{pmatrix} \xi_1 \\ \ldots \\ \xi_n \end{pmatrix} \right)$$

has a nontrivial solution $\mathbf{x} \neq \mathbf{0}$ if and only if $\det \mathbf{B} = 0$. (The matrix $\mathbf{B}$ is then said to be singular.)

3) If $\det \mathbf{B} \neq 0$ ($\mathbf{B}$ is nonsingular), there exists a unique matrix $\mathbf{B}^{-1}$ such that

$$\mathbf{B}\mathbf{B}^{-1} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}_n.$$

This matrix is given by

$$\mathbf{B}^{-1}=\frac{1}{\det \mathbf{B}}\begin{Vmatrix} A_{11} & \dots & A_{n1} \\ \dots & \dots & \dots \\ A_{1n} & \dots & A_{nn} \end{Vmatrix},$$

where A_{ij} denotes the cofactor of the element b_{ij} $(i, j=1, \dots, n)$ in the determinant of $\mathbf{B}$, i. e., the determinant obtained from the latter by deleting its i-th row and j-th column and multiplying by $(-1)^{i+j}$. Hence it follows at once that in this case the inhomogeneous vector equation

$$\mathbf{Bx}=\mathbf{b}$$

has a unique solution $\mathbf{x}=\mathbf{B}^{-1}\mathbf{b}$, since $\mathbf{B}(\mathbf{B}^{-1}\mathbf{b})=\mathbf{I}_n\mathbf{b}=\mathbf{b}$.

4) $\det(\mathbf{AB})=\det\mathbf{A}\cdot\det\mathbf{B}$, $\det\mathbf{I}_n=1$;
if $\det\mathbf{B}\neq 0$, then $\det\mathbf{B}^{-1}=(\det\mathbf{B})^{-1}$.

5) $\det(\mathbf{A}_1+\mathbf{A}_2+\dots+\mathbf{A}_k)=\det\mathbf{A}_1\cdot\det\mathbf{A}_2\cdot\dots\cdot\det\mathbf{A}_k$.

6) Let the columns of the matrix $\mathbf{B}$ be vectors $\mathbf{b}_1, \dots, \mathbf{b}_n$ and write $\mathbf{B}$ as

$$\mathbf{B}=\|\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\|.$$

Suppose that $\mathbf{b}_1$ is a linear combination of vectors $\mathbf{b}'$ and $\mathbf{b}''$: $\mathbf{b}_1=\alpha\mathbf{b}'+\beta\mathbf{b}''$ (where α, β are complex numbers). Then we have the following representation of the determinant of $\mathbf{B}$:

$$\det\|\alpha\mathbf{b}'+\beta\mathbf{b}'', \mathbf{b}_2, \dots, \mathbf{b}_n\|=\alpha\det\|\mathbf{b}', \mathbf{b}_2, \dots, \mathbf{b}_n\|+\beta\det\|\mathbf{b}'', \mathbf{b}_2, \dots, \mathbf{b}_n\|.$$

7) If two columns are permuted, the determinant changes sign:

$$\det\|\mathbf{b}_1, \dots, \mathbf{b}_\nu, \dots, \mathbf{b}_\mu, \dots, \mathbf{b}_n\|=-\det\|\mathbf{b}_1, \dots, \mathbf{b}_\mu, \dots, \mathbf{b}_\nu, \dots, \mathbf{b}_n\|.$$

It follows from properties 4 and 5 that if a column of the matrix is a linear combination of other columns, say $\mathbf{b}_3=\alpha\mathbf{b}_1+\beta\mathbf{b}_2$, then the determinant vanishes. By property 1, properties 4 and 5 are also valid for rows.

Consider any r rows and r columns of a matrix $\mathbf{B}$. The elements at the intersections of these rows and columns form an $r\times r$ matrix whose determinant is known as an r-th order minor of $\mathbf{B}$. The rank of an $n\times n$ matrix $\mathbf{B}$ is the number r such that all minors of $\mathbf{B}$ of order greater than r (if $n>r$) vanish, but there is at least one minor of order r ("basis minor") which does not vanish. For a nonsingular matrix, $r=n$, and $\det\mathbf{B}$ is a basis minor.

The columns of $\mathbf{B}$ whose elements figure in a basis minor are called basis columns. In a singular matrix $\mathbf{B}$ $(\det\mathbf{B}=0)$ any column is a linear combination of basis columns (/103a/, Chapter I, §9). The difference $d=n-r$ between the order and rank of a matrix is known as its nullity.

8) If $\mathbf{A}$ is a nonsingular matrix, the ranks of the matrices $\mathbf{AB}$ and $\mathbf{BA}$ are both equal to that of $\mathbf{B}$.

1.4. Linearly dependent and independent vectors. Gram matrix

Vectors $\mathbf{f}_1, \ldots, \mathbf{f}_k$ are said to be linearly independent if any relation

$$\alpha_1\mathbf{f}_1+\ldots+\alpha_k\mathbf{f}_k=\mathbf{0}, \tag{1.7}$$

where $\alpha_1, \ldots, \alpha_k$ are complex numbers, implies that $\alpha_1=\ldots=\alpha_k=0$. Otherwise the vectors $\mathbf{f}_1, \ldots, \mathbf{f}_k$ are said to be linearly dependent. It is clear, for example, that the vectors

$$\mathbf{e}_1=\begin{pmatrix}1\\0\\\ldots\\0\end{pmatrix}, \quad \mathbf{e}_2=\begin{pmatrix}0\\1\\\ldots\\0\end{pmatrix}, \ \ldots, \ \mathbf{e}_n=\begin{pmatrix}0\\0\\\ldots\\1\end{pmatrix} \tag{1.8}$$

are linearly independent. Any n linearly independent vectors are called a basis of the space $\mathfrak{R}_n$. The basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is known as the standard basis.

Vectors $\mathbf{f}_1, \ldots, \mathbf{f}_k$ are linearly independent if and only if the rank of the $n \times k$ matrix $\mathbf{F}=\|\mathbf{f}_1, \ldots, \mathbf{f}_k\|$ with columns $\mathbf{f}_1, \ldots, \mathbf{f}_k$ is k. In particular, for any matrix $\mathbf{B}$ we have $\det \mathbf{B}=0$ if and only if its columns (or its rows, since $\det \mathbf{B}^{\tau}=\det \mathbf{B}$) are linearly dependent (see /21/, Chapter III, § 1).

The matrix of scalar products

$$\mathbf{G}=\begin{Vmatrix}(\mathbf{f}_1, \mathbf{f}_1) & \ldots & (\mathbf{f}_k, \mathbf{f}_1)\\ \ldots & \ldots & \ldots\\ (\mathbf{f}_1, \mathbf{f}_k) & \ldots & (\mathbf{f}_k, \mathbf{f}_k)\end{Vmatrix}$$

is called the Gram matrix* of $\mathbf{f}_1, \ldots, \mathbf{f}_k$. It is easy to see that $\mathbf{G}^{\tau}=\mathbf{F}^*\mathbf{F}$.

We shall now show that

$$\det \mathbf{G} \neq 0$$

is a necessary and sufficient condition for the vectors $\mathbf{f}_1, \ldots, \mathbf{f}_k$ *to be linearly independent.* First suppose that (1.7) is true and $\mathbf{a}^{\tau}=\|\alpha_1, \ldots, \alpha_k\| \neq \mathbf{0}$. We have

$$\mathbf{Ga}=\left\|\sum_{\varkappa}(\mathbf{f}_{\varkappa}, \mathbf{f}_j)\alpha_{\varkappa}\right\|=\left\|\left(\sum_{\varkappa}\alpha_{\varkappa}\mathbf{f}_{\varkappa}, \mathbf{f}_j\right)\right\|=\|0\|=\mathbf{0}.$$

Thus $\det \mathbf{G}=0$.

Conversely, let $\det \mathbf{G}=0$. Then the rows of $\mathbf{G}$ are linearly dependent, so that for some $\mathbf{a} \neq \mathbf{0}$ we have $\mathbf{Ga}=\mathbf{0}$. Since $\mathbf{a}^*\mathbf{G}^{\tau}\mathbf{a}=\left|\sum_{\varkappa}\alpha_{\varkappa}\mathbf{f}_{\varkappa}\right|^2=0$, it follows that $\sum_{\varkappa}\alpha_{\varkappa}\mathbf{f}_{\varkappa}=\mathbf{0}$ and so the vectors $\mathbf{f}_1, \ldots, \mathbf{f}_k$ are linearly dependent.

* In some textbooks the Gram matrix is defined as the transpose $\mathbf{G}^{\tau}$ of our definition.

1.5. Linear subspaces

A set $\mathfrak{R}' = \{\mathbf{x}'\}$ of vectors $\mathbf{x}'$ in $\mathfrak{R}_n$ which is closed under the operations of addition and multiplication by numbers is called a subspace of $\mathfrak{R}_n$. Thus $\mathfrak{R}'$ is a subspace if $\mathbf{x}_1 \in \mathfrak{R}'$, $\mathbf{x}_2 \in \mathfrak{R}'$ imply $\alpha\mathbf{x}_1 + \beta\mathbf{x}_2 \in \mathfrak{R}'$ for any complex numbers α and β. The smallest number of linearly independent vectors in a subspace $\mathfrak{R}'$ is known as its dimension, and the vectors themselves are a basis for the subspace. For example, the set $\mathfrak{R}_2'$ of all vectors

$$\begin{pmatrix} \xi \\ \eta \\ 0 \\ \dots \\ 0 \end{pmatrix} = \xi\mathbf{e}_1 + \eta\mathbf{e}_2,$$

where ξ, η are arbitrary complex numbers, is a two-dimensional subspace with basis $\mathbf{e}_1$, $\mathbf{e}_2$ (1.8).

The subspace $\mathfrak{R}'$ of all vectors $\alpha_1\mathbf{f}_1 + \ldots + \alpha_k\mathbf{f}_k$, where $\mathbf{f}_1, \ldots, \mathbf{f}_k$ are given vectors and $\alpha_1, \ldots, \alpha_k$ arbitrary numbers, is known as the subspace spanned by the vectors $\mathbf{f}_1, \ldots, \mathbf{f}_k$ (or their span). If $\mathbf{f}_1, \ldots, \mathbf{f}_k$ are linearly independent, they form a basis for $\mathfrak{R}'$.

Let $\mathfrak{R}'$ and $\mathfrak{R}''$ be disjoint subspaces (i. e., having only the zero vector in common) of $\mathfrak{R}_n$. The set of all vectors $\mathbf{x}' + \mathbf{x}''$, where $\mathbf{x}' \in \mathfrak{R}'$, $\mathbf{x}'' \in \mathfrak{R}''$, is readily seen to be a subspace, known as the direct sum of the subspaces and denoted by $\mathfrak{R}' \dotplus \mathfrak{R}''$. If

$$\mathfrak{R}' \dotplus \mathfrak{R}'' = \mathfrak{R}_n, \tag{1.9}$$

i. e., every vector $\mathbf{x} \in \mathfrak{R}_n$ is uniquely expressible as

$$\mathbf{x} = \mathbf{x}' + \mathbf{x}'',$$

where $\mathbf{x}' \in \mathfrak{R}'$, $\mathbf{x}'' \in \mathfrak{R}''$, we shall say that $\mathfrak{R}_n$ may be decomposed into the direct sum of subspaces $\mathfrak{R}'$ and $\mathfrak{R}''$.

Again, let $\mathfrak{R}'$ and $\mathfrak{R}''$ be arbitrary subspaces of $\mathfrak{R}_n$. They are said to be orthogonal ($\mathfrak{R}' \perp \mathfrak{R}''$) if the scalar product of any pair of vectors $\mathbf{x}' \in \mathfrak{R}'$, $\mathbf{x}'' \in \mathfrak{R}''$ vanishes:

$$(\mathbf{x}', \mathbf{x}'') = 0.$$

For example, the subspaces

$$\mathfrak{R}' = \{\xi_1\mathbf{e}_1 + \xi_2\mathbf{e}_2\} \quad \text{and} \quad \mathfrak{R}'' = \{\xi_3\mathbf{e}_3 + \xi_4\mathbf{e}_4\},$$

where ξ_1, ξ_2, ξ_3, ξ_4 are arbitrary complex numbers and $\mathbf{e}_j$ the vectors of the standard basis, are orthogonal. A direct sum of orthogonal subspaces is denoted by $\mathfrak{R}' \oplus \mathfrak{R}''$. Thus, if the subspaces $\mathfrak{R}'$ and $\mathfrak{R}''$ in (1.9) are orthogonal, we write

$$\mathfrak{R}_n = \mathfrak{R}' \oplus \mathfrak{R}''.$$

In this case $\mathfrak{R}''$ is called the orthogonal complement of $\mathfrak{R}'$ (and $\mathfrak{R}'$ is the orthogonal complement of $\mathfrak{R}''$). We write $\mathfrak{R}''=(\mathfrak{R}')^{\perp}$. Similarly one defines a decomposition of $\mathfrak{R}_n$ into a direct sum of l subspaces:

$$\mathfrak{R}_n=\mathfrak{R}'\dot{+}\mathfrak{R}''\dot{+}\ldots\dot{+}\mathfrak{R}^{(l)},$$

and if moreover any two of the summands are orthogonal subspaces, then

$$\mathfrak{R}_n=\mathfrak{R}'\oplus\mathfrak{R}''\oplus\ldots\oplus\mathfrak{R}^{(l)}.$$

Lemma I (orthogonal complement). *If $\mathfrak{R}'$ is an m-dimensional subspace, $0<m<n$, then any vector $\mathbf{x}\in\mathfrak{R}_n$ admits a unique expression as*

$$\mathbf{x}=\mathbf{y}+\mathbf{z}, \text{ where } \mathbf{y}\in\mathfrak{R}', \quad \mathbf{z}\in(\mathfrak{R}')^{\perp}. \tag{1.10}$$

Proof. Let $\mathbf{f}_1, \ldots, \mathbf{f}_m$ be a basis for $\mathfrak{R}'$. Assuming that (1.10) holds, we have

$$\mathbf{x}=\alpha_1\mathbf{f}_1+\ldots+\alpha_m\mathbf{f}_m+\mathbf{z}. \tag{1.11}$$

Multiplying (1.11) by $\mathbf{f}_1, \ldots, \mathbf{f}_m$ in succession, we obtain

$$\left.\begin{aligned}(\mathbf{x}, \mathbf{f}_1)&=\alpha_1(\mathbf{f}_1, \mathbf{f}_1)+\ldots+\alpha_m(\mathbf{f}_m, \mathbf{f}_1),\\ &\cdots\cdots\cdots\cdots\cdots\\ (\mathbf{x}, \mathbf{f}_m)&=\alpha_1(\mathbf{f}_1, \mathbf{f}_m)+\ldots+\alpha_m(\mathbf{f}_m, \mathbf{f}_m).\end{aligned}\right\} \tag{1.12}$$

Conversely, if (1.12) is true for some $\alpha_1, \ldots, \alpha_m$, we may define $\mathbf{z}$ as in (1.11), and then $(\mathbf{f}_1, \mathbf{z})=\ldots=(\mathbf{f}_m, \mathbf{z})=0$, so that $\mathbf{z}\perp(\mathfrak{R}')^{\perp}$ and (1.10) is true. Thus, to prove (1.10) it will suffice to find $\alpha_1, \ldots, \alpha_m$ satisfying the system of equations (1.12). But the determinant of the latter is the Gram determinant of the vectors $\mathbf{f}_1, \ldots, \mathbf{f}_m$ and so does not vanish (see § 1.4). Thus the required coefficients $\alpha_1, \ldots, \alpha_m$ may be determined uniquely. Q. E. D.

The assertion of the lemma may also be written

$$\mathfrak{R}_n=\mathfrak{R}'\oplus(\mathfrak{R}')^{\perp}, \tag{1.13}$$

in other words, any m-dimensional subspace $(0<m<n)$ has an orthogonal complement.

Corollary. Any system of $m<n$ linearly independent vectors $\mathbf{f}_1, \ldots, \mathbf{f}_m$ in $\mathfrak{R}_n$ may be completed to a system of n linearly independent vectors, and the vectors $\mathbf{f}_{m+1}, \ldots, \mathbf{f}_n$ completing a basis $\mathbf{f}_1, \ldots, \mathbf{f}_m$ of $\mathfrak{R}'$ to a basis of the entire space $\mathfrak{R}_n$ may be chosen orthogonal to $\mathbf{f}_1, \ldots, \mathbf{f}_m$:

$$(\mathbf{f}_\mu, \mathbf{f}_\nu)=0 \qquad (\mu=1, \ldots, m;\ \nu=m+1, \ldots, n).$$

Indeed, we need only define $\mathbf{f}_{m+1}, \ldots, \mathbf{f}_n$ as a basis of $(\mathfrak{R}')^{\perp}$.

Lemma II. *Any subspace $\mathfrak{R}'$ has an orthogonal (and also orthonormal) basis.*

Proof.* Let $\mathbf{f}_1, \ldots, \mathbf{f}_m$ be any basis of $\mathfrak{R}'$. We shall look for the stipulated basis $\mathbf{g}_1, \ldots, \mathbf{g}_m$ in the form

$$\begin{aligned}
\mathbf{g}_1 &= \mathbf{f}_1, \\
\mathbf{g}_2 &= \mathbf{f}_2 - \alpha_{21}\mathbf{g}_1, \\
\mathbf{g}_3 &= \mathbf{f}_3 - \alpha_{31}\mathbf{g}_1 - \alpha_{32}\mathbf{g}_2, \\
&\cdots\cdots\cdots\cdots\cdots \\
\mathbf{g}_m &= \mathbf{f}_m - \alpha_{m1}\mathbf{g}_1 - \ldots - \alpha_{m,\,m-1}\mathbf{g}_{m-1},
\end{aligned}$$

selecting the coefficients α_{jh} in such a way that each vector $\mathbf{g}_\mu$ is orthogonal to the vectors $\mathbf{g}_1, \ldots, \mathbf{g}_{\mu-1}$ $(\mu = 2, \ldots, m)$. Thus we successively find

$$\begin{aligned}
&\mathbf{g}_1 \neq \mathbf{0}, \quad \alpha_{21} = \frac{(\mathbf{f}_2, \mathbf{g}_1)}{(\mathbf{g}_1, \mathbf{g}_1)}, \\
&\mathbf{g}_2 \neq \mathbf{0}, \quad \alpha_{3h} = \frac{(\mathbf{f}_3, \mathbf{g}_h)}{(\mathbf{g}_h, \mathbf{g}_h)} \qquad (h = 1, 2), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\mathbf{g}_{m-1} \neq \mathbf{0}, \quad \alpha_{mh} = \frac{(\mathbf{f}_m, \mathbf{g}_h)}{(\mathbf{g}_h, \mathbf{g}_h)} \qquad (h = 1, \ldots, m-1), \quad \mathbf{g}_m \neq \mathbf{0}.
\end{aligned}$$

The orthogonality of the vectors $\mathbf{g}_\mu \neq \mathbf{0}$ implies that they are linearly independent, and so $\mathbf{g}_1, \ldots, \mathbf{g}_m$ form an orthogonal basis. The vectors $\mathbf{g}_\mu / |\mathbf{g}_\mu|$ form an orthonormal basis.

The procedure whereby the vectors $\mathbf{g}_1, \ldots, \mathbf{g}_m$ are constructed in terms of $\mathbf{f}_1, \ldots, \mathbf{f}_m$ is known as orthogonalization of $\mathbf{f}_1, \ldots, \mathbf{f}_m$.

Corollary. Any system of orthogonal vectors $\mathbf{y}_1 \neq \mathbf{0}, \ldots, \mathbf{y}_d \neq \mathbf{0}$ may be completed orthogonally** to a basis $\mathbf{y}_1, \ldots, \mathbf{y}_d, \mathbf{y}_{d+1}, \ldots, \mathbf{y}_n$. Indeed, by Lemma II we need only take an orthogonal basis $\mathbf{y}_{d+1}, \ldots, \mathbf{y}_n$ in the orthogonal complement of the subspace spanned by $\mathbf{y}_1, \ldots, \mathbf{y}_d$.

1.6. Systems of linear algebraic equations

An inhomogeneous system of linear equations with complex coefficients,

$$\begin{aligned}
&\alpha_{11}\xi_1 + \ldots + \alpha_{1n}\xi_n = \beta_1, \\
&\cdots\cdots\cdots\cdots \\
&\alpha_{n1}\xi_1 + \ldots + \alpha_{nn}\xi_n = \beta_n,
\end{aligned}$$

may be written in vector notation as

$$\boxed{\mathbf{Ax} = \mathbf{b},} \tag{1.14}$$

where

$$\mathbf{A} = \|\alpha_{ij}\|_1^n, \quad \mathbf{x} = \begin{pmatrix} \xi_1 \\ \ldots \\ \xi_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \beta_1 \\ \ldots \\ \beta_n \end{pmatrix}.$$

* The reader may also deduce Lemma II from Lemma I.

** I.e., in such a way that the vectors $\mathbf{y}_1, \ldots, \mathbf{y}_n$ form an orthogonal basis: $(\mathbf{y}_j, \mathbf{y}_h) = 0$ $(j, h = 1, \ldots, n;$ $j \neq h)$.

The corresponding homogeneous equation is

$$\mathbf{Ay}=0 \tag{1.15}$$

and the adjoint homogeneous equation

$$\mathbf{A}^*\mathbf{z}=0 \qquad (\mathbf{A}^*=\|\bar{\alpha}_{ji}\|_1^n). \tag{1.16}$$

In the trivial case ($\det \mathbf{A} \neq 0$), equation (1.14) has a unique solution $\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}$.

Now consider the case when the matrix $\mathbf{A}$ is singular ($\det \mathbf{A}=0$). Let $r<n$ denote its rank and $d=n-r$ its nullity. It is easy to see that the solutions of equation (1.15) form a subspace. The dimension of this subspace is the nullity of $\mathbf{A}$ (/103a/, Chapter 3), so that equation (1.15) has d linearly independent solutions $\mathbf{y}_1, \ldots, \mathbf{y}_d$ and any solution $\mathbf{y}$ of the equation may be expressed as $\mathbf{y}=\alpha_1\mathbf{y}_1+\ldots+\alpha_d\mathbf{y}_d$, where $\alpha_1, \ldots, \alpha_d$ are complex numbers. It follows from the definition of nullity (see §1.3) that the nullity of $\mathbf{A}^*$ is also d, so that equation (1.16) also has d linearly independent solutions $\mathbf{z}_1, \ldots, \mathbf{z}_d$.

Lemma. *(a) Equation (1.14) has a solution if and only if*

$$(\mathbf{b}, \mathbf{z}_\delta)=0 \qquad (\delta=1, \ldots, d). \tag{1.17}$$

(b) Let (1.17) hold and let $\mathbf{u}_1, \ldots, \mathbf{u}_d$ be arbitrary vectors such that

$$\det\|(\mathbf{y}_j, \mathbf{u}_h)\|_1^d \neq 0. \tag{1.18}$$

Then equation (1.14) has a unique solution $\mathbf{x}$ satisfying the conditions

$$(\mathbf{x}, \mathbf{u}_j)=0 \qquad (j=1, \ldots, d). \tag{1.19}$$

This solution is determined by

$$\mathbf{x}=\mathbf{C}^{-1}\mathbf{b}, \quad \mathbf{C}=\mathbf{A}-\mathbf{VU}^*, \tag{1.20}$$

where $\mathbf{U}=\|\mathbf{u}_1, \ldots, \mathbf{u}_d\|$, $\mathbf{V}=\|\mathbf{z}_1, \ldots, \mathbf{z}_d\|$ are $n\times d$ matrices with columns $\mathbf{u}_1, \ldots, \mathbf{u}_d$ and $\mathbf{z}_1, \ldots, \mathbf{z}_d$, and $\det \mathbf{C} \neq 0$.

(c) The general solution of equation (1.14) is

$$\mathbf{x}=\mathbf{x}_0+\sum_{\delta=1}^{d}\gamma_\delta\mathbf{y}_\delta, \tag{1.21}$$

where $\mathbf{x}_0$ is some (particular) solution of equation (1.14) and $\gamma_1, \ldots, \gamma_d$ arbitrary complex numbers.

Remarks. (a) It follows from (1.20) that there is a constant $\gamma=|C^{-1}|>0$, independent of the vector $\mathbf{b}$, such that for any solution $\mathbf{x}$ satisfying (1.19)

$$|\mathbf{x}| \leqslant \gamma|\mathbf{b}|. \tag{1.22}$$

(b) Since the vectors $\mathbf{y}_1, \ldots, \mathbf{y}_d$ are linearly independent, it follows that $\det \|(\mathbf{y}_j, \mathbf{y}_k)\|_1^d \neq 0$ (§1.4), so that we may take $\mathbf{u}_1 = \mathbf{y}_1, \ldots, \mathbf{u}_d = \mathbf{y}_d$.

Proof. (a) Necessity. Let $\mathbf{x}$ be a solution of equation (1.14). Then by (1.14) and (1.16)

$$(\mathbf{b}, \mathbf{z}_\delta) = (\mathbf{Ax}, \mathbf{z}_\delta) = (\mathbf{x}, \mathbf{A}^*\mathbf{z}_\delta) = 0 \qquad (\delta = 1, \ldots, d),$$

and (1.17) holds.

Sufficiency. Using the corollary in § 1.5 to complete each of the systems $\mathbf{y}_1, \ldots, \mathbf{y}_d$ and $\mathbf{z}_1, \ldots, \mathbf{z}_d$ orthogonally to systems of n linearly independent vectors, we obtain new bases and corresponding nonsingular matrices:

$$\mathbf{Y} = \|\mathbf{y}_1, \ldots, \mathbf{y}_n\|, \quad \mathbf{Z} = \|\mathbf{z}_1, \ldots, \mathbf{z}_n\|.$$

Here $\mathbf{Y}$ is the matrix with columns $\mathbf{y}_1, \ldots, \mathbf{y}_n$ and $\mathbf{Z}$ the matrix with columns $\mathbf{z}_1, \ldots, \mathbf{z}_n$. Since the two systems $\mathbf{y}_1, \ldots, \mathbf{y}_n$ and $\mathbf{z}_1, \ldots, \mathbf{z}_n$ are linearly independent, we have

$$\det \mathbf{Y} \neq 0, \quad \det \mathbf{Z} \neq 0.$$

Multiplying equation (1.14) on the left by $\mathbf{Z}^*$, we get the equivalent equation

$$\mathbf{A}'\mathbf{x}' = \mathbf{b}' \qquad (\mathbf{A}' = \mathbf{Z}^*\mathbf{AY}, \ \mathbf{x}' = \mathbf{Y}^{-1}\mathbf{x}, \ \mathbf{b}' = \mathbf{Z}^*\mathbf{b}).$$

Since $\mathbf{Ay}_\delta = 0$ and $\mathbf{A}^*\mathbf{z}_\delta = 0$ $(\delta = 1, \ldots, d)$, it follows that the first d columns of the matrix $\mathbf{AY}$ vanish, as do the first d rows of $\mathbf{Z}^*\mathbf{A}$. Thus

$$\mathbf{A}' = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{A}_0 \end{pmatrix},$$

where $\mathbf{A}_0$ is an $r \times r$ matrix. Since $\mathbf{A}$ and $\mathbf{A}'$ have the same ranks (§1.3, property 8), it follows that $\mathbf{A}_0$ is nonsingular. We now express $\mathbf{x}'$ and $\mathbf{b}'$ as direct sums of a d-vector and an r-vector:

$$\mathbf{x}' = \mathbf{x}_1' + \mathbf{x}_2' = \begin{pmatrix} \mathbf{x}_1' \\ \mathbf{x}_2' \end{pmatrix}, \quad \mathbf{b}' = \mathbf{b}_1' + \mathbf{b}_2' = \begin{pmatrix} \mathbf{b}_1' \\ \mathbf{b}_2' \end{pmatrix}.$$

Then the equation $\mathbf{A}'\mathbf{x}' = \mathbf{b}'$ may be written

$$\mathbf{b}_1' = 0, \quad \mathbf{A}_0\mathbf{x}_2' = \mathbf{b}_2'. \tag{1.23}$$

The first equation of (1.23) is satisfied since, using (1.1) and (1.7), we have

$$\mathbf{b}_1' = \begin{pmatrix} \mathbf{z}_1^*\mathbf{b} \\ \cdots \\ \mathbf{z}_d^*\mathbf{b} \end{pmatrix} = \begin{pmatrix} (\mathbf{b}, \mathbf{z}_1) \\ \cdots \\ (\mathbf{b}, \mathbf{z}_d) \end{pmatrix} = 0.$$

The second equation of (1.23) uniquely determines the vector $\mathbf{x}_2' = \mathbf{A}_0^{-1}\mathbf{b}_2'$. The vector $\mathbf{x}_1'$, which does not figure in equation (1.23), may be arbitrary:

$$\mathbf{x}_1' = \begin{pmatrix} \beta_1 \\ \cdots \\ \beta_d \end{pmatrix}.$$

The solution $\mathbf{x}$ of equation (1.14) is determined by

$$\mathbf{x} = \mathbf{Y}\mathbf{x}', \quad \text{where} \quad \mathbf{x}' = \mathbf{x}_1' + \mathbf{A}_0^{-1}\mathbf{b}_2'.$$

This proves the sufficiency of conditions (1.17).

(b) It is easy to see that the vector (1.21) satisfies equation (1.14). Conversely, if $\mathbf{x}$, $\mathbf{x}_0$ are solutions of equation (1.14), then $\mathbf{y} = \mathbf{x} - \mathbf{x}_0$ is a solution of equation (1.15), so that $\mathbf{y} = \sum_{\delta} \gamma_\delta \mathbf{y}_\delta$, and hence (1.21) is true.

(c) Substituting (1.21) into (1.19), we obtain a system of equations in the unknowns $\gamma_1, \ldots, \gamma_d$:

$$\sum_{\delta=1}^{d} (\mathbf{y}_\delta, \mathbf{u}_j)\gamma_\delta = -(\mathbf{x}_0, \mathbf{u}_j) \qquad (j = 1, \ldots, d),$$

whose determinant is nonzero by (1.18). Thus the solution $\mathbf{x}$ is uniquely determined by conditions (1.19).

We claim that $\det \mathbf{C} \neq 0$, where $\mathbf{C}$ is the matrix in (1.20). Suppose that $\mathbf{C}\mathbf{y}_0 = \mathbf{0}$ for some vector $\mathbf{y}_0$. We must show that $\mathbf{y}_0 = \mathbf{0}$. Denote $\mathbf{v} = \mathbf{V}\mathbf{U}^*\mathbf{y}_0$; then $\mathbf{A}\mathbf{y}_0 = \mathbf{v}$. Therefore $(\mathbf{z}_\delta, \mathbf{v}) = (\mathbf{z}_\delta, \mathbf{A}\mathbf{y}_0) = (\mathbf{A}^*\mathbf{z}_\delta, \mathbf{y}_0) = 0$ $(\delta = 1, \ldots, d)$. Since $\mathbf{v} = \sum_{h=1}^{d} \zeta_h \mathbf{z}_h$, where we have set $\zeta_j = (\mathbf{y}_0, \mathbf{u}_j)$, it follows from the conditions $(\mathbf{z}_\delta, \mathbf{v}) = 0$ $(\delta = 1, \ldots, d)$ and $\det \|(\mathbf{z}_\delta, \mathbf{z}_j)\|_1^d \neq 0$ that $\zeta_j = 0$ $(j = 1, \ldots, d)$, so that $\mathbf{v} = \mathbf{0}$. Consequently, $\mathbf{A}\mathbf{y}_0 = \mathbf{0}$, and so $\mathbf{y}_0 = \sum_{\delta=1}^{d} \eta_\delta \mathbf{y}_\delta$. Together with (1.18), the equations $\zeta_j = \sum_{\delta=1}^{d} \eta_\delta (\mathbf{y}_\delta, \mathbf{u}_j) = 0$ $(j = 1, \ldots, d)$ imply that $\eta_\delta = 0$, so that $\mathbf{y}_0 = \mathbf{0}$. Thus $\det \mathbf{C} \neq 0$. Let $\mathbf{x}$ be the solution of equations (1.14), (1.19) which we have proved to exist. In other words, $\mathbf{U}^*\mathbf{x} = 0$, $\mathbf{A}\mathbf{x} = \mathbf{b}$. Then $\mathbf{C}\mathbf{x} = (\mathbf{A} - \mathbf{V}\mathbf{U}^*)\mathbf{x} = \mathbf{b}$, and so $\mathbf{x} = \mathbf{C}^{-1}\mathbf{b}$. This completes the proof.

1.7. Linear operator and its matrix. Change of basis

Suppose that for each vector $\mathbf{x} \in \mathfrak{N}_n$ we have a vector $\mathbf{y} = \mathfrak{A}\mathbf{x} \in \mathfrak{N}_n$, in such a way that for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathfrak{N}_n$ and any (complex) numbers α, β,

$$\mathfrak{A}(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2) = \alpha\mathfrak{A}\mathbf{x}_1 + \beta\mathfrak{A}\mathbf{x}_2; \tag{1.24}$$

then we say that $\mathfrak{A}$ is a linear operator defined in $\mathfrak{N}_n$. It follows from (1.24) that a linear operator is uniquely determined by its values on any basis, say the standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ (1.18). Let

$$\mathfrak{A}\mathbf{e}_j = \sum_{i=1}^{n} \alpha_{ij}\mathbf{e}_i \qquad (j = 1, \ldots, n).$$

Then the matrix

$$\mathbf{A}=\begin{Vmatrix}\alpha_{11} & \dots & \alpha_{1n}\\ \dots & \dots & \dots\\ \alpha_{n1} & \dots & \alpha_{nn}\end{Vmatrix}=\|\alpha_{ij}\|_1^n$$

is known as the matrix of the operator $\mathfrak{A}$ relative to the basis $\mathbf{e}_1, \dots, \mathbf{e}_n$.

Thus, to each linear operator in $\mathfrak{R}_n$ there corresponds a unique $n\times n$ matrix — the matrix of the operator relative to a fixed basis. It follows from the definition that for the standard basis

$$\mathbf{A}=\|\mathfrak{A}\mathbf{e}_1, \dots, \mathfrak{A}\mathbf{e}_n\|, \quad \mathfrak{A}\mathbf{x}=\mathbf{A}\mathbf{x}. \tag{1.25}$$

Let $\tilde{\mathbf{A}}=\|\tilde{\alpha}_{kj}\|$ be the matrix of $\mathfrak{A}$ relative to some other basis $\mathbf{g}_1, \dots, \mathbf{g}_n$:

$$\mathfrak{A}\mathbf{g}_j=\sum_{k=1}^{n}\tilde{\alpha}_{kj}\mathbf{g}_k \qquad (j=1, \dots, n). \tag{1.26}$$

Let us express $\tilde{\mathbf{A}}$ in terms of $\mathbf{A}$. Let

$$\mathbf{g}_j=\sum_{m=1}^{n}\sigma_{mj}\mathbf{e}_m \quad \text{and} \quad \mathbf{e}_j=\sum_{k=}^{n}\sigma_{kj}^{(-1)}\mathbf{g}_k \qquad (j=1, \dots, n)$$

be the formulas defining the change from the standard basis to $\mathbf{g}_1, \dots, \mathbf{g}_n$ and back. Consider the nonsingular matrix

$$\mathbf{T}=\begin{Vmatrix}\sigma_{11} & \dots & \sigma_{1n}\\ \dots & \dots & \dots\\ \sigma_{n1} & \dots & \sigma_{nn}\end{Vmatrix}.$$

Then $\sigma_{kj}^{(-1)}$ are the elements of the inverse

$$\mathbf{T}^{-1}=\begin{Vmatrix}\sigma_{11}^{(-1)} & \dots & \sigma_{1n}^{(-1)}\\ \dots & \dots & \dots\\ \sigma_{n1}^{(-1)} & \dots & \sigma_{nn}^{(-1)}\end{Vmatrix}.$$

We have

$$\mathfrak{A}\mathbf{g}_j=\sum_{m=1}^{n}\sigma_{mj}\mathfrak{A}\mathbf{e}_m=\sum_{m=1}^{n}\sigma_{mj}\sum_{i=1}^{n}\alpha_{im}\mathbf{e}_i=\sum_{k=1}^{n}\left[\sum_{i,\,m=1}^{n}\sigma_{mj}\alpha_{im}\sigma_{ki}^{(-1)}\right]\mathbf{g}_k.$$

Comparing this equality with (1.26), we obtain

$$\tilde{\alpha}_{kj}=\sum_{i,\,m=1}^{n}\sigma_{ki}^{(-1)}\alpha_{im}\sigma_{mj},$$

or, in matrix notation,

$$\boxed{\tilde{\mathbf{A}}=\mathbf{T}^{-1}\mathbf{A}\mathbf{T}.} \tag{1.27}$$

This formula defines the transformation of the matrix of an operator under a change of basis.

1.8. Eigenvalues and eigenvectors. Eigensubspaces

An eigenvector $\mathbf{A}$ of an $n\times n$ matrix $\mathbf{a}$ is a nonzero vector such that

$$\mathbf{Aa}=\lambda\mathbf{a}, \quad \text{or} \quad (\mathbf{A}-\lambda\mathbf{I}_n)\,\mathbf{a}=\mathbf{0}, \tag{1.28}$$

where λ is some number, called an eigenvalue of $\mathbf{A}$. It follows from §1.3, (2), that the eigenvalues are roots of the equation

$$\det(\mathbf{A}-\lambda\mathbf{I}_n)=0 \tag{1.29}$$

and that, conversely, every root of this equation is an eigenvalue of $\mathbf{A}$.

Equation (1.29) is called the characteristic equation of $\mathbf{A}$. Let λ be some eigenvalue. Since a system with zero determinant always has a nontrivial solution, it follows that for each eigenvalue (root of (1.29)) there is at least one eigenvector $\mathbf{a}\neq\mathbf{0}$. The set $\mathfrak{S}_\lambda$ of all $\mathbf{a}\in\mathfrak{R}_n$ satisfying (1.28) is clearly a subspace, known as the eigensubspace of $\mathbf{A}$ belonging to the eigenvalue λ.

The null-space of a matrix $\mathbf{A}$ is the set of all vectors $\mathbf{f}\in\mathfrak{R}_n$ such that $\mathbf{Af}=\mathbf{0}$. If $\lambda=0$ is not an eigenvalue (that is to say, $\mathbf{A}$ is a nonsingular matrix), the null-space consists of the zero vector alone; if $\lambda=0$ is an eigenvalue ($\mathbf{A}$ is a singular matrix), the null-space is simply the eigensubspace belonging to the eigenvalue $\lambda=0$.

The set of vectors $\mathbf{Ax}$, $\mathbf{x}\in\mathfrak{R}_n$, is easily seen to be a subspace, called the range of the matrix $\mathbf{A}$. It is the whole space if and only if $\mathbf{A}$ is nonsingular. Later we shall need the following simple proposition.

Lemma. *If all the eigenvalues of a matrix $\mathbf{A}$ are distinct, the corresponding eigenvectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are linearly independent.*

Proof. Suppose the contrary:

$$\gamma_1\mathbf{a}_1+\ldots+\gamma_n\mathbf{a}_n=\mathbf{0}, \tag{1.30}$$

where, say, $\gamma_1\neq 0$. By definition,

$$\mathbf{Aa}_\nu=\lambda_\nu\mathbf{a}_\nu \qquad (\nu=1, \ldots, n;\ \mathbf{a}_\nu\neq\mathbf{0}). \tag{1.31}$$

Multiplying (1.30) by λ_n and on the left by $\mathbf{A}$ and then subtracting, we get

$$\gamma_1'\mathbf{a}_1+\ldots+\gamma_{n-1}'\mathbf{a}_{n-1}=\mathbf{0},$$

where $\gamma_j'=(\lambda_n-\lambda_j)\gamma_j$ $(j=1, \ldots, n-1)$. Continuing this elimination procedure $n-1$ times, we obtain

$$\gamma^{(n)}\mathbf{a}_1 = 0 \qquad (\gamma^{(n)} = (\lambda_n - \lambda_1)(\lambda_{n-1} - \lambda_1) \ldots (\lambda_2 - \lambda_1)\gamma_1).$$

This contradicts the assumption $\gamma_1 \neq 0$ and proves the lemma.

Assume again that the eigenvalues of $\mathbf{A}$ are distinct. We define a matrix $S = \|\mathbf{a}_1, \ldots, \mathbf{a}_n\|$, and write (1.31) as

$$\mathbf{AS} = \mathbf{S}\begin{Vmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{Vmatrix}.$$

By the above lemma, $\mathbf{S}$ is a nonsingular matrix. Therefore,

$$\mathbf{A} = \mathbf{S}\begin{Vmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{Vmatrix}\mathbf{S}^{-1}. \tag{1.32}$$

This equality may also hold in the case of multiple eigenvalues. If (1.32) is true, we shall say that the matrix $\mathbf{S}$ reduces $\mathbf{A}$ to diagonal form. Formula (1.32) has various corollaries in matrix theory.

If $\mathbf{A}$ has multiple eigenvalues, it need not be reducible to diagonal form, i. e., it may not admit a representation (1.32). Even then, however, there is a special form of the matrix — the Jordan form — to which any matrix is reducible: Every matrix has a representation similar to (1.32) except that the numbers λ_ν along the diagonal are replaced by "blocks" (elementary Jordan matrices) of a special structure. We now present a detailed account.

1.9. Reduction of a matrix to block-diagonal form. Invariant subspaces

Suppose that for some nonsingular matrix $\mathbf{S}$

$$\mathbf{A} = \mathbf{S}\begin{pmatrix} \mathbf{A}_1 & 0 \\ 0 & \mathbf{A}_2 \end{pmatrix}\mathbf{S}^{-1}, \tag{1.33}$$

where $\mathbf{A}_k$ are $m_k \times m_k$ matrices ($k = 1, 2$; $m_1 + m_2 = n$) and

$$\mathbf{A}_1 = \|\alpha'_{ij}\|_1^{m_1}, \quad \mathbf{A}_2 = \|\alpha''_{ij}\|_1^{m_2}.$$

Let $\mathbf{s}_1, \ldots, \mathbf{s}_n$ denote the columns of $\mathbf{S}$:

$$\mathbf{S} = \|\mathbf{s}_1, \ldots, \mathbf{s}_{m_1}, \mathbf{s}_{m_1+1}, \ldots, \mathbf{s}_n\|. \tag{1.34}$$

It follows from (1.33) that $\mathbf{AS} = \mathbf{S}(\mathbf{A}_1 + \mathbf{A}_2)$, i. e.,

$$\mathbf{A}\|\mathbf{s}_1, \ldots, \mathbf{s}_{m_1}\| = \|\mathbf{s}_1, \ldots, \mathbf{s}_{m_1}\|\mathbf{A}_1, \quad \mathbf{A}\|\mathbf{s}_{m_1+1}, \ldots, \mathbf{s}_n\| = \|\mathbf{s}_{m_1+1}, \ldots, \mathbf{s}_n\|\mathbf{A}_2,$$

or

$$\mathbf{A}\mathbf{s}_\nu = \sum_{i=1}^{m_1} \alpha'_{i\nu}\mathbf{s}_i \quad (\nu = 1, \ldots, m_1), \qquad \mathbf{A}\mathbf{s}_\mu = \sum_{i=1}^{m_2} \alpha''_{i\mu}\mathbf{s}_i \quad (\mu = 1, \ldots, m_2). \tag{1.35}$$

The vectors $\mathbf{s}_1, \ldots, \mathbf{s}_n$ are linearly independent (because $\det \mathbf{S} \neq 0$). Consequently, the same is true of the vectors $\mathbf{s}_1, \ldots, \mathbf{s}_{m_1}$ and $\mathbf{s}_{m_1+1}, \ldots, \mathbf{s}_n$.

Consider the subspaces $\mathfrak{M}_1$ and $\mathfrak{M}_2$ with bases $\mathbf{s}_1, \ldots, \mathbf{s}_{m_1}$ and $\mathbf{s}_{m_1+1}, \ldots, \mathbf{s}_n$, respectively ($\mathfrak{M}_1$ is the set of all vectors $\gamma_1\mathbf{s}_1 + \ldots + \gamma_{m_1}\mathbf{s}_{m_1}$, and $\mathfrak{M}_2$ the set of vectors $\gamma_{m_1+1}\mathbf{s}_{m_1+1} + \ldots + \gamma_n\mathbf{s}_n$, where γ_j are arbitrary complex numbers).

A subspace $\mathfrak{M}$ is said to be invariant under $\mathbf{A}$ if $\mathbf{x} \in \mathfrak{M}$ implies that $\mathbf{A}\mathbf{x} \in \mathfrak{M}$. It follows from (1.35) that each of the subspaces $\mathfrak{M}_1$, $\mathfrak{M}_2$ is invariant under $\mathbf{A}$.

Since $\mathbf{s}_1, \ldots, \mathbf{s}_n$ is a basis for $\mathfrak{N}_n$, any vector $\mathbf{x}$ is uniquely expressible as $\mathbf{x} = \mathbf{x}' + \mathbf{x}''$, where $\mathbf{x}' \in \mathfrak{M}_1$, $\mathbf{x}'' \in \mathfrak{M}_2$, i. e.,

$$\mathfrak{N}_n = \mathfrak{M}_1 \dotplus \mathfrak{M}_2.$$

Now suppose that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are arbitrary $\mathbf{A}$-invariant subspaces of dimensions m_1 and m_2, respectively, such that the above decomposition holds. It is easily shown that then (1.33) is true for some $\mathbf{S}$, $\mathbf{A}_1$, $\mathbf{A}_2$. To this end, we simply note that $m_1 + m_2 = n$, take an arbitrary basis $\mathbf{s}_1, \ldots, \mathbf{s}_{m_1}$ in $\mathfrak{M}_1$ and an arbitrary basis $\mathbf{s}_{m_1+1}, \ldots, \mathbf{s}_n$ in $\mathfrak{M}_2$, and define $\mathbf{S}$ by (1.34). The invariance of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ implies (1.35), which is equivalent to (1.33).

All this remains valid when applied to a block-diagonal matrix with any number of blocks. In fact, if there is a matrix $\mathbf{S}$ such that

$$\mathbf{A} = \mathbf{S}\begin{pmatrix} \mathbf{A}_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{A}_p \end{pmatrix}\mathbf{S}^{-1} = \mathbf{S}(\mathbf{A}_1 \dotplus \ldots \dotplus \mathbf{A}_p)\mathbf{S}^{-1},$$

where $\mathbf{A}_j$ are matrices of order m_j ($j = 1, \ldots, p$; $m_1 + \ldots + m_p = n$), then the space $\mathfrak{N}_n$ splits into a direct sum of $\mathbf{A}$-invariant subspaces $\mathfrak{M}_j$ of dimension m_j:

$$\mathfrak{N}_n = \mathfrak{M}_1 \dotplus \ldots \dotplus \mathfrak{M}_p, \tag{1.36}$$

and conversely, if (1.36) is true we have a representation of the above type. The basis for each subspace $\mathfrak{M}_j$ will be the corresponding group of columns of $\mathbf{S}$. Thus reduction of a matrix $\mathbf{A}$ to block-diagonal form is equivalent to decomposition of the space $\mathfrak{N}_n$ as a direct sum of $\mathbf{A}$-invariant subspaces.

1.10. Root subspaces and simple cyclic subspaces. Decomposition of the space into a direct sum of simple cyclic A-invariant subspaces

A vector $\mathbf{b} \neq \mathbf{0}$ is called a root vector* of a matrix $\mathbf{A}$ if there exists a complex number λ_0 such that for some natural number k

* [Or generalized eigenvector.]

$$\boxed{(\mathbf{A}-\lambda_0\mathbf{I}_n)^k\,\mathbf{b}=0.} \tag{1.37}$$

It follows from (1.37) (§ 1.6) that

$$\boxed{\det[(\mathbf{A}-\lambda_0\mathbf{I}_n)^k]=[\det(\mathbf{A}-\lambda_0\mathbf{I}_n)]^k=0.}$$

Thus the number λ_0 in (1.37) must be a root of the characteristic equation (1.29), i. e., an eigenvalue of $\mathbf{A}$. And then for any $k\geqslant 1$ there is a vector $\mathbf{b}\neq\mathbf{0}$ satisfying (1.37).

The set $\mathfrak{L}_{\lambda_0}$ of root vectors $\mathbf{b}\in\mathfrak{R}_n$ corresponding to a fixed root λ_0 is clearly a subspace, known as a root subspace. Thus the matrix $\mathbf{A}$ has as many root subspaces as the characteristic equation has roots.

Every eigenvector is of course a root vector (with $k=1$). Conversely, for every root vector $\mathbf{b}$ we can find a corresponding eigenvector. Indeed, in the sequence of vectors

$$\mathbf{b},\ (\mathbf{A}-\lambda_0\mathbf{I}_n)\,\mathbf{b},\ \ldots,\ (\mathbf{A}-\lambda_0\mathbf{I}_n)^k\,\mathbf{b},\ \ldots$$

the first is not zero, but from some point on all vectors in the sequence vanish, by (1.37). Denote the last nonzero vector by $\mathbf{a}$. Then for some natural $p\leqslant k$

$$\mathbf{a}=(\mathbf{A}-\lambda_0\mathbf{I}_n)^{p-1}\,\mathbf{b}\neq 0,\qquad (\mathbf{A}-\lambda_0\mathbf{I}_n)^p\,\mathbf{b}=0.$$

Thus $(\mathbf{A}-\lambda_0\mathbf{I}_n)\,\mathbf{a}=0$, so that $\mathbf{a}$ is an eigenvector.

Denote $\mathbf{f}_1=\mathbf{a}=(\mathbf{A}-\lambda_0\mathbf{I}_n)^{p-1}\,\mathbf{b},\ \mathbf{f}_2=(\mathbf{A}-\lambda_0\mathbf{I}_n)^{p-2}\,\mathbf{b},\ \ldots,\ \mathbf{f}_{p-1}=(\mathbf{A}-\lambda_0\mathbf{I}_n)\,\mathbf{b},\ \mathbf{f}_p=\mathbf{b}$. With this notation, we have

$$(\mathbf{A}-\lambda_0\mathbf{I}_n)\,\mathbf{f}_1=0,\ (\mathbf{A}-\lambda_0\mathbf{I}_n)\,\mathbf{f}_2=\mathbf{f}_1,\ \ldots,\ (\mathbf{A}-\lambda_0\mathbf{I}_n)\,\mathbf{f}_p=\mathbf{f}_{p-1},$$

or

$$\mathbf{A}\mathbf{f}_1=\lambda_0\mathbf{f}_1,\quad \mathbf{A}\mathbf{f}_2=\lambda_0\mathbf{f}_2+\mathbf{f}_1,\ \ldots,\quad \mathbf{A}\mathbf{f}_p=\lambda_0\mathbf{f}_p+\mathbf{f}_{p-1}. \tag{1.38}$$

The vectors $\mathbf{f}_1,\ \ldots,\mathbf{f}_p$ are linearly independent. Indeed, in the case $p=3$, say, if

$$\alpha_1\mathbf{f}_1+\alpha_2\mathbf{f}_2+\alpha_3\mathbf{f}_3=0$$

then

$$(\mathbf{A}-\lambda_0\mathbf{I}_n)(\alpha_1\mathbf{f}_1+\alpha_2\mathbf{f}_2+\alpha_3\mathbf{f}_3)=\alpha_2\mathbf{f}_1+\alpha_3\mathbf{f}_2=0,$$
$$(\mathbf{A}-\lambda_0\mathbf{I}_n)(\alpha_2\mathbf{f}_1+\alpha_3\mathbf{f}_2)=\alpha_3\mathbf{f}_1=0.$$

Since $\mathbf{f}_1\neq\mathbf{0}$, we successively obtain $\alpha_3=0,\ \alpha_2=0,\ \alpha_1=0$.

A subspace $\mathfrak{M}$ which has a basis $\mathbf{f}_1,\ \ldots,\ \mathbf{f}_p$ satisfying (1.38) is known as a simple cyclic subspace (relative to $\mathbf{A}$). * This subspace is obviously invariant under $\mathbf{A}$: if $\mathbf{x}\in\mathfrak{M}$, then $\mathbf{A}\mathbf{x}\in\mathfrak{M}$.

* The definition of a cyclic subspace will not be needed below, and is therefore omitted.

We have shown that to each root vector **b** corresponds a simple cyclic subspace $\mathfrak{M}$.

Note that whereas the root λ_0 uniquely determines a root subspace of **A**, there may be more than one simple cyclic subspace — it depends on the choice of the vector **b**.

Let $\lambda^{(1)}, \ldots, \lambda^{(q)}$ $(q \leqslant n)$ be all the distinct eigenvalues of a matrix **A** and $\mathfrak{L}_1, \ldots, \mathfrak{L}_q$ the corresponding root subspaces. (We are interested in the case $q < n$, i. e., some of the numbers $\lambda^{(\varkappa)}$ are multiple roots of the characteristic equation.)

Theorem. (1) The dimension of the root subspace $\mathfrak{L}_\varkappa$ is equal to the multiplicity of the corresponding root $\lambda^{(\varkappa)}$ of the equation $\det(\mathbf{A}-\lambda\mathbf{I}_n)=0$ $(\varkappa=1, \ldots, q)$.

(2) *The space $\mathfrak{R}_n$ is a direct sum of root subspaces:*

$$\mathfrak{R}_n = \mathfrak{L}_1 \dotplus \ldots \dotplus \mathfrak{L}_q, \tag{1.39}$$

i.e., any vector $\mathbf{x} \in \mathfrak{R}_n$ is uniquely expressible as

$$\mathbf{x} = \mathbf{x}_1 + \ldots + \mathbf{x}_q, \text{ where } \mathbf{x}_\varkappa \in \mathfrak{L}_\varkappa \qquad (\varkappa = 1, \ldots, q).$$

For every root subspace $\mathfrak{L}$, there exist simple cyclic subspaces $\mathfrak{M}_1, \ldots, \mathfrak{M}_j$ such that

$$\mathfrak{L} = \mathfrak{M}_1 \dotplus \ldots \dotplus \mathfrak{M}_j. \tag{1.40}$$

A proof of this theorem may be found in /63/, Chapter III, §6.49.

A simple cyclic subspace is no longer expressible as a direct sum of **A**-invariant subspaces.

1.11. Jordan form

We now consider the reduction of a matrix to block-diagonal form corresponding (see §1.9) to the direct-sum decompositions (1.39) and (1.40).

Suppose that the total number of simple cyclic subspaces into which the space $\mathfrak{R}_n$ splits is s. Substituting (1.40) into (1.39) and changing the notation, we have

$$\mathfrak{R}_n = \mathfrak{M}_1 \dotplus \ldots \dotplus \mathfrak{M}_s.$$

Let the dimension of $\mathfrak{M}_\sigma$ be m_σ. For each $\mathfrak{M}_\sigma$ we have (1.38), with $p = m_\sigma$ and λ_0 some root of the characteristic equation (1.29). Let $\lambda_{(\sigma)}$ denote the corresponding roots and $\mathbf{f}_1^{(\sigma)}, \ldots, \mathbf{f}_{m_\sigma}^{(\sigma)}$ a basis for $\mathfrak{M}_\sigma$ $(\sigma = 1, \ldots, s)$. (Generally speaking, some of the s numbers $\lambda_{(\sigma)}$ may now coincide — each of the numbers $\lambda_{(\sigma)}$ corresponds to a simple cyclic subspace $\mathfrak{M}_\sigma$.)

We rewrite (1.38) as

$$\mathbf{A}\mathbf{f}_1^{(\sigma)} = \lambda_{(\sigma)}\mathbf{f}_1^{(\sigma)}, \quad \mathbf{A}\mathbf{f}_2^{(\sigma)} = \lambda_{(\sigma)}\mathbf{f}_2^{(\sigma)} + \mathbf{f}_1^{(\sigma)}, \quad \ldots, \quad \mathbf{A}\mathbf{f}_{m_\sigma}^{(\sigma)} = \lambda_{(\sigma)}\mathbf{f}_{m_\sigma}^{(\sigma)} + \mathbf{f}_{m_\sigma-1}^{(\sigma)}, \tag{1.41}$$

then as

$$\mathbf{A}\left\|\mathbf{f}_1^{(\sigma)}, \ldots, \mathbf{f}_{m_\sigma}^{(\sigma)}\right\| = \left\|\mathbf{f}_1^{(\sigma)}, \ldots, \mathbf{f}_{m_\sigma}^{(\sigma)}\right\| \begin{Vmatrix} \lambda_{(\sigma)} & 1 & 0 & \ldots & 0 \\ 0 & \lambda_{(\sigma)} & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & \lambda_{(\sigma)} \end{Vmatrix} \tag{1.42}$$

and finally

$$\mathbf{AF}_\sigma = \mathbf{F}_\sigma \mathbf{Q}_\sigma(\lambda_\sigma) \qquad (\sigma = 1, \ldots, s). \tag{1.43}$$

Here $\mathbf{F}_\sigma = \left\|\mathbf{f}_1^{(\sigma)}, \ldots, \mathbf{f}_{m_\sigma}^{(\sigma)}\right\|$ is a rectangular $n \times m_\sigma$ matrix whose columns form a basis for the σ-th cyclic subspace, and

$$\mathbf{Q}_\sigma(\lambda_{(\sigma)}) = \begin{Vmatrix} \lambda_{(\sigma)} & 1 & 0 & \ldots & 0 \\ 0 & \lambda_{(\sigma)} & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & \lambda_{(\sigma)} \end{Vmatrix} \tag{1.44}$$

is a square $m_\sigma \times m_\sigma$ matrix, called an elementary Jordan matrix $m_1 + \ldots + m_s = n$. We can combine (1.43) into a single equality:

$$\mathbf{AS} = \mathbf{SQ}, \tag{1.45}$$

where $\mathbf{S}$ and $\mathbf{Q}$ are $n \times n$ matrices:

$$\mathbf{S} = \|\mathbf{F}_1, \ldots, \mathbf{F}_s\|,$$
$$\mathbf{Q} = \begin{pmatrix} \mathbf{Q}_1(\lambda_{(1)}) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{Q}_s(\lambda_{(s)}) \end{pmatrix} = \mathbf{Q}_1(\lambda_{(1)}) \dotplus \ldots \dotplus \mathbf{Q}_s(\lambda_{(s)}). \tag{1.46}$$

The columns of the matrix $\mathbf{S}$,

$$\mathbf{f}_1^{(1)}, \ldots, \mathbf{f}_{m_1}^{(1)}, \mathbf{f}_1^{(2)}, \ldots, \mathbf{f}_{m_2}^{(2)}, \ldots, \mathbf{f}_1^{(s)}, \ldots, \mathbf{f}_{m_s}^{(s)} \tag{1.47}$$

form a basis for $\mathfrak{R}_n$ (theorem in §1.10) and so $\det \mathbf{S} \neq 0$. Consequently, we may write (1.45) as

$$\mathbf{S}^{-1}\mathbf{AS} = \mathbf{Q}, \quad \text{or} \quad \mathbf{A} = \mathbf{SQS}^{-1}. \tag{1.48}$$

The matrix $\mathbf{Q}$, which is a direct sum of elementary Jordan matrices, is called the Jordan (or normal, or canonical) form of the matrix $\mathbf{A}$. Formula (1.48) generalizes (1.32) and is one of the fundamental formulas of matrix theory.

We shall refer to (1.47) as the basis relative to which $\mathbf{A}$ is in Jordan form.

Formulas (1.48) and (1.27) together describe the structure of an arbitrary linear operator $\mathfrak{A}$: the space $\mathfrak{R}_n$ may always be expressed as a direct sum of simple cyclic subspaces, in each of which the action of the operator is determined by strikingly simple formulas (1.43). In other words, relative to the special basis (1.47) the matrix $\mathbf{Q}$ of the operator $\mathfrak{A}$ (§1.7) is a direct sum of elementary Jordan matrices.

Remark. In many cases, one defines an elementary Jordan matrix as

$$\mathbf{Q}(\lambda_{(\sigma)}, \delta_\sigma) = \begin{Vmatrix} \lambda_{(\sigma)} & \delta_\sigma & 0 & \dots & 0 \\ 0 & \lambda_{(\sigma)} & \delta_\sigma & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \delta_\sigma \\ 0 & 0 & 0 & \dots & \lambda_{(\sigma)} \end{Vmatrix},$$

where $\delta_\sigma \neq 0$ is an arbitrary number. If $\mathfrak{A}$ is an operator defined by its matrix $\mathbf{A}$ relative to the standard basis, then $\mathbf{Q}_\sigma(\lambda_{(\sigma)}, \delta_\sigma)$ is the matrix of this operator in the subspace $\mathfrak{M}_\sigma$, relative to the basis

$$\mathbf{g}_1^{(\sigma)} = \mathbf{f}_1^{(\sigma)}, \quad \mathbf{g}_2^{(\sigma)} = \delta_\sigma \mathbf{f}_2^{(\sigma)}, \quad \mathbf{g}_3^{(\sigma)} = \delta_\sigma^2 \mathbf{f}_3^{(\sigma)}, \quad \dots, \quad \mathbf{g}_{m_\sigma}^{(\sigma)} = \delta_\sigma^{m_\sigma - 1} \mathbf{f}_{m_\sigma}^{(\sigma)},$$

where $\mathbf{f}_1^{(\sigma)}, \dots, \mathbf{f}_{m_\sigma}^{(\sigma)}$ are the vectors defined above. Indeed, multiplying (1.41) in succession by $1, \delta_\sigma, \delta_\sigma^2, \dots, \delta_\sigma^{m_\sigma - 1}$, we get

$$\mathbf{A}\mathbf{g}_1^{(\sigma)} = \lambda_{(\sigma)} \mathbf{g}_1^{(\sigma)}, \quad \mathbf{A}\mathbf{g}_2^{(\sigma)} = \lambda_{(\sigma)} \mathbf{g}_2^{(\sigma)} + \delta_\sigma \mathbf{g}_1^{(\sigma)}, \quad \dots, \quad \mathbf{A}\mathbf{g}_{m_\sigma}^{(\sigma)} = \lambda_{(\sigma)} \mathbf{g}_{m_\sigma}^{(\sigma)} + \delta_\sigma \mathbf{g}_{m_\sigma - 1}^{(\sigma)}.$$

These equalities may be written

$$\mathbf{A}\mathbf{G}_\sigma = \mathbf{G}_\sigma(\lambda_{(\sigma)}, \delta_\sigma),$$

where

$$\mathbf{G}_\sigma = \left\| \mathbf{g}_1^{(\sigma)}, \dots, \mathbf{g}_{m_\sigma}^{(\sigma)} \right\|,$$

proving the assertion.

1.12. Elementary divisors. Minimum polynomial

Let us associate the elementary Jordan matrix $\mathbf{Q}_\sigma(\lambda_{(\sigma)})$ in (1.46) with the binomial

$$(\lambda - \lambda_{(\sigma)})^{m_\sigma} = \det [\lambda \mathbf{I}_{m_\sigma} - \mathbf{Q}_\sigma(\lambda_{(\sigma)})],$$

where m_σ is the dimension of the square matrix $\mathbf{Q}_\sigma(\lambda_{(\sigma)})$.

It follows from (1.48) that

$$\det(\lambda \mathbf{I}_n - \mathbf{A}) = \prod_{\sigma=1}^{s} (\lambda - \lambda_{(\sigma)})^{m_\sigma}. \tag{1.49}$$

The binomials

$$(\lambda-\lambda_{(1)})^{m_1}, \ldots, (\lambda-\lambda_{(s)})^{m_s}, \tag{1.50}$$

whose number is precisely the number of cyclic subspaces, are known as the elementary divisors of $\mathbf{A}$. * Given the matrix, they may be determined directly (/21/, Chapter VI, § 3). The set of elementary divisors (1.50) uniquely determines the decomposition (1.46), up to permutations of the diagonal blocks. Recall that some of the numbers $\lambda_{(s)}$ in the sequence (1.50) may be equal. ** Thus, to each root of the characteristic equation (1.29) there will in general correspond several elementary divisors (several elementary Jordan matrices $\mathbf{Q}_\sigma(\lambda^{(\varkappa)})$ in (1.46)). However, it should be clear from (1.49) that the sum of exponents of the elementary divisors belonging to any root (the sum of orders of the blocks $\mathbf{Q}_\sigma(\lambda^{(\varkappa)})$) is equal to its multiplicity as a root of the characteristic equation.

If all the elementary divisors are simple ($s=n$, $m_1=m_2=\ldots=m_n=1$), the second of formulas (1.48) reduces to (1.32), so that $\mathbf{A}$ may be reduced to diagonal form. This situation occurs, as we saw in subsection 1.8, if (but not only if) all the roots of the characteristic equation are distinct.

Let $\lambda^{(1)}, \ldots, \lambda^{(q)}$ be all distinct eigenvalues of $\mathbf{A}$. Consider the elementary divisors corresponding to an eigenvalue $\lambda^{(\varkappa)}$. Let

$$(\lambda-\lambda^{(\varkappa)})^{e_\varkappa}$$

be one of these elementary divisors, with maximum exponent. The polynomial

$$\varphi(\lambda)=(\lambda-\lambda^{(1)})^{e_1}\ldots(\lambda-\lambda^{(q)})^{e_q} \tag{1.51}$$

is known as the minimum polynomial of the matrix $\mathbf{A}$. It can be shown (see, e.g., /21/, Chapter IV, § 6) that

$$\varphi(\mathbf{A})\equiv(\mathbf{A}-\lambda^{(1)}\mathbf{I}_n)^{e_1}\ldots(\mathbf{A}-\lambda^{(q)}\mathbf{I}_n)^{e_q}=\mathbf{0}$$

and if $\varphi_1(\lambda)$ is any other polynomial such that $\varphi_1(\mathbf{A})=\mathbf{0}$ then $\varphi_1(\lambda)$ divides $\varphi(\lambda)$, in the sense that

$$\varphi_1(\lambda)=\text{const}\,(\lambda-\lambda^{(1)})^{p_1}\ldots(\lambda-\lambda^{(q)})^{p_q},$$

where

$$p_1\geqslant e_1, \ldots, p_q\geqslant e_q.$$

* Frequently called the elementary divisors of $\lambda\mathbf{I}_n-\mathbf{A}$ (see /21/).

** A word of explanation about the notation. Apart from the notation $\lambda_1, \ldots, \lambda_n$ for the eigenvalues of an $n\times n$ matrix $\mathbf{A}$, we shall use the notation $\lambda^{(1)}, \ldots, \lambda^{(q)}$ when listing all the distinct eigenvalues, and $\lambda_{(1)}, \ldots, \lambda_{(s)}$ when each of the eigenvalues is listed as many times as there are elementary divisors (cyclic subspaces) corresponding to it. Clearly, $q\leqslant s\leqslant n$.

Let $\mathfrak{M}$ be an invariant subspace for $\mathbf{A}$ ($\mathbf{x} \in \mathfrak{M}$ implies $\mathbf{Ax} \in \mathfrak{M}$). Letting $\mathbf{f}_1, \ldots, \mathbf{f}_m$ be any basis of $\mathfrak{M}$, we have

$$\mathbf{Af}_\mu = \sum_{j=1}^{n} \alpha_{\mu j} \mathbf{f}_j \qquad (\mu = 1, \ldots, m).$$

The $m \times m$ matrix

$$\mathbf{A}_{\mathfrak{M}} = \| \alpha_{\mu j} \|_1^m$$

is known as the matrix induced by $\mathbf{A}$ in $\mathfrak{M}$ relative to the basis $\mathbf{f}_1, \ldots, \mathbf{f}_m$. It is easy to see that the eigenvalues of $\mathbf{A}_{\mathfrak{M}}$ are necessarily eigenvalues of $\mathbf{A}$. The minimum polynomial of $\mathbf{A}_{\mathfrak{M}}$ may be written

$$\varphi_{\mathfrak{M}}(\lambda) = (\lambda - \lambda^{(1)})^{\mu_1} \ldots (\lambda - \lambda^{(q)})^{\mu_q},$$

where some of the numbers $\mu_1, \ldots, \mu_q$ may vanish. The polynomial $\varphi_{\mathfrak{M}}(\lambda)$ is independent of the choice of the basis $\mathbf{f}_1, \ldots, \mathbf{f}_m$ and is called the minimum polynomial of $\mathbf{A}$ in $\mathfrak{M}$. It can be shown (/21/, Chapter IV, §6) that $\varphi(\lambda)$ is divisible by $\varphi_{\mathfrak{M}}(\lambda)$, i. e.,

$$e_1 \geqslant \mu_1, \ldots, e_q \geqslant \mu_q.$$

1.13. Root subspaces of adjoint matrices

Consider an arbitrary matrix $\mathbf{A}$ and its adjoint $\mathbf{A}^*$. Let $\lambda^{(1)}, \ldots, \lambda^{(q)}$ be all the distinct eigenvalues of $\mathbf{A}$, i. e. (§1.8), the roots of the characteristic equation $\det(\mathbf{A} - \lambda \mathbf{I}_n) = 0$, and $r_1, \ldots, r_q$ their respective multiplicities. Then

$$\boxed{(-1)^n \det(\mathbf{A} - \lambda \mathbf{I}) = \prod_{j=1}^{q} (\lambda - \lambda^{(j)})^{r_j}.}$$

Since

$$\det(\mathbf{A}^* - \lambda \mathbf{I}) = \det(\overline{\mathbf{A}} - \lambda \mathbf{I})^{\mathrm{t}} = \overline{\det(\mathbf{A} - \bar{\lambda} \mathbf{I})},$$

we have

$$(-1)^n \det(\mathbf{A}^* - \lambda \mathbf{I}) = \prod_{j=1}^{q} (\lambda - \bar{\lambda}^{(j)})^{r_j}.$$

This shows that the distinct roots of the equation $\det(\mathbf{A}^* - \lambda \mathbf{I}) = 0$, in other words the eigenvalues of $\mathbf{A}^*$, are the numbers $\bar{\lambda}^{(1)}, \ldots, \bar{\lambda}^{(q)}$, and their multiplicities are $r_1, \ldots, r_q$.

Let $\mathfrak{L}_j$ denote the root subspace of $\mathbf{A}$ belonging to $\lambda^{(j)}$, and $\mathfrak{L}_j^*$ the root subspace of $\mathbf{A}^*$ belonging to $\bar{\lambda}^{(j)}$. By part 1 of the theorem in §1.10, $\mathfrak{L}_j$ and $\mathfrak{L}_j^*$ have the same dimension r_j. By part 2 of the same theorem,

$$\mathfrak{R}_n = \mathfrak{L}_1 \dotplus \ldots \dotplus \mathfrak{L}_q, \quad \mathfrak{R}_n = \mathfrak{L}_1^* \dotplus \ldots \dotplus \mathfrak{L}_q^*.$$

Theorem. *The subspaces $\mathfrak{L}_j$ and $\mathfrak{L}_h^*$, $j \neq h$, are orthogonal.*

Proof. Let $\mathbf{a} \in \mathfrak{L}_j$ and $\mathbf{b} \in \mathfrak{L}_h^*$, $j \neq h$. We have to show that $(\mathbf{a}, \mathbf{b}) = 0$. For some $m_1 \geqslant 1$, $m_2 \geqslant 1$, we have

$$(\mathbf{A} - \lambda^{(j)}\mathbf{I})^{m_1}\mathbf{a} = 0, \quad (\mathbf{A}^* - \bar{\lambda}^{(h)}\mathbf{I})^{m_2}\mathbf{b} = 0. \tag{1.52}$$

The proof will now proceed by induction on $m_1 + m_2$. If $m_1 + m_2 = 2$, then $m_1 = m_2 = 1$, so that both $\mathbf{a}$ and $\mathbf{b}$ are eigenvectors:

$$\mathbf{Aa} = \lambda^{(j)}\mathbf{a}, \quad \mathbf{A}^*\mathbf{b} = \bar{\lambda}^{(h)}\mathbf{b}.$$

We have

$$\lambda^{(j)}(\mathbf{a}, \mathbf{b}) = (\mathbf{Aa}, \mathbf{b}) = (\mathbf{a}, \mathbf{A}^*\mathbf{b}) = \lambda^{(h)}(\mathbf{a}, \mathbf{b}),$$

so that

$$(\lambda^{(j)} - \lambda^{(h)})(\mathbf{a}, \mathbf{b}) = 0. \tag{1.53}$$

But by assumption $\lambda^{(j)} \neq \lambda^{(h)}$, and so $(\mathbf{a}, \mathbf{b}) = 0$.

Let $m \geqslant 3$ be an arbitrary integer; suppose the assertion true for all natural numbers m_1, such that $m_1 + m_2 < m$. We shall prove that it is also true for all m_1, m_2 such that $m_1 + m_2 = m$.

Denote

$$\mathbf{a}' = (\mathbf{A} - \lambda^{(j)}\mathbf{I})\mathbf{a}, \quad \mathbf{b}' = (\mathbf{A}^* - \lambda^{(h)}\mathbf{I})\mathbf{b}.$$

It follows from (1.52) that

$$(\mathbf{A} - \lambda^{(j)}\mathbf{I})^{m_1 - 1}\mathbf{a}' = 0, \quad (\mathbf{A}^* - \bar{\lambda}^{(h)}\mathbf{I})^{m_2 - 1}\mathbf{b}' = 0.$$

By the inductive hypothesis,

$$(\mathbf{a}', \mathbf{b}) = 0, \quad (\mathbf{a}, \mathbf{b}') = 0.$$

Substituting $\mathbf{a}'$ and $\mathbf{b}'$ into these equalities, we get

$$(\mathbf{Aa}, \mathbf{b}) = \lambda^{(j)}(\mathbf{a}, \mathbf{b}), \quad (\mathbf{a}, \mathbf{A}^*\mathbf{b}) = \bar{\lambda}^{(h)}(\mathbf{a}, \mathbf{b}).$$

The left-hand sides of these equalities are equal and so $(\mathbf{a}, \mathbf{b}) = 0$. Q. E. D.

Corollary 1. The orthogonal complement of the root subspace $\mathfrak{L}_j$ of $\mathbf{A}$ belonging to an eigenvalue $\lambda^{(j)}$ is the sum of the root subspaces of $\mathbf{A}^*$ belonging to all its eigenvalues other than $\bar{\lambda}^{(j)}$:

$$\mathfrak{L}_j^{\perp} = \sum_{h \neq j} \mathfrak{L}_h^*. \tag{1.54}$$

Indeed, by the above theorem the subspaces $\mathfrak{L}_j$ and $\sum_{h\neq j}\mathfrak{L}_h^*$ are orthogonal. In addition, the sum of their dimensions is the dimension n of the entire space, so that

$$\boxed{\mathfrak{R}_n=\mathfrak{L}_j\oplus\left(\sum_{h\neq j}\mathfrak{L}_h^*\right),} \tag{1.55}$$

which is equivalent to (1.54).

Corollary 2. Let r be the dimension of some root subspace $\mathfrak{L}_j$, hence also of $\mathfrak{L}_j^*$. Let $\mathbf{a}_1, \ldots, \mathbf{a}_r$ be an arbitrary basis of $\mathfrak{L}_j$ and $\mathbf{b}_1, \ldots, \mathbf{b}_r$ an arbitrary basis of $\mathfrak{L}_j^*$. Then

$$\det\|(\mathbf{a}_k, \mathbf{b}_h)\|_{k,\,h=1}^{r}\neq 0. \tag{1.56}$$

Indeed, supposing the contrary, we see that for some numbers $\zeta_1, \ldots, \zeta_r$

$$\sum_h(\mathbf{a}_k, \mathbf{b}_k)\zeta_h=0 \qquad \left(k=1, \ldots, r;\ \sum_h|\zeta_h|\neq 0\right).$$

Setting $\mathbf{b}=\zeta_1\mathbf{b}_1+\ldots+\zeta_r\mathbf{b}_r$, we obtain

$$\mathbf{b}\neq 0, \quad \mathbf{b}\in\mathfrak{L}_j^*, \quad \mathbf{b}\perp\mathfrak{L}_j.$$

It follows from the last two relations and from (1.55) that $\mathbf{b}=0$ — contradiction.

1.14. Hermitian, antihermitian, symmetric, skew-symmetric, unitary and orthogonal matrices. Reduction of hermitian and unitary matrices to diagonal form

The definitions of the matrices listed in the heading are as follows:
hermitian matrix: $\mathbf{A}^*=\mathbf{A}$,
antihermitian (or skew-hermitian) matrix: $\mathbf{A}^*=-\mathbf{A}$,
symmetric matrix: $\mathbf{A}^\tau=\mathbf{A}$,
skew-symmetric matrix: $\mathbf{A}^\tau=-\mathbf{A}$,
unitary matrix: $\mathbf{A}^*\mathbf{A}=\mathbf{I}$,
orthogonal* matrix: $\mathbf{A}^\tau\mathbf{A}=\mathbf{I}$.
(For the definitions of the operators $(\ )^*$ and $(\ )^\tau$, see §1.2.) A real symmetric matrix $\mathbf{A}$ is obviously hermitian. It is easy to see that the columns of a unitary matrix $\mathbf{A}$ constitute an orthonormal basis.

For any $\mathbf{x}, \mathbf{y}\in\mathfrak{R}_n$, we have $(\mathbf{Ax}, \mathbf{y})=(\mathbf{x}, \mathbf{Ay})$ if $\mathbf{A}$ is a hermitian matrix, and $(\mathbf{Ax}, \mathbf{Ay})=(\mathbf{x}, \mathbf{y})$ if $\mathbf{A}$ is a unitary matrix.

All the eigenvalues of a hermitian matrix are real. All the eigenvalues of a unitary matrix have absolute value 1.

Indeed, let $\mathbf{Aa}=\lambda\mathbf{a}$, where $\mathbf{A}$ is hermitian. We have

$$\lambda=\frac{(\mathbf{Aa}, \mathbf{a})}{(\mathbf{a}, \mathbf{a})}=\frac{(\mathbf{a}, \mathbf{Aa})}{(\mathbf{a}, \mathbf{a})}=\overline{\lambda}.$$

* An orthogonal matrix is sometimes defined as a real matrix $\mathbf{S}$ such that $\mathbf{S}^\tau\mathbf{S}=\mathbf{I}$.

If $\mathbf{A}$ is unitary, then $(\mathbf{Aa}, \mathbf{Aa}) = \lambda\bar{\lambda}(\mathbf{a}, \mathbf{a})$ and $(\mathbf{Aa}, \mathbf{Aa}) = (\mathbf{a}, \mathbf{a})$, so that $\lambda\bar{\lambda} = |\lambda|^2 = 1$.

Eigenvectors of a hermitian (or unitary) matrix belonging to different eigenvalues are orthogonal.

Indeed, let $\mathbf{Aa}_1 = \lambda_1\mathbf{a}_1$, $\mathbf{Aa}_2 = \lambda_2\mathbf{a}_2$. For a hermitian matrix,

$$\lambda_1(\mathbf{a}_1, \mathbf{a}_2) = (\mathbf{Aa}_1, \mathbf{a}_2) = (\mathbf{a}_1, \mathbf{Aa}_2) = \lambda_2(\mathbf{a}_1, \mathbf{a}_2),$$

whence $(\mathbf{a}_1, \mathbf{a}_2) = 0$. For a unitary matrix,

$$\lambda_1\bar{\lambda}_2(\mathbf{a}_1, \mathbf{a}_2) = (\mathbf{Aa}_1, \mathbf{Aa}_2) = (\mathbf{a}_1, \mathbf{a}_2)$$

and since $\bar{\lambda}_2 = \lambda_2^{-1}$, it follows again that $(\mathbf{a}_1, \mathbf{a}_2) = 0$.

Hermitian and unitary matrices have only simple elementary divisors.

Indeed, suppose, say, that a hermitian matrix $\mathbf{A}$ has a multiple elementary divisor, so that

$$\mathbf{Af}_1 = \lambda\mathbf{f}_1, \quad \mathbf{Af}_2 = \lambda\mathbf{f}_2 + \mathbf{f}_1 \qquad (\mathbf{f}_1 \neq \mathbf{0}).$$

Then

$$(\mathbf{f}_2, \mathbf{Af}_1) = \lambda(\mathbf{f}_2, \mathbf{f}_1), \quad (\mathbf{Af}_2, \mathbf{f}_1) = \lambda(\mathbf{f}_2, \mathbf{f}_1) + (\mathbf{f}_1, \mathbf{f}_1).$$

The left-hand sides of these equalities are equal. Comparing their right-hand sides, we obtain $(\mathbf{f}_1, \mathbf{f}_1) = 0$ or $\mathbf{f}_1 = \mathbf{0}$, contradicting our assumption.

Hence it follows that *hermitian and unitary matrices may be reduced to diagonal form* (1.32). We shall show that, in addition, *the matrix* $\mathbf{S}$ *in* (1.32) *may be chosen to be unitary.* Indeed, the columns of the matrix $\mathbf{S}$ are eigenvectors of $\mathbf{A}$. We showed above that if the eigenvalues are distinct the eigenvectors are orthogonal. If we normalize them, the resulting matrix $\mathbf{S}$ will be unitary. In the case of multiple eigenvalues, we choose an orthogonal basis (see §1.5) in each eigensubspace; $\mathbf{S}$ will then be a unitary matrix.

Thus any hermitian or unitary matrix $\mathbf{A}$ may be expressed as

$$\mathbf{A} = \mathbf{S}\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\mathbf{S}^*, \tag{1.57}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\mathbf{A}$ and $\mathbf{S}$ is a unitary matrix.

If $\mathbf{A}$ is a real symmetric matrix ($\mathbf{A}^\tau = \mathbf{A}^* = \bar{\mathbf{A}} = \mathbf{A}$), then $\mathbf{S}$ may also be chosen to be real and therefore orthogonal. Indeed, the columns of $\mathbf{S}$ are eigenvectors and these, as solutions of the equation $\mathbf{Bx} = \mathbf{0}$ with a real matrix $\mathbf{B} = \mathbf{A} - \lambda_j\mathbf{I}_n$, may be assumed real.

A function $f(\mathbf{x}) = (\mathbf{Ax}, \mathbf{x})$, where $\mathbf{A}$ is some hermitian matrix, is called a hermitian form.

A hermitian matrix $\mathbf{A} = \|\alpha_{ij}\|_1^n$ (hermitian form $(\mathbf{Ax}, \mathbf{x})$) is said to be positive definite (notation: $\mathbf{A} > 0$) if $(\mathbf{Ax}, \mathbf{x}) > 0$ for any vector $\mathbf{x} \in \mathfrak{R}_n$, $\mathbf{x} \neq \mathbf{0}$.

A hermitian matrix $\mathbf{A}$ *is positive definite if and only if all its eigenvalues are positive.* Necessity follows from the formula $\lambda_\nu = (\mathbf{Aa}_\nu, \mathbf{a}_\nu)$, where $\mathbf{a}_\nu$ is a normalized eigenvector, and sufficiency from (1.57).

Formula (1.57) also yields Sylvester's test:

A hermitian matrix $\mathbf{A}$ *is positive definite if and only if all its principal minors are positive:*

$$\alpha_{11} > 0, \quad \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} > 0, \quad \ldots, \quad \begin{vmatrix} \alpha_{11} & \ldots & \alpha_{1n} \\ \cdot & \cdot & \cdot \\ \alpha_{n1} & \ldots & \alpha_{nn} \end{vmatrix} > 0.$$

1.15. Inequalities for hermitian forms

Let $\mathbf{H}$ *be a hermitian matrix,* $h_{\min}$ *and* $h_{\max}$ *its smallest and largest eigenvalues. Then for any vector* $\mathbf{x} \in \mathfrak{R}_n$

$$h_{\min}(\mathbf{x}, \mathbf{x}) \leqslant (\mathbf{Hx}, \mathbf{x}) \leqslant h_{\max}(\mathbf{x}, \mathbf{x}). \tag{1.58}$$

Proof. (a) We first assume that $\mathbf{H}$ is a diagonal hermitian matrix, $\mathbf{H} = \operatorname{diag}(h_1, \ldots, h_n)$. Then

$$(\mathbf{Hx}, \mathbf{x}) = h_1 |\xi_1|^2 + \ldots + h_n |\xi_n|^2$$

and clearly

$$h_{\min} \sum |\xi_\nu|^2 \leqslant \sum h_\nu |\xi_\nu|^2 \leqslant h_{\max} \sum |\xi_\nu|^2,$$

proving (1.58).

(b) In the general case, we have from (1.57)

$$\mathbf{H} = \mathbf{SQS}^*, \quad \text{where} \quad \mathbf{Q} = \operatorname{diag}(h_1, \ldots, h_n), \quad \mathbf{S}^*\mathbf{S} = \mathbf{I}_n.$$

Hence

$$(\mathbf{Hx}, \mathbf{x}) = (\mathbf{SQS}^*\mathbf{x}, \mathbf{x}) = (\mathbf{Qy}, \mathbf{y}),$$

where $\mathbf{y} = \mathbf{S}^*\mathbf{x}$. By case (a), $h_{\min}(\mathbf{y}, \mathbf{y}) \leqslant (\mathbf{Qy}, \mathbf{y}) \leqslant h_{\max}(\mathbf{y}, \mathbf{y})$. But

$$(\mathbf{y}, \mathbf{y}) = (\mathbf{S}^*\mathbf{x}, \mathbf{S}^*\mathbf{x}) = (\mathbf{SS}^*\mathbf{x}, \mathbf{x}) = (\mathbf{x}, \mathbf{x}),$$

and this proves (1.58) for an arbitrary hermitian matrix.

1.16. Two lemmas on decomposition of $\mathfrak{R}_n$. Test for simplicity of elementary divisors

Lemma I. Let $\lambda = 0$ *be an eigenvalue of a matrix* $\mathbf{A}$, $\mathfrak{N}$ *the corresponding eigensubspace (null-space of the matrix* $\mathbf{A}$*),* $\mathfrak{D}$ *the range of* $\mathbf{A}$ *(§ 1.8). Then all elementary divisors of* $\mathbf{A}$ *belonging to the eigenvalue* $\lambda = 0$ *are simple if and only if the whole space is the direct sum of* $\mathfrak{N}$ *and* $\mathfrak{D}$*:*

$$\mathfrak{R}_n = \mathfrak{N} \dotplus \mathfrak{D}. \tag{1.59}$$

Proof. Necessity. We first prove the assertion for a matrix in Jordan form:

$$\mathbf{Q} = \mathbf{Q}^{(1)} \dotplus \mathbf{Q}^{(2)} \dotplus \ldots \dotplus \mathbf{Q}^{(q)}, \tag{1.60}$$

where each $\mathbf{Q}^{(s)}$ is a direct sum of elementary Jordan matrices corresponding to one eigenvalue. Since all the elementary divisors belonging to $\lambda = 0$ (whose multiplicity we denote by p) are simple, it follows that one of the matrices $\mathbf{Q}^{(s)}$ in (1.60) must vanish and the others are nonsingular. We may assume that $\mathbf{Q}^{(1)}$ is the $p \times p$ zero matrix. Formula (1.60) defines a decomposition (1.39) of $\mathfrak{R}_n$ as a direct sum of root subspaces, with $\mathfrak{L}_1$ the null-space of $\mathbf{A}$. It is clear that $\mathbf{x} \in \mathfrak{L}_1$ if and only if all the components of $\mathbf{x}$ except the first p vanish. Analogous statements hold for $\mathbf{x} \in \mathfrak{L}_2, \ldots, \mathbf{x} \in \mathfrak{L}_q$.

We claim that the sum $\mathfrak{L}_2 \dotplus \ldots \dotplus \mathfrak{L}_q = \mathfrak{L}$ in (1.39) coincides with $\mathfrak{D}$. It will then follow from (1.39) that (1.59) is valid. Now $\mathfrak{L}$ is the span of the $n-p$ vectors $\mathbf{e}_{p+1}, \mathbf{e}_{p+2}, \ldots, \mathbf{e}_n$. Multiplying the matrix $\mathbf{Q}$ by an arbitrary vector $\mathbf{x} \in \mathfrak{R}_n$, we see that the first p components of $\mathbf{Qx}$ vanish, so that $\mathbf{Qx} \in \mathfrak{L}$. Conversely, if $\mathbf{y} \in \mathfrak{L}$, i. e., the first p components of $\mathbf{y}$ vanish, then since the matrices $\mathbf{Q}^{(2)}, \ldots, \mathbf{Q}^{(q)}$ are nonsingular there is a vector $\mathbf{x} \in \mathfrak{R}_n$ such that $\mathbf{Qx} = \mathbf{y}$. Thus $\mathfrak{L} = \mathfrak{D}$.

We now prove necessity for an arbitrary matrix $\mathbf{A}$. Let $\mathbf{Q}$ be its Jordan form. Denote the corresponding subspaces by $\mathfrak{N}_\mathbf{A}, \mathfrak{D}_\mathbf{A}, \mathfrak{N}_\mathbf{Q}, \mathfrak{D}_\mathbf{Q}$. Then by (1.48)

$$\mathbf{AS} = \mathbf{SQ} \qquad (\det \mathbf{S} \neq 0). \tag{1.61}$$

Let $\mathbf{S}\mathfrak{R}'$ denote the subspace of all vectors $\mathbf{Sx}$, $\mathbf{x} \in \mathfrak{R}'$. It follows from (1.61) that

$$\mathfrak{N}_\mathbf{A} = \mathbf{S}\mathfrak{N}_\mathbf{Q}, \quad \mathfrak{D}_\mathbf{A} = \mathbf{S}\mathfrak{D}_\mathbf{Q}.$$

By the case treated previously,

$$\mathfrak{R}_n = \mathfrak{N}_\mathbf{Q} \dotplus \mathfrak{D}_\mathbf{Q}$$

and so

$$\mathfrak{R}_n = \mathbf{S}\mathfrak{R}_n = \mathbf{S}\mathfrak{N}_\mathbf{Q} \dotplus \mathbf{S}\mathfrak{D}_\mathbf{Q} = \mathfrak{N}_\mathbf{A} \dotplus \mathfrak{D}_\mathbf{A}.$$

Sufficiency. Suppose that $\mathbf{A}$ has a multiple elementary divisor λ^{e_0} $(e_0 > 1)$. Then there is a vector $\mathbf{f}_0 \neq \mathbf{0}$ such that

$$\mathbf{Af}_0 = \mathbf{0}, \quad \mathbf{Af}_1 = \mathbf{f}_0.$$

The first equality means that $\mathbf{f}_0 \in \mathfrak{N}$, the second that $\mathbf{f}_0 \in \mathfrak{D}$. Thus the intersection $\mathfrak{N} \cap \mathfrak{D}$ of $\mathfrak{N}$ and $\mathfrak{D}$ contains a vector $\mathbf{f}_0 \neq \mathbf{0}$, and (1.59) cannot hold.

Corollary. A necessary and sufficient condition for all elementary divisors of a matrix $\mathbf{A}$ belonging to the eigenvalue $\lambda = 0$ to be simple is that the subspaces $\mathfrak{N}$ and $\mathfrak{D}$ be disjoint (i. e., their only common vector is the zero vector).

Indeed, we have just shown that the existence of multiple elementary divisors implies the existence of a vector $\mathbf{f}_0 \neq \mathbf{0}$ such that $\mathbf{f}_0 \in \mathfrak{N}$, $\mathbf{f}_0 \in \mathfrak{D}$. Conversely, if the elementary divisors are simple, it follows from (1.59) that $\mathfrak{N}$ and $\mathfrak{D}$ are disjoint.

Lemma II. *Let $\mathfrak{N}$, $\mathfrak{D}$ and $\mathfrak{N}_*$, $\mathfrak{D}_*$ be the null-spaces and ranges of the matrices $\mathbf{A}$ and $\mathbf{A}^*$, respectively. Then $\mathfrak{R}_n$ is an orthogonal direct sum*

$$\mathfrak{R}_n = \mathfrak{N} \oplus \mathfrak{D}_* = \mathfrak{N}_* \oplus \mathfrak{D}. \tag{1.62}$$

Proof. 1) We shall prove that the subspaces $\mathfrak{N}$ and $\mathfrak{D}_*$ intersect only in the zero vector: $\mathfrak{N} \cap \mathfrak{D}_* = \mathbf{0}$. Let $\mathbf{f} \in \mathfrak{N} \cap \mathfrak{D}_*$. We have $\mathbf{f} \in \mathfrak{N}$, so that $\mathbf{Af} = \mathbf{0}$ and $\mathbf{f} \in \mathfrak{D}_*$, so that there is a vector $\mathbf{g}$ such that $\mathbf{A}^*\mathbf{g} = \mathbf{f}$. Therefore,

$$(\mathbf{f}, \mathbf{f}) = (\mathbf{A}^*\mathbf{g}, \mathbf{f}) = (\mathbf{g}, \mathbf{Af}) = 0,$$

whence it follows that $\mathbf{f} = \mathbf{0}$.

2) We show that $\mathfrak{N} \perp \mathfrak{D}_*$. Consider arbitrary vectors $\mathbf{f} \in \mathfrak{N}$, $\mathbf{g} \in \mathfrak{D}_*$. Since $\mathbf{g} \in \mathfrak{D}_*$, there exists a vector $\mathbf{h}$ such that $\mathbf{A}^*\mathbf{h} = \mathbf{g}$. Then

$$(\mathbf{f}, \mathbf{g}) = (\mathbf{f}, \mathbf{A}^*\mathbf{h}) = (\mathbf{Af}, \mathbf{h}) = 0,$$

as required.

3) There is no vector $\mathbf{f} \neq \mathbf{0}$ orthogonal to both $\mathfrak{N}$ and $\mathfrak{D}_*$: $\mathbf{f} \perp \mathfrak{N}$, $\mathbf{f} \perp \mathfrak{D}_*$. For if there were, it would follow from $\mathbf{f} \perp \mathfrak{D}_*$ that $\mathbf{f} \perp \mathbf{A}^*\mathbf{x}$ for any $\mathbf{x} \in \mathfrak{R}_n$, so that

$$0 = (\mathbf{f}, \mathbf{A}^*\mathbf{x}) = (\mathbf{Af}, \mathbf{x}).$$

Hence $\mathbf{Af} = \mathbf{0}$, $\mathbf{f} \in \mathfrak{N}$. Since $\mathbf{f} \perp \mathfrak{N}$, this means that $\mathbf{f} = \mathbf{0}$.

4) Denote $\mathfrak{R}' = \mathfrak{N} \oplus \mathfrak{D}_*$. By Lemma I in §1.5, $\mathfrak{R}_n = \mathfrak{R}' \oplus (\mathfrak{R}')^\perp$, where $(\mathfrak{R}')^\perp$ is the set of all vectors orthogonal to $\mathfrak{R}'$. But by part 3 above, $(\mathfrak{R}')^\perp = \mathbf{0}$, so that $\mathfrak{R}_n = \mathfrak{R}' = \mathfrak{N} \oplus \mathfrak{D}_*$. Q. E. D.

Note that Lemma II is equivalent to the lemma of §1.6.

Lemmas I and II easily yield the following test for simplicity of elementary divisors.

Lemma III. *Let $\mathbf{y}_\delta \in \mathfrak{N}$, $\mathbf{z}_\delta \in \mathfrak{N}_*$ $(\delta = 1, \ldots, d)$ be bases for the subspaces $\mathfrak{N}$ and $\mathfrak{N}_*$, i.e., complete sets of linearly independent solutions of the equations**

$$\mathbf{Ay} = \mathbf{0}, \quad \mathbf{A}^*\mathbf{z} = \mathbf{0}$$

for a singular matrix $\mathbf{A}$. Then all elementary divisors of $\mathbf{A}$ belonging to the eigenvalue $\lambda = 0$ are simple if and only if

$$\det \| (\mathbf{y}_j, \mathbf{z}_h) \|_1^d \neq 0. \tag{1.63}$$

* By § 1.6, these equations have the same number of linearly independent solutions.

Proof. Let $\det\|(\mathbf{y}_j, \mathbf{z}_h)\|_1^d = 0$. Then there are numbers $\gamma_1, \ldots, \gamma_d$ such that $|\gamma_1| + \ldots + |\gamma_d| \neq 0$ and $\gamma_1(\mathbf{y}_1, \mathbf{z}_h) + \ldots + \gamma_d(\mathbf{y}_d, \mathbf{z}_h) = 0$ $(h = 1, \ldots, d)$. Setting $\mathbf{y} = \gamma_1\mathbf{y}_1 + \ldots + \gamma_d\mathbf{y}_d$, we have: (a) $\mathbf{y} \neq \mathbf{0}$, since $\mathbf{y}_1, \ldots, \mathbf{y}_d$ are linearly independent; (b) $\mathbf{y} \in \mathfrak{R}$, since $\mathbf{y}_k \in \mathfrak{R}$; (c) $(\mathbf{y}, \mathbf{z}_h) = 0$ $(h = 1, \ldots, d)$, so that by Lemma II $\mathbf{y} \in \mathfrak{D}$. By the corollary to Lemma I, not all the elementary divisors are simple.

Now let (1.63) be true. Let $\mathbf{y} \in \mathfrak{D} \cap \mathfrak{R}$. Since $\mathbf{y} \in \mathfrak{R}$, it follows that $\mathbf{y} = \gamma_1\mathbf{y}_1 + \ldots + \gamma_d\mathbf{y}_d$. Since $\mathbf{y} \in \mathfrak{D}$, Lemma II implies that $(\mathbf{y}, \mathbf{z}_h) = 0$, so that $\gamma_1(\mathbf{y}_1, \mathbf{z}_h) + \ldots + \gamma_d(\mathbf{y}_d, \mathbf{z}_h) = 0$ $(h = 1, \ldots, d)$. It follows from (1.63) that $\gamma_1 = \gamma_2 = \ldots = \gamma_d = 0$, $\mathbf{y} = \mathbf{0}$. By the corollary to Lemma I, all the elementary divisors λ^q are simple and therefore have the form $\lambda, \ldots, \lambda$.

1.17. Complex conjugate and real subspaces

A matrix or vector is said to be real if all its components are real. This is true if and only if $\overline{\mathbf{A}} = \mathbf{A}$ or $\overline{\mathbf{x}} = \mathbf{x}$. It is clear that for any complex α, β and any matrix $\mathbf{A}$

$$\overline{\alpha\mathbf{x}_1 + \beta\mathbf{x}_2} = \overline{\alpha}\overline{\mathbf{x}_1} + \overline{\beta}\overline{\mathbf{x}_2}, \quad \overline{\mathbf{A}\mathbf{x}} = \overline{\mathbf{A}}\,\overline{\mathbf{x}}.$$

Lemma I. A matrix $\mathbf{A}$ *is real if and only if for any vector* $\mathbf{x} \in \mathfrak{R}_n$

$$\overline{\mathbf{A}\mathbf{x}} = \mathbf{A}\overline{\mathbf{x}}. \tag{1.64}$$

Indeed, if $\mathbf{A}$ is a real matrix, then $\mathbf{A}\overline{\mathbf{x}} = \overline{\mathbf{A}}\,\overline{\mathbf{x}} = \overline{\mathbf{A}\mathbf{x}}$. Conversely, if (1.64) is true, we see by successively setting $\mathbf{x} = \mathbf{e}_1, \ldots, \mathbf{x} = \mathbf{e}_n$ (where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is the standard basis) that the vectors $\mathbf{A}\mathbf{e}_1, \ldots, \mathbf{A}\mathbf{e}_n$, the columns of $\mathbf{A}$, are real. This proves Lemma I.

If $\mathfrak{R}' = \{\mathbf{x}'\}$ is some subspace of $\mathfrak{R}_n$, then the set $\overline{\mathfrak{R}'} = \{\overline{\mathbf{x}'}\}$ of all complex conjugates of the vectors $\mathbf{x}' \in \mathfrak{R}'$ is also a subspace. Indeed, if $\mathbf{x}_1 \in \overline{\mathfrak{R}'}$, $\mathbf{x}_2 \in \overline{\mathfrak{R}'}$, then $\mathbf{x}_1 = \overline{\mathbf{x}_1'}$, $\mathbf{x}_2 = \overline{\mathbf{x}_2'}$, where $\mathbf{x}_1'$ and $\mathbf{x}_2'$ are vectors of $\mathfrak{R}'$. Thus $\overline{\alpha}\mathbf{x}_1 + \overline{\beta}\mathbf{x}_2 \in \mathfrak{R}'$ for all complex α and β. Consequently, $\overline{\alpha\mathbf{x}_1' + \beta\mathbf{x}_2'} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2 \in \overline{\mathfrak{R}'}$, as required. The subspace $\overline{\mathfrak{R}'}$ is said to be complex conjugate to $\mathfrak{R}'$. A subspace $\mathfrak{R}'$ is said to be real if $\overline{\mathfrak{R}'} = \mathfrak{R}'$, i. e., $\mathfrak{R}'$ is invariant under complex conjugation. A real subspace (like any other nontrivial subspace of a complex space) contains vectors with complex components.

If $\mathfrak{L}_\lambda$ is the root subspace of a real matrix $\mathbf{A}$ belonging to an eigenvalue λ, then $\overline{\mathfrak{L}}_\lambda = \mathfrak{L}_{\overline{\lambda}}$ is the root subspace belonging to the eigenvalue $\overline{\lambda}$. Indeed, if $(\mathbf{A} - \lambda\mathbf{I}_n)^k\mathbf{x} = \mathbf{0}$, then $\overline{\mathbf{A} - \lambda\mathbf{I}_n)^k\mathbf{x}} = \mathbf{0}$, i. e., $(\mathbf{A} - \overline{\lambda}\mathbf{I}_n)^k\overline{\mathbf{x}} = \mathbf{0}$, since $\overline{\mathbf{A}} = \mathbf{A}$.

Let $\mathfrak{L}_\lambda^{(\alpha)}$ be all the root subspaces of a real matrix $\mathbf{A}$. Set

$$\mathfrak{R}_0 = \sum_{\operatorname{Im}\lambda^{(\alpha)} = 0} \mathfrak{L}_\lambda^{(\alpha)}, \quad \mathfrak{R}^{(+)} = \sum_{\operatorname{Im}\lambda^{(\alpha)} > 0} \mathfrak{L}_\lambda^{(\alpha)}, \quad \mathfrak{R}^{(-)} = \sum_{\operatorname{Im}\lambda^{(\alpha)} < 0} \mathfrak{L}_\lambda^{(\alpha)}.$$

Combining the appropriate terms in (1.39), we obtain a decomposition of $\mathfrak{R}$ into three $\mathbf{A}$-invariant subspaces:

$$\mathfrak{R} = \mathfrak{R}_0 \dotplus \mathfrak{R}^{(+)} \dotplus \mathfrak{R}^{(-)}, \tag{1.65}$$

where

$$\overline{\mathfrak{R}_0}=\mathfrak{R}_0, \quad \overline{\mathfrak{R}^{(-)}}=\mathfrak{R}^{(+)}. \tag{1.66}$$

The converse is also valid:

Lemma II. Suppose we have a decomposition (1.65), where $\mathfrak{R}_0$, $\mathfrak{R}^{(+)}$, $\mathfrak{R}^{(-)}$ *are* $\mathbf{A}$*-invariant subspaces satisfying (1.66), and (1.64) is true for vectors* $\mathbf{x}'\in\mathfrak{R}_0$ *and* $\mathbf{x}''\in\mathfrak{R}^{(+)}$. *Then* $\mathbf{A}$ *is a real matrix.*

Proof. Equality (1.64) is also true for vectors $\mathbf{x}'''\in\mathfrak{R}^{(-)}$. Indeed, if $\mathbf{x}'''\in\mathfrak{R}^{(-)}$, then $\mathbf{x}''=\overline{\mathbf{x}'''}\in\mathfrak{R}^{(+)}$. It follows from (1.64) for $\mathbf{x}=\mathbf{x}''$ that $\mathbf{A}\mathbf{x}''=\overline{\mathbf{A}\overline{\mathbf{x}''}}$, i. e., $\mathbf{A}\overline{\mathbf{x}'''}=\overline{\mathbf{A}\mathbf{x}'''}$.

Thus (1.64) is true for each of the subspaces $\mathfrak{R}_0$, $\mathfrak{R}^{(+)}$, $\mathfrak{R}^{(-)}$. (1.65) any vector $\mathbf{x}$ is expressible as

$$\mathbf{x}=\mathbf{x}'+\mathbf{x}''+\mathbf{x}''', \text{ where } \mathbf{x}'\in\mathfrak{R}_0, \quad \mathbf{x}''\in\mathfrak{R}^{(+)}, \quad \mathbf{x}'''\in\mathfrak{R}^{(-)}.$$

Thus

$$\overline{\mathbf{A}\mathbf{x}}=\overline{\mathbf{A}\mathbf{x}'}+\overline{\mathbf{A}\mathbf{x}''}+\overline{\mathbf{A}\mathbf{x}'''}=\mathbf{A}\overline{\mathbf{x}'}+\mathbf{A}\overline{\mathbf{x}''}+\mathbf{A}\overline{\mathbf{x}'''}=\mathbf{A}\overline{\mathbf{x}}.$$

It follows that (1.64) is true for all $\mathbf{x}\in\mathfrak{R}$ and so, by Lemma I, $\mathbf{A}$ is a real matrix.

Now let $\mathbf{A}$ be an arbitrary matrix. It may be regarded as the matrix (relative to the standard basis) of some operator $\mathfrak{A}$. We shall say that the operator $\mathfrak{A}$ is r e a l if its matrix relative to the standard basis is real.

Lemma III. Let $\mathfrak{A}$ *be a real operator and* $\mathfrak{M}$ *an invariant subspace. Then the* $m\times m$ *matrix* $\mathbf{A}$ *of* $\mathfrak{A}$ *in the subspace* $\mathfrak{M}$, *relative to an arbitrary real basis* $\mathbf{g}_1, \ldots, \mathbf{g}_m$, *is real.*

Indeed, $\mathbf{A}=\|\alpha_{jh}\|$, where the numbers α_{jh} are defined by (1.26):

$$\mathfrak{A}\mathbf{g}_j=\sum_{h=1}^{m}\alpha_{hj}\mathbf{g}_h \qquad (j=1, \ldots, m).$$

We have

$$\overline{\mathfrak{A}\mathbf{g}_j}=\sum\overline{\alpha_{hj}}\mathbf{g}_h,$$

since $\overline{\mathbf{g}_h}=\mathbf{g}_h$. In addition, $\overline{\mathfrak{A}\mathbf{g}_j}=\overline{\mathfrak{A}\overline{\mathbf{g}_j}}=\mathfrak{A}\mathbf{g}_j$. Therefore $\sum\alpha_{hj}\mathbf{g}_h=\sum\overline{\alpha_{hj}}\mathbf{g}_h$, whence it follows that $\alpha_{hj}=\overline{\alpha_{hj}}$ $(j, h=1, \ldots, m)$.

Lemma IV. Suppose we have a direct sum decomposition

$$\mathfrak{R}_n=\sum_{j=1}^{p}\dot{+}\,\mathfrak{R}_j^{(0)}\dot{+}\sum_{j=1}^{r}\dot{+}\,\mathfrak{R}_j^{(+)}\dot{+}\sum_{j=1}^{r}\dot{+}\,\mathfrak{R}_j^{(-)},$$

where $\mathfrak{R}_j^{(0)}$ *are real subspaces and* $\overline{\mathfrak{R}_j^{(+)}}=\mathfrak{R}_j^{(-)}$. *Pick a basis for each of the subspaces: for each* $\mathfrak{R}_j^{(0)}$ *a real basis* $\{\mathbf{a}_h'\}$, *for each* $\mathfrak{R}_j^{(+)}$ *an arbitrary basis* $\{\mathbf{b}_h'\}$, *and for each* $\mathfrak{R}_j^{(-)}$ *the complex conjugate basis* $\{\overline{\mathbf{b}_h'}\}$. *Let each of the subspaces* $\mathfrak{R}_j^{(0)}$, $\mathfrak{R}_j^{(+)}$, $\mathfrak{R}_j^{(-)}$ *be invariant under* $\mathfrak{A}$ *and let* $\mathbf{A}_j^{(0)}$, $\mathbf{A}_j^{(+)}$, $\mathbf{A}_j^{(-)}$ *be the matrices of the operator in these subspaces relative to the selected bases. Then* $\mathfrak{A}$ *is a real operator if and only if*

$$\overline{\mathbf{A}_j^{(0)}}=\mathbf{A}_j^{(0)}, \quad \overline{\mathbf{A}_j^{(+)}}=\mathbf{A}_j^{(-)}.$$

For simplicity's sake we give the proof only for $p=r=1$, omitting the indices of the subspaces, the basis vectors and the matrices $\mathbf{A}_j^{(0)}$, $\mathbf{A}_j^{(+)}$, $\mathbf{A}_j^{(-)}$.

Necessity. Let

$$\mathfrak{A}\mathbf{a}_h=\sum\alpha_{kh}\mathbf{a}_k,\quad \mathfrak{A}\mathbf{b}_h=\sum\beta_{kh}\mathbf{b}_k,\qquad \mathfrak{A}\bar{\mathbf{b}}_h=\sum\gamma_{kh}\bar{\mathbf{b}}_k. \tag{1.67}$$

If $\mathfrak{A}$ is a real operator, then $\overline{\mathfrak{A}\mathbf{a}_h}=\mathfrak{A}\mathbf{a}_h$, $\overline{\mathfrak{A}\mathbf{b}_h}=\mathfrak{A}\bar{\mathbf{b}}_h$. Substituting (1.67) into these equalities and comparing coefficients, we find that

$$\alpha_{kh}=\bar{\alpha}_{kh},\quad \bar{\beta}_{kh}=\gamma_{kh}, \tag{1.68}$$

i.e., $\overline{\mathbf{A}^{(0)}}=\mathbf{A}^{(0)}$, $\overline{\mathbf{A}^{(+)}}=\mathbf{A}^{(-)}$.

Sufficiency. If conditions (1.68) are satisfied, then for any vector $\mathbf{x}$

$$\mathbf{x}=\sum\xi_h\mathbf{a}_h+\sum\eta_h\mathbf{b}_h+\sum\zeta_h\bar{\mathbf{b}}_h.$$

Using (1.67), (1.68), we obtain

$$\overline{\mathfrak{A}\mathbf{x}}=\sum\bar{\xi}_h\alpha_{kh}\mathbf{a}_k+\sum\bar{\eta}_h\gamma_{kh}\bar{\mathbf{b}}_k+\sum\bar{\zeta}_h\bar{\gamma}_{kh}\mathbf{b}_k=\mathfrak{A}\bar{\mathbf{x}}.$$

By Lemma I, $\mathfrak{A}$ is a real operator.

Lemma V. A subspace $\mathfrak{M}$ is real if and only if it has a real basis.

Proof. If $\mathbf{f}_1,\ \ldots,\ \mathbf{f}_m$ is a real basis in $\mathfrak{M}$, then for any vector $\mathbf{x}=\alpha_1\mathbf{f}_1+\ldots+\ldots+\alpha_m\mathbf{f}_m\in\mathfrak{M}$ we have $\bar{\mathbf{x}}=\bar{\alpha}_1\mathbf{f}_1+\ldots+\bar{\alpha}_m\mathbf{f}_m\in\mathfrak{M}$, so that $\mathfrak{M}$ is real.

Let $\mathfrak{M}=\bar{\mathfrak{M}}$. We have to show that $\mathfrak{M}$ has a real basis. Let $\mathbf{g}_1,\ \ldots,\ \mathbf{g}_m$ be an arbitrary basis. Then $\bar{\mathbf{g}}_j\in\mathfrak{M}$, i.e.,

$$\begin{aligned}\bar{\mathbf{g}}_1&=\gamma_{11}\mathbf{g}_1+\ldots+\gamma_{1m}\mathbf{g}_m,\\ &\cdots\cdots\cdots\cdots\\ \bar{\mathbf{g}}_m&=\gamma_{m1}\mathbf{g}_1+\ldots+\gamma_{mm}\mathbf{g}_m.\end{aligned}$$

Taking complex conjugates, substituting the values of $\bar{\mathbf{g}}_j$ and comparing coefficients, we see that the matrix $\mathbf{C}=\|\gamma_{jh}\|_{j,h=1}^{m}$ has the property

$$\bar{\mathbf{C}}\mathbf{C}=\mathbf{I}_m.$$

We seek a real basis $\mathbf{f}_1,\ \ldots,\ \mathbf{f}_m$ in the form

$$\begin{aligned}\mathbf{f}_1&=\beta_{11}\mathbf{g}_1+\ldots+\beta_{1m}\mathbf{g}_m,\\ &\cdots\cdots\cdots\cdots\\ \mathbf{f}_m&=\beta_{m1}\mathbf{g}_1+\ldots+\beta_{mm}\mathbf{g}_m,\end{aligned}$$

or

$$\begin{pmatrix}\mathbf{f}_1\\ \ldots\\ \mathbf{f}_m\end{pmatrix}=\mathbf{B}\begin{pmatrix}\mathbf{g}_1\\ \ldots\\ \mathbf{g}_m\end{pmatrix},\quad\text{where}\quad \mathbf{B}=\|\beta_{jh}\|_{j,h=1}^{m}.$$

Since the space is real,

$$\begin{pmatrix} \bar{\mathbf{f}}_1 \\ \cdots \\ \bar{\mathbf{f}}_m \end{pmatrix} = \bar{\mathbf{B}} \begin{pmatrix} \bar{\mathbf{g}}_1 \\ \cdots \\ \bar{\mathbf{g}}_m \end{pmatrix} = \bar{\mathbf{B}}\mathbf{C} \begin{pmatrix} \mathbf{g}_1 \\ \cdots \\ \mathbf{g}_m \end{pmatrix} = \bar{\mathbf{B}}\mathbf{C}\mathbf{B}^{-1} \begin{pmatrix} \mathbf{f}_1 \\ \cdots \\ \mathbf{f}_m \end{pmatrix}.$$

Thus it will suffice to find a nonsingular matrix $\mathbf{B}$ such that $\bar{\mathbf{B}}\mathbf{C}\mathbf{B}^{-1} = \mathbf{I}_m$, or $\mathbf{B} = \bar{\mathbf{B}}\mathbf{C}$. Set $\mathbf{B} = \alpha\mathbf{C} + \beta\mathbf{I}_m$. Substituting this into $\mathbf{B} = \bar{\mathbf{B}}\mathbf{C}$ and using the fact that $\bar{\mathbf{C}}\mathbf{C} = \mathbf{I}_m$, we get

$$\alpha\mathbf{C} + \beta\mathbf{I}_m = (\bar{\alpha}\bar{\mathbf{C}} + \bar{\beta}\mathbf{I}_m)\mathbf{C} = \bar{\alpha}\mathbf{I}_m + \bar{\beta}\mathbf{C}.$$

Consequently, we shall have $\mathbf{B} = \bar{\mathbf{B}}\mathbf{C}$ if $\beta = \bar{\alpha}$. Set $\beta = \bar{\alpha} = \exp(i\varphi/2)$. Then $\mathbf{B} = e^{\frac{i\varphi}{2}}(\mathbf{C} + e^{i\varphi}\mathbf{I}_m)$. We may clearly choose φ in such a way that $\det \mathbf{B} \neq 0$. The basis $\mathbf{f}_1, \ldots, \mathbf{f}_m$ will then be real, as required.

An operator $\mathfrak{A}$ is said to be real if its matrix $\mathbf{A}$ relative to the standard basis is real. This will be the case if and only if, for any $\mathbf{x} \in \mathfrak{R}$

$$\mathbf{A}\bar{\mathbf{x}} = \overline{\mathbf{A}\mathbf{x}}.$$

1.18. Jordan form of a real matrix

Let $\mathbf{A}$ be a real matrix. Its nonreal eigenvalues, as roots of the equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, which has real coefficients, may be grouped in pairs of complex conjugates with the same multiplicity. Let λ_1 and $\lambda_2 = \bar{\lambda}_1$ be one of these pairs, p their multiplicity, and $\mathfrak{L}_1$, $\mathfrak{L}_2$ corresponding root subspaces. It is clear that $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are complex conjugate subspaces. Indeed, if $\mathbf{x} \in \mathfrak{L}_1$, then $(\mathbf{A} - \lambda_1\mathbf{I})^p \mathbf{x} = \mathbf{0}$. Taking complex conjugates, we see that $(\mathbf{A} - \lambda_2\mathbf{I})^p \bar{\mathbf{x}} = \mathbf{0}$, so that $\bar{\mathbf{x}} \in \mathfrak{L}_2$. Similarly, if $\mathbf{x} \in \mathfrak{L}_2$ then $\bar{\mathbf{x}} \in \mathfrak{L}_1$. Thus $\bar{\mathfrak{L}}_1 = \mathfrak{L}_2$. Expressing $\mathfrak{L}_1$ as a direct sum of simple cyclic subspaces and taking a cyclic basis in each, we obtain a basis

$$\mathbf{f}_1^{(1)}, \ldots, \mathbf{f}_{m_1}^{(1)}, \quad \mathbf{f}_1^{(2)}, \ldots, \mathbf{f}_{m_2}^{(2)}, \ldots \tag{1.69}$$

for $\mathfrak{L}_1$, where the vectors $\mathbf{f}_k^{(j)}$ with the same superscript form a basis for one of the simple cyclic subspaces, so that they satisfy (1.41) with $\lambda_{(\sigma)} = \lambda_1$. The complex conjugates of the vectors (1.69) form a cyclic basis for $\mathfrak{L}_2$. Indeed, taking complex conjugates in (1.41), we obtain the necessary conditions for the vectors $\bar{\mathbf{f}}_k^{(j)}$. Let

$$\mathbf{f}_k^{(j)} = \mathbf{u}_k^{(j)} + i\mathbf{v}_k^{(j)}, \tag{1.70}$$

where $\mathbf{u}_k^{(j)}$, $\mathbf{v}_k^{(j)}$ are real vectors. It is clear that the vectors $\mathbf{u}_k^{(j)}$, $\mathbf{v}_k^{(j)}$ form a (real) basis in the subspace $\mathfrak{L}_1 \dotplus \mathfrak{L}_2$. Indeed, $\mathbf{u}_k^{(j)}$, $\mathbf{v}_k^{(j)}$ may be expressed as linear combinations of $\mathbf{f}_k^{(j)}$, $\bar{\mathbf{f}}_k^{(j)}$. Since each of the systems $\{\mathbf{f}_k^{(j)}, \bar{\mathbf{f}}_k^{(j)}\}$ and $\{\mathbf{u}_k^{(j)}, \mathbf{v}_k^{(j)}\}$ is linearly dependent on the other, they are related through some nonsingular transformation.

Let us determine the form of the matrix of the operator $\mathbf{A}$ in the subspace $\mathfrak{L}_1 \dot{+} \mathfrak{L}_2$ relative to the basis $\{\mathbf{u}_k^{(j)}, \mathbf{v}_k^{(j)}\}$. Let

$$\lambda_{(\sigma)} = \alpha + i\beta,$$

where α and β are real numbers, $\beta \neq 0$. Separating (1.41) (with δ chosen to be real) into real and imaginary parts and omitting the index σ, we obtain

$$\left.\begin{array}{l} \mathbf{Au}_1 = \alpha\mathbf{u}_1 - \beta\mathbf{v}_1, \quad \mathbf{Av}_1 = \beta\mathbf{u}_1 + \alpha\mathbf{v}_1, \quad \mathbf{Au}_2 = \alpha\mathbf{u}_2 + \beta\mathbf{v}_2 + \delta\mathbf{u}_1, \\ \mathbf{Av}_2 = -\beta\mathbf{u}_2 + \alpha\mathbf{v}_2 + \delta\mathbf{v}_2, \ \ldots, \ \mathbf{Au}_m = \alpha\mathbf{u}_m + \beta\mathbf{v}_m + \delta\mathbf{u}_{m-1}, \\ \mathbf{Av}_m = -\beta\mathbf{u}_m + \alpha\mathbf{v}_m + \delta\mathbf{v}_{m-1}. \end{array}\right\} \tag{1.71}$$

Here $\delta \neq 0$ is arbitrary. We may write these relations as

$$\mathbf{AW} = \mathbf{WQ}(\alpha, \beta), \tag{1.72}$$

where

$$\mathbf{W} = \|\mathbf{u}_1, \mathbf{v}_1, \ldots, \mathbf{u}_m, \mathbf{v}_m\|, \tag{1.73}$$

and $\mathbf{Q}(\alpha, \beta)$ is a $2m \times 2m$ matrix:

$$\mathbf{Q}(\alpha, \beta) = \left\|\begin{array}{cccccc} \alpha & \beta & \delta & 0 & 0 & 0 & & 0 & 0 \\ -\beta & \alpha & 0 & \delta & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \alpha & \beta & \delta & 0 & \ldots & 0 & 0 \\ 0 & 0 & -\beta & \alpha & 0 & \delta & & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & & \delta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \delta \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & \alpha & \beta \\ 0 & 0 & 0 & 0 & 0 & 0 & & -\beta & \alpha \end{array}\right\|. \tag{1.74}$$

This is the required matrix. We shall call it the real elementary Jordan matrix belonging to the pair of elementary divisors $(\lambda - \lambda_{(\sigma)})^m$, $(\lambda - \overline{\lambda}_{(\sigma)})^m$, where $\lambda_{(\sigma)} = \alpha + i\beta$.

Doing this for all pairs of simple cyclic subspaces belonging to all the nonreal root subspaces, we obtain a sequence of relations (1.72) and matrices (1.73), (1.74). For the real eigenvalues, we again have relations (1.42), with $\mathbf{f}^{(\sigma)}$ real vectors. The system of all vectors $\{\mathbf{f}_j^{(\sigma)}\}$, $\{\mathbf{u}_j^{(\sigma)}, \mathbf{v}_j^{(\sigma)}\}$ is obviously a basis for the whole space. As in §1.11, we consider the matrix

$$\mathbf{S} = \|\mathbf{F}_1, \ldots, \mathbf{F}_p, \mathbf{W}_1, \ldots, \mathbf{W}_r\|, \tag{1.75}$$

where $\mathbf{F}_\sigma$, $\mathbf{W}_\sigma$ are rectangular matrices, $\mathbf{F}_\sigma = \|\mathbf{f}_1^{(\sigma)}, \ \ldots, \ \mathbf{f}_{m_\sigma}^{(\sigma)}\|$, $\mathbf{W}_\sigma = \|\mathbf{u}_1^{(\sigma)}, \mathbf{v}_1^{(\sigma)}, \ \ldots, \ \mathbf{u}_{m_\sigma}^{(\sigma)}, \mathbf{v}_{m_\sigma}^{(\sigma)}\|$, and combine all the relations (1.72), and also (1.42) (or (1.43)):

$$\mathbf{AS} = \mathbf{SQ}, \tag{1.76}$$

where $\mathbf{Q}$ is the block-diagonal matrix

$$\mathbf{Q}=\mathbf{Q}_1(\lambda_{(1)})\dotplus\ldots\dotplus\mathbf{Q}_p(\lambda_{(s)})\dotplus\mathbf{Q}_1(\alpha_{(1)},\ \beta_{(1)})\dotplus\ldots\dotplus\mathbf{Q}_r(\alpha_{(r)},\ \beta_{(r)}),\qquad(1.77)$$

$\mathbf{Q}_{(1)}(\lambda_\sigma)$ being elementary Jordan matrices (1.44) with real $\lambda_{(\sigma)}$ and $\mathbf{Q}_\sigma(\alpha_{(\sigma)},\ \beta_{(\sigma)})$ real elementary Jordan matrices of type (1.74). Some of the numbers $\lambda_{(\sigma)}$ and $\alpha_{(\sigma)}$, $\beta_{(\sigma)}$ may coincide.

We have proved the following theorem: for any real matrix $\mathbf{A}$, we can find a nonsingular real matrix $\mathbf{S}$ such that $\mathbf{S}^{-1}\mathbf{AS}=\mathbf{Q}$, where $\mathbf{Q}$ is a block-diagonal matrix of the form (1.77).

We shall call $\mathbf{Q}$ the real Jordan form of the matrix $\mathbf{A}$.

The summands in (1.77) may be permuted arbitrarily. Any such permutation will of course change the matrix $\mathbf{S}$— the new matrix is obtained from (1.75) by a suitable permutation of the rectangular matrices $\mathbf{F}_\sigma$, $\mathbf{W}_\sigma$.

In the special case when all the elementary divisors of $\mathbf{A}$ are simple (for example, if all the eigenvalues of $\mathbf{A}$ are distinct), the result is that there exists a nonsingular real matrix $\mathbf{S}$ such that $\mathbf{S}^{-1}\mathbf{AS}=\mathbf{Q}$, where $\mathbf{Q}$ is a block-diagonal matrix of the form

$$\mathbf{Q}=\lambda_1\dotplus\ldots\dotplus\lambda_{(s)}\dotplus\begin{Vmatrix}\alpha_{(1)} & \beta_{(1)}\\ -\beta_{(1)} & \alpha_{(1)}\end{Vmatrix}\dotplus\ldots\dotplus\begin{Vmatrix}\alpha_{(r)} & \beta_{(r)}\\ -\beta_{(r)} & \alpha_{(r)}\end{Vmatrix}.$$

Here $\lambda_{(1)},\ \ldots,\ \lambda_{(s)}$ are the real eigenvalues of $\mathbf{A}$, and $\alpha_{(1)}\pm i\beta_{(1)},\ \ldots,$ $\ldots,\ \alpha_{(r)}\pm i\beta_{(r)}$ the pairs of complex conjugate eigenvalues of $\mathbf{A}$.

1.19. Examples

The examples given below will illustrate the various concepts discussed above. For brevity's sake, we shall use the term cyclic space for simple cyclic space.

Example 1. Calculate the eigenvectors and root vectors and reduce to Jordan form:

$$\mathbf{A}=\begin{Vmatrix}-3 & 2 & -4\\ 1 & -1 & 1\\ 4 & -3 & 5\end{Vmatrix}.$$

Solution. The characteristic equation (1.29) is

$$\det(\mathbf{A}-\lambda\mathbf{I}_3)=\begin{vmatrix}-3-\lambda & 2 & -4\\ 1 & -1-\lambda & 1\\ 4 & -3 & 5-\lambda\end{vmatrix}=-(\lambda-1)\lambda^2=0.$$

Consequently, the eigenvalues of $\mathbf{A}$ are $\lambda_1=1$, $\lambda_2=\lambda_3=0$. We determine an eigenvector $\mathbf{a}_1$ belonging to the simple eigenvalue $\lambda_1=1$ from the homogeneous system of equations (1.28):

$$\begin{Vmatrix} -4 & 2 & -4 \\ 1 & -2 & -1 \\ 4 & -3 & 4 \end{Vmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0, \qquad \begin{cases} -4\alpha_1 + 2\alpha_2 - 4\alpha_3 = 0, \\ \alpha_1 - 2\alpha_2 + \alpha_3 = 0, \\ 4\alpha_1 - 3\alpha_2 + 4\alpha_3 = 0. \end{cases}$$

Solving, we obtain $\alpha_2 = 0$, $\alpha_3 = -\alpha_1$, and so

$$\mathbf{a}_1 = \alpha_1 (\mathbf{e}_1 - \mathbf{e}_3) = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Thus the eigensubspace $\mathfrak{S}_1$ (see § 1.8) is the set of vectors collinear with $\mathbf{e}_1 - \mathbf{e}_3$. Since $\lambda_1 = 1$ is a simple eigenvalue, the vector equation (1.37) is not solvable for any $k > 1$ (this is also easy to check directly). It is also obvious that only the first of the vector equations (1.38) is solvable.

Similarly, we find an eigenvector $\mathbf{a}_2$ for the double eigenvalue $\lambda_2 = 0$:

$$\mathbf{a}_2 = \alpha \left(\mathbf{e}_1 + \frac{1}{2} \mathbf{e}_2 - \frac{1}{2} \mathbf{e}_3 \right) = \alpha \begin{pmatrix} 1 \\ 1/2 \\ -1/2 \end{pmatrix}.$$

A root vector $\mathbf{b}$ for $k = 2$ is determined from the homogeneous system of equations (1.37); in our case:

$$(\mathbf{A} - 0 \cdot \mathbf{I}_3)^2 \mathbf{b} = 0, \qquad \begin{cases} -5\beta_1 + 4\beta_2 - 6\beta_3 = 0, \\ 0 = 0, \\ 5\beta_1 - 4\beta_2 + 6\beta_3 = 0. \end{cases}$$

Hence

$$\mathbf{b} = \beta_1 \left(\mathbf{e}_1 - \frac{5}{6} \mathbf{e}_3 \right) + \beta_2 \left(\mathbf{e}_2 + \frac{2}{3} \mathbf{e}_3 \right),$$

so that the corresponding root subspace $\mathfrak{L}_2$ is spanned by the vectors $\mathbf{e}_1 - \frac{5}{6} \mathbf{e}_3$ and $\mathbf{e}_2 + \frac{2}{3} \mathbf{e}_3$. Formula (1.39) is

$$\mathfrak{R}_3 = \mathfrak{S}_1 \dotplus \mathfrak{L}_2.$$

We now seek a cyclic basis for $\mathfrak{L}_2$. From the first two vector equations (1.38) for $\lambda = 0$,

$$\mathbf{A}\mathbf{f}_1 = 0, \quad \mathbf{A}\mathbf{f}_2 = \mathbf{f}_1,$$

we see, taking $\mathbf{f}_1 = 2\mathbf{a}_2/\sigma$, that

$$\mathbf{f}_1 = -2\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3, \quad \mathbf{f}_2 = 3\mathbf{e}_2 + 2\mathbf{e}_3.$$

The chain of equations (1.38) breaks off at this point, since $\lambda=0$ is a double eigenvalue. However, it may be verified directly that the equation $\mathbf{A}\mathbf{f}_3=\mathbf{f}_2$ is unsolvable. Thus the Jordan form of the matrix $\mathbf{A}$ is

$$\mathbf{Q}=\|1\|\dotplus\begin{Vmatrix}0&1\\0&0\end{Vmatrix}.$$

A matrix $\mathbf{S}$ whose columns form a cyclic basis is determined by the first formula of (1.46):

$$\mathbf{S}=\|\mathbf{a}_1,\ \mathbf{f}_2^{(1)},\ \mathbf{f}_2^{(2)}\|=\begin{Vmatrix}1&-2&0\\0&-1&3\\-1&1&2\end{Vmatrix}.$$

By (1.48), we finally obtain $\mathbf{A}$:

$$\mathbf{A}=\begin{Vmatrix}1&-2&0\\0&-1&3\\-1&1&2\end{Vmatrix}\begin{Vmatrix}1&0&0\\0&0&1\\0&0&0\end{Vmatrix}\begin{Vmatrix}1&-2&0\\0&-1&3\\-1&1&2\end{Vmatrix}^{-1}.$$

This is easily checked by calculating $\mathbf{S}^{-1}$ and carrying out the matrix multiplication.

The elementary divisors of $\mathbf{A}$ are clearly $(\lambda-1)$, λ^2, and its minimum polynomial $\varphi(\lambda)=(\lambda-1)\lambda^2$ coincides with its characteristic polynomial.

Example 2. Reduce the matrices

$$\text{a) }\mathbf{A}=\begin{Vmatrix}\alpha-2&-4&-2\\4&\alpha+8&3\\-9&-18&\alpha-6\end{Vmatrix},\quad \text{b) }\mathbf{B}=\begin{Vmatrix}\alpha-2&-4&-2\\1&\alpha+2&1\\0&0&\alpha\end{Vmatrix}$$

to Jordan form (α is a complex parameter).

Solution. a) The characteristic equation (1.29) of $\mathbf{A}$ is

$$\det[\mathbf{A}-\lambda\mathbf{I}_3]=\begin{vmatrix}\alpha-2-\lambda&-4&-2\\4&\alpha+8-\lambda&3\\-9&-18&\alpha-6-\lambda\end{vmatrix}=-(\lambda-\alpha)^3=0$$

and there is just one (threefold) eigenvalue $\lambda=\alpha$. We determine a cyclic basis from equations (1.41):

$$(\mathbf{A}-\alpha\mathbf{I}_3)^3\mathbf{f}_3=0,\quad \mathbf{f}_1=(\mathbf{A}-\alpha\mathbf{I}_3)^2\mathbf{f}_3,\quad \mathbf{f}_2=(\mathbf{A}-\alpha\mathbf{I}_3)\mathbf{f}_3.$$

Since $(\mathbf{A}-\alpha\mathbf{I}_3)^3=0$ and $(\mathbf{A}-\alpha\mathbf{I}_3)^2\neq 0$, the vector $\mathbf{f}_3$ is arbitrary; setting $\mathbf{f}_3=\mathbf{e}_3$, we solve the last two equations, to get

$$\mathbf{f}_1=4\mathbf{e}_1-2\mathbf{e}_2,\quad \mathbf{f}_2=-2\mathbf{e}_1+3\mathbf{e}_2-6\mathbf{e}_3,\quad \mathbf{f}_3=\mathbf{e}_3.$$

Thus the Jordan form of $\mathbf{A}$ is simply an elementary Jordan matrix with $\lambda=\alpha$:

$$\mathbf{Q}=\begin{Vmatrix}\alpha & 1 & 0\\ 0 & \alpha & 1\\ 0 & 0 & \alpha\end{Vmatrix}.$$

The matrix $\mathbf{S}$ is determined from the first formula of (1.46):

$$\mathbf{S}=\|\mathbf{f}_1,\ \mathbf{f}_2,\ \mathbf{f}_3\|=\begin{Vmatrix}4 & -2 & 0\\ -2 & 3 & 0\\ 0 & -6 & -1\end{Vmatrix}.$$

By (1.48),

$$\mathbf{A}=\begin{Vmatrix}4 & -2 & 0\\ -2 & 3 & 0\\ 0 & -6 & 1\end{Vmatrix}\begin{Vmatrix}\alpha & 1 & 0\\ 0 & \alpha & 1\\ 0 & 0 & \alpha\end{Vmatrix}\begin{Vmatrix}4 & -2 & 0\\ -2 & 3 & 0\\ 0 & -6 & 1\end{Vmatrix}^{-1}.$$

This is easily checked directly. The matrix $\mathbf{A}$ has exactly one elementary divisor

$$(\lambda-\alpha)^3$$

and its minimum polynomial

$$\varphi(\lambda)=(\lambda-\alpha)^3$$

coincides with its characteristic polynomial.

b) The characteristic equation of $\mathbf{B}$

$$\det(\mathbf{B}-\lambda\mathbf{I}_3)=\begin{vmatrix}\alpha-2-\lambda & -4 & -2\\ 1 & \alpha+2-\lambda & 1\\ 0 & 0 & \alpha-\lambda\end{vmatrix}=-(\lambda-\alpha)^3=0$$

again determines a single (threefold) eigenvalue $\lambda=\alpha$. Since now $(\mathbf{B}-\lambda\mathbf{I}_3)^2=\mathbf{0}$, there is no three-dimensional cyclic subspace. We determine a two-dimensional cyclic subspace from the equations

$$(\mathbf{B}-\alpha\mathbf{I}_3)^2\mathbf{f}_2=\mathbf{0},\quad \mathbf{f}_1=(\mathbf{B}-\alpha\mathbf{I}_3)\,\mathbf{f}_2.$$

Setting $\mathbf{f}_2=\mathbf{e}_2$, we get

$$\mathbf{f}_1=\begin{Vmatrix}-2 & -4 & -2\\ 1 & 2 & 1\\ 0 & 0 & 0\end{Vmatrix}\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}=\begin{pmatrix}-4\\ 2\\ 0\end{pmatrix}$$

and find a one-dimensional cyclic basis

$$\mathbf{B}\mathbf{f}_3=\alpha\mathbf{f}_3,\ \text{ i. e., }\ (\mathbf{B}-\alpha\mathbf{I}_3)\,\mathbf{f}_3=\mathbf{0}.$$

Clearly, $\mathbf{f}_3$ is an eigenvector of $\mathbf{B}$, and we may put, for example,

$$\mathbf{f}_3=\mathbf{e}_1+\mathbf{e}_2-\mathbf{e}_3.$$

The vectors $\mathbf{f}_3$ and $\mathbf{f}_1$, $\mathbf{f}_2$ must be linearly independent. The nonsingular matrix $\mathbf{S}$ is

$$\mathbf{S}=\|\mathbf{f}_1,\ \mathbf{f}_2,\ \mathbf{f}_3\|=\begin{Vmatrix}-4 & 0 & 1\\ 2 & 1 & 1\\ 0 & 0 & -3\end{Vmatrix}.$$

The Jordan form of $\mathbf{B}$ is

$$\mathbf{Q}_B=\mathbf{S}^{-1}\mathbf{B}\mathbf{S}=\frac{1}{12}\begin{Vmatrix}-3 & 0 & -1\\ 6 & 12 & 6\\ 0 & 0 & 4\end{Vmatrix}\begin{Vmatrix}\alpha-2 & -4 & -2\\ 1 & \alpha+2 & 1\\ 0 & 0 & \alpha\end{Vmatrix}\begin{Vmatrix}-4 & 0 & 1\\ 2 & 1 & 1\\ 0 & 0 & 3\end{Vmatrix}=\begin{Vmatrix}\alpha & 1 & 0\\ 0 & \alpha & 0\\ 0 & 0 & \alpha\end{Vmatrix}.$$

The elementary divisors of $\mathbf{B}$ are

$$(\lambda-\alpha)^2,\quad (\lambda-\alpha)$$

and its minimum polynomial is

$$\varphi(\lambda)=(\lambda-\alpha)^2.$$

A simpler solution of Example 2b is obtained by setting $\mathbf{B}=\alpha\mathbf{I}_3+\mathbf{B}_1$. Reducing $\mathbf{B}_1$ to Jordan form, we obtain

$$\mathbf{B}=\alpha\mathbf{I}_3+\mathbf{S}\mathbf{Q}_1\mathbf{S}^{-1}=\mathbf{S}(\alpha\mathbf{I}_3+\mathbf{Q}_1)\mathbf{S}^{-1}=\mathbf{S}\mathbf{Q}_B\mathbf{S}^{-1}.$$

In the last two examples, which we shall need in §§ IV.1 and IV. 5, we shall only outline the solution in brief.

Example 3. Calculate the eigenvectors and reduce to Jordan form:

$$\mathbf{C}=\begin{Vmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{Vmatrix}\quad\text{and}\quad \mathbf{C}^*=\begin{Vmatrix}0 & 0 & 0\\ 1 & 0 & -1\\ 0 & 1 & 0\end{Vmatrix}.$$

Solution. The eigenvalues of the matrices $\mathbf{C}$ and $\mathbf{C}^*$ are $\lambda_1=0$, $\lambda_2=i$, $\lambda_3=-i$. As eigenvectors we can take

$$\mathbf{f}_1=\begin{pmatrix}\eta_1\\ 0\\ 0\end{pmatrix},\ \mathbf{f}_2=\overline{\mathbf{f}}_3=\begin{pmatrix}-\vartheta_3\\ -i\vartheta_3\\ \vartheta_3\end{pmatrix}\quad\text{and}\quad \mathbf{g}_1=\begin{pmatrix}\chi_1\\ 0\\ \chi_1\end{pmatrix},\ \mathbf{g}_2=\overline{\mathbf{g}_3}=\begin{pmatrix}0\\ -i\psi_3\\ \psi_3\end{pmatrix}.$$

Hence the following matrix $\mathbf{S}$ for $\mathbf{C}$:

$$\mathbf{S}=\begin{Vmatrix}1 & 1 & -1\\ 0 & i & i\\ 0 & -1 & 1\end{Vmatrix}.$$

The Jordan form of $\mathbf{C}$ is

$$\mathbf{Q}=\mathbf{S}^{-1}\mathbf{C}\mathbf{S}=\begin{Vmatrix}1 & 0 & 1\\ 0 & -i/2 & -1/2\\ 0 & -i/2 & 1/2\end{Vmatrix}\begin{Vmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{Vmatrix}\begin{Vmatrix}1 & 1 & -1\\ 0 & i & i\\ 0 & -1 & 1\end{Vmatrix}=\begin{Vmatrix}0 & 0 & 0\\ 0 & i & 0\\ 0 & 0 & -i\end{Vmatrix}.$$

Example 4. Reduce the following matrix to Jordan form:

$$\mathbf{C}=\begin{pmatrix}\mathbf{0} & \mathbf{I}_3\\ -\mathbf{P}_0 & \mathbf{0}\end{pmatrix},\quad \text{where}\quad \mathbf{P}_0=\begin{Vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{Vmatrix}.$$

Solution. We evaluate the eigenvalues of $\mathbf{C}$ as roots of the equation $\det(\mathbf{C}-\lambda\mathbf{I}_6)=0$: $\lambda_{1,2}=0$, $\lambda_{3,4}=i$, $\lambda_{5,6}=-i$. We now determine a cyclic basis for the root subspace belonging to the double eigenvalue 0:

$$\mathbf{C}\mathbf{f}_1=\mathbf{0},\quad \mathbf{C}\mathbf{f}_2=\mathbf{f}_1.$$

These two vector equations have the solution

$$\mathbf{f}_1=\xi_3\mathbf{e}_3,\quad \mathbf{f}_2=\eta_3\mathbf{e}_3+\xi_3\mathbf{e}_6,$$

and we henceforth put $\mathbf{f}_1^{(1)}=\mathbf{e}_3$, $\mathbf{f}_2^{(1)}=\mathbf{e}_6$.

However, for the root subspace belonging to the double eigenvalue i there is no two-dimensional cyclic subspace, since there is no vector $\mathbf{f}_3\neq\mathbf{0}$ such that

$$\mathbf{C}\mathbf{f}_3=i\mathbf{f}_3,\quad \mathbf{C}\mathbf{f}_4=i\mathbf{f}_4+\mathbf{f}_3.$$

We must therefore look for one-dimensional cyclic subspaces. In other words, we find an eigenvector $\mathbf{a}$ for which

$$\mathbf{C}\mathbf{a}=i\mathbf{a}.$$

The calculation gives

$$\mathbf{a}=\alpha_1\mathbf{e}_1+\alpha_2\mathbf{e}_2+i\alpha_1\mathbf{e}_4+i\alpha_2\mathbf{e}_5,$$

and we now put

$$\mathbf{f}_1^{(2)}=\mathbf{e}_1+i\mathbf{e}_4,\quad \mathbf{f}_2^{(2)}=\mathbf{e}_2+i\mathbf{e}_5.$$

For the eigenvalue $\lambda=-i$, we have

$$\mathbf{f}_1^{(3)}=\overline{\mathbf{f}}_1^{(2)},\quad \mathbf{f}_2^{(3)}=\overline{\mathbf{f}}_2^{(2)}.$$

The matrix $\mathbf{S}$ (§1.11) has the form

$$\mathbf{S}=\| \mathbf{f}_1^{(1)},\ \mathbf{f}_2^{(1)},\ \mathbf{f}_1^{(2)},\ \mathbf{f}_2^{(2)},\ \mathbf{f}_1^{(3)},\ \mathbf{f}_2^{(3)} \|=\begin{Vmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & -i & 0 \\ 0 & 0 & 0 & i & 0 & -i \\ 0 & 1 & 0 & 0 & 0 & 0 \end{Vmatrix}.$$

In accordance with §1.11, we have

$$\mathbf{Q}=\mathbf{S}^{-1}\mathbf{C}\mathbf{S}=\begin{Vmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \end{Vmatrix}.$$

The inverse $\mathbf{S}^{-1}$ is

$$\mathbf{S}^{-1}=\begin{Vmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1/2 & 0 & 0 & -i/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & -i/2 & 0 \\ 1/2 & 0 & 0 & i/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & i/2 & 0 \end{Vmatrix}.$$

The elementary divisors of $\mathbf{C}$ are

$$\lambda^2,\quad (\lambda-i),\quad (\lambda-i),\quad (\lambda+i),\quad (\lambda+i),$$

and its minimum polynomial is

$$\varphi(\lambda)=\lambda^2(\lambda-i)(\lambda+i)=\lambda^2(\lambda^2+1).$$

§2. FUNCTIONS OF MATRICES

2.1. Holomorphic functions of matrices

Let $f(z)$ be a function of the complex variable z, holomorphic in a disk $|z|<R$; in other words, we have an expansion

$$f(z)=\alpha_0+\alpha_1 z+\alpha_2 z^2+\ldots, \tag{2.1}$$

convergent for $|z|<R$, where $\alpha_0, \alpha_1, \alpha_2, \ldots$ are complex numbers. We shall assume that R is the radius of convergence of the series (2.1). Formal substitution of an $n\times n$ matrix $\mathbf{Z}$ for z on the right of (2.1) yields a matrix series which we denote by $f(\mathbf{Z})$:

$$f(\mathbf{Z})=\alpha_0\mathbf{I}_n+\alpha_1\mathbf{Z}+\alpha_2\mathbf{Z}^2+\ldots. \tag{2.2}$$

We shall say that this series is convergent if all n^2 scalar series of the elements of $f(\mathbf{Z})$ are convergent. By property 4 in §1.2, each element of the general term in (2.2) — the matrix $\alpha_n \mathbf{Z}^n$ —is bounded in absolute value by

$$|\alpha_n \mathbf{Z}^n| \leqslant |\alpha_n| |\mathbf{Z}|^n$$

(this inequality follows from properties 3,6, §1.2). Thus all n^2 scalar series (2.2) are majorized by the series

$$\alpha_0 |\mathbf{I}_n| + \alpha_1 |\mathbf{Z}| + \alpha_2 |\mathbf{Z}|^2 + \ldots,$$

which is convergent for $|\mathbf{Z}| < R$. Consequently, the series (2.2) itself is convergent for $|\mathbf{Z}| < R$. We shall show later that the series (2.2) is convergent if all the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix $\mathbf{Z}$ lie in the interior of the disk of convergence of (2.1), i.e., $|\lambda_\nu| < R$ $(\nu = 1, \ldots, n)$, and divergent if $|\lambda_\nu| > R$ for at least one ν.

It follows that the series

$$\left.\begin{aligned} e^{\mathbf{Z}} &= \mathbf{I}_n + \sum_{m=1}^{\infty} \frac{1}{m!} \mathbf{Z}^m, \\ \cos \mathbf{Z} &= \mathbf{I}_n + \sum_{m=1}^{\infty} (-1)^m \frac{1}{(2m)!} \mathbf{Z}^{2m}, \\ \sin \mathbf{Z} &= \sum_{m=1}^{\infty} (-1)^{m-1} \frac{1}{(2m-1)!} \mathbf{Z}^{2m-1} \end{aligned}\right\} \qquad (2.3)$$

are convergent for any square matrix $\mathbf{Z}$, so that these functions may be defined for all square matrices. Using the series, one proves the identity

$$e^{i\mathbf{Z}} \equiv \cos \mathbf{Z} + i \sin \mathbf{Z}$$

in the usual way.

Similarly, the matrix series

$$\ln \mathbf{Z} = \ln [\mathbf{I}_n + (\mathbf{Z} - \mathbf{I}_n)] = \sum_{m=1}^{\infty} (-1)^{m-1} \frac{1}{m} (\mathbf{Z} - \mathbf{I}_n)^m \qquad (2.4)$$

is convergent for $|\mathbf{Z} - \mathbf{I}_n| < 1$. In fact, the series (2.4) will converge if the eigenvalues of $\mathbf{Z}$ lie in the disk $|\lambda - 1| < 1$. Indeed, the corresponding scalar series converges for $|z - 1| < 1$. On the other hand, it follows from the definition of eigenvalues that the eigenvalues of $\mathbf{Z} - \mathbf{I}_n$ are $\lambda_\nu - 1$ $(\nu = 1, \ldots, n)$, proving our assertion.

We list a few properties of functions of matrices:*

* The proofs of these properties do not use the statement that matrix series are convergent when $|\lambda_\nu| < R$ $(\nu = 1, \ldots, n)$, which will be proved later on the basis of these properties.

1) If $\mathbf{B}=\mathbf{SAS}^{-1}$ and $f(\mathbf{A})$ exists, i. e., the corresponding matrix series is convergent, then $f(\mathbf{B})$ exists and is equal to $\mathbf{S}f(\mathbf{A})\mathbf{S}^{-1}$:

$$f(\mathbf{B})=f(\mathbf{SAS}^{-1})=\mathbf{S}f(\mathbf{A})\mathbf{S}^{-1}. \tag{2.5}$$

2) If $\mathbf{B}$ is a direct sum of matrices $\mathbf{A}_1, \ldots, \mathbf{A}_k$ $(\mathbf{B}=\mathbf{A}_1\dot{+}\ldots\dot{+}\mathbf{A}_k)$ and the functions $f(\mathbf{A}_1), \ldots, f(\mathbf{A}_k)$ exist, then $f(\mathbf{B})$ exists and is the direct sum of the matrices $f(\mathbf{A}_x)$:

$$f(\mathbf{B})=f(\mathbf{A}_1)\dot{+}\ldots\dot{+}f(\mathbf{A}_k). \tag{2.6}$$

3) If $\mathbf{Q}$ is a $k\times k$ elementary Jordan matrix,

$$\mathbf{Q}=\begin{Vmatrix}\lambda & 1 & 0 & \ldots & 0\\ 0 & \lambda & 1 & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & 0 & \ldots & 1\\ 0 & 0 & 0 & \ldots & \lambda\end{Vmatrix},$$

and $|\lambda|<R$, where R is the radius of convergence of (2.1), then $f(\mathbf{Q})$ exists and is defined by

$$f(\mathbf{Q})=\begin{Vmatrix}f(\lambda) & f'(\lambda) & \frac{1}{2!}f''(\lambda) & \ldots & \frac{1}{(k-1)!}f^{(k-1)}(\lambda)\\ 0 & f(\lambda) & f'(\lambda) & \ldots & \frac{1}{(k-2)!}f^{(k-2)}(\lambda)\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & 0 & \ldots & f'(\lambda)\\ 0 & 0 & 0 & \ldots & f(\lambda)\end{Vmatrix}. \tag{2.7}$$

Proof. 1) Clearly,

$$\mathbf{B}^2=\mathbf{SAS}^{-1}\mathbf{SAS}^{-1}=\mathbf{SA}^2\mathbf{S}^{-1},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\mathbf{B}^m=\mathbf{B}^{m-1}\mathbf{B}=\mathbf{SA}^m\mathbf{S}^{-1}\cdot\mathbf{SAS}^{-1}=\mathbf{SA}^m\mathbf{S}^{-1}.$$

Substituting into (2.2), we obtain

$$f(\mathbf{B})=\mathbf{S}(\alpha_0\mathbf{I}_n+\alpha_1\mathbf{A}+\alpha_2\mathbf{A}^2+\ldots)\mathbf{S}^{-1}=\mathbf{S}f(\mathbf{A})\mathbf{S}^{-1}.$$

2) Using the block multiplication rule (§1.2), we find that

$$\mathbf{B}^2=\mathbf{A}_1^2\dot{+}\ldots\dot{+}\mathbf{A}_k^2, \ \ldots, \ \mathbf{B}^m=\mathbf{A}_1^m\dot{+}\ldots\dot{+}\mathbf{A}_k^m$$

and substitution into (2.2) gives

$$f(\mathbf{B})=\sum_{m=0}^{\infty}\alpha_m(\mathbf{A}_1^m\dot{+}\ldots\dot{+}\mathbf{A}_k^m)=f(\mathbf{A}_1)\dot{+}\ldots\dot{+}f(\mathbf{A}_k).$$

3) To fix ideas, we shall prove (2.7) for a 4×4 matrix. We have

$$\mathbf{Q}^2 = \begin{Vmatrix} \lambda^2 & 2\lambda & 1 & 0 \\ 0 & \lambda^2 & 2\lambda & 1 \\ 0 & 0 & \lambda^2 & 2\lambda \\ 0 & 0 & 0 & \lambda^2 \end{Vmatrix}, \quad \mathbf{Q}^3 = \mathbf{Q}^2\mathbf{Q} = \begin{Vmatrix} \lambda^3 & 3\lambda^2 & 3\lambda & 1 \\ 0 & \lambda^3 & 3\lambda^2 & 3\lambda \\ 0 & 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & 0 & \lambda^3 \end{Vmatrix},$$

$$\mathbf{Q}^m = \mathbf{Q}^{m-1}\mathbf{Q} = \begin{Vmatrix} \lambda^m & m\lambda^{m-1} & \frac{1}{2!}m(m-1)\lambda^{m-2} & \frac{1}{3!}m(m-1)(m-2)\lambda^{m-3} \\ 0 & \lambda^m & m\lambda^{m-1} & \frac{1}{2!}m(m-1)\lambda^{m-2} \\ 0 & 0 & \lambda^m & m\lambda^{m-1} \\ 0 & 0 & 0 & \lambda^m \end{Vmatrix}.$$

The last formula may be proved by induction on m. Substituting the powers of $\mathbf{Q}$ into (2.2), we get a matrix for $f(\mathbf{Q})$ in which all the elements on the principal diagonal are

$$\sum_{m=0}^{\infty} \alpha_m \lambda^m = f(\lambda),$$

the elements on the superdiagonal are

$$\sum_{m=1}^{\infty} m\alpha_m \lambda^{m-1} = f'(\lambda),$$

those directly above the superdiagonal are

$$\sum_{m=2}^{\infty} \frac{1}{2!}m(m-1)\alpha_m \lambda^{m-2} = \frac{1}{2!}f''(\lambda)$$

and finally the upper right element is

$$\sum_{m=3}^{\infty} \frac{1}{3!}m(m-1)(m-2)\alpha_m \lambda^{m-3} = \frac{1}{3!}f'''(\lambda).$$

All these series are convergent for $|\lambda| < R$. Thus, for a fourth-order matrix,

$$f(\mathbf{Q}) = \begin{Vmatrix} f(\lambda) & f'(\lambda) & \frac{1}{2!}f''(\lambda) & \frac{1}{3!}f'''(\lambda) \\ 0 & f(\lambda) & f'(\lambda) & \frac{1}{2!}f''(\lambda) \\ 0 & 0 & f(\lambda) & f'(\lambda) \\ 0 & 0 & 0 & f(\lambda) \end{Vmatrix}.$$

It follows from properties 1, 2, 3 that evaluation of $f(\mathbf{Z})$ reduces to evaluating functions of the elementary Jordan matrices in the canonical form of $\mathbf{Z}$. Indeed, if we have (1.48)

$$\mathbf{Z} = \mathbf{S}(\mathbf{Q}_1 \dotplus \ldots \dotplus \mathbf{Q}_s)\mathbf{S}^{-1}, \tag{2.8}$$

then

$$f(\mathbf{Z}) = \mathbf{S}[f(\mathbf{Q}_1) \dotplus \ldots \dotplus f(\mathbf{Q}_s)]\,\mathbf{S}^{-1}, \tag{2.9}$$

where each $f(\mathbf{Q}_\sigma)$ $(\sigma = 1, \ldots, s)$ has the form (2.7).

We can now prove the assertion at the beginning of this subsection, concerning the radius of convergence of the series (2.2). Let $f_k(\mathbf{Z})$ denote a partial sum of the series:

$$f_k(\mathbf{Z}) = \sum_{\varkappa=0}^{k} \alpha_\varkappa \mathbf{Z}^\varkappa.$$

Substituting from (2.8), we write

$$f_k(\mathbf{Z}) = \mathbf{S}\left[\sum_{\varkappa=0}^{k} \alpha_\varkappa \mathbf{Q}_1^\varkappa \dotplus \ldots \dotplus \sum_{\varkappa=0}^{k} \alpha_\varkappa \mathbf{Q}_s^\varkappa\right]\mathbf{S}^{-1}. \tag{2.10}$$

The matrices

$$\sum_{\varkappa=0}^{k} \alpha_\varkappa \mathbf{Q}_\sigma^\varkappa$$

$(\sigma = 1, \ldots, s)$ have the following form. The diagonal elements are $f_k(\lambda_{(\sigma)})$ (the partial sums of the series (2.1)), above them we have $f_k'(\lambda_{(\sigma)})$, next the elements $f_k''(\lambda_{(\sigma)})/2!$, and so on; all subdiagonal elements are zero. Thus all the series obtained from the partial sums in the expression bracketed in (2.10) *are convergent, provided* $|\lambda_{(\sigma)}| < R$. *If at least one of the* $\lambda_{(\sigma)}$*'s is outside the disk of convergence, the corresponding series is divergent, and so the matrix series (2.2) is divergent.*

Formula (2.9) for functions of matrices, as well as (2.7), are often extremely useful in proofs. As an example, we prove a simple proposition which will be needed below.

Lemma. *Let* $\mathbf{R}$ *be a singular matrix. If there is a vector* $\mathbf{c} \neq \mathbf{0}$ *such that*

$$(e^{\mathbf{R}} - \mathbf{I}_n)\mathbf{c} = \mathbf{0}, \tag{2.11}$$

then

$$\mathbf{R}\mathbf{c} = \mathbf{0}. \tag{2.12}$$

Note that the converse is obvious, and so (2.11) and (2.12) are equivalent.

Proof. Let

$$\tilde{\mathbf{R}} = \mathbf{S}^{-1}\mathbf{R}\mathbf{S} = \mathbf{Q}_1 \dotplus \mathbf{Q}_2 \dotplus \ldots \dotplus \mathbf{Q}_s$$

be the Jordan form of the matrix $\mathbf{R}$, where $\mathbf{Q}_\sigma$ is an elementary Jordan matrix of order k_σ $(\sigma = 1, \ldots, s)$. Multiplying equation (2.11) on the left by $\mathbf{S}^{-1}$ and using the relation $\mathbf{S}^{-1}e^{\mathbf{R}}\mathbf{S} = \exp(\mathbf{S}^{-1}\mathbf{R}\mathbf{S})$, we get

$$(e^{\tilde{\mathbf{R}}} - \mathbf{I}_n)\tilde{\mathbf{c}} = \mathbf{0}, \tag{2.13}$$

where $\tilde{\mathbf{c}}=\mathbf{S}^{-1}\mathbf{c}$. We express $\tilde{\mathbf{c}}$ as a direct sum

$$\tilde{\mathbf{c}}=\mathbf{c}_1\dot{+}\ldots\dot{+}\mathbf{c}_s,$$

where $\mathbf{c}_\sigma$ is a k_σ-vector. Since

$$(e^{\tilde{\mathbf{R}}}-\mathbf{I}_n)=(e^{\mathbf{Q}_1}-\mathbf{I}_{k_1})\dot{+}\ldots\dot{+}(e^{\mathbf{Q}_s}-\mathbf{I}_{k_s}),$$

where $\mathbf{I}_{k_\sigma}$ is the identity matrix of order k_σ, it follows from (2.13) that

$$(e^{\mathbf{Q}_\sigma}-\mathbf{I}_{k_\sigma})\,\mathbf{c}_\sigma=\mathbf{0}\qquad(\sigma=1,\ \ldots,\ s). \tag{2.14}$$

By (2.7),

$$e^{\mathbf{Q}}=e^{\lambda_{(\sigma)}}\left\|\begin{array}{cccc}1 & 1 & \frac{1}{2!} & \ldots\\ 0 & 1 & 1 & \ldots\\ 0 & 0 & 1 & \ldots\\ \ldots & \ldots & \ldots & \ldots\end{array}\right\|.$$

Therefore,

$$\det\,[e^{\mathbf{Q}_\sigma}-\mathbf{I}_\sigma]=(e^{\lambda_{(\sigma)}}-1)^{k_\sigma}.$$

By (2.14), we have $\mathbf{c}_\sigma=\mathbf{0}$ for $\lambda_{(\sigma)}\neq 0$, i. e.,

$$\mathbf{Q}_\sigma\mathbf{c}_\sigma=\mathbf{0}\quad\text{for}\quad\lambda_{(\sigma)}\neq 0.$$

For $\lambda_{(\sigma)}=0$, it follows from (2.14) that

$$\mathbf{c}_\sigma=(\xi,\ 0,\ 0,\ \ldots\ ,\ 0)^{\tau},$$

where ξ is a number. Thus

$$\mathbf{Q}_\sigma\mathbf{c}_\sigma=\mathbf{0}\quad\text{for}\quad\lambda_{(\sigma)}=0.$$

Since $\mathbf{Q}_\sigma\mathbf{c}_\sigma=\mathbf{0}$ $(\sigma=1,\ \ldots,\ s)$, we have $\tilde{\mathbf{R}}\tilde{\mathbf{c}}=\mathbf{0}$, and so also $\mathbf{R}\mathbf{c}=\mathbf{0}$. Q. E. D.

2.2. Piecewise-analytic functions of matrices

A different definition of functions of matrices, in terms of contour integrals, is frequently more convenient. This definition also yields a larger class of functions. Let G be the union of a finite number of disjoint simply-connected domains* G_h in the complex plane (for example, a union

* A set of points on the plane is said to be connected if any two of its points may be joined by a curve lying entirely in the set (a curve is any continuous image of a closed interval). A set in the plane is open if each of its points has a neighborhood lying entirely in the set. By a domain we mean a connected open set. If any closed curve in the domain may be contracted to a point without crossing the boundary of the domain, the domain is said to be simply-connected.

of finitely many disjoint disks), in whose interior a single-valued analytic function $f(z)$ is defined. Note that we do not require the analytic continuation of the function (if it exists) from one domain to another to yield any prescribed value, so that $f(z)$ does not in general define an analytic function in the sense of Weierstrass: $f(z)$ is a piecewise-analytic function. For instance, let G be the union of two disjoint disks G_1, G_2, and

$$f(z)=\begin{cases}\sin z & \text{for } z\in G_1,\\ \cos z & \text{for } z\in G_2.\end{cases}$$

Let $\mathfrak{M}_G$ denote the set of all matrices $\mathbf{A}$ whose spectrum (= set of eigenvalues) lies in G. Let $\mathbf{A}$ be a fixed matrix in $\mathfrak{M}_G$. Surround all the eigenvalues of $\mathbf{A}$ in G_k by a closed curve* Γ_k lying entirely in G_k. We now define the matrix-function $f(\mathbf{A})$ by the Cauchy formula

$$f(\mathbf{A})=\frac{1}{2\pi i}\sum_k\int_{\Gamma_k}(\zeta\mathbf{I}_n-\mathbf{A})^{-1}f(\zeta)\,d\zeta, \tag{2.15}$$

where the integration along Γ_k is performed in the positive direction.

We shall now verify that properties 1, 2 and 3 of §2.1 hold for this definition. Properties 1 and 2 are obvious. In regard to property 3, if $\mathbf{Q}$ is a $k\times k$ elementary Jordan matrix, then

$$(\zeta\mathbf{I}_k-\mathbf{Q})^{-1}=\begin{Vmatrix}(\zeta-\lambda)^{-1} & (\zeta-\lambda)^{-2} & \dots & (\zeta-\lambda)^{-k}\\ 0 & (\zeta-\lambda)^{-1} & \dots & (\zeta-\lambda)^{-k+1}\\ \dots & \dots & \dots & \dots\\ 0 & 0 & & (\zeta-\lambda)^{-1}\end{Vmatrix}.$$

Let Γ be a closed contour around the point λ in the complex plane. By the Cauchy formula for analytic functions,

$$\frac{1}{2\pi i}\int_\Gamma(\zeta\mathbf{I}_k-\mathbf{Q})^{-1}f(\zeta)\,d\zeta=\begin{Vmatrix}f(\lambda) & f'(\lambda) & \dots & \frac{1}{(k-1)!}f^{(k-1)}(\lambda)\\ 0 & f(\lambda) & \dots & \frac{1}{(k-2)!}f^{(k-2)}(\lambda)\\ \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & f(\lambda)\end{Vmatrix}, \tag{2.16}$$

and this is precisely (2.7).

We claim that the definition in §2.1 is a special case of our new definition. Reduce the matrix $\mathbf{A}$ to Jordan form:

$$\mathbf{A}=\mathbf{S}(\mathbf{Q}_1\dotplus\dots\dotplus\mathbf{Q}_s)\mathbf{S}^{-1}.$$

Let R be so large that all the eigenvalues $\lambda_{(1)}, \dots, \lambda_{(s)}$ of $\mathbf{A}$ lie in the disk $|z|<R$, which we denote by G. Let $f(z)$ be a holomorphic function defined in G. By properties 1 and 2, which hold for both definitions, we need only

* We shall assume that Γ_k is a piecewise-smooth (or rectifiable) curve and that it "encircles" each eigenvalue $\lambda_j\in G_k$ of $\mathbf{A}$ exactly once: if a point ζ traverses Γ_k in the positive direction, then $\Delta\operatorname{Arg}(\zeta-\lambda_j)=2\pi$.

prove (2.15) for an elementary Jordan matrix $\mathbf{Q}_\sigma$, of order $k_\sigma \times k_\sigma$, say. In this case the fact that the function has the same value according to both definitions follows from the fact that the right-hand sides of (2.7) and (2.16) are identical. Thus, definition (2.15) extends that of §2.1, since it yields a larger class of functions but gives the same value if all the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix $\mathbf{Z}$ lie in a disk $|z| < R$ in which $f(z)$ is holomorphic.

2.3. Interpolation polynomials of matrices

Actual computation of the value of a function using the series (2.2) or the integral (2.15) is usually impossible. In this subsection we shall derive formulas which are convenient for practical computation of functions of matrices.

Definition. Let $(\lambda - \lambda_{(\sigma)})^{k_\sigma}$ $(\sigma = 1, \ldots, s)$ be the elementary divisors of a matrix $\mathbf{A}$. We shall say that functions $f_1(z)$ and $f_2(z)$, analytic at the points $\lambda_{(1)}, \ldots, \lambda_{(s)}$, coincide on the spectrum of $\mathbf{A}$ if

$$f_1(\lambda_{(\sigma)}) = f_2(\lambda_{(\sigma)}),\ f_1'(\lambda_{(\sigma)}) = f_2'(\lambda_{(\sigma)}),\ \ldots,\ f_1^{(k_\sigma - 1)}(\lambda_{(\sigma)}) = f_2^{(k_\sigma - 1)}(\lambda_{(\sigma)}) \quad (\sigma = 1, \ldots, s).$$

Here s is the number of elementary divisors of $\mathbf{A}$ and $k_1, \ldots, k_s$ are their degrees, so that each of the above rows of equalities corresponds to an elementary divisor $(\lambda - \lambda_{(\sigma)})^{k_\sigma}$ of $\mathbf{A}$.

In particular, if all the eigenvalues λ_ν of $\mathbf{A}$ are distinct or there are no multiple roots with multiple elementary divisors, then $f_1(z)$ and $f_2(z)$ coincide on the spectrum of $\mathbf{A}$ if and only if $f_1(\lambda_\nu) = f_2(\lambda_\nu)$ $(\nu = 1, \ldots, n)$.

This definition and properties 1, 2, 3 imply:

Lemma. *If $f_1(z)$ and $f_2(z)$ coincide on the spectrum of $\mathbf{Z}$, then*

$$f_1(\mathbf{Z}) = f_2(\mathbf{Z}).$$

Using this lemma, given any $n \times n$ matrix $\mathbf{Z}$ and a function $f(z)$, we can construct a polynomial $P(z)$ of degree at most $n-1$ such that

$$f(\mathbf{Z}) = P(\mathbf{Z}).$$

This we now proceed to do.

We first consider the case that all the elementary divisors of the matrix $\mathbf{Z}$ are simple (in particular, the case of distinct eigenvalues). Let $\lambda^{(1)}, \ldots, \lambda^{(q)}$ be all distinct eigenvalues of $\mathbf{Z}$ $(q \leqslant n)$. The polynomial $P(z)$ is defined by the conditions

$$P(\lambda^{(\varkappa)}) = f(\lambda^{(\varkappa)}) \qquad (\varkappa = 1, \ldots, q)$$

and is given by the Lagrange interpolation formula (see, e. g., /85a/, Vol. I, § 158):

$$P(z) = \sum_{\varkappa=1}^{q} \frac{(z-\lambda^{(1)}) \ldots (z-\lambda^{(\varkappa-1)})(z-\lambda^{(\varkappa+1)}) \ldots (z-\lambda^{(q)})}{(\lambda^{(\varkappa)}-\lambda^{(1)}) \ldots (\lambda^{(\varkappa)}-\lambda^{(\varkappa-1)})(\lambda^{(\varkappa)}-\lambda^{(\varkappa+1)}) \ldots (\lambda^{(\varkappa)}-\lambda^{(q)})} f(\lambda^{(\varkappa)}),$$

as is readily checked directly. Replacing z by $\mathbf{Z}$ and $P(z)$ by $f(\mathbf{Z})$, we get

$$f(\mathbf{Z})=\sum_{\varkappa=1}^{q}\frac{(\mathbf{Z}-\lambda^{(1)}\mathbf{I}_n)\ \dots\ (\mathbf{Z}-\lambda^{(\varkappa-1)}\mathbf{I}_n)\,(\mathbf{Z}-\lambda^{(\varkappa+1)}\mathbf{I}_n)\ \dots\ (\mathbf{Z}-\lambda^{(q)}\mathbf{I}_n)}{(\lambda^{(\varkappa)}-\lambda^{(1)})\ \dots\ (\lambda^{(\varkappa)}-\lambda^{(\varkappa-1)})\,(\lambda^{(\varkappa)}-\lambda^{(\varkappa+1)})\ \dots\ (\lambda^{(\varkappa)}-\lambda^{(q)})}f(\lambda^{(\varkappa)}). \quad (2.17)$$

This formula is known as Sylvester's formula; it expresses a function of an $n\times n$ matrix $\mathbf{Z}$ all of whose elementary divisors are simple as a polynomial of degree $q-1$ in $\mathbf{Z}$. It is convenient for use in actual computation of the function $f(\mathbf{Z})$. We illustrate the general form of the matrix $f(\mathbf{Z})$ for matrices of orders 2, 3 and 4 with distinct eigenvalues:

$$n=2: \quad f(\mathbf{Z})=\frac{1}{\lambda_1-\lambda_2}f(\lambda_1)(\mathbf{Z}-\lambda_2\mathbf{I}_2)+\frac{1}{\lambda_2-\lambda_1}f(\lambda_2)(\mathbf{Z}-\lambda_1\mathbf{I}_2), \quad (2.18)$$

$$n=3: \quad f(\mathbf{Z})=\frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}f(\lambda_1)(\mathbf{Z}-\lambda_2\mathbf{I}_3)(\mathbf{Z}-\lambda_3\mathbf{I}_3)+$$
$$+\frac{1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}f(\lambda_2)(\mathbf{Z}-\lambda_1\mathbf{I}_3)(\mathbf{Z}-\lambda_3\mathbf{I}_3)+$$
$$+\frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}f(\lambda_3)(\mathbf{Z}-\lambda_1\mathbf{I}_3)(\mathbf{Z}-\lambda_2\mathbf{I}_3), \quad (2.19)$$

$$n=4: \quad f(\mathbf{Z})=\frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}f(\lambda_1)(\mathbf{Z}-\lambda_2\mathbf{I}_4)(\mathbf{Z}-\lambda_3\mathbf{I}_4)(\mathbf{Z}-\lambda_4\mathbf{I}_4)+$$
$$+\frac{1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}f(\lambda_2)(\mathbf{Z}-\lambda_1\mathbf{I}_4)(\mathbf{Z}-\lambda_3\mathbf{I}_4)(\mathbf{Z}-\lambda_4\mathbf{I}_4)+$$
$$+\frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)}f(\lambda_3)(\mathbf{Z}-\lambda_1\mathbf{I}_4)(\mathbf{Z}-\lambda_2\mathbf{I}_4)(\mathbf{Z}-\lambda_4\mathbf{I}_4)+$$
$$+\frac{1}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}f(\lambda_4)(\mathbf{Z}-\lambda_1\mathbf{I}_4)(\mathbf{Z}-\lambda_2\mathbf{I}_4)(\mathbf{Z}-\lambda_3\mathbf{I}_4). \quad (2.20)$$

Formulas (2.18), (2.19) and (2.20) are also valid for matrices of arbitrary order having 2, 3 and 4 distinct eigenvalues, respectively (and no others!), with simple elementary divisors.

We now consider the general case, in which not all the elementary divisors of $\mathbf{Z}$ are simple. Again, let $\lambda^{(1)}, \dots, \lambda^{(q)}$ denote all distinct eigenvalues of an $n\times n$ matrix $\mathbf{Z}$ $(q<n)$. For each $\lambda^{(\varkappa)}$, consider the elementary divisor with maximum exponent $p_\varkappa$:

$$(\lambda-\lambda^{(\varkappa)})^{p_\varkappa} \qquad (p_\varkappa=1, 2, \dots;\ \varkappa=1, \dots, q).$$

The function $f(\mathbf{Z})$ will be expressed through the Lagrange-Sylvester interpolation polynomial (/21/, Chapter V, § 3) of degree $m-1$ $(m=p_1+\dots+p_q)$:

$$f(\mathbf{Z})=\sum_{\varkappa=1}^{1}[f(\lambda^{(\varkappa)})\mathbf{Z}_{\varkappa 1}+f'(\lambda^{(\varkappa)})\mathbf{Z}_{\varkappa 2}+\dots+f^{(p_\varkappa-1)}(\lambda^{(\varkappa)})\mathbf{Z}_{\varkappa p_\varkappa}]. \quad (2.21)$$

To determine the matrices $\mathbf{Z}_{\varkappa\rho}$ $(\rho=1, \dots, p_\varkappa;\ \varkappa=1, \dots, q)$ in the basic formula (2.21), one replaces $f(\lambda)$ successively by suitable simple polynomials of degree $0, 1, 2, \dots, m-1$, and uses the resulting linear equations to determine the matrices $\mathbf{Z}_{\varkappa\rho}$. Referring the reader for details to /21/, let us consider the case of second- and third-order matrices. For 2×2

matrices, the only remaining case is that of a quadratic elementary divisor $(\lambda-\lambda_1)^2$, and then formula (2.21) is

$$f(\mathbf{Z})=f(\lambda_1)\,\mathbf{I}_2+f'(\lambda_1)\,(\mathbf{Z}-\lambda_1\mathbf{I}_2). \tag{2.22}$$

For 3×3 matrices, there are three possibilities:

a) The elementary divisors are $(\lambda-\lambda_1)$, $(\lambda-\lambda_1)^2$. $f(\mathbf{Z})$ is then given by formula (2.22) with $\mathbf{I}_2$ replaced by $\mathbf{I}_3$.

b) The elementary divisors are $(\lambda-\lambda_1)$, $(\lambda-\lambda_2)^2$, $\lambda_1\neq\lambda_2$. Evaluation of $\mathbf{Z}_{11}$, $\mathbf{Z}_{21}$, $\mathbf{Z}_{22}$ gives

$$f(\mathbf{Z})=f(\lambda_2)\,\mathbf{I}_3+f'(\lambda_2)\,(\mathbf{Z}-\lambda_2\mathbf{I}_3)+ \\ +\frac{1}{(\lambda_1-\lambda_2)^2}\,[f(\lambda_1)-f(\lambda_2)-(\lambda_1-\lambda_2)\,f'(\lambda_2)]\,(\mathbf{Z}-\lambda_2\mathbf{I}_3)^2. \tag{2.23}$$

c) There is only one elementary divisor, $(\lambda-\lambda_1)^3$. Using (2.21), we find

$$f(\mathbf{Z})=f(\lambda_1)\,\mathbf{I}_3+f'(\lambda_1)\,(\mathbf{Z}-\lambda_1\mathbf{I}_3)+\frac{1}{2}f''(\lambda_1)\,(\mathbf{Z}-\lambda_1\mathbf{I}_3)^2. \tag{2.24}$$

2.4. Multiplicative property

Let $\varphi(z)$ and $\psi(z)$ be functions analytic in a set G which, as in §2.2, is the union of finitely many simply-connected domains, and let $\mathbf{A}$ be a matrix whose spectrum lies in G. Construct the matrices $\varphi(\mathbf{A})$, $\psi(\mathbf{A})$, and also $f(\mathbf{A})$, where $f(z)=\varphi(z)\,\psi(z)$. We claim that

$$f(\mathbf{A})=\varphi(\mathbf{A})\,\psi(\mathbf{A}). \tag{2.25}$$

To prove this, we first deform the contour Γ_k in (2.15) into a contour Γ_k', lying inside Γ_k, which has the same properties as Γ_k. Applying formula (2.15) to the function $\psi(\mathbf{A})$ and the "deformed" formula (2.15) to $\varphi(\mathbf{A})$, we obtain

$$\varphi(\mathbf{A})\,\psi(\mathbf{A})=\left(\frac{1}{2\pi i}\right)^2\sum_{h,\,k}\int_{\Gamma_h'}\int_{\Gamma_k}(\zeta\mathbf{I}_n-\mathbf{A})^{-1}\,(\eta\mathbf{I}_n-\mathbf{A})^{-1}\,\varphi(\zeta)\,\psi(\eta)\,d\zeta\,d\eta.$$

Here $\zeta\in\Gamma_h'$, $\eta\in\Gamma_k$. The following identity is easily verified:

$$(\zeta\mathbf{I}_n-\mathbf{A})^{-1}(\eta\mathbf{I}_n-\mathbf{A})^{-1}=\frac{1}{\zeta-\eta}\,[(\zeta\mathbf{I}_n-\mathbf{A})^{-1}-(\eta\mathbf{I}_n-\mathbf{A})^{-1}].$$

Next, for $\zeta\in\Gamma_h'$, $\eta\in\Gamma_k$ $(k\neq h)$ we have

$$\int_{\Gamma_k}\frac{1}{\zeta-\eta}\,(\eta\mathbf{I}_n-\mathbf{A})^{-1}\,\varphi(\zeta)\,\psi(\eta)\,d\eta=0,\qquad \int_{\Gamma_h'}\frac{1}{\zeta-\eta}\,(\zeta\mathbf{I}_n-\mathbf{A})^{-1}\,\varphi(\zeta)\,\psi(\eta)\,d\zeta=0,$$

since the integrands are analytic functions of the variable of integration inside the integration contour. Therefore

$$\varphi(\mathbf{A})\psi(\mathbf{A})=\left(\frac{1}{2\pi i}\right)^2\sum_h\int_{\Gamma_h'}\int_{\Gamma_h}\frac{1}{\zeta-\eta}\left[(\zeta\mathbf{I}_n-\mathbf{A})^{-1}-(\eta\mathbf{I}_n-\mathbf{A})^{-1}\right]\varphi(\zeta)\psi(\eta)\,d\zeta\,d\eta.$$

We have

$$\int_{\Gamma_h'}\frac{1}{\zeta-\eta}(\eta\mathbf{I}_n-\mathbf{A})^{-1}\varphi(\zeta)\,d\zeta=0,$$

since the integrand is an analytic function of ζ inside Γ_h'. Consequently,

$$\varphi(\mathbf{A})\psi(\mathbf{A})=\left(\frac{1}{2\pi i}\right)^2\sum_h\int_{\Gamma_h'}\int_{\Gamma_h}\frac{1}{\zeta-\eta}(\zeta\mathbf{I}_n-\mathbf{A})^{-1}\varphi(\zeta)\psi(\eta)\,d\zeta\,d\eta.$$

Since

$$\frac{1}{2\pi i}\int_{\Gamma_h}\frac{\psi(\eta)}{\zeta-\eta}\,d\eta=\psi(\zeta),$$

we finally have

$$\varphi(\mathbf{A})\psi(\mathbf{A})=\frac{1}{2\pi i}\sum_h\int_{\Gamma_h}(\zeta\mathbf{I}_n-\mathbf{A})^{-1}\varphi(\zeta)\psi(\zeta)\,d\zeta=f(\mathbf{A}),$$

Q. E. D.

It follows from the multiplicative property that $\varphi(\mathbf{A})\psi(\mathbf{A})=\psi(\mathbf{A})\varphi(\mathbf{A})$, i. e., different functions of the same matrix commute.

Setting $\varphi(z)=e^{rz}$, $\psi(z)=e^{sz}$ (where r, s are arbitrary complex numbers) in (2.25), we obtain the matrix identity $e^{r\mathbf{A}}e^{s\mathbf{A}}=e^{(r+s)\mathbf{A}}$, which is also readily proved directly (by multiplying the corresponding series).

Another result of the multiplicative property is that the function $\psi(z)=z^m$ (where m is an integer) corresponds to $\varphi(\mathbf{A})=\mathbf{A}^m$, which is the ordinary -th power of the matrix $\mathbf{A}$. More important corollaries of the multiplicative property will be established below.

2.5. Composite function lemma

Lemma. Let $\varphi(z)$ be a single-valued analytic function on a set G which is the union of finitely many simply-connected domains; denote $\varphi(G)=G_1$. Let $\psi(w)$ be a single-valued analytic function in G_1, so that $f(z)=\psi[\varphi(z)]$ is single-valued and analytic in G. Suppose that the spectrum of the matrix $\mathbf{A}$ is in G (so that the spectrum of $\mathbf{B}=f(\mathbf{A})$ is in G_1). Then $f(\mathbf{A})=\psi(\mathbf{B})$. In other words, stepwise calculation $\mathbf{B}=\varphi(\mathbf{A})$, $\psi(\mathbf{B})=\psi[\varphi(\mathbf{A})]$ and direct calculation $f(\mathbf{A})=\psi[\varphi(\mathbf{A})]$ yield the same result.

Proof. Given $\mathbf{A}$ and $\varphi(z)$, we can construct a polynomial (§2.3)

$$\Phi(z)=\sum_{\mu=0}^{m}\alpha_\mu z^\mu$$

such that $\varphi(\mathbf{A})\equiv\sum\alpha_\mu\mathbf{A}^\mu$. To this end, it suffices to determine $\Phi(z)$ from the conditions

$$\Phi(\lambda^{(\varkappa)})=\varphi(\lambda^{(\varkappa)}),\quad \Phi'(\lambda^{(\varkappa)})=\varphi'(\lambda^{(\varkappa)}),\ \ldots,\ \Phi^{(p_\varkappa-1)}(\lambda^{(\varkappa)})=\varphi^{(p_\varkappa-1)}(\lambda^{(\varkappa)}),$$

where $\lambda^{(\varkappa)}$ $(\varkappa=1,\ldots,q)$ are the distinct eigenvalues of $\mathbf{A}$ and $p_\varkappa$ is the order of the largest elementary Jordan matrix corresponding to the eigenvalue $\lambda^{(\varkappa)}$ $(p_1+\ldots+p_q\leqslant n)$. We construct a polynomial $\Psi(w)$ in similar fashion:

$$\Psi(\mu_\varkappa)=\psi(\mu_\varkappa),\ \Psi'(\mu_\varkappa)=\psi'(\mu_\varkappa),\ \ldots,\ \Psi^{(p_\varkappa-1)}(\mu_\varkappa)=\psi^{(p_\varkappa-1)}(\mu_\varkappa).$$

Here $\mu_\varkappa=\varphi(\lambda^{(\varkappa)})$ $(\varkappa=1,\ldots,q)$ are the eigenvalues of $\mathbf{B}$. We claim that the polynomial $F(z)\equiv\Psi[\Phi(z)]$ and the function $f(z)$ coincide on the spectrum of $\mathbf{A}$. Indeed,

$$\begin{aligned}
&F(\lambda^{(\varkappa)})=\Psi[\Phi(\lambda^{(\varkappa)})]=\Psi(\mu_\varkappa)=\psi(\mu_\varkappa)=f(\lambda^{(\varkappa)}),\\
&F'(\lambda^{(\varkappa)})=\Psi'(\mu_\varkappa)\Phi'(\lambda^{\varkappa})=\psi'(\mu_\varkappa)\varphi'(\lambda^{(\varkappa)})=f'(\lambda^{(\varkappa)}),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&F^{(p_\varkappa-1)}(\lambda^{(\varkappa)})=f^{(p_\varkappa-1)}(\lambda^{(\varkappa)})\quad(\varkappa=1,\ldots,q).
\end{aligned}$$

Thus $F(z)$ and $f(z)$ coincide on the spectrum of $\mathbf{A}$ and by the lemma of §2.3, $F(\mathbf{A})=f(\mathbf{A})$. Letting $\mathbf{C}_1$ denote the result of the stepwise calculation and $\mathbf{C}_2$ that of the direct calculation of $\psi[\varphi(\mathbf{A})]$, we have

$$\mathbf{B}=\varphi(\mathbf{A})=\Phi(\mathbf{A}),\ \mathbf{C}_1=\psi(\mathbf{B})=\Psi(\mathbf{B})=\Psi[\Phi(\mathbf{A})]=F(\mathbf{A}).$$

On the other hand, $\mathbf{C}_2=f(\mathbf{A})=F(\mathbf{A})$, so that $\mathbf{C}_1=\mathbf{C}_2$, Q. E. D.

2.6. Logarithm of a matrix

Later we shall need the function $\ln\mathbf{Z}$. Let $\rho^{(j)}$ $(j=1,\ldots,q\leqslant n)$ be the distinct eigenvalues of an $n\times n$ matrix $\mathbf{Z}$. Assume that $\rho^{(j)}\neq 0$ $(j=1,\ldots,q)$, so that $\det\mathbf{Z}\neq 0$. Slit the complex plane along a ray going from the origin $\rho=0$ to infinity, not containing any of the points $\rho^{(1)},\ldots,\rho^{(q)}$. In the slit plane we can now define single-valued branches of the function $\ln(z)$, which we denote by $(\ln z)_m$, $m=0,\pm 1,\pm 2,\ldots$:

$$(\ln z)_m=\ln|z|+i(\arg z+2\pi m).$$

Here $\arg z$ is a single-valued function of z in the slit plane. If the plane is slit along the ray $\arg z=\varphi_0$, we can stipulate, for example, that $\varphi_0\leqslant\arg z<\varphi_0+2\pi$.

Let Γ_j be circles centered at the points $\rho^{(j)}$ $(j=1, \ldots, q)$ which do not intersect one another or the ray along which the plane has been slit. Let G_j denote the open disk with circumference Γ_j, and G the union of these disks. We define a piecewise-analytic function in G by

$$\ln z = (\ln z)_{m_j} \quad \text{for} \quad z \in G_j \qquad (j=1, \ldots, q).$$

For the moment, the integers m_j are arbitrary. Using the Cauchy formula (2.15), we can now define a piecewise-analytic function $\ln \mathbf{Z}$ of the matrix $\mathbf{Z}$ by

$$\ln \mathbf{Z} = \frac{1}{2\pi i} \sum_{j=1}^{q} \int_{\Gamma_j} (\zeta \mathbf{I}_n - \mathbf{Z})^{-1} (\ln \zeta)_{m_j} \, d\zeta. \tag{2.26}$$

We shall show that the matrix $\mathbf{K} = \ln \mathbf{Z}$ satisfies the equation

$$e^{\mathbf{K}} = \mathbf{Z}. \tag{2.27}$$

Setting

$$\varphi(z) = (\ln z)_{m_j} \quad \text{for} \quad z \in G_j, \quad \psi(w) = e^w,$$

we see that the composite function $f(z) = \psi[\varphi(z)]$ is simply $f(z) \equiv z$. By the composite function lemma, the same result is obtained whether one directly takes $f(\mathbf{Z}) = \mathbf{Z}$ or first calculates $\mathbf{K} = \ln \mathbf{Z}$ and then $e^{\mathbf{K}} = f(\mathbf{Z})$, so that (2.27) is valid. Thus there exists an infinite set of matrices $\mathbf{K}$ satisfying equation (2.27), since the choice of the numbers m_j was arbitrary.

The eigenvalues of the matrices $\mathbf{K} = \ln \mathbf{Z}$ are the numbers $[\ln \rho^{(j)}]_{m_j}$. Indeed, let $\mathbf{Z}\mathbf{a}_j = \rho^{(j)} \mathbf{a}_j$. Then $(\zeta \mathbf{I}_n - \mathbf{Z})^{-1} \mathbf{a}_j = (\zeta - \rho^{(j)})^{-1} \mathbf{a}_j$. Substituting into formula (2.26), and using the Cauchy formula, we get

$$(\ln \mathbf{Z})\, \mathbf{a}_j = [\ln \rho^{(j)}]_{m_j} \mathbf{a}_j.$$

Let α_j be arbitrary numbers such that $e^{\alpha_j} = \rho^{(j)}$. There always exists a matrix $\mathbf{K} = \ln \mathbf{Z}$ with eigenvalues α_j. This matrix is determined by (2.26), with the numbers m_j chosen subject to the conditions

$$\alpha_j = [\ln \rho^{(j)}]_{m_j}.$$

2.7. Logarithm of a real matrix

We now consider the case of a real matrix $\mathbf{Z}$. The matrix $\mathbf{K} = \ln \mathbf{Z}$ defined by (2.26), which satisfies equation (2.27), is generally complex.

Lemma 1. Let $\mathbf{X}$ be a real nonsingular $n \times n$ matrix. There is a real $n \times n$ matrix $\mathbf{K}$ satisfying the equation

$$e^{2\mathbf{K}} = \mathbf{X}^2. \tag{2.28}$$

Proof. Let $\rho^{(1)}, \ldots, \rho^{(q)}$ denote the distinct eigenvalues of $\mathbf{X}$. Slit the complex plane along the negative real axis and define corresponding branches of the logarithm $(\ln z)_{m_j}$. Let

$$(\ln z)_m = \ln|z| + i(\arg z + 2\pi m), \text{ where } -\pi < \arg z < \pi.$$

Let Γ_j be circles about the points $\rho^{(j)}$, whose radii are so small that each contains only one point $\rho^{(j)}$, they do not contain the origin, are disjoint and do not intersect the real axis if $\operatorname{Im}\rho^{(j)} \neq 0$. Letting G_j denote the disks bounded by the circles Γ_j, we define a piecewise-analytic function $\varphi(z)$ as follows:

1) If $\operatorname{Im}\rho^{(j)} \neq 0$, then for $z \in G_j$

$$\varphi(z) = (\ln z)_{m_j};$$

the numbers m_j are determined by the condition that complex conjugate numbers $\rho^{(j)}$ correspond to numbers m_j of opposite signs.

2) If $\operatorname{Im}\rho^{(j)} = 0$, $\rho^{(j)} > 0$, then

$$\varphi(z) = (\ln z)_0.$$

3) If $\operatorname{Im}\rho^{(j)} = 0$, $\rho^{(j)} < 0$, then

$$\varphi(z) = [\ln(-z)]_0.$$

Note that

$$e^{2\varphi(z)} = z^2. \tag{2.29}$$

Now set

$$\mathbf{K} = \varphi(\mathbf{X}).$$

By (2.29) and the lemma of §2.5, we obtain (2.28). It remains to show that $\mathbf{K}$ is a real matrix. Denote the real negative $\rho^{(j)}$s by $-\mu_j$ and the corresponding disks by $\Gamma_j^{(-)}$. By the Cauchy formula (2.15),

$$\mathbf{K} = \frac{1}{2\pi i}\left[\sum_j \oint_{\Gamma_j^{(-)}} [\zeta \mathbf{I}_n - \mathbf{X}]^{-1} [\ln(-\zeta)]_0 \, d\zeta + \sum_j{}' \oint_{\Gamma_j} [\zeta \mathbf{I}_n - \mathbf{X}]^{-1} (\ln \zeta)_{m_j} \, d\zeta\right], \tag{2.30}$$

$$\overline{\mathbf{K}} = -\frac{1}{2\pi i}\left[\sum_j \oint_{\Gamma_j^{(-)}} [\bar{\zeta} \mathbf{I}_n - \mathbf{X}]^{-1} \overline{[\ln(-\zeta)]_0} \, d\bar{\zeta} + \sum_j{}' \oint_{\Gamma_j} [\bar{\zeta} \mathbf{I}_n - \mathbf{X}]^{-1} \overline{(\ln \zeta)_{m_j}} \, d\bar{\zeta}\right],$$

where $\overline{\mathbf{K}}$ is the complex conjugate matrix and $\sum'$ stands for summation over the nonreal and the positive real eigenvalues. In the expression for $\overline{\mathbf{K}}$, the point $\overline{\zeta}$ describes the integration contour in the negative direction. By our choice of the branches of the logarithm, we have

$$\overline{(\ln \zeta)_m} = \ln|\zeta| - i(\arg \zeta + 2\pi m) = (\ln \bar{\zeta})_{-m}.$$

Denoting the variable of integration in the expression for $\bar{\mathbf{K}}$ by ζ and reversing the direction in which the integration contour is described, we get

$$\bar{\mathbf{K}}=\frac{1}{2\pi i}\left[\sum_j \oint_{\Gamma_j^{(-)}} [\zeta \mathbf{I}_n-\mathbf{X}]^{-1}\,[\ln(-\zeta)]_0\,d\zeta+\sum_j{}' \oint_{\Gamma_j} [\zeta \mathbf{I}_n-\mathbf{X}]^{-1}\,(\ln\zeta)_{-m_j}\,d\zeta\right]. \quad (2.31)$$

In view of our choice of the numbers m_j, we see that the second sum in (2.31) differs from the second sum in (2.30) only in the order of summands, while the first sums coincide. Thus

$$\bar{\mathbf{K}}=\mathbf{K},$$

Q. E. D.

Lemma II. Let $\mathbf{X}$ *be a real nonsingular matrix. There exist real matrices* $\mathbf{K}$ *and* $\mathbf{R}$ *such that*

$$e^{\mathbf{K}}=\mathbf{XR}=\mathbf{RX},\ \mathbf{RK}=\mathbf{KR},\ \mathbf{R}^2=\mathbf{I}_n.$$

Proof. We define $\mathbf{K}$ as in the proof of Lemma I, $\mathbf{K}=\varphi(\mathbf{X})$. We now define a function $\psi(z)$ in the disks $\mathbf{G}_j$ (see proof of Lemma I) by

$$\psi(z)=-1, \quad \text{if} \quad \operatorname{Im}\rho^{(j)}=0,\ \rho^{(j)}<0,$$
$$\psi(z)=1, \quad \text{if} \quad \operatorname{Im}\rho^{(j)}=0,\ \rho^{(j)}>0 \quad \text{or if} \quad \operatorname{Im}\rho^{(j)}\neq 0.$$

Let $\mathbf{R}=\psi(\mathbf{X})$. It is obvious that $\mathbf{XR}=\mathbf{RX}$, $\mathbf{RK}=\mathbf{KR}$. In addition, for any $z\in G_j$.

$$e^{\varphi(z)}=z\psi(z),\quad \psi(z)^2=1.$$

Hence, via the lemmas of §§ 2.4, 2.5,

$$e^{\mathbf{K}}=\mathbf{XR},\quad \mathbf{R}^2=\mathbf{I}_n,$$

Q. E. D.

Similarly, one proves

Lemma III. If $\mathbf{X}$ *is a real nonsingular matrix having no negative real eigenvalues, there exists a real matrix* $\mathbf{K}=\ln\mathbf{X}$ *such that*

$$e^{\mathbf{K}}=\mathbf{X}. \quad (2.32)$$

The matrix $\mathbf{K}$ is defined in the same way as before, but case 3 is omitted. Therefore $\exp\varphi(z)\equiv z$, and so (2.32) holds.

Corollary. If $\mathbf{X}$ *is a nonsingular real matrix which has no positive real eigenvalues, there exists a real matrix* $\mathbf{K}=\ln(-\mathbf{X})$ *such that*

$$e^{\mathbf{K}}=-\mathbf{X}.$$

2.8. Examples

Example 1. Compute the matrix $e^{t\mathbf{C}}$, where $\mathbf{C}$ is the matrix in Example 4, §1.19.

Solution. By (1.48),

$$\mathbf{C}=\mathbf{S}\begin{Vmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \end{Vmatrix}\mathbf{S}^{-1},$$

where $\mathbf{S}$ and $\mathbf{S}^{-1}$ are as determined in Example 4 of §1.19. By (2.5) and (2.7), we have

$$e^{t\mathbf{C}}=\mathbf{S}\begin{Vmatrix} 1 & t & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{it} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{it} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-it} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-it} \end{Vmatrix}\mathbf{S}^{-1}.$$

Carrying out the matrix multiplication, we find that

$$e^{t\mathbf{C}}=\begin{Vmatrix} \cos t & 0 & 0 & \sin t & 0 & 0 \\ 0 & \cos t & 0 & 0 & \sin t & 0 \\ 0 & 0 & 1 & 0 & 0 & t \\ -\sin t & 0 & 0 & \cos t & 0 & 0 \\ 0 & -\sin t & 0 & 0 & \cos t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix}. \qquad \text{(a)}$$

There is a simpler derivation of this result, utilizing the special form of the matrix $\mathbf{C}$. It is readily proved by induction that

$$\mathbf{C}^{2k}=(-1)^k\begin{pmatrix} \mathbf{P}_0^k & 0 \\ 0 & \mathbf{P}_0^k \end{pmatrix},\quad \mathbf{C}^{2k+1}=(-1)^k\begin{pmatrix} 0 & \mathbf{P}_0^k \\ -\mathbf{P}_0^{k+1} & 0 \end{pmatrix}.$$

Hence, substituting into the series (2.3) for $e^{t\mathbf{C}}$, we obtain

$$e^{t\mathbf{C}}=\begin{pmatrix} \cos\left(t\mathbf{P}_0^{1/2}\right) & \mathbf{P}_0^{-1/2}\sin\left(t\mathbf{P}_0^{1/2}\right) \\ -\mathbf{P}_0^{1/2}\left(\sin t\mathbf{P}_0^{1/2}\right) & \cos\left(t\mathbf{P}_0^{1/2}\right) \end{pmatrix}, \qquad \text{(b)}$$

where the block elements are entire functions of the matrix $\mathbf{P}_0$* corresponding to the entire functions $f_1(z)=\cos(t\sqrt{z})$, $f_2(z)=\frac{1}{z}\sin(t\sqrt{z})$, $f_3(z)=$ $=-\sqrt{z}\sin(t\sqrt{z})$, namely:

$$\cos t\mathbf{P}_0^{1/2}=\mathbf{I}_3-\frac{t^2}{2!}\mathbf{P}_0+\frac{t^4}{4!}\mathbf{P}_0^2-\ldots,$$

$$\mathbf{P}_0^{-1/2}\sin t\mathbf{P}_0^{1/2}=t\mathbf{I}_3-\frac{t^3}{3!}\mathbf{P}_0+\frac{t^5}{5!}\mathbf{P}_0^2-\ldots,$$

$$\mathbf{P}_0^{1/2}\sin t\mathbf{P}_0^{1/2}=t\mathbf{P}_0-\frac{t^3}{3!}\mathbf{P}_0^2+\frac{t^5}{5!}\mathbf{P}_0^3-\ldots$$

Formula (a) now follows from (b).

* The matrix $\mathbf{P}_0$ is singular and has no inverse, so that $\mathbf{P}_0^{-1/2}$ itself is meaningless. See also (II.1.34).

Example 2. Compute $\exp(t\mathbf{J}_2)$, where

$$\mathbf{J}_2=\begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix}.$$

Solution. It is easy to see that $\mathbf{J}_2^2=-\mathbf{I}_2$, and so, by analogy with the powers of $\sqrt{-1}$, we have

$$\mathbf{J}_2^3=-\mathbf{J}_2, \quad \mathbf{J}_2^4=\mathbf{I}_2, \quad \mathbf{J}_2^5=\mathbf{J}_2, \text{ etc.}$$

Using (2.3), we get

$$e^{t\mathbf{J}_2}=\mathbf{I}_2+t\mathbf{J}_2-\frac{1}{2!}t^2\mathbf{I}_2-\frac{1}{3!}t^3\mathbf{J}_2+\frac{1}{4!}t^4\mathbf{I}_2+\frac{1}{5!}t^5\mathbf{J}_2+\ldots=$$
$$=\left(1-\frac{1}{2!}t^2+\frac{1}{4!}t^4-\ldots\right)\mathbf{I}_2+\left(t-\frac{1}{3!}t^3+\frac{1}{5!}t^5\ldots\right)\mathbf{J}_2,$$

or, finally,

$$e^{t\mathbf{J}_2}=\cos t\mathbf{I}_2+\sin t\mathbf{J}_2.$$

This formula is true for the corresponding $2k\times 2k$ matrix:

$$\mathbf{J}_{2k}=\begin{pmatrix} 0 & -\mathbf{I}_k \\ \mathbf{I}_k & 0 \end{pmatrix}.$$

§ 3. LOGARITHM OF AN ANALYTIC MATRIX-FUNCTION*

3.1. Lemma on the logarithm of an analytic matrix-function

Let $\mathbf{Y}_0$ be a nonsingular $n\times n$ matrix, $\rho^{(1)}, \ldots, \rho^{(q)}$ its distinct eigenvalues. Enclose these eigenvalues in nonintersecting circles (or any other closed curves) $\Gamma_1, \ldots, \Gamma_q$, of such small diameter that they do not contain the point $\rho=0$ and each point of the spectrum of $\mathbf{Y}_0$ lies in the interior of exactly one circle $\Gamma_\varkappa$. Let $G_\varkappa$ denote the open disk with circumference $\Gamma_\varkappa$, and G the union of all these disks. Slit the complex plane along a ray going from the origin to infinity which does not intersect any of the circles $\Gamma_\varkappa$ ($\varkappa=1, \ldots, \ldots, q$). This ray defines branches of the logarithm

$$(\ln\zeta)_m=\ln|\zeta|+i(\arg\zeta+2\pi m).$$

Let $\mathfrak{M}_G$ be the set of all $n\times n$ matrices $\mathbf{Y}$ whose spectrum lies in G. For $\mathbf{Y}\in\mathfrak{M}_G$, we set (see (2.26))

$$\ln\mathbf{Y}=\frac{1}{2\pi i}\sum_{\varkappa=1}^{q}\int_{\Gamma_\varkappa}(\zeta\mathbf{I}_n-\mathbf{Y})^{-1}(\ln\zeta)_{m_\varkappa}d\zeta, \tag{3.1}$$

where $m_\varkappa$ are for the moment arbitrary integers ($m_\varkappa=0, \pm1, \pm2, \ldots$).

* This section will be needed only in § IV.4 and may be omitted on a first reading.

Lemma. Suppose that the matrix

$$\mathbf{Y}(\varepsilon)=\mathbf{Y}_0+\varepsilon\mathbf{Y}_1+\varepsilon^2\mathbf{Y}_2+\ldots \tag{3.2}$$

(where $\mathbf{Y}_0, \mathbf{Y}_1, \mathbf{Y}_2, \ldots$ *are* $n\times n$ *matrices) is an analytic function of* ε *at* $\varepsilon=0$; $\mathbf{Y}_0=e^{\mathbf{A}_0}$ *and* $\mathbf{A}_0$ *has no distinct eigenvalues which are congruent modulo* $2\pi i$*:*

$$\alpha^{(g)}-\alpha^{(h)}\neq 2\pi m i \qquad (m=\pm 1, \pm 2, \ldots), \tag{3.3}$$

where $\alpha^{(g)}$ *and* $\alpha^{(h)}$ *are any of the distinct eigenvalues* $\alpha^{(1)}, \ldots, \alpha^{(q)}$ *of the matrix* $\mathbf{A}_0$. *Define* $\ln\mathbf{Y}(\varepsilon)$ *by (3.1), selecting the integers* $m_\varkappa$ *according to the conditions*

$$(\ln\rho^{(\varkappa)})_{m_\varkappa}=\alpha^{(\varkappa)} \qquad (\varkappa=1, \ldots, q), \tag{3.4}$$

where $\rho^{(1)}, \ldots, \rho^{(q)}$ *are the distinct eigenvalues of* $\mathbf{Y}_0$. *Then:*

1) $\ln\mathbf{Y}(\varepsilon)$ *is an analytic function of* ε *at* $\varepsilon=0$.

2) $\ln\mathbf{Y}(0)=\ln e^{\mathbf{A}_0}=\mathbf{A}_0$.

3) *The matrix* $A(\varepsilon)=\ln\mathbf{Y}(\varepsilon)$, *as a solution of the equation* $\mathbf{Y}(\varepsilon)=e^{\mathbf{A}}$, *is uniquely determined by conditions 1 and 2 for sufficiently small* ε.

Note that the choice of the numbers $m_\varkappa$ in (3.4) is possible only if for equal numbers $\rho_\nu=\exp\alpha_\nu$ we have equal numbers α_ν $(\nu=1, \ldots, n)$. That this is indeed the case follows from (3.3).

Proof. Part 1 follows at once from (3.1), since the integrand is an analytic function of ε at $\varepsilon=0$ on the integration contour. To prove part 2, we observe that the definition of the logarithm in (3.1) is the definition of a matrix-function by the Cauchy formula (2.15), starting from the scalar function $\psi(w)=(\ln w)_{m_\varkappa}$, $w\in G_\varkappa$. Define the function $f(z)=\psi(e^z)$ in the neighborhood of the spectrum of $\mathbf{A}_0$. It follows from (3.4) that $f(z)\equiv z$. By the composite function lemma (§2.5), the sequence of computations

$$\mathbf{Y}_0=\varphi(\mathbf{A}_0)=e^{\mathbf{A}_0}, \quad \psi(\mathbf{Y}_0)=\ln\mathbf{Y}_0$$

yields the same result as the direct computation $f(\mathbf{A}_0)=\mathbf{A}_0$, so that $\ln\mathbf{Y}_0=\mathbf{A}_0$, as required.

It remains to prove part 3. Let $\mathbf{A}_1(\varepsilon)=\ln\mathbf{Y}(\varepsilon)$ satisfy conditions 1 and 2. Since $\mathbf{A}_1(0)=\mathbf{A}_0$, the spectrum of $\mathbf{A}_1(\varepsilon)$ may be made arbitrarily close to that of $\mathbf{A}_0$, provided ε is sufficiently small. For sufficiently small ε,

$$\mathbf{A}_1(\varepsilon)=f[\mathbf{A}_1(\varepsilon)],$$

since $f(z)\equiv z$ in the neighborhood of the spectrum of $\mathbf{A}_0$. But

$$f[\mathbf{A}_1(\varepsilon)]=\psi[e^{\mathbf{A}_1(\varepsilon)}]=\psi[\mathbf{Y}(\varepsilon)]\equiv\mathbf{A}(\varepsilon),$$

and so $\mathbf{A}_1(\varepsilon)\equiv\mathbf{A}(\varepsilon)$ for sufficiently small ε.

Note that in the uniqueness proof it is sufficient to stipulate that $\mathbf{A}(\varepsilon)$ be continuous in ε, not necessarily analytic.

Remark. If condition (3.3) fails to hold, the assertion of the lemma becomes meaningless. In fact, there exist matrices $\mathbf{A}_0$ (not satisfying condition (3.3)) such that

a) for any matrix $\mathbf{Y}$ sufficiently close to $\mathbf{Y}_0 = \exp \mathbf{A}_0$ there exists a matrix $\mathbf{A} = \ln \mathbf{Y}$ such that $e^{\mathbf{A}} = \mathbf{Y}$;

b) in the neighborhood of $\mathbf{Y}_0$ one cannot define a function $\ln \mathbf{Y}$ such that $\ln \mathbf{Y}_0 = \ln \exp \tilde{\mathbf{A}}_0 = \tilde{\mathbf{A}}_0$ and the function $\ln \mathbf{Y}$ is continuous at $\mathbf{Y}_0$.

Example. Counterexamples are easily constructed. * For example, let

$$\tilde{\mathbf{A}}_0 = -2\pi \mathbf{J}_2 = 2\pi \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix};$$

then by Example 2, §2.1,

$$\mathbf{Y}_0 = e^{\tilde{\mathbf{A}}_0} = \mathbf{I}_2 = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}.$$

For any matrix sufficiently near $\mathbf{I}_2$ we can define the logarithm, e. g., by a series (2.4) or by an interpolating polynomial (§2.3). Let

$$\mathbf{Q}(\mu) = \begin{Vmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{Vmatrix} \qquad (0 < \mu < 1).$$

We shall show that all the values of $\ln \mathbf{Q}(\mu)$ $(\mu \neq 1)$, i. e., all solutions $\mathbf{A}$ of the equation $e^{\mathbf{A}} = \mathbf{Q}(\mu)$, are determined by the formula

$$\ln \mathbf{Q}(\mu) = \begin{Vmatrix} (\ln \mu)_0 + 2p\pi i & 0 \\ 0 & -(\ln \mu)_0 + 2q\pi i \end{Vmatrix} \quad (p, q = 0, \pm 1, \pm 2, \ldots). \tag{3.5}$$

Let α_1 and α_2 be two eigenvalues of $\mathbf{A}$. Since $e^{\mathbf{A}} = \mathbf{Q}(\mu)$, we have $e^{\alpha_1} = \mu$, $e^{\alpha_2} = \mu^{-1}$, or

$$\alpha_1 = (\ln \mu)_0 + 2p\pi i; \quad \alpha_2 = -(\ln \mu)_0 + 2q\pi i \qquad (p, q = 0, \pm 1, \pm 2, \ldots).$$

If $\mu \neq \pm 1$, then $\alpha_1 \neq \alpha_2$, i. e.,

$$\mathbf{A} = \mathbf{S}^{-1} \begin{Vmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{Vmatrix} \mathbf{S} \qquad (\det \mathbf{S} \neq 0). \tag{3.6}$$

Substituting this into $e^{\mathbf{A}} = \mathbf{Q}$, we obtain

$$\begin{Vmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{Vmatrix} \mathbf{S} = \mathbf{S} \begin{Vmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{Vmatrix}.$$

* In general, matrices $\mathbf{Y}_0$ with multiple eigenvalues play the role of essential singularities for multiple-valued functions $f(\mathbf{Z})$: as $\mathbf{Y} \longrightarrow \mathbf{Y}_0$ the set of limit values of $f(\mathbf{Y})$ has the cardinal of the continuum (/85a/, Volume III, §§ 91, 92). This set is generally the union of several "surfaces" and isolated points. A singularity may be eliminated if the value $f(\mathbf{Y})$ for $\mathbf{Y}$ sufficiently close to $\mathbf{Y}_0$ may be taken from the neighborhood of one of these points; this value is said to be regular. Of course, these statements must be formulated more rigorously (see /85a/, Volume III, §§91, 92).

Let

$$S=\begin{Vmatrix}\alpha & \beta\\ \gamma & \delta\end{Vmatrix}.$$

Carrying out the matrix multiplication, we have

$$\begin{Vmatrix}\mu\alpha & \mu\beta\\ \mu^{-1}\gamma & \mu^{-1}\delta\end{Vmatrix}=\begin{Vmatrix}\alpha\mu & \beta\mu^{-1}\\ \gamma\mu & \delta\mu^{-1}\end{Vmatrix}.$$

Comparing the superdiagonal elements, we see that $\beta\mu^{-1}=\mu\beta$, whence, recalling that $\mu\neq\pm1$, we obtain $\beta=0$. Similarly, $\gamma=0$. Thus S is a diagonal matrix, and (3.6) becomes

$$A=\begin{Vmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{Vmatrix},$$

which is precisely (3.5).

If μ is sufficiently close to 1, we can make the matrix $Q(\mu)$ arbitrarily close to $Y_0=I_2$, but the matrix $\ln Q(\mu)$ cannot be arbitrarily close to $\tilde{A}_0=\ln Y_0$, and this proves the above assertion (b). *

3.2. Lemma on radius of convergence

We have shown that if condition (3.3) is satisfied the matrix $A(\varepsilon)=\ln Y(\varepsilon)$ is an analytic function of ε at $\varepsilon=0$, in other words, the series

$$A(\varepsilon)=A_0+\varepsilon A_1+\varepsilon^2A_2+\dots \tag{3.7}$$

is convergent for sufficiently small ε. We shall now determine the radius of convergence r of this series.

Let us assume that the matrix-function

$$Y(\varepsilon)=Y_0+\varepsilon Y_1+\varepsilon^2Y_2+\dots \tag{3.8}$$

is an analytic function of ε in the disk $|\varepsilon|<r_0$, and that

$$\det Y(\varepsilon)\neq0 \text{ for } |\varepsilon|<r_0.$$

Let (3.7) be a series such that for ε of sufficiently small absolute value, identically,

$$\exp A(\varepsilon)=Y(\varepsilon). \tag{3.9}$$

Let $\rho_1(\varepsilon),\dots,\rho_n(\varepsilon)$ denote the eigenvalues of the matrix $Y(\varepsilon)$ $(|\varepsilon|<r_0)$. Let L be a curve on the ε-plane in the disk $|\varepsilon|<r_0$, issuing from the origin;

* This situation is obviously impossible in the scalar case.

on this curve we can define a unique function $\ln\rho_\nu(\varepsilon)$ by continuity, given the values $\ln\rho_\nu(0)=\alpha_\nu$, where α_ν are the eigenvalues of the matrix $\mathbf{A}_0$ ($\nu=1, \ldots, n$). Let $\rho^{(0)}$ be a multiple eigenvalue of $\mathbf{Y}(\varepsilon_0)$ for some ε_0, $|\varepsilon_0|<r_0$. Let $l>1$ be the multiplicity of $\rho^{(0)}$. Then exactly l of the numbers $\ln\rho_\nu(\varepsilon_0)$ are congruent modulo $2\pi i$ and such that $\rho_\nu(\varepsilon_0)=\rho^{(0)}$. Let p be the maximum multiplicity of any of these numbers, i. e., the greatest number of equal terms in the group of numbers $\ln\rho_\nu(\varepsilon_0)$ such that $\rho_\nu(\varepsilon_0)=\rho^{(0)}$. Let q be the maximum degree of the elementary divisors $(\rho-\rho^{(0)})^k$ of the matrix $\mathbf{Y}(\varepsilon_0)$. The numbers l, q depend only on the choice of $\rho^{(0)}$. The number p depends on the choice of $\rho^{(0)}$, the curve L and the branch of $\ln\rho_\nu(0)$. The latter is determined by $\mathbf{A}_0$.

Lemma. Suppose that the series (3.7), (3.8) satisfy the identity (3.9) for sufficiently small $|\varepsilon|$. Then

1) *If $r<r_0$ is the radius of convergence of (3.7), the eigenvalues $\alpha_1(\varepsilon), \ldots, \alpha_n(\varepsilon)$ of the matrix $\mathbf{A}(\varepsilon)$, defined for $|\varepsilon|<r$, may be defined by continuity on the circle $|\varepsilon|=r$, and there is a point ε_0 such that for some $g, h=1, \ldots, n$*

$$\alpha_g(\varepsilon_0)-\alpha_h(\varepsilon_0)=2\pi mi,$$

where $m\neq 0$ is an integer.

2) *Suppose that ε_0 is the value of ε of smallest absolute value ($\varepsilon_0\neq 0$, $|\varepsilon_0|<r_0$), such that the matrix $\mathbf{Y}(\varepsilon_0)$ has an eigenvalue $\rho^{(0)}$ of multiplicity $l>1$. Suppose that there is at least one such point* $\rho^{(0)}$ and at least one curve in the disk $|\varepsilon|<|\varepsilon_0|$, joining the origin $\varepsilon=0$ to the point $\varepsilon=\varepsilon_0$, such that $p<q$, where p and q are the integers defined above. Then $r=|\varepsilon_0|$ is the radius of convergence of the series (3.7).*

Recall that the number p depends on the choice of the branches $(\ln\rho_\nu(0))_m$, i. e., on the matrix $\mathbf{A}_0=\ln\mathbf{Y}_0$. If $\mathbf{A}(\varepsilon)$ is defined by (3.1), then the values of the branches $(\ln\rho_\nu(0))_m$ are determined by the choice of the m's.

Proof. 1) The eigenvalues $\rho_\nu(\varepsilon)$ of the matrix $\mathbf{Y}(\varepsilon)$ are defined and continuous in $|\varepsilon|<r_0$. Since by assumption $\rho_\nu(\varepsilon)\neq 0$, it follows that $\ln\rho_\nu(\varepsilon)$ ($\nu=1, \ldots, n$) are continuous functions of ε in the interior of the disk (but generally multiple-valued); the relations

$$\alpha_\nu(\varepsilon)=\ln\rho_\nu(\varepsilon) \qquad (\nu=1, \ldots, n)$$

define $\alpha_\nu(\varepsilon)$ on the circumference $|\varepsilon|=r$. Suppose that for any ε_0 such that $|\varepsilon_0|<r$

$$\alpha_g(\varepsilon_0)-\alpha_h(\varepsilon_0)\neq 2\pi mi \tag{3.10}$$

for any $g, h=1, \ldots, n$ and $m=\pm 1, \pm 2, \ldots$ To prove part 1, it will suffice to show that in the neighborhood of any ε_0, $|\varepsilon_0|=r$, the matrix-function $\mathbf{A}(\varepsilon)$ admits an analytic continuation to $|\varepsilon|>r$. The proof is almost the same as that of the lemma of §3.1.

Define the numbers m_h by

$$\alpha_h(\varepsilon_0)=[\ln\rho_h(\varepsilon_0)]_{m_h}.$$

* There may be several multiple roots $\rho^{(0)}$ at $\varepsilon=\varepsilon_0$.

Then, in view of (3.10), equal $\rho_h(\varepsilon_0)$ determine equal m_h. Thus formula (3.1) is meaningful, with Γ_h circles of sufficiently small radii about the points $\rho_h(\varepsilon_0)$ and $\mathbf{Y}=\mathbf{Y}(\varepsilon)$. In that case, $\mathbf{A}_1(\varepsilon)=\ln \mathbf{Y}(\varepsilon)=\psi[\mathbf{Y}(\varepsilon)]$ is an analytic function at $\varepsilon=\varepsilon_0$. We claim that $\mathbf{A}_1(\varepsilon)=\mathbf{A}(\varepsilon)$ for $|\varepsilon|<|\varepsilon_0|$ and sufficiently small $|\varepsilon-\varepsilon_0|$. As in the proof of the lemma in §3.1, it follows from the composite function lemma (§2.5) that $\mathbf{A}(\varepsilon)=\psi[e^{\mathbf{A}(\varepsilon)}]=\psi[\mathbf{Y}(\varepsilon)]=\mathbf{A}_1(\varepsilon)$ for these values of ε, completing the proof of part 1.

2) By assumption, if $|\varepsilon|<|\varepsilon_0|$ the eigenvalues $\rho_1(\varepsilon), \ldots, \rho_n(\varepsilon)$ of $\mathbf{Y}(\varepsilon)$ are distinct. If $r<|\varepsilon_0|$, it follows from part 1 that there is a point ε on the circle $|\varepsilon|=r$ such that some of the numbers $\rho_\nu(\varepsilon)=\exp\alpha_\nu(\varepsilon)$ are multiple. Hence $r\geqslant|\varepsilon_0|$: in other words, the series (3.7) is convergent for $|\varepsilon|<|\varepsilon_0|$.

We claim that the series is divergent at $\varepsilon=\varepsilon_0$. Suppose the contrary. For every elementary divisor $[\rho-\rho_j(\varepsilon_0)]^\mu$ of the matrix $\mathbf{Y}(\varepsilon)$ we have (§2.1) an elementary divisor $[\alpha-\alpha_j(\varepsilon_0)]^\mu$ *. Therefore, to at least one of the numbers $\alpha_j(\varepsilon_0)$ $(j=j_1,\ldots,j_l)$ there corresponds an elementary divisor $[\alpha-\alpha_j(\varepsilon_0)]^q$, i. e., the eigenvalue $\alpha_j(\varepsilon)$ of the matrix $\mathbf{A}(\varepsilon)$ has multiplicity at least q: $p\geqslant q$, and this contradicts our assumptions, proving the lemma.

Remark I. A perusal of the proof shows that we can in fact prove more than part 2 (the formulation is, however, rather cumbersome, and we have therefore not done this). Sometimes the series (3.7) can be shown to diverge at $\varepsilon=\varepsilon_0$ even when $p\geqslant q$. Suppose that the eigenvalue $\rho^{(0)}$ of $\mathbf{Y}(\varepsilon_0)$, of multiplicity l, has elementary divisors of degrees $q_1\geqslant q_2\geqslant\ldots\geqslant q_b$ (so that $q_1+\ldots+q_b=l$). If the series (3.7) is convergent, then for each elementary divisor of $\mathbf{Y}(\varepsilon_0)$ we have an elementary divisor of $\mathbf{A}(\varepsilon_0)$ of the same degree. Thus the l numbers $\alpha_j^{(0)}=\ln\rho_j(\varepsilon_0)$ with indices j for which $\rho_j(\varepsilon_0)=\rho^{(0)}$ may be divided into b groups, containing $q_1, q_2, \ldots, q_b$ numbers, in such a way that each group contains identical numbers (the numbers of different groups may also coincide). But if no such partition is possible, the series (3.7) diverges for $\varepsilon=\varepsilon_0$.

For example, suppose that for some eigenvalue of $\mathbf{Y}(\varepsilon_0)$ there are three elementary divisors $(\rho-\rho^{(0)})^2$, $(\rho-\rho^{(0)})^2$, $(\rho-\rho^{(0)})^2$, the corresponding values of $\alpha_j^{(0)}=\ln\rho_j(\varepsilon_0)$ being

$$\alpha_1^{(0)}=\alpha_2^{(0)}=\alpha_3^{(0)},\ \alpha_4^{(0)}=\alpha_1^{(0)}+2\pi i,\ \alpha_5^{(0)}=\alpha_1^{(0)}+4\pi i,$$
$$\alpha_6^{(0)}=\alpha_1^{(0)}+6\pi i \quad (\text{i. e., } p=3,\ q=2).$$

These numbers cannot be partitioned into three groups each of which contains two identical numbers. Thus the series (3.7) is divergent at $\varepsilon=\varepsilon_0$.

Remark II. In many cases with $p\geqslant q$, the series (3.7) may converge at $\varepsilon=\varepsilon_0$ and the radius of convergence r is greater than $|\varepsilon_0|$. This will occur, for example, if $\mathbf{Y}(\varepsilon)$ is a diagonal matrix whose elements $\rho_\nu(\varepsilon)$ are nonvanishing entire functions of ε. In that case $\ln\mathbf{Y}(\varepsilon)$ is an entire function, though for some choice of $\ln\rho_\nu(\varepsilon)$ there may exist values of ε_0 of the above type for which $p\geqslant q=1$.

* If $f'(\lambda)\neq 0$, the matrix (2.7) has one elementary divisor $[\mu-f(\lambda)]^k$. Thus, if $f'(\lambda)\neq 0$ on the spectrum of $\mathbf{A}$, each elementary divisor $[\mu-\lambda_{(\sigma)}]^{k_\sigma}$ of $\mathbf{A}$ determines an elementary divisor $[\mu-f(\lambda_{(\sigma)})]^{k_\sigma}$ of $f(\mathbf{A})$. This implies our assertion.

The cases indicated in Remark II are in a certain case exceptional, as we shall now explain. Let the series (3.7) converge at $\varepsilon=\varepsilon_0$. To simplify matters, let us assume that $p=1$, i. e., the numbers $\alpha_j(\varepsilon_0)$ $(j=j_1,\ldots, j_t)$ are distinct. The number $\rho^{(0)}=\exp\alpha_j(\varepsilon_0)$ will then be an r-fold eigenvalue of $\mathbf{Y}(\varepsilon_0)$, with simple elementary divisors, i. e., $q=1$. Let $\mathfrak{M}$ denote the set of nonsingular $n\times n$ matrices having $\rho^{(0)}$ as a multiple eigenvalue, and $\mathfrak{M}_0\subset\mathfrak{M}$ the set of nonsingular $n\times n$ matrices having a multiple eigenvalue $\rho^{(0)}$ with simple elementary divisors. It can be shown that the set $\mathfrak{M}_0$ has lower dimension (in the topological sense) than $\mathfrak{M}$. The "trajectory" $\mathbf{Y}(\varepsilon)$ passes through $\mathfrak{M}_0$ at $\varepsilon=\varepsilon_0$.

Such trajectories are exceptional with respect to the set of all trajectories passing through $\mathfrak{M}$. The same conclusion is reached if some of the numbers $\alpha_j(\varepsilon_0)$ $(j=j_1, \ldots, j_t)$ are multiple $(p>1)$.

In this sense, the value of $|\varepsilon_0|$ in the lemma is "almost always" the radius of convergence of the series (3.7).

3.3. Choice of branch of the logarithm guaranteeing maximum radius of convergence of the series (3.7)

The various series (3.7) that identically satisfy the relation $\exp\mathbf{A}(\varepsilon)=\mathbf{Y}(\varepsilon)$ generally have different radii of convergence. The lemma of §3.2 makes it possible to select the branch of the logarithm in (3.1) in such a way that the resulting series (3.7) has the largest possible radius of convergence.

Indeed, when $\varepsilon=\varepsilon_0$ the matrix $\mathbf{Y}(\varepsilon_0)$ has multiple eigenvalues, i. e., the values of ε_0 are roots of the equation

$$\Delta(\varepsilon)=0, \tag{3.11}$$

where $\Delta(\varepsilon)$ is the discriminant of the characteristic equation

$$\det[\mathbf{Y}(\varepsilon)-\rho\mathbf{I}_n]=0.$$

Number the roots of equation (3.11) in order of nondecreasing absolute value:

$$\varepsilon_0^{(1)}, \varepsilon_0^{(2)}, \varepsilon_0^{(3)}, \ldots \tag{3.12}$$

Consider the circle $|\varepsilon|=|\varepsilon_0^{(1)}|$ on the complex ε-plane. Arrange the points $\rho_\nu(0)$ on the Riemann surface of $\ln z$ in accordance with the selected branches of the logarithms $(\ln\rho_\nu(0))_m=\alpha_\nu$ (recall that the α_ν's are the eigenvalues of $\mathbf{A}_0$). Let $\rho_0^{(1)}$ denote a multiple eigenvalue of the matrix $\mathbf{Y}(\varepsilon_0^{(1)})$. Let ε move along some fixed curve L from $\varepsilon=0$ to $\varepsilon=\varepsilon_0^{(1)}$ in the circle $|\varepsilon|\leqslant|\varepsilon_0^{(1)}|$. Some of the roots $\rho_j(\varepsilon)$ will then tend to $\rho_0^{(1)}$ at $\varepsilon=\varepsilon_0^{(1)}$. If these points lie at $\varepsilon=\varepsilon_0^{(1)}$ on the same sheet of the Riemann surface, and this is true for all multiple eigenvalues $\rho_0^{(1)}$, we shall call $\varepsilon_0^{(1)}$ an extraneous root of equation (3.11). Otherwise, $\varepsilon_0^{(1)}$ is called a nonextraneous root.

This classification of the roots is independent of the choice of the curve L. Indeed, by the lemma of §3.1, when $|\varepsilon| < |\varepsilon_0^{(1)}|$ the series (3.7) for $\mathbf{A}(\varepsilon)$ is certainly convergent, so that the numbers $\ln \rho_\nu(\varepsilon)$ are uniquely determined up to multiplicity as eigenvalues of the matrix $\mathbf{A}(\varepsilon)$.

If $\varepsilon_0^{(1)}$ is an extraneous root, we proceed to the next root in the sequence (3.12). We continue this classification until a nonextraneous root is obtained.

A nonextraneous root ε_0 will generally correspond to several values of ρ_0 (multiple eigenvalues of $\mathbf{Y}(\varepsilon_0)$), and for each ρ_0 there are integers p and q (see the beginning of §3.2). The statement of the lemma of §3.2 may be formulated as follows.

If $r \neq \infty$ is the radius of convergence of the series (3.7), there is at least one nonextraneous root of equation (3.11) on the circle $|\varepsilon| = r$. If this root is ε_0 and $p < q$ for some corresponding $\rho^{(0)}$, then r is the radius of convergence of the series (3.7).

Which of the roots of equation (3.11) are extraneous and which are not depends on the choice of the branches of the logarithms $\ln \rho_j(0)$ or, in other words, the choice of the numbers $m_\varkappa$ in (3.1) — the choice of the branch of the matrix logarithm $\mathbf{A}_0 = \ln \mathbf{Y}(0)$. The branches should be chosen in such a way that the roots of equation (3.11) closest to the origin be "as extraneous as possible." The radius of convergence of (3.7) will then be the largest possible.

3.4. Case of 2×2 matrix

We shall now clarify the discussion at the end of §3.3, taking 2×2 matrices as an example. In this case equation (3.11) is

$$\Delta(\varepsilon) \equiv [\mathrm{Tr}\,\mathbf{Y}(\varepsilon)]^2 - 4 \det \mathbf{Y}(\varepsilon) = 0. \tag{3.13}$$

If $\mathbf{Y}(\varepsilon_0) \neq \rho \mathbf{I}_2$, (i. e., $\mathbf{Y}(\varepsilon_0)$ is not a multiple of the identity matrix), then $q = 2$. The only possible values for p are $p = 1, 2$. If $p = 2$, i. e., $\alpha_1(\varepsilon_0) = \alpha_2(\varepsilon_0)$, then ε_0 is an extraneous root. If $p = 1$, it follows from part 2 of the lemma in §3.2 that $r = |\varepsilon_0|$. Thus:

Let $\mathbf{Y}(\varepsilon)$ be a 2×2 matrix, ε_0 a nonextraneous root of equation (3.13) of least absolute value, and $\mathbf{Y}(\varepsilon_0) \neq \rho \mathbf{I}_2$. Then $r = |\varepsilon_0|$ is the radius of convergence of the series (3.7).

Example 1. Let

$$\mathbf{Y}(\varepsilon) = \begin{Vmatrix} 1 & \sqrt{2}\,\varepsilon \\ \sqrt{2}\,\varepsilon & 1 + 2\varepsilon^2 \end{Vmatrix}.$$

In this case $\mathbf{Y}(0) = \mathbf{I}_2$, $\rho_1(0) = \rho_2(0) = 1$. It is therefore natural to take the principal values of the logarithms: $\ln \rho_1(0) = \ln \rho_2(0) = 0$, $\mathbf{A}_0 = 0$. This defines a series (3.7) which coincides with the usual "principal value" of the logarithm of the matrix (2.4):

$$\ln \mathbf{Y} = \mathbf{Y} - \mathbf{I}_2 - \frac{1}{2}(\mathbf{Y} - \mathbf{I}_2)^2 + \frac{1}{3}(\mathbf{Y} - \mathbf{I}_2)^3 - \ldots$$

The series (3.7) is obtained from this series by expansion in powers of ε. We wish to determine its radius of convergence. Equation (3.13) is

$$[2(1+\varepsilon^2)]^2-4=0$$

and its roots are $\varepsilon_0=0, \pm i\sqrt{2}$. The root $\varepsilon_0=0$ is clearly extraneous. As ε varies along the imaginary axis from $\varepsilon=0$ to $\varepsilon=i\sqrt{2}$, the eigenvalues of the matrix $\mathbf{Y}(\varepsilon)$,

$$\rho_{1,2}(\varepsilon)=1+\varepsilon^2 \pm \sqrt{(1+\varepsilon)^2-1},$$

move along the unit circle, one clockwise and the other counterclockwise, from $\rho_{1,2}(0)=1$ to $\rho_{1,2}(i\sqrt{2})=-1$. Thus $\rho_1(i\sqrt{2})$ and $\rho_2(i\sqrt{2})$ will lie on different sheets of the Riemann surface, so that $i\sqrt{2}$ is a nonextraneous root ($p=1$). Since $\mathbf{Y}(i\sqrt{2}) \neq \rho\mathbf{I}_2$ (i. e., $q=2$), the radius of convergence of (3.7) is

$$|i\sqrt{2}|=\sqrt{2}.$$

Example 2. Let

$$\mathbf{Y}(\varepsilon)=\begin{Vmatrix} \cos c+\varepsilon+16\varepsilon^4 & \sin c+\varepsilon \\ -\sin c+\varepsilon & \cos c-\varepsilon+16\varepsilon^4 \end{Vmatrix} \quad (\sin c>0).$$

The characteristic equation of $\mathbf{Y}(\varepsilon)$ is

$$\rho^2-2(\cos c+16\varepsilon^4)\rho-(1-2\varepsilon^2+32\varepsilon^4\cos c+256\,\varepsilon^8)=0,$$

so that its eigenvalues are $\rho_{1,2}(\varepsilon)=\cos c+4(\sin^2 c-t^2)^2 \pm it$, where t stands for the parameter

$$t=\sqrt{\sin^2 c-2\varepsilon^2}.$$

As ε varies along the real axis from 0 to $\varepsilon_0=\sin c/\sqrt{2}>0$, the parameter t varies along the real axis from $\sin c$ to 0, and $\rho_1(\varepsilon)$, $\rho_2(\varepsilon)$ move in the complex plane from $\rho_1(0)=e^{ic}$, $\rho_2(0)=e^{-ic}$ to $\rho_1(\varepsilon_0)=\rho_2(\varepsilon_0)=\cos c+4\sin^4 c$, as illustrated in Figure 1. As ε varies along the real axis from zero to $\varepsilon_0'=-\sin c/\sqrt{2}<0$, the roots $\rho_1(\varepsilon)$, $\rho_2(\varepsilon)$ perform the same motion.

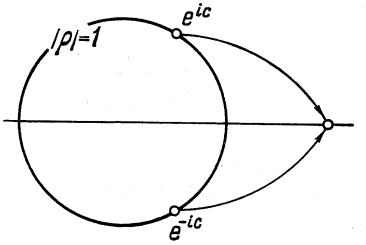


FIGURE 1.

We wish to determine the value of $\ln \mathbf{Y}(0)$ in such a way that the radius of convergence of the series (3.7) will be as large as possible. We must determine the branches of the logarithm so as to make ε_0 an extraneous root, i. e.,

$$\ln \rho_1(\varepsilon_0) = \ln \rho_2(\varepsilon_0).$$

Slit the complex plane along the negative real axis, i. e.,

$$(\ln \rho)_\mu = \ln|\rho| + i(\arg \rho + 2\mu\pi) \qquad (-\pi < \arg \rho < \pi).$$

Express c as

$$c = 2m\pi + \delta \qquad (0 < \delta < \pi)$$

and set

$$\ln \rho_1(0) = ic - 2k\pi i = i[2\pi(m-k) + \delta], \quad \ln \rho_2(0) = -ic + 2l\pi i = i[2\pi(l-m) - \delta].$$

As the representative point moves from $\varepsilon = 0$ to $\varepsilon = \varepsilon_0$, the argument of ρ_1 decreases by δ, while that of ρ_2 increases by δ; therefore

$$\ln \rho_1(\varepsilon_0) = \ln|\rho_1(\varepsilon_0)| + i\,2\pi(m-k), \quad \ln \rho_2(\varepsilon_0) = \ln|\rho_2(\varepsilon_0)| + i\,2\pi(l-m).$$

The root ε_0 will be extraneous if and only if $\ln \rho_1(\varepsilon_0) = \ln \rho_2(\varepsilon_0)$, i. e.,

$$k + l = 2m. \tag{3.14}$$

Then, by the lemma of §3.2, the number $|\varepsilon_0|$ is not the radius of convergence of the series (3.7), and since equation (3.11) has no other roots the radius of convergence must be $r = \infty$. If condition (3.14) holds, then ε_0 is a nonextraneous root. Obviously,

$$\mathbf{Y}(\varepsilon_0) \neq \text{const} \cdot \mathbf{I}_2$$

and, by §3.3, the radius of convergence of the series (3.7) is $|\varepsilon_0| = \sin c/\sqrt{2}$.

In particular, a sufficient condition for the radius of convergence to be $r = \infty$ is $l = k = m$, i. e.,

$$\ln \rho_1(0) = i(c - 2m\pi), \quad \ln \rho_2(0) = -i(c - 2m\pi).$$

Then we have

$$\mathbf{Y}(0) = \begin{Vmatrix} \cos c & \sin c \\ -\sin c & \cos c \end{Vmatrix} = e^{-c\mathbf{J}_2} \qquad \left(\mathbf{J}_2 = \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix}\right),$$

and so we can take

$$\mathbf{A}_0 = \ln \mathbf{Y}_0 = (2\pi m - c)\,\mathbf{J}_2.$$

By part 2 of the lemma in §3.2, the matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots$ in (3.7) are uniquely determined and, as we have proved, the series is an entire function of ε.

Chapter II

GENERAL THEORY OF SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

§§1 through 3 of this chapter contain a systematic exposition of some general, fairly elementary material on linear systems of differential equations with periodic coefficients. The central result of §2 is the Floquet-Lyapunov theorem. This theorem combines two propositions: Floquet's theorem /106/ on the general solution of equations of this type, and Lyapunov's reducibility theorem /60/. In a modern account the distinction between these two results is so insignificant that it is better to combine them. Moreover, this is apparently the first explicit formulation of the sharpened Floquet-Lyapunov theorem for equations with real coefficients (§2.3), despite its simplicity. More specialized material is presented in §4. We show there how it is possible, without integrating a system, to estimate the order of growth or attenuation of its solutions. The derivation of these estimates is quite simple, although many of them are sharp (in a sense to be clarified). Even sharper estimates for the characteristic exponents, based on more profound arguments, will be established in §§VII.2, VIII.2, VIII.4.

§1. SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

1.1. Existence and uniqueness of solutions

We shall be concerned with systems

$$\frac{d\xi_j}{dt} = \alpha_{j1}(t)\xi_1 + \ldots + \alpha_{jn}(t)\xi_n \qquad (j = 1, \ldots, n), \tag{1.1}$$

where the independent variable t is real $(-\infty < t < \infty)$ and the coefficients $\alpha_{jh}(t)$ $(j, h = 1, \ldots, n)$ are generally complex-valued, integrable, piecewise-continuous functions in any finite interval (t_1, t_2). *

We recall the definitions. A function $f(t)$ on $(-\infty, \infty)$ is said to be piecewise-continuous if it has only finitely many discontinuity points on any finite interval.

* For the reader familiar with the Lebesgue integral, we remark that the coefficients $\alpha_{jh}(t)$ in (1.1) may be any functions summable on every finite interval; the solution $\xi_j(t)$ then consists of absolutely continuous functions. All our results carry over in a natural manner to this case.

A piecewise-continuous function is said to be integrable if, at any discontinuity point t^*, the limits

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \int_{t_1}^{t^*-\varepsilon} |f(t)|\,dt = \int_{t_1}^{t^*} |f(t)|\,dt, \quad \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \int_{t^*+\varepsilon}^{t_2} |f(t)|\,dt = \int_{t^*}^{t_2} |f(t)|\,dt$$

exist, where t_1, t_2 are so chosen that the intervals (t_1, t^*), (t^*, t_2) contain no other discontinuity points.

In matrix notation, system (1.1) becomes a single vector differential equation

$$\frac{dx}{dt} = A(t)x, \tag{1.2}$$

where $\mathbf{x}$ is a column-vector with components $\xi_1, \ldots, \xi_n$ and $A(t)$ the $n\times n$ matrix with elements $\alpha_{jk}(t)$:

$$\mathbf{x} = (\xi_1, \ldots, \xi_n)^T, \quad A(t) = \|\alpha_{jk}(t)\|_1^n$$

(see §§ I.1.1, I.1.2).

When dealing with matrices, we shall say that a matrix-function $F(t)$ is continuous, piecewise-continuous, integrable, etc. if the appropriate condition holds for each of its elements. The discontinuity points of $F(t)$ are the discontinuity points of its elements. Integrability of a matrix-function $F(t)$ implies integrability of the scalar function $|F(t)|$ (its norm). (For the definition of the norm, see § I.1.2.)

We shall now outline the standard proof that the solution of the vector equation (1.2) with initial conditions $\mathbf{x}(0) = \mathbf{x}_0$, defined as the solution of the vector integral equation

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_0^t A(t)\,\mathbf{x}(t)\,dt \tag{1.3}$$

in the class of continuous vector-functions with integrable piecewise-continuous derivative, exists and is unique.

We shall look for a solution to equation (1.3) by Picard's method of successive approximations:

$$\mathbf{x}_{k+1}(t) = \mathbf{x}_0 + \int_0^t A(t_1)\,\mathbf{x}_k(t_1)\,dt_1 \quad (k = 0, 1, 2, \ldots, \mathbf{x}_0(t) \equiv \mathbf{x}_0). \tag{1.4}$$

It follows from the definition of an integrable piecewise-continuous function that any sum of functions with these properties again has the same properties, and that the integral of a piecewise-continuous function is continuous. It therefore follows by induction that the functions $\mathbf{x}_k(t)$ are continuous for all t.

We claim that the sequence $\{\mathbf{x}_k(t)\}$, or, equivalently, the series of vector-functions

$$\mathbf{x}_0(t)+[\mathbf{x}_1(t)-\mathbf{x}_0(t)]+[\mathbf{x}_2(t)-\mathbf{x}_1(t)]+\ldots$$

is uniformly convergent on any finite interval of the t-axis. To prove this, we shall show that for any t

$$|\mathbf{x}_k(t)-\mathbf{x}_{k-1}(t)|\leqslant\frac{1}{k!}\left(\int_0^t|\mathbf{A}(t_1)|\,dt_1\right)^k|\mathbf{x}_0| \qquad (k=1,2,\ldots). \tag{1.5}$$

For $k=1$,

$$|\mathbf{x}_1(t)-\mathbf{x}_0(t)|\leqslant\int_0^t|\mathbf{A}(t_1)|\,dt_1\cdot|\mathbf{x}_0|,$$

so that (1.5) is true (we have used properties 6—8 of the norm; see §I.1.2).

Assuming now that inequality (1.5) holds for some k, we see from (1.4) that

$$|\mathbf{x}_{k+1}(t)-\mathbf{x}_k(t)|\leqslant\int_0^t|\mathbf{A}(t_1)|\,|\mathbf{x}_k(t_1)-\mathbf{x}_{k-1}(t_1)|\,dt_1\leqslant$$
$$\leqslant\int_0^t\frac{1}{k!}\left(\int_0^{t_1}|\mathbf{A}(t_2)|\,dt_2\right)^k|\mathbf{x}_0|\,d\left(\int_0^{t_1}|\mathbf{A}(t_2)|\,dt_2\right)=\frac{1}{(k+1)!}\left(\int_0^t|\mathbf{A}(t_1)|\,dt_1\right)^{k+1}|\mathbf{x}_0|.$$

Thus inequality (1.5) is true for any k by induction, and this proves our assertion.

Denote

$$\mathbf{x}(t)=\lim_{k\to\infty}\mathbf{x}_k(t).$$

As the limit of a uniformly convergent sequence of functions continuous on every finite interval, the vector-function $\mathbf{x}(t)$ is continuous for all t. Letting $k\to\infty$ in (1.4) (this is legitimate by virtue of uniform convergence), we see that $\mathbf{x}(t)$ is a solution of equation (1.3). It follows directly from this equation that $\mathbf{x}(t)$ has an integrable piecewise-continuous derivative. The derivative may have discontinuities only at the discontinuity points of $\mathbf{A}(t)$. Differentiating (1.3), we see that $\mathbf{x}(t)$ satisfies equation (1.2) at all points which are not discontinuity points of $\mathbf{A}(t)$.

We now turn to the question of uniqueness. We shall need the following simple proposition (/72/, Chapter I, § 2).

Bellman-Gronwall

Lemma. *Let $\xi(t)\geqslant 0$ and $\varphi(t)\geqslant 0$ be integrable piecewise-continuous functions such that for $t>0$*

$$\xi(t)\leqslant\xi_0+\int_0^t\varphi(t_1)\,\xi(t_1)\,dt_1, \tag{1.6}$$

where $\xi_0>0$. Then

$$\xi(t) \leqslant \xi_0 \exp \int_0^t \varphi(s)\, ds. \tag{1.7}$$

Proof. Multiply (1.6) by $\varphi(t)$ and write the inequality thus:

$$\dot{\psi}(t) \leqslant \varphi(t)\,[\xi_0 + \psi(t)] \qquad \left(\psi(t) \equiv \int_0^t \varphi(s)\,\xi(s)\, ds\right),$$

or, dividing both sides by $\xi_0 + \psi(t) > 0$,

$$\frac{\dot{\psi}(t)}{\xi_0 + \psi(t)} \leqslant \varphi(t).$$

Integrating from 0 to t, we get

$$\xi_0 + \psi(t) \leqslant \xi_0 \exp \int_0^t \varphi(s)\, ds.$$

Substituting the expression for $\psi(t)$ into this inequality and using (1.6), we obtain

$$\xi(t) \leqslant \xi_0 + \int_0^t \varphi(s)\,\xi(s)\, ds \leqslant \xi_0 \exp \int_0^t \varphi(s)\, ds.$$

Q. E. D.

Assuming that a solution exists (not necessarily the series solution constructed above), we deduce from (1.3), using the properties of the norm (§I.1.2), that

$$|\mathbf{x}(t)| \leqslant |\mathbf{x}(0)| + \int_0^t |\mathbf{A}(t_1)|\,|\mathbf{x}(t_1)|\, dt_1.$$

By the lemma, we have

$$|\mathbf{x}(t)| \leqslant |\mathbf{x}(0)| \exp \int_0^t |\mathbf{A}(t_1)|\, dt_1. \tag{1.8}$$

If $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are solutions satisfying the same initial conditions, then $\mathbf{x}(t) = \mathbf{x}^{(1)}(t) - \mathbf{x}^{(2)}(t)$ is also a solution and $\mathbf{x}(0) = \mathbf{0}$. It follows from (1.8) that $\mathbf{x}(t) \equiv \mathbf{0}$, proving the uniqueness of the solution.

1.2. Matrizant and fundamental set of solutions

Let $\mathbf{x}_j(t)$ $(j = 1, \ldots, k)$ be k arbitrary solutions of equation (1.2) ($\mathbf{x}_j(t)$ is a column vector) with initial conditions $\mathbf{x}_j(0) = \mathbf{a}_j$ $(j = 1, \ldots, k)$. Then the $n \times k$ matrix

$$\mathbf{X}(t) = \|\mathbf{x}_1(t), \ldots, \mathbf{x}_k(t)\|$$

satisfies the matrix differential equation

$$\frac{d\mathbf{X}}{dt}=\mathbf{A}(t)\mathbf{X} \tag{1.9}$$

and

$$\mathbf{X}(0)=\|\mathbf{a}_1, \ldots, \mathbf{a}_k\|.$$

Indeed, the j-th column of $\dot{\mathbf{X}}$ is $\dot{\mathbf{x}}_j$, and the j-th column of the matrix $\mathbf{AX}$ is $\mathbf{Ax}_j$ (§I.1.2). Hence the matrix equality (1.9) is equivalent to the k vector equalities $\dot{\mathbf{x}}_j=\mathbf{Ax}_j$ $(j=1, \ldots, k)$. Thus (1.9) is true.

Conversely, equating the columns of equation (1.9), where $\mathbf{X}$ is some $n\times k$ matrix satisfying it, we see that these columns satisfy the vector equation (1.2). Thus the solution of equation (1.9) with initial conditions $\mathbf{X}(0)=\mathbf{C}$, where $\mathbf{C}$ is any constant $n\times k$ matrix, exists and is unique.

Let $k=n$ and $\det \mathbf{X}(0)\neq 0$. Any other solution $\mathbf{X}_1(t)$ of equation (1.9) may be expressed as*

$$\mathbf{X}_1(t)=\mathbf{X}(t)\mathbf{C}, \tag{1.10}$$

where $\mathbf{C}=\mathbf{X}^{-1}(0)\mathbf{X}_1(0)$. Indeed, for any constant matrix $\mathbf{C}$ the matrix $\mathbf{X}_2(t)=\mathbf{X}(t)\mathbf{C}$ satisfies equation (1.9), and if $\mathbf{C}$ is defined as shown then $\mathbf{X}_1(0)=\mathbf{X}_2(0)$. Since equation (1.9) is uniquely solvable, it follows that $\mathbf{X}_1(t)\equiv\mathbf{X}_2(t)$, so that (1.10) is true for any $t>0$.

When $k=n$, we have the Liouville-Jacobi formula (/21/, Chapter XIV, §1):

$$\det \mathbf{X}_1(t)=\det \mathbf{X}_1(0)\exp\int_0^t \mathrm{Tr}\mathbf{A}(t)\,dt. \tag{1.11}$$

Definition. The $n\times n$ matrix $\mathbf{X}(t)$ satisfying equation (1.9) and the initial condition $\mathbf{X}(0)=\mathbf{I}_n$ (where $\mathbf{I}_n$ is the $n\times n$ identity matrix) is called the matrizant** of equation (1.2).

The columns of the matrizant are vectors satisfying equation (1.2). If $\mathbf{X}(t)$ is the matrizant, the Liouville formula becomes

$$\det \mathbf{X}(t)=\exp\int_0^t \mathrm{Tr}\mathbf{A}(t)\,dt. \tag{1.12}$$

Any $k\times n$ matrix solution $\mathbf{X}_1(t)$ of equation (1.9) is derived from the matrizant by the formula

$$\mathbf{X}_1(t)=\mathbf{X}(t)\mathbf{X}_1(0),$$

* In (1.10), $\mathbf{X}(t)$ is an $n\times n$ matrix, $\mathbf{X}_1(t)$ and $\mathbf{C}$ are $n\times k$ matrices.

** [This term was apparently introduced by H.F. Baker, On the integration of linear differential equations. — Proc. London Math. Soc. 35 (1903), 333.]

and any solution of equation (1.2) by

$$\mathbf{x}(t)=\mathbf{X}(t)\,\mathbf{x}(0). \tag{1.13}$$

Both these formulas are special cases of (1.10).

Definition. A fundamental set of solutions of equation (1.2) is any set of n solutions $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ which are linearly independent for any t. The square matrix $\mathbf{X}_1(t)$ with columns $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ is called a fundamental matrix.

It follows from the Liouville formula (1.11) that $\det \mathbf{X}_1(t) \neq 0$ if $\det \mathbf{X}_1(0) \neq 0$. In other words, if solutions $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ of equation (1.2) are linearly independent at $t=0$, they are linearly independent for any $t \neq 0$. Of course, instead of $t=0$ we may have any point $t=t_0$. Thus, in order to determine whether a set of solutions $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ is fundamental, we need only check that the vectors $\mathbf{x}_1(t_0), \ldots, \mathbf{x}_n(t_0)$, where t_0 is any fixed number, are linearly independent.

Any solution $\mathbf{x}(t)$ of equation (1.2) may be expressed in terms of a fundamental set

$$\mathbf{x}(t)=\sum_{\nu=1}^{n} \alpha_\nu \mathbf{x}_\nu(t) \tag{1.14}$$

(α_ν are scalars). Indeed, defining a vector $\mathbf{a}$ with components $\alpha_1, \ldots, \alpha_n$ from the formula

$$\mathbf{X}_1(0)\,\mathbf{a}=\mathbf{x}(0),$$

or, equivalently,

$$\sum_{\nu=1}^{n} \alpha_\nu \mathbf{x}_\nu(0)=\mathbf{x}(0),$$

we see that both sides of (1.14) are solutions that coincide at the initial point. The uniqueness of the solution implies (1.14).

Thus, formula (1.14) with α_ν running through all numbers, gives the general solution of equation (1.2).

1.3. Matrizant as an analytic function of a parameter

Consider a system of linear differential equations with coefficients which are analytic functions of some parameter, written in vector notation as

$$\frac{d\mathbf{x}}{dt}=[\mathbf{A}_0(t)+\varepsilon\mathbf{A}_1(t)+\varepsilon^2\mathbf{A}_2(t)+\ldots]\,\mathbf{x}. \tag{1.15}$$

Here $\mathbf{x}$ is an n-vector, $\mathbf{A}_0(t)$, $\mathbf{A}_1(t)$, $\mathbf{A}_2(t)$, ... are piecewise-continuous $n \times n$ matrix-functions, integrable on $(0, T)$, and having discontinuities at most at finitely many points $t_1, \ldots, t_h \in [0, T]$; we assume that if $|\varepsilon| < r_0$ the series

$$\sum_{k=0}^{\infty} \varepsilon^k \int_0^T |\mathbf{A}_k(t)|\,dt,$$

where $T>0$ is a fixed number, is convergent.

We shall determine the matrizant of equation (1.15) by successive approximations,

$$\mathbf{X}_0(t,\varepsilon)=I_n,\quad \mathbf{X}_l(t,\varepsilon)=\int_0^t\left[\sum_{k=0}^{\infty}\varepsilon^k\mathbf{A}_k(t_1)\right]\mathbf{X}_{l-1}(t_1,\varepsilon)\,dt_1 \qquad (l=1,2,\ldots).$$

By assumption, the series

$$\sum_{k=0}^{\infty}\varepsilon^k\int_0^t|\mathbf{A}_k(t_1)|\,dt_1$$

is convergent for $|\varepsilon|<r_0$, $0<t\leqslant T$, and so one can prove by induction that the functions $\mathbf{X}_1(t,\varepsilon)$, $\mathbf{X}_2(t,\varepsilon)$, ... are analytic. Since the sequence $\{\mathbf{X}_l(t,\varepsilon)\}$ converges to the matrizant $\mathbf{X}(t,\varepsilon)$ uniformly in t, ε on any finite t-interval (§1.1) and for $|\varepsilon|\leqslant\rho_0<r_0$, it follows from a theorem of Weierstrass (/85a/, Volume III, Chapter IV, § 86) that $\mathbf{X}(t,\varepsilon)$ is an analytic function of ε for $|\varepsilon|<r_0$. In particular, if the matrix coefficient $\mathbf{A}_0+\varepsilon\mathbf{A}_1(t)+\varepsilon^2\mathbf{A}_2(t)+\ldots$ is an entire function of ε, then $\mathbf{X}(t,\varepsilon)$ is an entire function of ε for fixed t.

1.4. Inhomogeneous systems

The solution of an inhomogeneous vector equation

$$\boxed{\frac{d\mathbf{z}}{dt}=\mathbf{A}(t)\mathbf{z}+\mathbf{f}(t)\qquad\left(\mathbf{f}(t)=\begin{pmatrix}f_1(t)\\ \vdots\\ f_n(t)\end{pmatrix}\right)} \tag{1.16}$$

may be expressed in terms of the matrizant $\mathbf{X}(t)$ of the corresponding homogeneous equation as follows (/21/, Chapter XIV, § 5):

$$\boxed{\mathbf{z}(t)=\mathbf{X}(t)\left[\mathbf{x}(0)+\int_0^t\mathbf{X}(t_1)^{-1}\mathbf{f}(t_1)\,dt_1\right].} \tag{1.17}$$

For the corresponding matrix equation

$$\frac{d\mathbf{Z}}{dt}=\mathbf{A}(t)\mathbf{Z}+\mathbf{F}(t) \tag{1.18}$$

we have

$$\mathbf{Z}(t)=\mathbf{X}(t)\left[\mathbf{Z}(0)+\int_0^t\mathbf{X}(t_1)^{-1}\mathbf{F}(t_1)\,dt_1\right]. \tag{1.19}$$

Formulas (1.17) and (1.19) may be verified by direct substitution into equations (1.16) and (1.18).

1.5. First-order equations with constant coefficients

We consider system (1.1) with constant coefficients; in vector notation,

$$\frac{d\mathbf{x}}{dt}=\mathbf{A}\mathbf{x}, \tag{1.20}$$

where $\mathbf{A}$ is a constant $n\times n$ matrix. We shall look for the solution as

$$\mathbf{x}=e^{\lambda t}\mathbf{a}, \tag{1.21}$$

where λ is a scalar and $\mathbf{a}\neq\mathbf{0}$ a constant vector. Substituting this into (1.20), we obtain

$$\mathbf{A}\mathbf{a}=\lambda\mathbf{a}, \tag{1.22}$$

so that λ and $\mathbf{a}$ must be an eigenvalue and an eigenvector of $\mathbf{A}$ (§I.1.8). The values of λ are determined from the equation

$$\det(\mathbf{A}-\lambda\mathbf{I}_n)=0 \tag{1.23}$$

(§I.1.8), called the characteristic equation of equation (1.20). To each root of equation (1.23) corresponds a solution $\mathbf{a}\neq\mathbf{0}$ of equation (1.22), i. e., a solution (1.21) of equation (1.20). If all the roots $\lambda_1, \ldots, \lambda_n$ of equation (1.23) are distinct, the corresponding eigenvectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are linearly independent (lemma, §I.1.8) and the vector-functions

$$\mathbf{x}_\nu(t)=e^{\lambda_\nu t}\mathbf{a}_\nu \qquad (\nu=1, \ldots, n) \tag{1.24}$$

form a fundamental set of solutions for equation (1.20), so that the general solution

$$\mathbf{x}(t)=\sum_{\nu=1}^{n}\alpha_\nu e^{\lambda_\nu t}\mathbf{a}_\nu,$$

where α_ν are arbitrary constants. It will follow from the sequel that a similar formula holds when the roots of equation (1.23) are multiple but the matrix $\mathbf{A}$ has only simple elementary divisors.

Let us determine the matrizant of equation (1.20), imposing no restrictions on the roots of the characteristic equation (1.23).

In the scalar case ($n=1$), the solution is $\mathbf{x}(t)=e^{\mathbf{A}t}\mathbf{x}(0)$. By analogy with this case, if $n>1$ we consider the matrix

$$\mathbf{X}(t)=e^{t\mathbf{A}},$$

defined, say, by the series (I.2.3)

$$e^{t\mathbf{A}} = \mathbf{I}_n + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \dots$$

where $\mathbf{Z} = t\mathbf{A}$. Differentiating with respect to t (this is legitimate because the elements of the matrix $e^{t\mathbf{A}}$ are entire functions of t), we see that the matrix $\mathbf{X}(t) = e^{t\mathbf{A}}$ satisfies the equation

$$\frac{d\mathbf{X}}{dt} = \mathbf{A}\mathbf{X}.$$

Since $\mathbf{X}(0) = e^0 = \mathbf{I}_n$, it follows that $e^{t\mathbf{A}}$ is the matrizant of equation (1.20) and its columns are linearly independent solutions of (1.20). By (1.13), the solution of equation (1.20) with initial conditions $\mathbf{x}(0)$ is

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}(0).$$

By the Liouville formula (1.12),

$$\det e^{t\mathbf{A}} = e^{t\,\mathrm{Tr}\,\mathbf{A}}. \tag{1.25}$$

We shall need this formula later.

Remark. It is sometimes easier to find the matrix $e^{t\mathbf{C}}$ by direct determination of the matrizant of the equation $\dot{\mathbf{x}} = \mathbf{C}\mathbf{x}$. The following example illustrates this.

Example. Compute the matrix $e^{t\mathbf{C}}$, where

$$\mathbf{C} = \begin{Vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{Vmatrix}.$$

Solution. Write the vector equation $\dot{\mathbf{x}} = \mathbf{C}\mathbf{x}$ as a system of linear differential equations

$$\dot{\xi}_1 = \xi_2, \quad \dot{\xi}_2 = \xi_3, \quad \dot{\xi}_3 = -\xi_2.$$

The initial conditions for the matrizant are defined by the identity matrix:

$$\begin{Vmatrix} \xi_1^{(1)}(0) & \xi_1^{(2)}(0) & \xi_1^{(3)}(0) \\ \xi_2^{(1)}(0) & \xi_2^{(2)}(0) & \xi_2^{(3)}(0) \\ \xi_3^{(1)}(0) & \xi_3^{(2)}(0) & \xi_3^{(3)}(0) \end{Vmatrix} = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}.$$

By the original system of equations,

$$\ddot{\xi}_2 + \xi_2 = 0,$$

i. e.,

$$\xi_2 = \xi_2(0)\cos t + \dot{\xi}_2(0)\sin t = \xi_2(0)\cos t + \xi_3(0)\sin t, \quad \xi_3 = \dot{\xi}_2, \quad \xi_1 = \xi_1(0) + \int_0^t \xi_2\,dt.$$

Thus we successively find

$$\begin{array}{lll} \xi_2^{(1)} \equiv 0, & \xi_3^{(1)} \equiv 0, & \xi_1^{(1)} \equiv 1, \\ \xi_2^{(2)} = \cos t, & \xi_3^{(2)} = -\sin t, & \xi_1^{(2)} = \sin t, \\ \xi_2^{(3)} = \sin t, & \xi_3^{(3)} = \cos t, & \xi_1^{(3)} = 1 - \cos t. \end{array}$$

It follows that the matrizant of the equation $\dot{\mathbf{x}} = \mathbf{C}\mathbf{x}$ is

$$e^{t\mathbf{C}} = \begin{Vmatrix} \xi_1^{(1)} & \xi_1^{(2)} & \xi_1^{(3)} \\ \xi_2^{(1)} & \xi_2^{(2)} & \xi_2^{(3)} \\ \xi_3^{(1)} & \xi_3^{(2)} & \xi_3^{(3)} \end{Vmatrix} = \begin{Vmatrix} 1 & \sin t & 1-\cos t \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{Vmatrix}.$$

We leave it to the reader to evaluate $e^{t\mathbf{C}}$ by the method of Example 1 in § I.2.1 and to check that the results coincide.

We now wish to determine the structure of the solution of equation (1.20) when the roots of the characteristic equation are multiple. Let $\lambda_{(1)}, \ldots, \lambda_{(s)}$ be the eigenvalues of the matrix $\mathbf{A}$, each multiple eigenvalue being written as many times as it has elementary divisors. Let m_σ denote the degree of the σ-th elementary divisor of $\mathbf{A}$ ($\sigma = 1, \ldots, s$). By (I. 1. 48) (reduction of a matrix to Jordan form), we have

$$\mathbf{A} = \mathbf{S}\left[\mathbf{Q}_1(\lambda_{(1)}) \dotplus \ldots \dotplus \mathbf{Q}_s(\lambda_{(s)})\right]\mathbf{S}^{-1} = \mathbf{S}\mathbf{Q}\mathbf{S}^{-1},$$

where

$$\mathbf{Q}_{(\sigma)}(\lambda_{(\sigma)}) = \begin{Vmatrix} \lambda_{(\sigma)} & 1 & 0 & \ldots & 0 \\ 0 & \lambda_{(\sigma)} & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & \lambda_{(\sigma)} \end{Vmatrix}$$

is an $m_\sigma \times m_\sigma$ matrix (elementary Jordan matrix) and $\mathbf{S}$ a nonsingular matrix ($\dotplus$ denotes direct summation of matrices). Therefore

$$\mathbf{X}(t) = e^{t\mathbf{A}} = \mathbf{S}\left[\exp\left(t\mathbf{Q}_1(\lambda_{(1)})\right) \dotplus \ldots \dotplus \exp\left(t\mathbf{Q}_s(\lambda_{(s)})\right)\right]\mathbf{S}^{-1} = \mathbf{S}e^{t\mathbf{Q}}\mathbf{S}^{-1} \qquad (1.26)$$

(see formulas (I. 2.5), (I. 2.6)). By (I. 2.7) (functions of an elementary Jordan matrix),

$$\exp\left(t\mathbf{Q}_\sigma(\lambda_{(\sigma)})\right) = e^{\lambda_{(\sigma)}t}\begin{Vmatrix} 1 & t & \frac{1}{2}t^2 & \ldots & \frac{1}{(m_\sigma-1)!}t^{m_\sigma-1} \\ 0 & 1 & t & \ldots & \frac{1}{(m_\sigma-2)!}t^{m_\sigma-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 \end{Vmatrix}. \qquad (1.27)$$

Formulas (1.26) and (1.27) clearly characterize the dependence of the matrizant $\mathbf{X}(t)$ on the time t. They show, in particular, that (in the most general case) the elements of the matrizant of equation (1.20) are linear combinations of functions of the type $p_\sigma(t)\exp\lambda_{(\sigma)}t$, where $\lambda_{(\sigma)}$ are the roots of the characteristic equation and $p_\sigma(t)$ are polynomials of degrees q_σ such that

$q_\sigma \leqslant \max_h m_\sigma^{(h)} - 1$, where $m_\sigma^{(h)}$ are the degrees of the elementary divisors of $\mathbf{A}$ belonging to $\lambda_{(\sigma)}$.

We now formulate a similar result in vector notation. Let $(\lambda - \lambda_{(\sigma)})^m$ $(\sigma = 1, \ldots, s)$ be the elementary divisors of the matrix $\mathbf{A}$ (§I. 1.12). We claim that to each elementary divisor $(\lambda - \lambda_{(\sigma)})^{m_\sigma}$ corresponds a group of m_σ linearly independent solutions

$$\left.\begin{aligned} \mathbf{x}_1^{(\sigma)} &= e^{\lambda_{(\sigma)} t} \mathbf{a}_1^{(\sigma)}, \\ \mathbf{x}_2^{(\sigma)} &= e^{\lambda_{(\sigma)} t} (t\mathbf{a}_1^{(\sigma)} + \mathbf{a}_2^{(\sigma)}), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \mathbf{x}_{m_\sigma}^{(\sigma)} &= e^{\lambda_{(\sigma)} t} \left(\frac{1}{(m_\sigma - 1)!} t^{m_\sigma - 1} \mathbf{a}_1^{(\sigma)} + \ldots + \mathbf{a}_{m_\sigma}^{(\sigma)} \right), \end{aligned}\right\} \tag{1.28}$$

where $\mathbf{a}_h^{(\sigma)}$ are linearly independent constant vectors. Moreover, combining the solutions of all the groups (1.28) (there are obviously n in all, since $m_1 + \ldots + m_s = n$), we obtain a complete system of linearly independent solutions of equation (1.20).

Let the vectors $\mathbf{a}_h^{(\sigma)}$ be the columns of the matrix $\mathbf{S}$ of (1.26), where the first m_1 columns (from the left) are taken as the vectors $\mathbf{a}_h^{(1)}$, the next m_2 as $\mathbf{a}_h^{(2)}$, and so on. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis (all components of the column vector $\mathbf{e}_\nu$ except the ν-th from the top are zeros, the ν-th component is 1). We may write the j-th column of any matrix $\mathbf{Z}$ as $\mathbf{Z}\mathbf{e}_j$. In particular,

$$\mathbf{a}_h^{(1)} = \mathbf{S}\mathbf{e}_h \quad (h = 1, \ldots, m_1), \qquad \mathbf{a}_h^{(2)} = \mathbf{S}\mathbf{e}_{m_1 + h} \quad (h = 1, \ldots, m_2), \quad \ldots$$

On the other hand, denoting the bracketed matrix in (1.26) by $e^{t\mathbf{Q}}$, we have

$$\left.\begin{aligned} e^{t\mathbf{Q}} \mathbf{e}_1 &= e^{\lambda_{(1)} t} \mathbf{e}_1, \\ e^{t\mathbf{Q}} \mathbf{e}_2 &= e^{\lambda_{(1)} t} (t\mathbf{e}_1 + \mathbf{e}_2), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ e^{t\mathbf{Q}} \mathbf{e}_{m_1} &= e^{\lambda_{(1)} t} \left(\frac{1}{(m_1 - 1)!} t^{m_1 - 1} \mathbf{e}_1 + \ldots + \mathbf{e}_{m_1} \right), \\ e^{t\mathbf{Q}} \mathbf{e}_{m_1 + 1} &= e^{\lambda_{(2)} t} \mathbf{e}_{m_1 + 1}, \\ e^{t\mathbf{Q}} \mathbf{e}_{m_1 + 2} &= e^{\lambda_{(2)} t} (t\mathbf{e}_{m_1 + 1} + \mathbf{e}_{m_1 + 2}), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ e^{t\mathbf{Q}} \mathbf{e}_{m_1 + m_2} &= e^{\lambda_{(2)} t} \left(\frac{1}{(m_2 - 1)!} t^{m_2 - 1} \mathbf{e}_{m_1 + 1} + \ldots + \mathbf{e}_{m_1 + m_2} \right), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{1.29}$$

The vectors $\mathbf{S}\mathbf{e}_1, \ldots, \mathbf{S}\mathbf{e}_n$ (the columns of $\mathbf{S}$) are of course linearly independent. Hence the vectors $\mathbf{S}e^{t\mathbf{Q}}\mathbf{e}_\nu = \mathbf{X}(t)\mathbf{S}\mathbf{e}_\nu$ $(\nu = 1, \ldots, n)$ form a fundamental set of solutions of equation (1.20) (§1.2). Multiplying (1.29) on the left by $\mathbf{S}$, we deduce that this fundamental set splits into s groups of type (1.28). Note that the vectors $\mathbf{a}_h^{(\sigma)}$ in (1.28) form a cyclic basis for the matrix $\mathbf{A}$ (§I. 1.10).

In scalar notation, formula (1.28) becomes

$$\left.\begin{aligned}
&\xi_\nu^{(1\sigma)}(t) = e^{\lambda_{(\sigma)}t}\xi_\nu^{(1\sigma)}(0),\\
&\xi_\nu^{(2\sigma)}(t) = e^{\lambda_{(\sigma)}t}\left(t\,\xi_\nu^{(1\sigma)}(0) + \xi_\nu^{(2\sigma)}(0)\right),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\xi_\nu^{(m_\sigma\sigma)}(t) = e^{\lambda_{(\sigma)}t}\left(\frac{1}{(m_\sigma-1)!}t^{m_\sigma-1}\xi_\nu^{(1\sigma)}(0) + \ldots + \xi_\nu^{(m_\sigma\sigma)}(0)\right)
\end{aligned}\right\} \qquad (1.30)$$

($\nu = 1, \ldots, n$), where $\xi_1^{(h\sigma)}, \ldots, \xi_n^{(h\sigma)}$ are the components of the vector $\mathbf{x}_h^{(\sigma)}$. There may be more than one group of solutions for each eigenvalue of the matrix $\mathbf{A}$ — as many as there are elementary divisors.

The total number of solutions corresponding to one eigenvalue is equal to its multiplicity. In particular, if the root $\lambda_{(1)}$, say, is simple, the corresponding group consists of a single solution

$$\mathbf{x}^{(1)}(t) = e^{\lambda_{(1)}t}\mathbf{x}^{(1)}(0),$$

or, in scalar notation,

$$\xi_1^{(1)}(t) = e^{\lambda_{(1)}t}\xi_1^{(1)}(0), \ \ldots, \ \xi_n^{(1)}(t) = e^{\lambda_{(1)}t}\xi_n^{(1)}(0).$$

Thus, if all the elementary divisors of $\mathbf{A}$ are simple, equation (1.20) has a fundamental set of solutions (1.24), where some of the numbers λ_ν may coincide.

1.6. Second-order system of equations with constant coefficients

Consider a system of k second-order linear differential equations with constant coefficients,

$$\begin{aligned}
&\frac{d^2\eta_1}{dt^2} + p_{11}\eta_1 + \ldots + p_{1k}\eta_k = 0,\\
&\cdots\cdots\cdots\cdots\\
&\frac{d^2\eta_k}{dt^2} + p_{k1}\eta_1 + \ldots + p_{kk}\eta_k = 0,
\end{aligned}$$

or, in vector notation,

$$\frac{d^2\mathbf{y}}{dt^2} + \mathbf{P}\mathbf{y} = 0, \qquad (1.31)$$

where

$$\mathbf{y} = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_k \end{pmatrix}, \quad \mathbf{P} = \| p_{jh} \|_1^k$$

($\mathbf{P}$ is a constant matrix). If so desired, equation (1.31) may be reduced to (1.20). Setting

$$x_1 = y, \quad x_2 = \frac{dy}{dt} \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

we see that equation (1.31) is equivalent to equation (1.20) with a $2k \times 2k$ matrix

$$A = \begin{pmatrix} 0 & I_k \\ -P & 0 \end{pmatrix},$$

and so the solution of equation (1.31) is uniquely determined by specifying the vectors

$$y(0), \quad \dot{y}(0)$$

and it has the form

$$\begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix} = e^{t \begin{pmatrix} 0 & I_k \\ -P & 0 \end{pmatrix}} \begin{pmatrix} y(0) \\ \dot{y}(0) \end{pmatrix}. \tag{1.32}$$

It is more convenient to tackle equation (1.31) directly. In the scalar case ($k = 1$), the solution is

$$y = \cos(\sqrt{P}\, t)\, y(0) + \frac{1}{\sqrt{P}} \sin(\sqrt{P}\, t)\, \dot{y}(0). \tag{1.33}$$

We shall show that the solution of system (1.31) in the general case ($k > 1$) is similar in form. To this end, we consider

$$\varphi_1(z) = \cos(\sqrt{z}\, t), \quad \varphi_2(z) = \frac{1}{\sqrt{z}} \sin(\sqrt{z}\, t),$$

which are entire functions of a complex variable, and construct the matrices

$$Y_1(t) = \cos(t\sqrt{P}), \quad Y_2(t) = \frac{1}{\sqrt{P}} \sin(t\sqrt{P}),$$

which are entire functions of t (§ I. 2. 1). Differentiating the corresponding series, we see that $Y_1(t)$ and $Y_2(t)$ satisfy the matrix equation

$$\frac{d^2Y}{dt^2} + PY = 0.$$

In checking this equation, one uses identities of the type

$$P \frac{1}{\sqrt{P}} \sin(t\sqrt{P}) = \sqrt{P} \sin(t\sqrt{P})$$

(the right-hand side is the matrix $\varphi_3(P)$, where $\varphi_3(z) = \sqrt{z} \sin(\sqrt{z}\, t)$ is an entire function of z), which follow from the multiplicative property (§ I. 2.4) or more simply from consideration of the appropriate series. The vector-function

$$\mathbf{y}(t)=\mathbf{Y}_1(t)\,\mathbf{a}+\mathbf{Y}_2(t)\,\mathbf{b}$$

will satisfy equation (1.31) for any constant vectors $\mathbf{a}$ and $\mathbf{b}$. Since $\mathbf{a}=\mathbf{y}(0)$, $\mathbf{b}=\dot{\mathbf{y}}(0)$, it follows that formula (1.33) (with $\mathbf{P}$ a matrix and $\mathbf{y}$ a vector) gives the solution of the vector equation (1.31). Differentiating (1.33), comparing with (1.32), and using the uniqueness of the solution, we find

$$e^{t\left(\begin{smallmatrix}0 & I_k\\ -P & 0\end{smallmatrix}\right)}=\begin{pmatrix}\cos(t\sqrt{\mathbf{P}}) & \frac{1}{\sqrt{\mathbf{P}}}\sin(t\sqrt{\mathbf{P}})\\ -\sqrt{\mathbf{P}}\sin(t\sqrt{\mathbf{P}}) & \cos(t\sqrt{\mathbf{P}})\end{pmatrix}. \tag{1.34}$$

Example. Find the solution of the system

$$\frac{d^2\eta_1}{dt^2}+q\eta_1+r\eta_2=0,\quad \frac{d^2\eta_2}{dt^2}+r\eta_1+q\eta_2=0 \qquad (r>q>0)$$

with initial conditions

$$\eta_1(0)=\eta_1^0,\quad \eta_2(0)=\eta_2^0,\quad \dot{\eta}_1(0)=\dot{\eta}_1^0,\quad \dot{\eta}_2(0)=\dot{\eta}_2^0.$$

Solution. Let us evaluate the matrix-function on the right of (1.23). We begin by reducing the matrix

$$\mathbf{P}=\begin{Vmatrix} q & r\\ r & q\end{Vmatrix}$$

to diagonal form. Its eigenvalues are the roots of the equation

$$\begin{vmatrix} q-\lambda & r\\ r & q-\lambda\end{vmatrix}=\lambda^2-2q\lambda+q^2-r^2=0$$

so that they are $\lambda_1=q+r>0$, $\lambda_2=-(r-q)<0$. The columns of the matrix $\mathbf{S}$ transforming the symmetric matrix $\mathbf{P}$ to diagonal form are eigenvectors and we may assume that $\mathbf{S}$ is orthogonal (§I. 1. 14). Finding the eigenvectors, we obtain

$$\mathbf{S}=\frac{\sqrt{2}}{2}\begin{Vmatrix}1 & -1\\ 1 & 1\end{Vmatrix},\quad \mathbf{S}^{\mathrm{T}}=\mathbf{S}^{-1}=\frac{\sqrt{2}}{2}\begin{Vmatrix}1 & 1\\ -1 & 1\end{Vmatrix}.$$

Therefore

$$\mathbf{P}=\mathbf{S}\begin{Vmatrix}q+r & 0\\ 0 & -(r-q)\end{Vmatrix}\mathbf{S}^{-1},$$

and, using (I. 2. 5) and (I. 2. 7), we get

$$\cos(t\sqrt{\mathbf{P}})=\mathbf{S}\begin{Vmatrix}\cos\sqrt{q+r}\,t & 0\\ 0 & \operatorname{ch}\sqrt{r-q}\,t\end{Vmatrix}\mathbf{S}^{-1}=$$
$$=\frac{1}{2}\begin{Vmatrix}\cos\sqrt{q+r}\,t+\operatorname{ch}\sqrt{r-q}\,t & \cos\sqrt{q+r}\,t-\operatorname{ch}\sqrt{r-q}\,t\\ \cos\sqrt{q+r}\,t-\operatorname{ch}\sqrt{r-q}\,t & \cos\sqrt{q+r}\,t+\operatorname{ch}\sqrt{r+q}\,t\end{Vmatrix},$$

$$\frac{1}{\sqrt{\mathbf{P}}}\sin(t\sqrt{\mathbf{P}})=\mathbf{S}\begin{Vmatrix}\frac{1}{\sqrt{q+r}}\sin\sqrt{q+r}\,t & 0\\ 0 & \frac{1}{\sqrt{r-q}}\operatorname{sh}\sqrt{r-q}\,t\end{Vmatrix}\mathbf{S}^{-1}=$$

$$=\frac{1}{2}\begin{Vmatrix}\frac{1}{\sqrt{q+r}}\sin\sqrt{q+r}\,t+\frac{1}{\sqrt{r-q}}\operatorname{sh}\sqrt{r-q}\,t & \frac{1}{\sqrt{q+r}}\sin\sqrt{q+r}t-\frac{1}{\sqrt{r-q}}\operatorname{sh}\sqrt{r-q}t\\ \frac{1}{\sqrt{q+r}}\sin\sqrt{q+r}\,t-\frac{1}{\sqrt{r-q}}\operatorname{sh}\sqrt{r-q}\,t & \frac{1}{\sqrt{q+r}}\sin\sqrt{q+r}t+\frac{1}{\sqrt{r-q}}\operatorname{sh}\sqrt{r-q}t\end{Vmatrix}.$$

By (1.33), we have

$$2\eta_1=(\eta_1^0+\eta_2^0)\cos\sqrt{q+r}\,t+\frac{1}{\sqrt{q+r}}(\dot\eta_1^0+\dot\eta_2^0)\sin\sqrt{q+r}\,t+$$
$$+(\eta_1^0-\eta_2^0)\operatorname{ch}\sqrt{r-q}t+\frac{1}{\sqrt{r-q}}(\dot\eta_1^0-\dot\eta_2^0)\operatorname{sh}\sqrt{r-q}\,t,$$
$$2\eta_2=(\eta_1^0+\eta_2^0)\cos\sqrt{q+r}t+\frac{1}{\sqrt{q+r}}(\dot\eta_1^0+\dot\eta_2^0)\sin\sqrt{q+r}t-$$
$$-(\eta_1^0-\eta_2^0)\operatorname{ch}\sqrt{r-q}t-\frac{1}{\sqrt{r-q}}(\dot\eta_1^0-\dot\eta_2^0)\operatorname{sh}\sqrt{r-q}t,$$

where $\eta_{1,2}^0=\eta_{1,2}(0)$, $\dot\eta_{1,2}^0=\dot\eta_{1,2}(0)$.

Finally, we consider a slightly more general system of k second-order linear differential equations with constant coefficients: in vector notation,

$$\boxed{\mathbf{M}\ddot{\mathbf{y}}+\mathbf{Q}\dot{\mathbf{y}}+\mathbf{C}\mathbf{y}=\mathbf{0}.} \tag{1.35}$$

Here, as before, $\mathbf{y}$ is a k-vector; $\mathbf{M}$, $\mathbf{Q}$ and $\mathbf{C}$ are constant $k\times k$ matrices. We shall look for a solution

$$\mathbf{y}=e^{\lambda t}\mathbf{y}_0.$$

Substituting this into the original system, we obtain

$$e^{\lambda t}(\lambda^2\mathbf{M}+\lambda\mathbf{Q}+\mathbf{C})\,\mathbf{y}_0=\mathbf{0}.$$

The solution exists if

$$\boxed{\det(\lambda^2\mathbf{M}+\lambda\mathbf{Q}+\mathbf{C})=0.} \tag{1.36}$$

This is the characteristic equation of system (1.35). Suppose that $\mathbf{M}$ is nonsingular. We reduce equation (1.35) to (1.20) by putting

$$\mathbf{x}_1=\mathbf{y},\quad \mathbf{x}_2=\frac{d\mathbf{y}}{dt},\qquad \mathbf{x}=\begin{pmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{pmatrix}.$$

The result is

$$\mathbf{A}=\begin{pmatrix}\mathbf{0} & \mathbf{I}_k\\ -\mathbf{M}^{-1}\mathbf{C} & -\mathbf{M}^{-1}\mathbf{Q}\end{pmatrix}.$$

The characteristic equation must be (see §1.5)

$$\det\left[\begin{pmatrix} 0 & \mathbf{I}_k \\ -\mathbf{M}^{-1}\mathbf{C} & -\mathbf{M}^{-1}\mathbf{Q} \end{pmatrix} - \lambda \mathbf{I}_{2k}\right] = 0.$$

Thus the structure of the solution displays all the features determined in §1.5.

1.7. Continuity of solution with respect to coefficients

Let $\mathbf{X}(t)$ be a nonsingular continuous $n \times n$ matrix-function, defined on an interval $[0, T]$ $(T > 0)$, having an integrable piecewise-continuous derivative, $\mathbf{X}(0) = \mathbf{I}_n$. We shall show that $\mathbf{X}(t)$ is the matrizant of some equation of type (1.2), where the coefficient matrix $\mathbf{A}(t)$ is an integrable piecewise-continuous matrix-function.

To this end, we define

$$\mathbf{A}(t) = \frac{d\mathbf{X}}{dt}\mathbf{X}^{-1}(t). \tag{1.37}$$

The matrix-function $\mathbf{X}^{-1}(t)$ is continuous and hence bounded on $[0, T]$. Thus $\mathbf{A}(t)$ is integrable and piecewise-continuous. By the definition in §1.2, $\mathbf{X}(t)$ is the matrizant of the corresponding equation (1.2).

Let $\mathfrak{X} = \{\mathbf{X}(t)\}$ be the set of all matrix-functions of this type and $\mathfrak{A} = \{\mathbf{A}(t)\}$ the set of all integrable piecewise-continuous matrix-functions on $[0, T]$. As we have seen, $\mathfrak{X}$ is the set of all matrizants of equations (1.2), where $\mathbf{A}$ ranges over the set $\mathfrak{A}$. Define a norm in $\mathfrak{A}$ and a metric in $\mathfrak{X}$ by

$$\|\mathbf{A}\| = \int_0^T |\mathbf{A}(t)|\,dt, \quad \rho(\mathbf{X}_1, \mathbf{X}_2) = \sup_{0 \le t \le T} |\mathbf{X}_1(t) - \mathbf{X}_2(t)| + \int_0^T |\dot{\mathbf{X}}_1(t) - \dot{\mathbf{X}}_2(t)|\,dt, \tag{1.38}$$

where $|\mathbf{Z}|$ denotes the norm of the matrix $\mathbf{Z} = \|\zeta_{jh}\|$:

$$|\mathbf{Z}| = \sqrt{\sum_{j,h} |\zeta_{jh}|^2}.$$

(Concerning the properties of norms used below, see §I. 1. 2.)

We shall show that coefficient matrices $\mathbf{A}_1(t)$ and $\mathbf{A}_2(t)$ which are "close" together in the norm $\|\mathbf{A}\|$ determine matrizants $\mathbf{X}_1(t)$ and $\mathbf{X}_2(t)$ which are "close" together in the metric $\rho(\mathbf{X}_1, \mathbf{X}_2)$, and conversely. More precisely:

Theorem. The correspondence defined above between $\mathfrak{A}$ and $\mathfrak{X}$ is a homeomorphism (one-to-one bicontinuous mapping).

Proof. For each matrix-function $\mathbf{X}(t) \in \mathfrak{X}$ we have a unique matrix-function $\mathbf{A}(t) \in \mathfrak{A}$ defined by (1.37). The matrizant $\mathbf{X}(t) \in \mathfrak{X}$ of an equation (1.2) is also uniquely determined by $\mathbf{A}(t) \in \mathfrak{A}$. Thus the correspondence $\mathfrak{A} \sim \mathfrak{X}$ is indeed one-to-one. By the theorem of §1.1, we need only prove now that

$\mathbf{A}(t)$ is a continuous function of $\mathbf{X}(t)$. The proof is almost obvious and follows from (1.37), (1.38). Consider a matrix-function $\mathbf{X}_1(t)$ in the neighborhood of $\mathbf{X}(t)$. Let $\mathbf{A}_1(t)=\dot{\mathbf{X}}_1(t)\,\mathbf{X}_1(t)^{-1}$. Since $\rho(\mathbf{X}_1, \mathbf{X})$ is small, it follows by (1.38) that $\sup\limits_{0\leqslant t\leqslant T}|\mathbf{X}_1(t)-\mathbf{X}(t)|$ is also small. Since $\det\mathbf{X}(t)\geqslant\delta>0$ $(0\leqslant t\leqslant T)$ for $\mathbf{X}\in\mathfrak{X}$, it also follows that $\sup|\mathbf{X}_1(t)^{-1}-\mathbf{X}(t)^{-1}|$ is small. For the norm $\|\mathbf{A}_1-\mathbf{A}\|$, we now have

$$\|\mathbf{A}_1-\mathbf{A}\|=\int_0^T|\mathbf{A}_1(t)-\mathbf{A}(t)|\,dt=\int_0^T|\dot{\mathbf{X}}(\mathbf{X}^{-1}-\mathbf{X}_1^{-1})+(\dot{\mathbf{X}}-\dot{\mathbf{X}}_1)\,\mathbf{X}_1^{-1}|\,dt\leqslant$$

$$\leqslant\sup_{0\leqslant t\leqslant T}|\mathbf{X}^{-1}-\mathbf{X}_1^{-1}|\cdot\int_0^T|\dot{\mathbf{X}}|\,dt+\sup_{0\leqslant t\leqslant T}|\mathbf{X}_1^{-1}|\cdot\int_0^T|\dot{\mathbf{X}}-\dot{\mathbf{X}}_1|\,dt.$$

The right-hand side of this inequality may be made arbitrarily small for sufficiently small $\rho(\mathbf{X}_1, \mathbf{X})$, and this completes the proof.

It follows from the theorem that if $\|\mathbf{A}_1-\mathbf{A}\|$ is sufficiently small, then so are $\rho(\mathbf{X}_1, \mathbf{X})$ and, consequently, $\sup|\mathbf{X}_1-\mathbf{X}|$; in other words, the solution of equation (1.2) undergoes an arbitrarily small variation if the variation of the coefficients is sufficiently small.

It is useful, however, to have an explicit estimate for $\sup|\mathbf{X}_1-\mathbf{X}|$ in terms of $\|\mathbf{A}_1-\mathbf{A}\|$. It follows from the equations $\dot{\mathbf{X}}=\mathbf{A}\mathbf{X}$, $\dot{\mathbf{X}}_1=\mathbf{A}\mathbf{X}_1$ that

$$\frac{d(\mathbf{X}_1-\mathbf{X})}{dt}=\mathbf{A}_1(t)(\mathbf{X}_1-\mathbf{X})+[\mathbf{A}_1(t)-\mathbf{A}_2(t)]\,\mathbf{X},\quad[\mathbf{X}_1-\mathbf{X}]_{t=0}=0. \tag{1.39}$$

Integrating the resulting inhomogeneous matrix differential equation with initial conditions (1.19), we obtain

$$\mathbf{X}_1(t)-\mathbf{X}(t)=\mathbf{X}_1(t)\int_0^t\mathbf{X}_1(t_1)^{-1}\,[\mathbf{A}_1(t_1)-\mathbf{A}(t_1)]\,\mathbf{X}(t_1)\,dt_1$$

and so

$$|\mathbf{X}_1(t)-\mathbf{X}(t)|\leqslant|\mathbf{X}_1(t)|\int_0^t|\mathbf{X}_1(t_1)^{-1}||\mathbf{A}_1(t_1)-\mathbf{A}(t_1)||\mathbf{X}(t_1)|\,dt_1.$$

The matrizant $\mathbf{X}_1(t)$ satisfies the equation

$$\mathbf{X}_1(t)=\mathbf{I}_n+\int_0^t\mathbf{A}_1(t_1)\,\mathbf{X}_1(t_1)\,dt_1,$$

whence

$$|\mathbf{X}_1(t)|\leqslant|\mathbf{I}_n|+\int_0^t|\mathbf{A}_1(t_1)||\mathbf{X}_1(t_1)|\,dt_1.$$

By the lemma of §1.1, we obtain the estimate (1.7)

$$|\mathbf{X}_1(t)|\leqslant|\mathbf{I}_n|\exp\int_0^t|\mathbf{A}_1(t_1)|\,dt_1\leqslant|\mathbf{I}_n|\exp\int_0^T|\mathbf{A}_1(t_1)|\,dt_1=\sqrt{n}e^{\|\mathbf{A}_1\|}$$
$$(0\leqslant t\leqslant T,\quad|\mathbf{I}_n|=\sqrt{n}).$$

Differentiating the identity $\mathbf{X}_1(t)^{-1}\mathbf{X}_1(t) \equiv \mathbf{I}_n$, we see after some obvious manipulations that

$$\frac{d\mathbf{X}_1^{-1}}{dt} = -\mathbf{X}_1^{-1}\frac{d\mathbf{X}_1^{-1}}{dt}\mathbf{X}_1^{-1} = -\mathbf{X}_1^{-1}\mathbf{A}_1(t).$$

Proceeding similarly, we obtain

$$|\mathbf{X}_1(t)^{-1}| \leqslant \sqrt{n}e^{\|\mathbf{A}_1\|} \qquad (0 \leqslant t \leqslant T).$$

These inequalities imply the required estimate:

$$|\mathbf{X}_1(t) - \mathbf{X}(t)| \leqslant n\sqrt{n}e^{2\|\mathbf{A}_1\|+\|\mathbf{A}\|}\int_0^T |\mathbf{A}_1(t) - \mathbf{A}(t)|\,dt,$$

i. e.,

$$\sup_{0 \leqslant t \leqslant T} |\mathbf{X}_1(t) - \mathbf{X}(t)| \leqslant n\sqrt{n}e^{2\|\mathbf{A}_1\|+\|\mathbf{A}\|}\,\|\mathbf{A}_1 - \mathbf{A}\|. \tag{1.40}$$

1.8. Adjoint systems

The system of linear differential equations

$$\frac{d\zeta_\nu}{dt} = -\bar{\alpha}_{1\nu}(t)\zeta_1 - \ldots - \bar{\alpha}_{n\nu}(t)\zeta_n \qquad (\nu = 1, \ldots, n)$$

or, in vector notation,

$$\boxed{\frac{d\mathbf{z}}{dt} = -\mathbf{A}(t)^*\mathbf{z}} \quad \left(\mathbf{z} = \begin{pmatrix}\zeta_1\\ \ldots\\ \zeta_n\end{pmatrix}, \quad \mathbf{A}(t)^* = \|\bar{\alpha}_{ji}(t)\|_1^n\right) \tag{1.41}$$

$\dot{x} = A(t)x$

is said to be adjoint to system (1.1) (or (1.2)). If $\mathbf{A}(t)$ is a real matrix, then $\mathbf{A}(t)^* = \mathbf{A}(t)^T$. In this case system (1.41) is said to be associated with (1.1). Let $\mathbf{Z}(t)$ be the matrizant of system (1.41). Since it follows from (1.9) that

$$\frac{d\mathbf{X}^*}{dt} = \mathbf{X}^*\mathbf{A}(t)^*,$$

we have $\mathbf{X}(t)^*\mathbf{Z}(t) \equiv \mathbf{I}_n$. Indeed, $\mathbf{X}(0)^*\mathbf{Z}(0) = \mathbf{I}_n$, and differentiation gives

$$\frac{d}{dt}(\mathbf{X}^*\mathbf{Z}) = \frac{d\mathbf{X}^*}{dt}\mathbf{Z} + \mathbf{X}^*\frac{d\mathbf{Z}}{dt} = \mathbf{X}^*\mathbf{A}(t)^*\mathbf{Z} - \mathbf{X}^*\mathbf{A}(t)^*\mathbf{Z} \equiv 0.$$

Thus, the matrizants of systems (1.41) and (1.1) satisfy the identity

$$\mathbf{Z}(t) = [\mathbf{X}(t)^*]^{-1}. \tag{1.42}$$

For any two solutions $\mathbf{x}(t)$ and $\mathbf{z}(t)$ of equations (1.1) and (1.41), respectively,

$$(\mathbf{x}(t),\ \mathbf{z}(t)) \equiv \text{const.} \tag{1.43}$$

Indeed,

$$\frac{d}{dt}(\mathbf{x},\ \mathbf{z}) = \left(\frac{d\mathbf{x}}{dt},\ \mathbf{z}\right) + \left(\mathbf{x},\ \frac{d\mathbf{z}}{dt}\right) = (\mathbf{A}\mathbf{x},\ \mathbf{z}) - (\mathbf{x},\ \mathbf{A}^*\mathbf{z}) \equiv 0.$$

It is evident from (1.43) that for any solution $\mathbf{z}(t)$ of equation (1.41) the function $\psi(t,\ \mathbf{x}) = (\mathbf{x},\ \mathbf{z}(t))$ is an integral of equation (1.1). The converse is also true: if $\psi(t,\ \mathbf{x}) = (\mathbf{x},\ \mathbf{z}(t))$ is an integral of equation (1.1), the vector-function $\mathbf{z}(t)$ satisfies equation (1.41). Indeed, by definition, $\dot{\psi} \equiv 0$, i. e.,

$$(\mathbf{x},\ \dot{\mathbf{z}}) + (\mathbf{A}\mathbf{x},\ \mathbf{z}) = (\mathbf{x},\ \dot{\mathbf{z}} + \mathbf{A}^*\mathbf{z}) \equiv 0.$$

Since the vector $\mathbf{x}(t)$ is arbitrary, we have

$$\dot{\mathbf{z}} + \mathbf{A}^*\mathbf{z} \equiv 0,$$

as required.

§ 2. SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS OF ARBITRARY TYPE

2.1. Monodromy matrix. Multipliers

Throughout the sequel we shall assume that for some $T > 0$ and all t

$$\mathbf{A}(t+T) = \mathbf{A}(t). \tag{2.1}$$

In other words, the coefficients of system (1.1) are T-periodic functions of t. The matrizant of equation (1.2) will then satisfy the identity

$$\mathbf{X}(t+T) = \mathbf{X}(t)\,\mathbf{X}(T) \quad (-\infty < t < \infty). \tag{2.2}$$

Indeed, both sides of this equality satisfy equation (1.9) (with $k = n$) and coincide at $t = 0$. By the uniqueness of the solution to equation (1.9), both sides of (2.2) are identically equal.

We now prove the following converse of this statement: Let $\mathbf{X}(t) \in \mathfrak{X}$ (see § 1.7; $0 \leqslant t \leqslant T$) satisfy (2.2) identically. Then $\mathbf{X}(t)$ is the matrizant of some equation (2.2) with T-periodic coefficients.

As follows from § 1.7, it will suffice to prove that $\mathbf{A}(t)$ satisfies (2.1). Using (2.2), we find

$$\mathbf{A}(t+T) = \frac{d\mathbf{X}(t+T)}{dt}\,\mathbf{X}(t+T)^{-1} = \frac{d\mathbf{X}(t)}{dt}\,\mathbf{X}(T)\,\mathbf{X}(T)^{-1}\,\mathbf{X}(t)^{-1} = \mathbf{A}(t),$$

as required.

Definition. $\mathbf{X}(T)$ is called the monodromy matrix and its eigenvalues the multipliers of system (1.1) (equation (1.2)). The set of all multipliers is called the spectrum of equation (1.2).

Thus, the monodromy matrix is the value at $t=T$ (the "end" of a period) of the fundamental matrix $\mathbf{X}(T)$ defined by the initial condition $\mathbf{X}(0)=\mathbf{I}_n$ (i. e., the matrizant), and the multipliers are the roots of the equation

$$\det[\mathbf{X}(T)-\rho\mathbf{I}_n]=0. \tag{2.3}$$

Equation (2.3) is known as the characteristic equation of system (1.1).

Let $\mathbf{a}$ be an eigenvector of the monodromy matrix belonging to some multiplier ρ,

$$\mathbf{X}(T)\,\mathbf{a}=\rho\mathbf{a}. \tag{2.4}$$

Consider the solution $\mathbf{x}(t)$ of equation (1.2) with initial condition $\mathbf{x}(0)=\mathbf{a}$ (the vector $\mathbf{a}$ will generally be complex, even when $\mathbf{A}(t)$ is a real matrix); it is expressed in terms of the matrizant (see (1.13)) by

$$\mathbf{x}(t)=\mathbf{X}(t)\,\mathbf{a}.$$

Using (2.2) and (2.4), we have

$$\mathbf{x}(t+T)=\mathbf{X}(t+T)\,\mathbf{a}=\mathbf{X}(t)\,\mathbf{X}(T)\,\mathbf{a}=\rho\mathbf{X}(t)\,\mathbf{a},$$

i. e.,

$$\mathbf{x}(t+T)=\rho\mathbf{x}(t). \tag{2.5}$$

Thus, for each multiplier there is a solution of equation (1.2) satisfying conditions (2.5). We can derive the characteristic equation (2.3) by determining a solution $\mathbf{x}(t)\not\equiv\mathbf{0}$ of equation (1.2) with the property (2.5), where ρ is some number. Indeed, the vector-functions $\mathbf{x}(t+T)$ and $\rho\mathbf{x}(t)$ satisfy equation (1.2). If they coincide at $t=0$,

$$\mathbf{x}(T)=\rho\mathbf{x}(0), \tag{2.6}$$

then it follows from uniqueness considerations that (2.5) is true. It is thus sufficient to find a solution satisfying (2.6). Since $\mathbf{x}(T)=\mathbf{X}(T)\,\mathbf{x}(0)$, it follows that (2.6) is equivalent to

$$[\mathbf{X}(T)-\rho\mathbf{I}_n]\,\mathbf{x}(0)=\mathbf{0},$$

so that ρ is a root of the characteristic equation (i. e., a multiplier) and $\mathbf{x}(0)$ is an eigenvector of the monodromy matrix.

2.2. Floquet-Lyapunov theorem

Let $\mathbf{K}$ denote the constant $n\times n$ matrix

$$\mathbf{K}=\frac{1}{T}\ln\mathbf{X}(T), \qquad \Leftrightarrow \quad \mathbf{X}(T)=e^{\mathbf{K}T}$$

where $\mathbf{Y}=\ln\mathbf{X}(T)$ is one of the solutions of the matrix equation $e^{\mathbf{Y}}=\mathbf{X}(T)$ (§I. 2.6).

If the matrix $\mathbf{A}(t)$ is real, then $\mathbf{X}(T)$ is also a real matrix, but $\mathbf{K}$ will generally be complex.

Let $\mathbf{F}(t)$ denote the matrix-function

$$\mathbf{F}(t)=\mathbf{X}(t)e^{-t\mathbf{K}}. \tag{2.7}$$

Using the identity (2.2), we obtain

$$\mathbf{F}(t+T)=\mathbf{X}(t+T)e^{-T\mathbf{K}}e^{-t\mathbf{K}}=\mathbf{X}(t)\mathbf{X}(T)\mathbf{X}(T)^{-1}e^{-t\mathbf{K}}=\mathbf{F}(t).$$

Thus, the matrizant of equation (1.2) is

$$\boxed{\mathbf{X}(t)=\mathbf{F}(t)e^{t\mathbf{K}},} \tag{2.8}$$

where $\mathbf{F}(t)$ is an $n\times n$, T-periodic matrix. In addition, as is evident from (2.7) the matrix-function $\mathbf{F}(t)$ is nonsingular, continuous and has an integrable piecewise-continuous derivative; $\mathbf{F}(0)=\mathbf{I}_n$.

Conversely, any matrix $\mathbf{X}(t)$ of type (2.8), where $\mathbf{F}(t)$ has the properties just listed and $\mathbf{K}$ is a constant $n\times n$ matrix, is the matrizant of some equation (1.2) with T-periodic coefficient matrix $\mathbf{A}(t)$.

Indeed, by (2.8) we have $\mathbf{X}(t)\in\mathfrak{X}$ (§1.7) and

$$\mathbf{X}(t+T)=\mathbf{F}(t+T)e^{(t+T)\mathbf{K}}=F(t)e^{t\mathbf{K}}e^{T\mathbf{K}}=\mathbf{X}(t)\mathbf{X}(T),$$

so that $\mathbf{X}(t)$ satisfies identity (2.2). But, as mentioned previously, this means that the coefficient matrix $\mathbf{A}(t)$ of equation (1.2), for which $\mathbf{X}(t)$ is the matrizant, is periodic. We have thus proved:

Floquet-Lyapunov theorem. *The matrizant of a system of differential equations (1.1) (equation 1.2) with T-periodic coefficients may be expressed in the form (2.8), where $\mathbf{F}(t)$ is a T-periodic $n\times n$ matrix-function, nonsingular for all t, continuous, with an integrable piecewise-continuous derivative, and such that $\mathbf{F}(0)=\mathbf{I}_n$, and $\mathbf{K}$ is a constant $n\times n$ matrix. Conversely, if $\mathbf{F}(t)$ is a matrix-function with these properties and $\mathbf{K}$ an arbitrary constant matrix, then the matrix $\mathbf{X}(t)$ defined by (2.8) is the matrizant of some system (1.1) (equation (1.2)) with T-periodic coefficients.*

We can rephrase the Floquet-Lyapunov theorem so as to refer not to the matrizant but to an arbitrary fundamental matrix $\mathbf{X}_1(t)$ of equation (1.2) ($\mathbf{X}_1(0)=\mathbf{C}\neq\mathbf{I}_n$, $\det\mathbf{C}\neq 0$). Formulas (1.10) and (2.7) give

$$\mathbf{X}_1(t)=\mathbf{X}(t)\mathbf{C}=\mathbf{F}(t)e^{t\mathbf{K}}\mathbf{C}=\mathbf{F}(t)\mathbf{C}e^{t\mathbf{C}^{-1}\mathbf{K}\mathbf{C}},$$

since by formula (I.2.5)

$$e^{tC^{-1}KC} = C^{-1}e^{tK}C.$$

Thus,

$$X_1(t) = F_1(t)e^{tK_1}, \tag{2.9}$$

where

$$F_1(t) = F_1(t+T) = F(t)C, \quad K_1 = C^{-1}KC.$$

Conversely, if $X_1(t)$ may be expressed as (2.9), then

$$X(t) = X_1(t)C^{-1} = F_1(t)e^{tK_1}C^{-1} = F_1(t)C^{-1}e^{tCK_1C^{-1}} = F(t)e^{tK},$$

where

$$F(t) = F_1(t)C^{-1}, \quad K = CK_1C^{-1} \quad \text{and} \quad F(0) = CC^{-1} = I_n.$$

Thus the matrizant $X(t)$ may be expressed as (2.8), so that (1.1) is a system with periodic coefficients. Hence the following version of the Floquet-Lyapunov theorem.

Any fundamental matrix $X_1(t)$ of an equation (1.2) with T-periodic coefficients is expressible in the form (2.9), with $F_1(t)$ a nonsingular, continuous, T-periodic $n \times n$ matrix-function whose derivative is an integrable piecewise-continuous function, and K_1 some constant matrix. Conversely, any matrix (2.9) for which $F_1(t)$ and K_1 have the above properties is a fundamental matrix for some equation (1.2) with T-periodic coefficients.

2.3. Case of real coefficients

If the matrix $A(t)$ in (1.2) is real for all $t \geqslant 0$, the matrizant $X(t)$ will also be real for all $t \geqslant 0$. Moreover, if none of the multipliers of equation (1.2) (the roots of the characteristic equation (2.3)) is a negative real number, we may assume that the matrices K and $F(t)$ in (2.8) are real (the latter — for all $t \geqslant 0$).

Indeed, in this case we can define the value of $K = T^{-1} \ln X(T)$ in such a way that K is a real matrix (see Lemma III, § I. 2. 7). Then $F(t) = X(t)e^{-tK}$ will also be a real matrix.

In the general case of real periodic coefficients, we have the following proposition:

The matrizant $X(t)$ may be expressed as

$$X(t) = F_0(t)e^{tK_0}, \tag{2.10}$$

where K_0 is a real constant matrix, $F_0(t)$ a real matrix-function such that

$$\mathbf{F}_0(t+T)=\mathbf{F}_0(t)\,\mathbf{R}, \tag{2.11}$$

and $\mathbf{R}$ *some real matrix such that*

$$\mathbf{R}^2=\mathbf{I}_n, \quad \mathbf{K}_0\mathbf{R}=\mathbf{R}\mathbf{K}_0. \tag{2.12}$$

In particular,

$$\mathbf{F}_0(t+2T)=\mathbf{F}_0(t) \qquad (0\leqslant t\leqslant 2T).$$

The function $\mathbf{F}_0(t)$ *has an integrable piecewise-continuous derivative.*

Conversely, let $\mathbf{F}_0(t)$, $\mathbf{K}_0$ *and* $\mathbf{R}$ *be arbitrary real matrices satisfying conditions (2.11), (2.12),* $\det \mathbf{F}_0(t)\neq 0$, *and let* $\mathbf{F}_0(t)$ *have an integrable piecewise-continuous derivative. Then (2.10) is a fundamental matrix for some equation (1.2) with real T-periodic coefficient matrix* $\mathbf{A}(t)$.

Proof. By Lemma II, §2.7, there exist real constant matrices $\mathbf{K}_0$ and $\mathbf{R}$ satisfying (2.12) and the condition

$$e^{T\mathbf{K}_0}=\mathbf{X}(T)\,\mathbf{R}=\mathbf{R}\mathbf{X}(T).$$

By the first equality of (2.12), $\mathbf{R}^{-1}=\mathbf{R}$. Therefore

$$\mathbf{X}(T)\,e^{-T\mathbf{K}_0}=\mathbf{R}. \tag{2.13}$$

By the second equality of (2.12), for any t,

$$\mathbf{R}e^{-t\mathbf{K}_0}=e^{-t\mathbf{K}_0}\mathbf{R}. \tag{2.14}$$

Define a matrix-function $\mathbf{F}_0(t)$ by the formula

$$\mathbf{F}_0(t)=\mathbf{X}(t)\,e^{-t\mathbf{K}_0}.$$

Using the identity (2.2) and conditions (2.13) and (2.14), we obtain

$$\mathbf{F}_0(t+T)=\mathbf{X}(t)\,\mathbf{X}(T)\,e^{-(t+T)\mathbf{K}_0}=\mathbf{X}(t)\,\mathbf{R}e^{-t\mathbf{K}_0}=\mathbf{X}(t)\,e^{-t\mathbf{K}_0}\mathbf{R}=\mathbf{F}_0(t)\,\mathbf{R}.$$

Thus condition (2.11) holds. It is obvious that $\det \mathbf{F}_0(t)\neq 0$ and that $d\mathbf{F}_0(t)/dt$ (and also $\mathbf{X}(t)$) has the required properties.

Now for the converse. Let $\mathbf{X}(t)$ be defined by (2.10). Clearly, $\mathbf{X}(t)$ is a fundamental matrix for some equation (1.2) with a real, piecewise-continuous and integrable coefficient matrix $\mathbf{A}(t)$. To prove that $\mathbf{A}(t+T)\equiv\mathbf{A}(t)$, it will suffice to prove the identity (2.2) for the matrizant $\mathbf{X}(t)\,\mathbf{X}(0)^{-1}$, i. e., to show that $\mathbf{X}(t+T)=\mathbf{X}(t)\,\mathbf{X}(0)^{-1}\,\mathbf{X}(T)$. Using (2.11), and (2.12), we have

$$\mathbf{X}(t+T)\,\mathbf{X}(T)^{-1}=\mathbf{F}_0(t)\,\mathbf{R}e^{t\mathbf{K}_0}\mathbf{R}^{-1}\mathbf{F}_0(0)^{-1}=\mathbf{X}(t)\,\mathbf{X}(0)^{-1},$$

Q. E. D.

2.4. Reducibility of systems with periodic coefficients

Differentiating (2.9) and using the identity $\dot{\mathbf{X}}_1 = \mathbf{A}(t)\mathbf{X}_1$, we see that the matrix $\mathbf{F}_1(t)$ satisfies the equation

$$\frac{d\mathbf{F}_1}{dt} = \mathbf{A}(t)\mathbf{F}_1 - \mathbf{F}_1\mathbf{K}_1. \tag{2.15}$$

We claim that integration of equation (1.2) is equivalent to solution of the following problem: Find a constant $n \times n$ matrix $\mathbf{K}_1$ such that the matrix equation (2.15) has a T-periodic solution $\mathbf{F}_1(t)$ for which

$$\det \mathbf{F}_1(t) \neq 0. \tag{2.16}$$

Indeed, suppose we have found a solution with these properties. A direct check shows that the matrix

$$\mathbf{X}_1(t) = \mathbf{F}_1(t)e^{t\mathbf{K}_1} \tag{2.17}$$

satisfies equation (1.9), and by (2.16) $\det \mathbf{X}_1(t) \neq 0$, so that $\mathbf{X}_1(t)$ is a fundamental matrix for equation (2.2), Q. E. D.

It follows also from (2.17) that

$$\det \mathbf{F}_1(t) = \det \mathbf{X}_1(t)e^{-t\operatorname{Tr}\mathbf{K}_1} = \det \mathbf{C} \exp\left[\operatorname{Tr}\int_0^t \mathbf{A}(t_1)\,dt_1 - t\operatorname{Tr}\mathbf{K}_1\right].$$

Here we have used (1.10), (1.11) and (1.25). Therefore, instead of condition (2.16) we need only verify that

$$\det \mathbf{F}_1(t_0) \neq 0 \tag{2.18}$$

where t_0 is any fixed number.

Yet another version of the Floquet-Lyapunov theorem is the so-called

Lyapunov reducibility theorem:

Given a system (1.1) with T-periodic coefficients, there exists a continuous nonsingular (for all t) T-periodic matrix-function $\mathbf{F}_1(t)$, having an integrable piecewise-continuous derivative, such that the substitution

$$\mathbf{x} = \mathbf{F}_1(t)\mathbf{y} \tag{2.19}$$

transforms equation (1.2) to the form

$$\frac{d\mathbf{y}}{dt} = \mathbf{K}_1\mathbf{y}. \tag{2.20}$$

Conversely, if $\mathbf{F}_1(t)$ is an arbitrary continuous nonsingular (for all t) T-periodic matrix-function such that the substitution (2.19) reduces equation (1.2) to equation (2.20), then equation (1.2) represents a system of equations with T-periodic coefficients for which (2.9) is a fundamental matrix.

Indeed, applying the substitution (2.19) to (1.2), with $\mathbf{F}_1(t)$ the matrix defined by (2.9), we obtain

$$\dot{\mathbf{F}}_1(t)\,\mathbf{y}+\mathbf{F}_1(t)\frac{d\mathbf{y}}{dt}=\mathbf{A}(t)\,\mathbf{F}_1(t)\,\mathbf{y}.$$

Substituting $\dot{\mathbf{F}}_1(t)$ from equation (2.15), we get

$$\mathbf{F}_1(t)\frac{d\mathbf{y}}{dt}=\mathbf{F}_1(t)\,\mathbf{K}_1\mathbf{y},$$

and since $\mathbf{F}_1(t)$ is nonsingular it follows that $\mathbf{y}$ satisfies the vector equation (2.20) with constant coefficients.

Conversely, suppose we have a substitution (2.19), where $\mathbf{F}_1(t)$ has the stipulated properties, which reduces equation (1.2) to (2.20). Then if $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are solutions of equations (1.20) and (2.20), respectively, we have $\mathbf{x}(t)=\mathbf{F}_1(t)\,\mathbf{y}(t)$. Replacing $\mathbf{y}(t)$ in this formula by the columns $\mathbf{y}_1(t), \ldots, \mathbf{y}_n(t)$ of the matrizant $\mathbf{Y}(t)=\exp(t\mathbf{K}_1)$ of system (2.20) (which are of course solutions of that system), we obtain n solutions $\mathbf{x}_\nu(t)=\mathbf{F}_1(t)\,\mathbf{y}_\nu(t)$ $(\nu=1, \ldots, n)$ of equation (1.2). These vector equalities may be combined into a single matrix equality:

$$\mathbf{X}_1(t)=\mathbf{F}_1(t)\,\mathbf{Y}(t)=\mathbf{F}_1(t)\,e^{t\mathbf{K}_1},$$

where $\mathbf{X}_1(t)=\|\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)\|$ is the matrix with columns $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$. Since

$$\det\mathbf{X}_1(t)=\det\mathbf{F}_1(t)\det e^{t\mathbf{K}_1}=\det\mathbf{F}_1(t)\,e^{t\,\mathrm{sp}\,\mathbf{K}_1}\neq 0,$$

it follows that $\mathbf{X}_1(t)$ is a fundamental matrix of equation (1.2). Thus the existence of the substitution implies that a fundamental matrix $\mathbf{X}_1(t)$ of equation (1.2) may be expressed as (2.9), proving this version of the Floquet-Lyapunov theorem.

Note that the matrices $\mathbf{K}$, $\mathbf{F}_1(t)$, $\mathbf{K}_1$, $\mathbf{F}_1(t)$ in (2.8) and (2.9) are generally complex, even for a real matrix $\mathbf{A}(t)$, when the matrizant $\mathbf{X}(t)$ is real. If $\mathbf{A}(t)$ is real, there exists a substitution $\mathbf{x}=\mathbf{F}_0(t)$ that reduces (2.1) to an equation $d\mathbf{y}/dt=\mathbf{K}_0\mathbf{y}$, where $\mathbf{F}_0(t)$, $\mathbf{K}_0$ are real matrices satisfying (2.11), (2.12).

2.5. Characteristic exponents. General conclusions on stability

It is evident from (2.8) that such properties as stability or instability of the trivial solution (that is to say, boundedness or unboundedness of all solutions as $t\to+\infty$), rate of increase or decrease of solutions of equation (2.1) as $t\to+\infty$, depend entirely on the matrix $\mathbf{K}$. For practical purposes, therefore, it is often sufficient to characterize the matrix $\mathbf{K}$ or even only its eigenvalues.

Definition. The eigenvalues of the matrix $\mathbf{K}$ are known as the characteristic exponents of system (1.1).

In practice, however, it is frequently more convenient to start with an arbitrary fundamental set of solutions, considering instead of $\mathbf{K}$ some similar matrix $\mathbf{K}_1$, which has the same eigenvalues as $\mathbf{K}$ since $\mathbf{K}_1=\mathbf{C}^{-1}\mathbf{K}\mathbf{C}$ (§I.1.11).

It follows from (2.8) that to each characteristic exponent α correspond solutions of equation (1.2) of the form

$$\mathbf{x}(t)=e^{\alpha t}\mathbf{f}(t), \quad \text{where} \quad \mathbf{f}(t+T)=\mathbf{f}(t). \tag{2.21}$$

Indeed, let $\mathbf{K}\mathbf{a}=\alpha\mathbf{a}$; then by (1.13), (2.8) the solution satisfying the initial condition $\mathbf{x}(0)=\mathbf{a}$ is

$$\mathbf{x}(t)=\mathbf{F}(t)e^{t\mathbf{K}}\mathbf{a}=\mathbf{F}(t)e^{\alpha t}\mathbf{a}=e^{\alpha t}\mathbf{f}(t),$$

where $\mathbf{f}(t)=\mathbf{F}(t)\mathbf{a}$. Conversely, if we have a solution of type (2.21), then α is a characteristic exponent. Indeed, it follows from (2.21) that $\mathbf{x}(T)=e^{\alpha T}\mathbf{x}(0)$, so that the number $\rho=e^{\alpha T}$ is a multiplier of equation (1.2). By §2.1, ρ is an eigenvalue of the matrix $\mathbf{X}(T)$. Since there is always a matrix $\mathbf{K}=\frac{1}{T}\ln\mathbf{X}(T)$ with eigenvalue $\alpha=\frac{1}{T}\ln\rho$ (see end of §I.2.6), it follows that α is indeed a characteristic exponent.

Thus the characteristic exponents α_ν may be expressed in terms of the multipliers ρ_ν:

$$\alpha_\nu=\frac{1}{T}\ln\rho_\nu \qquad (\nu=1, \ldots, n), \tag{2.22}$$

where the branches of the logarithm are arbitrary. The characteristic exponents are therefore determined uniquely up to an additive term $2\pi im/T$, where m is an integer.

2.6. Structure of solutions of a system with periodic coefficients

Let us assume that the eigenvalues α_ν of the matrix $\mathbf{K}$ are distinct and let $\mathbf{a}_\nu$ be corresponding eigenvectors. As we have just shown, there exist n solutions of system (1.1) of the form

$$\mathbf{x}_\nu(t)=e^{\lambda_\nu t}\mathbf{f}_\nu(t) \qquad (\mathbf{f}_\nu(t+T)=\mathbf{f}_\nu(t)). \tag{2.23}$$

Here $\mathbf{f}_\nu(t)=\mathbf{F}(t)\mathbf{a}_\nu$. Since $\mathbf{F}(0)=\mathbf{I}_n$ and the vectors $\mathbf{a}_\nu$ are linearly independent (§I.1.8, lemma), the vectors $\mathbf{x}_\nu(0)=\mathbf{F}(0)\mathbf{a}_\nu=\mathbf{a}_\nu(\nu=1, \ldots, n)$ are linearly independent. Consequently, the vectors (2.23) form a fundamental set of solutions (§I.2).

Consider the general case. Use of the substitution (2.19) yields an equation (2.20) with a fundamental set of solutions (1.28), where $\sigma=1, \ldots, s$, which we denote by $\{\mathbf{y}_h^{(\sigma)}\}$. Since $\det\mathbf{F}_1(t)\neq 0$, the vectors $\mathbf{x}_h^{(\sigma)}=\mathbf{F}_1(t)\mathbf{y}_h^{(\sigma)}$ form a fundamental set of solutions to equation (1.2). Setting $\mathbf{f}_h^{(\sigma)}(t)=\mathbf{F}(t)\mathbf{a}_h^{(\sigma)}$, we get the following proposition.

Equation (1.2) has a fundamental set of solutions which splits into s groups, each of the form

$$\left.\begin{aligned} &\mathbf{x}_1^{(\sigma)} = e^{\alpha_{(\sigma)}t}\,\mathbf{f}_1^{(\sigma)}(t),\\ &\mathbf{x}_2^{(\sigma)} = e^{\alpha_{(\sigma)}t}\left[t\mathbf{f}_1^{(\sigma)}(t)+\mathbf{f}_2^{(\sigma)}(t)\right],\\ &\qquad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &\mathbf{x}_{m_\sigma}^{(\sigma)} = e^{\alpha_{(\sigma)}t}\left[\frac{1}{(m_\sigma-1)!}t^{m_\sigma-1}\mathbf{f}_1^{(\sigma)}(t)+\ldots+\mathbf{f}_{m_\sigma}^{(\sigma)}(t)\right],\\ &\qquad\qquad(\sigma=1,\ \ldots,\ s;\ m_1+\ldots+m_s=n), \end{aligned}\right\} \tag{2.24}$$

where $\mathbf{f}_k^{(\sigma)}(t)$ are T-periodic vector-functions with piecewise-continuous integrable derivatives. Each group of solutions belongs to an elementary divisor $(\lambda-\alpha_{(\sigma)})^{m_\sigma}$ of the matrix $\mathbf{K}$.

Note that there may be several groups of solutions for each characteristic exponent. The total number of solutions belonging to a characteristic exponent is equal to its multiplicity as a root of the equation $\det[\mathbf{K}-\lambda\mathbf{I}_n]=0$. In particular, if the root α_1, say, is simple, it determines one group of solutions, containing one solution:

$$\mathbf{x}_1(t)=e^{\alpha_1 t}\mathbf{f}_1(t) \qquad (\mathbf{f}_1(t+T)=\mathbf{f}_1(t)).$$

In case the characteristic exponents $\alpha_1, \ldots, \alpha_n$ are distinct (or some of them are multiple but the corresponding elementary divisors are simple), we again get a fundamental set of type (2.23).

2.7. Properties of solutions depending on characteristic exponents and multipliers

According to the general definition of Lyapunov stability, the trivial solution of system (1.1) is stable if and only if all solutions of system (1.1) are bounded as $t\to+\infty$, and asymptotically stable if all solutions of the system tend to zero as $t\to+\infty$. By (1.13) and (2.8), the first situation occurs if the matrix $e^{t\mathbf{K}}$ is bounded as $t\to+\infty$, and the second if $e^{t\mathbf{K}}\to 0$ as $t\to+\infty$.

Thus the trivial solution of system (1.1) is stable if and only if its characteristic exponents have nonpositive real parts, with pure imaginary or zero characteristic exponents corresponding to simple elementary divisors of the matrix $\mathbf{K}-\lambda\mathbf{I}_n$ (so that the corresponding group of solutions (2.24) has the form (2.23)). Otherwise — when the real part of at least one characteristic exponent is positive or there is a multiple pure imaginary or zero characteristic exponent with a multiple elementary divisor of $\mathbf{K}-\lambda\mathbf{I}_n$ — the trivial solution of system (1.1) is unstable.

The trivial solution of system (1.1) will be asymptotically stable if and only if all the characteristic exponents of the system have negative real parts.

Instead of the characteristic exponents α_ν, we can use the multipliers (see §2.1):

$$\rho_\nu=e^{\alpha_\nu T} \qquad (\nu=1,\ \ldots,\ n).$$

The table given below correlates properties of solutions with properties of the characteristic exponents and multipliers.

To check the trivial solution of system (1.1) for stability, it is sufficient to have the matrix $\mathbf{K}$ or the matrix $\mathbf{X}(T)$, or even only the properties of their eigenvalues. Using approximate methods, one can always determine the monodromy matrix $\mathbf{X}(T)$, accurate to the same degree as the coefficient matrix $\mathbf{A}(t)$. In applications, however, it often happens that the coefficients of system (1.1) also depend on certain parameters, as will also the matrices $\mathbf{K}$ and $\mathbf{X}(T)$. These parameters will govern the behavior of the solutions; by suitable choice of their values, one may be able to ensure that all solutions of system (1.1) be bounded as $t \to +\infty$, that some of the solutions be unbounded (unstable trivial solution), or that the solutions satisfy the estimate

$$|\mathbf{x}(t)| \leqslant ce^{\mu t} \qquad (0 \leqslant t < \infty)$$

for some given μ, and so on.

TABLE

Property of solutions	Characteristic exponents	Multipliers
Stability of trivial solution (all solutions bounded on $[0, \infty)$)	Real parts nonpositive; zero or pure imaginary characteristic exponents (if present) correspond to simple elementary divisors	Inside or on the unit circle; latter case corresponds to simple elementary divisors
Asymptotic stability	Real parts negative	Inside unit circle
Instability of trivial solution	At least one characteristic exponent with positive real part, or a pure imaginary (or zero) exponent with multiple (elementary divisor	At least one multiplier either outside the unit circle, or on the unit circle with multiple elementary divisor
Existence of T-periodic solution	For at least one characteristic exponent α_ν: $\alpha_\nu T = 2m\pi i$ (m an integer)	At least one multiplier equal to 1
Existence of T-antiperiodic solution $\mathbf{x}(t+T) = -\mathbf{x}(t)$	For at least one characteristic exponent α_ν: $\alpha_\nu T = (2m+1)\pi i$ (m an integer)	At least one multiplier $\rho = -1$

Sometimes one has to construct the corresponding domains in the parameter space, or at least to describe their relative positions in general terms. Frequently, exact values of (all or some of) the coefficients of system (1.1) are not available; for example, only upper and lower bounds may be prescribed. One must nonetheless draw definite conclusions concerning the boundedness or unboundedness of the solutions, or their rate of increase as $t \to +\infty$.

Problems of this kind present considerable difficulties, even for systems with constant coefficients, if one demands effective solutions. Later (§§4, III.6 and Chapters IV, VII, VIII) we shall devote special attention to some methods for their solution.

2.8. Remark: application of Lyapunov's second method

Consider an equation (1.2) with real coefficients depending also on a real parameter ε (for simplicity's sake we shall assume that the dependence is linear):

$$\frac{dx}{dt} = [\mathbf{C} + \varepsilon \mathbf{B}(t)]\,\mathbf{x}. \tag{2.25}$$

Here $\mathbf{B}(t)$ is an $n \times n$ matrix-function with continuous T-periodic elements. We assume in addition that all the eigenvalues of the matrix $\mathbf{C}$ are in the left half-plane. We wish to determine a value of ε for which the trivial solution of system (2.25) is asymptotically stable.

To solve this problem, we use Lyapunov's second method, in a form first proposed by Chetaev /102/.

Consider system (2.25) with $\varepsilon = 0$:

$$\frac{dx}{dt} = \mathbf{C}x. \tag{2.26}$$

Since the trivial solution of system (2.26) is by assumption asymptotically stable, there exists a positive definite quadratic form (in Lyapunov's sense) with constant coefficients:

$$V = (\mathbf{H}x,\ x) \qquad (\mathbf{H}^* = \mathbf{H} > 0),$$

whose derivative along trajectories of system (2.26), i. e.,

$$\frac{dV}{dt} = (\mathbf{H}\mathbf{C}x,\ x) + (\mathbf{H}x,\ \mathbf{C}x) = ([\mathbf{H}\mathbf{C} + \mathbf{C}^*\mathbf{H}]\,x,\ x),$$

is negative definite. The matrix $\mathbf{H}$ may be found from the equation

$$\mathbf{H}\mathbf{C} + \mathbf{C}^*\mathbf{H} = -\mathbf{G}_0 < 0,$$

where $\mathbf{G}_0$ is any positive definite matrix, say $\mathbf{G}_0 = \beta \mathbf{I}_n$ $(\beta > 0)$. The derivative $\dot{V}$ along trajectories of system (2.25) is

$$\dot{V} = -([\mathbf{G}_0 + \varepsilon \mathbf{G}_1(t)]\,x,\ x), \tag{2.27}$$

where $\mathbf{G}_1(t)$ is a T-periodic matrix function

$$\mathbf{G}_1(t) = -\mathbf{H}\mathbf{B}(t) - \mathbf{B}(t)^*\mathbf{H}.$$

Let $\Delta_j(t, \varepsilon)$ denote the principal minors of the matrix $\mathbf{G}_0+\varepsilon\mathbf{G}_1(t)$. Since $\mathbf{G}_0$ is positive definite, it follows from Sylvester's test (§I. 1.14) that $\Delta_j(t, 0)\equiv\Delta_j>0$ $(j=1, \ldots, n)$. The minors $\Delta_j(t, \varepsilon)$ are continuous functions of t, ε. Hence, for sufficiently small $\varepsilon_0\geqslant 0$,

$$\Delta_j(t, \varepsilon)>0 \quad (j=1, \ldots, n) \quad \text{for} \quad 0\leqslant t\leqslant T, \quad |\varepsilon|\leqslant\varepsilon_0, \tag{2.28}$$

i. e., $\mathbf{G}_0+\varepsilon\mathbf{G}_1(t)>0$ for $0\leqslant t\leqslant T$, $|\varepsilon|\leqslant\varepsilon_0$. By continuity, there exists $\delta>0$ such that all the principal minors of the matrix

$$\mathbf{G}_0+\varepsilon\mathbf{G}_1(t)-\delta\mathbf{I}_n$$

are positive for $0\leqslant t\leqslant T$, $|\varepsilon|\leqslant\varepsilon_0$. This means that $\dot{V}$ satisfies inequality (2.27):

$$\dot{V}\leqslant-\delta(\mathbf{x}, \mathbf{x}).$$

By Lyapunov's theorem, the trivial solution of system (2.25) is asymptotically stable for $|\varepsilon|\leqslant\varepsilon_0$. The value of ε_0 is determined by Sylvester's conditions (2.28).

2.9. T-periodic solutions of the inhomogeneous equation when the homogeneous equation has no T-periodic solutions

Consider the inhomogeneous equation

$$\frac{d\mathbf{x}}{dt}=\mathbf{A}(t)\mathbf{x}+\mathbf{f}(t), \tag{2.29}$$

where

$$\mathbf{A}(t+T)=\mathbf{A}(t), \quad \mathbf{f}(t+T)=\mathbf{f}(t). \tag{2.30}$$

We shall show that when the corresponding homogeneous equation

$$\frac{d\mathbf{y}}{dt}=\mathbf{A}(t)\mathbf{y} \tag{2.31}$$

has no T-periodic solutions, equation (2.29) has a unique T-periodic solution. Let $\mathbf{Y}(t)$ be the matrizant of equation (2.31). By assumption (see table, p. 97), none of the multipliers of equation (2.31) are equal to 1, so that $[\mathbf{Y}(T)-\mathbf{I}_n]^{-1}$ exists. In order that

$$\mathbf{x}(t+T)=\mathbf{x}(t) \qquad (-\infty<t<\infty) \tag{2.32}$$

it is sufficient that

$$\mathbf{x}(T)=\mathbf{x}(0). \tag{2.33}$$

Indeed, by the periodicity conditions (2.30), the vector-function $\mathbf{x}(t+T)$ satisfies equation (2.29) and coincides with $\mathbf{x}(t)$ at $t=0$ (by (2.33)). The uniqueness of the solution implies (2.32).

By (1.17),

$$\mathbf{x}(T)=\mathbf{Y}(T)\left[\mathbf{x}(0)+\int_0^T \mathbf{Y}(t)^{-1}\mathbf{f}(t)\,dt\right].$$

Thus (2.33) is equivalent to the equation

$$[\mathbf{Y}(T)-\mathbf{I}_n]\,\mathbf{x}(0)=-\mathbf{Y}(T)\int_0^T \mathbf{Y}(t)^{-1}\mathbf{f}(t)\,dt.$$

Consequently, the required value of $\mathbf{x}(0)$ exists and is unique.

We now want to find an explicit expression for the T-periodic solution $\mathbf{x}(t)$. By (1.17),

$$\mathbf{x}(t+T)=\mathbf{Y}(t+T)\left[\mathbf{x}(0)+\int_0^{t+T} \mathbf{Y}(s)^{-1}\mathbf{f}(s)\,ds\right].$$

Setting

$$\mathbf{x}(t+T)=\mathbf{x}(t),\quad \int_0^{t+T}=\int_0^t+\int_t^{t+T},$$

and again using (1.17), we obtain

$$\mathbf{x}(t)=\mathbf{Y}(t+T)\,\mathbf{Y}(t)^{-1}\left[\mathbf{x}(t)+\mathbf{Y}(t)\int_t^{t+T}\mathbf{Y}(s)^{-1}f(s)\,ds\right]. \tag{2.34}$$

Consider the matrix-function

$$\mathbf{R}(t,\ s)=\mathbf{Y}(t+T)\,\mathbf{Y}(t+s)^{-1}; \tag{2.35}$$

$\mathbf{R}(t,s)$ is a T-periodic function of t. Indeed, by (2.2),

$$\mathbf{R}(t,\ s)=\mathbf{Y}(t)\,\mathbf{Y}(T)\,\mathbf{Y}(t+s)^{-1},$$

whence, again by (2.2),

$$\mathbf{R}(t+T,\ s)=\mathbf{Y}(t)\,\mathbf{Y}(T)^2\,\mathbf{Y}(T)^{-1}\,\mathbf{Y}(t+s)^{-1}=\mathbf{R}(t,\ s).$$

By assumption, $\mathbf{Y}(T)$ has no eigenvalues equal to unity; hence the same is true of the similar matrix $\mathbf{R}(t,\ 0)$:

$$\det[\mathbf{R}(t,\ 0)-\mathbf{I}_n]\neq 0.$$

Thus the substitution $\tau=t+T-s$ in (2.34) gives

$$\mathbf{x}(t)=[I_n-R(t,\ 0)]^{-1}\int_0^T \mathbf{R}(t,\ T-\tau)\,\mathbf{f}(t-\tau)\,d\tau. \tag{2.36}$$

Formulas (2.36) and (2.35) give an explicit expression of the T-periodic solution $\mathbf{x}(t)$ in terms of the "extremal force" $\mathbf{f}(t)$ and the matrizant $\mathbf{Y}(t)$ of the homogeneous equation.

When the matrix $\mathbf{A}(t)=\mathbf{A}$ is constant, we have $\mathbf{Y}(t)=e^{t\mathbf{A}}$, $\mathbf{R}(t, s)=e^{(T-s)\mathbf{A}}$. Formula (2.36) then becomes

$$\mathbf{x}(t)=\left[\mathbf{I}_n-e^{T\mathbf{A}}\right]^{-1}\int_0^T e^{\tau\mathbf{A}}\,\mathbf{f}(t-\tau)\,d\tau. \tag{2.37}$$

It is easy to derive analogous formulas for the T-antiperiodic solution $(\mathbf{x}(t+T)=-\mathbf{x}(t))$ when $\mathbf{f}(t)$ is an antiperiodic vector-function $(\mathbf{f}(t+T)=-\mathbf{f}(t))$ and the homogeneous equation has no T-antiperiodic solutions.

Instead of (2.36), we can deduce another formula, in which $\mathbf{R}(t, s)$ is replaced by the matrices $\mathbf{F}_1(t)$ and $\mathbf{K}_1$ of (2.9). To this end, we change variables in (2.29),

$$\mathbf{x}=\mathbf{F}_1(t)\,\mathbf{u},$$

where $\mathbf{F}_1(t)$ is the T-periodic matrix-function in (2.9).

Using the equality $\dot{\mathbf{F}}_1=\mathbf{A}(t)\,\mathbf{F}_1-\mathbf{F}_1\mathbf{K}_1$ (2.15), we see that $\mathbf{u}$ satisfies the equation

$$\frac{d\mathbf{u}}{dt}=\mathbf{K}_1\mathbf{u}+\mathbf{F}_1(t)^{-1}\,\mathbf{f}(t). \tag{2.38}$$

The corresponding homogeneous equation $\dot{\mathbf{v}}=\mathbf{K}_1\mathbf{v}$ has no T-periodic solutions, since otherwise, setting $\mathbf{f}(t)\equiv\mathbf{0}$, we would have a T-periodic solution $\mathbf{x}=\mathbf{F}_1(t)\,\mathbf{v}(t)$ of the equation $\dot{\mathbf{x}}=\mathbf{A}(t)\,\mathbf{x}$. Thus the T-periodic solution of equation (2.38) is deduced from (2.37):

$$\mathbf{u}(t)=\left[\mathbf{I}_n-e^{T\mathbf{K}_1}\right]^{-1}\int_0^T e^{\tau\mathbf{K}_1}\,\mathbf{F}_1(t-\tau)^{-1}\,\mathbf{f}(t-\tau)\,d\tau.$$

Consequently, the T-periodic solution of equation (2.29) is

$$\mathbf{x}(t)=\mathbf{F}_1(t)\left[\mathbf{I}_n-e^{T\mathbf{K}_1}\right]^{-1}\int_0^T e^{\tau\mathbf{K}_1}\,\mathbf{F}_1(t-\tau)^{-1}\,\mathbf{f}(t-\tau)\,d\tau. \tag{2.39}$$

These formulas are also readily established directly from (2.36), but the derivation is then a little longer. To end this subsection, we use (2.37) to find a T-periodic solution of the inhomogeneous second-order vector equation

$$\ddot{\mathbf{v}}+\mathbf{P}\mathbf{v}=\mathbf{h}(t),\quad \mathbf{P}=\|p_{ij}\|_1^k,\quad \mathbf{h}(t)=\mathbf{h}(t+T) \tag{2.40}$$

when the characteristic equation

$$\det\left[\omega^2\mathbf{I}_k+\mathbf{P}\right]=0$$

has no roots of the form $\omega = 2\pi pi/T$ $(p = 0, \pm 1, \pm 2, \ldots)$, in other words, when the homogeneous equation $\ddot{\mathbf{v}} + \mathbf{P}\mathbf{v} = \mathbf{0}$ has no T-periodic solutions. Setting

$$\mathbf{x} = \begin{pmatrix} \mathbf{v} \\ \dot{\mathbf{v}} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & \mathbf{I}_k \\ -\mathbf{P} & 0 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 0 \\ \mathbf{h}(t) \end{pmatrix},$$

we obtain equation (2.29). The matrizant of the homogeneous equation is given by (1.34). The first factor in (2.37) is

$$[\mathbf{I}_{2k} - e^{T\mathbf{A}}]^{-1} = \frac{1}{2}\begin{pmatrix} \mathbf{I}_k & (\sqrt{\mathbf{P}})^{-1} \operatorname{ctg} \frac{T\sqrt{\mathbf{P}}}{2} \\ \sqrt{\mathbf{P}} \operatorname{ctg} \frac{T\sqrt{\mathbf{P}}}{2} & \mathbf{I}_k \end{pmatrix},$$

as can be checked through multiplication by $\mathbf{I}_{2k} - e^{T\mathbf{A}}$. Note that in accordance with the general theory of functions of matrices (§ I. 2.1) the matrices $(\sqrt{\mathbf{P}})^{\pm 1} \operatorname{ctg} \frac{T\sqrt{\mathbf{P}}}{2}$ should be understood as the appropriate functions of a complex variable:

$$\sqrt{z} \operatorname{ctg}(\sqrt{z}\, t), \quad \frac{1}{\sqrt{z}} \operatorname{ctg}(\sqrt{z}\, t).$$

Since $\omega \neq 2\pi pi/T$, these functions are analytic on the spectrum of $\mathbf{P}$. Carrying out the matrix multiplication in (2.37), we get formulas for the unique (under our assumptions) T-periodic solution and its derivative:

$$\mathbf{v}(t) = \frac{1}{2}\int_0^T (\sqrt{\mathbf{P}})^{-1} \sin(\tau\sqrt{\mathbf{P}})\, \mathbf{h}(t-\tau)\, d\tau + \\ + \frac{1}{2}(\sqrt{\mathbf{P}})^{-1} \operatorname{ctg} \frac{T\sqrt{\mathbf{P}}}{2} \int_0^T \cos(\tau\sqrt{\mathbf{P}})\, \mathbf{h}(t-\tau)\, d\tau, \qquad (2.41)$$

$$\dot{\mathbf{v}}(t) = -\frac{1}{2}\int_0^T \operatorname{ctg} \frac{T\sqrt{\mathbf{P}}}{2} \sin(\tau\sqrt{\mathbf{P}})\, \mathbf{h}(t-\tau)\, d\tau + \\ + \frac{1}{2}\int_0^T \cos(\tau\sqrt{\mathbf{P}})\, \mathbf{h}(t-\tau)\, d\tau. \qquad (2.42)$$

The validity of these formulas may also be verified by a direct check. In fact, to verify the last formula we differentiate the right-hand side of (2.41). Differentiating the right-hand side of (2.42) with respect to t and substituting into the original equation, we obtain an identity. A look at the shape of the formula is sufficient to see that it is T-periodic.

2.10. T-periodic solutions of the inhomogeneous equation when the homogeneous equation has T-periodic solutions

We shall use the following notation. Given any vector-functions $\mathbf{u}(t)$, $\mathbf{v}(t)$ on $(0, T)$, we set

$$((\mathbf{u},\ \mathbf{v}))=\frac{1}{T}\int_0^T(\mathbf{u}(t),\ \mathbf{v}(t))\,dt,$$

assuming that the integral on the right exists. We let $\|\mathbf{u}\|$ denote

$$\|\mathbf{u}\|=\sqrt{((\mathbf{u},\mathbf{u}))}=\sqrt{\frac{1}{T}\int_0^T|\mathbf{u}(t)|^2dt}.$$

We shall need the Cauchy-Bunyakovskii inequality

$$|((\mathbf{u},\mathbf{v}))|\leqslant\|\mathbf{u}\|\cdot\|\mathbf{v}\|$$

and the triangle inequality

$$\|\mathbf{u}+\mathbf{v}\|\leqslant\|\mathbf{u}\|+\|\mathbf{v}\|.$$

For proofs, see, e. g., /103a/, Chapter 7, § 50.

By assumption, the monodromy matrix $\mathbf{Y}(T)$ of equation (2.31) satisfies the equality

$$\det\ [\mathbf{Y}(T)-\mathbf{I}_n]=0.$$

Together with the homogeneous equation (2.31), let us consider the adjoint equation

$$\frac{d\mathbf{z}}{dt}=-\mathbf{A}(t)^*\mathbf{z}. \tag{2.43}$$

If $\mathbf{x}(t)$, $\mathbf{z}(t)$ are any solutions of equations (2.29), (2.43), then

$$\frac{d}{dt}(\mathbf{x},\ \mathbf{z})=(\mathbf{A}(t)\mathbf{x}+\mathbf{f}(t),\ \mathbf{z})+(\mathbf{x},-\mathbf{A}(t)^*\mathbf{z})=(\mathbf{f}(t),\ \mathbf{z}).$$

Integrating this identity and using the notation introduced above, we get

$$(\mathbf{x}(T),\ \mathbf{z}(T))-(\mathbf{x}(0),\ \mathbf{z}(0))=T((\mathbf{f},\ \mathbf{z})). \tag{2.44}$$

Let d be the nullity of the matrix $\mathbf{Y}(T)-\mathbf{I}_n$ (§ I. 1.3). Then equation (2.31) has d linearly independent T-periodic solutions $\mathbf{y}_\delta(t)=\mathbf{Y}(t)\mathbf{c}_\delta$ $(\delta=1,\ \ldots,\ d)$, where $\mathbf{c}_1,\ \ldots,\ \mathbf{c}_d$ are linearly independent solutions of the equation $[\mathbf{Y}(T)-\mathbf{I}_n]\mathbf{c}=0$. The general T-periodic solution of equation (2.31) is $\mathbf{y}=\gamma_1\mathbf{y}_1+\ldots+\gamma_d\mathbf{y}_d$, where $\gamma_1,\ \ldots,\ \gamma_d$ are arbitrary constant scalars.

Lemma 1. Assume that the homogeneous system (2.31) has d linearly independent T-periodic solutions $\mathbf{y}_1(t),\ \ldots,\ \mathbf{y}_d(t)$. *Then:*

1) *The adjoint system (2.43) also has d linearly independent T-periodic solutions* $\mathbf{z}_1(t),\ \ldots,\ \mathbf{z}_d(t)$.

2) *The inhomogeneous system (2.29) has T-periodic solutions if and only if*

$$((\mathbf{f},\ \mathbf{z}_\delta)) \equiv \frac{1}{T}\int_0^T (\mathbf{f}(t),\ \mathbf{z}_\delta(t))\,dt = 0 \qquad (\delta = 1,\ \ldots,\ d). \tag{2.45}$$

3) *If conditions (2.45) are satisfied, any T-periodic solution of the inhomogeneous system (2.29) is given by*

$$\mathbf{x} = \mathbf{x}_0(t) + \sum_{\delta=1}^{d} \gamma_\delta \mathbf{y}_\delta(t), \tag{2.46}$$

where $\mathbf{x}_0(t)$ is some particular T-periodic solution of (2.29) and $\gamma_1,\ \ldots,\ \gamma_d$ are arbitrary complex numbers.

Proof. 1) Let $\mathbf{Y}(t)$ and $\mathbf{Z}(t)$ be the matrizants of systems (2.31) and (2.43). By (1.42), $\mathbf{Z}(t) = [\mathbf{Y}(t)^*]^{-1}$. A solution $\mathbf{z}(t)$ of system (2.43) will be T-periodic if (by uniqueness) and only if $\mathbf{z}(T) = \mathbf{z}(0)$, i. e.,

$$[\mathbf{Z}(T) - \mathbf{I}_n]\,\mathbf{z}(0) = 0.$$

Multiplying on the left by the nonsingular matrix $-\mathbf{Y}(T)^* = -\mathbf{Z}(T)^{-1}$, we obtain an equivalent equation

$$[\mathbf{Y}(T)^* - \mathbf{I}_n]\,\mathbf{z}(0) = \mathbf{0}.$$

But by assumption the equation

$$[\mathbf{Y}(T) - \mathbf{I}_n]\,\mathbf{y}(0) = \mathbf{0}$$

has d linearly independent solutions $\mathbf{y}_1(0),\ \ldots,\ \mathbf{y}_d(0)$. The preceding vector equation also has d linearly independent solutions $\mathbf{z}_1(0),\ \ldots,\ \mathbf{z}_d(0)$ (cf. beginning of §I. 1.6). Consequently, system (2.43) has d linearly independent T-periodic solutions

$$\mathbf{z}_\delta(t) = \mathbf{Z}(t)\,\mathbf{z}_\delta(0) \qquad (\delta = 1,\ \ldots,\ d),$$

Q. E. D.

2) Conditions (2.45) follow from (2.44), if we let $\mathbf{x}$ be a T-periodic solution of equation (2.29) and set $\mathbf{z}$ equal successively to $\mathbf{z}_1(t),\ \ldots,\ \mathbf{z}_d(t)$. This proves necessity.

To prove sufficiency, we complete the d solutions $\mathbf{z}_1(t),\ \ldots,\ \mathbf{z}_d(t)$ to a system of n linearly independent solutions $\mathbf{z}_1(t),\ \ldots,\ \mathbf{z}_n(t)$ of system (2.43) and let $\mathbf{Z}_1(t)$ denote the corresponding fundamental matrix:

$$\mathbf{Z}_1(t) = \|\mathbf{z}_1(t),\ \ldots,\ \mathbf{z}_n(t)\|.$$

We first show that $\mathbf{x}(t)$ is a T-periodic solution of system (2.29) if and only if

$$(\mathbf{x}(0),\ \mathbf{z}_\nu(T) - \mathbf{z}_\nu(0)) = T((\mathbf{f},\ \mathbf{z}_\nu)) \qquad (\nu = 1,\ \ldots,\ n). \tag{2.47}$$

By (2.44),

$$(\mathbf{x}(T),\ \mathbf{z}_\nu(T))-(\mathbf{x}(0),\ \mathbf{z}_\nu(0))=T((\mathbf{f},\ \mathbf{z}_\nu)) \qquad (\nu=1,\ \ldots,\ n). \tag{2.48}$$

If $\mathbf{x}$ is a T-periodic solution of (2.29), these equalities yield (2.47). Conversely, if equalities (2.47) hold, we subtract them from (2.48), to get

$$(\mathbf{x}(T)-\mathbf{x}(0),\ \mathbf{z}_\nu(T))=0 \qquad (\nu=1,\ \ldots,\ n),$$

and since the vectors $\mathbf{z}_1(T),\ \ldots,\ \mathbf{z}_n(T)$ are linearly independent it follows that $\mathbf{x}(T)=\mathbf{x}(0)$. Thus conditions (2.47) are necessary and sufficient for the solution $\mathbf{x}(t)$ of system (2.29) to be T-periodic. We can now prove that conditions (2.45) are sufficient. We show that the system of algebraic equations for the components of $\mathbf{a}$,

$$(\mathbf{a},\ \mathbf{z}_\nu(T)-\mathbf{z}_\nu(0))=T((\mathbf{f},\ \mathbf{z}_\nu)) \qquad (\nu=1,\ \ldots,\ n),$$

which is system (2.47) with $\mathbf{a}=\mathbf{x}(0)$, is solvable if conditions (2.45) are satisfied. Let $\mathbf{g}$ denote the vector with components $T((\mathbf{f},\ \mathbf{z}_1)),\ \ldots,\ T((\mathbf{f},\ \mathbf{z}_n))$, and write the above system first as

$$[\mathbf{z}_\nu(T)-\mathbf{z}_\nu(0)]^*\,\mathbf{a}=T((\mathbf{f},\ \mathbf{z}_\nu)) \qquad (\nu=1,\ \ldots,\ n),$$

and then as

$$[\mathbf{Z}_1(T)-\mathbf{Z}_1(0)]^*\,\mathbf{a}=\mathbf{g}. \tag{2.49}$$

Consider the homogeneous system of algebraic equations adjoint to (2.49):

$$[\mathbf{Z}_1(T)-\mathbf{Z}_1(0)]\,\mathbf{c}=0.$$

By assumption, the matrix $\mathbf{Z}_1(T)-\mathbf{Z}_1(0)$ has nullity d, and this equation has d linearly independent solutions $\mathbf{c}_1=\mathbf{e}_1,\ \ldots,\ \mathbf{c}_d=\mathbf{e}_d$ (where $\mathbf{e}_\delta$ are the elements of the standard basis — see (I. 1.8)). By part (a) of the lemma in §I. 1.6, system (2.47) will have a solution if and only if

$$(\mathbf{g},\ \mathbf{e}_\delta)=0 \qquad (\delta=1,\ \ldots,\ d).$$

But it follows from the definition of $\mathbf{g}$ that $(\mathbf{g},\ \mathbf{e}_\delta)\equiv T((\mathbf{f},\ \mathbf{z}_\delta))$, so that these equalities are simply conditions (2.45). Thus, if the latter hold, system (2.47) is solvable and, letting the solution be $\mathbf{x}(0)$, we see that $\mathbf{x}(t)$ is a T-periodic solution of system (2.29), proving part 2.

3) Let $\mathbf{x}_0(t)$ be some particular T-periodic solution of system (2.29), determined, say, by (1.17). Then $\mathbf{y}(t)=\mathbf{x}(t)-\mathbf{x}_0(t)$ satisfies the homogeneous system (2.31). Any T-periodic solution of the latter may be expressed, by assumption, as

$$\mathbf{y}=\mathbf{x}-\mathbf{x}_0=\sum_{\delta=1}^{d}\gamma_\delta\mathbf{y}_\delta(t),$$

and this implies the desired assertion.

Lemma II. Under the assumptions of Lemma I, let conditions (2.45) hold. Let $\mathbf{u}_1, \ldots, \mathbf{u}_d$ *be arbitrary vectors such that* $\Delta_1 = \det \| (\mathbf{y}_j(0), \mathbf{u}_h) \|_1^d \neq 0$ *and* $\mathbf{v}_1(t), \ldots, \mathbf{v}_d(t)$ *arbitrary continuous vector-functions such that* $\Delta_2 = \det \| ((\mathbf{y}_j, \mathbf{v}_h)) \|_1^d \neq 0$. *The T-periodic solution* $\mathbf{x}(t)$ *of equation (2.29) is uniquely determined by any one of the following four conditions:*

$$(\mathbf{x}(0), \mathbf{y}_\delta(0)) = 0 \tag{2.50}$$

$$(\mathbf{x}(0), \mathbf{u}_\delta) = 0, \tag{2.51}$$

$$((\mathbf{x}, \mathbf{y}_\delta)) = 0, \tag{2.52}$$

$$((\mathbf{x}, \mathbf{v}_\delta)) = 0 \qquad (\delta = 1, \ldots, d). \tag{2.53}$$

In each of these four cases, there exists a constant υ, *independent of the vector-function* $\mathbf{f}(t)$, *such that the above solution satisfies the estimate*

$$|\mathbf{x}(t)| \leqslant \upsilon \|\mathbf{f}\| \qquad (0 \leqslant t \leqslant T). \tag{2.54}$$

Proof. Conditions (2.50) are a special case of (2.51) with $\mathbf{u}_\delta = \mathbf{y}_\delta(0)$, and conditions (2.52) a special case of (2.53) with $\mathbf{v}_\delta(t) = \mathbf{y}_\delta(t)$. Indeed, in this case $\Delta_1 \neq 0$, for the vectors $\mathbf{y}_1(0), \ldots, \mathbf{y}_d(0)$ are linearly independent (§ I. 1.4). Similarly, $\Delta_2 \neq 0$, because the vector-functions $\mathbf{y}_1(t), \ldots, \mathbf{y}_d(t)$ are linearly independent. This is verified in exactly the same way as the statement for Δ_1. It will thus suffice to consider conditions (2.51) and (2.53).

As shown in the proof of Lemma I, a solution $\mathbf{x}(t)$ is T-periodic if and only if the vector $\mathbf{a} = \mathbf{x}(0)$ is a solution of equation (2.49).

By (1.10) and (1.42), we have $\mathbf{Z}_1(t) = [\mathbf{Y}(t)^*]^{-1} \mathbf{Z}_1(0)$. Substituting this expression for $\mathbf{Z}_1(T)$ into (2.49), we get

$$[\mathbf{Y}(T)^{-1} - \mathbf{I}_n] \mathbf{a} = [\mathbf{Z}_1(0)^*]^{-1} \mathbf{g},$$

or

$$[\mathbf{Y}(T) - \mathbf{I}_n] \mathbf{a} = -\mathbf{Y}(T) [\mathbf{Z}_1(0)^*]^{-1} \mathbf{g}. \tag{2.55}$$

By assumption, the vectors $\mathbf{y}_1(0), \ldots, \mathbf{y}_d(0)$ form a set of linearly independent solutions of the homogeneous equation

$$[\mathbf{Y}(T) - \mathbf{I}_n] \mathbf{a} = 0.$$

By the lemma of § I. 1.6, the solution $\mathbf{x}(0) = \mathbf{a}$ of equation (2.55) is uniquely determined by (2.51). Moreover, we then have an estimate (I.1.22), i. e.,

$$|\mathbf{x}(0)| \leqslant \gamma |\mathbf{Y}(T) \mathbf{Z}_1(0)^{*-1} \mathbf{g}| \leqslant \upsilon_1 |\mathbf{g}|, \tag{2.56}$$

where the numbers γ and $\upsilon_1 = \gamma |\mathbf{Y}(T) \mathbf{Z}_1(0)^{*-1}|$ are independent of $\mathbf{f}(t)$. We claim that (2.56) implies (2.54). Indeed,

$$|g|^2=\sum_{\nu=1}^{n}|((f,\ z_\nu))|^2.$$

By the Cauchy-Bunyakovskii inequality,

$$|((f,\ z_\nu))|^2\leqslant\|f\|^2\varkappa_\nu^2 \qquad (\nu=1,\ \ldots,\ n),$$

where the numbers $\varkappa_\nu^2=\|z_\nu\|^2$ are independent of $f(t)$. Therefore,

$$|g|^2\leqslant\|f\|^2\upsilon_2^2, \quad \text{where} \quad \upsilon_2^2=\varkappa_1^2+\ldots+\varkappa_n^2.$$

Substituting into (2.55), we obtain

$$|x(0)|\leqslant\upsilon_1\upsilon_2\|f\|. \tag{2.57}$$

By (1.17),

$$x(t)=Y(t)x(0)+\int_0^t Y(t)\,Y(t_1)^{-1}\ f(t_1)\,dt_1.$$

Using the properties of the norm (§§ I. 1.1, I. 1.2), we obtain

$$|x(t)|\leqslant\sup_{0\leqslant t\leqslant T}|Y(t)|\cdot|x(0)|+|Y(t)|\int_0^t|Y(t_1)^{-1}\|f(t_1)|\,dt_1. \tag{2.58}$$

We now estimate the second term. By the Cauchy-Bunyakovskii inequality,

$$\int_0^t|Y(t_1)^{-1}\|f(t_1)|\,dt_1\leqslant\int_0^T|Y(t)^{-1}\|f(t)|\,dt\leqslant\upsilon_3\|f\|,$$

where

$$\upsilon_3=\sqrt{T\int_0^T|Y(t)^{-1}|^2\,dt}.$$

Using (2.57), we deduce the required estimate (2.54) from (2.58), with

$$\upsilon=\upsilon_1\upsilon_2\sup_{0\leqslant t\leqslant T}|Y(t)|+\upsilon_3.$$

We now consider conditions (2.53). If $x(t)$ is a T-periodic solution of equation (2.29) satisfying conditions (2.53), it follows from part 3 of Lemma I that $x(t)$ may be expressed as (2.46), where $x_0(t)$ is the solution defined by (2.50) and the numbers $\gamma_1,\ \ldots,\ \gamma_d$, as follows from (2.53), satisfy the equations

$$0=((x_0,\ v_\delta))+\sum_{j=1}^{d}((y_j,\ v_\delta))\,\gamma_j \qquad (\delta=1,\ \ldots,\ d).$$

By assumption, this system is uniquely solvable:

$$\begin{pmatrix} \gamma_1 \\ \dots \\ \gamma_d \end{pmatrix} = -U \begin{pmatrix} ((x_0, v_1)) \\ \dots \\ ((x_0, v_d)) \end{pmatrix}, \quad \text{where} \quad U = [\,\|((y_j, v_\delta))\|_1^d]^{-1}. \tag{2.59}$$

Thus conditions (2.53) determine a unique T-periodic solution of equation (2.29). It remains to prove estimate (2.54). We have

$$|x(t)| \leqslant |x_0(t)| + \sum_{\delta=1}^{d} |\gamma_\delta| v^{(\delta)},$$

where the numbers

$$v^{(\delta)} = \sup_{0 \leqslant t \leqslant T} |y_\delta(t)| \qquad (\delta = 1, \dots, d)$$

are independent of $f(t)$. The first term was estimated above. For the second, we again use the Cauchy-Bunyakovskii inequality:

$$\left(\sum_{\delta=1}^{d} |\gamma_\delta| v^{(\delta)} \right)^2 \leqslant v' \sum_{\delta=1}^{d} |\gamma_\delta|^2,$$

where

$$v' = \sum_{\delta=1}^{d} (v^{(\delta)})^2$$

is independent of $f(t)$. It follows from (2.59) (see the properties of the norm in §§I.1.1, I.1.2) that

$$\sqrt{\sum_{\delta=1}^{d} |\gamma_\delta|^2} \leqslant |U| \sqrt{\sum_{\delta=1}^{d} |((x_0, v_\delta))|^2}.$$

On the other hand, by the Cauchy-Bunyakovskii inequality,

$$\sum_{\delta=1}^{d} ((x_0, v_\delta))^2 \leqslant \|x_0\|^2 \sum_{\delta=1}^{d} \|v_\delta\|^2.$$

Hence, by part 1 of Lemma II,

$$\sqrt{\sum_{\delta=1}^{d} |\gamma_\delta|^2} \leqslant v'' \|x_0\| \leqslant v'' v_0 \|f\|,$$

where

$$v'' = \|U\| \sqrt{\sum_{\delta=1}^{d} \|v_\delta\|^2}$$

and v_0 is the constant in (2.54) for the solution $\mathbf{x}_0(t)$. Finally,

$$|\mathbf{x}(t)| \leqslant v_0 \|\mathbf{f}\| + \sqrt{v'v''}v_0\|\mathbf{f}\| = v_0(1+\sqrt{v'v''})\|\mathbf{f}\| \qquad (0 \leqslant t \leqslant T),$$

completing the proof of Lemma II.

Note that (2.54) implies another estimate:

$$\|\mathbf{x}(t)\| = \sqrt{\frac{1}{T}\int_0^T |\mathbf{x}(t)|^2\,dt} \leqslant v\|\mathbf{f}\|. \tag{2.60}$$

§3. CANONICAL AND HAMILTONIAN SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

3.1. Basic definitions

Let the Hamiltonian of a holonomic mechanical system with k degrees of freedom be

$$H = \frac{1}{2}\sum_{i,j=1}^{2k} h_{ij}(t)\,\xi_i\xi_j \qquad (h_{ji}(t) = h_{ij}(t);\ i,\ j = 1,\ \ldots,\ 2k;\ t_0 \leqslant t < \infty),$$

i. e., a quadratic form in the generalized coordinates $\xi_\varkappa$ and generalized momenta $\xi_{k+\varkappa}$ ($\varkappa = 1,\ \ldots,\ k$). Hamilton's equations

$$\frac{d\xi_\varkappa}{dt} = \frac{\partial H}{\partial \xi_{k+\varkappa}}, \quad \frac{d\xi_{k+\varkappa}}{dt} = -\frac{\partial H}{\partial \xi_\varkappa} \qquad (\varkappa = 1,\ \ldots,\ k)$$

for this system may be written

$$\left.\begin{aligned} \frac{d\xi_\varkappa}{dt} &= \sum_{i=1}^{2k} h_{i,\,k+\varkappa}(t)\,\xi_i, \\ \frac{d\xi_{k+\varkappa}}{dt} &= -\sum_{i=1}^{2k} h_{i\varkappa}(t)\,\xi_i \qquad (\varkappa = 1,\ \ldots,\ k). \end{aligned}\right\} \tag{3.1}$$

System (3.1) may be written as a single vector differential equation

$$\mathbf{J}_{2k}\frac{d\mathbf{x}}{dt} = \mathbf{H}(t)\,\mathbf{x}, \tag{3.2}$$

where $\mathbf{x}$ is a column-vector with components $\xi_1,\ \ldots,\ \xi_{2k}$; $\mathbf{H}(t)$ is the matrix of the quadratic form $2H$:

$$\mathbf{x} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{2k} \end{pmatrix}, \quad \mathbf{H}(t) = \|h_{ij}(t)\|_1^{2k}$$

and $\mathbf{J}_{2k}$ is a matrix of the special form

$$\mathbf{J}_{2k}=\begin{pmatrix}0 & -\mathbf{I}_k\\ \mathbf{I}_k & 0\end{pmatrix}.$$

It is easy to see that

$$\mathbf{J}_{2k}^{\mathrm{T}}=\mathbf{J}_{2k}^{-1}=-\mathbf{J}_{2k},\quad \mathbf{J}_{2k}^{2}=-\mathbf{I}_{2k}. \tag{3.3}$$

Clearly, equation (3.2) may be written as (1.2) with $n=2k$ and $\mathbf{A}(t)=-\mathbf{J}_{2k}\mathbf{H}(t)$. Had we defined the vector $\mathbf{x}$ in some other way, the result would have been equation (3.2) with some other nonsingular skew-symmetric (see §I. 1.14) matrix $\mathbf{J}_{2k}$. Hence the following definition:

Definition. Any equation

$$\tilde{\mathbf{J}}\frac{dz}{dt}=\tilde{\mathbf{H}}(t)\,z, \tag{3.4}$$

where $\tilde{\mathbf{H}}(t)$ is a real symmetric matrix-function: $\tilde{\mathbf{H}}(t)^{\mathrm{T}}=\tilde{\mathbf{H}}(t)$, and $\tilde{\mathbf{J}}$ a constant real nonsingular skew-symmetric matrix, will be called a canonical equation.

Any canonical system may be reduced to the form (3.2). Indeed, we shall show later (see §III. 2.3) that any real nonsingular skew-symmetric matrix $\tilde{\mathbf{J}}$ may be expressed as

$$\tilde{\mathbf{J}}=\mathbf{S}^{\mathrm{T}}\mathbf{J}_{2k}\mathbf{S}\qquad(\det\mathbf{S}\neq 0), \tag{3.5}$$

where $\mathbf{S}$ is a real matrix. The substitution

$$\mathbf{x}=\mathbf{S}z \tag{3.6}$$

brings equation (3.4) to the form (3.2), where

$$\mathbf{H}(t)=(\mathbf{S}^{-1})^{\mathrm{T}}\tilde{\mathbf{H}}(t)\,\mathbf{S}^{-1} \tag{3.7}$$

is a real symmetric matrix.

We now consider the case of complex coefficients.

Definition. Any equation (3.4), with complex matrices $\tilde{\mathbf{J}}$, $\tilde{\mathbf{H}}(t)$, where $\tilde{\mathbf{J}}$ is a constant nonsingular antihermitian matrix, i. e.,

$$\tilde{\mathbf{J}}^{*}=-\tilde{\mathbf{J}},\quad \det\tilde{\mathbf{J}}\neq 0,$$

and $\mathbf{H}(t)$ a hermitian matrix-function,

$$\tilde{\mathbf{H}}(t)^{*}=\tilde{\mathbf{H}}(t).$$

will be called a Hamiltonian equation.

A canonical equation is of course also Hamiltonian. The matrix-function $\tilde{\mathbf{H}}(t)$ in equation (3.4) (for both canonical and Hamiltonian equations) will be called a Hamiltonian.

Let $\mathbf{G}=i\tilde{\mathbf{J}}$ be a nonsingular hermitian matrix with p positive and q negative eigenvalues, $p+q=2k$. It is easy to see, using (I. 1.57), that a Hamiltonian equation may be reduced by a nonsingular transformation (3.6) to the form

$$i^{-1}\mathbf{G}_0\frac{d\mathbf{x}}{dt}=\mathbf{H}_0(t)\,\mathbf{x},$$

where $\mathbf{H}_0(t)$ is a hermitian matrix-function and

$$\mathbf{G}_0=\begin{pmatrix}\mathbf{I}_p & 0\\ 0 & -\mathbf{I}_q\end{pmatrix}$$

($\mathbf{I}_p$, $\mathbf{I}_q$ are the $p\times p$ and $q\times q$ identity matrices).

3.2. Second-order canonical equations and their reduction to first-order equations

Consider a holonomic conservative mechanical system with k degrees of freedom, in which the kinetic energy K and potential energy Π are quadratic forms with constant and variable coefficients, respectively:

$$K=\frac{1}{2}\sum_{i,\,j=1}^{k}m_{ij}\dot{\eta}_i\dot{\eta}_j,\quad \Pi=\frac{1}{2}\sum_{i,\,j=1}^{k}c_{ij}(t)\,\eta_i\eta_j,$$

where $\eta_1, \ldots, \eta_k$ and $\dot{\eta}_1, \ldots, \dot{\eta}_k$ are generalized coordinates and velocities, and

$$m_{ji}=m_{ij},\ c_{ji}(t)=c_{ij}(t)\qquad (i,\ j=1,\ \ldots,\ k;\ t_0\leqslant t<\infty).$$

Assume that the system is in the field of action of gyrosocopic forces

$$\tilde{Q}_i=\sum_{j=1}^{k}q_{ij}\dot{\eta}_j\qquad (q_{ji}=q_{ij};\ i,\ j=1,\ \ldots,\ k),$$

whose work in actual displacement of the system is of course zero.

Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial K}{\partial\dot{\eta}_i}\right)-\frac{\partial K}{\partial\eta_i}+\frac{\partial\Pi}{\partial\eta_i}-\tilde{Q}_i=0\qquad (i=1,\ \ldots,\ k)$$

for the system are

$$\sum_{j=1}^{k}m_{ij}\ddot{\eta}_j+\sum_{j=1}^{k}q_{ij}\dot{\eta}_j+\sum_{j=1}^{k}c_{ij}(t)\,\eta_j=0\qquad (i=1,\ \ldots,\ k).\tag{3.8}$$

Defining matrices $\mathbf{M}$, $\mathbf{Q}$, $\mathbf{C}(t)$ and a vector $\mathbf{y}$ by

$$\mathbf{M}=\|m_{ij}\|_1^k, \quad \mathbf{Q}=\|q_{ij}\|_1^k, \quad \mathbf{C}(t)=\|c_{ij}(t)\|_1^k, \quad \mathbf{y}=\begin{pmatrix}\eta_1\\ \cdot\\ \cdot\\ \eta_k\end{pmatrix},$$

we write system (3.8) as a single vector equation

$$\mathbf{M}\ddot{\mathbf{y}}+\mathbf{Q}\dot{\mathbf{y}}+\mathbf{C}(t)\,\mathbf{y}=\mathbf{0}. \tag{3.9}$$

The matrices $\mathbf{M}$ and $\mathbf{C}(t)$ are symmetric, $\mathbf{Q}$ is skew-symmetric, and in addition the matrix $\mathbf{M}$ of the quadratic form K is positive definite.

(a) We first treat the case in which there are no gyroscopic forces ($\mathbf{Q}=\mathbf{0}$). Then Lagrange's equations are

$$\mathbf{M}\ddot{\mathbf{y}}+\mathbf{C}(t)\,\mathbf{y}=\mathbf{0}. \tag{3.10}$$

Equation (3.10) may be reduced to equation (3.2) by defining a $2k$-vector $\mathbf{x}$ as the direct sum of the k-vectors $\mathbf{y}$ and $\mathbf{M}\dot{\mathbf{y}}$:

$$\mathbf{x}=\mathbf{y}\dotplus\mathbf{M}\frac{d\mathbf{y}}{dt}.$$

It is easily checked that the vector $\mathbf{x}$ satisfies equation (3.2) with

$$\mathbf{H}(t)=\begin{pmatrix}\mathbf{C}(t) & \mathbf{0}\\ \mathbf{0} & \mathbf{M}^{-1}\end{pmatrix}$$

a real symmetric matrix-function.

Using the fact that $\mathbf{M}$ is positive definite, we can reduce equation (3.10) to a form in which $\mathbf{M}=\mathbf{I}_k$. To do this, we express $\mathbf{M}$ as (I.1.52)

$$\mathbf{M}=\mathbf{S}\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_k\end{pmatrix}\mathbf{S}^{-1} \qquad (\lambda_1>0,\ \ldots,\ \lambda_k>0),$$

where $\mathbf{S}$ is an orthogonal matrix, $\mathbf{S}^{-1}=\mathbf{S}^{\tau}$. Setting

$$\mathbf{N}=\sqrt{\mathbf{M}}=\mathbf{S}\begin{pmatrix}\sqrt{\lambda_1} & & 0\\ & \ddots & \\ 0 & & \sqrt{\lambda_k}\end{pmatrix}\mathbf{S}^{-1}$$

and using §I.1.15, we see that $\mathbf{N}$ is positive definite and $\mathbf{N}^2=\mathbf{M}$. We now write equation (3.10) as

$$\mathbf{N}^2\ddot{\mathbf{y}}+\mathbf{C}(t)\,\mathbf{N}^{-1}\mathbf{N}\mathbf{y}=\mathbf{0}.$$

Multiply on the left by N^{-1}, and denote

$$\tilde{y} = Ny, \quad \tilde{P}(t) = N^{-1}C(t)N^{-1}.$$

Hence the following equation for the vector $\tilde{y}$:

$$\frac{d^2\tilde{y}}{dt^2} + \tilde{P}(t)\tilde{y} = 0, \tag{3.11}$$

where $\tilde{P}(t)$ is a symmetric real matrix-function.

(b) We now consider the case of nonvanishing gyroscopic forces. Equation (3.9) is a special case of the equation

$$\frac{d}{dt}\left[R(t)\frac{dy}{dt} + \tilde{Q}(t)y\right] - \tilde{Q}(t)^{\tau}\frac{dy}{dt} + P(t)y = 0, \tag{3.12}$$

where $R(t) = R(t)^{\tau}$, $P(t) = P(t)^{\tau}$, $\tilde{Q}(t)$ are real matrix-functions, $\det R(t) \neq 0$ $(t_0 \leqslant t < \infty)$.

Equation (3.12) may be reduced to canonical form (3.2). Indeed, setting

$$v = R(t)\frac{dy}{dt} + \tilde{Q}(t)y, \quad x = y + v = \begin{pmatrix} y \\ v \end{pmatrix}, \tag{3.13}$$

we get equation (3.2) for x, with

$$H(t) = \begin{pmatrix} P + \tilde{Q}^{\tau}R^{-1}\tilde{Q} & -\tilde{Q}^{\tau}R^{-1} \\ -R^{-1}\tilde{Q} & R^{-1} \end{pmatrix}. \tag{3.14}$$

Equation (3.9) is the special case of (3.12) in which $R(t)$, $\tilde{Q}(t)$ are constant matrices.

Equation (3.9) may also be reduced by the substitution

$$z = \begin{pmatrix} y \\ M\dot{y} \end{pmatrix}$$

to equation (3.4) with

$$\tilde{H}(t) = \begin{pmatrix} C(t) & 0 \\ 0 & M^{-1} \end{pmatrix}, \quad \tilde{J} = \begin{pmatrix} -Q & -I_k \\ I_k & 0 \end{pmatrix}. \tag{3.15}$$

We then have an equality of type (3.5), with

$$S = \begin{pmatrix} I_k & 0 \\ \frac{1}{2}Q & I_k \end{pmatrix},$$

as is readily checked by direct substitution of $\tilde{J}$ and S into (3.5).

Thus the substitution (3.6) reduces equation (3.9) to the form (3.2) with

$$x = Sz = \begin{pmatrix} y \\ \frac{1}{2}Qy + M\dot{y} \end{pmatrix}, \quad H = H^{\tau} = \begin{pmatrix} C - \frac{1}{4}QM^{-1}Q & \frac{1}{2}QM^{-1} \\ -\frac{1}{2}M^{-1}Q & M^{-1} \end{pmatrix}. \tag{3.16}$$

Formulas (3.16) also follow from (3.13) and (3.14), by putting

$$\mathbf{R}(t)=\mathbf{M}, \quad \mathbf{Q}(t)=\frac{1}{2}\mathbf{Q}, \quad \mathbf{P}(t)=\mathbf{C}(t).$$

3.3. Canonical equations with constant coefficients

We consider the case in which the real matrix $\mathbf{H}(t)$ is constant:

$$\mathbf{H}(t)=\mathbf{H}_0=\text{const}.$$

The characteristic equation (§I. 1.5) for equation (3.2) in this case is

$$\det(\mathbf{J}_{2k}^{-1}\mathbf{H}_0-\lambda\mathbf{I}_{2k})=0. \tag{3.17}$$

The left-hand side of this equation may be transformed as follows:

$$\det(\mathbf{J}_{2k}^{-1}\mathbf{H}_0-\lambda\mathbf{I}_{2k})=\det(\mathbf{J}_{2k}^{-1}\mathbf{H}_0-\lambda\mathbf{I}_{2k})^{\mathrm{T}}=\det(-\mathbf{H}_0\mathbf{J}_{2k}^{-1}-\lambda\mathbf{I}_{2k})=$$
$$=\det[\mathbf{J}_{2k}^{-1}(-\mathbf{H}_0\mathbf{J}_{2k}^{-1}-\lambda\mathbf{I}_{2k})\mathbf{J}_{2k}]=\det(\mathbf{J}_{2k}^{-1}\mathbf{H}_0+\lambda\mathbf{I}_{2k}).$$

Hence it follows that if $\lambda_\varkappa$ is a root of equation (3.17), then so is $-\lambda_\varkappa$ ($\varkappa=1, \ldots, k$), so that, when developed, equation (3.17) cannot contain odd powers of λ:

$$\det(\mathbf{J}_{2k}^{-1}\mathbf{H}_0-\lambda\mathbf{I}_{2k})=\lambda^{2k}+a_1\lambda^{2k-2}+\ldots+a_{k-1}\lambda^2+a_k=0. \tag{3.18}$$

Since the coefficients of equation (3.18) are real, its roots (counting multiplicities) lie symmetrically about both the real axis and the imaginary axis.

If all the roots of equation (3.18) are pure imaginary and belong to simple elementary divisors of the matrix $\mathbf{J}_{2k}^{-1}\mathbf{H}_0$, then all solutions of the equation

$$\mathbf{J}_{2k}\frac{d\mathbf{x}}{dt}=\mathbf{H}_0\mathbf{x} \tag{3.19}$$

are bounded on $(-\infty, \infty)$ (§2.5). Moreover, it can be seen by analyzing the roots of equation (3.19) that the solutions may be bounded only in this case. *

The solutions of equation (3.19) are certainly bounded if $\mathbf{H}_0>0$ (or $\mathbf{H}_0<0$). Indeed, in that case the quadratic form $V=(\mathbf{H}_0\,\mathbf{x},\,\mathbf{x})$ is positive definite (negative definite) in Lyapunov's sense, and the derivative along the trajectories of equation (3.19) vanishes identically:

$$\frac{d}{dt}(\mathbf{H}_0\mathbf{x},\,\mathbf{x})=(-\mathbf{H}_0\mathbf{J}_{2k}\mathbf{H}_0\mathbf{x},\,\mathbf{x})+(\mathbf{H}_0\mathbf{x},\,-\mathbf{J}_{2k}\mathbf{H}_0\mathbf{x})=$$
$$=(-\mathbf{H}_0\mathbf{J}_{2k}\mathbf{H}_0\mathbf{x},\,\mathbf{x})+(-\mathbf{H}_0\mathbf{J}_{2k}^{\mathrm{T}}\mathbf{H}_0\mathbf{x},\,\mathbf{x})\equiv 0.$$

* In particular, the solutions cannot be asymptotically stable. This is also valid for Hamiltonian equations with periodic coefficients (see below, §3.5).

It therefore follows from Lyapunov's theorem that the trivial solution of equation (3.19) is stable, i. e., all its solutions are bounded.*

It may however happen that $(\mathbf{H}_0\mathbf{x}, \mathbf{x})$ is neither positive nor negative definite but the solutions are nevertheless stable. This is the case for equation (3.9) with $\mathbf{C}(t)=\mathbf{C}=\text{const}$ and a special choice of the matrices $\mathbf{Q}$, $\mathbf{C}$, as follows from Kelvin's theorems on gyroscopic stabilization (see, e. g., /102/, Chapter V, subsection 40). It is evident from the first formula of (3.15) that here both $\mathbf{C}$ and $\tilde{\mathbf{H}}(t)$ are constant matrices, so that $\mathbf{C}$ and $\mathbf{H}_0$ certainly have the same number of negative eigenvalues.

3.4. Properties of solutions of canonical and Hamiltonian equations

Proposition. The matrizant $\mathbf{Z}(t)$ of a Hamiltonian equation (3.4) satisfies the identity

$$\mathbf{Z}^*(t)\tilde{\mathbf{J}}\mathbf{Z}(t)\equiv\tilde{\mathbf{J}}. \tag{3.20}$$

Conversely, if a matrix $\mathbf{Z}(t)$ satisfies condition (3.20), is continuous, has an integrable piecewise-continuous derivative, and $\mathbf{Z}(0)=\mathbf{I}_{2k}$, then it is the matrizant of some Hamiltonian equation (3.4).

Proof. Let $\mathbf{Z}(t)$ be the matrizant of a Hamiltonian equation. Then (3.20) is true at $t=0$, because $\mathbf{Z}(0)=\mathbf{I}_{2k}$. The derivative of the left-hand side of the equality is zero:

$$\frac{d}{dt}(\mathbf{Z}^*\tilde{\mathbf{J}}\mathbf{Z})=\frac{d\mathbf{Z}^*}{dt}\tilde{\mathbf{J}}\mathbf{Z}+\mathbf{Z}^*\tilde{\mathbf{J}}\frac{d\mathbf{Z}}{dt}=-\mathbf{Z}^*\tilde{\mathbf{H}}\tilde{\mathbf{J}}^{-1}\tilde{\mathbf{J}}\mathbf{Z}+\mathbf{Z}^*\tilde{\mathbf{H}}\mathbf{Z}=0,$$

since

$$(\tilde{\mathbf{J}}\dot{\mathbf{Z}})^*=(\tilde{\mathbf{H}}\mathbf{Z})^*,\quad -\dot{\mathbf{Z}}^*\tilde{\mathbf{J}}^*=\mathbf{Z}^*\tilde{\mathbf{H}},\quad \dot{\mathbf{Z}}^*=-\mathbf{Z}^*\tilde{\mathbf{H}}\tilde{\mathbf{J}}^{-1}.$$

Consequently, (3.20) is valid for all t.

Now for the converse. Taking determinants on both sides of (3.20), we obtain

$$\det\mathbf{Z}^*\det\mathbf{Z}=|\det\mathbf{Z}|^2=1,$$

so that $\mathbf{Z}(t)^{-1}$ exists. Differentiating (3.20), we get

$$\frac{d\mathbf{Z}^*}{dt}\tilde{\mathbf{J}}\mathbf{Z}+\mathbf{Z}^*\tilde{\mathbf{J}}\frac{d\mathbf{Z}}{dt}=0.$$

* In other words: by (I.1.58), we have

$$(\mathbf{H}_0\mathbf{x}, \mathbf{x})=\text{const}\geqslant h_{\min}|\mathbf{x}|^2,$$

whence $|\mathbf{x}(t)|\leqslant\sqrt{\frac{(\mathbf{H}_0\mathbf{x}(0), \mathbf{x}(0))}{h_{\min}}}$. Thus this argument furnishes an explicit upper bound for the absolute value of the solution.

Multiplying this identity on the left by $(Z^*)^{-1}$ and on the right by Z^{-1}, we obtain

$$(Z^{-1})^* \left(\frac{dZ}{dt}\right)^* \tilde{J}^* = \tilde{J} \frac{dZ}{dt} Z^{-1}, \tag{3.21}$$

since $(Z^*)^{-1} = (Z^{-1})^*$. This identity means that the matrix on the right of (3.21) is hermitian; denoting it by $\tilde{H}(t)$, we get

$$\tilde{J} \frac{dZ}{dt} = \tilde{H}(t) Z,$$

Q. E. D.

Definition. A real matrix Z such that

$$Z^T \tilde{J} Z = \tilde{J}$$

where $\tilde{J}$ is a real nonsingular skew-symmetric matrix, is said to be symplectic (or $\tilde{J}$-orthogonal).

Thus, the matrizant of a canonical equation (3.4) is a symplectic matrix for any t.

The identity (3.20) yields several properties of a Hamiltonian equation, first pointed out by Poincaré.

I. For any two solutions $z_1(t)$, $z_2(t)$ of a Hamiltonian equation,

$$(\tilde{J} z_1(t),\ z_2(t)) \equiv \text{const}. \tag{3.22}$$

II. Suppose that a Hamiltonian equation (3.4) has a linear first integral, i. e., an integral of the form $\psi(t, z) = (z, v(t))$. Then the vector-function $w(t) = \tilde{J}^{-1} v(t)$ is a solution of equation (3.4).

Property I is proved directly:

$$(\tilde{J} z_1(t),\ z_2(t)) = (\tilde{J} Z(t) z_1(0), Z(t) z_2(0)) = (Z(t)^* \tilde{J} Z(t) z_1(0),\ z_2(0)) = (\tilde{J} z_1(0),\ z_2(0)).$$

It is easy to see that, conversely, property I implies (3.20).

We now prove property II. By the assertion at the end of §1.8, the vector-function $v(t)$ is a solution of the adjoint equation

$$\frac{dv}{dt} = -(\tilde{J}^{-1} \tilde{H}(t))^* v = \tilde{H}(t) \tilde{J}^{-1} v.$$

Since $v = \tilde{J} w$, we have

$$\tilde{J} \frac{dw}{dt} = \tilde{H}(t) w,$$

as required.

In the case $\tilde{J} = J_{2k}$, this tells us that the vector $J_{2k}^{-1} v = -J_{2k} v$, i. e., the vector with components $\vartheta_{k+1}, \ldots, \vartheta_{2k}, -\vartheta_1, \ldots, -\vartheta_k$ (where $\vartheta_1, \ldots, \vartheta_{2k}^{-1}$ are the components of v), is a solution of equation (3.2).

The left-hand side of (3.22) for two real solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ of equation (3.2) is known as a Poincaré invariant; when $\tilde{\mathbf{J}} = \mathbf{J}_{2k}$, (3.22) becomes

$$\sum_{\varkappa=1}^{k} (\xi_{\varkappa}^{(1)} \xi_{k+\varkappa}^{(2)} - \xi_{k+\varkappa}^{(1)} \xi_{\varkappa}^{(2)}) = \text{const}, \tag{3.23}$$

where $\xi_1^{(1)}, \ldots, \xi_{2k}^{(1)}$ are the components of the vector $\mathbf{x}_1$ and $\xi_1^{(2)}, \ldots, \xi_{2k}^{(2)}$ those of $\mathbf{x}_2$.

3.5. Lyapunov-Poincaré theorem

Up to this point we have not assumed that the coefficients of the various systems of linear differential equations considered are periodic — they were arbitrary integrable piecewise-continuous functions of time (except in §3.3, where we stipulated that they be constants). We now consider a canonical equation (3.4) with T-periodic coefficients:

$$\tilde{\mathbf{J}} \frac{d\mathbf{x}}{dt} = \tilde{\mathbf{H}}(t)\mathbf{x} \qquad (\tilde{\mathbf{J}}^{\mathrm{T}} = -\tilde{\mathbf{J}},\ \tilde{\mathbf{H}}(t+T) = \tilde{\mathbf{H}}(t) = \tilde{\mathbf{H}}(t)^{\mathrm{T}},\ 0 \leqslant t \leqslant T), \tag{3.24}$$

where $\tilde{\mathbf{H}}(t)$ is a real symmetric T-periodic matrix-function.

It follows from the identity (3.20) that

$$\mathbf{X}(T) = \tilde{\mathbf{J}}^{-1} [\mathbf{X}(T)^{\mathrm{T}}]^{-1} \tilde{\mathbf{J}}, \tag{3.25}$$

where $\mathbf{X}(T)$ is the monodromy matrix (value of the matrizant at the end of a period) of equation (3.24). Let $\{\rho_\nu\}$ be a complete set of multipliers (§2.1) for equation (3.24). The matrices $\mathbf{X}(T)$ and $\mathbf{X}(T)^{\mathrm{T}}$ have the same eigenvalues. Thus the eigenvalues of the matrix $[\mathbf{X}(T)^{\mathrm{T}}]^{-1}$ are the numbers ρ_ν^{-1}. Similar matrices have the same eigenvalues. Thus it follows from (3.25) that the sets $\{\rho_\nu\}$ and $\{\rho_\nu^{-1}\}$ coincide.

This proves the following theorem of Lyapunov and Poincaré:

Theorem. *The characteristic equation of the canonical equation (3.24)*

$$\det [\mathbf{X}(T) - \rho \mathbf{I}_{2k}] = \rho^{2k} - a_1 \rho^{2k-1} + \ldots + a_{2k-1}\rho + a_{2k} = 0 \tag{3.26}$$

is reversible, i.e.,

$$a_{2k} = 1; \quad a_{2k-\varkappa} = a_\varkappa \qquad (\varkappa = 1, \ldots, k).$$

In the case of complex coefficients and a hermitian T-periodic matrix-function $\mathbf{H}(t)$, so that we are dealing with a Hamiltonian equation

$$\tilde{\mathbf{J}} \frac{d\mathbf{z}}{dt} = \tilde{\mathbf{H}}(t)\mathbf{z} \qquad (\mathbf{J}^* = -\mathbf{J},\ \tilde{\mathbf{H}}(t+T) = \tilde{\mathbf{H}}(t) = \tilde{\mathbf{H}}(t)^*,\ 0 \leqslant t \leqslant T), \tag{3.27}$$

we have instead of (3.25)

$$\mathbf{Z}(T) = \tilde{\mathbf{J}}^{-1} [\mathbf{Z}(T)^*]^{-1} \tilde{\mathbf{J}}, \tag{3.28}$$

and $\mathbf{Z}(T)$ is in general a complex matrix. The set of eigenvalues of $\mathbf{Z}(T)$* is now $\{\bar{\rho}_\nu\}$. Thus it follows from (3.28), as before, that the sets $\{\rho_\nu\}$ and $\{\bar{\rho}_\nu^{-1}\}$ coincide. If $|\rho| \neq 1$, the points ρ and $\bar{\rho}^{-1}$ are symmetric (in the sense of inversion) about the unit circle. Thus the spectrum of a Hamiltonian equation is symmetric about the unit circle (Figure 2a). In the case of a canonical equation, the spectrum is also symmetric about the real axis (Figure 2b), so that the nonreal multipliers not on the unit circle may be partitioned into quadruples ρ, $\bar{\rho}^{-1}$, $\bar{\rho}$, ρ^{-1}.

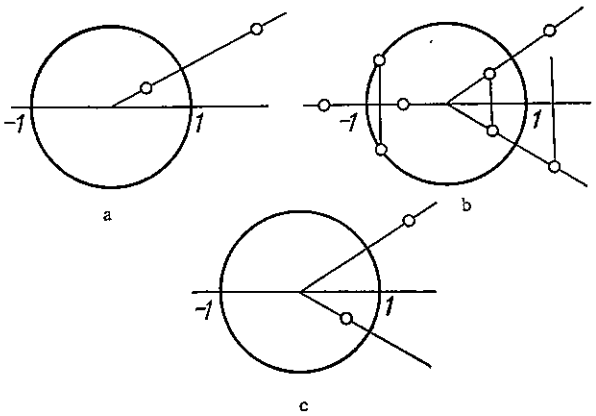


FIGURE 2.

The above arguments remain valid if we consider instead of ρ_ν the elementary divisors $(\lambda-\rho_{(\sigma)})^{m_\sigma}$ of $\mathbf{Z}(T)$. The elementary divisors of $\mathbf{Z}(T)^*$ are $(\lambda-\bar{\rho}_{(\sigma)})^{m_\sigma}$ (§I.1.12). The set of elementary divisors of the matrix $[\mathbf{Z}(T)^*]^{-1}$, and hence also of the matrix on the left of (3.28), consists of the polynomials $(\lambda-\bar{\rho}_{(\sigma)}^{-1})^{m_\sigma}$ (§II.2.1). We thus obtain an improved version of the Lyapunov-Poincaré theorem:

*Theorem. The multipliers of a Hamiltonian (canonical) equation (3.27) ((3.24)), allowing for their multiplicities and the structure of the elementary divisors, are symmetric (in the sense of inversion) about the unit circle.**

Remark. For equation (3.24) with complex matrices $\tilde{\mathbf{J}}$, $\tilde{\mathbf{H}}(t)$ satisfying the conditions

$$\tilde{\mathbf{J}}^\tau = -\tilde{\mathbf{J}}, \quad \det \tilde{\mathbf{J}} \neq 0, \quad \tilde{\mathbf{H}}(t+T) = \tilde{\mathbf{H}}(t) = \tilde{\mathbf{H}}(t)^\tau,$$

we have a proposition similar to the theorem of §3.4, with (3.20) replaced by the identity

$$\mathbf{X}(t)^\tau \tilde{\mathbf{J}} \mathbf{X}(t) \equiv \tilde{\mathbf{J}},$$

* This assertion remains valid for an arbitrary hermitian (or real symmetric) matrix-function $\mathbf{H}(t)$ (not necessarily T-periodic) in equation (3.27) (or (3.24)), if $T > 0$ is some fixed value of t, provided we replace the word "multipliers" by "roots of equation (3.26)." In this situation, equation (3.26) is no longer the characteristic equation, and its roots are not the multipliers in the sense of definition (2.5).

which implies (3.25). Together with any eigenvalue ρ, the monodromy matrix will then have an eigenvalue ρ^{-1} with the same structure of elementary divisors. (The spectrum of the corresponding equation is "skew-symmetric" about the unit circle; see Figure 2c). The characteristic equation (3.26) has the same symmetry property as before, though its coefficients are complex.

The Lyapunov-Poincaré theorem implies the following proposition:

The trivial solution of a canonical system (3.24) (Hamiltonian system (3.27)) is stable if and only if all the multipliers (i.e., roots of the characteristic equation (3.26)) lie on the unit circle and all elementary divisors of the matrix $\mathbf{X}(T)-\rho\mathbf{I}_{2k}$ *are simple (see table, p. 97).*

Indeed, if there is a multiplier ρ with $|\rho| \neq 1$, then by the Lyapunov-Poincaré theorem equation (3.24) also has a multiplier ρ^{-1} (and equation (3.27) a multiplier $\bar{\rho}^{-1}$). One of these is necessarily outside the unit circle, implying instability.

It follows from §2.7 that the trivial solutions of canonical and Hamiltonian systems of linear differential equations are never asymptotically stable, and that if all solutions of either system (3.24) or (3.27) are bounded for $0 < t < \infty$ they are also bounded for all $t < 0$.

3.6. Conservation of stability under small perturbations of the Hamiltonian

Theorem I. Suppose that a Hamiltonian equation (3.27) with Hamiltonian $\tilde{\mathbf{H}}_0(t)$ *has distinct multipliers, all on the unit circle. Then there exists* $\varepsilon > 0$ *such that the trivial solution of any equation (3.27) with Hamiltonian* $\tilde{\mathbf{H}}(t)$ *such that*

$$\| \tilde{\mathbf{H}} - \tilde{\mathbf{H}}_0 \| = \int_0^T |\tilde{\mathbf{H}}(t) - \tilde{\mathbf{H}}_0(t)|\, dt < \varepsilon \qquad (3.29)$$

is stable. *

Proof. Describe disjoint circles Γ_ν around each of the multipliers $\rho_\nu^{(0)}$ ($\nu = 1, \ldots, 2k$) of the original system. If the matrix $\tilde{\mathbf{H}}_0(t)$ changes to $\tilde{\mathbf{H}}(t)$ continuously in the sense of the norm (I.1.5), the corresponding change in the matrizant $\mathbf{Z}(t)$ is also continuous (§1.7).

Consequently, the monodromy matrix also changes continuously, and hence so do the multipliers (as the roots of the characteristic equation (3.26)). Thus, for sufficiently small $\varepsilon > 0$, the interior of each circle Γ_ν ($\nu = 1, \ldots, 2k$) will contain exactly one multiplier of the equation (3.27) with Hamiltonian $\tilde{\mathbf{H}}(t)$. By the symmetry properties of the spectrum, it can lie only on the unit circle. In fact, if there is a multiplier ρ, say, inside the unit circle (Figure 3), then, since the spectrum of $\mathbf{Z}(T)$ is symmetric with respect to the unit circle, if ε is sufficiently small there is another multiplier outside the unit circle, $\bar{\rho}^{-1}$. But this contradicts the

* If the matrices $\tilde{\mathbf{H}}(t)$ and $\tilde{\mathbf{H}}_0(t)$ are continuous, inequality (3.29) may be written more simply as $|\tilde{\mathbf{H}}(t) - \tilde{\mathbf{H}}_0(t)| < \varepsilon$ $(0 \leqslant t \leqslant T)$.

assumption that equation (3.27) has only one multiplier inside Γ. Thus all the multipliers of equation (3.27) are distinct and lie on the unit circle, proving Theorem I.

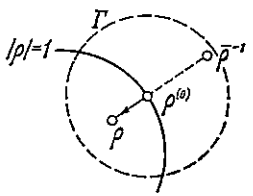


FIGURE 3.

Remark 1. We have assumed that when the matrix $\tilde{\mathbf{H}}_0(t)$ is changed to $\tilde{\mathbf{H}}(t)$ by a perturbation, equation (3.27) preserves its form; that is to say, the matrix $\tilde{\mathbf{H}}(t)$ (like $\tilde{\mathbf{H}}_0(t)$) is hermitian (or real symmetric) with T-periodic elements. If we allow perturbations under which equation (3.27) is no longer Hamiltonian the statement of the theorem is of course false.

Remark 2. The assumptions of Theorem I are satisfied if $\tilde{\mathbf{H}}_0(t) = \tilde{\mathbf{H}}_0$ is a constant matrix and the characteristic equation

$$\det\left[\tilde{\mathbf{J}}^{-1}\tilde{\mathbf{H}}_0 - \lambda \mathbf{I}_{2k}\right] = 0 \tag{3.30}$$

has only distinct pure imaginary roots which are not congruent modulo $2\pi i/T$:*

$$\lambda_{\pm\varkappa} = \pm i h_\varkappa \ (h_\varkappa > 0) \quad h_\varkappa \pm h_\mu \neq m\frac{2\pi}{T} \ (\varkappa, \mu = 1, \ldots, k;\ m = 1, 2, \ldots). \tag{3.31}$$

Indeed, the monodromy matrix of the equation

$$\frac{dz}{dt} = \tilde{\mathbf{J}}^{-1}\tilde{\mathbf{H}}_0 z \qquad (\tilde{\mathbf{H}}_0 = \text{const}) \tag{3.32}$$

is, as follows from §1.5,

$$\mathbf{Z}(T) = e^{T\tilde{\mathbf{J}}^{-1}\tilde{\mathbf{H}}_0}.$$

The multipliers $\rho_{\pm\varkappa}$ $(\varkappa = 1, \ldots, k)$ of equation (3.32) are the eigenvalues of the monodromy matrix $\mathbf{Z}(T)$ and so, if $\pm i h_\varkappa T$ are the eigenvalues of $T\tilde{\mathbf{J}}^{-1}\tilde{\mathbf{H}}_0$, then

$$\rho_{\pm\varkappa} = e^{\pm i h_\varkappa T} \qquad (\varkappa = 1, \ldots, k),$$

as follows from (I.2.7). All these multipliers lie on the unit circle and by (3.31) they are distinct, so that the assumptions of Theorem I hold.

* We are considering T-periodic perturbations of equation (3.27). In accordance with Theorem I, therefore, the constant Hamiltonian $\tilde{\mathbf{H}}$ is assigned a period T.

In applications one encounters systems of second-order linear differential equations with periodic coefficients, represented by the vector equation

$$\frac{d^2\mathbf{y}}{dt^2}+[\mathbf{P}_0+\varepsilon\mathbf{P}(t)]\,\mathbf{y}=0. \tag{3.33}$$

Here $\mathbf{y}$ is a column-vector of order k, $\mathbf{P}_0$ a real symmetric matrix with distinct positive eigenvalues $\sigma_1^2, \ldots, \sigma_k^2$, $\mathbf{P}(t)$ a symmetric matrix whose elements are real functions of period $T=2\pi/\sigma$, integrable and piecewise-continuous on $[0, T]$, and ε is a "small" real parameter.

Theorem II. If the numbers $\sigma_\varkappa>0$ satisfy the conditions

$$\sigma_\varkappa\pm\sigma_\mu\neq m\sigma \qquad (\varkappa, \mu=1, \ldots, k;\ m=1, 2, \ldots), \tag{3.34}$$

then for ε of sufficiently small absolute value all the solutions of system (3.33) are bounded on $(-\infty, \infty)$.

Proof. Together with equation (3.33), let us consider the equation

$$\frac{d^2\mathbf{y}}{dt^2}+\mathbf{P}_0\mathbf{y}=0. \tag{3.35}$$

As in §3.2, we write equations (3.33) and (3.35) as canonical equations (3.2), with the respective matrices

$$\mathbf{H}_0=\begin{pmatrix}\mathbf{P}_0 & 0\\ 0 & \mathbf{I}_k\end{pmatrix}, \quad \mathbf{H}(t)=\begin{pmatrix}\mathbf{P}_0+\varepsilon\mathbf{P}(t) & 0\\ 0 & \mathbf{I}_k\end{pmatrix},$$

and

$$\mathbf{x}=\begin{pmatrix}\mathbf{y}\\ \dot{\mathbf{y}}\end{pmatrix}=\begin{pmatrix}\mathbf{y}\\ \frac{d\mathbf{y}}{dt}\end{pmatrix}. \tag{3.36}$$

Let us find the multipliers of equation (3.35) (or of the corresponding equation (3.2)). By assumption,

$$\mathbf{P}_0\mathbf{a}_\varkappa=\sigma_\varkappa^2\mathbf{a}_\varkappa \qquad (\varkappa=1, \ldots, k),$$

where $\mathbf{a}_1, \ldots, \mathbf{a}_k$ are eigenvectors of $\mathbf{P}_0$. Obviously, the $2k$ vectors

$$\mathbf{y}_\varkappa^\pm(t)=e^{\pm i\sigma_\varkappa t}\mathbf{a}_\varkappa$$

are solutions of equation (3.35), and

$$\mathbf{y}_\varkappa^\pm(t+T)=e^{\pm i\sigma_\varkappa T}\mathbf{y}_\varkappa(t).$$

Thus the corresponding $2k$ vectors (3.36) satisfy (2.5) with

$$\rho=\rho_\varkappa^\pm=e^{\pm i\sigma_\varkappa T} \qquad (\varkappa=1, \ldots, k). \tag{3.37}$$

By §2.1, these are multipliers of equation (3.35). It follows from (3.34) that none of the numbers $\rho_\varkappa^\pm$ are equal, and since there are in all $2k$ these are all the multipliers.

By Theorem I, the trivial solution of system (3.35) is stable, provided that

$$\int_0^T |\mathbf{H}(t)-\mathbf{H}_0|\,dt_1=\int_0^T\left|\begin{pmatrix}\varepsilon\mathbf{P}(t) & 0\\ 0 & 0\end{pmatrix}\right|dt=|\varepsilon|\int|\mathbf{P}(t)\|\,dt$$

is sufficiently small. This proves Theorem II.

Note that we could have found the $\rho_\varkappa^\pm$ directly, as the eigenvalues of the matrix $\exp(T\mathbf{J}_{2k}^{-1}\mathbf{H}_0)$, after first finding eigenvectors $\mathbf{b}_\varkappa^\pm$ and eigenvalues $\lambda_\varkappa^\pm$ of the matrix $T\mathbf{J}_{2k}^{-1}\mathbf{H}_0$. It is easy to see that

$$\mathbf{b}_\varkappa^\pm=\begin{pmatrix}\mathbf{a}_\varkappa\\ \mathbf{a}_\varkappa^\pm\end{pmatrix},\quad \text{where}\quad \mathbf{a}_\varkappa^\pm=i\frac{1}{T}\lambda_\varkappa^\pm\mathbf{a}_\varkappa,\quad \lambda_\varkappa^\pm=\pm iT\sigma_\varkappa\qquad(\varkappa=1,\ldots,k),$$

and this again implies (3.37).

We return again to the corollary that follows from the symmetry of the spectrum of the monodromy matrix $\mathbf{X}(T)$ for canonical systems (3.24). If we deform the coefficient matrix $\mathbf{H}(t)$ continuously, preserving its form, the monodromy matrix $\mathbf{X}(T)$ and its eigenvalues (the multipliers of system (3.24)) will also undergo a continuous deformation, preserving their symmetry with respect to the unit circle and the real axis. This implies that the multipliers move over the complex plane in a special manner.

If all the multipliers of system (3.24) lie on the unit circle, and the deformation produces a nonreal multiplier of absolute value $\neq 1$, then there necessarily appear four multipliers ρ, $\bar\rho^{-1}$, $\bar\rho$, ρ^{-1}; in the case of a real multiplier $\neq\pm1$, we will have a pair of multipliers ρ, ρ^{-1}. And this means that the first new multipliers to appear will be multiple roots lying on the unit circle, produced by merging (coincidence) of several simple roots, for in general such roots may only move away from the unit circle.

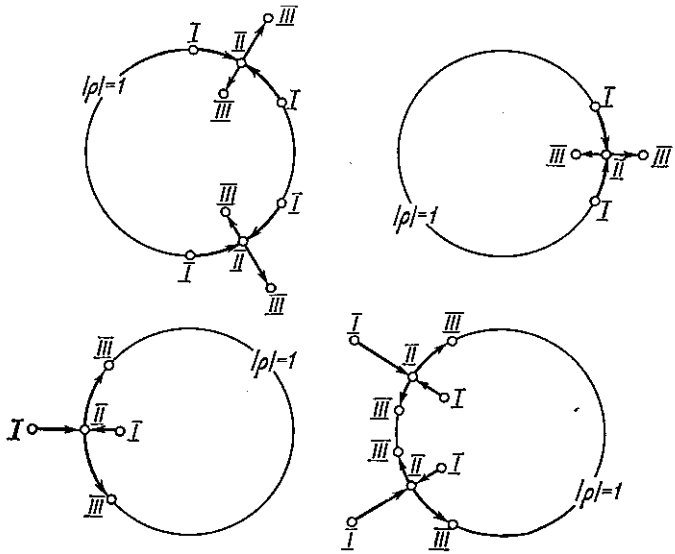


FIGURE 4.

This process is not always possible, and depends on the properties of the coinciding roots; this was first proved by Krein (§ III. 1.4).

The possible motions* of the multipliers of a fourth-order system (3.24) in the complex plane are shown in Figure 4; the numerals I, II, III indicate three successive positions of multipliers.

3.7. Poincaré variational equations and their properties

The system of equations (3.1) may be derived via another approach. Consider a holonomic mechanical system in which the applied forces admit a force function. Denote the coordinates and moments by $q_\varkappa$ and $p_\varkappa$ ($\varkappa = 1, \ldots, k$), and the Hamiltonian by $H(q_1, \ldots, q_k; p_1, \ldots, p_k)$. Our point of departure are the equations of motion in canonical (Hamiltonian) form

$$\frac{dq_\varkappa}{dt} = \frac{\partial H}{\partial p_\varkappa}, \quad \frac{dp_\varkappa}{dt} = -\frac{\partial H}{\partial q_\varkappa} \quad (\varkappa = 1, \ldots, k). \tag{3.38}$$

Suppose that system (3.38) has a T-periodic solution

$$q_\varkappa = q_\varkappa^{(0)}(t), \quad p_\varkappa = p_\varkappa^0(t) \qquad (\varkappa = 1, \ldots, k), \tag{3.39}$$

in some neighborhood of which the Hamiltonian H is holomorphic. Let us consider the stability of the corresponding motion with respect to the variables $q_1, \ldots, q_k; p_1, \ldots, p_k$. The values of the coordinates and momenta for the perturbed motion

$$q_\varkappa = q_\varkappa^{(0)}(t) + \xi_\varkappa, \quad p_\varkappa = p_\varkappa^{(0)}(t) + \xi_{k+\varkappa} \qquad (\varkappa = 1, \ldots, k) \tag{3.40}$$

satisfy the same system of differential equations (3.38). Substituting (3.40) into (3.38) and expanding the right-hand sides in Taylor series with respect to the perturbations $\xi_1, \ldots, \xi_k; \xi_{k+1}, \ldots, \xi_{2k+1}$, we obtain a system of differential equations for the perturbed motion:

$$\left.\begin{aligned} \frac{d\xi_\varkappa}{dt} &= \sum_{j=1}^{k}\left[\left(\frac{\partial^2 H}{\partial p_\varkappa \partial q_j}\right)_0 \xi_j + \left(\frac{\partial^2 H}{\partial p_\varkappa \partial p_j}\right)_0 \xi_{k+j}\right] + \Xi_\varkappa, \\ \frac{d\xi_{k+\varkappa}}{dt} &= -\sum_{j=1}^{k}\left[\left(\frac{\partial^2 H}{\partial q_\varkappa \partial q_j}\right)_0 \xi_j + \left(\frac{\partial^2 H}{\partial q_\varkappa \partial p_j}\right)_0 \xi_{k+j}\right] + \Xi_{k+\varkappa} \quad (\varkappa = 1, 2, \ldots, k). \end{aligned}\right\} \tag{3.41}$$

Here $\Xi_1, \ldots, \Xi_{2k}$ contain the perturbation terms of second and higher order. If they are dropped, we get a first approximation to the equations of the perturbed motion:

$$\left.\begin{aligned} \frac{d\xi_\varkappa}{dt} &= \sum_{j=1}^{k}\left[\left(\frac{\partial^2 H}{\partial p_\varkappa \partial q_j}\right)_0 \xi_j + \left(\frac{\partial^2 H}{\partial p_\varkappa \partial p_j}\right)_0 \xi_{k+j}\right], \\ \frac{d\xi_{k+\varkappa}}{dt} &= -\sum_{j=1}^{k}\left[\left(\frac{\partial^2 H}{\partial q_\varkappa \partial q_j}\right)_0 \xi_j + \left(\frac{\partial^2 H}{\partial q_\varkappa \partial p_j}\right)_0 \xi_{k+j}\right] \quad (\varkappa = 1, \ldots, k) \end{aligned}\right\} \tag{3.42}$$

* It can be shown /224m/ that to any motion of the multipliers that preserves their symmetry with respect to the unit circle and the real axis there corresponds some deformation of the matrix $\tilde{\mathbf{H}}(t)$.

called the Poincaré variational equations. In the case at hand (periodic unperturbed solution), (3.42) is a system of linear differential equations with periodic coefficients of type (3.1).

Thus the variational equations may be expressed in the form (3.2), with

$$\mathbf{H}(t)=\left\|\begin{array}{cccc}\left(\frac{\partial^2 H}{\partial q_1^2}\right)_0 \cdots \left(\frac{\partial^2 H}{\partial q_k \partial q_1}\right)_0 & \left(\frac{\partial^2 H}{\partial p_1 \partial q_1}\right)_0 \cdots & \left(\frac{\partial^2 H}{\partial p_k \partial q_1}\right)_0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & \\ \left(\frac{\partial^2 H}{\partial q_1 \partial q_k}\right)_0 \cdots \left(\frac{\partial^2 H}{\partial q_k^2}\right)_0 & \left(\frac{\partial^2 H}{\partial p_1 \partial q_k}\right)_0 \cdots & \left(\frac{\partial^2 H}{\partial p_k \partial q_k}\right)_0 \\ \left(\frac{\partial^2 H}{\partial q_1 \partial p_1}\right)_0 \cdots \left(\frac{\partial^2 H}{\partial q_k \partial p_1}\right)_0 & \left(\frac{\partial^2 H}{\partial p_1^2}\right)_0 \cdots & \left(\frac{\partial^2 H}{\partial p_k \partial p_1}\right)_0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & \\ \left(\frac{\partial^2 H}{\partial q_1 \partial p_k}\right)_0 \cdots \left(\frac{\partial^2 H}{\partial q_k \partial p_k}\right)_0 & \left(\frac{\partial^2 H}{\partial p_1 \partial p_k}\right)_0 \cdots & \left(\frac{\partial^2 H}{\partial p_k^2}\right)_0\end{array}\right\|,$$

where the zero subscript means that the partial derivatives are evaluated at the point (3.39).

Apart from properties I and II of § 3.2, which are valid for any system of type (3.2), the Poincaré variational equations also have certain other properties.

III. Equations (3.42) possess a periodic particular solution:

$$\xi_\varkappa=\dot{q}_\varkappa^{(0)}(t), \quad \xi_{k+\varkappa}=\dot{p}_\varkappa^{(0)}(t) \qquad (\varkappa=1, \ldots, k), \tag{3.43}$$

as may be verified by substituting $q_1^{(0)}(t), \ldots, q_k^{(0)}(t)$ and $p_1^{(0)}(t), \ldots, p_k^{(0)}(t)$ into system (3.38) and differentiating with respect to t.

IV. Let

$$\varphi(t, q_1, \ldots, q_k, p_1, \ldots, p_k)=a \tag{3.44}$$

be a first integral of the original canonical system (3.38), holomorphic in the neighborhood of the solution (3.39). Then the variation of (3.44),

$$\delta\varphi=\sum_{j=1}^{k}\left[\left(\frac{\partial\varphi}{\partial q_j}\right)_0\xi_j+\left(\frac{\partial\varphi}{\partial p_j}\right)_0\xi_{k+j}\right]$$

is a linear first integral of the variational equations (3.42); the symbols

$$\left(\frac{\partial\varphi}{\partial q_j}\right)_0, \qquad \left(\frac{\partial\varphi}{\partial p_j}\right)_0 \qquad (j=1, \ldots, k)$$

mean, as before, that the partial derivatives are evaluated at the point

$$q_\varkappa=q_\varkappa^{(0)}(t), \quad p_\varkappa=p_\varkappa^0(t) \qquad (\varkappa=1, \ldots, k).$$

Proof. For the perturbed motion, we have

$$\varphi(t, q_1^{(0)}+\xi_1, \ldots, q_k^{(0)}+\xi_k, p_1^{(0)}+\xi_{k+1}, \ldots, p_k^{(0)}+\xi_{2k})=a,$$

and expansion in powers of the perturbations gives

$$\varphi(t, q_1^{(0)}, \ldots, p_k^{(0)}) + \sum_{j=1}^{k} \left[\left(\frac{\partial \varphi}{\partial q_j} \right)_0 \xi_j + \left(\frac{\partial \varphi}{\partial p_j} \right)_0 \xi_{k+j} \right] + [2] = a,$$

where [2] stands for second- and higher-order terms. Since $q_1^{(0)}, \ldots, p_k^{(0)}$ is a solution of system (3.38), it follows that $\varphi(t, q_1^{(0)}, \ldots, p_k^{(0)}) = a_0$ is a constant, and we have

$$\delta\varphi + [2] = a - a_0.$$

Differentiating with respect to t, we have

$$\frac{d\delta\varphi}{dt} + \frac{d[2]}{dt} \equiv 0. \tag{3.45}$$

The values of the derivatives $d\xi_\nu/dt$ ($\nu = 1, \ldots 2k$) must be taken from the equations of the perturbed motion (3.41); for brevity, we write these as

$$\frac{d\xi_\nu}{dt} = L_\nu + \Xi_\nu \qquad (\nu = 1, \ldots, 2k),$$

where L_ν denote the terms linear in the perturbations. We may therefore write (3.45) as

$$\frac{\partial\delta\varphi}{\partial t} + \sum_{\nu=1}^{2k} \left[\frac{\partial\delta\varphi}{\partial\xi_\nu} (L_\nu + \Xi_\nu) \right] + \frac{d[2]}{dt} \equiv 0,$$

or, separating out the linear terms,

$$\frac{\partial\delta\varphi}{\partial t} + \sum_{\nu=1}^{2k} \frac{\partial\delta\varphi}{\partial\xi_\nu} L_\nu + [\tilde{2}] \equiv 0.$$

Since this is an identically vanishing holomorphic function, the terms of each order must vanish separately, so that

$$\frac{\partial\delta\varphi}{\partial t} + \sum_{\nu=1}^{2k} \frac{\partial\delta\varphi}{\partial\xi_\nu} L_\nu \equiv 0.$$

But the right-hand side here is the derivative of $\delta\varphi$ along the trajectories of the variational equations (3.42), which we have written

$$\frac{d\xi_\nu}{dt} = L_\nu \qquad (\nu = 1, \ldots, 2k),$$

so that $\delta\varphi$ is a linear first integral of equations (3.42), completing the proof (the proof follows Chetaev).

V. As a corollary of property IV and property II in §3.4, we see at once that if $\varphi(t, q_1, \ldots, q_k; p_1, \ldots, p_k) = a$ is a first integral of the original canonical system (3.38), then the functions

$$\left(\frac{\partial\varphi}{\partial p_1}\right)_0,\ \ldots,\ \left(\frac{\partial\varphi}{\partial p_k}\right)_0,\ -\left(\frac{\partial\varphi}{\partial q_1}\right)_0,\ \ldots,\ -\left(\frac{\partial\varphi}{\partial q_k}\right)_0 \tag{3.46}$$

form a particular solution of the variational equations (3.42). Since the equations of motion (3.38) have a T-periodic particular solution (3.39), the function φ on the left of (3.44) is periodic in the independent variable t, of period T/m (where m is a natural number). Thus the particular solution (3.46) is T-periodic.

It follows from Lyapunov's Theorem II on stability in the first approximation (/60/, Chapter III, subsection 55) that if not all the multipliers (roots of the characteristic equation; see § 2.1) of the variational equations (3.42) lie on the unit circle, then the unperturbed motion (3.39) is unstable. Generally speaking, however, stability of the trivial solution of the Poincaré variational equations does not imply stability of the unperturbed motion (/60/, Chapter III, subsections 56—65).

3.8. Example: flexible shaft with heavy symmetrically attached flywheel

As an example (to be investigated further in §§ IV.2.5, IV.5.6), let us consider a horizontal flexible shaft (absolutely rigid in torsion) with a heavy flywheel attached midway between its supports. When the shaft is bent, the plane of the orthogonally attached flywheel remains unchanged. In this plane we fix a coordinate system Oxy with origin O at the intersection of the unbent shaft with the plane and horizontal axis Ox. Let S be the center of gravity of the flywheel, W the point at which it is attached to the shaft, ψ the angle of rotation of the flywheel, $SW = e$ the eccentricity, m and k the mass and radius of inertia, respectively, of the flywheel, $K = \alpha OW$ the elastic force due to bending of the flywheel (Figure 5). We shall assume that the angular velocity ω_0 of the shaft is less than the critical angular velocity*

$$\omega_k = \sqrt{\frac{\alpha}{m}}$$

and constant. This will be the case if the resultant of the external torques and all resistances is zero. It remains to add that if, say, the inclination of the shaft at the supports varies freely, then

$$\alpha = \frac{48EJ}{(2l)^3},$$

where J is the equatorial moment of inertia of the shaft, E the elastic modulus and $2l$ the length of the shaft.

* The angular velocity is said to be critical if near it the deflection of the shaft becomes infinite.

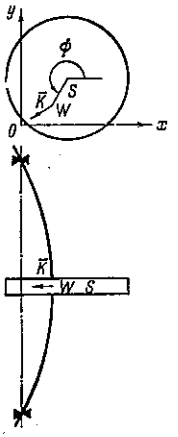


FIGURE 5.

The flywheel has three degrees of freedom. As generalized coordinates we take the polar coordinates q_1, q_2 of the point S and the angle $q_3=\psi-q_2$ (Figure 5). The kinetic and potential energies of the flywheel are

$$K=\frac{1}{2}m(\dot{q}_1^2+q_1^2\dot{q}_2^2)+\frac{1}{2}mk^2\dot{\psi}^2,\quad \Pi=\frac{1}{2}\alpha OW^2+mgq_1\sin q_2.$$

Replacing OW^2 and ψ by their values $OW^2=q_1^2+e^2+2eq_1\cos q_3$, $\psi=q_2+q_3$, and dropping the inessential constant term in Π we obtain

$$K=\frac{1}{2}m\dot{q}_1^2+\frac{1}{2}m(k^2+q_1^2)\dot{q}_2^2+mk^2\dot{q}_2\dot{q}_3+\frac{1}{2}mk^2\dot{q}_3^2,$$
$$\Pi=\frac{1}{2}\alpha(q_1^2+2eq_1\cos q_3)+mgq_1\sin q_2.$$

The momenta $p_i=\partial K/\partial\dot{q}_i$ $(i=1, 2, 3)$ are

$$p_1=m\dot{q}_1,\quad p_2=m[(k^2+q_1^2)\dot{q}_2+k_2\dot{q}_3],\quad p_3=mk^2(\dot{q}_2+\dot{q}_3).$$

In the case considered here, the system does not depend explicitly on time, and so the Hamiltonian H is equal to the total energy $K+\Pi$ of the system, in which the derivatives of the generalized coordinates are expressed in terms of the momenta:

$$2H=\frac{1}{m}\left[p_1^2+\frac{1}{q_1^2}p_2^2+\left(\frac{1}{k^2}+\frac{1}{q_1^2}\right)p_3^2-\frac{2}{q_1^2}p_2p_3\right]+m\omega_k^2(q_1^2+2eq_1\cos q_3)+2mgq_1\sin q_2.$$

The canonical Hamilton's equations (3.38) are

$$\left.\begin{aligned}&\frac{dq_1}{dt}=\frac{1}{m}p_1,\quad \frac{dq_2}{dt}=\frac{1}{mq_1^2}(q_2-q_3),\quad \frac{dq_3}{dt}=\frac{k^2+q_1^2}{mk^2q_1^2}p_3-\frac{1}{mq_1^2}p_2,\\&\frac{dp_1}{dt}=-m\omega_k^2(q_1+e\cos q_3)-mg\sin q_2+\frac{1}{mq_1^3}(p_2-p_3)^2,\\&\frac{dp_2}{dt}=-mgq_1\cos q_2,\quad \frac{dp_3}{dt}=m\omega_k^2eq_1\sin q_3.\end{aligned}\right\}\qquad(3.47)$$

Stodola /113/ found a particular solution of the equations of motion of a heavy flywheel, which for system (3.47) is

$$\left.\begin{aligned} &q_1 = q_{10}(t) = \frac{4}{3}e - \frac{2g}{\omega_k^2}\sin\frac{1}{2}\omega_k t, \quad q_2 = \frac{1}{2}\omega_k t, \quad q_3 = \pi, \\ &p_1 = -\frac{mg}{\omega_k}\cos\frac{1}{2}\omega_k t, \quad p_2 = \frac{1}{2}m\omega_k\left[k^2 + \left(\frac{4}{3}e - \frac{2g}{\omega_k^2}\sin\frac{1}{2}\omega_k t\right)^2\right], \\ &p_3 = \frac{1}{2}m\omega_k k^2. \end{aligned}\right\} \qquad (3.48)$$

Taking (3.48) as the equations of unperturbed motion and proceeding to investigate their stability with respect to the variables q_1, q_2, q_3, p_1, p_2, p_3, we obtain the Poincaré variational equations corresponding to (3.47):

$$\left.\begin{aligned} &\frac{d\xi_1}{dt} = \frac{1}{m}\xi_4, \quad \frac{d\xi_2}{dt} = -\frac{\omega_k}{q_{10}(t)}\xi_1 + \frac{1}{mq_{10}^2(t)}\xi_5 - \frac{1}{mq_{10}^2(t)}\xi_6, \\ &\frac{d\xi_3}{dt} = \frac{\omega_k}{q_{10}(t)}\xi_1 - \frac{1}{mq_{10}^2(t)}\xi_2 + \frac{1}{mk^2}\left(1 + \frac{k^2}{q_{10}^2(t)}\right)\xi_6, \\ &\frac{d\xi_4}{dt} = -\frac{7}{4}m\omega_k^2\xi_1 - mg\cos\frac{1}{2}\omega_k t\,\xi_2 + \frac{\omega_k}{q_{10}(t)}\xi_5 - \frac{\omega_k}{q_{10}(t)}\xi_6, \\ &\frac{d\xi_5}{dt} = -mg\cos\frac{1}{2}\omega_k t\,\xi_1 + mgq_{10}(t)\sin\frac{1}{2}\omega_k t\,\xi_2, \quad \frac{d\xi_6}{dt} = -m\omega_k^2 e q_{10}(t)\,\xi_3. \end{aligned}\right\} \qquad (3.49)$$

To transform system (3.49) to the form (3.2), we compute the matrix $\mathbf{H}(t) = \mathbf{J}_6^{-1}\mathbf{A}(t) = -\mathbf{J}_6\mathbf{A}(t)$, where $\mathbf{A}(t)$ is the coefficient matrix of (3.49). This gives the following expression for $\mathbf{H}(t)$:

$$\begin{Vmatrix} \frac{7}{4}m\omega_k^2 & mg\cos\frac{1}{2}\omega_k t & 0 & 0 & -\frac{\omega_k}{q_{10}(t)} & \frac{\omega_k}{q_{10}(t)} \\ mg\cos\frac{1}{2}\omega_k t & -mgq_{10}(t)\sin\frac{1}{2}\omega_k t & 0 & 0 & 0 & 0 \\ 0 & 0 & m\omega_k^2 e q_{10}(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{m} & 0 & 0 \\ -\frac{\omega_k}{q_{10}(t)} & 0 & 0 & 0 & \frac{1}{mq_{10}^2(t)} & -\frac{1}{mq_{10}^2(t)} \\ \frac{\omega_k}{q_{10}(t)} & 0 & 0 & 0 & -\frac{1}{mq_{10}^2(t)} & \frac{1}{mk^2}\left(1 + \frac{k^2}{q_{10}^2(t)}\right) \end{Vmatrix}. \qquad (3.50)$$

It follows from the expression for $q_{10}(t)$ that the elements of $\mathbf{H}(t)$ are periodic of period $T = 4\pi$, continuous if $\frac{4}{3}e > 2g/\omega_k^2$ and piecewise-continuous if $\frac{4}{3}e \leqslant 2g/\omega_k^2$, i. e., if the eccentricity is small enough. In the latter case, however, regardless of the infinite discontinuity, the coefficients are integrable.

We shall show that by testing for stability with respect to certain other variables, we may get the equations of perturbed motion with continuous coefficients. The equations of motion of the center of gravity $S(x_s, y_s)$ of the flywheel and the equation of rotation about the center of gravity are

$$m\ddot{x}_s = -\alpha(x_s + e\cos\psi), \quad m\ddot{y}_s = -\alpha(y_s + e\sin\psi) - mg,$$
$$mk^2\ddot{\psi} = \alpha e(x_s\sin\psi - y_s\cos\psi).$$

Introducing dimensionless variables

$$x_1=\frac{1}{k}x_s,\quad x_2=\frac{1}{k}\left(y_s+\frac{g}{\omega_k^2}\right)$$

and dimensionless time $\tau=\omega_k t$, we write the equations of motion as

$$x_1''+x_1+\varepsilon\cos\psi=0,\quad x_2''+x_2+\varepsilon\sin\psi=0,\quad \psi''-\varepsilon x_1\sin\psi+\varepsilon x_2\cos\psi=\varepsilon\delta\cos\psi; \quad (3.51)$$

here we have two dimensionless parameters

$$\varepsilon=\frac{e}{k},\quad \delta=\frac{g}{k\omega_k^2},$$

and the prime denotes differentiation with respect to τ.

Stoloda's particular solution (3.48) for system (3.51) is

$$x_1^{(0)}(\tau)=\frac{4}{3}\varepsilon\cos\frac{1}{2}\tau-\delta\sin\tau,\quad x_2^{(0)}(\tau)=\frac{4}{3}\varepsilon\sin\frac{1}{2}\tau+\delta\cos\tau,\quad \psi^{(0)}(\tau)=\pi+\frac{\tau}{2}. \quad (3.52)$$

It is significant that a particular solution of this type exists only at one value of the angular velocity of the shaft — half the critical value. The trajectory of the flywheel's center of gravity S is a limaçon of Pascal whose precise shape depends on the parameters ε and δ.

To test for stability with respect to the variables $x_1, x_2, \psi, x_1', x_2', \psi'$, we put

$$x_1=x_1^{(0)}(\tau)+\eta_1,\quad x_2=x_2^{(0)}(\tau)+\eta_2,\quad \psi=\psi^{(0)}(\tau)+\eta_3.$$

Substituting into (3.51) and dropping terms of second and higher order in the perturbations, we obtain the differential equations of the perturbed motion in the first approximation (variational equations):

$$\left.\begin{aligned}&\frac{d^2\eta_1}{d\tau^2}+\eta_1+\varepsilon\eta_3\sin\frac{1}{2}\tau=0,\qquad \frac{d^2\eta_2}{d\tau^2}+\eta_2-\varepsilon\eta_3\cos\frac{1}{2}\tau=0,\\&\frac{d^2\eta_3}{d\tau^2}+\varepsilon\eta_1\sin\frac{1}{2}\tau-\varepsilon\eta_2\cos\frac{1}{2}\tau+\varepsilon\left(\frac{4}{3}\varepsilon-2\delta\sin\frac{1}{2}\tau\right)\eta_3=0.\end{aligned}\right\} \quad (3.53)$$

It is natural to write equations (3.53) in the form (3.11), with

$$\mathbf{y}=\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix},\quad P(\tau)=\begin{Vmatrix}1 & 0 & \varepsilon\sin\frac{1}{2}\tau\\ 0 & 1 & -\varepsilon\cos\frac{1}{2}\tau\\ \varepsilon\sin\frac{1}{2}\tau & -\varepsilon\cos\frac{1}{2}\tau & \varepsilon\left(\frac{4}{3}\varepsilon-2\delta\sin\frac{1}{2}\tau\right)\end{Vmatrix}. \quad (3.54)$$

3.9. Systems of linear differential equations invariant under time reversal (t-invariant)

We shall show that analogs of the Lyapunov-Poincaré theorem (§3.5) are valid for certain other (not necessarily canonical or Hamiltonian) systems. We again consider the vector equation

$$\boxed{\frac{dx}{dt} = \mathbf{A}(t)\,\mathbf{x},} \tag{3.55}$$

where $\mathbf{A}(t)$ is a T-periodic $n \times n$ matrix-function:

$$\boxed{\mathbf{A}(t+T) = \mathbf{A}(t)} \tag{3.56}$$

with integrable piecewise-continuous elements (generally complex-valued). Transform variables in (3.56) by

$$t = -\tau, \quad \mathbf{x} = \mathbf{G}\mathbf{z} \qquad \left(\mathbf{z} = \begin{pmatrix} \zeta_1 \\ \dots \\ \zeta_n \end{pmatrix},\ \mathbf{G} = \|g_{ij}\|_1^n\right), \tag{3.57}$$

where $\mathbf{G}$ is some nonsingular constant $n \times n$ matrix. The vector $\mathbf{z}$ satisfies the equation

$$\frac{d\mathbf{z}}{d\tau} = -\mathbf{G}^{-1}\mathbf{A}(-\tau)\,\mathbf{G}\mathbf{z}.$$

We demand that this equation be identical to the original equation (3.55), i. e.,

$$\mathbf{A}(t) \equiv -\mathbf{G}^{-1}\mathbf{A}(-t)\,\mathbf{G}$$

or

$$\boxed{\mathbf{A}(-t)\,\mathbf{G} + \mathbf{G}\mathbf{A}(t) \equiv \mathbf{0}.} \tag{3.58}$$

The vector equation (3.55) is said to be invariant under time reversal (or t-invariant) if there exists a nonsingular constant matrix $\mathbf{G}$ such that equation (3.55) is invariant under the substitution (3.57), in other words, such that the identity (3.58) is valid.

Let us determine what restrictions must be imposed on the matrix $\mathbf{A}(t)$ in order to satisfy conditions (3.56) and (3.58). Since (3.58) is valid for all t, substitution of $-t$ for t shows that, again for all t,

$$\mathbf{A}(-t) = -\mathbf{G}^{-1}\mathbf{A}(t)\,\mathbf{G}$$

and, again substituting into (3.58), we see that

$$\mathbf{A}(t)\,\mathbf{G}^2 = \mathbf{G}^2\mathbf{A}(t). \tag{3.59}$$

It follows from (3.56) and (3.58) that in order to evaluate $\mathbf{A}(t)$ for any t we need only know its values on the interval $[0, T/2]$. Indeed, from (3.58) we obtain the values on $[-T/2, 0]$ and (3.56) then yields the values on the whole t-axis. Thus the matrix $\mathbf{A}(t)$ may be defined arbitrarily on $[0, T/2]$ (provided (3.59) is satisfied), and then extended to the entire t-axis by (3.56) and (3.58). Usually, $\mathbf{G}^2 = \text{const} \cdot \mathbf{I}$; the definition of $\mathbf{A}(t)$ on $[0, T/2]$ is then entirely arbitrary.

As we saw in §2.1, condition (3.56) is equivalent to

$$\mathbf{X}(t+T) \equiv \mathbf{X}(t)\,\mathbf{X}(T) \tag{3.60}$$

where $\mathbf{X}(t)$ is the matrizant of equation (3.55). We claim that (3.58) is equivalent to

$$\mathbf{X}(-t) \equiv \mathbf{G}^{-1}\mathbf{X}(t)\,\mathbf{G}. \tag{3.61}$$

Indeed, the matrix $\mathbf{X}_1(t) = \mathbf{G}\mathbf{X}(-t)\,\mathbf{G}^{-1}$ satisfies the equation

$$\frac{d\mathbf{X}}{dt} = \mathbf{A}(t)\,\mathbf{X}$$

with initial condition $\mathbf{X}_1(0) = \mathbf{I}_n$. Consequently, $\mathbf{X}_1(t) \equiv \mathbf{X}(t)$, and we have the identity (3.61).

Conversely, let $\mathbf{X}(t)$ be a nonsingular continuous matrix-function with integrable piecewise-continuous derivative $\mathbf{X}(0) = \mathbf{I}_n$, and let (3.61) hold. Then (§2.1) $\mathbf{X}(t)$ is the matrizant of an equation (3.55) with coefficient matrix

$$\mathbf{A}(t) = \frac{d\mathbf{X}(t)}{dt}\,\mathbf{X}(t)^{-1}.$$

It follows from (3.61) that

$$\mathbf{A}(-t) = -\frac{d\mathbf{X}(-t)}{dt}\,\mathbf{X}(-t)^{-1} = -\mathbf{G}\frac{d\mathbf{X}(t)}{dt}\,\mathbf{X}(t)^{-1}\,\mathbf{G}^{-1}$$

and thus we see that $\mathbf{A}(t)$ satisfies (3.58).

Like $\mathbf{A}(t)$, the matrizant $\mathbf{X}(t)$ may be defined at will on the interval $(0, T/2)$ (provided this is done in such a way that $\mathbf{X}(t)$ has the properties stipulated above). Conditions (3.60) and (3.61) then define $\mathbf{X}(t)$ on the entire t-axis.

The identities (3.60) and (3.61) yield two equalities which we shall need later. First, setting $t = -T/2$ we find from (3.60) that

$$\mathbf{X}(T) = \mathbf{X}\left(-\tfrac{1}{2}T\right)^{-1}\mathbf{X}\left(\tfrac{1}{2}T\right),$$

and, using (3.61), we obtain

$$\mathbf{X}(T) = \mathbf{G}\mathbf{X}\left(\tfrac{1}{2}T\right)^{-1}\mathbf{G}^{-1}\mathbf{X}\left(\tfrac{1}{2}T\right). \tag{3.62}$$

Second, setting $t = -T$ in (3.60) and using (3.61), we obtain

$$\mathbf{X}(T)^{-1} = \mathbf{G}\mathbf{X}(T)\,\mathbf{G}^{-1}. \tag{3.63}$$

This implies the following result.

Lyapunov's theorem (/60/, Chapter III, subsection 51). If ρ is an eigenvalue of the monodromy matrix $\mathbf{X}(T)$ of a t-invariant system, then ρ^{-1} is also an eigenvalue, with the same structure of elementary divisors. Consequently, the spectrum of the monodromy matrix of a t-invariant system is skew-symmetric with respect to the unit circle.

If $\mathbf{A}(t)$ is a real matrix-function, then the matrizant $\mathbf{X}(t)$ is also real; thus, together with each eigenvalue ρ, it also has an eigenvalue $\bar{\rho}$, and so:

The spectrum of the monodromy matrix of a real t-invariant system is symmetric with respect to both the unit circle and the real axis.

In either case, the characteristic equation is reversible, so that the Lyapunov-Poincaré theorem (§3.5) is valid for t-invariant systems.

The next theorem, which follows from the Lyapunov-Poincaré theorem, is analogous to Theorem I of §3.6.

Theorem. Suppose that the characteristic equation (2.3) of the system

$$\frac{d\mathbf{x}}{dt}=\mathbf{A}_0(t)\,\mathbf{x} \qquad (\mathbf{A}_0(t+T)=\mathbf{A}_0(t)) \tag{3.64}$$

is reversible and that all its roots (the multipliers of the system) lie on the unit circle and are distinct. Then there exists $\varepsilon>0$ such that all solutions of the system

$$\frac{d\mathbf{x}}{dt}=\mathbf{A}(t)\,\mathbf{x} \qquad (\mathbf{A}(t+T)=\mathbf{A}(t))$$

are bounded for all $t>0$ provided the characteristic equation of the latter system is reversible and moreover

$$\int_0^T |\mathbf{A}(t)-\mathbf{A}_0(t)|\,dt<\varepsilon.$$

Example. Consider the system of second-order linear differential equations

$$\frac{d^2\eta_i}{dt^2}+q_{i1}(t)\frac{d\eta_1}{dt}+\ldots+q_{ik}(t)\frac{d\eta_k}{dt}+p_{i1}(t)\,\eta_1+\ldots+p_{ik}(t)\,\eta_k=0 \qquad (i=1,\ \ldots,\ k),$$

where the functions $q_{ij}(t)$, $p_{ij}(t)$ $(i,\ j=1,\ \ldots,\ k)$ are complex-valued, integrable, piecewise-continuous and T-periodic, with

$$q_{ij}(-t)=-q_{ij}(t), \quad p_{ij}(-t)=p_{ij}(t) \qquad (0\leqslant t\leqslant T;\ i,\ j=1,\ \ldots,\ k).$$

This system may be expressed as a second-order vector equation

$$\frac{d^2\mathbf{y}}{dt^2}+\mathbf{Q}(t)\frac{d\mathbf{y}}{dt}+\mathbf{P}(t)\,\mathbf{y}=\mathbf{0}, \tag{3.65}$$

where $\mathbf{y}=\begin{pmatrix}\eta_1\\ \vdots\\ \eta_k\end{pmatrix}$ and the matrices $\mathbf{Q}(t)=\|q_{ij}(t)\|_1^k$, $\mathbf{P}(t)=\|p_{ij}(t)\|_1^k$ satisfy the conditions

$$\mathbf{Q}(-t)=-\mathbf{Q}(t), \quad \mathbf{P}(-t)=\mathbf{P}(t). \tag{3.66}$$

Note that in general the matrices $\mathbf{Q}(t)$ and $\mathbf{P}(t)$ are not symmetric.

Equation (3.65) remains invariant under the substitution $t \to -t$. If we write equation (3.65) in the form (3.55), where

$$\mathbf{x}=\begin{pmatrix} \mathbf{y} \\ \frac{d\mathbf{y}}{dt} \end{pmatrix}, \quad \mathbf{A}(t)=\begin{pmatrix} \mathbf{0} & \mathbf{I}_k \\ -\mathbf{P}(t) & -\mathbf{Q}(t) \end{pmatrix},$$

then the substitution $t=-\tau$ corresponds to

$$\mathbf{x}(t)=\begin{pmatrix} \mathbf{y}(-\tau) \\ -\frac{d\mathbf{y}(-\tau)}{d\tau} \end{pmatrix}=\begin{pmatrix} \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_k \end{pmatrix}\mathbf{z}(\tau),$$

where

$$\mathbf{z}(t)=\begin{pmatrix} \mathbf{y}(-\tau) \\ \frac{d\mathbf{y}(-\tau)}{d\tau} \end{pmatrix}.$$

In other words, equation (3.65), expressed as (3.55), satisfies the identity (3.58) with

$$\mathbf{G}=\begin{pmatrix} \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_k \end{pmatrix}.$$

Thus, equation (3.65) is t-invariant.

§4. ESTIMATES FOR THE CHARACTERISTIC EXPONENTS OF A SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

4.1. Theorem on estimates for characteristic exponents

As usual, we consider a system of linear differential equations in vector notation

$$\frac{d\mathbf{x}}{dt}=\mathbf{A}(t)\mathbf{x}, \tag{4.1}$$

where $\mathbf{x}$ is an n-vector, $\mathbf{A}(t+T)=\mathbf{A}(t)$ an $n\times n$ matrix-function whose elements are T-periodic integrable piecewise-continuous functions. The components of $\mathbf{x}$ and elements of $\mathbf{A}(t)$ are generally complex-valued functions.

The aim of this section is to establish effective estimates for the characteristic exponents of system (4.1), i. e., estimates that can be evaluated directly in terms of the coefficients of system (4.1).

We shall say that an $n\times n$ matrix-function $\mathbf{G}(t)$ satisfies conditions (A) if it is hermitian $(\mathbf{G}^*(t)=\mathbf{G}(t))$, T-periodic, differentiable and positive definite. The last condition means that $(\mathbf{G}(t)\mathbf{a}, \mathbf{a})>0$ for any vector $\mathbf{a}\neq 0$ and any t. *

If α is a characteristic exponent of system (4.1), the latter has a solution $\mathbf{x}=e^{\alpha t}\mathbf{u}(t)$, where $\mathbf{u}(t)$ is a periodic vector, $\mathbf{u}(t+T)=\mathbf{u}(t)$ (§ 2.5). Denote $\xi(t)=(\mathbf{G}(t)\mathbf{x}, \mathbf{x})$. Then

$$\left.\begin{aligned}\xi(t)&=e^{(\alpha+\overline{\alpha})t}(\mathbf{Gu}, \mathbf{u})=e^{2t\operatorname{Re}\alpha}(\mathbf{Gu}, \mathbf{u}),\\ \ln\xi(t)&=2t\operatorname{Re}\alpha+\ln(\mathbf{Gu}, \mathbf{u}).\end{aligned}\right\} \tag{4.2}$$

The function $(\mathbf{Gu}, \mathbf{u})$ is positive, T-periodic and continuous. Consequently, $\ln(\mathbf{Gu}, \mathbf{u})$ is bounded. Hence it follows from (4.2) that**

$$\operatorname{Re}\alpha=\frac{1}{2}\lim_{t\to\infty}\frac{\ln\xi(t)}{t}. \tag{4.3}$$

This readily implies estimates for $\operatorname{Re}\alpha$. Indeed, for the above solution $\mathbf{x}$ it follows from (4.1) that

$$\dot{\xi}=(\dot{\mathbf{G}}\mathbf{x}, \mathbf{x})+(\mathbf{GAx}, \mathbf{x})+(\mathbf{Gx}, \mathbf{Ax})=(\mathbf{Qx}, \mathbf{x}),$$

where

$$\mathbf{Q}=\dot{\mathbf{G}}+\mathbf{GA}+\mathbf{A}^*\mathbf{G}\qquad(\mathbf{Q}^*=\mathbf{Q}). \tag{4.4}$$

Let $q_1(t)$ and $q_2(t)$ denote the smallest and largest roots of the equation

$$\det(\mathbf{Q}-q\mathbf{G})=0 \tag{4.5}$$

($q_1(t)$ and $q_2(t)$ are real T-periodic functions). Then (see §I.1.11)

$$q_1(\mathbf{y}, \mathbf{y})\leqslant(\mathbf{QG}^{-1/2}\mathbf{y}, \mathbf{G}^{-1/2}\mathbf{y})\leqslant q_2(\mathbf{y}, \mathbf{y})$$

and, setting $\mathbf{y}=\mathbf{G}^{1/2}\mathbf{x}$, we get

$$q_1(t)\leqslant\frac{1}{\xi}\dot{\xi}\leqslant q_2(t). \tag{4.6}$$

Consequently,

* If $\mathbf{G}(t)=\|g_{ij}(t)\|_1^n$, these conditions mean that $g_{ij}(t+T)=g_{ij}(t)$ are differentiable functions, $\overline{g_{ji}}(t)=g_{ij}(t)$ and $\Delta_i(t)>0$, where $\Delta_i(t)$ are the principal minors of $\mathbf{G}(t)$ $(i, j=1, \ldots, n;\ 0\leqslant t\leqslant T)$. If $\mathbf{F}(t)$ is a differentiable T-periodic matrix-function, the matrix-function $\mathbf{G}(t)=\mathbf{F}^*(t)\mathbf{F}(t)$ satisfies the conditions. The relation $\mathbf{G}=(\mathbf{U}\sqrt{\mathbf{G}})^*\mathbf{U}\sqrt{\mathbf{G}}$, where $\mathbf{U}$ is any T-periodic unitary matrix-function, shows that any matrix-function $\mathbf{G}(t)$ satisfying conditions (A) may be expressed in infinitely many ways as $\mathbf{G}=\mathbf{F}^*\mathbf{F}$.

** The numbers $-\operatorname{Re}\alpha$ are obviously the Lyapunov characteristic numbers [eigenvalues] of system (4.1).

$$\int_0^t q_1(t_1)\,dt_1 \leqslant \ln\xi - \ln\xi_0 \leqslant \int_0^t q_2(t_1)\,dt_1. \tag{4.6a}$$

For any T-periodic function $q(t)$,

$$\int_0^t q(t_1)\,dt_1 = \frac{t}{T}\int_0^T q(t)\,dt + r(t),$$

where $r(t)$ is some T-periodic function. Therefore

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t q(t_1)\,dt_1 = \frac{1}{T}\int_0^T q(t)\,dt.$$

Dividing (4.6a) by $2t$ and letting $t\longrightarrow\infty$, we obtain

$$\frac{1}{2T}\int_0^T q_1(t)\,dt \leqslant \operatorname{Re}\alpha \leqslant \frac{1}{2T}\int_0^T q_2(t)\,dt. \tag{4.7}$$

Thus, if $\mathbf{G}(t)$ is an arbitrary matrix-function satisfying conditions (A) and the functions $q_1(t)$, $q_2(t)$ are defined as above, then the characteristic exponents of system (4.1) satisfy the estimate (4.7). *

The expressions $\frac{1}{2T}\int_0^T q_i(t)\,dt$ $(i=1, 2)$ in (4.7) clearly depend on the choice of $\mathbf{G}(t)$.

The next theorem shows that by suitable choice of $\mathbf{G}(t)$ the estimate (4.7) may be made arbitrarily sharp.

Theorem. Let α_1 and α_n be characteristic exponents of system (4.1) with the minimum and maximum real parts, respectively. For any $\varepsilon>0$, there is a matrix-function $\mathbf{G}(t)$ satisfying conditions (A) such that

$$\frac{1}{2T}\int_0^T q_1\,dt \leqslant \operatorname{Re}\alpha_1 < \frac{1}{2T}\int_0^T q_1\,dt+\varepsilon,\quad \frac{1}{2T}\int_0^T q_2\,dt-\varepsilon < \operatorname{Re}\alpha_n \leqslant \frac{1}{2T}\int_0^T q_2\,dt. \tag{4.8}$$

If the coefficient matrix $\mathbf{A}(t)$ is real, then $\mathbf{G}(t)$ may also be assumed real.

Proof. Perform the substitution

$$\mathbf{x} = \mathbf{P}(t)\,\mathbf{y} \tag{4.9}$$

in equation (4.1), where $\mathbf{P}(t)$ is a nonsingular differentiable matrix-function.

* The equality sign in (4.6) is attained for vectors $\mathbf{x}_1$, $\mathbf{x}_2$ such that $\mathbf{Q}\mathbf{x}_i \equiv q_i\mathbf{G}\mathbf{x}_i$ $(i=1, 2)$, which are moreover solutions of equation (4.1). Otherwise (as is usually the case), the sign $\leqslant$ in (4.7) should be replaced by $<$.

Note that the entire reasoning carries over to an arbitrary reducible system. Instead of (4.7), we then have

$$\frac{1}{2}\overline{\lim}\,\frac{1}{t}\int_0^t q_1(t_1)\,dt_1 \leqslant \operatorname{Re}\alpha \leqslant \frac{1}{2}\underline{\lim}\,\frac{1}{t}\int_0^t q_2(t_1)\,dt_1.$$

The result is

$$\frac{dy}{dt} = \mathbf{B}(t)\,\mathbf{y} \tag{4.10}$$

where

$$\mathbf{B}(t) = \mathbf{P}(t)^{-1}\,\mathbf{A}(t)\,\mathbf{P}(t) - \mathbf{P}(t)^{-1}\,\frac{d\mathbf{P}}{dt}. \tag{4.11}$$

If $\mathbf{P}(t+T) \equiv \mathbf{P}(t)$, then clearly also $\mathbf{B}(t+T) \equiv \mathbf{B}(t)$. For any solution $\mathbf{x}(t)$ of type $\mathbf{x} = \mathbf{u}(t)e^{\lambda t}$, where $\mathbf{u}(t+T) \equiv \mathbf{u}(t)$, we have a solution $\mathbf{y}(t) = \mathbf{v}(t)e^{\lambda t}$, where $\mathbf{v}(t+T) \equiv \mathbf{v}(t)$, and conversely. Thus equations (4.1) and (4.10) have the same characteristic exponents. *

It may happen that $\mathbf{B}(t+T) \equiv \mathbf{B}(t)$ although $\mathbf{P}(t)$ is not T-periodic. (This is the case, for example, when $\mathbf{A}(t)$ is a real matrix, $\mathbf{B}(t) \equiv \mathbf{K}$ is a constant real matrix, and some of the multipliers of equation (4.1) are real and negative; see §2.3.) However, if $\mathbf{P}(t)$ and $\mathbf{P}(t)^{-1}$ are bounded, then, repeating the simple reasoning given above, we see that the real parts of the characteristic exponents of equations (4.1) and (4.10) coincide.

We now consider the connection between the estimates (4.7) for equations (4.1) and (4.10).

The forms $\xi = (\mathbf{G}\mathbf{x}, \mathbf{x})$, $\dot{\xi} = (\mathbf{Q}\mathbf{x}, \mathbf{x})$ become $\xi = (\mathbf{G}_1\mathbf{y}, \mathbf{y})$, $\dot{\xi} = (\mathbf{Q}_1\mathbf{y}, \mathbf{y})$, where

$$\mathbf{G}_1(t) = \mathbf{P}(t)^*\,\mathbf{G}(t)\,\mathbf{P}(t), \quad \mathbf{Q}_1(t) = \mathbf{P}(t)^*\,\mathbf{Q}(t)\,\mathbf{P}(t). \tag{4.12}$$

Since

$$\det(\mathbf{Q}_1 - q\mathbf{G}_1) = |\det \mathbf{P}|^2 \det(\mathbf{Q} - q\mathbf{G}),$$

it follows that the equation

$$\det(\mathbf{Q}_1 - q\mathbf{G}_1) = 0$$

has the same roots as equation (4.5). Thus the integrals on the left and right of inequalities (4.7) for equations (4.1) and (4.10) are the same, provided that the corresponding matrices $\mathbf{G}$ and $\mathbf{G}_1$ are related as in (4.12), where $\mathbf{P}(t)$ is the matrix in (4.9).

The fact that these integrals are the same for equations (4.1) and (4.10) and that the real parts of the characteristic exponents coincide implies the following conclusion. Suppose we have been able to choose a matrix $\mathbf{G}_1(t)$ for equation (4.10) so as to satisfy (4.8). Then there is a matrix $\mathbf{G}(t)$ for equation (4.1) (defined by the first half of (4.12)) such that (4.8) is valid for equation (4.1) as well.

Suppose now that $\mathbf{A}(t)$ and $\mathbf{G}(t)$ are complex matrices. By the Floquet-Lyapunov theorem, there exists a T-periodic matrix $\mathbf{P}(t)$ such that $\mathbf{B}(t) = \mathbf{K}$ is a constant matrix (see (4.10)). It follows that we need only prove the theorem for equations with constant coefficients.

* Recall that the characteristic exponents are determined up to an additive term which is a multiple of $2\pi i/T$.

Thus, we consider the equation

$$\frac{dy}{dt} = \mathbf{K}y. \tag{4.13}$$

Assume first that $\mathbf{K}$ may be reduced to diagonal form:

$$\mathbf{K} = \mathbf{S}^{-1}(\alpha_1 \dotplus \ldots \dotplus \alpha_n)\mathbf{S}, \tag{4.14}$$

where $\mathbf{S}$ is a nonsingular (complex) matrix, and the eigenvalues $\alpha_1, \ldots, \alpha_n$ are so numbered that $\operatorname{Re}\alpha_1 \leqslant \ldots \leqslant \operatorname{Re}\alpha_n$.

Let

$$\mathbf{G}_1 = \mathbf{S}^*\mathbf{S}. \tag{4.15}$$

It is clear that $\mathbf{G}$ is positive definite. By (4.4),

$$\mathbf{Q}_1 = \mathbf{G}_1\mathbf{K} + \mathbf{K}^*\mathbf{G}_1 = \mathbf{S}^*(2\operatorname{Re}\alpha_1 \dotplus \ldots \dotplus 2\operatorname{Re}\alpha_n)\mathbf{S}. \tag{4.16}$$

Since

$$\det(\mathbf{Q}_1 - q\mathbf{G}_1) = |\det \mathbf{S}|^2 \prod_{j=1}^{n}(2\operatorname{Re}\alpha_j - q),$$

it follows that $q_1 = 2\operatorname{Re}\alpha_1$, $q_2 = 2\operatorname{Re}\alpha_n$. Thus we have (4.8) for equation (4.10), with $\varepsilon = 0$. It follows from the foregoing that for equation (4.1) and $\mathbf{G} = (\mathbf{P}(t)^*)^{-1}\mathbf{S}^*\mathbf{S}\mathbf{P}(t)^{-1}$ we again have $q_1(t) \equiv 2\operatorname{Re}\alpha_1$, $q_2(t) \equiv 2\operatorname{Re}\alpha_n$, so that (4.8) holds with $\varepsilon = 0$ for equation (4.1) too.

Now suppose that $\mathbf{K}$ is not reducible to diagonal form.

The proof follows the same general lines, but is rather more complicated. For any $\delta > 0$, there is a nonsingular matrix $\mathbf{S}_\delta$ such that

$$\mathbf{K} = \mathbf{S}_\delta^{-1}[\mathbf{Q}_\delta(\alpha_{(1)}) \dotplus \ldots \dotplus \mathbf{Q}_\delta(\alpha_{(s)})]\mathbf{S}_\delta,$$

where $\alpha_{(j)}$ are the eigenvalues of $\mathbf{K}$ and $\mathbf{Q}_\delta(\alpha)$ are the elementary Jordan matrices

$$\mathbf{Q}_\delta(\alpha) = \begin{Vmatrix} \alpha & \delta & 0 & \ldots & 0 & 0 \\ 0 & \alpha & \delta & \ldots & 0 & 0 \\ 0 & 0 & \alpha & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \alpha & \delta \\ 0 & 0 & 0 & \ldots & 0 & \alpha \end{Vmatrix}$$

(see §I.1.13). We may assume that

$$\operatorname{Re}\alpha_{(1)} \leqslant \ldots \leqslant \operatorname{Re}\alpha_{(s)}.$$

Let

$$\mathbf{G}_1 = \mathbf{S}_\delta^*\mathbf{S}_\delta. \tag{4.17}$$

Then

$$\mathbf{Q}_1=\mathbf{G}_1\mathbf{K}+\mathbf{K}^*\mathbf{G}_1=\mathbf{S}_\delta^*\left[2\operatorname{Re}\mathbf{Q}_\delta(\alpha_{(1)})\dotplus\ldots\dotplus 2\operatorname{Re}\mathbf{Q}_\delta(\alpha_{(s)})\right]\mathbf{S}_\delta,$$

where $\operatorname{Re}\mathbf{A}$ denotes the matrix

$$\operatorname{Re}\mathbf{A}=\frac{\mathbf{A}+\mathbf{A}^*}{2}.$$

The roots of the equation

$$\det\left[\mathbf{Q}_1-q\mathbf{G}_1\right]=|\det\mathbf{S}|^2\prod_{j=1}^{s}\det\left[2\operatorname{Re}\mathbf{Q}_\delta(\alpha_{(j)})-q\mathbf{I}_j\right]=0 \tag{4.18}$$

(where $\mathbf{I}_j$ are identity matrices) are the eigenvalues of all the matrices $2\operatorname{Re}\mathbf{Q}_\delta(\alpha_{(j)})$. For small δ, the eigenvalues of the matrix

$$2\operatorname{Re}\mathbf{Q}_\delta(\alpha)=\begin{Vmatrix} 2\operatorname{Re}\alpha & \delta & 0 & \ldots \\ \delta & 2\operatorname{Re}\alpha & \delta & \ldots \\ 0 & \delta & 2\operatorname{Re}\alpha & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & 2\operatorname{Re}\alpha \end{Vmatrix}$$

are arbitrarily close to $2\operatorname{Re}\alpha$. Hence the roots of equation (4.18) are arbitrarily close to the numbers $2\operatorname{Re}\alpha_{(j)}$ for sufficiently small δ. Consequently, for any $\varepsilon>0$, we can find $\delta>0$ such that the choice of (4.17) gives condition (4.8) for equation (4.10). As before, we arrive at the same conclusion for equation (4.1) by setting

$$\mathbf{G}=\left[\mathbf{P}(t)^*\right]^{-1}\mathbf{S}_\delta^*\mathbf{S}_\delta\mathbf{P}(t)^{-1}.$$

Now suppose that $\mathbf{A}(t)$ is a real matrix. It remains to prove that we can then choose $\mathbf{G}(t)$ as a real and, as before, T-periodic matrix. By §I. 1.18, in this case there exists a real nonsingular matrix $\mathbf{P}(t)$ such that $\mathbf{B}(t)=\mathbf{K}$ is a constant real matrix in (4.10) and

$$\mathbf{P}(t+T)=\mathbf{P}(t)\mathbf{R}, \tag{4.19}$$

where $\mathbf{R}^2=\mathbf{I}$. We consider the case when $\mathbf{K}$ has simple elementary divisors.

To fix ideas, let α_1 be real and $\alpha_n=\beta+i\gamma$ complex eigenvalues. Then (see §I.1.18) there exists a real nonsingular matrix $\mathbf{S}$ such that $\mathbf{K}=\mathbf{S}^{-1}\mathbf{Q}\mathbf{S}$, where

$$\mathbf{Q}=\alpha_1\dotplus\ldots\dotplus\begin{Vmatrix}\beta & \gamma\\ -\gamma & \beta\end{Vmatrix}.$$

As before, we define $\mathbf{G}$ by (4.15). Then if

$$\mathbf{Q}_1=\mathbf{G}_1\mathbf{K}+\mathbf{K}^*\mathbf{G}_1=\mathbf{S}^*(\mathbf{Q}+\mathbf{Q}^*)\mathbf{S}$$

we again get (4.16). As before, we see that $q_1 = \operatorname{Re}\alpha_1$, $q_2 = \operatorname{Re}\alpha_n$ for equation (4.10), hence also for (4.1), and (4.8) is satisfied. Moreover

$$\mathbf{G}(t) = (\mathbf{P}(t)^*)^{-1}\mathbf{S}^*\mathbf{S}\mathbf{P}(t)^{-1}$$

is a real matrix. We claim that it is also T-periodic. Using (4.19), we see that it will suffice to prove the equality

$$(\mathbf{R}^*)^{-1}\mathbf{S}^*\mathbf{S}\mathbf{R}^{-1} = \mathbf{S}^*\mathbf{S}. \tag{4.20}$$

It follows from §I. 1.18 that the matrix $\mathbf{R}$ may be defined as $+\mathbf{I}$ or $-\mathbf{I}$ on each of the root subspaces of $\mathbf{K}$. This means that

$$\mathbf{R} = \mathbf{S}^{-1}\mathbf{Q}_0\mathbf{S},$$

where $\mathbf{Q}_0$ is a diagonal matrix with diagonal elements ± 1. Thus $\mathbf{Q}_0^2 = \mathbf{I}$. The desired relation (4.20) now follows from the expression for $\mathbf{R}$, and so $\mathbf{G}(t+T) = \mathbf{G}(t)$.

If $\mathbf{K}$ has multiple elementary divisors, the reasoning is similar to that in the complex case with multiple elementary divisors. We then use the real Jordan form (1.74), (1.77) of §I. 1.18. That $\mathbf{G}(t)$ is T-periodic is proved as above. Q. E. D.

4.2. Second-order systems. Example

For a second-order system (4.1) with real coefficients, the matrix $\mathbf{G}(t)$ has the form

$$\mathbf{G}(t) = \begin{Vmatrix} a(t) & b(t) \\ b(t) & c(t) \end{Vmatrix}.$$

By the theorem of §4. 1, we may assume that $\mathbf{G}(t)$ is a real matrix-function.

Conditions (A) entail that $a(t)$, $b(t)$, $c(t)$ are differentiable and T-periodic functions, with

$$a(t) > 0, \quad c(t) > 0, \quad a(t)c(t) - b^2(t) > 0 \qquad (0 \leqslant t \leqslant T).$$

Otherwise these functions are arbitrary. Denote

$$e^{2\delta} = \det \mathbf{G} = ac - b^2, \quad \Delta = \det \mathbf{Q}. \tag{4.21}$$

Equation (4.5) may now be written

$$\det(\mathbf{Q}\mathbf{G}^{-1} - q\mathbf{I}_2) = 0,$$

or

$$q^2 - \operatorname{Tr}(\mathbf{Q}\mathbf{G}^{-1})q + \det(\mathbf{Q}\mathbf{G}^{-1}) = 0.$$

We have

$$\operatorname{Tr}(\mathbf{Q}\mathbf{G}^{-1}) = \operatorname{Tr}(\dot{\mathbf{G}}\mathbf{G}^{-1} + \mathbf{G}\mathbf{A}\mathbf{G}^{-1} + \mathbf{A}^*) = \operatorname{Tr}(\dot{\mathbf{G}}\mathbf{G}^{-1}) + 2\operatorname{Tr}\mathbf{A}.$$

We shall show that

$$\operatorname{Tr}(\dot{\mathbf{G}}\mathbf{G}^{-1}) = 2\dot{\delta}. \tag{4.22}$$

Indeed, denoting $\mathbf{B} = \dot{\mathbf{G}}\mathbf{G}^{-1}$ and using the Liouville-Jacobi formula (1.11), we get

$$e^{2\delta} = \det \mathbf{G}(t) = \det \mathbf{G}(0) \exp \int_0^t \operatorname{Tr}\mathbf{B}(t_1)\, dt_1,$$

i. e.,

$$2\delta = \int_0^t \operatorname{Tr}\mathbf{B}(t_1)\, dt_1 + \text{const},$$

whence follows (4.22).

Thus the equation for q becomes

$$q^2 - 2(\dot{\delta} + \operatorname{Tr}\mathbf{A})\, q + e^{-2\delta}\Delta = 0.$$

It follows from the theorem of §4.1 that

$$\operatorname{Re}\alpha_{1,2} = \frac{1}{2T}\left[\int_0^T \operatorname{Tr}\mathbf{A}\, dt \mp \inf \int_0^T \sqrt{(\dot{\delta} + \operatorname{Tr}\mathbf{A})^2 - e^{-2\delta}\Delta}\, dt\right]. \tag{4.23}$$

The infimum on the right extends over all functions a, b, c with the necessary properties. The functions $\delta(t)$ and $\Delta(t)$ are defined by (4.21) and (4.4). Taking arbitrary functions a, b, c of this type in (4.23), we obtain estimates for the numbers $\operatorname{Re}\alpha_1$ and $\operatorname{Re}\alpha_2$.

Example. Consider the system

$$\dot{x}_1 = \mu(-1 + 2\sin t)x_1 + \mu x_2, \quad \dot{x}_2 = \mu x_1 - x_2 \quad (\mu > 0), \tag{4.24}$$

used as an example by Malkin in /62b/ in the section dealing with the method of small parameters (§84, p. 353). Using the small parameter method, Malkin shows that the solutions of system (4.24) are asymptotically stable provided

$$|\mu^2\psi_{sf}| < \frac{1}{2}\mu. \tag{4.25}$$

He also gives explicit expressions for the functions ψ_{sf}. Based on a rough estimate for these quantities, Malkin shows that inequalities (4.25) hold for $\mu < 1/9$. The expression for $\psi_{12}(t)$, for example, shows that

$$|\mu^2\psi_{12}(0)| < (1/2)\mu$$

is no longer true for $\mu = 0.178$. It follows that a more thorough analysis of inequalities (4.25) cannot yield a significant sharpening of the estimate $\mu < 1/9 = 0.111\ldots$.

We shall show that the solutions of system (4.24) are asymptotically stable for $\mu < 2/3 = 0.666\ldots$. Define $\mathbf{G} = \mathbf{I}_2$. Then, using (4.21), we obtain

$$\mathbf{Q} = \mathbf{A} + \mathbf{A}^* = 2\begin{Vmatrix} \mu\psi & \mu \\ \mu & -1 \end{Vmatrix},$$
$$\varphi = -1 + 2\sin t, \qquad \delta = 0,$$
$$\Delta = -4(\mu\varphi + \mu^2).$$

Formula (4.23) gives

$$\operatorname{Re}\alpha_{1,2} \leqslant \frac{1}{2}\left[-1-\mu+\frac{1}{2\pi}\int_0^{2\pi}\sqrt{(1+\mu\varphi)^2+4\mu^2}\,dt\right].$$

For any function $\psi \geqslant 0$, we have, by the Cauchy-Bunyakovskii inequality,

$$\frac{1}{T}\int_0^T \sqrt{\psi}\,dt \leqslant \left(\frac{1}{T}\int_0^T \psi\,dt\right)^{1/2}.$$

Therefore,

$$\operatorname{Re}\alpha_{1,2} \leqslant \frac{1}{2}(-1-\mu+\sqrt{1-2\mu+7\mu^2}).$$

The right-hand side of this inequality is negative for $\mu < 2/3$. Consequently, the trivial solution of system (4.24) is asymptotically stable for $\mu < 2/3$. *

4.3. Estimates for characteristic exponents of second-order equations

Consider the scalar differential equation

$$\ddot{x} + P(t)\dot{x} + Q(t)x = 0, \tag{4.26}$$

where $P(t)$ and $Q(t)$ are T-periodic functions, $\dot{P}(t) = dP/dt$ and $Q(t)$ are assumed to be integrable and piecewise-continuous. The usual substitution

$$x = y\exp\left[-\frac{1}{2}\int_0^t P(t_1)\,dt_1\right] \tag{4.27}$$

reduces equation (4.26) to Hill's equation

$$\ddot{y} + p(t)y = 0 \tag{4.28}$$

* The above argument shows in fact that we have asymptotic stability for $\mu < 2/3 + \varepsilon$, $\varepsilon > 0$.

where

$$\boxed{p(t)=Q(t)-\frac{1}{4}P^2(t)-\frac{1}{2}\dot{P}(t),} \tag{4.29}$$

is under our assumptions T-periodic, integrable and piecewise-continuous. Denote

$$\frac{1}{T}\int_0^T P(t)\,dt=2\mu_0. \tag{4.30}$$

Then

$$x=e^{-\mu_0 t}e^{r(t)}y,$$

where

$$r(t)=\frac{1}{2}\left[\frac{t}{T}\int_0^T P(t)\,dt-\int_0^t P(t_1)\,dt_1\right]$$

is continuous and T-periodic. Hence it follows that in order to find bounds for the characteristic exponents of equation (4.26) we need only do this for Hill's equation (4.28).

Theorem I. Let $\operatorname{Re}\alpha\geqslant 0$ *be the real part of a characteristic exponent of equation (4.28), where* $p(t)$ *is a real* T*-periodic integrable piecewise-continuous function.*

1) *Let* $c(t)>0$ *be any positive* T*-periodic function with integrable piecewise-continuous derivative* $\dot{c}(t)$. *Then*

$$\operatorname{Re}\alpha\leqslant\frac{1}{2T}\int_0^T\sqrt{\left[\frac{\dot{c}(t)}{c(t)}\right]^2+\left[c(t)-\frac{p(t)}{c(t)}\right]^2}\,dt. \tag{4.31}$$

In particular, if $c>0$ *is any positive constant, then*

$$\operatorname{Re}\alpha\leqslant\frac{1}{2Tc}\int_0^T|p(t)-c^2|\,dt. \tag{4.32}$$

2) *If*

$$p_{\mathrm{av}}\equiv\frac{1}{T}\int_0^T p(t)\,dt=-m^2<0$$

and

$$p(t)\leqslant+m^2,$$

then

$$\operatorname{Re}\alpha \leqslant m.$$

3) *Let $p(t)$ be bounded above and*

$$p_{\max} = \max_{0 \leqslant t \leqslant T} p(t).$$

Then

$$\operatorname{Re}\alpha \leqslant \begin{cases} \dfrac{1}{2\sqrt{p_{\max}}}(p_{\max} - p_{\text{av}}) & \text{if} \quad p_{\max} + p_{\text{av}} > 0, \\ \sqrt{-p_{\text{av}}} & \text{if} \quad p_{\max} + p_{\text{av}} \leqslant 0. \end{cases} \tag{4.33}$$

4) *Let $p(t) > 0$ have an integrable piecewise-continuous derivative. Let $t_1^+, \ldots, t_n^+$ denote the maximum points and $t_1^-, \ldots, t_n^-$ the minimum points of $p(t)$ on the half-open interval $[t_0, t_0+T)$. Then*

$$\operatorname{Re}\alpha \leqslant \frac{1}{T} \ln \frac{p(t_1^+) \ldots p(t_n^+)}{p(t_1^-) \ldots p(t_n^-)}. \tag{4.34}$$

Proof. 1) We apply inequality (4.23) to equation (4.28). As follows from §§2.5, 3.5, the characteristic exponents α_1, α_2 of Hill's equation are either both real or both pure imaginary, and $\alpha_1 + \alpha_2 = 0$. Let $\operatorname{Re}\alpha \geqslant 0$. Both estimates (4.7) give an upper bound for $\operatorname{Re}\alpha$. Let

$$\mathbf{x} = \begin{pmatrix} y \\ \dot{y} \end{pmatrix},$$

then

$$\mathbf{A}(t) = \begin{Vmatrix} 0 & 1 \\ -p(t) & 0 \end{Vmatrix}.$$

Define the matrix $\mathbf{G}(t)$ by

$$\mathbf{G}(t) = \begin{Vmatrix} c(t) & 0 \\ 0 & c^{-1}(t) \end{Vmatrix}, \quad c(t+T) = c(t) > 0.$$

By (4.4) and (4.21), we have

$$\mathbf{Q} = \begin{Vmatrix} \dot{c} & c - \frac{p}{c} \\ c - \frac{p}{c} & -c^{-2}\dot{c} \end{Vmatrix}, \quad \delta \equiv 0, \quad \Delta = -\frac{\dot{c}^2}{c^2} - \left(c - \frac{p}{c}\right)^2.$$

Inequality (4.23) now gives (4.31). In particular, if $c > 0$ is any constant, we get (4.32).

2) Setting

$$c = m = \sqrt{-p_{av}}$$

in (4.32), we see that $\operatorname{Re}\alpha \leqslant m$. Note that this estimate is not valid for positive definite or semidefinite functions $p(t)$.

3) It is readily seen that if $p(t) \equiv \text{const}$ the estimate (4.33) is valid. If $p(t) \not\equiv \text{const}$, we have $p_{max} > p_{av}$, and so $p_{max} > 0$ if $p_{max} + p_{av} > 0$, $p_{av} < 0$ if $p_{max} + p_{av} \leqslant 0$. We may therefore put

$$c^2 = \begin{cases} p_{max} & \text{if} \quad p_{max} + p_{av} > 0, \\ -p_{av} & \text{if} \quad p_{max} + p_{av} \leqslant 0. \end{cases}$$

Inequality (4.33) then follows from (4.32).

4) If $p(t) > 0$ is a differentiable function, we may put $c(t) = \sqrt{p(t)}$ in (4.31). Assuming that $p(t)$ has finitely many maxima and minima in each period, we deduce (4.34) from (4.31). This completes the proof of Theorem I.

The following problem is important in applications. Suppose that the coefficients $P(t)$ and $Q(t)$ of equation (4.26) depend on certain parameters (determining the structure of the system). We wish to determine these parameters in such a way that $x = O(e^{-kt})$, where $k > 0$ is a prescribed number. The substitution (4.27) reduces equation (4.26) to the form (4.28) with the function (4.29). The problem is thus reduced to finding sufficient conditions for equation (4.28) such that $y = O(e^{\lambda_0 t})$, $\lambda_0 \geqslant 0$ (it is clear that λ_0 cannot be negative).

Example. Considering the special case

$$p(t) = \xi + \eta \cos 2t + \zeta \cos 4t, \tag{4.35}$$

of equation (4.28), where ξ, η, ζ are constants, $\eta \geqslant 0$, $\zeta \geqslant 0$, it is required to establish sufficient conditions guaranteeing that

$$y(t) = O(e^{\lambda_0 t}), \qquad \lambda_0 > 0. \tag{4.36}$$

Setting $c^2 = \xi + \eta + \zeta$ (if $\xi + \eta + \zeta > 0$), we obtain a first condition

$$(\eta + \zeta)^2 \leqslant 4\lambda_0^2 (\xi + \eta + \zeta). \tag{4.37}$$

Inequality (4.33) gives

$$0 < -\xi \leqslant \lambda_0^2, \quad \eta + \zeta \leqslant -2\xi. \tag{4.38}$$

To use inequality (4.34), we must distinguish between the cases $4\zeta \leqslant \eta$, when $p(t)$ has one maximum and one minimum per period, and $4\zeta > \eta$, when $p(t)$ has two maxima and two minima per period. Inequality (4.34) gives

$$4\zeta \leqslant \eta, \ \xi - \eta + \zeta > 0, \ \xi + \eta + \zeta \leqslant e^{2\pi\lambda_0} (\xi - \eta + \zeta) \tag{4.39}$$

and

$$4\zeta > \eta, \ \xi - \frac{\eta^2}{8\zeta} - \zeta > 0, \ (\xi + \eta + \zeta)(\xi - \eta + \zeta) \leqslant e^{2\pi\lambda_0} \left(\xi - \frac{\eta^2}{8\zeta} - \zeta\right)^2. \tag{4.40}$$

Each of the inequalities (4.37) – (4.40) may serve as the required sufficient condition.

When $\zeta=0$, we get Mathieu's equation. The appropriate conditions are

$$\xi \geqslant \frac{\eta^2}{4\lambda_0^2}-\eta, \tag{4.41}$$

$$2\xi+\eta \leqslant 0, \quad \xi \geqslant -\lambda_0^2, \tag{4.42}$$

and

$$\eta \leqslant \xi \operatorname{th} \pi\lambda_0. \tag{4.43}$$

Figure 6 illustrates the resulting region on the (ξ, η)-plane for $\lambda_0=1$. Also shown in the figure are the stability and instability regions, according to Strutt /89/. It is clear that none of the three regions (4.41), (4.42), (4.43) entirely contains any of the others; thus each of the estimates (4.32), (4.33), (4.34) "works." Note that these conditions may be considerably improved by an accurate (though rather tedious) calculation.

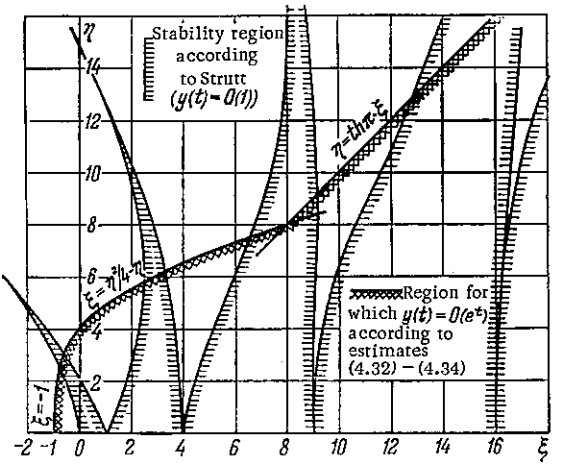


FIGURE 6.

The estimate (4.33) is in a certain sense best possible, as the following theorem shows.

Theorem II. Let P_M *be the set of* 2π*-periodic integrable piecewise-continuous functions* $p(s)$, *and set*

$$\sup_{0 \leqslant s \leqslant 2\pi} p(s) = p_{\max},$$

$$\frac{1}{2\pi}\int_0^{2\pi} p(s)\,ds = p_{\text{av}}.$$

Then the solutions of the equation

for any $\vartheta > 0$ *and any function* $p(s) \in P_M$ *satisfy the estimate*

$$\ddot{y} + p(\vartheta t)\, y = 0 \tag{4.44}$$

$$y = O(e^{\lambda t}), \qquad t \to +\infty, \tag{4.45}$$

where $\lambda > 0$ *is some fixed number, if and only if*

$$\lambda \geqslant \begin{cases} \dfrac{1}{2\sqrt{p_{\max}}}(p_{\max} - p_{av}) & \text{if} \quad p_{\max} + p_{av} > 0, \\ \sqrt{-p_{av}} & \text{if} \quad p_{\max} + p_{av} \leqslant 0. \end{cases} \tag{4.46}$$

Proof. Sufficiency follows from part 3 of Theorem I. To prove necessity, we must show that there exist $\vartheta > 0$, $p(s) \in P_M$, such that the number $\operatorname{Re}\alpha$ for equation (4.44) is arbitrarily close to the expression on the right of inequality (4.46).

First let $p_{\max} + p_{av} > 0$. We shall show that there exists an equation of type (4.44) such that

$$\operatorname{Re}\alpha = \frac{1}{2\sqrt{p_{\max}}}(p_{\max} - p_{av}). \tag{4.47}$$

Letting $q > 0$, $T > 0$, $M > 0$ be arbitrary numbers, construct the function

$$y_0(t) = \begin{cases} \frac{1}{M}\sin Mt & \text{for} \quad 0 \leqslant t \leqslant \xi, \\ Ae^{Bt} & \text{for} \quad \xi \leqslant t \leqslant \eta, \\ \frac{q}{M}\sin M(T-t) & \text{for} \quad \eta \leqslant t \leqslant T. \end{cases}$$

The constants A, B, ξ, η are determined by the condition that $y_0(t)$ and $\dot{y}_0(t)$ be continuous for $0 \leqslant t \leqslant T$. This construction is possible only if

$$T > \frac{\pi}{M}, \tag{4.48}$$

which we shall indeed assume to be the case. The continuity conditions are

$$Ae^{B\xi} = \frac{1}{M}\sin M\xi, \qquad ABe^{B\xi} = \cos M\xi, \tag{4.49}$$

$$Ae^{B\eta} = \frac{q}{M}\sin M(T-\eta), \quad ABe^{B\eta} = -q\cos M(T-\eta). \tag{4.50}$$

Multiplying the first equation in (4.49) by the second equation in (4.50), the second equation in (4.49) by the first equation in (4.50), and then comparing, we get

$$\sin M(T - \eta + \xi) = 0.$$

Since $0<T-\eta<\pi/M$, $0<\xi<\pi/M$, so that $0<M(T-\eta+\xi)<2\pi$, it follows that

$$M(T-\eta+\xi)=\pi, \qquad \eta-\xi=T-\frac{\pi}{M}. \tag{4.51}$$

The first equations in (4.49) and (4.50) give

$$e^{B(\eta-\xi)}=\frac{Ae^{B\eta}}{Ae^{\alpha\xi}}=q$$

and, substituting from (4.51), we obtain

$$B=\frac{M\ln q}{TM-\pi}. \tag{4.52}$$

Letting $y_0(t)$ be a solution of equation (4.44), we obtain the following expression for the corresponding function $p(\vartheta t)$:

$$p(\vartheta t)=-\frac{\ddot{y}_0}{y_0}=\begin{cases} M^2 & \text{for } 0<t<\xi, \\ -B^2 & \text{for } \xi<t<\eta, \\ M^2 & \text{for } \eta<t<T. \end{cases} \tag{4.53}$$

Extending $p(\vartheta t)$ to a periodic function, of period $T=2\pi/\vartheta$, we use the condition $\dot{y}_0(T)=T-q\dot{y}_0(0)$ to deduce that $y_0(t+T)=-qy_0(t)$. Thus the characteristic exponent of equation (4.44) is (§2.5) $\alpha=\frac{1}{T}\ln q+\pi i/T$, i.e.,

$$\operatorname{Re}\alpha=\frac{1}{T}\ln q. \tag{4.54}$$

From (4.53) we get

$$p_{\max}=M^2, \quad p_{\text{av}}=M^2-\frac{1}{T}(\eta-\xi)(M^2+B^2). \tag{4.55}$$

It follows from (4.51), (4.52), (4.54), (4.55) that

$$\operatorname{Re}\alpha=M\sqrt{\zeta(2\zeta_0-\zeta)}, \tag{4.56}$$

where

$$2\zeta_0=1-\frac{1}{M^2}p_{\text{av}}>0, \qquad \zeta=1-\frac{\pi}{TM}.$$

Since the quantity $\vartheta=2\pi/T>0$ may be varied at will, provided only that (4.48) holds, we may assume that ζ is any number in the interval $0<\zeta<1$. The condition $p_{\max}+p_{\text{av}}>0$ is equivalent to the inequality $\zeta_0<1$. The right-hand side of (4.56) attains a maximum at $\zeta=\zeta_0$, and then (4.47) holds and our assertion is proved.

Now let $p_{max}+p_{av}\leqslant 0$. Then $p_{av}<0$. The characteristic exponent α of equation (4.44) is a continuous functional of $p(s)$ (for fixed ϑ) on the set of all T-periodic integrable piecewise-continuous functions $p(s)$ with norm

$$\|p\|=\sqrt{\frac{1}{2\pi}\int_0^{2\pi}|p(s)|^2\,ds}\,.$$

For $p_0(s)\equiv -a^2$, the characteristic exponent is $\alpha=a=\sqrt{-p_{av}}$. Thus, for any function $p(s)\in P_M$ sufficiently close in norm to $p_0(s)\equiv -a^2$ (and such functions $p(s)$ clearly exist), the characteristic exponent is arbitrarily close to $\sqrt{-p_{av}}$. This completes the proof of Theorem II.

Remark. It can be shown that if $p_{max}+p_{av}\leqslant 0$, in contrast to the case $p_{max}+p_{av}>0$, there are no functions $p(s)\in P_M$ for which $\operatorname{Re}\alpha=\sqrt{-p_{av}}$. Hence we can sharpen the estimate (4.33) for the characteristic exponent α of equation (4.44):

$$\left.\begin{aligned}&\operatorname{Re}\alpha\leqslant\frac{1}{2\sqrt{p_{max}}}(p_{max}-p_{av}) && \text{if} \quad p_{max}+p_{av}>0,\\ &\operatorname{Re}\alpha<\sqrt{-p_{av}} && \text{if} \quad p_{max}+p_{av}\leqslant 0.\end{aligned}\right\} \tag{4.57}$$

4.4. Unbounded stability of dynamic system described by a second-order linear differential equation

We shall first show that the equation

$$\ddot{z}+P(t)\dot{z}+Q(t)z=0,$$

where $P(t)$, $Q(t)$ are T-periodic functions with $\dot{P}(t)$, $Q(t)$ integrable and piecewise-continuous, may be reduced to an analogous equation with $P(t)=\text{const}$ that has the same characteristic exponents. Let

$$\int_0^t P(t_1)\,dt_1=\nu t+P_1(t),$$

where

$$\nu=\frac{1}{T}\int_0^T P(t)\,dt,\quad P_1(t)=\int_0^t[P(t_1)-\nu]\,dt_1,$$

with $P_1(t)$ a T-periodic function. Setting

$$z=e^{-\frac{1}{2}P_1(t)}x,$$

we get

$$\ddot{x}+\nu\dot{x}+F(t)x=0,$$

in which

$$F(t)=Q(t)-\frac{1}{4}P^2(t)-\frac{1}{2}\dot{P}(t)+\frac{1}{4}\nu^2$$

is T-periodic, integrable and piecewise-continuous. Consider the equation

$$\ddot{x}+\nu\dot{x}+\varphi(\vartheta t)x=0, \tag{4.58}$$

where $\nu>0$, $\vartheta>0$, and $\varphi(s)$ is a 2π-periodic integrable piecewise-continuous function.

Definition. Following Aizerman /116/, we shall say that the dynamic system described by equation (4.58) is unboundedly stable if for any $\vartheta>0$

$$x\to 0,\quad \dot{x}=\frac{dx}{dt}\to 0 \quad \text{as} \quad t\to+\infty. \tag{4.59}$$

Theorem. Let $\Phi_{M,\sigma}$ denote the set of 2π-periodic integrable piecewise-continuous functions such that

$$\sup_{0\leqslant s\leqslant 2\pi}\varphi(s)=M^2,\quad \frac{1}{2\pi}\int_0^{2\pi}\varphi(s)\,ds=\sigma^2\geqslant 0$$

are fixed numbers. Then the system described by equation (4.58) with any function $\varphi(s)\in\Phi_{M,\sigma}$ is unboundedly stable if and only if

$$\nu^2>\frac{1}{2}\left(M^2-\sigma\sqrt{2M^2-\sigma^2}\right). \tag{4.60}$$

Proof. Sufficiency. Setting

$$x=e^{-\nu t}y,$$

we obtain an equation (4.44) for y, with

$$p(\vartheta t)=\varphi(\vartheta t)-\nu^2.$$

Using the notation of the theorem in Theorem II of §4.3 (see also (4.33)), we obtain the following estimate for the characteristic exponent of equation (4.44):

$$\operatorname{Re}\alpha\leqslant\begin{cases}\dfrac{1}{2\sqrt{M^2-\nu^2}}(M^2-\sigma^2) & \text{if} \quad 2\nu^2<M^2+\sigma^2,\\ \sqrt{\nu^2-\sigma^2} & \text{if} \quad 2\nu^2\geqslant M^2+\sigma^2.\end{cases}$$

The system will be unboundedly stable if $\operatorname{Re}\alpha<\nu$, i. e.,

$$\frac{1}{2\sqrt{M^2-\nu^2}}(M^2-\sigma^2)<\nu \quad \text{if} \quad 2\nu^2<M^2+\sigma^2, \tag{4.61}$$

$$\sqrt{\nu^2-\sigma^2}<\nu \quad \text{if} \quad 2\nu^2\geqslant M^2+\sigma^2. \tag{4.62}$$

Condition (4.62) is always satisfied. Condition (4.61) gives

$$\nu_1^2 < \nu^2 < \nu_2^2, \quad \text{if} \quad \nu^2 < \frac{1}{2}(M^2+\sigma^2),$$

where

$$\nu_{1,2}^2 = \frac{1}{2}(M^2 \mp \sigma\sqrt{2M^2-\sigma^2}).$$

Since $\nu_2^2 > \frac{1}{2}(M^2+\sigma^2)$, we have stability if $\nu^2 > \nu_1^2$, which is condition (4.60).

Necessity. Suppose that (4.60) fails to hold, $\nu^2 \leqslant \nu_1^2$. Then

$$\frac{1}{2\sqrt{M^2-\nu^2}}(M^2-\sigma^2) \geqslant \nu$$

and obviously $2\nu^2 < M^2+\sigma^2$. We thus have for equation (4.44) $p_{\max} + p_{\text{av}} = M^2 + \sigma^2 - 2\nu^2 > 0$. It follows from the proof of Theorem II in §4.3 that for any fixed

$$p_{\max} = M^2-\nu^2, \quad p_{\text{av}} = \sigma^2-\nu^2$$

there is a function $p(s) \in P_M$ such that

$$\operatorname{Re}\alpha = \frac{1}{2\sqrt{p_{\max}}}(p_{\max}+p_{\text{av}}) = \frac{1}{2\sqrt{M^2-\nu^2}}(M^2-\sigma^2).$$

Then $\operatorname{Re}\alpha \geqslant \nu$ and condition (4.59) fails to hold for equation (4.58) with the corresponding function $\varphi(s) = p(s) + \nu^2$ and the appropriate ϑ. This proves the theorem.

Remark. It follows from the proof that if

$$\nu^2 < \frac{1}{2}(M^2-\sigma\sqrt{2M^2-\sigma^2})$$

there exist a function $\varphi(s) \in \Phi_{M,\sigma}$ and a number ϑ such that equation (4.58) has characteristic exponents with positive real part. But if

$$\nu^2 = \frac{1}{2}(M^2-\sigma\sqrt{2M^2-\sigma^2}),$$

the characteristic exponents lie in the closed left half-plane and there is an equation (4.58) one of whose characteristic exponents is pure imaginary.

Example. Consider the equation

$$\ddot{x} + a\dot{x} + (1+q\cos\vartheta t)\,x = 0. \tag{4.63}$$

This equation was studied in /2/ in connection with the theory of electrical machines with parametric excitation. It is also encountered in various problems on the dynamic stability of elastic systems in which friction is proportional to velocity /7/. We have

$$M^2 = 1+q, \quad \sigma^2 = 1.$$

Inequality (4.60) gives a condition for unbounded stability:

$$a^2 > \frac{1}{2}(1+q-\sqrt{1+2q}).$$

4.5. Estimates for characteristic exponents of a system of second-order equations

Consider the vector differential equation

$$\frac{d^2\mathbf{y}}{dt^2}+\mathbf{P}(t)\,\mathbf{y}=\mathbf{0}, \tag{4.64}$$

where

$$\mathbf{P}(t+T)=\mathbf{P}(t) \tag{4.65}$$

is an $n\times n$ matrix-function and $\mathbf{y}$ (the solution-vector) is an n-vector.

Theorem I. Let $\mathbf{C}(t)=\mathbf{C}(t)^*$ be an arbitrary $n\times n$ positive definite matrix-function having an integrable piecewise-continuous derivative $\dot{\mathbf{C}}(t)$. Let $q(t)$ denote the norm of the matrix*

$$\tilde{\mathbf{Q}}(t)=\begin{pmatrix}\mathbf{C}(t)^{-1/2}\dot{\mathbf{C}}(t)\,\mathbf{C}(t)^{-1/2} & \mathbf{C}(t)-\mathbf{C}(t)^{-1/2}\,\mathbf{P}(t)^*\,\mathbf{C}(t)^{-1/2}\\ \mathbf{C}(t)-\mathbf{C}(t)^{-1/2}\,\mathbf{P}(t)\,\mathbf{C}(t)^{-1/2} & -\mathbf{C}(t)^{-1/2}\,\dot{\mathbf{C}}(t)\,\mathbf{C}(t)^{-1/2}\end{pmatrix}. \tag{4.66}$$

a) *Then the characteristic exponents α of equation (4.64) satisfy the estimate*

$$-\frac{1}{2T}\int_0^T q(t)\,dt\leqslant \operatorname{Re}\alpha\leqslant\frac{1}{2T}\int_0^T q(t)\,dt. \tag{4.67}$$

In particular, if $\mathbf{P}(t)=\mathbf{P}(t)^*$ is hermitian and $\mathbf{C}=\mathbf{C}^*$ is an arbitrary constant positive definite matrix, then

$$\operatorname{Re}\alpha\leqslant\frac{1}{2T}\int_0^T|\,\mathbf{C}-\mathbf{C}^{-1/2}\mathbf{P}(t)\,\mathbf{C}^{-1/2}\,|\,dt. \tag{4.68}$$

b) *Let $\mathbf{P}(t)=\mathbf{P}(t)^*$ be a hermitian matrix-function and $\beta_{\min}(t)$ its smallest eigenvalue. Suppose that*

$$\mathbf{P}(t)\leqslant m^2\mathbf{I}_n,$$

where $m>0$ is some number. Then

* Here and below we are speaking of the operator norm $q=\sup\frac{|\tilde{\mathbf{Q}}\mathbf{x}|}{|\mathbf{x}|}$ corresponding to the vector norm (1.1.2), i.e., $|\mathbf{x}|=\sqrt{\sum_{j=1}^{n}|\xi_j|^2}$.

$$\operatorname{Re}\alpha \leqslant \frac{1}{2}\left[m - \frac{1}{mT}\int_0^T \beta_{\min}(t)\,dt\right]. \tag{4.69}$$

c) *Let* $\mathbf{P}(t)=\mathbf{P}(t)^*$ *and let* $\beta_{\min}(t)$, $\beta_{\max}(t)$ *be respectively the smallest and largest eigenvalues of* $\mathbf{P}(t)$. *Denote*

$$\beta = \max_{0<t<T} \beta_{\max}(t), \quad \beta_{\text{av}}^+ = \frac{1}{T}\int_0^T \beta_{\max}(t)\,dt, \quad \beta_{\text{av}}^- = \frac{1}{T}\int_0^T \beta_{\min}(t)\,dt. \tag{4.70}$$

Then

$$\operatorname{Re}\alpha \leqslant \begin{cases} \dfrac{\beta - \beta_{\text{av}}^-}{2\sqrt{\beta}}, & \text{if} \quad \beta + \beta_{\text{av}}^+ > 0, \\ \dfrac{-\beta_{\text{av}}^+ + \beta_{\text{av}}^-}{2\sqrt{-\beta_{\text{av}}^+}}, & \text{if} \quad \beta + \beta_{\text{av}}^+ \leqslant 0. \end{cases} \tag{4.71}$$

Another valid estimate is

$$\operatorname{Re}\alpha \leqslant \begin{cases} \dfrac{\beta - \beta_{\text{av}}^-}{\sqrt{\beta}}, & \text{if} \quad \beta + \beta_{\text{cp}}^- > 0, \\ \sqrt{-\beta_{\text{av}}^-}, & \text{if} \quad \beta + \beta_{\text{av}}^- \leqslant 0. \end{cases} \tag{4.72}$$

d) *Let* $\mathbf{P}(t)=\mathbf{P}(t)^* > 0$ *and suppose that the derivative* $\dot{\mathbf{P}}(t)$ *exists and is piecewise-continuous and integrable. Then*

$$\operatorname{Re}\alpha \leqslant \frac{1}{2T}\int_0^T |\mathbf{P}(t)^{-1/2}\dot{\mathbf{P}}(t)\mathbf{P}(t)^{-1/2}|\,dt. \tag{4.73}$$

Proof. a) We write equation (4.64) as a system (4.1) with

$$\mathbf{x} = \begin{pmatrix} \mathbf{y} \\ \dot{\mathbf{y}} \end{pmatrix}, \quad \mathbf{A}(t) = \begin{pmatrix} 0 & \mathbf{I}_n \\ -\mathbf{P}(t) & 0 \end{pmatrix}.$$

Apply the theorem of §4.1 with

$$\mathbf{G}(t) = \begin{pmatrix} \mathbf{C}(t) & 0 \\ 0 & \mathbf{C}(t)^{-1} \end{pmatrix}.$$

Equation (4.5) becomes $\det(\tilde{\mathbf{Q}} - q\mathbf{I}_{2n}) = 0$, where $\tilde{\mathbf{Q}} = \mathbf{G}^{-1/2}\mathbf{Q}\mathbf{G}^{-1/2}$. In our case the matrix $\tilde{\mathbf{Q}} = \tilde{\mathbf{Q}}(t)$ has the form (4.66). If q_2 is the largest root of equation (4.5), then $q_2 \leqslant q$. Indeed, for the corresponding eigenvector

$$\tilde{\mathbf{Q}}\mathbf{z} = q_2\mathbf{z}, \quad q = |\tilde{\mathbf{Q}}| = \sup \frac{|\tilde{\mathbf{Q}}\mathbf{x}|}{|\mathbf{x}|} \geqslant \frac{|\tilde{\mathbf{Q}}\mathbf{z}|}{|\mathbf{z}|} = |q_2| \geqslant q_2.$$

Similarly, $q \geqslant |q_1|$, $q_1 \geqslant -q$. Inequality (4.67) now follows from (4.7).

If $\mathbf{P}(t)=\mathbf{P}(t)^*$ and $\mathbf{C}(t)=\mathbf{C}$ is a constant matrix, it follows from (4.66) that

$$\tilde{\mathbf{Q}} = \begin{pmatrix} 0 & \mathbf{R} \\ \mathbf{R} & 0 \end{pmatrix},$$

where $\mathbf{R}=\mathbf{C}-\mathbf{C}^{-1/2}\mathbf{P}\mathbf{C}^{-1/2}=\mathbf{R}^*$. Setting $\mathbf{x}=\mathbf{y}+\mathbf{z}$, where $\mathbf{y}$ and $\mathbf{z}$ are n-vectors, we get

$$|\tilde{\mathbf{Q}}|=\sup_{x,y}\frac{\sqrt{|\mathbf{Ry}|^2+|\mathbf{Rx}|^2}}{\sqrt{|\mathbf{y}|^2+|\mathbf{x}|^2}}=|\mathbf{R}|.$$

This proves (4.68).

b) Estimate (4.69) follows at once from (4.68) for $\mathbf{C}=m\mathbf{I}_n$, if we note that

$$|m^2\mathbf{I}_n-\mathbf{P}|=m^2-\beta_{\min}(t).$$

c) Taking (4.68) with

$$\mathbf{C}^2=\begin{cases}\beta\mathbf{I}_n, & \text{if} \quad \beta+\beta_{av}^+>0,\\ -\beta_{av}^+\mathbf{I}_n, & \text{if} \quad \beta+\beta_{av}^+\leqslant 0,\end{cases}$$

we obtain (4.71). Taking (4.68) with

$$\mathbf{C}^2=\begin{cases}\beta\mathbf{I}_n, & \text{if} \quad \beta+\beta_{av}^-<0,\\ -\beta_{av}^-\mathbf{I}_n, & \text{if} \quad \beta+\beta_{av}^-\geqslant 0,\end{cases}$$

we obtain (4.72).

d) The proof is similar to that of case (a) for

$$\mathbf{C}(t)=\begin{pmatrix}\mathbf{I}_n & 0\\ 0 & \mathbf{P}(t)^{-1}\end{pmatrix}.$$

Q. E. D.

Remark. Let $\mathbf{P}(t)=\mathbf{P}(t)^*$. Set $\mathbf{C}(t)=c(t)\mathbf{I}_n$, where $c(t)>0$ is a scalar function. It is readily checked that the norm of the matrix (4.66) is now

$$q(t)=\sqrt{\left[\frac{\dot{c}(t)}{c(t)}\right]^2+\left|c(t)\mathbf{I}_n-\frac{1}{c(t)}\mathbf{P}(t)\right|^2}. \tag{4.74}$$

Thus estimate (4.67) is valid for $q(t)$ of this type with an arbitrary function $c(t)>0$.

Example. Consider the system

$$\begin{cases}\dfrac{d^2y_1}{dt^2}+\omega_1^2y_1-\varphi(\theta t)\gamma_1y_2=0,\\ \dfrac{d^2y_2}{dt^2}+\omega_2^2y_2-\varphi(\theta t)\gamma_2y_1=0,\end{cases}$$

where $\varphi(\theta t)$ is a 2π-periodic function. This system is of type (4.64) for $n=2$, where

$$\mathbf{P}(t)=\begin{Vmatrix}\omega_1^2 & 0\\ 0 & \omega_2^2\end{Vmatrix}-\varphi(\theta t)\begin{Vmatrix}0 & \gamma_1\\ \gamma_2 & 0\end{Vmatrix}.$$

a) Letting C in (4.68) be the matrix

$$C=\begin{Vmatrix}\omega_1 & 0\\ 0 & \omega_2\end{Vmatrix},$$

we see that the characteristic exponents α of the system satisfy the inequality

$$|\operatorname{Re}\alpha|\leqslant\frac{\max(|\gamma_1|,|\gamma_2|)}{2\sqrt{\omega_1\omega_2}}|\varphi|_{av}, \tag{4.75}$$

where

$$|\varphi|_{av}=\frac{1}{2\pi}\int_0^{2\pi}|\varphi(t)|\,dt.$$

Let $\varphi(t)=\lambda+\beta\cos t$, $\gamma_1=\gamma_2=\gamma$. Set

$$C^2=\begin{Vmatrix}\omega_1^2 & -\lambda\gamma\\ -\lambda\gamma & \omega_2^2\end{Vmatrix}.$$

This matrix is positive definite if

$$\omega_1^2\omega_2^2>\lambda^2\gamma^2,$$

and we shall assume that this is so. It follows readily from (4.68) that

$$\operatorname{Re}\alpha\leqslant\frac{\sqrt{2}}{\pi}\frac{|\beta\gamma|}{\sqrt{\omega_1^2+\omega_2^2-\sqrt{(\omega_1^2-\omega_2^2)^2+4\lambda^2\gamma^2}}}. \tag{4.76}$$

Estimate (4.76) is sometimes superior to (4.75).

b) Let $\gamma_1=\gamma_2=\gamma$; $\varphi(t)=\lambda+\beta\cos t$; $\lambda>\beta>0$, $\theta>0$,

$$\omega_1^2\omega_2^2>\gamma^2(\lambda+\beta)^2.$$

Then $P(t)=P(t)^*>0$. Using formula (4.74), we obtain

$$\operatorname{Re}\alpha\leqslant\frac{|\gamma\theta\beta|}{4\pi}\int_0^{2\pi}\frac{|\sin t|\,dt}{\omega_1\omega_2|\gamma(\lambda+\beta\cos t)|}.$$

Note that in contradistinction to our previous estimates this estimate depends explicitly on the perturbation frequency θ.

Chapter III

HAMILTONIAN SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

This chapter presents the most important results from the theoretical standpoint. A detailed account will be given of Hamiltonian and canonical vector equations (see §II.3.1). A Hamiltonian $\mathbf{H}(t)$ is said to be strongly stable if, first, all the solutions of the equation are bounded on $(-\infty, +\infty)$ (i.e., the trivial solution is stable) and, second, this property is preserved by any small deformation of the Hamiltonian (the rigorous definition will be given below). The core of this chapter is the necessary and sufficient condition for strong stability (§3) due to Krein, Gel'fand and Lidskii. This condition is linked to certain properties of the multipliers of a canonical equation. Namely, the multipliers of a canonical equation on the unit circle are classified according to a certain rule (due to Krein) into multipliers of the first and the second kinds; the system is strongly stable if (Krein /166a, f/) and only if (Gel'fand and Lidskii /132/) all the multipliers lie on the unit circle and no multipliers of different kinds coincide. The rigorous statment and proof of this rule requires certain simple but rather specialized results from linear algebra, established in §§1 and 2. A more delicate discussion of the analytic properties of the multipliers of Hamiltonian equations, following Krein and Lyubarskii /167b/, is given in §4. Our exposition differs in methodological respects from that of the original papers.

Since the multipliers of a canonical equation are usually unknown, the Krein-Gel'fand-Lidskii strong stability test cannot be applied directly to determine whether the system is strongly stable (or unstable). It is nevertheless highly important and may be used to deduce effective sufficient conditions (i.e., conditions expressible in terms of the coefficients) for strong stability (§§6, 7). It is also the basis for our analysis of the qualitative structure of dynamic instability domains in Chapter V, §§1, 3.

In the space of all Hamiltonians, the set of all strongly stable Hamiltonians may be partitioned into countably many connected domains such that the Hamiltonians in each domain (more precisely, the solutions of the corresponding equations) have certain characteristic common properties (see /132/). For systems of two equations, there is an analogous result for the set of all strongly unstable Hamiltonians (see /224a, g/). For systems of higher order, $2k > 2$, the set of strongly unstable Hamiltonians is connected. An analogous partition problem is formulated and solved in regard to the set of Hamiltonians whose solutions $\mathbf{x}(t)$ satisfy an estimate $|\mathbf{x}(t)| < \text{const} \cdot e^{\alpha t}$ with prescribed $\alpha > 0$, and also for other similarly defined sets. These results are described briefly in §5. The mathematical

tools necessary for their proofs are beyond the scope of this book. We only sketch the proofs; for details the reader may consult the original papers (see the bibliographical notes at the end of Volume 2). The important case of systems of two equations will be discussed in detail in Chapter VIII.

§1. BEHAVIOR OF MULTIPLIERS UNDER PERTURBATION OF THE HAMILTONIAN. KREIN'S THEOREM ON MULTIPLIERS OF THE FIRST AND SECOND KINDS ON THE UNIT CIRCLE

1.1. Linear space with indefinite metric

Let $\mathfrak{R}_n$ be complex n-space, as defined in § I. 1.1, and $\mathbf{G}$ a nonsingular hermitian $n\times n$ matrix (i. e., $\det \mathbf{G}\neq 0$, $\mathbf{G}^*=\mathbf{G}$) which is neither positive nor negative definite (§ I. 1.14). Together with the standard scalar product $(\mathbf{x}, \mathbf{y})$ (see § I.1.1), we define a new, indefinite scalar product (the significance of the term will be clarified below):

$$\langle \mathbf{x}, \mathbf{y}\rangle = (\mathbf{Gx}, \mathbf{y}). \tag{1.1}$$

In other words, $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{y}^*\mathbf{Gx}$ (for the notation, see § I. 1.1). Two vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be $\mathbf{G}$-orthogonal if $\langle \mathbf{x}, \mathbf{y}\rangle = 0$ (notation: $\mathbf{x}\langle\perp\rangle\mathbf{y}$).

The scalar product $\langle \mathbf{x}, \mathbf{y}\rangle$ has the following properties.

1. $\langle \mathbf{x}, \mathbf{y}\rangle$ is a linear function of $\mathbf{x}$, i. e., for any complex numbers α, β and any $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}\in\mathfrak{R}_n$

$$\langle \alpha\mathbf{x}_1+\beta\mathbf{x}_2, \mathbf{y}\rangle = \alpha\langle \mathbf{x}_1, \mathbf{y}\rangle+\beta\langle \mathbf{x}_2, \mathbf{y}\rangle.$$

2. $\langle \mathbf{y}, \mathbf{x}\rangle = \overline{\langle \mathbf{x}, \mathbf{y}\rangle}$ (hermitian property). (As usual, the bar denotes complex conjugation.)

3. There is no vector $\mathbf{x}_0\neq\mathbf{0}$ in $\mathfrak{R}_n$ which is $\mathbf{G}$-orthogonal to every other vector (nondegeneracy property). In other words, if $\langle \mathbf{x}_0, \mathbf{x}\rangle = 0$ for every $\mathbf{x}\in\mathfrak{R}_n$, then $\mathbf{x}_0=\mathbf{0}$.

The first two properties are obvious. We prove the third. Let $\langle \mathbf{x}_0, \mathbf{x}\rangle \doteq (\mathbf{Gx}_0, \mathbf{x}) = 0$ for all $\mathbf{x}$. Setting $\mathbf{x}=\mathbf{Gx}_0$, we get $|\mathbf{Gx}_0|^2=0$. Therefore $\mathbf{Gx}_0=\mathbf{0}$, but $\det\mathbf{G}\neq 0$ and so $\mathbf{x}_0=\mathbf{0}$.

Note that the number $\langle \mathbf{x}, \mathbf{x}\rangle$ is real (property 2) but unlike $(\mathbf{x}, \mathbf{x})$ it need not be positive. This is why $\langle \mathbf{x}, \mathbf{y}\rangle$ is called an indefinite scalar product. If $\langle \mathbf{x}, \mathbf{x}\rangle = 0$, it need not be true that $\mathbf{x}=\mathbf{0}$ (whereas $(\mathbf{x}, \mathbf{x})=0$ always implies $\mathbf{x}=\mathbf{0}$). A vector $\mathbf{x}$ for which $\langle \mathbf{x}, \mathbf{x}\rangle = 0$ is said to be isotropic.

Let $\mathbf{W}$ be some $n\times n$ matrix. The matrix $\mathbf{W}^+$ defined by

$$\langle \mathbf{Wx}, \mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{W}^+\mathbf{y}\rangle \tag{1.2}$$

($\mathbf{x}$,$\mathbf{y}$ are arbitrary vectors in $\mathfrak{R}_n$) is said to be G-adjoint to $\mathbf{W}$. Since

$$\langle \mathbf{Wx}, \mathbf{y}\rangle = (\mathbf{GWx}, \mathbf{y}) = (\mathbf{x}, \mathbf{W}^*\mathbf{Gy}) = (\mathbf{Gx}, \mathbf{G}^{-1}\mathbf{W}^*\mathbf{Gy}) = \langle \mathbf{x}, \mathbf{G}^{-1}\mathbf{W}^*\mathbf{Gy}\rangle,$$

the explicit formula for $\mathbf{W}^+$ is

$$\boxed{\mathbf{W}^+ = \mathbf{G}^{-1}\mathbf{W}^*\mathbf{G}.} \tag{1.3}$$

It follows from (1.3) that

$$\boxed{(\mathbf{W}_1+\mathbf{W}_2)^+ = \mathbf{W}_1^+ + \mathbf{W}_2^+, \quad (\mathbf{W}_1\mathbf{W}_2)^+ = \mathbf{W}_2^+\mathbf{W}_1^+.} \tag{1.4}$$

The table given below lists the definitions of G-unitary, G-hermitian and G-antihermitian (G-Hamiltonian) matrices.

G-unitary matrix	G-antihermitian (G-Hamiltonian) matrix	G-hermitian matrix
$\langle \mathbf{Wx}, \mathbf{Wy}\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$	$\langle \mathbf{Wx}, \mathbf{y}\rangle = -\langle \mathbf{x}, \mathbf{Wy}\rangle$	$\langle \mathbf{Wx}, \mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{Wy}\rangle$
$\mathbf{W}^+\mathbf{W} = \mathbf{I}_n$	$\mathbf{W}^+ = -\mathbf{W}$	$\mathbf{W}^+ = \mathbf{W}$
$\mathbf{W}^*\mathbf{GW} = \mathbf{G}$	$\mathbf{W} = i\mathbf{G}^{-1}\mathbf{H}$, where $\mathbf{H}^* = \mathbf{H}$	$\mathbf{W} = \mathbf{G}^{-1}\mathbf{H}$, where $\mathbf{H}^* = \mathbf{H}$.

In the first row of the table, $\mathbf{x}$ and $\mathbf{y}$ are arbitrary vectors. Any of the three (equivalent) conditions in each column may serve as the definition of the appropriate type of matrix. The last row of the table gives the usual definitions.

Recall that according to the definition in § II. 3.1 a Hamiltonian equation has the form

$$i^{-1}\mathbf{G}\frac{d\mathbf{z}}{dt} = \mathbf{H}(t)\,\mathbf{z}, \tag{1.5}$$

where $\mathbf{G}$ is a nonsingular hermitian matrix and $\mathbf{H}(t)^* = \mathbf{H}(t)$. Thus a G-Hamiltonian matrix-function $\mathbf{W}(t) = i\mathbf{G}^{-1}\mathbf{H}(t)$ is the coefficient matrix of a Hamiltonian equation in matrix notation: $\dot{\mathbf{z}} = \mathbf{W}(t)\,\mathbf{z}$.

The matrizant (§II. 1. 2) $\mathbf{Z}(t)$ of a Hamiltonian equation is a G-unitary matrix for any t.

The proof is the same as in § II. 3.4:

$$\frac{d}{dt}(\mathbf{Z}^+\mathbf{Z}) = \mathbf{Z}^+\frac{d\mathbf{Z}}{dt} + \left(\frac{d\mathbf{Z}}{dt}\right)^+\mathbf{Z} = \mathbf{Z}^+\mathbf{WZ} - \mathbf{Z}^+\mathbf{WZ} = \mathbf{0},$$

since

$$\dot{\mathbf{Z}} = \mathbf{WZ} \quad \text{and} \quad \mathbf{W}^+ = -\mathbf{W}.$$

Therefore $\mathbf{Z}(t)^+\mathbf{Z}(t) = \text{const}$, and since $\mathbf{Z}(0) = \mathbf{I}_n$ it follows that $\mathbf{Z}(t)^+\mathbf{Z}(t) = \mathbf{I}_n$.

The formulas in the last row of the table yield certain conclusions concerning the spectra of matrices of these types.

Thus, for a G-unitary matrix we have

$$\mathbf{W}^* = \mathbf{G}\mathbf{W}^{-1}\mathbf{G}^{-1}. \tag{1.6}$$

Let $\{\rho_\nu\}$ be the spectrum of the G-unitary matrix $\mathbf{W}$. Then the spectrum of $\mathbf{W}^*$ is $\{\bar{\rho}_\nu\}$. The spectrum of $\mathbf{W}^{-1}$, hence also of $\mathbf{G}\mathbf{W}^{-1}\mathbf{G}^{-1}$, is the set $\{\rho_\nu^{-1}\}$. It follows from (1.6) that the sets $\{\bar{\rho}_\nu\}$ and $\{\rho_\nu^{-1}\}$ coincide. Consequently, so do the sets $\{\rho_\nu\}$ and $\{\bar{\rho}_\nu^{-1}\}$. Thus, if ρ is an eigenvalue of $\mathbf{W}$, then so is $\bar{\rho}^{-1}$. The complex numbers ρ and $\bar{\rho}^{-1}$ are symmetric with respect to the unit circle. Thus:

The spectrum of a G-unitary matrix is symmetric about the unit circle.

If $\mathbf{W}$ is a G-Hamiltonian matrix, then $\mathbf{W} = -\mathbf{G}^{-1}\mathbf{W}^*\mathbf{G}$, and so the matrices $-\mathbf{W}^*$ and $\mathbf{W}$ have the same spectrum. If $\{\lambda_\nu\}$ is the spectrum of $\mathbf{W}$, then that of $-\mathbf{W}^*$ is $\{-\bar{\lambda}_\nu\}$. The numbers λ and $-\bar{\lambda}$ are symmetric about the imaginary axis. Thus:

The spectrum of a G-Hamiltonian matrix is symmetric about the imaginary axis.

To obtain more precise information, we must consider the elementary divisors instead of the spectrum. Let $\{(\lambda-\rho_j)^{m_j}\}$ be the set of elementary divisors of $\mathbf{W}$. Then the set of elementary divisors of $\mathbf{W}^*$ is $\{(\lambda-\bar{\rho}_j)^{m_j}\}$ (§I. 1.12) and that of $\mathbf{W}^{-1}$ is $\{(\lambda-\rho_j^{-1})^{m_j}\}$. Repeating the above arguments, we see that *the symmetry properties of the spectrum remain valid when the elementary divisors are taken into consideration.* Let us explain this statement. If ρ is an eigenvalue of a G-unitary (G-Hamiltonian) matrix $\mathbf{W}$ not on the unit circle (not on the imaginary axis), and the corresponding elementary divisors are $(\lambda-\rho)^{m_1}, \ldots, (\lambda-\rho)^{m_s}$, then the number $\rho_1 = \bar{\rho}^{-1}$ ($\rho_1 = -\bar{\rho}$) is also an eigenvalue, with elementary divisors $(\lambda-\rho_1)^{m_1}, \ldots, (\lambda-\rho_1)^{m_s}$.

The operation $^+$ is continuous, as follows at once from the definition (1.3). Using this property and properties (1.4), we get

$$(e^{\mathbf{A}})^+ = \left(\sum \frac{1}{m!}\mathbf{A}^m\right)^+ = \sum \frac{1}{m!}(\mathbf{A}^m)^+ = \sum \frac{1}{m!}(\mathbf{A}^+)^m,$$

i. e.,

$$(e^{\mathbf{A}})^+ = e^{\mathbf{A}^+}. \tag{1.7}$$

It follows that if $\mathbf{A}$ is G-Hamiltonian, then $e^{\mathbf{A}}$ is G-unitary. Indeed, if $\mathbf{W} = e^{\mathbf{A}}$, then by (1.7) we have $\mathbf{W}^+ = e^{-\mathbf{A}} = \mathbf{W}^{-1}$, so that $\mathbf{W}^+\mathbf{W} = \mathbf{I}_n$, as claimed. The converse (the logarithm of a G-unitary matrix is G-Hamiltonian) is generally false. It is valid only for a special choice of the branch of the logarithm (see below, §4.4).

Throughout the sequel, a major role will be played by a special matrix $\mathbf{G}$:

$$\mathbf{G} = i\mathbf{J},$$

where $\mathbf{J}$ is the matrix defined in §II. 3.1:

$$\mathbf{J}=\begin{pmatrix}\mathbf{0} & -\mathbf{I}_k\\ \mathbf{I}_k & \mathbf{0}\end{pmatrix},$$

with $\mathbf{0}$ and $\mathbf{I}_k$ the $k\times k$ zero and identity matrices (so that the dimension in this case must be even: $n=2k$). Recall that

$$\mathbf{J}^2=-\mathbf{I}_{2k},\quad \mathbf{J}^*=\mathbf{J}^{-1}=-\mathbf{J}.$$

It follows from the first of these equalities that $(\det \mathbf{J})^2=\det(-\mathbf{I}_{2k})=1$, so that $\det \mathbf{J}=\pm 1$. It is easy to see that in fact $\det \mathbf{J}=1$. *

With this choice of $\mathbf{G}$, equation (1.5) becomes

$$\mathbf{J}\frac{d\mathbf{x}}{dt}=\mathbf{H}(t)\,\mathbf{x} \tag{1.8}$$

so that it is canonical provided $\mathbf{H}(t)$ is real. The matrizant of the canonical equation (1.8) is a real $(i\mathbf{J})$-unitary matrix. The property proved above for the spectrum of the monodromy matrix (symmetry with respect to the unit circle) is precisely the statement of the Lyapunov-Poincaré theorem (§ II.3.5).

A real $(i\mathbf{J})$-unitary matrix $\mathbf{X}$, i. e., a matrix $\mathbf{X}$ such that

$$\mathbf{X}^*\mathbf{J}\mathbf{X}=\mathbf{J},$$

is said to be symplectic.

Henceforth, for the sake of simplicity, we shall say simply J-orthogonal, J-unitary instead of $(i\mathbf{J})$-orthogonal, $(i\mathbf{J})$-unitary. For example, vectors $\mathbf{x}$ and $\mathbf{y}$ are J-orthogonal if $(\mathbf{Jx}, \mathbf{y})=0$.

We now show that the set of all G-unitary matrices is a group, i. e., any G-unitary matrix $\mathbf{W}$ has an inverse $\mathbf{W}^{-1}$ which is also G-unitary, and the product $\mathbf{VW}$ of any G-unitary matrices $\mathbf{V}$ and $\mathbf{W}$ is G-unitary. It follows from the G-unitarity condition $\mathbf{W}^*\mathbf{GW}=\mathbf{G}$ that $|\det \mathbf{W}|^2=1$, so that $\mathbf{W}^{-1}$ exists. Multiplying the equality $\mathbf{W}^*\mathbf{GW}=\mathbf{G}$ on the left by $(\mathbf{W}^*)^{-1}=(\mathbf{W}^{-1})^*$ and on the right by $\mathbf{W}^{-1}$, we get $\mathbf{G}=(\mathbf{W}^{-1})^*\mathbf{G}\mathbf{W}^{-1}$, so that $\mathbf{W}^{-1}$ is a G-unitary matrix. It follows from the G-unitarity condition in the first row of the table on p. 138, by the G-unitarity first of $\mathbf{V}$ and then of $\mathbf{W}$, that

$$\langle(\mathbf{VW})\mathbf{x}, (\mathbf{VW})\mathbf{y}\rangle=\langle\mathbf{V}(\mathbf{Wx}), \mathbf{V}(\mathbf{Wy})\rangle=\langle\mathbf{Wx}, \mathbf{Wy}\rangle=\langle\mathbf{x}, \mathbf{y}\rangle.$$

Thus $\mathbf{VW}$ is a G-unitary matrix.

1.2. Eigenvalues of the first and second kinds

Definition. a) Let ρ' be a simple eigenvalue of a G-unitary matrix $\mathbf{X}$, lying on the unit circle ($|\rho'|=1$) and $\mathbf{a}$ a corresponding eigenvector:

* For example, this follows without computations from formulas of Chapter I: If $\mathbf{U}(t)=e^{t\mathbf{J}}$, then by (I.2.3) $\mathbf{U}(t)=\mathbf{I}_{2k}\cos t+\mathbf{J}\sin t$, so that $\mathbf{J}=\mathbf{U}(\pi/2)$. On the other hand, by (II.1.25), $\det \mathbf{U}(t)=\exp(t\,\mathrm{Tr}\,\mathbf{J})\equiv 1$. Hence $\det \mathbf{J}=\det \mathbf{U}(\pi/2)=1$.

$\mathbf{Xa}=\rho'\mathbf{a}$. Then ρ' is called an eigenvalue of the first kind if $\langle \mathbf{a}, \mathbf{a}\rangle > 0$ and of the second kind if $\langle \mathbf{a}, \mathbf{a}\rangle < 0$. *

b) Let $\rho'(|\rho'|=1)$ be an r-fold eigenvalue of a G-unitary matrix $\mathbf{X}$, and let $\mathfrak{S}_{\rho'}$ be the corresponding eigensubspace (see § I. 1.8). If $\langle \mathbf{x}, \mathbf{x}\rangle > 0$ for any $\mathbf{x} \neq 0$ in this subspace, then ρ' is called an r-fold eigenvalue of the first kind, and if $\langle \mathbf{x}, \mathbf{x}\rangle < 0$ then ρ' is an r-fold eigenvalue of the second kind.

Eigenvalues of the first and second kinds are said to be definite. **

c) Let ρ' be an r-fold eigenvalue on the unit circle such that $\langle \mathbf{x}, \mathbf{x}\rangle$ is not of fixed sign on the eigensubspace $\mathfrak{S}_{\rho'}$, i. e., there exists a vector $\mathbf{x} \in \mathfrak{S}_{\rho'}$, $\mathbf{x} \neq 0$, such that $\langle \mathbf{x}, \mathbf{x}\rangle = 0$. Then we shall say that the eigenvalue is indefinite or, alternatively, an eigenvalue of mixed kind.

Below (§2.5) we shall present an equivalent definition, in some respects more convenient. We shall also show in §2.5 that in case (c) there exist vectors $\mathbf{x} \in \mathfrak{S}_{\rho'}$ such that $\langle \mathbf{x}, \mathbf{x}\rangle > 0$ and also vectors $\mathbf{x} \in \mathfrak{S}_{\rho'}$ such that $\langle \mathbf{x}, \mathbf{x}\rangle < 0$.

As of now, the definition refers only to eigenvalues on the unit circle. Eigenvalues of a G-unitary matrix with absolute value other than unity are classified according to the following simple rule.

d) Let $\rho(|\rho| \neq 1)$ be an r-fold eigenvalue of a G-unitary matrix $\mathbf{X}$. Then ρ is called an r-fold eigenvalue of the first kind if $|\rho| < 1$ and of the second kind if $|\rho| > 1$.

Lemma I. Let $\mathbf{X}$ be a G-unitary matrix having a multiple eigenvalue ρ' on the unit circle, with at least one multiple elementary divisor. Then there is an eigenvector $\mathbf{a} \neq 0$ belonging to the eigenvalue ρ' such that

$$\langle \mathbf{a}, \mathbf{a}\rangle = 0,$$

so that ρ' is an eigenvalue of mixed kind.

Proof. It follows from the assumptions of the lemma that the root subspace $\mathfrak{L}_{\rho'}$ has (§ I. 1.10) a cyclic basis $\mathbf{f}_1 = \mathbf{a} \neq 0$, $\mathbf{f}_2, \ldots$, of dimension at least 2, such that

$$\mathbf{Xa} = \rho'\mathbf{a}, \quad \mathbf{Xf}_2 = \rho'\mathbf{f}_2 + \mathbf{a}, \quad \ldots$$

Consider the scalar product

$$\langle \mathbf{Xa}, \mathbf{Xf}_2\rangle = |\rho'|^2 \langle \mathbf{a}, \mathbf{f}_2\rangle + \rho'\langle \mathbf{a}, \mathbf{a}\rangle.$$

Since $\mathbf{X}$ is G-unitary,

$$\langle \mathbf{Xa}, \mathbf{Xf}_2\rangle = \langle \mathbf{a}, \mathbf{X}^+\mathbf{Xf}_2\rangle = \langle \mathbf{a}, \mathbf{f}_2\rangle,$$

and the previous equality becomes

* It will be shown later (§ 2.3) that for a simple eigenvalue invariably $\langle \mathbf{a}, \mathbf{a}\rangle \neq 0$. For the moment we shall not assume this formally, but rather incorporate this case in the third part of the definition. We shall prove later that in case (c) the eigenvalue cannot be simple.

** It is clear that part (a) is a special case of part (b); it is included here purely for convenience.

$$(1-|\rho'|^2)\langle \mathbf{a}, \mathbf{f}_2\rangle = \rho'\langle \mathbf{a}, \mathbf{a}\rangle.$$

Since $|\rho'|=1$, the left-hand side vanishes, Q. E. D.

Lemma II. Let ρ be an eigenvalue, not on the unit circle, of a G-unitary matrix $\mathbf{X}$. Then any corresponding eigenvector $\mathbf{a}$ is isotropic: $\langle \mathbf{a}, \mathbf{a}\rangle = 0$.

Proof. By assumption, $\mathbf{Xa}=\rho\mathbf{a}$, and so

$$\langle \mathbf{Xa}, \mathbf{Xa}\rangle = \langle \rho\mathbf{a}, \rho\mathbf{a}\rangle = |\rho|^2\langle \mathbf{a}, \mathbf{a}\rangle.$$

Since $\mathbf{X}$ is G-unitary,

$$\langle \mathbf{Xa}, \mathbf{Xa}\rangle = \langle \mathbf{a}, \mathbf{X}^+\mathbf{Xa}\rangle = \langle \mathbf{a}, \mathbf{a}\rangle,$$

and the previous equality gives

$$(1-|\rho|^2)\langle \mathbf{a}, \mathbf{a}\rangle = 0.$$

Since $1-|\rho|^2 \neq 0$, we have $\langle \mathbf{a}, \mathbf{a}\rangle = 0$, Q. E. D.

1.3. First theorem on perturbation of G-unitary matrices

Theorem (Krein /166f/). Let $\mathbf{X}_0$ be a $2k\times 2k$ G-unitary matrix having a definite r-fold eigenvalue $\rho^{(0)}$ on the unit circle. Then:

a) *All elementary divisors belonging to $\rho^{(0)}$ are simple.*

b) *There exists numbers $\gamma > 0$, $\delta > 0$, such that for any $2k\times 2k$ G-unitary matrix $\mathbf{X}$ satisfying the inequality*

$$|\mathbf{X}-\mathbf{X}_0| < \delta,$$

all eigenvalues in the neighborhood $|\rho-\rho^{(0)}| < \gamma$ lie on the unit circle and all their elementary divisors are simple.

Proof. a) Suppose that the eigenvalue in question has a multiple elementary divisor. Then by Lemma I of §1.2 there is a corresponding eigenvector $\mathbf{a}_0$ such that $\langle \mathbf{a}_0, \mathbf{a}_0\rangle = 0$; but this is impossible, for $\rho^{(0)}$ is a definite eigenvalue.

b) Suppose the contrary. Then there is a sequence of matrices $\mathbf{X}_1, \mathbf{X}_2, \ldots \longrightarrow \mathbf{X}_0$ with respective eigenvalues $\rho_1, \rho_2, \ldots \longrightarrow \rho^{(0)}$ such that either $|\rho_m| \neq 1$, or $|\rho_m| = 1$ but $\mathbf{X}_m$ has a multiple elementary divisor belonging to ρ_m $(m=1, 2, \ldots)$. Let $\mathbf{a}_m$ be corresponding normalized eigenvectors,

$$\mathbf{X}_m\mathbf{a}_m = \rho_m\mathbf{a}_m \qquad (|\mathbf{a}_m| = 1),$$

so that by Lemmas I and II of §1.2,

$$\langle \mathbf{a}_m, \mathbf{a}_m\rangle = 0 \qquad (m=1, 2, \ldots).$$

We can select a subsequence such that $\mathbf{a}_{m_j} \longrightarrow \mathbf{a}$ as $m_j \longrightarrow \infty$; by suitably changing the notation, we may assume that $\mathbf{a}_m \longrightarrow \mathbf{a}$ as $m \longrightarrow \infty$. Letting $m \longrightarrow \infty$ in the above equalities, we obtain

$$\mathbf{X}_0\mathbf{a}=\rho^{(0)}\mathbf{a} \qquad (|\mathbf{a}|=1,\ \langle \mathbf{a},\ \mathbf{a}\rangle=0).$$

Hence $\rho^{(0)}$ is not a definite eigenvalue of $\mathbf{X}_0$, and this contradiction completes the proof.

Later (§3.1) we shall show that the converse is also valid.

1.4. Strong and weak stability of Hamiltonian systems

Let us return to the case of a Hamiltonian system of linear differential equations with periodic coefficients (§II. 3. 1), in vector notation:

$$i^{-1}\mathbf{G}\frac{d\mathbf{x}}{dt}=\mathbf{H}_0(t)\mathbf{x} \qquad (\mathbf{H}_0(t+T)=\mathbf{H}_0(t)=\mathbf{H}_0(t)^*). \tag{1.9}$$

As before, we shall call the hermitian matrix $\mathbf{H}_0(t)$ the Hamiltonian of the equation. As shown in § II. 3. 5, the trivial solution of this equation is stable (or, equivalently, all its solutions are bounded) if and only if all its multipliers, i. e., the roots of its characteristic equation

$$\det[\mathbf{X}(T)-\rho\mathbf{I}_{2k}]=0,$$

lie on the unit circle, and the multiple roots have only simple elementary divisors. (Here $\mathbf{X}(T)$ is the monodromy matrix of equation (1.9) — the value of its matrizant at the end of a period.)

As we shall see, however, these stability conditions are not sufficient for all solutions of the equation to remain bounded under sufficiently small perturbations of the Hamiltonian.

Definition. Suppose there is a number $\varepsilon>0$ such that all the solutions of any equation

$$i^{-1}\mathbf{G}\frac{d\mathbf{x}}{dt}=\mathbf{H}(t)\mathbf{x} \qquad (\mathbf{H}(t)+T)=\mathbf{H}(t)=\mathbf{H}(t)^*), \tag{1.10}$$

where

$$\|\mathbf{H}-\mathbf{H}_0\|\equiv\int_0^T|\mathbf{H}(t)-\mathbf{H}_0(t)|\,dt<\varepsilon, \tag{1.11}$$

are bounded on $(-\infty,\ +\infty)$. Then the Hamiltonian $\mathbf{H}_0(t)$ (or equation (1.9) itself) is said to be **strongly stable**.

If the Hamiltonian $\mathbf{H}_0(t)$ is stable (i. e., all solutions of (1.9) are bounded on $(-\infty,\ +\infty)$) but not strongly stable, we shall say that it (or the equation) is **weakly stable**.

Let us examine the property of strong stability more closely. As shown in § II. 3. 5, the multipliers of the Hamiltonian equation (1.10) are symmetric with respect to the unit circle. The matrizant of the system is a G-unitary matrix (§§ II. 3. 4 and 1.1). Thus its eigenvalues — the multipliers of equation (1.10) — satisfy the definitions of §1. 2, so that they may be classified as being of the first, second or mixed kind.

It was shown in §II. 3. 6 that if all the multipliers of equation (1.9) lie on the unit circle and are distinct, then equation (1.9) is strongly stable. This is a consequence of the symmetry of the multipliers with respect to the unit circle. Under a small deformation of the Hamiltonian (in the sense indicated above), the multipliers are also slightly displaced; thus they cannot move off the unit circle, for that would violate the symmetry (for the details, see §II. 3. 6).

On the other hand, if equation (1.9) has multipliers that coincide at some point ρ on the unit circle (hence, in the real case, also at $\bar{\rho}$), they can move off the unit circle without violating symmetry (Figure 7).

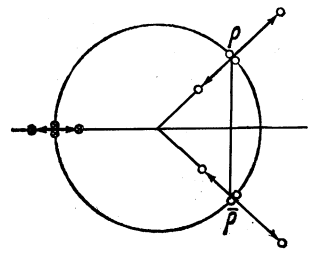


FIGURE 7.

In the neighborhood of ρ, the displaced multipliers will lie (by virtue of symmetry) on different sides of the circle. In the real case, a similar situation must hold in the neighborhood of $\bar{\rho}$, owing to the symmetry of the multipliers with respect to the real axis. Now our discussion of the deformation of multipliers interprets "multipliers" as any points on the complex plane possessing the required symmetry. In fact, however, these complex numbers must actually be the multipliers of some equation (1.10) with Hamiltonian $\mathbf{H}(t)$ close to $\mathbf{H}_0(t)$; in other words, the continuous deformation of the Hamiltonian should be such that the multipliers of the equation move as described, away from the unit circle (Figure 7). It turns out that the Hamiltonian cannot always be deformed in this way. Moreover, the very existence of a suitable deformation of the Hamiltonian depends only on the properties of the multipliers that coincide at ρ. Namely, if the set of multipliers that tend to ρ contains multipliers of *different* kinds (and then, as we shall presently see, the same is true for $\bar{\rho}$), there exists an appropriate deformation. But if all the multipliers coinciding at ρ are of *one* kind, i. e., ρ is a definite multiple multiplier (so that $\bar{\rho}$ is then also definite, though of the opposite kind), no such deformation is possible. To summarize: if any deformation of the Hamiltonian produces a "meeting" of multipliers of the *same* kind on the unit circle, they cannot subsequently move off the circle. Multipliers of *different* kinds that meet on the unit circle may move off the circle under a suitable deformation, and the latter always exists. This will be proved in §3. 1. The theorem, of paramount importance for our theory, is in fact a corollary of the theorem according to which the multipliers of the first (second) kind depend continuously on

the Hamiltonian. We shall prove in §2 that every equation (1.10) has exactly k multipliers of the first kind and k of the second kind, and that the multipliers of each kind may be indexed in such a way that under a continuous deformation (in the sense of the norm $\|\mathbf{H}\|$) of the Hamiltonian a multiplier with fixed index moves continuously in the complex plane. * Hence it follows at once that if two simple multipliers of the same, say first, kind, moving along the unit circle, meet at some point ρ, they cannot leave the unit circle. Indeed, by the symmetry of the spectrum they must lie on different sides of the circle, but multipliers of the first kind may lie only in the region $|\rho| \leqslant 1$. The reasoning is similar for multipliers of the second kind. On the other hand, multipliers of different kinds are not prevented by symmetry and continuity from leaving the unit circle after meeting (Figure 8). We shall show below that when multipliers of different kinds meet on the unit circle there is always a deformation of the Hamiltonian under which the multipliers indeed move off the unit circle (as illustrated in Figure 8).

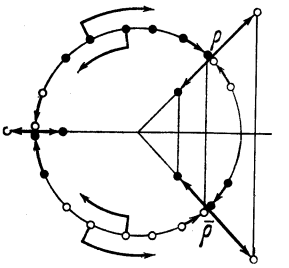


FIGURE 8.

The assertion that multipliers of the same kind which meet on the unit circle cannot move off it follows at once from the theorem of §1.3. Indeed, let $\rho^{(0)}$ be a definite multiple multiplier of an equation with Hamiltonian $\mathbf{H}_0(t)$ and monodromy matrix $\mathbf{X}_0(T)$ ($|\rho^{(0)}| = 1$). Let $\mathbf{H}(t)$ be a Hamiltonian with norm close to that of $\mathbf{H}_0(t)$ (in the sense of (1.11)) and $\mathbf{X}(T)$ the corresponding monodromy matrix. By the continuity theorem (§II. 1.7), as $\|\mathbf{H}(t) - \mathbf{H}_0(t)\| \to 0$ we have $|\mathbf{X}(T) - \mathbf{X}_0(T)| \to 0$. On the other hand, by the theorem of §1.3, there exist $\delta > 0$, $\gamma > 0$, such that if $|\mathbf{X}(T) - \mathbf{X}_0(T)| < \delta$ the eigenvalues of $\mathbf{X}(T)$ in the disk $|\rho - \rho^{(0)}| < \gamma$ lie on the unit circle and have simple elementary divisors. Consequently, under small deformations of $\mathbf{H}_0(t)$ the corresponding multipliers ρ_j (whose number, counting multiplicity, is equal to the multiplicity of $\rho^{(0)}$) move away from $\rho^{(0)}$ but remain on the unit circle, as asserted.

According to Krein's theorem (§1.3), a definite multiple multiplier $\rho^{(0)}$ has only simple elementary divisors. Thus the above assertion implies

* To describe the situation graphically, let black dots represent multipliers of the first kind and white dots multipliers of the second kind. The continuity theorem then implies that when the Hamiltonian is deformed continuously the multipliers vary continuously as to both position and "color" (there can be no sudden change of color).

the following important proposition, also due to Krein (who used a different method):

A sufficient condition for strong stability of equation (1.9) is that all its multipliers lie on the unit circle and be definite.

As mentioned above, this condition is not only sufficient but also necessary (Gel'fand-Lidskii, /132/). Later (§3.2) we shall present a simple proof of this statement, using certain algebraic theorems to be proved in §2. An alternative proof, not using the algebra of §2, may be based on perturbation theory (§V.1). *

Let us now consider a system (1.10) whose Hamiltonian depends on a parameter λ. Because of the special behavior of the two kinds of multipliers as described above, when the multipliers are treated as functions of λ they display several specific properties. We now proceed to an account of these properties.

1.5. Analytical properties of multipliers of Hamiltonian systems

We shall need the following simple lemma from the theory of analytic functions of several variables.

Lemma. Let $\varphi(\xi,\eta) \not\equiv 0$ be a holomorphic function of ξ and η at a point ξ_0, η_0, i.e., for sufficiently small $|\xi-\xi_0|, |\eta-\eta_0|$ the function may be represented by a convergent series

$$\varphi(\xi,\eta) = A_{00} + A_{10}(\xi-\xi_0) + A_{01}(\eta-\eta_0) + \ldots + \sum_{p+q=m} A_{pq}(\xi-\xi_0)^p(\eta-\eta_0)^q + \ldots$$

Let $A_{00} = \varphi(\xi_0,\eta_0) = 0$, $\varphi(\xi_0,\eta) \not\equiv 0$ and let $\eta = \eta(\xi)$ be the (generally many-valued) function defined in the neighborhood of $\xi = \xi_0$ by the equation $\varphi(\xi,\eta) = 0$ and the condition $\eta(\xi_0) = \eta_0$.

Let L be a curve on the complex ξ-plane through the point ξ_0, which has a tangent at ξ_0, M a curve on the complex η-plane through η_0, which has a tangent at η_0.

Suppose that when the point ξ is on the curve L (and $|\xi-\xi_0|$ is sufficiently small) all values of η such that $\varphi(\xi,\eta) = 0$ (which are sufficiently close to η_0) lie on M. Then all values of $\eta(\xi)$ (of which there are finitely many) are single-valued analytic functions at ξ_0:

$$\eta^{(j)}(\xi) = \eta_0 + \eta_1^{(j)}(\xi-\xi_0) + \eta_2^{(j)}(\xi-\xi_0)^2 + \ldots$$

Proof. By Weierstrass' lemma (/95/, p. 137), we can express $\varphi(\xi,\eta)$ as

$$\varphi(\xi,\eta) = (\xi-\xi_0)^\mu [\omega^m + A_1(\xi)\omega^{m-1} + \ldots + A_m(\xi)]\,\Omega(\xi,\eta),$$

* The reader interested only in a proof of the Krein-Gel'fand-Lidskii theorem (the main result of the first three sections) may proceed directly to §3.

where $w=(\eta-\eta_0)$; μ, m are integers; $A_k(\xi)$, $\Omega(\xi, \eta)$ are regular at $\xi=\xi_0$ and $\eta=\eta_0$; $A_k(\xi_0)=0$, $\Omega(\xi_0, \eta_0)\neq 0$ $(k=1, \ldots, m)$. Since $\varphi(\xi_0\eta)\not\equiv 0$, we have $\mu=0$. Thus the equation $\varphi(\xi, \eta)=0$ defines m values of $\eta=\eta(\xi)$, the roots of the equation $w^m+A_1(\xi)\,w^{m-1}+A_m(\xi)=0$. Consequently, all the values of $\eta=\eta(\xi)$ fall into several groups, the q values of any one group being defined by an expansion of the form

$$\eta(\xi)=\eta_0+\eta_1(\xi-\xi_0)^{p_1/q}+\eta_2(\xi-\xi_0)^{p_2/q}+\ldots \qquad (1.12)$$

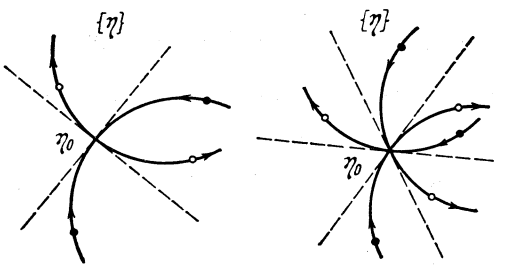
Here $p_1\leqslant p_2\leqslant\ldots$ are integers and $\eta_1\neq 0$, $\eta_2\neq 0$, ...

As ξ varies along the curve L in a sufficiently small neighborhood of ξ_0, the values (1.12) lie on q curves which intersect at η_0 and have tangents M_j there. The angles between the tangents M_j are equal to $\pi p_1/q$. By assumption, $\pi p_1/q$ is a multiple of π, so that $p_1/q=r_1$ is an integer. Similar reasoning holds for the function

$$\eta(\xi)-\eta_1(\xi-\xi_0)^{r_1}=\eta_0+\eta_2(\xi-\xi_0)^{p_2/q}+\ldots$$

Therefore p_2/q is an integer, and so on. Q. E. D.

An intuitive argument will clarify the main point in the above proof. Let $q=2$ in (1.12). As ξ varies along L, the two values of $\zeta=(\xi-\xi_0)^{1/2}$ trace out a curve which is the union of two smooth curves intersecting at η_0 at right angles. Consequently, the same is true of the locus M of the values of $\eta(\xi)=\eta_0+\eta_1\zeta^{p_1}+\eta_2\zeta^{p_2}+\ldots$, provided that at least one of the numbers p_j is odd (Figure 9a). Let us use the term "branches" for the connected parts of the curve L (or M) obtained after deletion of the point ξ_0 (or η_0). As ξ moves toward ξ_0, the two corresponding values of η lie on the branches of M, which have a common tangent (black points). At $\xi=\xi_0$ these come together at η_0, and then, when ξ crosses on L to the other branch, they move apart on the other branches of M (white points). If $q=3$ in (1.12) and at least one of the p_j's is not a multiple of 3, the locus of the values $\eta(\xi)$ (the curve M) will have the form shown in Figure 9b. The angle between successive tangents at η_0 is $\pi/3$. The pattern in the general case is analogous, with the angle between successive tangents equal to π/q. Since by assumption none of these situations can actually occur, it follows that (1.12) contains only integer powers of $\xi-\xi_0$.



a b

FIGURE 9.

We now consider a Hamiltonian equation whose Hamiltonian depends on a parameter:

$$i^{-1}\mathbf{G}\frac{d\mathbf{x}}{dt}=\mathbf{H}(t,\lambda)\,\mathbf{x}. \tag{1.13}$$

We shall assume that (1.13) is a Hamiltonian equation for real values of λ:

$$\mathbf{H}(t,\lambda)^{*}=\mathbf{H}(t,\lambda) \quad \text{for} \quad \operatorname{Im}\lambda=0,$$

and that $\mathbf{H}(t,\lambda)$ is an analytic function of λ in the neighborhood of a real $\lambda=\lambda_0$, in the sense outlined in §II. 1.3. This means that

$$\mathbf{H}(t,\lambda)=\mathbf{H}_0(t)+(\lambda-\lambda_0)\,\mathbf{H}_1(t)+\ldots, \tag{1.14}$$

and for sufficiently small $|\lambda-\lambda_0|$ the series

$$\sum_{l=0}^{\infty}|\lambda-\lambda_0|^{l}\int_0^T|\mathbf{H}_l(t)|\,dt$$

is convergent. In addition, we shall assume, as usual, that $\mathbf{H}_l(t+T)\equiv\mathbf{H}_l(t)=$ $=\mathbf{H}_l(t)^{*}$ $(l=0,1,2,\ldots)$.

As established in §II. 1.3, the matrizant $\mathbf{X}(T,\lambda)$ is analytic in λ at $\lambda=\lambda_0$. The multipliers $\rho_j(\lambda)$ of equation (1.13) are the roots of the equation.

$$\varphi(\lambda,\rho)\equiv\det[\mathbf{X}(T,\lambda)-\rho\mathbf{I}_{2k}]=0. \tag{1.15}$$

Since equation (1.13) is Hamiltonian for real λ, it follows that for such λ, including λ_0, we can meaningfully classify its multipliers as being of the first and second kinds. If some multiplier $\rho=\rho_0$ is r-fold at $\lambda=\lambda_0$, then the r multipliers $\rho_j=\rho_j(\lambda)$ which coincide with r at $\lambda=\lambda_0$ are ρ_0 values of a function $\rho(\lambda)$, which may be expanded in fractional powers of $\lambda-\lambda_0$. It turns out that if ρ_0 is an r-fold definite multiplier, then the corresponding r multipliers $\rho(\lambda)$ are analytic functions of λ, i. e., each of the r values of $\rho(\lambda)$ may be expanded in integer powers of $\lambda-\lambda_0$. In detail:

Theorem. Let ρ_0 be a point on the unit circle which is an r-fold definite multiplier of equation (1.13) for $\lambda=\lambda_0$. Then:

(I) *In some neighborhood $|\lambda-\lambda_0|<\delta$, we can determine r single-valued analytic branches $\rho_1(\lambda),\ldots,\rho_r(\lambda)$ of the function $\rho(\lambda)$ defined by (1.15), whose values are the r multipliers of equation (1.13) that coincide with ρ_0 at $\lambda=\lambda_0$.*

(II) *For all real λ sufficiently near λ_0, $|\rho_j(\lambda)|=1$ $(j=1,\ldots,r)$.*

(III) *The coefficients δ_j of the expansion*

$$\rho_j(\lambda)=\rho_0\{1+\delta_j(\lambda-\lambda_0)+O[(\lambda-\lambda_0)^2]\} \tag{1.16}$$

are given by

$$\delta_j=\frac{i}{\langle\mathbf{a}_j,\mathbf{a}_j\rangle}\int_0^T\left(\left(\frac{\partial\mathbf{H}}{\partial\lambda}\right)_{\lambda_0}\mathbf{x}_j,\mathbf{x}_j\right)dt, \tag{1.17}$$

where $\mathbf{a}_j$ *are suitably chosen eigenvectors of the monodromy matrix,* $\mathbf{X}(T, \lambda_0)\mathbf{a}_j = \rho_0 \mathbf{a}_j$, *and* $\mathbf{x}_j = \mathbf{X}(t, \lambda_0)\mathbf{a}_j$ *are the corresponding solutions of equation (1.13) at* $\lambda = \lambda_0$ $(j = 1, \ldots, r)$.

Proof. The truth of (II) follows from Krein's theorem (§1.3). The multipliers $\rho_1(\lambda), \ldots, \rho_r(\lambda)$ that coincide with ρ_0 at $\lambda = \lambda_0$ are all the values of the function $\rho(\lambda)$ defined by equation (1.15) and the condition $\rho(\lambda_0) = \rho_0$.

To prove (I), we apply the lemma to $\varphi(\lambda, \rho)$, with $\xi = \lambda$, $\eta = \rho$, $\xi_0 = \lambda_0$, $\eta_0 = \rho_0$, letting L be the real axis and M the unit circle. It follows from (1.15) that $\varphi(\lambda, \rho)$ is analytic at $\lambda = \lambda_0$, $\rho = \rho_0$. The condition $\varphi(\lambda_0, \rho) \not\equiv 0$ is obviously satisfied. Using (II) and the lemma, we deduce that all r values of $\rho_j(\lambda)$ are analytic functions of λ at $\lambda = \lambda_0$.

Finally, we prove (III). Since $\rho_1(\lambda), \ldots, \rho_r(\lambda)$ are analytic functions of λ and $\mathbf{X}(T, \lambda)$ is also analytic in λ in the neighborhood of $\lambda = \lambda_0$, it follows that suitably chosen eigenvectors $\mathbf{a}_1(\lambda), \ldots, \mathbf{a}_r(\lambda)$ are analytic functions. * Omitting the index j, we have the following identity for $\mathbf{a}_j(\lambda)$, $\rho_j(\lambda)$:

$$\mathbf{X}(T, \lambda)\mathbf{a}(\lambda) = \rho(\lambda)\mathbf{a}(\lambda). \tag{1.18}$$

Differentiating (1.18) and multiplying by $\mathbf{a}(\lambda)$ in the sense of the indefinite scalar product (§1.1), we obtain

$$\left\langle \frac{\partial \mathbf{X}}{\partial \lambda}\mathbf{a}, \mathbf{a} \right\rangle + \left\langle \mathbf{X}\frac{d\mathbf{a}}{d\lambda}, \mathbf{a} \right\rangle = \frac{d\rho}{d\lambda}\langle \mathbf{a}, \mathbf{a} \rangle + \rho\left\langle \frac{d\mathbf{a}}{d\lambda}, \mathbf{a} \right\rangle. \tag{1.19}$$

The matrix $\mathbf{X} = \mathbf{X}(T, \lambda)$ is G-unitary (§1.1). Therefore

$$\left\langle \mathbf{X}\frac{d\mathbf{a}}{d\lambda}, \mathbf{a} \right\rangle = \left\langle \frac{d\mathbf{a}}{d\lambda}, \mathbf{X}^{-1}\mathbf{a} \right\rangle = \left\langle \frac{d\mathbf{a}}{d\lambda}, \rho^{-1}\mathbf{a} \right\rangle = \bar{\rho}^{-1}\left\langle \frac{d\mathbf{a}}{d\lambda}, \mathbf{a} \right\rangle.$$

At $\lambda = \lambda_0$ we have $\rho_j(\lambda) = \rho_0$ and, since $|\rho_0| = 1$, it follows that $\bar{\rho}_0^{-1} = \rho_0$. Consequently,

$$\left\langle \mathbf{X}(T, \lambda_0)\left(\frac{d\mathbf{a}}{d\lambda}\right)_{\lambda_0}, \mathbf{a}(\lambda_0) \right\rangle = \rho_0\left\langle \left(\frac{d\mathbf{a}}{d\lambda}\right)_{\lambda_0}, \mathbf{a}(\lambda_0) \right\rangle,$$

i. e., at $\lambda = \lambda_0$ the last terms on the left and right of (1.19) cancel, and so

$$\left(\frac{d\rho}{d\lambda}\right)_{\lambda_0}\langle \mathbf{a}(\lambda_0), \mathbf{a}(\lambda_0) \rangle = \left\langle \frac{\partial \mathbf{X}(T, \lambda_0)}{\partial \lambda}\mathbf{a}(\lambda_0), \mathbf{a}(\lambda_0) \right\rangle. \tag{1.20}$$

We now transform the right-hand side of (1.20). Differentiating the identity

$$i^{-1}\mathbf{G}\frac{\partial \mathbf{X}(t, \lambda)}{\partial t} = \mathbf{H}(t, \lambda)\mathbf{X}(t, \lambda)$$

with respect to λ, we get

* The elements $\mathbf{a}_j(\lambda)$ may be defined as suitable minors of the matrix $\mathbf{X}(T, \lambda) - \rho_j(\lambda)\mathbf{I}_{2k}$. Since the vectors $\mathbf{a}_j(\lambda)$ are defined only up to a constant factor, "unsatisfactory" choice of this constant may make the vectors $\mathbf{a}_j(\lambda)$ nonanalytic functions of λ $(j = 1, \ldots, r)$.

$$i^{-1}\mathbf{G}\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{X}}{\partial \lambda}\right)=\mathbf{H}\frac{\partial \mathbf{X}}{\partial \lambda}+\frac{\partial \mathbf{H}}{\partial \lambda}\mathbf{X}.$$

Using formula (II. 1.17) to solve this inhomogeneous linear equation, we obtain

$$\frac{\partial \mathbf{X}(T, \lambda_0)}{\partial \lambda}=i\mathbf{X}(T, \lambda_0)\int_0^T \mathbf{X}(t, \lambda_0)^{-1}\mathbf{G}^{-1}\frac{\partial \mathbf{H}(t, \lambda_0)}{\partial \lambda}\mathbf{X}(t, \lambda_0)\,dt.$$

Substituting into (1.20) and using the fact that $\mathbf{X}(T, \lambda_0)$ is a G-unitary matrix, we obtain

$$\left(\frac{d\rho}{d\lambda}\right)_{\lambda_0}\langle \mathbf{a}(\lambda_0), \mathbf{a}(\lambda_0)\rangle =$$
$$=\left\langle i\int_0^T \mathbf{X}(t, \lambda_0)^{-1}\mathbf{G}^{-1}\frac{\partial \mathbf{H}(t, \lambda_0)}{\partial \lambda}\mathbf{X}(t, \lambda_0)\mathbf{a}(\lambda_0)\,dt,\ \mathbf{X}(T, \lambda_0)^{-1}\mathbf{a}(\lambda_0)\right\rangle.$$

Since

$$\mathbf{X}(T, \lambda_0)^{-1}\mathbf{a}(\lambda_0)=\rho_0^{-1}\mathbf{a}(\lambda_0),\ \langle \mathbf{u}, \mathbf{v}\rangle \equiv (\mathbf{Gu}, \mathbf{v})$$

and $\mathbf{X}(t, \lambda_0)$ is G-unitary, we have

$$\left(\frac{d\rho}{d\lambda}\right)_{\lambda_0}\langle \mathbf{a}(\lambda_0), \mathbf{a}(\lambda_0)\rangle = i\rho_0\int_0^T \left(\frac{\partial \mathbf{H}(t, \lambda_0)}{\partial \lambda}\mathbf{x}(t, \lambda_0), \mathbf{x}(t, \lambda_0)\right)dt,$$

where $\mathbf{x}(t, \lambda_0)=\mathbf{X}(t, \lambda_0)\mathbf{a}(\lambda_0)$ is the solution of system (1.13) at $\lambda=\lambda_0$ defined by the initial condition $\mathbf{x}(0, \lambda_0)=\mathbf{a}(\lambda_0)$. By definition, for a definite eigenvalue the product $\langle \mathbf{a}, \mathbf{a}\rangle$ does not vanish, and so

$$\left(\frac{d\rho}{d\lambda}\right)_{\lambda_0}=\frac{i\rho_0}{\langle \mathbf{a}(\lambda_0), \mathbf{a}(\lambda_0)\rangle}\int_0^T \left(\frac{\partial \mathbf{H}(t, \lambda_0)}{\partial \lambda}\mathbf{x}(t, \lambda_0), \mathbf{x}(t, \lambda_0)\right)dt. \qquad (1.20a)$$

Returning to the previous notation, we obtain (1.16) and (1.17), proving the theorem.

1.6. Behavior of multipliers of one kind under complex perturbations of the Hamiltonian

Let us consider equation (1.13) in the case that the Hamiltonian $\mathbf{H}(t, \lambda)$ is a linear function of λ:

$$i^{-1}\mathbf{G}\frac{d\mathbf{x}}{dt}=[\mathbf{H}(t)+\lambda\mathbf{Q}(t)]\mathbf{x}. \qquad (1.21)$$

Here $\mathbf{H}(t)$ and $\mathbf{Q}(t)$ are T-periodic hermitian matrix-functions. Assume, in addition, that $\mathbf{Q}(t)$ is positive definite for almost all t, i. e., $(\mathbf{Q}(t)\mathbf{c}, \mathbf{c}) > 0$ for any $\mathbf{c} \neq \mathbf{0}$. Let ρ_0 be an r-fold definite multiplier of the equation

$$i^{-1}\mathbf{G}\frac{d\mathbf{x}}{dt} = \mathbf{H}(t)\,\mathbf{x} \tag{1.22}$$

(so that $|\rho_0| = 1$). We shall treat $\lambda\,\mathbf{Q}(t)$ in (1.21) as a "perturbation" (small $|\lambda|$) of the Hamiltonian $\mathbf{H}(t)$ in equation (1.22). The multipliers of equation (1.21) that coincide with ρ_0 at $\lambda = \lambda_0$ move away from this point for $\lambda \neq 0$ of small absolute value, remaining in the neighborhood of ρ_0. We wish to examine their positions for small complex λ.

Written for equation (1.21) and $\lambda_0 = 0$, formulas (1.16) and (1.17) become

$$\rho_j(\lambda) = \rho_0[1 + i\sigma_j\lambda + O(|\lambda|^2)] \qquad (\sigma_j = -i\delta_j), \tag{1.23}$$

$$\sigma_j = \frac{1}{\langle \mathbf{a}_j, \mathbf{a}_j\rangle}\int_0^T (\mathbf{Q}(t)\,\mathbf{x}_j, \mathbf{x}_j)\,dt \qquad (j = 1, \ldots, r). \tag{1.24}$$

Since $\int_0^T (\mathbf{Q}(t)\,\mathbf{x}_j, \mathbf{x}_j)\,dt > 0$ and all the quantities $\langle \mathbf{a}_j, \mathbf{a}_j\rangle$ are real numbers of like sign (see the definition in §1.2), it follows that all the numbers (1.24) are nonzero, real and of like signs. For small $|\lambda|$, we have

$$|\rho_j(\lambda)|^2 = |1 + i\sigma_j\lambda + O(|\lambda|^2)|^2 = 1 - 2\sigma_j \operatorname{Im}\lambda + O(|\lambda|^2) \qquad (j = 1, \ldots, r).$$

Let ρ_0 be a definite multiplier of the first kind. Then $\sigma_1 > 0, \ldots, \sigma_r > 0$ (§ 1.2) and so

$$\begin{aligned} &|\rho_j(\lambda)| < 1 \quad \text{for} \quad \operatorname{Im}\lambda > 0 \qquad (j = 1, \ldots, r),\\ &|\rho_j(\lambda)| > 1 \quad \text{for} \quad \operatorname{Im}\lambda < 0 \qquad (j = 1, \ldots, r). \end{aligned}$$

We have thus proved that all the r multipliers which form an r-fold multiplier of the first kind at $\lambda = 0$ move into the unit circle as λ moves into the upper half-plane, and into the exterior of the unit circle as λ moves into the lower half-plane.

The behavior of multipliers which form a multiplier of the second kind at $\lambda = 0$ is analogous. For all j, we have $\sigma_j < 0$, and so

$$\begin{aligned} &|\rho_j(\lambda)| > 1 \quad \text{for} \quad \operatorname{Im}\lambda > 0 \qquad (j = 1, \ldots, r),\\ &|\rho_j(\lambda)| < 1 \quad \text{for} \quad \operatorname{Im}\lambda < 0 \qquad (j = 1, \ldots, r). \end{aligned}$$

Later (§4.1) we shall show that if ρ_0 is a multiplier of mixed kind, then when λ moves into the domain $\operatorname{Im}\lambda \neq 0$ some multipliers will move into $|\rho| > 1$ and others into $|\rho| < 1$. Moreover, there must be multipliers of equation (1.21 both inside and outside the unit circle in the neighborhood of ρ_0, and there are no multipliers on $|\rho| = 1$.

This proposition may also be derived directly from the formulas of perturbation theory (§ V.1).

1.7. Behavior of multipliers with increase in the Hamiltonian

We again consider the "unperturbed" equation (1.22) and the "perturbed" equation (1.21), where $\mathbf{H}(t)$ and $\mathbf{Q}(t)$ are T-periodic hermitian matrix-functions. (Equation (1.21) is therefore Hamiltonian for real λ.) Suppose that $\mathbf{Q}(t)$ is positive semidefinite for almost all t, i.e., $(\mathbf{Q}(t)\mathbf{c}, \mathbf{c}) \geqslant 0$ for every vector $\mathbf{c}$, paying special attention to the case in which $\mathbf{Q}(t)$ is positive definite for almost all t. Thus, the passage from equation (1.22) to an equation (1.21) with small real λ entails an increase in the Hamiltonian if $\lambda > 0$ and a decrease if $\lambda < 0$.

Let $\rho_0 = e^{i\varphi_0}$ be a definite r-fold multiplier of equation (1.22) on the unit circle. By the theorem of §1.5, for small real λ the corresponding r multipliers $\rho_j(\lambda)$ will lie on the unit circle. How do they behave as functions of λ? Set

$$\rho_j(\lambda) = e^{i\varphi_j(\lambda)} \qquad (j = 1, \ldots, r).$$

The functions $\varphi_j(\lambda)$ are uniquely determined by the conditions

$$\varphi_j(0) = \varphi_0 \qquad (j = 1, \ldots, r)$$

and by the theorem of §1.5 they are analytic in λ at $\lambda = 0$. We have

$$\left(\frac{d\varphi_j}{d\lambda}\right)_0 = -\frac{i}{\rho_0}\left(\frac{d\rho_j}{d\lambda}\right)_0.$$

It follows from (1.23) that

$$\varphi_j(\lambda) = \varphi_0 + \sigma_j\lambda + O(\lambda^2) \qquad (j = 1, \ldots, r),$$

where the σ_j are defined by (1.24). If $\mathbf{Q}(t) > 0$, the numbers $\sigma_1, \ldots, \sigma_r$ are positive if ρ_0 is of the first kind and negative if it is of the second kind. If $\mathbf{Q}(t) \geqslant 0$ we have $\sigma_j \geqslant 0$ or $\sigma_j \leqslant 0$ for all j, in accordance with the kind of ρ_0. We have thus proved:

Theorem (Krein). *With increase in the Hamiltonian $(\mathbf{Q}(t) > 0)$, the multipliers of the first kind on the unit circle move along the unit circle in the sense of increasing argument, while those of the second kind do so in the sense of decreasing argument.*

In particular, when multipliers of the same kind meet on the unit circle, they remain there, moving as just described. The same is true if the Hamiltonian is only nondecreasing $(\mathbf{Q}(t) \geqslant 0)$, provided the increase (or decrease) in the argument is understood in the weak sense:

$$\partial\varphi_j/\partial\lambda \geqslant 0 \qquad (\text{for } \partial\varphi_j/\partial\lambda \leqslant 0).$$

Thus, simple multipliers may leave the unit circle only after a "meeting" of multipliers of different kinds. Similarly, if multipliers reach the unit circle from its exterior or interior and subsequently remain there, they must be of different kinds, and at the point of contact with the unit circle they form a multiple multiplier of mixed kind.

A more detailed account of the motion of multipliers with increase in the Hamiltonian will be given in §4.3.

§2. LINEAR TRANSFORMATIONS IN A SPACE WITH INDEFINITE METRIC

In this section we present a few auxiliary algebraic propositions needed for our study of Hamiltonian systems. In particular, we shall establish some additional properties of the eigenvalues of G-unitary and G-Hamiltonian matrices.

2.1. Classification of subspaces in a space with indefinite metric

As before, let $\mathfrak{R}_n$ be a complex linear space with scalar product $\langle \mathbf{x}, \mathbf{y}\rangle$ (§1.1). A subspace $\mathfrak{L}\subset\mathfrak{R}_n$ is said to be positive definite (negative definite) if, for any vector $\mathbf{x}\in\mathfrak{L}$ $(\mathbf{x}\neq\mathbf{0})$,

$$\langle \mathbf{x}, \mathbf{x}\rangle > 0 \qquad (\langle \mathbf{x}, \mathbf{x}\rangle < 0).$$

Positive and negative definite subspaces are said to be definite. Recalling the definition of the kind of an eigenvalue of a G-unitary matrix (§1.2), we can say that a definite eigenvalue ρ ($|\rho|=1$) has a definite eigensubspace $\mathfrak{S}_\rho$, positive definite if ρ is of the first kind and negative definite if ρ is of the second kind. A subspace $\mathfrak{L}\in\mathfrak{R}_n$ is said to be degenerate if there is a vector $\mathbf{x}_0\in\mathfrak{L}$ $(\mathbf{x}_0\neq\mathbf{0})$ which is G-orthogonal to $\mathfrak{L}$: $\langle \mathbf{x}_0, \mathbf{x}\rangle=0$ for all $\mathbf{x}\in\mathfrak{L}$. Otherwise we shall say that $\mathfrak{L}$ is nondegenerate. Thus a subspace $\mathfrak{L}$ is nondegenerate if and only if the condition $\langle \mathbf{x}_0, \mathbf{x}\rangle=0$ ($\mathbf{x}_0\in\mathfrak{L}$) cannot hold for every $\mathbf{x}\in\mathfrak{L}$ unless $\mathbf{x}_0=\mathbf{0}$.

It is clear that definite subspaces are nondegenerate. By property 3 in §1.1, the space $\mathfrak{R}_n$ itself is nondegenerate.

Subspaces $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are said to be G-orthogonal if $\langle \mathbf{x}_1, \mathbf{x}_2\rangle=0$ for any $\mathbf{x}_1\in\mathfrak{L}_1$, $\mathbf{x}_2\in\mathfrak{L}_2$. A subspace $\mathfrak{L}$ may be G-orthogonal to itself; this simply means that $\langle \mathbf{x}, \mathbf{y}\rangle\equiv 0$ for any $\mathbf{x}\in\mathfrak{L}, \mathbf{y}\in\mathfrak{L}$. If $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are G-orthogonal, disjoint and $\mathfrak{L}$ is their direct sum (§I.1.9),

$$\mathfrak{L}=\mathfrak{L}_1\dotplus\mathfrak{L}_2,$$

we shall write

$$\mathfrak{L}=\mathfrak{L}_1\langle +\rangle\mathfrak{L}_2.$$

If $\langle \mathbf{x}_0, \mathbf{x}_1\rangle=0$ and $\langle \mathbf{x}_0, \mathbf{x}\rangle=0$ for any $\mathbf{x}\in\mathfrak{L}$, we write $\mathbf{x}_0\langle\perp\rangle\mathbf{x}_1$, $\mathbf{x}_0\langle\perp\rangle\mathfrak{L}$.

2.2. Gram matrix

Let $\mathfrak{L}$ be an $\mathfrak{R}_n$-dimensional subspace of m and $\mathbf{f}_1, \ldots, \mathbf{f}_m$ a basis in $\mathfrak{L}$. The $m \times m$ matrix

$$\mathbf{G}_{\mathfrak{L}} = \begin{Vmatrix} \langle \mathbf{f}_1, \mathbf{f}_1 \rangle \ldots \langle \mathbf{f}_m, \mathbf{f}_1 \rangle \\ \langle \mathbf{f}_1, \mathbf{f}_m \rangle \ldots \langle \mathbf{f}_m, \mathbf{f}_m \rangle \end{Vmatrix} \tag{2.1}$$

is called the Gram matrix of $\mathfrak{L}$ relative to the basis $\mathbf{f}_1, \ldots, \mathbf{f}_m$. * In particular, the original matrix $\mathbf{G}$ is the Gram matrix of $\mathfrak{R}_n$ relative to the standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ (§I. 1. 4). Indeed,

$$\begin{Vmatrix} \langle \mathbf{e}_1, \mathbf{e}_1 \rangle \ldots \langle \mathbf{e}_n, \mathbf{e}_1 \rangle \\ \cdots\cdots\cdots \\ \langle \mathbf{e}_1, \mathbf{e}_n \rangle \ldots \langle \mathbf{e}_n, \mathbf{e}_n \rangle \end{Vmatrix} = \begin{Vmatrix} (\mathbf{G}\mathbf{e}_1, \mathbf{e}_1) \ldots (\mathbf{G}\mathbf{e}_n, \mathbf{e}_1) \\ \cdots\cdots\cdots \\ (\mathbf{G}\mathbf{e}_1, \mathbf{e}_n) \ldots (\mathbf{G}\mathbf{e}_n, \mathbf{e}_n) \end{Vmatrix} = \mathbf{G}.$$

Consider the $n \times m$ matrix

$$\mathbf{S} = \| \mathbf{f}_1, \ldots, \mathbf{f}_m \|$$

whose μ-th column is formed by the components of $\mathbf{f}_\mu$ relative to the standard basis $(\mu = 1, \ldots, m)$. It is easy to see that

$$\mathbf{G}_{\mathfrak{L}} = \mathbf{S}^* \mathbf{G} \mathbf{S}. \tag{2.2}$$

The Gram matrix depends, of course, on the choice of basis. Let us go over to another basis in $\mathfrak{L}$, say

$$\mathbf{g}_\mu = \sum_{r=1}^{m} \sigma_{\mu r} \mathbf{f}_r \qquad (\mu = 1, \ldots, m).$$

We wish to determine the Gram matrix of $\mathfrak{L}$ relative to the basis $\mathbf{g}_1, \ldots, \mathbf{g}_m$.

Since $\mathbf{g}_1, \ldots, \mathbf{g}_m$ form a basis, the matrix

$$\mathbf{D} = \| \sigma_{r\mu} \|_1^m = \begin{Vmatrix} \sigma_{11} \ldots \sigma_{m1} \\ \cdots\cdots \\ \sigma_{1m} \ldots \sigma_{mm} \end{Vmatrix}$$

is nonsingular: $\det \mathbf{D} \neq 0$. Indeed, if $\det \mathbf{D} = 0$ there exist constants $\alpha_1, \ldots, \alpha_m$ $(|\alpha_1| + \ldots + |\alpha_m| \neq 0)$ such that $\alpha_1 \sigma_{1r} + \ldots + \alpha_m \sigma_{mr} = 0$ $(r = 1, \ldots, m)$. Multiplying these equalities by $\mathbf{f}_1, \ldots, \mathbf{f}_m$ and adding, we see that $\alpha_1 \mathbf{g}_1 + \ldots + \alpha_m \mathbf{g}_m = \mathbf{0}$, so that the vectors $\mathbf{g}_1, \ldots, \mathbf{g}_m$ are linearly dependent. We shall therefore assume that $\det \mathbf{D} \neq 0$. Then

$$\langle \mathbf{g}_h, \mathbf{g}_j \rangle = \sum_{r,t=1}^{m} \sigma_{hr} \langle \mathbf{f}_r, \mathbf{f}_t \rangle \bar{\sigma}_{jt} \qquad (j, h = 1, \ldots, m),$$

or, letting $\tilde{\mathbf{G}}_{\mathfrak{L}}$ denote the matrix

$$\tilde{\mathbf{G}}_{\mathfrak{L}} = \begin{Vmatrix} \langle \mathbf{g}_1, \mathbf{g}_1 \rangle \ldots \langle \mathbf{g}_m, \mathbf{g}_1 \rangle \\ \langle \mathbf{g}_1, \mathbf{g}_m \rangle \ldots \langle \mathbf{g}_m, \mathbf{g}_m \rangle \end{Vmatrix},$$

* In some texts the Gram matrix is defined as the transpose $\mathbf{G}_{\mathfrak{L}}^{\mathrm{T}}$.

we have

$$\tilde{\mathbf{G}}_{\mathfrak{L}} = \mathbf{D}^* \mathbf{G}_{\mathfrak{L}} \mathbf{D}. \tag{2.3}$$

This is the required formula for transformation of the Gram matrix under a change of basis.

Combining (2.2) and (2.3), we obtain

$$\tilde{\mathbf{G}}_{\mathfrak{L}} = (\mathbf{SD})^* \mathbf{G} (\mathbf{SD}).$$

On the other hand, if $m=n$ and the original basis is the standard one, we can derive (2.2) from (2.3), by virtue of the fact that in this case $\mathbf{S}=\mathbf{D}$.

The eigenvalues of a Gram matrix are real, since the Gram matrix is hermitian:

$$\mathbf{G}_{\mathfrak{L}}^* = \| \overline{\langle \mathbf{f}_j, \mathbf{f}_h \rangle} \|_1^m = \| \langle \mathbf{f}_h, \mathbf{f}_j \rangle \|_1^m = \mathbf{G}_{\mathfrak{L}}.$$

It is known from linear algebra (see /63/, Chapter V, §3) that the matrices $\mathbf{G}_{\mathfrak{L}}$ and $\tilde{\mathbf{G}}_{\mathfrak{L}}$ have the same number p of positive eigenvalues, the same number q of negative eigenvalues, and the same number s of zero eigenvalues. The converse is also valid (ibid.): if $\tilde{\mathbf{G}}_{\mathfrak{L}}$ is an arbitrary hermitian matrix with the same p, q, s as $\mathbf{G}_{\mathfrak{L}}$, there is a nonsingular matrix $\mathbf{D}$ such that (2.3) holds. Letting $\tilde{\mathbf{G}}_{\mathfrak{L}}$ be the diagonal matrix

$$\tilde{\mathbf{G}}_{\mathfrak{L}} = \begin{pmatrix} \mathbf{I}_p & 0 & 0 \\ 0 & -\mathbf{I}_q & 0 \\ 0 & 0 & \mathbf{0}_s \end{pmatrix}$$

(where $\mathbf{0}_s$ denotes the $s \times s$ zero matrix), we obtain:

Lemma I. Let p be the number of positive eigenvalues, q the number of negative eigenvalues and s the number of zero eigenvalues of a Gram matrix of a subspace $\mathfrak{L}$. There exists a basis $\mathbf{g}_j \in \mathfrak{L}$ such that

$$\langle \mathbf{g}_j, \mathbf{g}_h \rangle = 0 \quad \text{if} \quad j \neq h,$$

$$\langle \mathbf{g}_j, \mathbf{g}_j \rangle = \begin{cases} 1, & 1 \leqslant j \leqslant p, \\ -1, & p+1 \leqslant j \leqslant p+q, \\ 0, & p+q+1 \leqslant j \leqslant p+q+s. \end{cases}$$

The basis vectors $\mathbf{g}_j$ may be constructed effectively from the vectors of an arbitrary basis $\mathbf{f}_j$, by an orthogonalization process similar to that considered in § I. 1.5 but complicated by the fact that $\langle \mathbf{x}, \mathbf{y} \rangle$ is an indefinite scalar product; we shall not go into details.

The properties of nondegeneracy and definiteness for a subspace are readily expressed in terms of the Gram matrix:

Lemma II. A subspace $\mathfrak{L}$ is positive definite (negative definite) or nondegenerate if and only if its Gram matrix relative to some (and therefore relative to any) basis is positive definite (negative definite) or nonsingular, respectively.

Proof. 1) Let $\mathfrak{L}$ be a positive definite subspace and $\mathbf{f}_1, \ldots, \mathbf{f}_m$ a fixed basis. Any vector $\mathbf{x} \in \mathfrak{L}$ is expressible as

$$\mathbf{x} = \sum_{j=1}^{m} \xi_j \mathbf{f}_j.$$

The matrix of the quadratic form

$$\langle \mathbf{x}, \mathbf{x} \rangle = \sum_{j,h=1}^{m} \langle \mathbf{f}_j, \mathbf{f}_h \rangle \xi_j \xi_h$$

is the transpose $\mathbf{G}_{\mathfrak{L}}^{\tau}$ of the Gram matrix. Thus the condition $\mathbf{G}_{\mathfrak{L}} > \mathbf{0}$ ($\mathbf{G}_{\mathfrak{L}} < \mathbf{0}$) is equivalent to the conditions: $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ ($\langle \mathbf{x}, \mathbf{x} \rangle < 0$) for all $\mathbf{x} \in \mathfrak{L}$.

2) Let $\det \mathbf{G}_{\mathfrak{L}} = 0$. Then the columns of $\mathbf{G}_{\mathfrak{L}}$ are linearly dependent:

$$\alpha_1 \langle \mathbf{f}_1, \mathbf{f}_\mu \rangle + \ldots + \alpha_m \langle \mathbf{f}_m, \mathbf{f}_\mu \rangle = 0 \qquad (\mu = 1, \ldots, m), \tag{2.4}$$

where not all the constants $\alpha_1, \ldots, \alpha_m$ are zero. Then the vector

$$\mathbf{a} = \alpha_1 \mathbf{f}_1 + \ldots + \alpha_m \mathbf{f}_m \neq \mathbf{0}$$

and the relations (2.4) may be written as

$$\langle \mathbf{a}, \mathbf{f}_\mu \rangle = 0 \qquad (\mu = 1, \ldots, m). \tag{2.5}$$

Hence it follows that for any $\mathbf{x} \in \mathfrak{L}$, $\mathbf{x} = \xi_1 \mathbf{f}_1 + \ldots + \xi_m \mathbf{f}_m$, we have $\langle \mathbf{a}, \mathbf{x} \rangle = 0$, so that $\mathfrak{L}$ is a degenerate subspace.

If $\mathfrak{L}$ is nondegenerate, then (2.5) may hold only if $\mathbf{a} = \mathbf{0}$, and so (2.4) may hold only if all the constants $\alpha_1, \ldots, \alpha_m$ all vanish; thus the columns of the matrix are linearly independent, so that $\det \mathbf{G}_{\mathfrak{L}} \neq 0$, and the proof is complete.

We now express the G-orthogonality property in terms of Gram matrices. Let

$$\mathfrak{L} = \mathfrak{L}_1 \langle + \rangle \mathfrak{L}_2$$

(see end of §2.1) and let $\mathbf{G}_1$, $\mathbf{G}_2$ be the Gram matrices of $\mathfrak{L}_1$, $\mathfrak{L}_2$ relative to bases $\mathbf{f}_j' \in \mathfrak{L}_1$, $\mathbf{f}_j'' \in \mathfrak{L}_2$. The set of vectors $\{\mathbf{f}_j', \mathbf{f}_j''\}$ is a basis for $\mathfrak{L}$, relative to which the Gram matrix $\mathbf{G}_{\mathfrak{L}}$ is

$$\mathbf{G}_{\mathfrak{L}} = \begin{pmatrix} \mathbf{G}_1 & 0 \\ 0 & \mathbf{G}_2 \end{pmatrix} = \mathbf{G}_1 \dot{+} \mathbf{G}_2. \tag{2.6}$$

Conversely, let $\mathfrak{L} = \mathfrak{L}_1 \dot{+} \mathfrak{L}_2$ and suppose that the Gram matrix relative to the union of a basis $\{\mathbf{f}_j'\}$ of $\mathfrak{L}_1$ and a basis $\{\mathbf{f}_j''\}$ of $\mathfrak{L}_2$ has the form (2.6). Then it is clear that $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are G-orthogonal.

The set of vectors $\mathbf{x}$ which are G-orthogonal to all vectors in a subspace $\mathfrak{L}$ is clearly a subspace, known as the G-orthogonal complement of $\mathfrak{L}$ and denoted by $\mathfrak{L}^{\langle \perp \rangle}$.

The definition of the G-orthogonal complement of a subspace $\mathfrak{L}_1 \subset \mathfrak{L}$ in a nondegenerate subspace $\mathfrak{L}$ is similar (with $\mathfrak{R}_n$ replaced by $\mathfrak{L}$).

For a degenerate subspace, there exists by definition a vector $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{x} \in \mathfrak{L}^{\langle\perp\rangle}$ and $\mathbf{x} \in \mathfrak{L}$. Thus the G-orthogonal complement of a degenerate subspace has nonempty intersection with the subspace itself. Hence the existence of a decomposition

$$\mathfrak{R}_n = \mathfrak{L}_1 \langle + \rangle \mathfrak{L}_2$$

(which presupposes that the intersection of $\mathfrak{L}_1$ and $\mathfrak{L}_2$ contains only the zero vector and that $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ and $\mathbf{x} \in \mathfrak{L}_1$, $\mathbf{y} \in \mathfrak{L}_2$) implies that each of the subspaces $\mathfrak{L}_1$ and $\mathfrak{L}_2$ is nondegenerate. This is readily verified directly. Indeed, if $\mathbf{a} \langle\perp\rangle \mathfrak{L}_1$, $\mathbf{a} \in \mathfrak{L}_1$, then $\mathbf{a} \langle\perp\rangle \mathfrak{L}_2$, so that $\mathbf{a} \langle\perp\rangle \mathfrak{R}_n$ and thus $\mathbf{a} = \mathbf{0}$.

If $\mathfrak{L}$ is a nondegenerate subspace, it follows from the definition that the intersection of $\mathfrak{L}$ with its G-orthogonal complement $\mathfrak{L}^{\langle\perp\rangle}$ contains only $\mathbf{x} = \mathbf{0}$.

Lemma III (complement of a nondegenerate subspace). Let $\mathfrak{L}$ be a nondegenerate subspace which is not the entire space. Then any vector $\mathbf{x} \in \mathfrak{R}_n$ admits a unique decomposition

$$\mathbf{x} = \mathbf{y} + \mathbf{z}, \quad \textit{where } \mathbf{y} \in \mathfrak{L}, \quad \mathbf{z} \in \mathfrak{L}^{\langle\perp\rangle}. \tag{2.7}$$

In other words, for any nondegenerate subspace of $\mathfrak{R}_n$,

$$\mathfrak{R}_n = \mathfrak{L} \langle + \rangle \mathfrak{L}^{\langle\perp\rangle}.$$

The proof is word for word the same as that of Lemma I in § I. 1.5. Similar arguments yield the next lemma, which is also a corollary of Lemma III (simply replace $\mathfrak{R}_n$ by $\mathfrak{L}$).

Lemma IV. Let $\mathfrak{L}$, $\mathfrak{L}_1 \subset \mathfrak{L}$ be nondegenerate subspaces, $\mathfrak{L}_1 \neq \mathfrak{L}$. Let $\mathfrak{L}_2$ be the G-orthogonal complement of $\mathfrak{L}_1$ in $\mathfrak{L}$. Then

$$\mathfrak{L} = \mathfrak{L}_1 \langle + \rangle \mathfrak{L}_2.$$

If $\mathfrak{L}$ and $\mathfrak{L}_1$ are degenerate, these assertions fail to hold.

2.3. Lemma on real nonsingular skew-symmetric matrices

As a corollary of the lemmas of §2.2, we shall prove the following propositions, to be used in the sequel.

Lemma. a) Any nonsingular real skew-symmetric matrix $\mathbf{J}'$ is of even order. b) Let $\mathbf{J}'$ and $\mathbf{J}''$ be any two real nonsingular skew-symmetric matrices. There exists a real nonsingular matrix $\mathbf{S}$ such that

$$\mathbf{S}^* \mathbf{J}' \mathbf{S} = \mathbf{J}''. \tag{2.8}$$

Proof. a) Set $G = iJ'$. Obviously, $G^* = G$, $\det G \neq 0$. Let $\langle x, y\rangle = (Gx, y)$ be the corresponding nondegenerate scalar product. Let a_1 be an arbitrary real vector such that $J'a_1 \neq 0$. The equation $\langle a_1, x\rangle = i$ has a real solution $x = a_2 \neq 0$. For example, we can take $a_2 = J'a_1 / |J'a_1|^2$.

For any real vector a,

$$\langle a, a\rangle = 0. \tag{2.9}$$

Indeed,

$$(J'a, a) = -(a, J'a) = -(J'a, a),$$

so that $(J'a, a) = 0$ and (2.9) is true. It follows from (2.9) that the subspace $\mathfrak{L}_1 = \{a_1, a_2\}$ spanned by a_1, a_2 has the Gram matrix

$$\begin{Vmatrix} 0 & +i \\ -i & 0 \end{Vmatrix}. \tag{2.10}$$

Since the determinant of this matrix does not vanish, $\mathfrak{L}_1$ is a nondegenerate subspace. Hence, by Lemma III,

$$\mathfrak{R}_n = \mathfrak{L}_1 \langle + \rangle \mathfrak{L}, \tag{2.11}$$

where $\mathfrak{L} = \mathfrak{L}_1^{\langle \perp \rangle}$.

We claim that $\mathfrak{L}$ is a real subspace (see the definition in §I.1.17). Indeed, taking complex conjugates in the equalities $(J'a_1, x) = 0$, $(J'a_2, x) = 0$ and noting that $\overline{J'} = J'$, $\overline{a_1} = a_1$, $\overline{a_2} = a_2$, we obtain

$$(J'a_1, \overline{x}) = 0, \quad (J'a_2, \overline{x}) = 0.$$

Thus, if $x \in \mathfrak{L}$ then $\overline{x} \in \mathfrak{L}$.

Let $a_3 \neq 0$ be an arbitrary real vector in $\mathfrak{L}$. The equation $\langle a_3, x\rangle = i$ has a real solution $x = a_4$. Indeed, otherwise we would have $\langle a_3, x\rangle = 0$ for all $x \in \mathfrak{L}$, i. e., $(J'a_3, x) = 0$. Since $J'a_3$ is a real vector, this equality will also hold for all complex vectors $x \in \mathfrak{L}$. It follows, then, that $\mathfrak{L}$ is a degenerate subspace, contrary to (2.11). Let $\mathfrak{L}_2 = \{a_3, a_4\}$ be the subspace spanned by a_3, a_4. Its Gram matrix is again (2.10). Consequently, $\mathfrak{L}_2$ is a nondegenerate real subspace, G-orthogonal to $\mathfrak{L}_1$. Continuing to split off subspaces in this way, we finally obtain a subspace $\mathfrak{L}_p$, either one- or two-dimensional, which is G-orthogonal to each of the preceding two-dimensional subspaces $\mathfrak{L}_1, \mathfrak{L}_2, \ldots, \mathfrak{L}_{p-1}$. Let $\mathfrak{L}_p = \{a_{2p-1}\}$ be a one-dimensional real subspace and $a_{2p-1} \neq 0$ a real vector. By (2.9), we have $\langle a_{2p-1}, a_{2p-1}\rangle = 0$, and so a_{2p-1} is G-orthogonal to the entire space, which is impossible. Therefore $\mathfrak{L}_p = \{a_{2p-1}, a_{2p}\}$ must be two-dimensional. Consequently, J' has even order $2p$, proving part (a).

b) Relative to the above basis $a_1, a_2, \ldots, a_{2p-1}, a_{2p}$, the Gram matrix is

$$\|\langle a_j, a_h\rangle\| = \begin{Vmatrix} 0 & i \\ -i & 0 \end{Vmatrix} \dotplus \begin{Vmatrix} 0 & i \\ -i & 0 \end{Vmatrix} \dotplus \ldots \dotplus \begin{Vmatrix} 0 & i \\ -i & 0 \end{Vmatrix}. \tag{2.12}$$

Canceling the factor i, we get

$$\|(\mathbf{J}'\mathbf{a}_j, \mathbf{a}_h)\| = \tilde{\mathbf{J}}, \tag{2.13}$$

where

$$\tilde{\mathbf{J}} = \mathbf{J}_2 \dot{+} \mathbf{J}_2 \dot{+} \ldots \dot{+} \mathbf{J}_2, \quad \mathbf{J}_2 = \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix}.$$

Defining a real nonsingular matrix

$$\mathbf{S}_1 = \|\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{2p-1}, \mathbf{a}_{2p}\|$$

with columns $\mathbf{a}_1, \ldots, \mathbf{a}_{2p}$, we rewrite (2.13) as

$$\mathbf{S}_1^* \mathbf{J}' \mathbf{S}_1 = \tilde{\mathbf{J}}. \tag{2.14}$$

Here $\mathbf{J}'$ is an arbitrary real skew-symmetric nonsingular matrix. Thus, given $\mathbf{J}''$, we can again construct a real nonsingular matrix $\mathbf{S}_2$ such that

$$\mathbf{S}_2^* \mathbf{J}'' \mathbf{S}_2 = \tilde{\mathbf{J}}. \tag{2.15}$$

We now deduce (2.8), with $\mathbf{S} = \mathbf{S}_1 \mathbf{S}_2^{-1}$, from (2.14) and (2.15).

2.4. G-orthogonality of certain root subspaces of G-unitary and G-antihermitian matrices

Lemma. 1) Let ρ_1, ρ_2 be eigenvalues of a G-unitary matrix $\mathbf{X}$ (the case $\rho_1 = \rho_2$ is not excluded) and $\mathfrak{L}_{\rho_1}$, $\mathfrak{L}_{\rho_2}$ corresponding root subspaces (see §I.1.10). If

$$\rho_1 \bar{\rho}_2 \neq 1, \tag{2.16}$$

then $\mathfrak{L}_{\rho_1}$ and $\mathfrak{L}_{\rho_2}$ are G-orthogonal subspaces.

2) The same assertion is true for root subspaces $\mathfrak{L}_{\lambda_1}$, $\mathfrak{L}_{\lambda_2}$ of a G-Hamiltonian matrix $\mathbf{A}$, with condition (2.16) replaced by

$$\lambda_1 + \bar{\lambda}_2 \neq 0. \tag{2.17}$$

Remark. Condition (2.16) or (2.17) is automatically fulfilled if $|\rho_1| = 1$, $|\rho_2| \neq 1$ or $\operatorname{Re}\lambda_1 = 0$, $\operatorname{Re}\lambda_2 \neq 0$, respectively.

Proof. 1) Let $\mathbf{a} \in \mathfrak{L}_{\rho_1}$, $\mathbf{b} \in \mathfrak{L}_{\rho_2}$, i.e.,

$$(\mathbf{X} - \rho_1 \mathbf{I}_n)^{m_1} \mathbf{a} = 0, \quad (\mathbf{X} - \rho_2 \mathbf{I}_n)^{m_2} \mathbf{b} = 0$$

for some natural numbers m_1, m_2. Our assertion will be proved by induction on $m_1 + m_2$. If $m_1 + m_2 = 2$, then $m_1 = m_2 = 1$ and both root vectors $\mathbf{a}_1$ and $\mathbf{b}_1$ are eigenvectors:

$$\mathbf{X}\mathbf{a}_1 = \rho_1 \mathbf{a}_1, \quad \mathbf{X}\mathbf{b}_1 = \rho_2 \mathbf{b}_1.$$

Then, since $\mathbf{X}^+\mathbf{X}=\mathbf{I}_n$, we have

$$(\rho_1\bar{\rho}_2-1)\langle \mathbf{a}_1, \mathbf{b}_1\rangle=\langle \rho_1\mathbf{a}_1, \rho_2\mathbf{b}_1\rangle-\langle \mathbf{a}_1, \mathbf{b}_1\rangle= \\ =\langle \mathbf{Xa}_1, \mathbf{Xb}_1\rangle-\langle \mathbf{a}_1, \mathbf{b}_1\rangle=\langle \mathbf{X}^+\mathbf{Xa}_1, \mathbf{b}_1\rangle-\langle \mathbf{a}_1, \mathbf{b}_1\rangle=0,$$

proving the assertion for $m_1=m_2=1$.

Now let $m\geqslant 3$ and suppose the assertion true for all natural numbers m_1', m_2' such that $m_1'+m_2'<m$. We shall prove it true for any m_1, m_2 such that $m_1+m_2=m$.

Denote

$$\mathbf{a}'=(\mathbf{X}-\rho_1\mathbf{I}_n)\,\mathbf{a}, \quad \mathbf{b}'=(\mathbf{X}-\rho_2\mathbf{I}_n)\,\mathbf{b}$$

and write the assumptions of the lemma as

$$(\mathbf{X}-\rho_1\mathbf{I}_n)^{m_1-1}\,\mathbf{a}'=0, \quad (\mathbf{X}-\rho_2\mathbf{I}_n)^{m_2-1}\,\mathbf{b}'=0.$$

By the inductive hypothesis, $\langle \mathbf{a}', \mathbf{b}'\rangle=0$, $\langle \mathbf{a}', \mathbf{b}\rangle=0$, $\langle \mathbf{a}, \mathbf{b}'\rangle=0$. Substituting $\mathbf{a}'$ and $\mathbf{b}'$ into these equalities, we obtain

$$\langle \mathbf{Xa}, \mathbf{Xb}\rangle-\rho_1\langle \mathbf{a}, \mathbf{Xb}\rangle-\bar{\rho}_2\langle \mathbf{Xa}, \mathbf{b}\rangle+\rho_1\bar{\rho}_2\langle \mathbf{a}, \mathbf{b}\rangle=0, \\ \langle \mathbf{Xa}, \mathbf{b}\rangle-\rho_1\langle \mathbf{a}, \mathbf{b}\rangle=0, \quad \langle \mathbf{a}, \mathbf{Xb}\rangle-\bar{\rho}_2\langle \mathbf{a}, \mathbf{b}\rangle=0.$$

Substituting the values of $\langle \mathbf{Xa}, \mathbf{b}\rangle$, $\langle \mathbf{a}, \mathbf{Xb}\rangle$ from the last two equalities into the first one and using the relation $\langle \mathbf{Xa}, \mathbf{Xb}\rangle=\langle \mathbf{X}^+\mathbf{Xa}, \mathbf{b}\rangle=\langle \mathbf{a}, \mathbf{b}\rangle$, we have

$$(1-\rho_1\bar{\rho}_2)\langle \mathbf{a}, \mathbf{b}\rangle=0.$$

Since $\rho_1\bar{\rho}_2\neq 1$, it follows that $\langle \mathbf{a}, \mathbf{b}\rangle=0$, proving part (1) of the lemma.

Proceeding in exactly the same way for the proof of part 2, we obtain equalities $\langle \mathbf{a}', \mathbf{b}\rangle=0$, $\langle \mathbf{a}, \mathbf{b}'\rangle=0$, which we write as

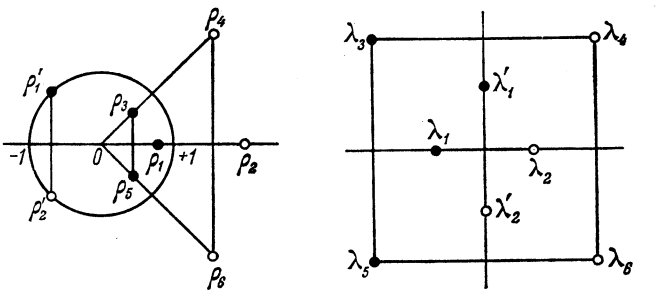
$$\langle \mathbf{Aa}, \mathbf{b}\rangle=\lambda_1\langle \mathbf{a}, \mathbf{b}\rangle, \quad \langle \mathbf{a}, \mathbf{Ab}\rangle=\bar{\lambda}_2\langle \mathbf{a}, \mathbf{b}\rangle.$$

Since $\mathbf{A}$ is now G-Hamiltonian, it follows that $\langle \mathbf{a}, \mathbf{Ab}\rangle=\langle \mathbf{A}^+\mathbf{a}, \mathbf{b}\rangle=-\langle \mathbf{Aa}, \mathbf{b}\rangle$; adding the above equalities, we see that

$$(\lambda_1+\bar{\lambda}_2)\langle \mathbf{a}, \mathbf{b}\rangle=0,$$

proving part 2 of the lemma.

An explanation of this lemma is in order. If $\rho_1\bar{\rho}_2=1$, then ρ_1 and ρ_2 are symmetric with respect to the unit circle. Thus the root subspace $\mathfrak{L}_{\rho_1}$ is G-orthogonal to all root subspaces $\mathfrak{L}_\rho$ for eigenvalues ρ which are not symmetric to ρ_1 with respect to the unit circle. For example, if the spectrum of a symplectic matrix $\mathbf{X}$ has the form illustrated in Figure 10a, then $\mathfrak{L}_{\rho_1}$ is G-orthogonal to all root subspaces except $\mathfrak{L}_{\rho_2}$; $\mathfrak{L}_{\rho_6}$ to all except $\mathfrak{L}_{\rho_5}$; $\mathfrak{L}_{\rho_4'}$ is G-orthogonal to all the other root subspaces; and so on.



a b

FIGURE 10.

Similarly, a root subspace $\mathfrak{L}_{\lambda_1}$ of a Hamiltonian matrix is G-orthogonal to all root subspaces $\mathfrak{L}_\lambda$ for which λ is not symmetric to λ_1 with respect to the imaginary axis. For example, if the spectrum of a G-Hamiltonian matrix $\mathbf{A}$ is as shown in Figure 10b, then $\mathfrak{L}_{\lambda_1}$ is G-orthogonal to all root subspaces except $\mathfrak{L}_{\lambda_2}$; $\mathfrak{L}_{\lambda_4}$ to all except $\mathfrak{L}_{\lambda_3}$; $\mathfrak{L}_{\lambda_1'}$ is G-orthogonal to all other root subspaces.

In particular, if $|\rho|=1$ ($\operatorname{Re}\lambda=0$), then $\mathfrak{L}_\rho(\mathfrak{L}_\lambda)$ is G-orthogonal to all other root subspaces.

Let $\mathbf{X}$ be a G-unitary matrix; let $\rho_1', \rho_2', \ldots$ be all its eigenvalues of absolute value 1, and $\rho_1, \bar{\rho}_1^{-1}, \rho_2, \bar{\rho}_2^{-1}, \ldots$ its eigenvalues of absolute value $\neq 1$ (only distinct eigenvalues are enumerated). We know (I. 1.39) that the whole space is the direct sum of its root subspaces:

$$\mathfrak{R}_n=\mathfrak{L}_{\rho_1'}\dot{+}\mathfrak{L}_{\rho_2'}\dot{+}\ldots\dot{+}(\mathfrak{L}_{\rho_1}\dot{+}\mathfrak{L}_{\bar{\rho}_1^{-1}})\dot{+}(\mathfrak{L}_{\rho_2}\dot{+}\mathfrak{L}_{\bar{\rho}_2^{-1}})\dot{+}\ldots$$

The parentheses pair off subspaces belonging to eigenvalues which are symmetric about the unit circle. Viewing each term in parentheses as a single term, we see from the lemma that each term in this sum is G-orthogonal to the others.

In the notation adopted previously, this equality is written

$$\mathfrak{R}_{2k}=\mathfrak{L}_{\rho_1'}\langle+\rangle\mathfrak{L}_{\rho_2'}\langle+\rangle\ldots\langle+\rangle(\mathfrak{L}_{\rho_1}\dot{+}\mathfrak{L}_{\bar{\rho}_1^{-1}})\langle+\rangle(\mathfrak{L}_{\rho_2}\dot{+}\mathfrak{L}_{\bar{\rho}_2^{-1}})\langle+\rangle\ldots \quad (2.18)$$

Note that by the lemma each of the subspaces $\mathfrak{L}_{\rho_1}, \mathfrak{L}_{\rho_2}, \ldots$ is G-orthogonal to itself.

We claim that each term in (2.18) (viewing each pair in parentheses as a separate term) is a nondegenerate subspace. Indeed, consider, say $\mathfrak{L}_{\rho_1'}$. Let $\mathbf{x}_0\in\mathfrak{L}_{\rho_1'}$ and $\mathbf{x}_0\langle\perp\rangle\mathfrak{L}_{\rho_1'}$. Since $\mathbf{x}_0$ is G-orthogonal to all the other subspaces, it follows that $\mathbf{x}_0\langle\perp\rangle\mathfrak{R}_{2k}$, i. e., $\langle\mathbf{x}_0,\mathbf{x}\rangle=0$ for all $\mathbf{x}\in\mathfrak{R}_{2k}$. Then $(\mathbf{G}\mathbf{x}_0,\ \mathbf{x})=0$, whence $\mathbf{G}\mathbf{x}_0=\mathbf{0}$ and $\mathbf{x}_0=\mathbf{0}$, and this means that $\mathfrak{L}_{\rho_1'}$ is nondegenerate. Thus the Gram matrix $\mathbf{G}_{\rho'}$ is nonsingular: $\det\mathbf{G}_{\rho'}\neq 0$.

In particular, this provides a proof of the assertion stated without proof in §1.2: If ρ' is a simple eigenvalue of a G-unitary matrix $\mathbf{X}$, $|\rho'|=1$ and $\mathbf{a}'$ is a corresponding eigenvector $(\mathbf{X}\mathbf{a}'=\rho'\mathbf{a}')$, then $\langle \mathbf{a}', \mathbf{a}'\rangle \neq 0$.

2.5. Canonical decompositions of G-unitary and G-Hamiltonian matrices with simple elementary divisors

Without assuming for the moment that a G-unitary matrix $\mathbf{X}$ has simple elementary divisors, let us choose bases in each of the root subspaces in (2.18) and combine them into a basis for the whole space. Let $\mathbf{G}_{\rho'}$ denote the Gram matrix of the root subspace $\mathfrak{L}_{\rho'}$. Since $\mathfrak{L}_{\rho'}$ is nondegenerate, $\det \mathbf{G}_{\rho'} \neq 0$. The Gram matrix of a subspace $\mathfrak{L}_{\rho} \dotplus \mathfrak{L}_{\bar{\rho}^{-1}}$, $|\rho| \neq 1$, is

$$\begin{pmatrix} 0 & \mathbf{G}_{\rho} \\ \mathbf{G}_{\rho}^{*} & 0 \end{pmatrix}, \tag{2.19}$$

since by the lemma in §2.4 each of the subspaces $\mathfrak{L}_{\rho}$ ($|\rho| \neq 1$) is G-orthogonal to itself. It follows from (2.18) that the Gram matrix $\tilde{\mathbf{G}}$ of the whole space relative to this basis is

$$\mathbf{G}=\begin{pmatrix} \mathbf{G}_{\rho_1'} & & & & & \\ & \mathbf{G}_{\rho_2'} & & & & \\ & & \ddots & & & \\ & & & \begin{pmatrix} 0 & \mathbf{G}_{\rho_1} \\ \mathbf{G}_{\rho_1}^{*} & 0 \end{pmatrix} & & \\ & & & & \begin{pmatrix} 0 & \mathbf{G}_{\rho_2} \\ \mathbf{G}_{\rho_2}^{*} & 0 \end{pmatrix} & \\ & & & & & \ddots \end{pmatrix} \tag{2.20}$$

(with zeros in the empty spaces). Alternatively, we can write (2.20) as

$$\tilde{\mathbf{G}}=\mathbf{G}_{\rho_1'} \dotplus \mathbf{G}_{\rho_2'} \dotplus \ldots \dotplus \begin{pmatrix} 0 & \mathbf{G}_{\rho_1} \\ \mathbf{G}_{\rho_1}^{*} & 0 \end{pmatrix} \dotplus \begin{pmatrix} 0 & \mathbf{G}_{\rho_2} \\ \mathbf{G}_{\rho_2}^{*} & 0 \end{pmatrix} \dotplus \ldots$$

Let $\mathbf{S}$ be the matrix whose columns are the components (relative to the standard basis) of the vectors of our basis. Then (I.1.48) the matrix

$$\tilde{\mathbf{X}}=\mathbf{S}^{-1}\mathbf{X}\mathbf{S}$$

is the direct sum of the matrices $\mathbf{Q}_{\rho}$ for distinct eigenvalues:

$$\tilde{\mathbf{X}}=\mathbf{Q}_{\rho_1'} \dotplus \mathbf{Q}_{\rho_2'} \dotplus \ldots \dotplus \begin{pmatrix} \mathbf{Q}_{\rho_1} & 0 \\ 0 & \mathbf{Q}_{\bar{\rho}_1^{-1}} \end{pmatrix} \dotplus \begin{pmatrix} \mathbf{Q}_{\rho_2} & 0 \\ 0 & \mathbf{Q}_{\bar{\rho}_2^{-1}} \end{pmatrix} \dotplus \ldots$$

If each root subspace is considered relative to a cyclic basis (though we do not assume that this is done), each of the matrices $\mathbf{Q}_{\rho}$ is a direct sum of elementary Jordan matrices (§I.1.11). In the general case, $\mathbf{Q}_{\rho}$ has a single eigenvalue ρ, and the order of $\mathbf{Q}_{\rho}$ is equal to the multiplicity of ρ.

By (2.2), the Gram matrix $\tilde{\mathbf{G}}$ relative to our basis is

$$\tilde{\mathbf{G}}=\mathbf{S}^{*}\mathbf{G}\mathbf{S},$$

where $\mathbf{G}$ is the Gram matrix relative to the standard basis. We have thus proved:

Lemma I. Let $\rho_1', \rho_2', \ldots$ *be the eigenvalues of a* G*-unitary matrix* $\mathbf{X}$ *on the unit circle, and* $\rho_1, \bar{\rho}_1^{-1}, \rho_2, \bar{\rho}_2^{-1}, \ldots$ *its eigenvalues not on the unit circle (only distinct eigenvalues are enumerated). There exists a nonsingular (generally complex) matrix* $\mathbf{S}$ *such that*

$$\left.\begin{aligned} \mathbf{S}^{-1}\mathbf{X}\mathbf{S} &= \mathbf{Q}_{\rho_1'} \dotplus \mathbf{Q}_{\rho_2'} \dotplus \ldots \dotplus \begin{pmatrix} \mathbf{Q}_{\rho_1} & 0 \\ 0 & \mathbf{Q}_{\bar{\rho}_1^{-1}} \end{pmatrix} \dotplus \begin{pmatrix} \mathbf{Q}_{\rho_2} & 0 \\ 0 & \mathbf{Q}_{\bar{\rho}_2^{-1}} \end{pmatrix} \dotplus \ldots, \\ \mathbf{S}^{*}\mathbf{G}\mathbf{S} &= \mathbf{G}_{\rho_1'} \dotplus \mathbf{G}_{\rho_2'} \dotplus \ldots \dotplus \begin{pmatrix} 0 & \mathbf{G}_{\rho_1} \\ \mathbf{G}_{\rho_1}^{*} & 0 \end{pmatrix} \dotplus \begin{pmatrix} 0 & \mathbf{G}_{\rho_2} \\ \mathbf{G}_{\rho_2} & 0 \end{pmatrix} \dotplus \ldots \end{aligned}\right\} \tag{2.21}$$

Now consider the case of simple eigenvalues. If ρ_1' ($|\rho_1'| = 1$) is a simple eigenvalue, then

$$\mathbf{G}_{\rho_1'} = \langle \mathbf{a}_1', \mathbf{a}_1' \rangle,$$

where $\mathbf{a}_1'$ is a corresponding eigenvector. Since the subspace $\mathfrak{L}_{\rho_1'}$ is nondegenerate, we must have $\langle \mathbf{a}_1', \mathbf{a}_1' \rangle \neq 0$, or, normalizing, $\langle \mathbf{a}_1', \mathbf{a}_1' \rangle = \pm 1$. By the definitions in §1.2, the eigenvalue ρ_1' is of the first kind (and we denote it by $\rho_1^{(+)}$) if $\langle \mathbf{a}_1', \mathbf{a}_1' \rangle = 1$ and of the second kind ($\rho_1^{(-)}$) if $\langle \mathbf{a}_1', \mathbf{a}_1' \rangle = -1$.

If ρ_1 ($|\rho_1| \neq 1$) is a simple eigenvalue, then

$$\mathbf{G}_{\rho_1} = \langle \mathbf{a}_2, \mathbf{a}_1 \rangle \neq 0,$$

where $\mathbf{a}_1$, $\mathbf{a}_2$ are eigenvectors:

$$\mathbf{X}\mathbf{a}_1 = \rho_1 \mathbf{a}_1, \quad \mathbf{X}\mathbf{a}_2 = \bar{\rho}_1^{-1}\mathbf{a}_2.$$

Indeed, since the subspace $\mathfrak{L}_{\rho_1} \dotplus \mathfrak{L}_{\bar{\rho}_1^{-1}}$ is nondegenerate,

$$\det \begin{pmatrix} 0 & \mathbf{G}_{\rho_1} \\ \mathbf{G}_{\rho_1}^{*} & 0 \end{pmatrix} = \begin{vmatrix} 0 & \langle \mathbf{a}_2, \mathbf{a}_1 \rangle \\ \overline{\langle \mathbf{a}_2, \mathbf{a}_1 \rangle} & 0 \end{vmatrix} = -|\langle \mathbf{a}_2, \mathbf{a}_1 \rangle|^2 \neq 0.$$

Normalizing, say, the vector $\mathbf{a}_1$, we may assume that $\langle \mathbf{a}_2, \mathbf{a}_1 \rangle = 1$.

Thus the decomposition (2.21) becomes

$$\left.\begin{aligned} \mathbf{S}^{-1}\mathbf{X}\mathbf{S} &= \ldots \dotplus \rho_j^{(+)} \dotplus \ldots \dotplus \rho_h^{(-)} \dotplus \ldots \dotplus \left\| \begin{matrix} \rho_l & 0 \\ 0 & \bar{\rho}_l^{-1} \end{matrix} \right\| \dotplus \ldots, \\ \mathbf{S}^{*}\mathbf{G}\mathbf{S} &= \ldots \dotplus (+1) \dotplus \ldots \dotplus (-1) \dotplus \ldots \dotplus \left\| \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right\| \dotplus \ldots \end{aligned}\right\} \tag{2.22}$$

Here $\rho_j^{(+)}$, $\rho_h^{(-)}$ are the eigenvalues of $\mathbf{X}$ on the unit circle, of first and second kinds, respectively, and ρ_l, $\bar{\rho}_l^{-1}$ are the eigenvalues not on the unit circle.

Conversely, if $\mathbf{S}$ is an arbitrary nonsingular matrix admitting the second decomposition in (2.22), then the matrix $\mathbf{X}$ defined by the first decomposition is G-unitary.

Indeed, denoting the matrices on the right of (2.22) by $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{G}}$ and using the block-multiplication rule for diagonal matrices,

$$(\mathbf{A}_1 \dotplus \mathbf{A}_2 \dotplus \ldots)(\mathbf{B}_1 \dotplus \mathbf{B}_2 \dotplus \ldots) = \mathbf{A}_1\mathbf{B}_1 \dotplus \mathbf{A}_2\mathbf{B}_2 \dotplus \ldots,$$

we see at once that $\tilde{\mathbf{X}}^*\tilde{\mathbf{G}}\tilde{\mathbf{X}}=\tilde{\mathbf{G}}$. Substituting the expressions for $\tilde{\mathbf{X}}$, $\tilde{\mathbf{G}}$, we obtain $\mathbf{X}^*\mathbf{G}\mathbf{X}=\mathbf{G}$.

We now consider the case that $\mathbf{X}$ has a multiple eigenvalue ρ' on the unit circle, with simple elementary divisors. Then, relative to any basis, the matrix $\mathbf{Q}_{\rho'}$ is a multiple of the identity matrix: $\mathbf{Q}_{\rho'}=\rho'\mathbf{I}$. Since the subspace $\mathfrak{L}_{\rho'}$ ($|\rho'|=1$) is nondegenerate, it follows by Lemma I of §2.2 that it has a basis for which $\mathbf{G}_{\rho'}$ is diagonal, with diagonal elements $+1$ or -1. Relative to this basis, the part of (2.21) belonging to eigenvalues on the unit circle has the form of the corresponding part of (2.22).

We now choose a special basis in each of the subspaces $\mathfrak{L}=\mathfrak{L}_\rho\dot{+}\mathfrak{L}_{\bar{\rho}^{-1}}$, $|\rho|\neq 1$. If we change the basis only in $\mathfrak{L}_{\bar{\rho}^{-1}}$, via some nonsingular matrix $\mathbf{K}^\tau$, the Gram matrix of $\mathfrak{L}$ is transformed (see (2.3)) according to the formula

$$\tilde{\mathbf{G}}_{\mathfrak{L}}=\begin{pmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{K}^*\end{pmatrix}\begin{pmatrix}\mathbf{0} & \mathbf{G}_\rho\\ \mathbf{G}_\rho^* & \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{K}\end{pmatrix}=\begin{pmatrix}\mathbf{0} & \tilde{\mathbf{G}}_\rho\\ \tilde{\mathbf{G}}_\rho^* & \mathbf{0}\end{pmatrix},$$

where $\tilde{\mathbf{G}}_\rho=\mathbf{G}_\rho\mathbf{K}$. Since $\det\mathbf{G}_\rho\neq 0$, we may assume by suitable choice of $\mathbf{K}$ that $\tilde{\mathbf{G}}_\rho$ is any prescribed nondegenerate matrix. Let $\mathbf{f}_1,\ldots,\mathbf{f}_m$ be any basis in $\mathfrak{L}_\rho$. We choose the basis $\mathbf{g}_1,\ldots,\mathbf{g}_m$ in $\mathfrak{L}_{\bar{\rho}^{-1}}$ so that $\mathbf{G}_\rho$ is the identity matrix. Then

$$\langle\mathbf{f}_j,\mathbf{f}_h\rangle=0,\quad\langle\mathbf{f}_j,\mathbf{g}_h\rangle=\delta_{jh},\quad\langle\mathbf{g}_j,\mathbf{g}_h\rangle=0\qquad(j,h=1,\ldots,m).\tag{2.23}$$

It follows from (2.23) that the Gram matrix of $\mathfrak{L}$ relative to the basis $\mathbf{f}_1,\mathbf{g}_1,\mathbf{f}_2,\mathbf{g}_2,\ldots,\mathbf{f}_m,\mathbf{g}_m$ is

$$\left\|\begin{matrix}0&1\\1&0\end{matrix}\right\|\dot{+}\left\|\begin{matrix}0&1\\1&0\end{matrix}\right\|\dot{+}\ldots\dot{+}\left\|\begin{matrix}0&1\\1&0\end{matrix}\right\|.$$

Moreover, $\mathbf{X}\mathbf{f}_j=\rho\mathbf{f}_j$, $\mathbf{X}\mathbf{g}_j=\bar{\rho}^{-1}\mathbf{g}_j$ ($j=1,\ldots,m$). Choosing the bases in each of the subspaces $\mathfrak{L}_{\rho'}$ and $\mathfrak{L}=\mathfrak{L}_\rho\dot{+}\mathfrak{L}_{\bar{\rho}^{-1}}$ in this way, we obtain (2.22). We have thus proved

Lemma II. Let $\mathbf{X}$ *be a* $\mathbf{G}$*-unitary matrix which can be reduced to diagonal form. Then there exists a nonsingular (generally complex) matrix* $\mathbf{S}$ *such that (2.22) holds, where* $\rho_j^{(+)}$, $\rho_j^{(-)}$ *are the eigenvalues on the unit circle of the first and second kinds, respectively, and* ρ_l, $\bar{\rho}_l^{-1}$ *the eigenvalues not on the unit circle. Conversely, if* $\mathbf{S}$ *is an arbitrary nonsingular matrix for which the second equality of (2.22) is valid, and* $\rho_j^{(+)}$, $\rho_j^{(-)}$, ρ_l *are arbitrary complex numbers having the above positions relative to the unit circle, then the matrix* $\mathbf{X}$ *defined by the first equality of (2.22) is* G*-unitary.*

The converse was proved previously.

The proof of the following lemma is exactly the same.

Lemma III. Let $\mathbf{A}$ *be a* G*-Hamiltonian matrix reducible to diagonal form. Then there exists a nonsingular matrix* $\mathbf{S}$ *such that*

$$\left.\begin{aligned}\mathbf{S}^{-1}\mathbf{A}\mathbf{S}&=\ldots\dot{+}\lambda_j^{(+)}\dot{+}\ldots\dot{+}\lambda_h^{(-)}\dot{+}\ldots\dot{+}\left\|\begin{matrix}\lambda_l&0\\0&-\bar{\lambda}_l\end{matrix}\right\|\dot{+}\ldots,\\ \mathbf{S}^*(i\mathbf{J})\mathbf{S}&=\ldots\dot{+}(+1)\dot{+}\ldots\dot{+}(-1)\dot{+}\ldots\dot{+}\left\|\begin{matrix}0&1\\1&0\end{matrix}\right\|\dot{+}\ldots,\end{aligned}\right\}\tag{2.24}$$

where $\lambda_j^{(+)}$, $\lambda_h^{(-)}$ *are the pure imaginary eigenvalues of the first and second kinds, respectively, and* $\lambda_l, -\bar{\lambda}_l$ *the eigenvalues with nonzero real part. Conversely, if* $\mathbf{S}$ *is any nonsingular matrix satisfying the second equality of (2.24) and* $\lambda_j^{(+)}$, $\lambda_h^{(-)}$, λ_l *are arbitrary complex numbers having the above positions relative to the imaginary axis, then the matrix* $\mathbf{A}$ *defined by the first equality of (2.24) is* G*-Hamiltonian.*

We prove only the converse. Denoting the right-hand sides of (2.24) by $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{G}}$, we have $(\tilde{\mathbf{G}}\tilde{\mathbf{A}})^* = -\tilde{\mathbf{G}}\tilde{\mathbf{A}}$. Substituting $\tilde{\mathbf{G}} = \mathbf{S}^*\mathbf{G}\mathbf{S}$, $\tilde{\mathbf{A}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$, we obtain $(\mathbf{G}\mathbf{A})^* = -\mathbf{G}\mathbf{A}$. This means (§1.1) that $\mathbf{A}$ is G-Hamiltonian.

2.6. Eigenvalues of mixed kind of G-unitary and G-Hamiltonian matrices

Again, let $\mathbf{G}_{\rho'}$ be the Gram matrix of a root subspace $\mathfrak{L}_{\rho'}$ belonging to an m-fold eigenvalue ρ' ($|\rho'| = 1$) of a G-unitary matrix $\mathbf{X}$. By Lemma II of §2.2, the fact that $\mathfrak{L}_{\rho'}$ is nondegenerate implies that $\mathbf{G}_{\rho'}$ has no vanishing eigenvalues. Let $\mathbf{G}_{\rho'}$ have $p \geqslant 0$ positive and $q \geqslant 0$ negative eigenvalues. Since all eigenvalues of $\mathbf{G}_{\rho'}$ are real,

$$p + q = m.$$

We complete the definition of an eigenvalue of mixed kind, given in §1.2, as follows.

Definition. Let ρ' ($|\rho'| = 1$) be an m-fold eigenvalue of a G-unitary matrix (let λ', $\operatorname{Re}\lambda' = 0$, be an m-fold eigenvalue of a G-Hamiltonian matrix $\mathbf{A}$) and $\mathbf{G}_{\rho'}$ ($\mathbf{G}_{\lambda'}$) the Gram matrix of the corresponding root subspace, having p positive and q negative eigenvalues ($p + q = m$). We shall then say that p eigenvalues of the first kind and q of the second coincide at ρ' (at λ').

In case $p = 0$, $q = m$ or $p = m$, $q = 0$, this definition agrees with the definition (§1.2) of a definite (m-fold) eigenvalue. We claim that when $p \neq 0$, $q \neq 0$, the number ρ' is an m-fold eigenvalue of mixed kind as defined in §1.2. Indeed, otherwise, let $\langle \mathbf{x}', \mathbf{x}' \rangle > 0$ (say) for $\mathbf{x}' \neq \mathbf{0}$ in the eigensubspace $\mathfrak{S}_{\rho'}$. Then by Lemma I of §1.2 $\mathfrak{S}_{\rho'}$ is the root subspace $\mathfrak{L}_{\rho'}$. Thus $\langle \mathbf{x}', \mathbf{x}' \rangle > 0$ for $\mathbf{x}' \in \mathfrak{L}_{\rho'}$, $\mathbf{x}' \neq \mathbf{0}$, so that $p = m$, $q = 0$, contrary to assumption.

2.7. Lemma on eigenvalues

Let p be the number of positive and q the number of negative eigenvalues of the matrix $\mathbf{G}$ defining the indefinite scalar product (1.1). Since by assumption $\mathbf{G}$ is nonsingular, it follows that $p + q = n$, where n is the order of $\mathbf{G}$. $\mathbf{G}$ is the Gram matrix relative to the standard basis (§2.2), i.e., $\mathbf{G} = \| \langle \mathbf{e}_h, \mathbf{e}_j \rangle \|_1^n$. Under any change of basis, the Gram matrix becomes $\tilde{\mathbf{G}} = \mathbf{S}^*\mathbf{G}\mathbf{S}$, where $\mathbf{S}$ is a nonsingular matrix. By Sylvester's law of inertia (/56/, Chapter VI, §2), the numbers p and q are the same for $\tilde{\mathbf{G}}$ as for $\mathbf{G}$.

Lemma 1. Any G*-unitary (*G*-Hamiltonian) matrix has* p *eigenvalues of the first kind and* q *eigenvalues of the second kind (counting multiplicities).*

Proof. To fix ideas, we prove the lemma for G-unitary matrices. As before, let $\rho_1', \rho_2', \ldots$ be the eigenvalues of a G-unitary matrix $\mathbf{X}$ on the unit circle, and $\rho_1, \bar{\rho}_1^{-1}, \rho_2, \bar{\rho}_2^{-1}, \ldots$ its eigenvalues not on the unit circle. Suppose that $|\rho_l| < 1$, so that ρ_l are the eigenvalues of the first kind, and $\bar{\rho}_l^{-1}$ those of the second kind (§1.2). Let m_l be the multiplicity of ρ_l, which by the theorem of § II. 3.5 is the same as that of $\bar{\rho}_l^{-1}$. Suppose that p_j multipliers of the first kind and q_j of the second coincide at ρ_j'. These numbers are equal respectively to the numbers of positive and negative eigenvalues of the matrix $\mathbf{G}_{\rho_j'}$ in (2.20). Let us determine the numbers of positive and negative eigenvalues of both sides of (2.20). It was shown above that for the matrix $\tilde{\mathbf{G}}$ these numbers are p and q, respectively. We now evaluate them for matrices of type (2.19), where $\rho = \rho_l$ is of absolute value $\neq 1$.

As shown in §2.6, we may assume (by a suitable change of basis in $\mathfrak{L}_{\bar{\rho}^{-1}}$) that $\mathbf{G}_\rho = \mathbf{I}_{m_l}$, so that the Gram matrix (2.19) in the subspace $\mathfrak{L}_l = \mathfrak{L}_\rho + \mathfrak{L}_{\bar{\rho}^{-1}}$ is

$$\mathbf{G}_l = \begin{pmatrix} 0 & \mathbf{I}_{m_l} \\ \mathbf{I}_{m_l} & 0 \end{pmatrix}.$$

It follows from the relation

$$\begin{pmatrix} \mathbf{I}_m & \mathbf{I}_m \\ \mathbf{I}_m & -\mathbf{I}_m \end{pmatrix}^{-1} \begin{pmatrix} 0 & \mathbf{I}_m \\ \mathbf{I}_m & 0 \end{pmatrix} \begin{pmatrix} \mathbf{I}_m & \mathbf{I}_m \\ \mathbf{I}_m & -\mathbf{I}_m \end{pmatrix} = \begin{pmatrix} \mathbf{I}_m & 0 \\ 0 & -\mathbf{I}_m \end{pmatrix}$$

that $\mathbf{G}_l$ has m_l positive and m_l negative eigenvalues.

We can now determine the required numbers for the matrix on the left of (2.20). The matrix $\mathbf{G}_{\rho_j'}$ has p_j' positive and q_j' negative eigenvalues, where p_j' and q_j' are the numbers of eigenvalues of the first and second kind, respectively, that coincide at ρ_j'. Therefore, comparing the numbers of positive and negative eigenvalues on both sides of (2.20), we get

$$p = p_1' + p_2' + \ldots + m_1 + m_2 + \ldots, \quad q = q_1' + q_2' + \ldots + m_1 + m_2 + \ldots,$$

proving Lemma I.

Corollary. Any $2k \times 2k$ J-unitary or J-Hamiltonian matrix has k eigenvalues of the first kind and k of the second (counting multiplicities).

Indeed, if

$$\mathbf{G} = i\mathbf{J} = i\begin{pmatrix} 0 & -\mathbf{I}_k \\ \mathbf{I}_k & 0 \end{pmatrix}$$

then

$$\mathbf{S}^{-1}(i\mathbf{J})\,\mathbf{S} = \begin{pmatrix} \mathbf{I}_k & 0 \\ 0 & -\mathbf{I}_k \end{pmatrix}, \quad \text{where} \quad \mathbf{S} = \begin{pmatrix} -i\mathbf{I}_k & i\mathbf{I}_k \\ \mathbf{I}_k & \mathbf{I}_k \end{pmatrix}. \tag{2.25}$$

Thus the eigenvalues of the matrix $i\mathbf{J}$ are the numbers $+1$ (multiplicity k) and -1 (multiplicity k). Hence in this case $p = q = k$, and the assertion follows from Lemma I.

Lemma II. a) If p eigenvalues of the first kind and q of the second of a symplectic matrix coincide at a point $\rho = e^{i\varphi}$, *then q eigenvalues of the first kind and p of the second coincide at* $\overline{\rho} = e^{-i\varphi}$. *In particular, simple eigenvalues* $\rho = e^{i\varphi}$ *and* $\overline{\rho} = e^{-i\varphi}$ *are of different kinds.*

b) *If p eigenvalues of the first kind and q of the second of a J-Hamiltonian matrix coincide at a point* λ_0, $\operatorname{Re}\lambda_0 = 0$, *then q eigenvalues of the first kind and p of the second coincide at* $\overline{\lambda}$.

Proof. Recall that a symplectic matrix is a real J-unitary matrix.

a) First let $\rho = e^{i\varphi}$ be a simple eigenvalue of a symplectic matrix $\mathbf{X}$, $\mathbf{Xa} = \rho\mathbf{a}$. Since $\overline{\mathbf{Xa}} = \overline{\rho}\overline{\mathbf{a}}$, i. e., $\mathbf{X}\overline{\mathbf{a}} = \overline{\rho}\overline{\mathbf{a}}$, it follows that $\overline{\mathbf{a}}$ is an eigenvector belonging to the eigenvalue $\overline{\rho} = e^{-i\varphi}$. Moreover,

$$\langle \overline{\mathbf{a}}, \overline{\mathbf{a}} \rangle = i(\mathbf{J}\overline{\mathbf{a}}, \overline{\mathbf{a}}) = \overline{[-i(\mathbf{Ja}, \mathbf{a})]} = -\overline{\langle \mathbf{a}, \mathbf{a} \rangle},$$

and since the number on the right is real it follows that $\langle \overline{\mathbf{a}}, \overline{\mathbf{a}} \rangle = -\langle \mathbf{a}, \mathbf{a} \rangle$. This means that the two eigenvalues in question are of different kinds.

Let $\rho = e^{i\varphi}$ be an m-fold eigenvalue of the symplectic matrix $\mathbf{X}$, and $\mathbf{f}_1, \ldots, \mathbf{f}_m$ a basis in the root subspace $\mathfrak{L}_\rho$. Then $\overline{\mathbf{f}}_1, \ldots, \overline{\mathbf{f}}_m$ is a basis in the root subspace $\mathfrak{L}_{\overline{\rho}}$. Indeed, $\mathbf{f}_1, \ldots, \mathbf{f}_m$ is a basis for $\overline{\mathfrak{L}}_\rho$ (§ I. 1.17). But $\overline{\mathfrak{L}}_\rho = \mathfrak{L}_{\overline{\rho}}$, since the equality

$$(\mathbf{X} - \rho\mathbf{I}_{2k})^\mu \mathbf{f}_j = 0$$

and the fact that $\mathbf{X}$ is real imply that

$$(\mathbf{X} - \overline{\rho}\mathbf{I}_{2k})^\mu \overline{\mathbf{f}}_j = 0.$$

The Gram matrix of $\mathfrak{L}_{\overline{\rho}}$ is

$$\mathbf{G}_{\overline{\rho}} = \| \langle \overline{\mathbf{f}}_h, \overline{\mathbf{f}}_j \rangle \|_1^m = \| -\overline{\langle \mathbf{f}_h, \mathbf{f}_j \rangle} \|_1^m = \| -\langle \mathbf{f}_j, \mathbf{f}_h \rangle \|_1^m = -\mathbf{G}_\rho^\tau,$$

where $\mathbf{G}_\rho$ is the Gram matrix of $\mathfrak{L}_\rho$. Therefore, if $\mathbf{G}_\rho$ has p positive and q negative eigenvalues, then $\mathbf{G}_{\overline{\rho}}$ has q positive and p negative eigenvalues, proving part (a).

Part (b) is proved in exactly the same way. This completes the proof of Lemma II.

Lemma III. An eigenvalue $\rho = 1$ *(or* $\rho = -1$*) of a symplectic matrix* $\mathbf{X}$ *and the eigenvalue* $\lambda = 0$ *of a real J-Hamiltonian matrix* $\mathbf{G}$ *have even multiplicity· if p eigenvalues of the first kind and q of the second coincide at these points, then* $p = q$.

Proof. Let $\mathfrak{L}$ be the root subspace of a symplectic matrix $\mathbf{X}$ belonging to the eigenvalue $\rho = 1$. Since $\mathbf{X}$ is a real matrix, we can choose a real basis $\mathbf{f}_1, \ldots, \mathbf{f}_m$ in $\mathfrak{L}$ (§ I. 1.17). As shown at the end of §2.3, $\mathfrak{L}$ is nondegenerate, i. e., the Gram matrix $\mathbf{G}_\mathfrak{L} = i\|(\mathbf{Jf}_h, \mathbf{f}_j)\|_1^m$ is nonsingular. The matrix $\mathbf{J}_\mathfrak{L} = \|(\mathbf{Jf}_h, \mathbf{f}_j)\|_1^m$ is real and skew-symmetric, and so (§2.3) it has even order, $2r$ say. As shown in the lemma of §2.3, any real skew-symmetric matrix, hence also $\mathbf{J}_\mathfrak{L}$, may be reduced to the form

$$\mathbf{J}_\mathfrak{L} = \mathbf{T}^\tau \begin{pmatrix} 0 & -\mathbf{I}_r \\ \mathbf{I}_r & 0 \end{pmatrix} \mathbf{T},$$

where $\mathbf{T}$ is a real nonsingular matrix. It follows from (2.25) that $\mathbf{G}_{\mathfrak{L}} = i\mathbf{J}_{\mathfrak{L}}$ has r positive and r negative eigenvalues. Hence (see the definition, §1.2) r eigenvalues of the first kind and r of the second coincide at $\rho = 1$, as claimed. The case $\rho = -1$ and the case of a real J-Hamiltonian matrix are treated in similar fashion. Q. E. D.

Let $\mathbf{X}$ be a $2k \times 2k$ symplectic matrix. By the corollary to Lemma I, when $\mathbf{G} = i\mathbf{J}$ we have $p_0 = q_0 = k$. Thus, by Lemma I, the matrix $\mathbf{X}$ has k eigenvalues of the first kind and k of the second (counting multiplicities).

Lemma IV. Let $\rho_1, \ldots, \rho_k$ be the eigenvalues of the first kind of a symplectic matrix $\mathbf{X}$ (each written according to its multiplicity). Then $\rho_1^{-1}, \ldots, \rho_k^{-1}$ are its eigenvalues of the second kind.

Indeed, if ρ_0 is a k-fold eigenvalue of the first kind and $|\rho_0| < 1$, then it is clear that ρ_0^{-1} is a k-fold eigenvalue of the second kind. Using Lemmas II and III, we see that the same is true if $|\rho_0| = 1$. Q. E. D.

2.8. Projections of invariant subspaces of G-unitary matrices

Let $\mathfrak{L}$ be a nondegenerate subspace, and consider the decomposition whose existence was established in Lemma III of §2.2. Formula (2.7) defines an operator $\mathbf{P}$, $\mathbf{y} = \mathbf{Px}$, called the projection of $\mathfrak{L}$. (It is essential here that the vector $\mathbf{y}$ in (2.7) is uniquely determined.) $\mathbf{P}$ is obviously a linear operator. Indeed, if $\mathbf{x} = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2$, where α_1, α_2 are scalars, then, by (2.7),

$$\mathbf{x}_j = \mathbf{y}_j + \mathbf{z}_j, \quad \text{where } \mathbf{y}_j \in \mathfrak{L}, \quad \mathbf{z}_j \in \mathfrak{L}^{\langle\perp\rangle} \qquad (j = 1, 2),$$

so that

$$\mathbf{x} = \mathbf{y} + \mathbf{z}, \quad \text{where } \mathbf{y} = \alpha_1\mathbf{y}_1 + \alpha_2\mathbf{y}_2, \quad \mathbf{z} = \alpha_1\mathbf{z}_1 + \alpha_2\mathbf{z}_2.$$

Moreover, $\mathbf{y} \in \mathfrak{L}$, $\mathbf{z} \in \mathfrak{L}^{\langle\perp\rangle}$. Thus $\mathbf{y}_j = \mathbf{Px}_j$, $\mathbf{y} = \mathbf{Px}$. Consequently, $\mathbf{Px} = \alpha_1\mathbf{y}_1 + \alpha_2\mathbf{y}_2 = \alpha_1\mathbf{Px}_1 + \alpha_2\mathbf{Px}_2$, and $\mathbf{P}$ is indeed a linear operator. The matrix of $\mathbf{P}$ (relative to a fixed basis) is known as a projection matrix $\mathbf{P}$ of $\mathfrak{L}$. It is proved in linear algebra that a projection (projection matrix) $\mathbf{P}$ satisfies the conditions*

$$\mathbf{P}^2 = \mathbf{P}, \quad \mathbf{P}^+ = \mathbf{P}. \tag{2.26}$$

The proof is quite simple. For $\mathbf{y} \in \mathfrak{L}$, the decomposition (2.7) has the form $\mathbf{y} = \mathbf{y} + \mathbf{0}$, so that $\mathbf{Py} = \mathbf{y}$. Thus for any $\mathbf{x}$ we have $\mathbf{Px} = \mathbf{Py} = \mathbf{y} = \mathbf{P}^2\mathbf{x}$, proving the first part of (2.26). Now let $\mathbf{x}_1, \mathbf{x}_2$ be any vectors and $\mathbf{x}_1 = \mathbf{y}_1 + \mathbf{z}_1$, $\mathbf{x}_2 = \mathbf{y}_2 + \mathbf{z}_2$ the corresponding decompositions (2.7). We have

$$\langle \mathbf{Px}_1, \mathbf{x}_2 \rangle = \langle \mathbf{y}_1, \mathbf{y}_2 + \mathbf{z}_2 \rangle = \langle \mathbf{y}_1, \mathbf{y}_2 \rangle, \quad \text{since } \langle \mathbf{y}_1, \mathbf{z}_2 \rangle = 0,$$
$$\langle \mathbf{x}_1, \mathbf{Px}_2 \rangle = \langle \mathbf{y}_1 + \mathbf{z}_1, \mathbf{y}_2 \rangle = \langle \mathbf{y}_1, \mathbf{y}_2 \rangle, \quad \text{since } \langle \mathbf{z}_1, \mathbf{y}_2 \rangle = 0.$$

Consequently, $\langle \mathbf{Px}_1, \mathbf{x}_2 \rangle = \langle \mathbf{x}_1, \mathbf{Px}_2 \rangle$, proving the second part of (2.26).

* It is usually assumed that the metric is definite, but this is unnecessary. Note that in this subsection, fixing the (standard) basis once and for all, we are identifying a projection and its matrix (the projection matrix).

The converse is also true: any linear operator satisfying (2.26) is the projection of some nondegenerate subspace $\mathfrak{L}$ —the set of all vectors $\mathbf{y}$ such that $\mathbf{Py}=\mathbf{y}$. Indeed, we can define a decomposition (2.7) by

$$\mathbf{x}=\mathbf{y}+\mathbf{z}, \text{ where } \mathbf{y}=\mathbf{Px}, \quad \mathbf{z}=(\mathbf{I}_n-\mathbf{P})\,\mathbf{x}. \tag{2.27}$$

It follows from (2.27) that $\mathbf{y}\in\mathfrak{L}$, $\langle \mathbf{y}, \mathbf{z}\rangle=0$. Clearly, $\mathbf{Pz}=\mathbf{0}$. Let $\mathfrak{L}_1$ denote the subspace of all vectors $\mathbf{z}$ such that $\mathbf{Pz}=\mathbf{0}$. Then we can write (2.27) as $\mathfrak{R}_n=\mathfrak{L}\langle+\rangle\mathfrak{L}_1$. Consequently (§2.2), $\mathfrak{L}$ is a nondegenerate subspace and $\mathfrak{L}_1=\mathfrak{L}^{\langle\perp\rangle}$.

Let p and q be the numbers defined in Lemma I of §2.2 for some nondegenerate subspace $\mathfrak{L}$ with projection matrix $\mathbf{P}$.

Lemma.

$$\mathbf{P^*GP}=\mathbf{GP}=\mathbf{P^*G}, \tag{2.28}$$

and the hermitian matrix (2.28) has exactly p positive, q negative and $n-m=n-(p+q)$ zero eigenvalues.

Proof. Let $\mathbf{f}_1, \ldots, \mathbf{f}_m$ be any basis in $\mathfrak{L}$ and $\mathbf{f}_{m+1}, \ldots, \mathbf{f}_n$ any basis in $\mathfrak{L}^{\langle\perp\rangle}=\mathfrak{M}$, and

$$\mathbf{G}_{\mathfrak{L}}=\|\langle \mathbf{f}_h, \mathbf{f}_j\rangle\|_1^m$$

the Gram matrix of $\mathfrak{L}$. It is easy to see that

$$\|\langle \mathbf{Pf}_h, \mathbf{Pf}_j\rangle\|_1^n=\begin{pmatrix}\mathbf{G}_{\mathfrak{L}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}.$$

Indeed, $\mathbf{Pf}_\mu=\mathbf{f}_\mu$ $(\mu=1, \ldots, m)$, $\mathbf{Pf}_\nu=\mathbf{0}$ $(\nu=m+1, \ldots, n)$. The matrix $\mathbf{G}_{\mathfrak{L}}$ has p positive and q negative eigenvalues. Consequently, the matrix

$$\|\langle \mathbf{Pf}_h, \mathbf{Pf}_j\rangle\|_1^n=\|\langle \mathbf{P}^+\mathbf{Pf}_h, \mathbf{f}_j\rangle\|_1^n=\|(\mathbf{P^*GPf}_h, \mathbf{f}_j\rangle\|_1^n$$

has p positive, q negative and $n-(p+q)$ zero eigenvalues. By Sylvester's law of inertia for quadratic forms (see /63/, Chapter VI, §2), these numbers are independent of the choice of basis. Thus the matrix

$$\|(\mathbf{P^*GPe}_h, \mathbf{e}_j)\|_1^n=\mathbf{P^*GP}$$

has exactly p positive, q negative and $n-(p+q)$ zero eigenvalues. It remains to prove (2.28). The matrix $\mathbf{P}$ is G-hermitian, i. e., $\mathbf{P}=\mathbf{P}^+=\mathbf{G}^{-1}\mathbf{P^*G}$, whence $\mathbf{GP}=\mathbf{P^*G}$. Hence, using the first equality of (2.26), we obtain $\mathbf{P^*GP}=\mathbf{GP}^2=\mathbf{GP}$. Q. E. D.

2.9. Projection matrix of a sum of root subspaces

Lemma. Let $\mathfrak{R}_{2k}=\mathfrak{L}\langle+\rangle\mathfrak{M}$, suppose that $\mathbf{P}$ is the projection matrix of $\mathfrak{L}$ and the subspaces $\mathfrak{L}$ and $\mathfrak{M}=\mathfrak{L}^{\langle+\rangle}$ are invariant under some $2k\times 2k$

matrix $\mathbf{A}$. *Let* $\lambda_1, \ldots, \lambda_m$ *be the eigenvalues of* $\mathbf{A}$ *in* $\mathfrak{L}$; $\lambda_{m+1}, \ldots, \lambda_{2k}$ *its eigenvalues in* $\mathfrak{M}, \lambda_\mu \neq \lambda_\nu$ $(\mu = 1, \ldots, m;\ \nu = m+1, \ldots, 2k)$. *Let* γ *be any rectifiable contour on the complex plane whose interior contains* $\lambda_1, \ldots, \lambda_m$ *but not* $\lambda_{m+1}, \ldots, \lambda_{2k}$. *Then*

$$\mathbf{P} = \frac{1}{2\pi i} \int_\gamma (\xi \mathbf{I}_{2k} - \mathbf{A})^{-1}\, d\xi. \tag{2.29}$$

Proof. Define the matrix $\mathbf{P}$ by (2.29); we shall show that it is the projection matrix of $\mathfrak{L}$. Let $\mathbf{s}'_1, \ldots, \mathbf{s}'_m \in \mathfrak{L}$ and $\mathbf{s}''_{m+1}, \ldots, \mathbf{s}''_{2k} \in \mathfrak{M}$ be bases (arbitrary for the moment) for $\mathfrak{L}$ and $\mathfrak{M}$. Since $\mathfrak{L}$ and $\mathfrak{M}$ are invariant subspaces, we have

$$\mathbf{A}\mathbf{s}'_j = \sum_h \alpha^{(1)}_{jh} \mathbf{s}'_h, \quad \mathbf{A}\mathbf{s}''_j = \sum_h \alpha^{(2)}_{jh} \mathbf{s}''_h, \tag{2.30}$$

where $\alpha^{(1)}_{jh}$, $\alpha^{(2)}_{jh}$ are constants. Defining matrices

$$\mathbf{S} = \| \mathbf{s}'_1, \ldots, \mathbf{s}'_m, \mathbf{s}''_{m+1}, \ldots, \mathbf{s}''_{2k} \|$$

and

$$\mathbf{A}_1 = \| \alpha^{(1)}_{jh} \|_1^m, \quad \mathbf{A}_2 = \| \alpha^{(2)}_{jh} \|_{m+1}^{2k},$$

we write (2.30) as

$$\mathbf{AS} = \mathbf{S} \begin{pmatrix} \mathbf{A}_1 & 0 \\ 0 & \mathbf{A}_2 \end{pmatrix}.$$

Since $\mathfrak{R}_{2k} = \mathfrak{L} \dot{+} \mathfrak{M}$, the matrix $\mathbf{S}$ is nonsingular. By suitable choice of the bases $\mathbf{s}'_j$ and $\mathbf{s}''_j$, we may reduce each of the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ to Jordan form in the corresponding subspace. We obtain

$$\mathbf{S}^{-1}\mathbf{AS} = \mathbf{Q}_1 \dot{+} \ldots \dot{+} \mathbf{Q}_r \dot{+} \mathbf{Q}_{r+1} \dot{+} \ldots \dot{+} \mathbf{Q}_s,$$

where

$$\mathbf{A}_1 = \mathbf{Q}_1 \dot{+} \ldots \dot{+} \mathbf{Q}_r, \quad \mathbf{A}_2 = \mathbf{Q}_{r+1} \dot{+} \ldots \dot{+} \mathbf{Q}_s.$$

For any $\mathbf{x} \in \mathfrak{L}$,

$$\mathbf{x} = \sum_{j=1}^{m} \xi_j \mathbf{s}'_j$$

or $\mathbf{x} = \mathbf{S}\mathbf{x}'$, where $\mathbf{x}'$ is the vector with components $\xi_1, \ldots, \xi_m, 0, \ldots, 0$. The converse is also true; thus $\mathbf{x} \in \mathfrak{L}$ if and only if the last $2k-m$ components of the vector $\mathbf{S}^{-1}\mathbf{x}$ vanish. Similarly, $\mathbf{z} \in \mathfrak{M}$ if and only if the first m components of $\mathbf{S}^{-1}\mathbf{z}$ vanish. The situation is clarified by the following diagram:

$$\left.\begin{matrix} m \\ 2k-m \end{matrix}\right\{ \left\langle \begin{matrix} \mathbf{Q}_1 & \cdots & \mathbf{Q}_r & & \\ & & & \mathbf{Q}_{r+1} & \\ & & & & \ddots \\ & & & & \mathbf{Q}_s \end{matrix} \right\rangle \left.\begin{matrix} \\ \\ \end{matrix}\right\} \begin{matrix} \mathfrak{L} \\ \mathfrak{M}. \end{matrix}$$

Thus the integrand in (2.29) may be written

$$\mathbf{S}^{-1}(\zeta \mathbf{I}_{2k}-\mathbf{A})^{-1}\mathbf{S} = (\zeta \mathbf{I}_{k_1}-\mathbf{Q}_1)^{-1} \dot{+} \ldots \dot{+} (\zeta \mathbf{I}_{k_s}-\mathbf{Q}_s)^{-1}. \tag{2.31}$$

Using formula (I.1.44) for the $k_\sigma \times k_\sigma$ elementary Jordan matrix $\mathbf{Q}_\sigma$, we obtain from (I. 2.7)

$$(\zeta \mathbf{I}_{k_\sigma}-\mathbf{Q}_\sigma)^{-1} = \left\| \begin{matrix} (\zeta-\lambda_{(\sigma)})^{-1} & (\zeta-\lambda_{(\sigma)})^{-2} & \ldots & (\zeta-\lambda_{(\sigma)})^{-k_\sigma} \\ 0 & (\zeta-\lambda_{(\sigma)})^{-1} & \ldots & (\zeta-\lambda_{(\sigma)})^{-k_\sigma+1} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & (\zeta-\lambda_{(\sigma)})^{-1} \end{matrix} \right\|.$$

Integrating this matrix element by element, we obtain

$$\frac{1}{2\pi i}\int_\gamma (\zeta \mathbf{I}_{2k}-\mathbf{Q}_\sigma)^{-1}\, d\zeta = \begin{cases} \mathbf{I}_{k_\sigma}, & \text{if } \lambda \text{ lies inside } \gamma, \\ \mathbf{0}, & \text{if } \lambda \text{ lies outside } \gamma. \end{cases}$$

Thus, integration of both sides of (2.31) gives

$$\mathbf{P} = \frac{1}{2\pi i}\int_\gamma (\zeta \mathbf{I}_{2k}-\mathbf{A})^{-1}\, d\zeta = \mathbf{S}\,[\mathbf{I}_{k_1} \dot{+} \ldots \dot{+} \mathbf{I}_{k_r} \dot{+} \mathbf{0} \dot{+} \ldots \dot{+} \mathbf{0}]\,\mathbf{S}^{-1}.$$

The terms $\mathbf{I}_{k_1}, \ldots, \mathbf{I}_{k_r}$ correspond to the cyclic subspaces which are summands of $\mathfrak{L}$, the terms $\mathbf{0}, \ldots, \mathbf{0}$ to the cyclic summands of $\mathfrak{M}$. Using the above expressions for vectors $\mathbf{x} \in \mathfrak{L}$ and $\mathbf{x} \in \mathfrak{M}$, we obtain

$$\mathbf{P}\mathbf{x} = \begin{cases} \mathbf{x}, & \text{if } \mathbf{x} \in \mathfrak{L}, \\ \mathbf{0}, & \text{if } \mathbf{x} \in \mathfrak{M}. \end{cases}$$

By assumption,

$$\mathbf{x} = \mathbf{y} + \mathbf{z} \quad (\mathbf{y} \in \mathfrak{L},\ \mathbf{z} \in \mathfrak{M},\ \mathbf{z} < \perp > \mathbf{y}).$$

Then $\mathbf{P}\mathbf{x} = \mathbf{y}$, so that by definition $\mathbf{P}$ is the projection matrix of $\mathfrak{L}$. Q. E. D.

2.10. Continuity of multipliers of first and second kinds with respect to the Hamiltonian

The multipliers of a Hamiltonian equation (1.22) are continuous functions of the Hamiltonian (§ II. 1.7), where the distance between Hamiltonians is defined by

$$\rho(\mathbf{H}_1, \mathbf{H}_2) = \int_0^T |\mathbf{H}_1(t) - \mathbf{H}_2(t)|\, dt.$$

We now wish to show that the multipliers of the first and second kinds, taken separately, are continuous functions of the Hamiltonian. Rigorously speaking, this assertion has the following meaning. Let $\mathbf{H}(t, s)$ be a Hamiltonian which is continuous (in the above sense) with respect to a parameter s. By Lemma I of §2.7, any Hamiltonian equation has p_0 multipliers of the first kind and q_0 of the second. Let $\rho_j^+(s)$ be the multipliers of the first kind and $\rho_j^-(s)$ those of the second of an equation with Hamiltonian $\mathbf{H}(t, s)$. Then, subject to suitable numbering, $\rho_j^+(s)$ and $\rho_j^-(s)$ are continuous functions of s. One might have expected some multiplier $\rho_j(s)$, defined by continuity and isolated for $s_1 \leqslant s \leqslant s_2$, to change its kind, i. e., to be a multiplier of the first kind for $s_1 \leqslant s < s_0$ and of the second kind for $s_0 \leqslant s \leqslant s_2$. If this were the case, then at least one of the functions $\rho_j^+(s)$ or $\rho_j^-(s)$ would be discontinuous at $s = s_0$.

Since the monodromy matrix of a Hamiltonian equation is a continuous function of the Hamiltonian (§II. 1.7), and is moreover a G-unitary matrix (§1.1), it will suffice to prove the following

Theorem. The eigenvalues of the first kind of a G-unitary matrix $\mathbf{X}$ are continuous functions of $\mathbf{X}$. The same holds for the eigenvalues of the second kind. *

Proof. Let $\mathbf{X}_0$ be any G-unitary matrix and ρ_j^+ $(j = 1, \ldots, p_0)$ its eigenvalues of the first kind, ρ_j^- $(j = 1, \ldots, q_0)$ its eigenvalues of the second kind. Describe disks γ_j around all the eigenvalues not on the unit circle, of radius so small that they do not interesect one another or the unit circle. Describe disks γ_j around all eigenvalues on the unit circle, which do not intersect one another or the previous disks. We may assume without loss of generality that all these disks are of the same radius ε. Let $\mathbf{X}$ be a G-unitary matrix "close" to $\mathbf{X}_0$ and $\tilde{\rho}_j^+$, $\tilde{\rho}_j^-$ its eigenvalues of the first and second kinds. We must show that for arbitrarily small $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that when $|\mathbf{X} - \mathbf{X}_0| < \delta$ the matrix $\mathbf{X}$ has the same number of eigenvalues of the first kind inside γ_j as $\mathbf{X}_0$, and the same number of eigenvalues of the second kind. Indeed, if this is true we can establish a correspondence between $\tilde{\rho}_j^+$, $\tilde{\rho}_j^-$ and ρ_j^+, ρ_j^- (e. g., the correspondence determined by the indices) such that

$$|\tilde{\rho}_j^+ - \rho_j^+| < \varepsilon, \quad |\tilde{\rho}_j^- - \rho_j^-| < \varepsilon.$$

Let ρ_0 be some k-fold eigenvalue of $\mathbf{X}_0$. Let $|\rho_0| < 1$, so that k eigenvalues ρ_j^+ coincide at ρ_0, and let γ_0 be the disk around ρ_0. If $\mathbf{X}$ is sufficiently close to $\mathbf{X}_0$, there will be exactly k eigenvalues of $\mathbf{X}$ in γ_0. Since by assumption γ_0 lies inside the unit circle, all these eigenvalues, both those of $\mathbf{X}_0$ and those of $\mathbf{X}$, are of the first kind. A similar statement holds for $|\rho_0| > 1$, in which case there will be exactly k eigenvalues of the second kind in γ_0 for each of the matrices $\mathbf{X}_0$ and $\mathbf{X}$. Let $|\rho_0| = 1$ and suppose that inside γ_0 there are p points ρ_j^+ and q points ρ_j^-. Let $\mathbf{P}_0$ and $\mathbf{P}$ denote the projection matrices of the subspaces defined for $\mathbf{X}_0$ and $\mathbf{X}$ respectively as the sums of the root subspaces $\mathfrak{L}_0$ and $\mathfrak{L}$ belonging to the eigenvalues lying inside γ_0.

* Thus, if $\mathbf{X} = \mathbf{X}(s)$ is a continuous "curve" in the set of G-unitary matrices, then the eigenvalues of the first (or second) kind may be numbered in such a way that they are continuous functions of s.

By the lemma of §2.9,

$$\mathbf{P}_0=\frac{1}{2\pi i}\int_{\tau_0}(\zeta\mathbf{I}_n-\mathbf{X}_0)^{-1}d\zeta,\quad \mathbf{P}=\frac{1}{2\pi i}\int_{\tau_0}(\zeta\mathbf{I}_n-\mathbf{X})^{-1}d\zeta, \tag{2.32}$$

where τ_0 is the circumference of γ_0. According to (2.32), the matrix $\mathbf{P}$ may be made as close as desired to $\mathbf{P}_0$, provided $\mathbf{X}$ is sufficiently close to $\mathbf{X}_0$. By the lemma of §2.8, the hermitian matrix $\mathbf{GP}_0$ has p positive, q negative and $n-(p+q)$ zero eigenvalues; $p+q$ is the dimension of $\mathfrak{L}_0$, i. e., $p+q=k$. If $\mathbf{P}$ is sufficiently close to $\mathbf{P}_0$, then the matrix $\mathbf{GP}$ will have at least p positive and q negative eigenvalues. By the same lemma, the rank of $\mathbf{GP}$ is equal to the dimension of $\mathfrak{L}$, i. e., to the sum of multiplicities of the eigenvalues of $\mathbf{X}$ within γ_0. Since the eigenvalues are continuous functions of the matrix, this number is equal to the multiplicity of ρ_0, i. e., to k. Thus $\mathbf{X}$ has exactly $k=p+q$ nonzero eigenvalues. Therefore, the number of positive eigenvalues of γ_0 in $\mathbf{X}$ is p, and the number of negative ones is q; thus these numbers are the same for the matrices $\mathbf{X}$ and $\mathbf{X}_0$, Q. E. D.

§3. KREIN-GEL'FAND-LIDSKII STRONG STABILITY THEOREM

3.1. Second theorem on perturbation of G-unitary matrices

The following proposition complements the theorem of §1.3.

Theorem. a) *Let $\mathbf{X}_0$ be a G-unitary matrix with an r-fold eigenvalue $\rho_0=e^{i\beta}$ $(\beta>0)$ of mixed kind on the unit circle, having simple elementary divisors. Then there is a G-unitary matrix $\mathbf{X}$ arbitrarily near $\mathbf{X}_0$ which has eigenvalues both outside and inside the unit circle in the neighborhood of ρ_0.*

b) *If $\mathbf{X}_0$ is symplectic, part (a) remains valid, and in addition $\mathbf{X}$ is also symplectic.*

Proof. We first note two readily verified propositions which we shall use below. 1) If $\tilde{\mathbf{X}}=\mathbf{S}^{-1}\mathbf{XS}$, $\tilde{\mathbf{G}}=\mathbf{S}^*\mathbf{GS}$, where $\mathbf{G}^*=\mathbf{G}$, $\det\mathbf{G}\neq 0$, and $\mathbf{X}$ is G-unitary ($\mathbf{X}^*\mathbf{GX}=\mathbf{G}$), then $\tilde{\mathbf{X}}$ is a $\tilde{G}$-unitary matrix, i. e., $\tilde{\mathbf{X}}^*\tilde{\mathbf{G}}\tilde{\mathbf{X}}=\tilde{\mathbf{G}}$. The converse is obviously valid. 2) Let $\mathbf{X}$, $\mathbf{G}$ be block-diagonal matrices,

$$\mathbf{X}=\mathbf{X}_1\dotplus\mathbf{X}_2\dotplus\ldots\dotplus\mathbf{X}_l,\quad \mathbf{G}=\mathbf{G}_1\dotplus\mathbf{G}_2\dotplus\ldots\dotplus\mathbf{G}_l,$$

where the matrices $\mathbf{G}_j$, $\mathbf{X}_j$ have the same order for each j.* If each matrix $\mathbf{X}_j$ is G_j-unitary, then $\mathbf{X}$ is G-unitary. If $\mathbf{X}$ is G-unitary, then each $\mathbf{X}_j$ is G_j-unitary.

We prove only the second part of proposition 2. Let $\mathbf{X}$ be G-unitary, $\mathbf{X}^*\mathbf{GX}=\mathbf{G}$. Since

$$\mathbf{X}^*\mathbf{GX}=\mathbf{X}_1^*\mathbf{G}_1\mathbf{X}_1\dotplus\ldots\dotplus\mathbf{X}_l^*\mathbf{G}_l\mathbf{X}_l,\quad \mathbf{G}=\mathbf{G}_1\dotplus\ldots\dotplus\mathbf{G}_l,$$

Comparison of the diagonal blocks shows that $\mathbf{X}_j^*\mathbf{G}_j\mathbf{X}_j=\mathbf{G}_j$, $j=1,\ \ldots,\ l$.

* Recall that $\mathbf{X}=\mathbf{X}_1\dotplus\ldots\dotplus\mathbf{X}_l$ means that $\mathbf{X}$ is a block-diagonal matrix with diagonal blocks $\mathbf{X}_1,\ \ldots,\ \mathbf{X}_l$.

We now proceed to part (a) of the theorem. Let $\mathfrak{L}$ be the root subspace (which is by assumption also an eigensubspace) belonging to the eigenvalue $\rho_0 = e^{i\beta}$. Let $\mathfrak{M}$ denote the sum of the remaining root subspaces. By (2.10),

$$\mathfrak{R}_n = \mathfrak{L} <+> \mathfrak{M},$$

where each of the subspaces $\mathfrak{L}$ and $\mathfrak{M}$ is nondegenerate (see end of § 2.3). By assumption, the Gram matrix $\mathbf{G}_{\mathfrak{L}}$ of the subspace has both positive and negative eigenvalues. Let $\mathbf{f}_1, \ldots, \mathbf{f}_r$ be a basis for $\mathfrak{L}$, relative to which $\mathbf{G}_{\mathfrak{L}}$ is a diagonal matrix with eigenvalues ± 1, and such that $\langle \mathbf{f}_1, \mathbf{f}_1 \rangle = 1$, $\langle \mathbf{f}_r, \mathbf{f}_r \rangle = -1$. Let $\mathbf{g}_1, \ldots, \mathbf{g}_{n-r}$ be any basis for $\mathfrak{M}$. Let $\mathbf{S}$ denote the matrix with columns $\mathbf{f}_1, \ldots, \mathbf{f}_r, \mathbf{g}_1, \ldots, \mathbf{g}_{n-r}$. Suppose first that $r = 2$. Since by assumption the matrix of the operator $\mathbf{X}_0$ in $\mathfrak{L}$ has the form $e^{i\beta}\mathbf{I}_2$ and the Gram matrix in $\mathfrak{L}$ is

$$\mathbf{E} = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}, \tag{3.1}$$

we have

$$\tilde{\mathbf{X}}_0 \equiv \mathbf{S}^{-1}\mathbf{X}_0\mathbf{S} = e^{i\beta}\mathbf{I}_2 \dot{+} \mathbf{Y}, \tag{3.2}$$

$$\tilde{\mathbf{G}} = \mathbf{S}^*\mathbf{G}\mathbf{S} = \mathbf{E} \dot{+} \mathbf{F}, \tag{3.3}$$

where* $\mathbf{Y}$, $\mathbf{F}$ are certain $(n-2)\times(n-2)$ matrices. Equality (3.2) follows from the $\mathbf{X}$-invariance of $\mathfrak{L}$ and $\mathfrak{M}$, while (3.3) is a consequence of the fact that $\mathfrak{L}$ and $\mathfrak{M}$ are G-orthogonal. By proposition 1, $\tilde{\mathbf{X}}_0$ is a $\tilde{G}$-unitary matrix. By proposition 2, the matrix $e^{i\beta}\mathbf{I}_2$ is E-unitary (this is also obvious directly) and $\mathbf{Y}$ is F-unitary. We can now construct the matrix $\mathbf{X}$.
Set

$$\mathbf{Q}(\alpha) = \begin{Vmatrix} \operatorname{ch}\alpha & \operatorname{sh}\alpha \\ \operatorname{sh}\alpha & \operatorname{ch}\alpha \end{Vmatrix}, \tag{3.4}$$

where α is a real number. It is easy to see that the eigenvalues of $\mathbf{Q}(\alpha)$ are $e^{\pm\alpha}$ and $\mathbf{Q}(\alpha)$ is E-unitary. This is verified by a direct check, which shows that

$$\mathbf{Q}(\alpha)^* \mathbf{E}\mathbf{Q}(\alpha) = \mathbf{E}.$$

Consequently, the eigenvalues of the matrix $e^{i\beta}\mathbf{Q}(\alpha)$ are $\exp(\pm\alpha + i\beta)$ and this matrix is also E-unitary. Define matrices $\tilde{\mathbf{X}}$ and $\mathbf{X}$ by

$$\tilde{\mathbf{X}} = \mathbf{S}^{-1}\mathbf{X}\mathbf{S} = e^{i\beta}\mathbf{Q}(\alpha) \dot{+} \mathbf{Y}, \tag{3.5}$$

where $\mathbf{S}$, $\mathbf{Y}$ are the matrices constructed above. Since the terms on the right of (3.5) are respectively E- and F-unitary, it follows that (3.3) holds, and by proposition 2 the matrix $\tilde{\mathbf{X}}$ is $\tilde{G}$-unitary. By proposition 1, $\mathbf{X}$ is G-unitary. This is true for any real α. If α is sufficiently small, $\mathbf{Q}(\alpha)$ is

* Recall that $\mathbf{I}_k$ denotes the $k\times k$ identity matrix. By the operator $\mathbf{X}_0$ we mean the operator whose matrix relative to the standard basis is $\mathbf{X}_0$.

arbitrarily close to $\mathbf{I}_2$. It follows from equations (3.2) and (3.5) that $\mathbf{X}$ will then be arbitrarily close to $\mathbf{X}_0$. In addition, as long as $\alpha \neq 0$ the eigenvalues of the matrix $\mathbf{X}$ are $\exp(\pm\alpha + i\beta)$, which are not on the unit circle. This proves the assertion for the case $r=2$.

Now let $r>2$. Instead of (3.2), (3.3), we then have

$$\tilde{\mathbf{X}}_0 \equiv \mathbf{S}^{-1}\mathbf{X}_0\mathbf{S} = (e^{i\beta}\mathbf{I}_2 \dotplus e^{i\beta}\mathbf{I}_{r-2}) \dotplus \mathbf{Y}, \tag{3.6}$$

$$\tilde{\mathbf{G}} = \mathbf{S}^*\mathbf{G}\mathbf{S} = (\mathbf{E} \dotplus \mathbf{E}') \dotplus \mathbf{F}, \tag{3.7}$$

where the terms enclosed in parentheses correspond to the subspace $\mathfrak{L}$. $\mathbf{E}'$ is a diagonal matrix of order $r-2$, with diagonal elements ± 1. Each term on the right of (3.6) is unitary relative to the corresponding term on the right of (3.7). Varying the first term in parenthesis in (3.6), we define matrices $\tilde{\mathbf{X}}$ and $\mathbf{X}$ by

$$\tilde{\mathbf{X}} = \mathbf{S}^{-1}\mathbf{X}\mathbf{S} = (e^{i\beta}\mathbf{Q}(\alpha) \dotplus e^{i\beta}\mathbf{I}_{r-2}) \dotplus \mathbf{Y}, \tag{3.8}$$

where, as before, α is a real number. Since each term on the right of (3.8) is unitary relative to the corresponding term in (3.7), $\tilde{\mathbf{X}}$ is a $\tilde{G}$-unitary matrix, and so $\mathbf{X}$ is G-unitary. As before, $\mathbf{X}$ has eigenvalues $\exp(\pm\alpha + i\beta)$ which, when $\alpha \neq 0$, do not lie on the unit circle; moreover, $\mathbf{X}$ is arbitrarily close to $\mathbf{X}_0$ if α is sufficiently small in absolute value. This proves part (a).

We now prove part (b), * using the material of § I. 1.17.

We begin with the case $\rho_0 = e^{i\beta} = \pm 1$. Since $\mathbf{X}_0$ is a real matrix and ρ_0 is real, it follows that $\mathfrak{L}$ is a real subspace and the vectors $\mathbf{f}_1, \ldots, \mathbf{f}_r$ may be assumed real. We claim that $\mathfrak{M}$ is a real subspace. $\mathfrak{M}$ is the sum of the real root subspaces $\mathfrak{L}_{\rho_j}$ with real eigenvalues ρ_j and the subspaces $\mathfrak{L}_{\rho_j} \dotplus \mathfrak{L}_{\bar{\rho}_j}$, where ρ_j are the nonreal eigenvalues of $\mathbf{X}_0$. Since $\mathfrak{L}_{\bar{\rho}_j} = \overline{\mathfrak{L}}_{\rho_j}$, the subspace $\mathfrak{L}_{\rho_j} \dotplus \mathfrak{L}_{\bar{\rho}_j}$ is real. Hence $\mathfrak{M}$, as a sum of real subspaces, is real. By Lemma II of § I. 1. 17, we may assume that $\mathbf{g}_1, \ldots, \mathbf{g}_{n-r}$ are real vectors. Consequently, $\mathbf{S}$ is a real matrix. In (3.6), $e^{i\beta} = \pm 1$, $\mathbf{X}_0$ and $\mathbf{S}$ are real matrices. Therefore the matrix $\mathbf{Y}$ in (3.6) is real. Formula (3.8) (with $e^{i\beta} = \pm 1$) shows that the matrix $\tilde{\mathbf{X}}$ is real, and hence so is $\mathbf{X}$.

Now let $\rho_0 = e^{i\beta} \neq \pm 1$. Then $\bar{\rho}_0 = e^{-i\beta}$ is also an eigenvalue of $\mathbf{X}_0$. Let $\mathfrak{L} = \mathfrak{L}_{\rho_0}$ and $\overline{\mathfrak{L}} = \mathfrak{L}_{\bar{\rho}_0}$ be the corresponding root subspaces (which are in fact eigensubspaces). It follows from (2.11) that

$$\mathfrak{N}_n = \mathfrak{L} \dotplus \overline{\mathfrak{L}} \dotplus \mathfrak{M},$$

where $\mathfrak{M}$ is the sum of the remaining root subspaces. As before, we see that $\mathfrak{M}$ is a real subspace. By Lemma II in § I. 1.17, we may choose a real basis in $\mathfrak{M}$. Choose some J-orthogonal basis $\mathbf{f}_1, \ldots, \mathbf{f}_r$ in $\mathfrak{L}$, such that (as before) $\langle \mathbf{f}_j, \mathbf{f}_j \rangle = \pm 1$ and $\langle \mathbf{f}_1, \mathbf{f}_1 \rangle = +1$, $\langle \mathbf{f}_r, \mathbf{f}_r \rangle = -1$. Then the vectors $\bar{\mathbf{f}}_1, \ldots, \bar{\mathbf{f}}_r$ form a J-orthogonal basis in $\overline{\mathfrak{L}}$. Indeed,

$$\mathbf{X}_0\bar{\mathbf{f}}_j = \overline{\mathbf{X}}_0\bar{\mathbf{f}}_j = \overline{\mathbf{X}_0\mathbf{f}_j} = \overline{\rho_0\mathbf{f}_j}, \text{ i. e., } \bar{\mathbf{f}}_j \in \mathfrak{L}_{\bar{\rho}_0} = \overline{\mathfrak{L}}$$

* In this case $\mathbf{G} = i\mathbf{J}$ as $\mathbf{X}_0$ is a symplectic matrix (see end of §1.1).

and

$$\langle \bar{\mathbf{f}}_j, \bar{\mathbf{f}}_h \rangle = i(\overline{\mathbf{J}\mathbf{f}_j, \mathbf{f}_h}) = -\langle \mathbf{f}_j, \mathbf{f}_h \rangle = 0 \text{ for } j \neq h.$$

Let $\mathbf{S}$ be the matrix with columns $\mathbf{f}_1, \ldots, \mathbf{f}_r, \bar{\mathbf{f}}_1, \ldots, \bar{\mathbf{f}}_r, \mathbf{g}_1, \ldots, \mathbf{g}_{2k-2r}$. The matrices of the operator $\mathbf{X}$ in $\mathfrak{L}$ and $\bar{\mathfrak{L}}$ are $e^{i\beta}\mathbf{I}_r$ and $e^{-i\beta}\mathbf{I}_r$. Therefore,

$$\tilde{\mathbf{X}}_0 \equiv \mathbf{S}^{-1}\mathbf{X}_0\mathbf{S} = (e^{i\beta}\mathbf{I}_2 \dotplus e^{i\beta}\mathbf{I}_{r-2}) \dotplus (e^{-i\beta}\mathbf{I}_2 \dotplus e^{-i\beta}\mathbf{I}_{r-2}) \dotplus \mathbf{Z}_0, \tag{3.9}$$

$$\tilde{\mathbf{G}} \equiv \mathbf{S}^*\mathbf{G}\mathbf{S} = \left(\begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix} \dotplus \mathbf{E}'\right) \dotplus \left(\begin{Vmatrix} -1 & 0 \\ 0 & 1 \end{Vmatrix} \dotplus \mathbf{E}''\right) \dotplus \mathbf{F}. \tag{3.10}$$

The terms in parentheses are the matrices of the operator $\mathbf{X}_0$ and the Gram matrices of the subspaces $\mathfrak{L}$ and $\bar{\mathfrak{L}}$, * while $\mathbf{E}'$, $\mathbf{E}''$ are diagonal matrices with diagonal elements ± 1. $\mathbf{Z}_0$ is the matrix of the operator $\mathbf{X}_0$ in $\mathfrak{M}$ and is therefore real. By proposition 2, each of the parenthetical terms in (3.9) is unitary relative to the corresponding term in (3.10), so that, in particular, $\mathbf{Z}_0$ is an F-unitary matrix. We now vary the first term of each pair in parentheses in (3.9), in such a way that the eigenvalues of the resulting matrix are complex conjugates, some of them lie outside the unit circle, and $\mathbf{X}$ is real and J-unitary. In other words, we define matrices $\mathbf{X}$ and $\tilde{\mathbf{X}}$ by

$$\tilde{\mathbf{X}} = \mathbf{S}^{-1}\mathbf{X}\mathbf{S} = (e^{i\beta}\mathbf{Q}(\alpha) \dotplus e^{i\beta}\mathbf{I}_{r-2}) \dotplus (e^{-i\beta}\mathbf{Q}(\alpha) \dotplus e^{-i\beta}\mathbf{I}_{r-2}) \dotplus \mathbf{Z}_0, \tag{3.11}$$

where α is a real number and $\mathbf{Q}(\alpha)$ the matrix (3.4). Since each term in (3.11) is unitary relative to the corresponding term in (3.10), it follows that $\tilde{\mathbf{X}}$ is G-unitary, and hence $\mathbf{X}$ is J-unitary. By construction, the matrices in parentheses in (3.11) are the matrices of the operator $\mathbf{X}$ in $\mathfrak{L}$ and $\bar{\mathfrak{L}}$, respectively. These matrices are complex conjugates. By Lemma II in §I. 1. 17, $\tilde{\mathbf{X}}$ is real, and hence so is $\mathbf{X}$. The set of eigenvalues of $\mathbf{X}$ contains the four numbers $\exp(\pm\alpha \pm i\beta)$, which are not on the unit circle when $\alpha \neq 0$. It follows by comparison of (3.11) and (3.9) that $\mathbf{X}$ may be made arbitrarily close to $\mathbf{X}_0$ if α is sufficiently small, proving part (b) of the theorem.

3.2. Krein-Gel'fand-Lidskii strong stability theorem

Lemma. Let $\mathbf{Z}$ *be a* G*-unitary matrix such that* $|\mathbf{Z}-\mathbf{I}_{2k}|<1$. *Then*

$$\mathbf{K} = \ln \mathbf{Z} = (\mathbf{Z}-\mathbf{I}_{2k}) - \tfrac{1}{2}(\mathbf{Z}-\mathbf{I}_{2k})^2 + \ldots \tag{3.12}$$

is a G*-Hamiltonian matrix.*

Proof. Since $\mathbf{Z}^+ = \mathbf{Z}^{-1}$ (§ 1.1), it follows from (3.12) that

$$\mathbf{K}^+ = (\mathbf{Z}^+ - \mathbf{I}_{2k}) - \tfrac{1}{2}(\mathbf{Z}^+ - \mathbf{I}_{2k})^2 + \ldots = \ln(\mathbf{Z}^+) = \ln(\mathbf{Z}^{-1}).$$

Now the function of a complex variable $\ln z = (z-1) - \frac{1}{2}(z-1)^2 + \ldots$, defined in the disk $|z-1|<1$ satisfies the identity $\ln(z^{-1}) = -\ln z$. By the composite

* If $r=2$ the second term in each of these pairs is absent.

function lemma (§I.2.5), the matrix logarithm $\ln \mathbf{Z}$ satisfies the identity $\ln(\mathbf{Z}^{-1}) = -\ln \mathbf{Z}$. Therefore $\mathbf{K}^+ = -\mathbf{K}$, as required.

We now consider a Hamiltonian (or canonical) system (1.9) with Hamiltonian $\mathbf{H}_0(t)$.

Krein-Gel'fand-Lidskii theorem. The Hamiltonian $\mathbf{H}_0(t)$ is strongly stable if and only if all the multipliers of equation (1.9) lie on the unit circle and are definite.

Proof. Sufficiency was proved above (§1.4).

Necessity. Let $\mathbf{X}_0(t)$ be the matrizant of equation (1.9) and suppose that the monodromy matrix $\mathbf{X}_0(T)$ has an eigenvalue of mixed kind on the unit circle. We shall show that for any $\varepsilon > 0$ there is a Hamiltonian $\mathbf{H}(t)$ $(0 \leqslant t \leqslant T)$ such that

$$\|\mathbf{H} - \mathbf{H}_0\| \equiv \int_0^T |\mathbf{H}(t) - \mathbf{H}_0(t)|\, dt < \varepsilon \tag{3.13}$$

and the "perturbed" equation (1.10)

$$i^{-1}\mathbf{G}\frac{d\mathbf{x}}{dt} = \mathbf{H}(t)\mathbf{x} \qquad (\mathbf{H}(t+T) = \mathbf{H}(t) = \mathbf{H}(t)^*)$$

has multipliers outside the unit circle (and hence has unbounded solutions). We shall also show that if $\mathbf{H}_0(t)$ is a real Hamiltonian then $\mathbf{H}(t)$ may also be assumed to be real.

It was shown in § II. 3.4 that any matrix-function $\mathbf{X}(t)$ which is G-unitary (or symplectic) for any $0 \leqslant t \leqslant T$ and has an integrable piecewise-continuous derivative is the matrizant of some G-Hamiltonian (or canonical) system (1.10). The Hamiltonian of this system is clearly

$$\mathbf{H}(t) = i^{-1}\mathbf{G}\frac{d\mathbf{X}(t)}{dt}\mathbf{X}(t)^{-1}.$$

We construct the matrizant $\mathbf{X}(t)$ of the required system (1.10) as follows. According to the theorem in §3.1, for arbitrarily small $\delta > 0$ we can find a G-unitary matrix $\mathbf{Y}$ with an eigenvalue outside the unit circle, such that $|\mathbf{Y} - \mathbf{X}_0(T)| < \delta$. (This is the crucial point of the proof!)

If $\mathbf{H}_0(t)$ is real so is $\mathbf{X}_0(T)$ and hence, by the theorem of §3.1, so is the matrix $\mathbf{Y}$ (which will be the monodromy matrix of the system (1.10) under construction).

Set $\mathbf{Z} = \mathbf{X}_0(T)^{-1}\mathbf{Y}$ and choose δ so small that $|\mathbf{Z} - \mathbf{I}_{2k}| < 1$. The set of G-unitary matrices is a group (closed under multiplication and matrix inversion; see § 1.1). Thus $\mathbf{Z}$ is a G-unitary matrix — this is also easy to verify directly:

$$\mathbf{Z}^*\mathbf{G}\mathbf{Z} = \mathbf{Y}^*\left[(\mathbf{X}_0(T)^{-1})^*\,\mathbf{G}\mathbf{X}_0(T)^{-1}\right]\mathbf{Y} = \mathbf{Y}^*\mathbf{G}\mathbf{Y} = \mathbf{G}.$$

By the above lemma, the matrix

$$\mathbf{K} = \frac{1}{T}\ln \mathbf{Z} = \frac{1}{T}\left[\mathbf{Z} - \mathbf{I}_{2k} - \frac{1}{2}(\mathbf{Z} - \mathbf{I}_{2k})^2 + \ldots\right]$$

is G-Hamiltonian. If the Hamiltonian is real, this matrix is also real. Then the matrix $e^{t\mathbf{K}}(-\infty<t<+\infty)$ is G-unitary:

$$(e^{t\mathbf{K}})^{+}=\mathbf{I}_{2k}+t\mathbf{K}^{+}+\frac{t^2}{2!}(\mathbf{K}^{+})^2+\ldots=e^{-t\mathbf{K}}=(e^{t\mathbf{K}})^{-1}.$$

Consequently, so is the matrix

$$\mathbf{X}(t)=\mathbf{X}_0(t)\,e^{t\mathbf{K}}, \tag{3.14}$$

which will be the required matrizant for $0\leqslant t\leqslant T$. We have thus determined a system (1.10) whose monodromy matrix is $\mathbf{X}(T)=\mathbf{X}_0(T)\mathbf{Z}=\mathbf{Y}$. Moreover, if $\mathbf{X}(t)$ is real, the matrices $\mathbf{H}(t)$ and (hence) $\mathbf{H}_0(t)$ will also be real. It remains to show that for sufficiently small $\delta>0$ we have (3.13). Substituting from (3.14), we obtain

$$\mathbf{H}(t)-\mathbf{H}_0(t)=i^{-1}\mathbf{G}\dot{\mathbf{X}}\mathbf{X}^{-1}-i^{-1}\mathbf{G}\dot{\mathbf{X}}_0\mathbf{X}_0^{-1}=i^{-1}\mathbf{G}\mathbf{X}_0(t)\,\mathbf{K}\mathbf{X}_0(t)^{-1}.$$

For sufficiently small $\delta>0$, $\mathbf{Y}$ is arbitrarily near $\mathbf{X}_0(t)$ and $\mathbf{Z}$ arbitrarily near $\mathbf{I}_{2k}$, so that $\mathbf{K}$ is arbitrarily small. Hence, for sufficiently small $\delta>0$, inequality (3.13) is indeed satisfied, completing the proof.

Remark. The necessity part of the Krein-Gel'fand-Lidskii theorem may also be deduced quite easily from certain formulas of perturbation theory to be established below (§ V.1).

§4. HAMILTONIAN SYSTEMS WITH A PARAMETER

4.1. Hamiltonian equations of positive type. Selfadjoint boundary-value problem for a Hamiltonian equation

Consider the Hamiltonian equation (1.21), rewritten as

$$\mathbf{L}\mathbf{x}=\lambda\mathbf{Q}(t)\,\mathbf{x}, \quad \text{where} \quad \mathbf{L}\mathbf{x}\equiv i^{-1}\mathbf{G}\frac{d\mathbf{x}}{dt}-\mathbf{H}(t)\,\mathbf{x}. \tag{4.1}$$

Recall that $\mathbf{G}$ is a nonsingular hermitian matrix, and $\mathbf{H}(t)$, $\mathbf{Q}(t)$ are hermitian matrix-functions:

$$\mathbf{H}(t)^{*}=\mathbf{H}(t), \quad \mathbf{Q}(t)^{*}=\mathbf{Q}(t) \qquad (0\leqslant t\leqslant T).$$

We shall examine two types of boundary conditions:

$$\mathbf{x}|_{t=T}=\mathbf{R}\mathbf{x}|_{t=0}, \tag{4.2}$$

where $\mathbf{R}$ is some G-unitary matrix, and

$$\mathbf{A}\mathbf{x}|_{t=T}+\mathbf{B}\mathbf{x}|_{t=0}=\mathbf{0}, \tag{4.3}$$

where $\mathbf{A}, \mathbf{B}$ are (not necessarily nonsingular) matrices such that

$$\mathbf{A}G^{-1}\mathbf{A}^* = \mathbf{B}G^{-1}\mathbf{B}^*. \tag{4.4}$$

In the latter case we shall assume that the rank of the $2n \times n$ matrix $\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}$ is n. If one of the matrices $\mathbf{A}$ or $\mathbf{B}$ in (4.3) is nonsingular, it follows from (4.4) that the second is also nonsingular. Condition (4.3) then becomes a special case of (4.2), with $\mathbf{R} = -\mathbf{A}^{-1}\mathbf{B}$, and it follows from (4.4) that $\mathbf{R}^*\mathbf{G}\mathbf{R} = \mathbf{G}$, so that $\mathbf{R}$ is G-unitary. On the other hand, condition (4.2) is a special case of (4.3) with $\mathbf{A} = -\mathbf{I}_n$, $\mathbf{B} = \mathbf{R}$. We may therefore restrict attention in the sequel to conditions (4.3).

We shall be particularly interested in a special case of (4.2):

$$\mathbf{x}|_{t=T} = \rho\mathbf{x}|_{t=0}, \tag{4.5}$$

where ρ is a complex number, $|\rho| = 1$.

For any vector-functions $\mathbf{u}(t)$, $\mathbf{v}(t)$, we denote

$$((\mathbf{u},\ \mathbf{v})) = \int_0^T (\mathbf{u}(t),\ \mathbf{v}(t))\, dt, \tag{4.6}$$

provided the integral is meaningful. We claim that the operator $\mathbf{L}$ in (4.1) is symmetric on the set of vector-functions satisfying (4.2) or (4.3), i. e.,

$$((\mathbf{Lu},\ \mathbf{v})) = ((\mathbf{u},\ \mathbf{Lv})). \tag{4.7}$$

Since clearly $((\mathbf{Hu},\ \mathbf{v})) = ((\mathbf{u},\ \mathbf{Hv}))$, it will suffice to prove (4.7) for $\mathbf{L} = i^{-1}\mathbf{G}\frac{d}{dt}$. We have

$$((\mathbf{Lu},\ \mathbf{v})) - ((\mathbf{u}, \mathbf{Lv})) = \int_0^T i^{-1} d\,(\mathbf{Gu},\ \mathbf{v}) = i^{-1}\,(\mathbf{Gu}(T),\ \mathbf{v}(T)) - i^{-1}\,(\mathbf{Gu}(0),\ \mathbf{v}(0)). \tag{4.8}$$

If $\mathbf{u}(t)$, $\mathbf{v}(t)$ satisfy conditions (4.2), the expression on the right clearly vanishes, so that (4.7) is true. Now consider (4.3). Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and $\mathbf{b}_1, \ldots, \mathbf{b}_n$ be the columns of $\mathbf{A}$ and $\mathbf{B}$, respectively. We define $2n$-vectors and a $2n \times 2n$ matrix by

$$\mathbf{c}_\nu = \begin{pmatrix} \mathbf{a}_\nu \\ \mathbf{b}_\nu \end{pmatrix} \quad (\nu = 1, \ldots, n), \quad \tilde{\mathbf{G}} = \begin{pmatrix} \mathbf{G} & 0 \\ 0 & -\mathbf{G} \end{pmatrix}.$$

By assumption, the vectors $\mathbf{c}_1, \ldots, \mathbf{c}_n$ are linearly independent. Let $\mathfrak{M}$ be the subspace of complex $2n$-space $\mathfrak{R}_{2n}$ spanned by $\mathbf{c}_1, \ldots, \mathbf{c}_n$ and $\mathfrak{N}$ the subspace spanned by $\tilde{\mathbf{G}}^{-1}\mathbf{c}_1, \ldots, \tilde{\mathbf{G}}^{-1}\mathbf{c}_n$ (the vectors $\tilde{\mathbf{G}}^{-1}\mathbf{c}_1, \ldots, \tilde{\mathbf{G}}^{-1}\mathbf{c}_n$ are also linearly independent, since $\det \tilde{\mathbf{G}} \neq 0$). We can write (4.4) as

$$\mathbf{a}_j^* G^{-1}\mathbf{a}_h - \mathbf{b}_j^* G^{-1}\mathbf{b}_h = 0 \qquad (j, h = 1, \ldots, n),$$

or

$$\mathbf{c}_j^*\tilde{\mathbf{G}}^{-1}\mathbf{c}_h = 0 \qquad (j, h = 1, \ldots, n). \tag{4.9}$$

It follows that $\mathfrak{N}$ is the orthogonal complement of $\mathfrak{M}$ in $\mathfrak{R}_{2n}$. Set

$$\tilde{\mathbf{u}} = \begin{pmatrix} \mathbf{u}\ (T) \\ \mathbf{u}\ (0) \end{pmatrix}, \quad \tilde{\mathbf{v}} = \begin{pmatrix} \mathbf{v}\ (T) \\ \mathbf{v}\ (0) \end{pmatrix}.$$

By assumption, condition (4.3) holds for $\mathbf{x} = \mathbf{u}(t)$, $\mathbf{x} = \mathbf{v}(t)$, so that

$$\mathbf{c}_j^*\tilde{\mathbf{u}} = 0, \quad \mathbf{c}_j^*\tilde{\mathbf{v}} = 0 \qquad (j = 1, \ldots, n).$$

These equalities show that the vectors $\tilde{\mathbf{u}}$, $\tilde{\mathbf{v}}$ are orthogonal to $\mathfrak{M}$, and so are elements of $\mathfrak{N}$, i. e., $\tilde{\mathbf{u}} = \sum_j \alpha_j \tilde{\mathbf{G}}^{-1}\mathbf{c}_j$, $\tilde{\mathbf{v}} = \sum_h \beta_h \tilde{\mathbf{G}}^{-1}\mathbf{c}_h$, where α_j, β_h are scalars. Using (4.9), we obtain

$$\tilde{\mathbf{u}}^*\tilde{\mathbf{G}}\tilde{\mathbf{v}} = \sum_{j,h} \alpha_j^*\beta_h \mathbf{c}_j^*\tilde{\mathbf{G}}^{-1}\mathbf{c}_h = 0.$$

On the other hand, the right-hand side of (4.8) is equal to $i^{-1}\tilde{\mathbf{u}}^*\tilde{\mathbf{G}}\tilde{\mathbf{v}}$. Thus the expression (4.8) vanishes, so that (4.7) is also true for boundary conditions (4.3).

Definition. A boundary-value problem (4.1), (4.3) is said to be selfadjoint if condition (4.4) holds and, in addition:

(I) the hermitian matrix-function $\mathbf{Q}(t)$ is nonnegative for almost all t, i. e., $(\mathbf{Q}(t)\mathbf{c}, \mathbf{c}) \geqslant 0$ for any vector $\mathbf{c}$;

(II) the equations

$$\mathbf{Lx} = 0, \quad \mathbf{Q}(t)\mathbf{x} = 0 \ \text{(almost everywhere)} \tag{4.10}$$

and (4.3) have only the trivial solution $\mathbf{x} \equiv 0$.

Equation (4.1) is said to be of positive type if the boundary-value problem (4.1), (4.5) is selfadjoint for any ρ, $|\rho| = 1$.

Stated in different terms, condition (II) means that problem (4.1), (4.3) has no nontrivial solution not depending on λ. If there is such a solution, so that condition (II) fails to hold, the spectrum of problem (4.1), (4.3) is the entire plane. Indeed, a nontrivial solution $\mathbf{x}(t)$ satisfying equations (4.10) and (4.3) is obviously a solution of problem (4.1), (4.3) for any λ.

We now present a few examples of selfadjoint boundary-value problems and equations (4.1) of positive type.

(A) An equation (4.1) in which $\mathbf{Q}(t)$ is a positive definite matrix-function for almost all t is clearly of positive type, and the boundary-value problem (4.1), (4.3) is selfadjoint.

(B) An equation (4.1) for which $\mathbf{H}(t) \equiv 0$, $\mathbf{Q}(t) \geqslant 0$, and

$$\mathbf{Q}_{\text{av}} \equiv \frac{1}{T}\int_0^T \mathbf{Q}(t)\,dt > 0$$

is of positive type, and problem (4.1), (4.3) is then selfadjoint. Indeed, it follows from the first equality of (4.10) that $\mathbf{x}=\text{const}$. Integrating the second equality of (4.10), we obtain $\mathbf{Q}_{av}\mathbf{x}=\mathbf{0}$, whence $\mathbf{x}=\mathbf{0}$.

Let $\mathbf{H}(t)\not\equiv\mathbf{0}$ and let $\mathbf{X}_0(t)$ be the matrizant of the equation $\mathbf{Lx}=\mathbf{0}$,

$$\mathbf{S}=\int_0^T \mathbf{X}_0(t)^*\mathbf{Q}(t)\mathbf{X}_0(t)\,dt.$$

If $\mathbf{Q}(t)\geqslant\mathbf{0}$, $\mathbf{S}>\mathbf{0}$, then equation (4.1) is of positive type and problem (4.1), (4.3) is selfadjoint. Indeed, by (4.10), we have

$$\mathbf{x}=\mathbf{X}_0(t)\mathbf{c}, \qquad \mathbf{X}_0(t)^*\mathbf{Q}(t)\mathbf{X}_0(t)\mathbf{c}\equiv\mathbf{0}.$$

Hence $\mathbf{Sc}=\mathbf{0}$, $\mathbf{c}=\mathbf{0}$, $\mathbf{x}\equiv\mathbf{0}$.

(C) The vector equation

$$\frac{d^2\mathbf{y}}{dt^2}+\lambda^2\mathbf{P}(t)\mathbf{y}=\mathbf{0}$$

with a positive definite hermitian matrix-function $\mathbf{P}(t)$ may be reduced to an equation (4.1) of positive type by the substitution

$$\mathbf{x}=\begin{pmatrix}\mathbf{y}\\ \frac{1}{\lambda}\frac{d\mathbf{y}}{dt}\end{pmatrix}.$$

Indeed, the result is equation (4.1) for $\mathbf{x}$, with

$$i^{-1}\mathbf{G}=\mathbf{J}=\begin{pmatrix}\mathbf{0} & -\mathbf{I}_k\\ \mathbf{I}_k & \mathbf{0}\end{pmatrix}, \quad \mathbf{Q}(t)=\begin{pmatrix}\mathbf{P}(t) & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_k\end{pmatrix}, \quad \mathbf{H}(t)\equiv\mathbf{0},$$

and $\mathbf{Q}(t)$ is positive definite. The corresponding problem (4.1), (4.3) is selfadjoint.

(D) The vector equation

$$\frac{d^2\mathbf{y}}{dt^2}+\lambda\mathbf{P}(t)\mathbf{y}=\mathbf{0}$$

where $\mathbf{P}(t)$ is a hermitian matrix-function (the form $(\mathbf{P}(t)\mathbf{c},\mathbf{c})$ need not be positive or negative definite), such that

$$\det\mathbf{P}_{av}\neq 0, \quad \text{where} \quad \mathbf{P}_{av}=\frac{1}{T}\int_0^T\mathbf{P}(t)\,dt, \tag{4.11}$$

may be reduced to an equation (4.1) of positive type by Krein's substitution

$$\mathbf{z}=\mathbf{Sy}+\frac{1}{\lambda}\frac{d\mathbf{y}}{dt}, \quad \mathbf{x}=\begin{pmatrix}\mathbf{y}\\ \mathbf{z}\end{pmatrix}, \quad \text{where} \quad \mathbf{S}=\int_0^t(\mathbf{P}-\mathbf{P}_{av})\,dt.$$

Indeed, this gives an equation (4.1) for $\mathbf{x}$ with $i^{-1}\mathbf{G}=\mathbf{J}$,

$$\mathbf{H}(t)=\begin{pmatrix}\mathbf{P}_{av} & 0\\ 0 & 0\end{pmatrix},\quad \mathbf{Q}(t)=\begin{pmatrix}\mathbf{S}^2 & -\mathbf{S}\\ -\mathbf{S} & \mathbf{I}_k\end{pmatrix}.$$

Since $(\mathbf{Q}(t)\mathbf{x},\ \mathbf{x})=|\mathbf{Sy}-\mathbf{z}|^2\geqslant 0$, condition (I) holds. The first equation of (4.10) is

$$\frac{d\mathbf{z}}{dt}=\mathbf{P}_{av}\mathbf{y},\quad \frac{d\mathbf{y}}{dt}=\mathbf{0}.$$

Hence $\mathbf{y}=\mathbf{c}=\text{const}$, $\mathbf{z}=\mathbf{P}_{av}\mathbf{c}t+\mathbf{c}_0$, where $\mathbf{c}_0=\text{const}$. Condition (4.5) means that

$$\mathbf{y}(T)=\rho\mathbf{y}(0),\quad \mathbf{z}(T)=\rho\mathbf{z}(0). \tag{4.12}$$

If $\rho\neq 1$, this gives $\mathbf{c}=\mathbf{0}$, $\mathbf{c}_0=\rho\mathbf{c}_0$, $\mathbf{c}_0=\mathbf{0}$, so that condition (II) holds. If $\rho=1$, the second equation (4.12) gives $T\mathbf{P}_{av}\mathbf{c}=\mathbf{0}$, $\mathbf{c}=\mathbf{0}$, $\mathbf{y}\equiv\mathbf{0}$, $\mathbf{z}\equiv\mathbf{c}_0$. The second equation of (4.10) yields $\mathbf{c}_0=\mathbf{0}$. Thus condition (II) holds in this case too.

Instead of (4.11), a weaker condition is sufficient: there is no constant vector $\mathbf{c}\neq\mathbf{0}$ such that

$$\mathbf{P}_{av}\mathbf{c}=\mathbf{0},\ \mathbf{P}(t)\mathbf{c}=\mathbf{0}\quad \text{a. e.} \tag{4.13}$$

Indeed, repeating the previous arguments for the case $\rho=1$, we get $\mathbf{P}_{av}\mathbf{c}=\mathbf{0}$, $\mathbf{y}\equiv\mathbf{c}$, $\mathbf{z}\equiv\mathbf{c}_0$. By assumption, $(\mathbf{Q}(t)\mathbf{x},\mathbf{x})\equiv 0$. Hence $|\mathbf{Sc}-\mathbf{c}_0|^2\equiv 0$, $\mathbf{S}(t)\mathbf{c}\equiv\mathbf{c}_0$. When $t=0$ we have $\mathbf{c}_0=\mathbf{0}$. Differentiating the identity $\mathbf{S}(t)\mathbf{c}\equiv\mathbf{0}$, we get $\mathbf{P}(t)\mathbf{c}\equiv\mathbf{0}$. By our assumption, $\mathbf{c}=\mathbf{0}$, so that condition (II) is again true.

(E) The Lagrange vector equation

$$\frac{d^2\mathbf{y}}{dt^2}+\mathbf{K}\frac{d\mathbf{y}}{dt}+[\mathbf{P}_0(t)+\lambda\mathbf{P}_1(t)]\mathbf{y}=\mathbf{0},$$

where $\mathbf{K}$ is a constant matrix with

$$\mathbf{K}^*=-\mathbf{K},\ \mathbf{P}_0(t)^*=\mathbf{P}_0(t),\ \mathbf{P}_1(t)^*=\mathbf{P}_1(t),$$

is reduced by the substitution

$$\mathbf{x}=\begin{pmatrix}\mathbf{y}\\ \frac{d\mathbf{y}}{dt}\end{pmatrix}$$

to equation (4.1). Indeed, the substitution gives equation (4.1) with

$$i^{-1}\mathbf{G}=\begin{pmatrix}-\mathbf{K} & -\mathbf{I}_k\\ \mathbf{I}_k & 0\end{pmatrix},\quad \mathbf{H}(t)=\begin{pmatrix}\mathbf{P}_0(t) & 0\\ 0 & \mathbf{I}_k\end{pmatrix},\quad \mathbf{Q}(t)=\begin{pmatrix}\mathbf{P}_1(t) & 0\\ 0 & 0\end{pmatrix}.$$

If $\mathbf{P}_1(t)$ is a positive definite matrix, this equation is of positive type and problem (4.1), (4.3) is selfadjoint. Indeed, condition (I) is obvious. The

solution of the second equation of (4.10) is $\mathbf{x}=\begin{pmatrix}\mathbf{0}\\ \mathbf{c}\end{pmatrix}$, where $\mathbf{c}$ is a constant vector. It follows from the first equation of (4.10) that $\mathbf{H}(t)\mathbf{x}\equiv\mathbf{0}$, so that $\mathbf{c}=\mathbf{0}$. Consequently, condition (II) is also satisfied.

(F) The scalar equation

$$\sum_{j=0}^{k}(-1)^j[\varphi_j(t)\,\eta^{(k-j)}]^{(k-j)}=(-1)^k\lambda\psi(t)\,\eta, \tag{4.14}$$

where $\varphi_0(t)\neq 0$, reduces to a vector equation (4.1). Indeed, setting

$$\left.\begin{aligned}&\xi_j=\eta^{(j-1)} \qquad (j=1,\ \ldots,\ k),\\ &\xi_{j+k}=\varphi_{k-j}(t)\,\xi_{j+1}-\frac{d\xi_{k+j+1}}{dt} \qquad (j=1,\ \ldots,\ k-1),\\ &\xi_{2k}=\varphi_0(t)\frac{d\xi_k}{dt},\end{aligned}\right\} \tag{4.15}$$

we write (4.14) as

$$\frac{d\xi_{k+1}}{dt}=[\varphi_k(t)-\lambda\psi(t)]\,\xi_1. \tag{4.16}$$

Omitting the first equation of (4.15) (the equation $\xi_1=\eta$, which may be viewed as a formal specification of notation), we can rewrite equations (4.15), (4.16) as a system of $2k$ differential equations:

$$\begin{aligned}&\dot\xi_1=\xi_2, && \dot\xi_{k+1}=[\varphi_k(t)-\lambda\psi(t)]\,\xi_1,\\ &\dot\xi_2=\xi_3, && \dot\xi_{k+2}=\varphi_{k-1}(t)\,\xi_2-\xi_{k+1},\\ &\cdots\cdots\cdots

We shall show that the Hamiltonian equation (4.1) corresponding to (4.14) is of positive type if $\psi(t) \geqslant 0$ in (4.14) and there is an interval on which $\psi(t) > 0$. Condition (I), i. e., the inequality $\mathbf{Q}(t) \geqslant 0$, is obviously satisfied, for $\mathbf{Q}(t)$ is a diagonal matrix with nonnegative elements $(0, \ldots, 0, \psi(t))$. To prove condition (II), it will suffice to show that the only solution $\mathbf{x}(t)$ of problem (4.1), (4.3) that does not depend on λ (and is a solution for all λ) is $\mathbf{x} \equiv \mathbf{0}$. Let $\mathbf{x} = \mathbf{x}(t)$ be a nonzero solution meeting these requirements. Then $\eta = \xi_1$ is a solution, independent of λ, of equation (4.14) (for all λ). Consequently, $\psi(t)\eta(t) \equiv 0$. Now, on the interval where $\psi(t) > 0$ we have $\eta(t) \equiv 0$, and so, for any interior point of this interval,

$$\eta(t_0) = 0, \quad \dot{\eta}(t_0) = 0, \quad \ldots, \quad \eta^{(2k-1)}(t_0) = 0.$$

Therefore $\eta(t) \equiv 0$. It follows from (4.15) that $\mathbf{x}(t) \equiv \mathbf{0}$, as required.

In entirely analogous fashion we can treat equation (4.14) (in which $\varphi_j(t)$, $\psi(t)$ are hermitian matrices), and the equation

$$\sum_{j=0} (-1)^j \{[\varphi_j(t) + \lambda\psi_j(t)]\eta^{(k-j)}\}^{(k-j)} = 0.$$

We shall confine ourselves, however, to the above examples.

We turn to consideration of the boundary-value problem (4.1), (4.3). The numbers λ for which this problem has a nontrivial solution are called its eigenvalues. Since the matrix $\mathbf{Q}(t)$ is singular, the problem may have no eigenvalues at all. (This is the case, for example, in the trivial instance when $\mathbf{Q}(t) \equiv 0$ and $\det[\mathbf{X}(T) - \mathbf{R}] \neq 0$, where $\mathbf{X}(t)$ is the matrizant of the equation $\mathbf{Lx} = \mathbf{0}$. *) If there are eigenvalues, we have the familiar situation:

Theorem I. A selfadjoint boundary-value problem (4.1), (4.3) has at most countably many eigenvalues λ, whose only point of accumulation (if at all) is $\lambda = \infty$. The eigenvalues (if they exist) are real and of finite multiplicity, and the corresponding eigenvectors $\mathbf{x}_j(t)$ are orthogonal, in the sense that

$$((\mathbf{Qx}_j, \mathbf{x}_h)) \equiv \int_0^T (\mathbf{Q}(t)\mathbf{x}_j(t), \mathbf{x}_h(t))\,dt = 0 \quad \text{for } \lambda_j \neq \lambda_h. \tag{4.17}$$

Proof. Let $\mathbf{X}(t, \lambda)$ be the matrizant of equation (4.1). Substituting $\mathbf{x}(T, \lambda) = \mathbf{X}(T, \lambda)\mathbf{x}(0, \lambda)$ into (4.3), we see that problem (4.1), (4.3) has a nontrivial solution if and only if λ satisfies the equation

$$F(\lambda) \equiv \det[\mathbf{AX}(T, \lambda) + \mathbf{B}] = 0. \tag{4.18}$$

These λ's are the eigenvalues of the problem. Now the function $F(\lambda)$ does not vanish identically, since otherwise any λ would be an eigenvalue, contrary to condition (II). Since $\mathbf{X}(T, \lambda)$ is an entire function of λ (§ II. 1. 3),

* As another example, note that the two-dimensional vector equation $\mathbf{Lx} = \lambda\mathbf{Qx}$, where $\mathbf{L} = \left\|\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right\|$, $\mathbf{Q} = \left\|\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right\|$ has no eigenvalues. This equation satisfies the selfadjointness conditions (I) and (II).

$F(\lambda)$ is an entire function, and so its zeros (provided they exist) form a countable set which cannot have a finite point of accumulation, and each zero has finite multiplicity.

Let λ_1 and λ_2 be any eigenvalues and $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ corresponding (nontrivial) solutions:

$$\mathbf{L}\mathbf{x}_j = \lambda_j \mathbf{Q}(t)\mathbf{x}_j, \quad \mathbf{A}\mathbf{x}_j(T) + \mathbf{B}\mathbf{x}_j(0) = \mathbf{0} \qquad (j = 1, 2). \tag{4.19}$$

(The case $\lambda_1 = \lambda_2$, $\mathbf{x}_1 = \mathbf{x}_2$ is not excluded.) The solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ satisfy the equality

$$((\mathbf{L}\mathbf{x}_1, \mathbf{x}_2)) = ((\mathbf{x}_1, \mathbf{L}\mathbf{x}_2)). \tag{4.20}$$

It follows from (4.19), via (4.20), that

$$\lambda_1((\mathbf{Q}\mathbf{x}_1, \mathbf{x}_2)) = ((\mathbf{L}\mathbf{x}_1, \mathbf{x}_2)) = ((\mathbf{x}_1, \mathbf{L}\mathbf{x}_2)) = \bar{\lambda}_2((\mathbf{Q}\mathbf{x}_1, \mathbf{x}_2)),$$

so that

$$(\lambda_1 - \bar{\lambda}_2)((\mathbf{Q}\mathbf{x}_1, \mathbf{x}_2)) = 0. \tag{4.21}$$

Taking $\lambda_1 = \lambda_2$, $\mathbf{x}_1 = \mathbf{x}_2$, we obtain

$$(\lambda_1 - \bar{\lambda}_1)((\mathbf{Q}\mathbf{x}_1, \mathbf{x}_1)) = 0. \tag{4.22}$$

We claim that $((\mathbf{Q}\mathbf{x}_1, \mathbf{x}_1)) \not\equiv 0$. Otherwise,

$$\int_0^T (\mathbf{Q}\mathbf{x}_1, \mathbf{x}_1)\, dt = 0,$$

so it follows from condition (I) that $(\mathbf{Q}\mathbf{x}_1, \mathbf{x}_1) \equiv 0$, $\mathbf{Q}\mathbf{x}_1 \equiv \mathbf{0}$. Thus the vector-function $\mathbf{x}_1(t) \not\equiv \mathbf{0}$ satisfies the equations $\mathbf{L}\mathbf{x}_1 = \mathbf{0}$, $\mathbf{Q}\mathbf{x}_1 = \mathbf{0}$ and condition (4.3). The contradiction to condition (II) shows that $((\mathbf{Q}\mathbf{x}_1, \mathbf{x}_1)) \not\equiv 0$. It follows from (4.22) that $\bar{\lambda}_1 = \lambda_1$, and so the eigenvalues are real and (4.21) may be written

$$(\lambda_1 - \lambda_2)((\mathbf{Q}\mathbf{x}_1, \mathbf{x}_2)) = 0.$$

This implies (4.17), completing the proof of Theorem I.

Theorem I may be sharpened in many respects. If λ_0 is a root of equation (4.18) of multiplicity p, there are exactly p linearly independent vectors $\mathbf{x}_j(t)$ belonging to λ_0. Problem (4.1), (4.3) may be reduced /166b/ to a weighted integral equation

$$\mathbf{x}(t) = \lambda \int_0^T \mathbf{P}(t, s)\mathbf{Q}(s)\mathbf{x}(s)\, ds$$

with hermitian kernel $\mathbf{P}(t, s)$, to which we can apply all the results of the theory of completely continuous* selfadjoint operators (/50/, Chapter IV,

* [Also known as compact operators.]

§ 6 and /78/). Hence, in particular, we deduce the above assertion concerning the existence of p linearly independent solutions $\mathbf{x}_j(t)$.

When the matrix $\mathbf{Q}(t)$ is singular, this integral equation may have no eigenvalues or only finitely many. Effective conditions guaranteeing this situation are not readily available. Krein /166f/ has shown that the problem

$$\mathbf{J}\frac{d\mathbf{x}}{dt}=\lambda\mathbf{Q}(t)\,\mathbf{x},\quad \mathbf{x}(T)+\mathbf{x}(0)=0$$

has at least one eigenvalue if and only if

$$\operatorname{Tr}(\mathbf{JQ}_{\mathrm{av}})^2\neq 0,\ \text{where}\ \mathbf{Q}_{\mathrm{av}}=\frac{1}{T}\int_0^T\mathbf{Q}(t)\,dt.$$

The following theorem, given without proof, is valid for the general case, though it does not furnish effective criteria.

Theorem II. The selfadjoint problem (4.1), (4.3) has at least one eigenvalue if and only if there exist absolutely continuous vector-functions $\mathbf{u}_0(t)$, $\mathbf{u}_1(t)$ *satisfying the boundary conditions (4.3) such that*

$$\mathbf{Lu}_1=\mathbf{Qu}_0,\qquad ((\mathbf{Qu}_1,\ \mathbf{u}_1))\neq 0.$$

4.2. Analytical properties of the multipliers of Hamiltonian equations of positive type

Consider a Hamiltonian equation (4.1) of positive type,

$$i^{-1}\mathbf{G}\frac{d\mathbf{x}}{dt}-\mathbf{H}(t)\,\mathbf{x}=\varepsilon\mathbf{Q}(t)\,\mathbf{x},\tag{4.23}$$

where the parameter λ has been replaced by ε in the interests of future reference. Our assumptions imply that $\mathbf{G}^*=\mathbf{G}$, $\mathbf{H}(t)^*=\mathbf{H}(t)$, $\mathbf{Q}(t)^*=\mathbf{Q}(t)\geqslant 0$ and that conditions (I), (II) in the definition of §4.1 (p. 199) are satisfied for boundary conditions (4.5), with any ρ, $|\rho|=1$.

Suppose that when $\varepsilon=0$ equation (4.23) has a multiplier ρ_0 of multiplicity $m'>1$ on the unit circle. We shall examine the multipliers of equation (4.23) near ρ_0 for small ε.

Let $\mathbf{X}(t,\ \varepsilon)$ denote the matrizant of equation (4.23). The multipliers $\rho_j(\varepsilon)$ of the equation are the roots of the equation

$$D(\varepsilon,\ \rho)\equiv\det[\mathbf{X}(T,\ \varepsilon)-\rho\mathbf{I}_{2k}]=0\tag{4.24}$$

and so they admit expansions in fractional powers of ε:

$$\rho_j(\varepsilon)=\rho_0+\sum_{m=1}^{\infty}c_{jm}\varepsilon^{m/q_j}\qquad (j=1,\ \ldots,\ d).\tag{4.25}$$

Here q_j are integers. It is obvious that for each expansion $\rho_j(\varepsilon)$ we have a solution $\mathbf{x}_j(t, \varepsilon)$:

$$\mathbf{x}_j(t, \varepsilon) = \sum_{m=0}^{\infty} \mathbf{x}_{jm}(t)\, \varepsilon^{m/q_j}, \tag{4.26}$$

satisfying the condition

$$\mathbf{x}_j(T, \varepsilon) \equiv \rho_j(\varepsilon)\mathbf{x}_j(0, \varepsilon). \tag{4.27}$$

The coefficients c_{jm} and vector-functions $\mathbf{x}_{jm}(t)$ may be determined successively by substituting the series (4.26), (4.25) into equations (4.23), (4.27) and comparing the coefficients of like powers. It is more difficult to characterize the expansions (4.25) as to the number d of expansions and the numbers $q_1, \ldots, q_d$. In this subsection we shall devote ourselves to this problem. Theoretically speaking, the numbers $d, q_1, \ldots, q_d$ may be determined from the coefficients of the expansion of $D(\varepsilon, \rho)$ in powers of ε and $\rho - \rho_0$. To this end, we must first determine the matrix coefficients in the series

$$\mathbf{X}(T, \varepsilon) = \mathbf{X}_0 + \varepsilon\mathbf{X}_1 + \varepsilon^2\mathbf{X}_2 + \ldots, \tag{4.28}$$

then substitute this series into (4.24), find the expansion coefficients of the function $D(\varepsilon, \rho)$, and finally determine $d, q_1, \ldots, q_d$ (e.g., using Newton diagrams).

This approach, however, is hardly practical, involving as it does a large amount of cumbersome algebraic calculations. In general, determination of the numbers $d, q_1, \ldots, q_d$ in the expansions (4.25) requires knowledge of several terms in (4.28), the actual number of terms being unknown in advance.

It turns out that if equation (4.23) is of positive type, the numbers $d, q_1, \ldots, q_d$ in (4.25) may be determined quite simply knowing only the matrix $\mathbf{X}_0$. The relevant theorem, which also has several important corollaries, is due to Krein and Lyubarskii /167b/.

Theorem (Krein-Lyubarskii). Let (4.23) be an equation of positive type. Then:

(a) *The number d of expansions is equal to the number of elementary Jordan matrices in the canonical form of $\mathbf{X}_0$ which belong to the eigenvalue ρ_0.*

(b) *d is also equal to the multiplicity of the root $\varepsilon = 0$ of the equation*

$$D(\varepsilon, \rho_0) = 0.$$

(c) *The numbers $q_1, \ldots, q_d$ are respectively the orders of the above-mentioned elementary Jordan matrices.*

(d) *The first coefficient c_{1m} of each of the expansions (4.25) is not zero.*

Proof. If ε and ρ satisfy equation (4.24), the boundary-value problem consisting of equation (4.23) with the boundary condition

$$\mathbf{x}(T) = \rho\mathbf{x}(0) \tag{4.29}$$

has a nontrivial solution. According to Weierstrass' lemma /95/, the function $D(\varepsilon, \rho)$, which is analytic at the point $\varepsilon=\varepsilon_0$, $\rho=\rho_0$, admits a series expansion

$$D(\varepsilon, \rho)=(\rho-\rho_0)^l[\varepsilon^p+\varepsilon^{p-1}A_1(\rho)+\ldots+A_p(\rho)]B(\varepsilon, \rho),$$

where l, p are integers, $A_j(\rho_0)=0$, $B(0, \rho_0)\neq 0$, and the functions $A_j(\rho)$ are analytic at $\rho=\rho_0$ $(j=1, \ldots, p)$, $B(\varepsilon, \rho)$ is analytic at $\varepsilon=0$, $\rho=\rho_0$.

We claim that $l=0$. Indeed, if $l\neq 0$, then $D(\varepsilon, \rho_0)\equiv 0$, so that the selfadjoint boundary-value problem (4.23), (4.29) with $\rho=\rho_0$ has a solution for all ε. This is impossible by the theorem of §4.1.

Let $\varepsilon_j(\rho)$ $(j=1, \ldots, p)$ denote the roots of the equation $D(\varepsilon, \rho)=0$, i.e., the equation $\varepsilon^p+\varepsilon^{p-1}A_1(\rho)+\ldots+A_p(\rho)=0$, and $\mathbf{x}_j(t, \rho)$ the corresponding solutions of the boundary-value problem (4.23), (4.29). As shown in §4.1, even when the roots $\varepsilon_j(\rho)$ are multiple we may assume that the solutions $\mathbf{x}_j(t, \rho)$ are linearly independent for any ρ, $|\rho|=1$. By the theorem of §4.1, the values of $\varepsilon_j(\rho)$ are real for $|\rho|=1$. By the lemma of §1.5, $\varepsilon_j(\rho)$ are analytic functions of ρ at $\rho=\rho_0$. This is the central point of the proof. Thus,

$$\varepsilon_j(\rho)=\sum_{h=q_j}^{\infty} a_{jh}(\rho-\rho_0)^h \qquad (j=1, \ldots, p), \tag{4.30}$$

where q_j is the index of the first nonvanishing coefficient of this series and p is the multiplicity of the root $\varepsilon=0$ of the equation $D(\varepsilon, \rho_0)=0$. Inverting the series (4.30), we get (4.25), where $d=p$.

Since $\varepsilon=\varepsilon_j(\rho)$ are all the solutions of the equation $D(\varepsilon, \rho)=0$, the series (4.25) assume all the values of the multipliers that coincide with ρ_0 at $\varepsilon=0$. There must be m' of these values, and so

$$q_1+\ldots+q_d=m'. \tag{4.31}$$

We now show that the solutions $\mathbf{x}_j(t, \rho)$ may be assumed to be analytic in ρ at $\rho=\rho_0$. Clearly, the solutions $\mathbf{c}_j=\mathbf{c}_j(\rho)$ of the equation

$$[\mathbf{X}(T, \varepsilon_j(\rho))-\rho\mathbf{I}]\mathbf{c}_j=0$$

may be taken to be analytic in ρ, since the bracketed matrix is analytic in ρ at $\rho=\rho_0$. Thus the solutions

$$\mathbf{x}_j(t, \rho)=\mathbf{X}(t, \varepsilon_j(\rho))\mathbf{c}_j(\rho) \tag{4.32}$$

are also analytic functions.

Substituting $\mathbf{x}=\mathbf{x}_j(t, \rho)$ into (4.29), we obtain the identity

$$\mathbf{X}(T, \varepsilon_j(\rho))\mathbf{c}_j(\rho)\equiv\rho\mathbf{c}_j(\rho). \tag{4.33}$$

Let

$$\mathbf{c}_j(\rho)=\mathbf{c}_0^{(j)}+(\rho-\rho_0)\mathbf{c}_1^{(j)}+\ldots$$

We substitute this into (4.33) and compare the coefficients of $(\rho-\rho_0)^h$ $(h=0, 1, \ldots, q_j-1)$. It follows from (4.28) and (4.30) that

$$\mathbf{X}(T, \varepsilon_j(\rho))=\mathbf{X}_0+\mathbf{O}((\rho-\rho_0)^{q_j}),$$

and so

$$\left.\begin{aligned}
&\mathbf{X}_0\mathbf{c}_0^{(j)}=\rho_0\mathbf{c}_0^{(j)},\\
&\mathbf{X}_0\mathbf{c}_1^{(j)}=\rho_0\mathbf{c}_1^{(j)}+\mathbf{c}_0^{(j)},\\
&\quad\cdots\cdots\cdots\\
&\mathbf{X}_0\mathbf{c}_{q_j-1}^{(j)}=\rho_0\mathbf{c}_{q_j-1}^{(j)}+\mathbf{c}_{q_j-2}^{(j)} \qquad (j=1, \ldots, d)
\end{aligned}\right\}. \tag{4.34}$$

We see from (4.31) that the equalities (4.34) involve m' vectors $\mathbf{c}_h^{(j)}$; recall that m' is the multiplicity of the multiplier ρ_0. If we can show that these vectors are linearly independent, the equalities (4.34) will imply the truth of parts (a) and (b) of the theorem, completing the proof. That the vectors $\mathbf{c}_0^{(j)}=\mathbf{c}_j(\rho_0)$ are linearly independent follows from the linear independence of the solutions (4.32). The linear independence of all the vectors $\mathbf{c}_h^{(j)}$ follows directly from (4.34). We demonstrate this for the case $d=3$, $q_1=q_2=2$, $q_3=3$. Let

$$\alpha_0\mathbf{c}_0^{(1)}+\alpha_1\mathbf{c}_1^{(1)}+\beta_0\mathbf{c}_0^{(2)}+\beta_1\mathbf{c}_1^{(2)}+\gamma_0\mathbf{c}_0^{(3)}+\gamma_1\mathbf{c}_1^{(3)}+\gamma_2\mathbf{c}_2^{(3)}=\mathbf{0}. \tag{4.35}$$

Multiplying both sides on the left by $(\mathbf{X}_0-\rho_0\mathbf{I})^2$, we see that $\gamma_2\mathbf{c}_0^{(3)}=\mathbf{0}$, whence $\gamma_2=0$. Multiplying (4.35) on the left by $(\mathbf{X}_0-\rho_0\mathbf{I})$, we get $\alpha_1\mathbf{c}_0^{(1)}+\beta_1\mathbf{c}_0^{(2)}+\gamma_1\mathbf{c}_0^{(3)}=\mathbf{0}$. so that $\alpha_1=\beta_1=\gamma_1=0$. It now follows from (4.35) that $\alpha_0=\beta_0=\gamma_0=0$. Thus the vectors $\mathbf{c}_h^{(j)}$ form a basis for the root subspace of the matrix $\mathbf{X}_0-\rho_0\mathbf{I}$. This completes the proof.

4.3. Behavior of multipliers with increase in the Hamiltonian (continuation of §1.7)

The Krein-Lyubarskii theorem enables us to perfect our knowledge of the behavior of the multipliers as established in §1.7.

Theorem I /167b/. Let (4.23) be a Hamiltonian system of positive type. Suppose that when $\varepsilon=0$, m_1 *multipliers of the first kind and* m_2 *of the second kind coincide at a point* ρ_0 *on the unit circle, where* ρ_0 *is an eigenvalue of the matrix* $\mathbf{X}(T,0)$ *with simple elementary divisors. Then there exists* $\delta>0$ *such that for any real* ε, $|\varepsilon|<\delta$, *all* $m=m_1+m_2$ *multipliers of system (4.23) in the neighborhood of* ρ_0 *lie on the unit circle and are analytic functions of* ε. *If* ε *increases monotonically in the neighborhood* $|\varepsilon|<\delta$, *the multipliers of the first kind move along the unit circle in the sense of increasing argument, those of the second in the sense of decreasing argument. Now suppose that when* $\varepsilon=0$ *the multiplier* ρ_0 *corresponds to at least one elementary Jordan matrix of order* $\geqslant 2$; *then in one of the intervals* $-\delta<\varepsilon<0$ *or* $0<\varepsilon<\delta$ *system (4.23) has multipliers in the neighborhood of* ρ_0 *which are not on the unit circle.*

Proof. The stipulated behavior of the arguments of the multipliers is verified exactly as in §1.7. One uses the fact that

$$(\mathbf{G}\mathbf{c}_j(\rho),\ \mathbf{c}_j(\rho))\frac{d}{d\varepsilon}\arg\rho_j(\varepsilon)=((\mathbf{Q}\mathbf{x}_j(t,\rho),\mathbf{x}_j(t,\rho))). \tag{4.36}$$

This equality differs only in notation from (1.20a) (§1.5). Since equation (4.23) is of positive type, it follows that when $\rho=\rho_0$ (hence, when $\varepsilon=0$) the right-hand side of (4.36) is positive. Indeed, it is nonnegative by condition (I) in the definition of §4.1. If it vanishes, then $\mathbf{Q}\mathbf{x}_j(t,\rho_0)=\mathbf{0}$. Since $\mathbf{x}_j(t,\rho_0)$ satisfies equation (4.23), we have $\mathbf{L}\mathbf{x}_j(t,\rho_0)=\mathbf{0}$. Since $\mathbf{x}_j(t,\rho_0)\neq\mathbf{0}$, we have a contradiction to condition (II) in the definition of §4.1. Thus

$$(\mathbf{G}\mathbf{c}_j(\rho_0),\ \mathbf{c}_j(\rho_0))\frac{d}{d\varepsilon}\arg\rho_j(\varepsilon)\Big|_{\varepsilon=0}>0.$$

This inequality is an analytical expression of the behavior of the multipliers as stipulated in the statement of our theorem.

We now prove the last assertion of the theorem. If $q_j\geqslant 2$, the values of (4.25) for $\varepsilon>0$ lie on q_j curves, having tangents at ρ_0, which form a "q_j-star," that is to say, the angles between successive tangents are all equal to $2\pi/q_j$. If $\varepsilon<0$ the corresponding q_j values are obtained from the previous ones by multiplication by $(-1)^{1/q_j}=\exp(\pi i/q_j)$. Thus, when $\varepsilon<0$ the appropriate "q_j-star" is obtained by rotating the previous one through an angle of π/q_j (see Figure 11, in which $q_j=2,3$ and the stars for $\varepsilon>0$ are depicted by solid curves, for $\varepsilon<0$ by dashed curves). It is clear that when $q_j=2$ the two multipliers will be off the unit circle either for $\varepsilon>0$ or for $\varepsilon<0$. When $q_j\geqslant 3$ at least two multipliers will be off the unit circle both for $\varepsilon>0$ and for $\varepsilon<0$. This completes the proof of Theorem I.

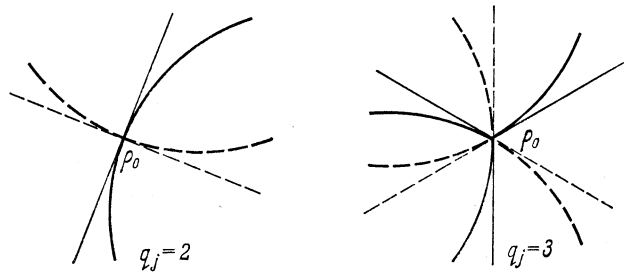


FIGURE 11.

To summarize: if the Hamiltonian increases (rigorously: equation (4.23) is of positive type), two multipliers of different kinds that meet on the unit circle continue to move along the unit circle unless a 2×2 elementary Jordan matrix is formed, in which case they move off the unit circle. If the number

of multipliers of different kinds that come together is greater than 2 and no multipliers approach the point of contact from outside (inside) the circle, only 2×2 elementary Jordan matrices may be formed, and the number of pairs that move off the circle is equal to the number of such matrices.

We again consider an equation (4.23) of positive type, assuming now that at $\varepsilon = 0$ it is stable but not strongly stable. This means that when $\varepsilon = 0$ all the multipliers of equation (4.23) lie on the unit circle, they have simple elementary divisors and there may be coincident multipliers of different kinds. If the Hamiltonian increases, or, more precisely, if the equation is of positive type, and $\varepsilon > 0$ is small, multipliers of different kinds move apart in different directions, and hence there will be no coincident multipliers of different kinds, provided only that $\varepsilon > 0$ is sufficiently small. Thus, for small $\varepsilon > 0$ equation (4.23) will be strongly stable. Strong stability will continue to hold until some of the multipliers of different kinds come together again. Denote the corresponding value of the parameter by ε_+; it may happen that $\varepsilon_+ = +\infty$. It is clear that the same is true for $\varepsilon < 0$. The first $\varepsilon < 0$ for which multipliers of different kinds meet, will be denoted by ε_- (possibly $\varepsilon_- = -\infty$). Thus equation (4.23) is strongly stable for $\varepsilon_- < \varepsilon < 0$, $0 < \varepsilon < \varepsilon_+$ but not for $\varepsilon = \varepsilon_-, 0, \varepsilon_+$.

We shall now show that in an important special case the parameter values $\varepsilon_\mp$ are eigenvalues of a selfadjoint boundary-value problem. Consider the canonical equation (4.23) with $\mathbf{H}(t) \equiv \mathbf{0}$, $i^{-1}\mathbf{G} = \mathbf{J}$, rewritten as

$$\mathbf{J}\frac{d\mathbf{x}}{dt} = \varepsilon \mathbf{Q}(t)\mathbf{x}. \tag{4.37}$$

Here $\mathbf{J}$ has the form (II. 3. 3) and $\mathbf{Q}(t+T) = \mathbf{Q}(t) = \mathbf{Q}(t)^\tau$ is a real matrix-function.

Assume that equation (4.37) is of positive type (§4. 1). In particular, this will be true (see Example (B) in §4. 1) if

(I′) $$\mathbf{Q}(t) \geqslant \mathbf{0},$$

(II′) $$\mathbf{Q}_{\text{av}} \equiv \frac{1}{T}\int_0^T \mathbf{Q}(t)\,dt > \mathbf{0}.$$

When $\varepsilon = 0$ all $2k$ multipliers are equal to unity. By Lemma III of §2. 6, k of these are of the first kind and k of the second. As established above, for small $\varepsilon > 0$ all k multipliers of the first kind lie on the upper half of the unit circle, all k of the second kind on the lower half. Since the coefficients of equation (4.37) are real, its multipliers are symmetric with respect to the real axis. With increasing ε, therefore, they may meet only at $\rho = -1$. Consequently, for $\varepsilon = \varepsilon_+$, there is a solution for which

$$\mathbf{x}(T) = -\mathbf{x}(0). \tag{4.38}$$

Similar arguments hold for small $\varepsilon < 0$, when only multipliers of the first kind lie on the lower semicircle, multipliers of the second kind on the upper semicircle. At $\varepsilon = \varepsilon_-$ we again have (4.38). In this special case, then, ε_+ is the first (least) positive eigenvalue of the boundary-value problem (4.37), (4.38), and ε_- the first (largest) negative eigenvalue of the same problem. We state this result as a theorem.

Theorem II. Let (4.37) be an equation of positive type, satisfying, say, conditions (I') and (II'). Let ε_+ denote the least positive eigenvalue and ε_- the largest negative eigenvalue of the boundary-value problem (4.37), (4.38). Then equation (4.37) is strongly stable for $0<\varepsilon<\varepsilon_+$, $\varepsilon_-<\varepsilon<0$.*

The import of this theorem is as follows. If we succeed in finding a lower bound for ε_+, say $\varepsilon_+>\varphi(\mathbf{Q})$, where $\varphi(\mathbf{Q})$ is some explicitly defined functional, we thereby have an effective test for strong stability of the canonical equation $\mathbf{J}\dot{\mathbf{x}}=\mathbf{Q}(t)\mathbf{x}$: the combination of conditions (I'), (II') and the condition $\varphi(\mathbf{Q})\geqslant 1$. Indeed, these conditions imply that for an equation (4.37) of positive type $1\in(0, \varepsilon_+)$, i. e., equation (4.37) is strongly stable for $\varepsilon=1$. This method of deriving stability tests for canonical equations was developed by Lidskii, Neigauz and (especially) Krein (§7).

4.4. Lemmas on logarithms of G-unitary and symplectic matrices

In the remainder of this section we wish to sharpen the Floquet-Lyapunov theorem (§ II. 2. 2) as applied to Hamiltonian systems (4.23). A crucial factor in the simple proof of the Floquet-Lyapunov theorem were the properties of the logarithm of the monodromy matrix. The monodromy matrix of equation (4.23) is G-unitary. We are thus interested in the specific properties of $\ln \mathbf{X}$ in case $\mathbf{X}$ is a G-unitary matrix. It is natural to expect $\ln\mathbf{X}$ to be G-antihermitian. Indeed, if $\mathbf{A}^+=-\mathbf{A}$ and $\mathbf{X}=e^{\mathbf{A}}$, then $\mathbf{X}^+=e^{\mathbf{A}^+}=e^{-\mathbf{A}}=\mathbf{X}^{-1}$, so that $\mathbf{X}$ is G-unitary. However, we need the converse, which is true only if the branch of the logarithm is suitably chosen.

Let $\Gamma_1, \ldots, \Gamma_q$ be nonintersecting circles on the complex plane, centered at points $\rho_1, \ldots, \rho_q$, and suppose that the entire spectrum of $\mathbf{X}$ lies in the circles $\Gamma_\varkappa$ (in other words, any eigenvalue of $\mathbf{X}$ is inside some $\Gamma_\varkappa$, but each circle may contain more than one eigenvalue). Recall the definition of $\ln\mathbf{X}$ (formula (I. 3.1)):

$$\ln \mathbf{X}=\frac{1}{2\pi i}\sum_{\varkappa=1}^{q}\int_{\Gamma_\varkappa}(\zeta\mathbf{I}-\mathbf{X})^{-1}(\ln\zeta)_{m_\varkappa}\,d\zeta. \tag{4.39}$$

Here $(\ln\zeta)_m$ denote branches of the logarithm defined by slitting the complex plane by some ray from $\zeta=0$ to $\zeta=\infty$ not intersecting any of the circles $\Gamma_1, \ldots, \Gamma_q$:

$$(\ln z)_m=\ln|z|+i(\arg z+2\pi m).$$

The integers $m_\varkappa$ in (4.39) are arbitrary, and their choice determines a branch of the matrix logarithm.

Lemma I. Let the centers $\rho_\varkappa$ ($\varkappa=1, \ldots, q$) of the circles $\Gamma_\varkappa$ in (4.39) be symmetric with respect to the unit circle, in such a way that to symmetric points $\rho_\varkappa$, $\bar{\rho}_\varkappa^{-1}$ correspond the same numbers $m_\varkappa$ in (4.39). If $\mathbf{X}$ is a G-unitary matrix, then the matrix $\ln\mathbf{X}$ defined by (4.39) is G-antihermitian.

* According to the above proof, we must put $\varepsilon_+=+\infty$ or $\varepsilon_-=-\infty$ if no such eigenvalue exists. Krein has shown however, that in fact both these eigenvalues always exist /166f/.

Proof. Since the spectrum of $\mathbf{X}$ is symmetric with respect to the unit circle, we can assure, by slightly deforming the circles $\Gamma_1, \ldots, \Gamma_q$ and (possibly) the ray along which the plane is slit, that the $\Gamma_\varkappa$'s will also be symmetric about the unit circle. In other words, each circle $\Gamma_\varkappa$ goes into some other $\Gamma_{\varkappa'}$ (possibly $\Gamma_{\varkappa'}=\Gamma_\varkappa$) under inversion. Assuming that the necessary deformation has been carried out, we deduce from (4.39) that

$$(\ln \mathbf{X})^+ = -\frac{1}{2\pi i}\sum_\varkappa \int_{\overline{\Gamma}_\varkappa} (\overline{\zeta}\mathbf{I}-\mathbf{X}^+)^{-1}(\overline{\ln \zeta})_{m_\varkappa} d\overline{\zeta}.$$

Here $\overline{\Gamma}_\varkappa$ denotes the complex conjugate of $\Gamma_\varkappa$ (i. e., the contour symmetric to $\Gamma_\varkappa$ with respect to the real axis), traversed in the negative sense. We now apply inversion $\zeta=\overline{\xi}^{-1}$ to the above integral; we have

$$\arg \xi = \arg \zeta,\ (\overline{\ln \zeta})_m = \ln|\zeta| - i(\arg \zeta + 2\pi m) = -(\ln \xi)_m.$$

Letting $\Gamma_{\varkappa'}$ denote the image of $\Gamma_\varkappa$ under inversion, we see, thanks to the G-unitarity of $\mathbf{X}$, that

$$(\ln \mathbf{X})^+ = -\frac{1}{2\pi i}\sum_\varkappa \int_{\Gamma_{\varkappa'}} (\xi^{-1}\mathbf{I}-\mathbf{X}^{-1})^{-1}\xi^{-2}(\ln \xi)_{m_\varkappa} d\overline{\xi}.$$

(The integration is performed along $\Gamma_{\varkappa'}$, since when ζ traverses $\Gamma_\varkappa$ positively, $\overline{\zeta}$ traverses $\overline{\Gamma}_\varkappa$ negatively and ξ traverses $\Gamma_{\varkappa'}$ negatively.) Since

$$\begin{aligned}\xi^{-2}(\xi^{-1}\mathbf{I}-\mathbf{X}^{-1})^{-1} &= \xi^{-2}[\xi^{-1}(\mathbf{X}-\xi\mathbf{I})\mathbf{X}^{-1}]^{-1} = \xi^{-1}\mathbf{X}(\mathbf{X}-\xi\mathbf{I})^{-1} - \\ &-\xi^{-1}(\mathbf{X}-\xi\mathbf{I})(\mathbf{X}-\xi\mathbf{I})^{-1}+\xi^{-1}\mathbf{I} = (\mathbf{X}-\xi\mathbf{I})^{-1}+\xi^{-1}\mathbf{I},\end{aligned}$$

it follows that

$$(\ln \mathbf{X})^+ = \frac{1}{2\pi i}\sum_\varkappa \int_{\Gamma_{\varkappa'}} [\xi^{-1}\mathbf{I}+(\mathbf{X}-\xi\mathbf{I})^{-1}](\ln \xi)_{m_\varkappa} d\xi.$$

The spectrum of $\mathbf{X}$ is symmetric about the unit circle, so there are no eigenvalues of $\mathbf{X}$ outside the contours $\Gamma_{\varkappa'}$. Since $\xi=0$ is not in any $\Gamma_{\varkappa'}$, we have

$$\int_{\Gamma_{\varkappa'}} \xi^{-1}(\ln \xi)_{m\varkappa} d\xi = 0.$$

Consequently,

$$(\ln \mathbf{X})^+ = -\frac{1}{2\pi i}\sum_\varkappa \int_{\Gamma_{\varkappa'}} (\xi\mathbf{I}-\mathbf{X})^{-1}(\ln \xi)_{m_\varkappa} d\xi.$$

The right-hand side of this expression differs from that of (4.39) only in the sign and order of its terms. Indeed, the center of $\Gamma_{\varkappa'}$ is $\overline{\rho}_\varkappa^{-1}$, and the points $\rho_\varkappa$, $\overline{\rho}_\varkappa^{-1}$ correspond by assumption to equal numbers $m_\varkappa$. Thus

$$(\ln \mathbf{X})^{+} = -\ln \mathbf{X},$$

Q. E. D.

Lemma II. Let the centers $\rho_\varkappa$ of the circles $\Gamma_\varkappa$ in (4.39) be symmetric with respect to both the unit circle and the real axis, and let $\mathbf{X}$ be a symplectic matrix.

1) *Assume that none of the numbers $\rho_\varkappa$ are real and negative. Define the branches of $\ln z$ in (4.39) by a slit along the real negative axis and choose the numbers $m_\varkappa$ so that:*

(a) *for numbers $\rho_\varkappa$ and $\rho_{\varkappa'} = \bar{\rho}_\varkappa^{-1}$ which are symmetric with respect to the unit circle, $m_{\varkappa'} = m_\varkappa$;*

(b) *if $\rho_\varkappa$ is real, then $m_\varkappa = 0$;*

(c) *for complex conjugate numbers $\rho_\varkappa$ and $\rho_{\varkappa_1} = \bar{\rho}_\varkappa$,*

$$m_{\varkappa_1} = -m_\varkappa.$$

Then the matrix $\mathbf{K} = \ln \mathbf{X}$ defined by (4.39) is real and J-antihermitian.

2) *Let $\mathbf{X}$ be an arbitrary symplectic matrix. Define a function $\varphi(z)$ in the neighborhood of the numbers $\rho_\varkappa$ as indicated in §I.2.7, with the additional condition that the numbers $m_\varkappa$ satisfy conditions (a) and (b) above. Then the matrix $\mathbf{K} = \varphi(\mathbf{X})$ is real and J-antihermitian, and $e^{2\mathbf{K}} = \mathbf{X}^2$.*

The proof duplicates the proofs of Lemmas I, II in §I.2.7 and Lemma I of this subsection. In fact, in case 1 it follows from the definition of $\varphi(z)$ that $\mathbf{K} = \varphi(\mathbf{X})$. Moreover, in case 1 we have $e^{\varphi(z)} \equiv z$ in the neighborhood of the spectrum of $\mathbf{X}$, so that $e^{\mathbf{K}} = \mathbf{X}$. It is evident from the proof of Lemma I in §I.2.7 that the matrix $\mathbf{K}$ whose existence is established there is precisely $\mathbf{K} = \varphi(\mathbf{X})$. Consequently, $\mathbf{K}$ is real. Repeating the previous arguments, we see that $\mathbf{K}$ is J-antihermitian.

4.5. Floquet-Lyapunov theorem for Hamiltonian systems

We again consider a Hamiltonian equation

$$i^{-1}\mathbf{G}\frac{d\mathbf{x}}{dt} = \mathbf{H}(t)\,\mathbf{x}, \tag{4.40}$$

and try to characterize the transformations

$$\mathbf{x} = \mathbf{F}(t)\,\mathbf{y} \tag{4.41}$$

which preserve the G-Hamiltonian property. In other words, the transformed equation in $\mathbf{y}$ must also be Hamiltonian, relative to the same matrix $\mathbf{G}$:

$$i^{-1}\mathbf{G}\frac{d\mathbf{y}}{dt} = \mathbf{H}_1(t)\,\mathbf{y}. \tag{4.42}$$

As usual, $\mathbf{F}(t)$ is a matrix-function, nonsingular for all t, having an integrable piecewise-continuous derivative, and $\mathbf{H}_1(t)$ is hermitian.

We claim that equation (4.42) will be G-Hamiltonian if the matrix $\mathbf{F}(t)$ is G-unitary for any t.

We know (§ II. 3. 4) that equation (4. 42) is G-Hamiltonian if and only if its matrizant is G-unitary for any t. Let $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ be the matrizants of equations (4.40) and (4.42), respectively. It follows from (4.41) that

$$\mathbf{X}(t)=\mathbf{F}(t)\,\mathbf{Y}(t)\,\mathbf{F}(0)^{-1} \tag{4.43}$$

(see § II. 1. 2 for details). Since the set of G-unitary matrices is a group (§1. 1), $\mathbf{Y}(t)=\mathbf{F}(t)^{-1}\mathbf{X}(t)\,\mathbf{F}(0)$ is a G-unitary matrix for any t. Hence equation (4.42) is Hamiltonian (i. e., $\mathbf{H}_1(t)=\mathbf{H}_1(t)^*$), as claimed.

Consider equation (4.40) with a T-periodic Hamiltonian $\mathbf{H}(t)$. We shall show that the matrix $\mathbf{F}(t)$ may be chosen in such a way that equation (4.42) is Hamiltonian with a constant Hamiltonian $\mathbf{H}_1(t)=\mathbf{C}$. Repeating the arguments of § II. 2.2, we define a matrix

$$\mathbf{K}=\frac{1}{T}\ln\mathbf{X}(T),$$

requiring in addition (this is the crucial point of the argument!) that $\mathbf{K}$ be a G-antihermitian matrix. By Lemma I, the necessary value of $\ln\mathbf{X}(T)$ exists and is defined by (4.39) in which $\Gamma_\varkappa$ are circles with centers at the distinct eigenvalues $\rho_\varkappa$ of $\mathbf{X}(T)$ and the numbers $m_\varkappa$ are chosen as stipulated in the lemma.

Defining $\mathbf{F}(t)$ by

$$\mathbf{F}(t)=\mathbf{X}(t)\,e^{-t\mathbf{K}}, \tag{4.44}$$

we see that $\mathbf{Y}(t)=e^{t\mathbf{K}}$. Since $\mathbf{Y}(t)$ is the matrizant of the equation $\dot{\mathbf{y}}=\mathbf{K}\mathbf{y}$, it follows that in (4.42) $i\mathbf{G}^{-1}\mathbf{H}_1(t)=\mathbf{K}$. Since $\mathbf{K}$ is G-antihermitian, equation (4.42) is Hamiltonian, with constant Hamiltonian $\mathbf{H}_1(t)=i\mathbf{G}\mathbf{K}$ (which is a hermitian matrix).

The matrix $\mathbf{Y}(t)^{-1}=e^{-t\mathbf{K}}$ is G-unitary for any real t. It follows from (4.44) that $\mathbf{F}(t)$ is also G-unitary. Since $\mathbf{X}(t+T)=\mathbf{X}(t)\,\mathbf{X}(T)$ (II. 2. 2), we infer from (4.44) that

$$\mathbf{F}(t+T)=\mathbf{X}(t+T)\,e^{-(t+T)\mathbf{K}}=\mathbf{X}(t)\,\mathbf{X}(T)\,e^{-T\mathbf{K}}\,e^{-t\mathbf{K}}=\mathbf{F}(t).$$

We have thus proved the first part of the following

Theorem I (Floquet-Lyapunov theorem for Hamiltonian systems).

(I) *The matrizant $\mathbf{X}(t)$ of a Hamiltonian equation (4.40) with T-periodic coefficients may be expressed as*

$$\mathbf{X}(t)=\mathbf{F}(t)\,e^{t\mathbf{K}}, \tag{4.45}$$

where $\mathbf{F}(t)$ is a T-periodic G-unitary matrix for any t and has an integrable piecewise-continuous derivative, and $\mathbf{K}$ is a constant G-antihermitian matrix.

(II) *Conversely, for any $\mathbf{F}(t)$ and $\mathbf{K}$ with the properties enumerated in part (I), the matrix-function $\mathbf{X}(t)$ defined by (4.45) is the matrizant of some Hamiltonian equation (4.40) with T-periodic Hamiltonian.*

The proof of the second part duplicates the arguments of § II. 2.2.

We now consider a canonical equation with T-periodic coefficients,

$$\mathbf{J}\frac{d\mathbf{x}}{dt}=\mathbf{H}(t)\,\mathbf{x}, \tag{4.46}$$

where $\mathbf{J}$ has the form (II. 3. 3). *

Theorem II. 1) *The matrizant* $\mathbf{X}(T)$ *of a canonical equation (4.46) with T-periodic Hamiltonian* $\mathbf{H}(t)$ *may be expressed as (4.45), with* $\mathbf{F}(t)$ *a 2T-periodic real symplectic matrix-function having an integrable piecewise-continuous derivative, and* $\mathbf{K}$ *a constant real J-antihermitian matrix.*

2) *If* $\mathbf{X}(T)$ *has no real negative eigenvalues, there exists a representation (4.45) in which the matrix-function* $\mathbf{F}(t)$ *is also T-periodic.*

3) *For any* $\mathbf{F}(t)$ *and* $\mathbf{K}$ *satisfying the conditions enumerated in part 1, the matrix-function* $\mathbf{X}(t)$ *defined by (4.45) is the matrizant of some canonical equation (4.46) with T-periodic Hamiltonian.*

Proof. The matrizant $\mathbf{X}(t)$ of equation (4.46) is symplectic for $0\leqslant t\leqslant T$. If $\mathbf{X}(T)$ has no real negative eigenvalues, it follows from Lemma II of §4. 4 that the value of the logarithm may be so chosen that $\mathbf{K}=\ln\mathbf{X}(T)$ is real and J-antihermitian. Then the matrix-function $\mathbf{F}(t)$ defined by (4.44) is T-periodic, and will also have the other properties stipulated in part 1. In the general case, it follows from part 2 of Lemma II, §4.3, that there is a real J-antihermitian matrix $\mathbf{K}$ for which $e^{2T\mathbf{K}}=\mathbf{X}(T)^2$. Then

$$\mathbf{F}(t+2T)=\mathbf{X}(t+2T)\,e^{-(t+2T)\mathbf{K}}=\mathbf{X}(t)\,\mathbf{X}(T)^2\,e^{-2T\mathbf{K}}e^{-t\mathbf{K}}=\mathbf{F}(t).$$

This proves part 1. If $\mathbf{X}(t)$ is defined by (4.45), where $\mathbf{F}(t)$ and $\mathbf{K}$ have the stipulated properties, then it has an integrable piecewise-continuous derivative and is symplectic for any t. Therefore (§§ II. 2.2, II.3.1), equation (4.46) is canonical. Since $\mathbf{X}(t+T)=\mathbf{X}(t)\,\mathbf{X}(T)$, the Hamiltonian is T-periodic (§ II. 2.1). This proves part 3. Q. E. D.

4.6. Floquet-Lyapunov theorem for Hamiltonian systems with a parameter

We consider systems (4.40) and (4.46) with a Hamiltonian $\mathbf{H}(t,\varepsilon)$ which is an analytic function of a small parameter ε:

$$\mathbf{H}(t,\varepsilon)=\mathbf{H}_0(t)+\varepsilon\mathbf{H}_1(t)+\ldots \tag{4.47}$$

The series (4.47) is convergent, in the sense that for sufficiently small ε the scalar series

$$\int_0^T|\mathbf{H}_0(t)|\,dt+\varepsilon\int_0^T|\mathbf{H}_1(t)|\,dt+\ldots$$

* Recall that by the definition (§ II.3.1) of a canonical equation the matrix $\mathbf{H}(t)$ in (4.46) is real and symplectic: $\mathbf{H}(t)^\tau=\mathbf{H}(t)$. In addition, $\mathbf{H}(t+T)=\mathbf{H}(t)$.

is convergent. It is also assumed that $\mathbf{H}(t, \varepsilon)$ is hermitian for real ε (equation (4.40)) or real and symmetric (equation (4.46)).

Theorem. Under these assumptions, part (I) of Theorem I and parts 1, 2 of Theorem II in §4.5 are valid; in addition, in (4.45) $\mathbf{F}(t) = \mathbf{F}(t, \varepsilon)$ *for each* t *and* $\mathbf{K} = \mathbf{K}(\varepsilon)$ *is analytic in* ε *at* $\varepsilon = 0$.

Proof. As shown in §II.1.3, the matrizant $\mathbf{X}(t, \varepsilon)$ is an analytic function of ε at $\varepsilon = 0$ for each t. Let the centers $\rho_\varkappa$ of the circles $\Gamma_\varkappa$ in (4.40) be the eigenvalues of the matrix $\mathbf{X}(T, 0)$. For small ε, the corresponding eigenvalues of $\mathbf{X}(T, \varepsilon)$ will lie inside the circles $\Gamma_\varkappa$ and formula (4.40) will remain valid for $\mathbf{X} = \mathbf{X}(T, \varepsilon)$. Since $(\zeta\mathbf{I} - \mathbf{X})^{-1}$ has no singularities for $\zeta \in \Gamma_\varkappa$, $|\varepsilon| \leqslant \varepsilon_0$, provided ε_0 is small enough, it follows that $\ln \mathbf{X}$ is an analytic function of ε at $\varepsilon = 0$. This is true for any choice of the numbers $m_\varkappa$. Thus all the matrices $\mathbf{K}$ defined in §4.5 are analytic functions of ε at $\varepsilon = 0$. Consequently, the matrix $\mathbf{F}(t, \varepsilon)$ defined by (4.44) is also an analytic function of ε at $\varepsilon = 0$. This completes the proof.

Suppose now that $\mathbf{H}_0(t) = \mathbf{C}$ in (4.47) is a constant matrix. This means that at $\varepsilon = 0$ equation (4.47) has constant coefficients. The matrizant is $\mathbf{X}(t, 0) = e^{it\mathbf{G}^{-1}\mathbf{C}}$. On the other hand, by (4.45), $\mathbf{X}(t, 0) = \mathbf{F}(t, 0)e^{t\mathbf{K}(0)}$, and $\mathbf{X}(T, 0) = e^{iT\mathbf{G}^{-1}\mathbf{C}} = e^{T\mathbf{K}(0)}$. It is natural to expect that in defining the matrix $\mathbf{K}(\varepsilon)$ we may choose the branch of the logarithm so that $\mathbf{K}(0) = i\mathbf{G}^{-1}\mathbf{C}$ (and then $\mathbf{F}(t, 0) = \mathbf{I}$). It turns out, however, that in many cases this is impossible. We shall consider this problem in detail in Chapter IV, showing that a suitable definition of $\mathbf{K}(\varepsilon)$ is possible in the so-called nonresonant case.

§5. STABILITY DOMAINS. STRUCTURE OF THE SPACE OF HAMILTONIANS

5.1. Statement of the problem

Let us return to a canonical equation with real, T-periodic, integrable, piecewise-continuous coefficients (§II.3.1),

$$\boxed{\mathbf{J}\frac{d\mathbf{x}}{dt} = \mathbf{H}(t)\mathbf{x}, \qquad (\mathbf{H}(t+T) = \mathbf{H}(t) = \mathbf{H}(t)^\tau).} \tag{5.1}$$

Let $\mathfrak{L} = \{\mathbf{H}(t)\}$ be the set of all $2k \times 2k$ real Hamiltonians of equations of this type, where k is fixed. $\mathfrak{L}$ is obviously a linear space, for if $\mathbf{H}_1 \in \mathfrak{L}$ and $\mathbf{H}_2 \in \mathfrak{L}$ then for any real α_1, α_2

$$\alpha_1\mathbf{H}_1 + \alpha_2\mathbf{H}_2 \in \mathfrak{L}.$$

We define a metric in $\mathfrak{L}$ by

$$\rho(\mathbf{H}_1, \mathbf{H}_2) = \int_0^T |\mathbf{H}_1(t) - \mathbf{H}_2(t)|\,dt. \tag{5.2}$$

We recall the definitions of §1.4. A Hamiltonian $\mathbf{H}(t)$ is **stable** if all solutions of system (5.1) are bounded, **strongly stable** if the same

holds for all Hamiltonians in some neighborhood of $\mathbf{H}(t)$ (in the sense of the above metric), and weakly stable if it is stable but has no such neighborhood.

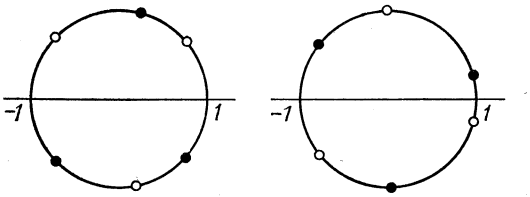
Thus the strongly stable Hamiltonians are interior points of the set of all stable Hamiltonians, while the weakly stable ones are boundary points. Let $\mathcal{G}$ denote the open set of all strongly stable Hamiltonians.

A problem of paramount importance, to be investigated in the next section, is to determine effective sufficient conditions for stability. In this connection, we would like to have information concerning the structure of the set $\mathcal{G}$ of all strongly stable Hamiltonians. The first and most natural question arising here is whether the set $\mathcal{G}$ is a domain (i. e., connected) or the union of several domains. In other words: given any two strongly stable Hamiltonians, we can deform one of them continuously into the other, preserving strong stability throughout the process? We shall show below that this is not always possible, for $\mathcal{G}$ is a union of several disjoint domains which we shall call stability domains. A second question suggests itself immediately: how can we characterize the Hamiltonians belonging to one stability domain? The complete solution to these problems, due to Gel'fand and Lidskii /132/, utilizes mathematical tools beyond the scope of this book, and we cannot present a full account here. Nonetheless, the concept of a stability domain proves highly useful in the derivation of sufficient conditions for stability. (For example, it would be natural to aim at a stability test that "picks out" Hamiltonians from each stability domain.) For this reason, in §§5.2 and 5.3 we shall state the Gel'fand-Lidskii result without proving the more difficult propositions. At the end of the section (§§5.4, 5.5) we shall outline the solution to a more general problem — the structure in the space $\mathfrak{L}$ of the set of Hamiltonians for which the solutions of the appropriate equations possess certain prescribed properties (for more details, see /224m/).

5.2. Multiplier types of strongly stable Hamiltonians

According to the Krein-Gel'fand-Lidskii theorem (§3.2), a Hamiltonian $\mathbf{H}(t)$ is strongly stable if and only if all the multipliers lie on the unit circle and there are no multipliers of mixed kind. If the Hamiltonian is deformed continuously (in the sense of the metric (5.2)) the monodromy matrix also varies continuously (§II.1.7). Consequently, the same is true of the multipliers. Even more: by the theorem of §2.8, each multiplier of the first or second kind, taken individually, varies continuously. Hence it follows at once that the set $\mathcal{G}$ cannot be connected. For example, consider two strongly stable sixth-order Hamiltonians $\mathbf{H}_1(t)$ and $\mathbf{H}_2(t)$, of which the first has multipliers in the relative positions illustrated in Figure 12a, the second as illustrated in Figure 12b. Black dots denote multipliers of the first kind, white dots — of the second kind. These Hamiltonians must belong to different stability domains. Indeed, if $\mathbf{H}_1(t)$ and $\mathbf{H}_2(t)$ are in the same stability domain, there is a continuous strongly stable "curve"

$\mathbf{H}(t, s)$ $(0 \leqslant s \leqslant 1)$ "joining" $\mathbf{H}_1(t)$ and $\mathbf{H}_2(t)$: $\mathbf{H}(t, 0) = \mathbf{H}_1(t)$, $\mathbf{H}(t, 1) = \mathbf{H}_2(t)$. As s varies continuously from 0 to 1, the corresponding multipliers (both those of the first kind and those of the second) move continuously along the unit circle without meeting multipliers of another kind. According to our assumption, the set of multipliers in Figure 12a will ultimately coincide with the set illustrated in Figure 12b. It is clear that this is impossible in a continuous process unless multipliers of a different kind are encountered "along the way." Thus $\mathbf{H}_1(t)$ and $\mathbf{H}_2(t)$ belong to different stability domains.



a) b)

FIGURE 12.

Generalizing this argument, we have a necessary condition for two Hamiltonians to belong to the same stability domain: the multipliers of the first and second kinds for the corresponding equations (5.1) must lie in the same relative positions on the unit circle. In other words, the black and white dots must alternate in the same sequence along the upper semicircle as the latter is traversed, say, in the counterclockwise sense.

Consider the spectrum of some strongly stable equation (5.1). Moving along the upper unit semicircle from +1 to −1, let us represent each multiplier of the first kind by a plus and each multiplier of the second kind by a minus. Any r-fold multiplier is represented by r pluses or r minuses, as the case may be. Since a strongly stable system has no multipliers of mixed kind (in particular, by Lemma III of §2.6 it has no multipliers at ±1), this representation by sequences of pluses and minuses is well-defined. In other words, for each strongly stable Hamiltonian $\mathbf{H}(t)$ we set up a sequence of k pluses or minuses

$$\mu = (+, +, -, +, -, -, \ldots, -, +),$$

which we shall call the multiplier type* of the Hamiltonian (or of the Hamiltonian equation (5.1)). Figure 13 shows all possible spectra and their types for $k = 1, 2$. To summarize: if two Hamiltonians $\mathbf{H}_1(t)$ and $\mathbf{H}_2(t)$ are in the same stability domain, their multiplier types must coincide.

Let us determine the number of multiplier types for any natural number k. On the upper semicircle, $\varkappa$ multipliers of the first kind may be distributed among $k-\varkappa$ multipliers of the second kind in $C_k^{\varkappa}$ ways ($\varkappa = 0, 1, \ldots, \ldots, k$). Since

* [Or "signature" according to some Western authors.]

$$\sum_{\varkappa=0}^{k} C_k^{\varkappa} = 2^k,$$

there are 2^k different multiplier types μ. We denote them by $\mu_1, \ldots, \mu_{2k}$.

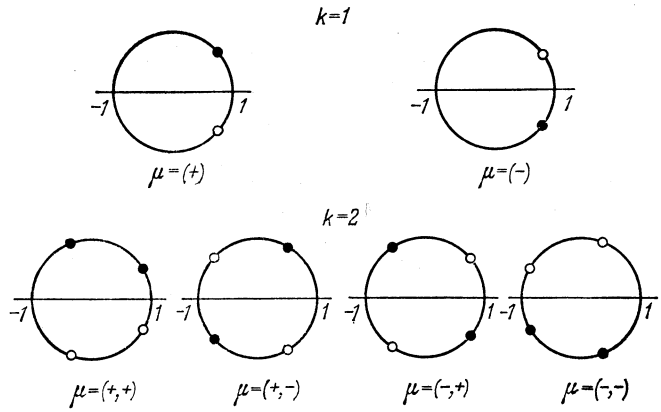


FIGURE 13.

5.3. Index of a strongly stable Hamiltonian. Gel'fand-Lidskii theorem

There is another invariant of continuous deformations of the Hamiltonian which preserve strong stability. Let $\mathbf{X}(t)$ be the matrizant of equation (5.1); by virtue of §II.3.4, the matrizant is a symplectic matrix for any t. Let $\rho_1(t), \ldots, \rho_k(t)$ be the eigenvalues of the first kind of $\mathbf{X}(t)$, defined by continuity (§1.2). (Then the numbers $\rho_1(t)^{-1}, \ldots, \rho_k(t)^{-1}$ are its eigenvalues of the second kind; see Lemma IV of §2.6.)

Since $\mathbf{X}(0) = \mathbf{I}_{2k}$, we have

$$\rho_1(0) = \rho_2(0) = \ldots = \rho_k(0) = 1.$$

The numbers $\rho_j(T)$ are the multipliers of the first kind of equation (5.1). Let $\mathbf{H}(t)$ be a strongly stable Hamiltonian. Then $|\rho_j(T)| = 1$ and $\rho_j(T) \neq \pm 1$ $(j = 1, \ldots, k)$. Note that for certain t in the interval $(0, T)$ the point $\rho_j(t)$ may not be on the unit circle.

We extend the functions $\rho_j(t)$ $(j = 1, \ldots, k)$ to the interval $T \leqslant t \leqslant 2T$, stipulating that when $T \leqslant t \leqslant 2T$ each of the points $\rho_j(t)$ $(j = 1, \ldots, k)$ moves along the unit circle to the point $\rho = 1$ without passing through $\rho = -1$. In other words, each point moves to $\rho = 1$ on the same (upper or lower) semicircle on which the point $\rho_j(T)$ lies. Since $\rho_j(t) \neq 0$ for $0 \leqslant t \leqslant 2T$, we can meaningfully speak of the increment $\Delta \operatorname{Arg} \rho_j(t)\big|_0^{2T}$, which must be a multiple of 2π because $\rho_j(0) = \rho_j(2T) = 1$. Consequently,

$$\sum_{j=1}^{k} \Delta \operatorname{Arg} \rho_j(t)\Big|_{t=0}^{2T} = 2n\pi, \tag{5.3}$$

where n is an integer.

Let

$$\rho_j(T)=e^{i\vartheta_j}, \quad \text{where} \quad -\pi<\vartheta_j<\pi \qquad (j=1, \ldots, k).$$

We may write (5.3) as

$$\sum_{j=1}^{k} \Delta \operatorname{Arg} \rho_j(t)\Big|_{t=0}^{T} - \sum_{j=1}^{k} \vartheta_j = 2n\pi. \tag{5.4}$$

The integer n in (5.3) (or (5.4)) will be called the index of the strongly stable Hamiltonian $\mathbf{H}(t)$.

Note that the left-hand side of (5.4), and therefore the index n, is independent of how we define the extension of $\rho_j(t)$ (provided the above condition is maintained) to $T \leqslant t \leqslant 2T$. *

It is almost self-evident that the index remains invariant under any continuous deformation of the Hamiltonian that preserves strong stability. To prove this, we first recall that the matrizant $\mathbf{X}(t)$ $(0 \leqslant t \leqslant T)$ is a continuous function, in the sense of the metric

$$\rho(\mathbf{X}_1, \mathbf{X}_2) = \sup_{0 \leqslant t \leqslant T} |\mathbf{X}_1 - \mathbf{X}_2| + \int_0^T |\dot{\mathbf{X}}_1 - \dot{\mathbf{X}}_2|\, dt$$

of the Hamiltonian $\mathbf{H}(t) \in \mathfrak{L}$ (§ II. 1.7). Let $\mathbf{H}(t, s)$ be the deformation in question, $\mathbf{X}(t, s)$ the corresponding matrizant and $\rho_j(t, s)$ the eigenvalues of the first kind. It follows from the continuity of the matrizant with respect to the Hamiltonian that $\mathbf{X}(t, s)$ is a continuous function of s for any $t \in [0, T]$. Hence, by the theorem of §2.8, each $\rho_j(t, s)$ is a continuous function of s for any $t \in [0, T]$. Consequently, the expression on the left of (5.3) is also continuous in s, and since n is an integer it must be independent of s.

We emphasize that only multipliers of the first kind figure in (5.3). For multipliers of the second kind, the increment of the argument will have the opposite sign (but the same absolute value).

Let $\mathfrak{G}_n^{(\mu)}$ denote the set of all strongly stable Hamiltonians of multiplier type μ and index n. Since neither μ nor n are changed by a continuous deformation of the Hamiltonian in $\mathfrak{G}$, Hamiltonians $\mathbf{H}_1(t)$ and $\mathbf{H}_2(t)$ from different sets $\mathfrak{G}_n^{(\mu)}$ cannot be joined in $\mathfrak{L}$ by a continuous curve preserving strong stability. It can be shown (though the proof is quite difficult /132/)

* Alternatively, we might define the index by (5.3), but letting the points $\rho_j(t)$, $T \leqslant t \leqslant 2T$, move to $\rho = 1$ in some other way, say in the clockwise sense (denote the corresponding index concept by $\tilde{n}$). Instead of (5.4), we then have

$$\sum_{j=1}^{n} \Delta \operatorname{Arg} \rho_j(t)\Big|_{t=0}^{T} - \sum_{j=1}^{n} \theta_j = 2\tilde{n}\pi,$$

where θ_j are the principal values of the arguments of the multipliers of the first kind $(0 < \theta_j < 2\pi)$. The definition given above (implying (5.4)) is in certain respects more convenient. Moreover, it leads to the same value for the index as in the original paper of Gel'fand and Lidskii /132/, who first introduced the index concept but defined it differently.

that each of the sets $\mathcal{O}_n^{(\mu)}$ is connected (so that Hamiltonians in one set $\mathcal{O}_n^{(\mu)}$ can be joined by a continuous curve as above).

Thus the sets $\mathcal{O}_n^{(\mu)}$ are domains (we shall call them stability domains), and their union is the whole set $\mathcal{O}$. We state this proposition in full.

Theorem (Gel'fand and Lidskii). The set $\mathcal{O}$ of all strongly Hamiltonians of canonical equations (5.1) of fixed order is the union in $\mathfrak{L}$ of countably many domains*

$$\mathcal{O}_n^{(\mu)} \qquad (n=0, \pm 1, \pm 2, \ldots; \mu=\mu_1, \ldots, \mu_{2^k}),$$

each of which is the set of all Hamiltonians with index n and multiplier type μ.

Henceforth, $\mathcal{O}_n^{(+)}$ will denote the stability domains corresponding to the multiplier type

$$\mu=(+, +, \ldots, +)$$

(all multipliers of the first kind lie on the upper unit semicircle) and $\mathcal{O}_n^{(-)}$ the domains with multiplier type

$$\mu=(-, -, \ldots, -)$$

(all multipliers of the first kind lie on the lower unit semicircle).

It will follow from some later results (end of §§6.5 and 6.6) that all sufficiently small positive definite Hamiltonians $\mathbf{H}(t)$ $(0 \leqslant t \leqslant T)$ belong to $\mathcal{O}_0^{(+)}$, and all sufficiently small negative definite $\mathbf{H}(t)$ $(0 \leqslant t \leqslant T)$ to $\mathcal{O}_0^{(-)}$.

5.4. Arguments of symplectic matrices. Other definitions of the index of a strongly stable Hamiltonian

In §6 we shall evolve various effective tests for strong stability. In the event that any of these tests is satisfied, it is quite easy to determine the index of the stability domain to which the Hamiltonian belongs. In this subsection we shall show (Theorem V) that the index may be defined in other ways. Together with the results of §6 and Chapter IV, this will yield various properties of the solutions of strongly stable Hamiltonian equations, provided the index of the stability domain is known.

Let $\mathfrak{G}$ denote the group (§1.1) of $2k \times 2k$ symplectic matrices, which we shall call the symplectic group. Let $\rho_1, \ldots, \rho_k$ be the eigenvalues of the first kind of a matrix $\mathbf{X} \in \mathfrak{G}$ and $\rho_1^{-1}, \ldots, \rho_k^{-1}$ its eigenvalues of the second kind (Lemma IV, §2.6). The definition of the index involved the quantity

$$\operatorname{Arg}_* \mathbf{X} = \sum_{j=1}^{k} \operatorname{Arg} \rho_j. \tag{5.5}$$

* The version of (5.3) given in /132/ is slightly different. The equivalence of our (5.3) and formula (6.9) in /132/ follows from /224 m/.

Here $\operatorname{Arg}\rho$ denotes the argument of the complex number ρ, assuming that $\operatorname{Arg}\rho$ is a single-valued continuous branch of the argument. We shall call $\operatorname{Arg}_*\mathbf{X}$ the argument of the symplectic matrix $\mathbf{X}$. By the theorem of §2.8, the numbers ρ_j are continuous functions of $\mathbf{X}$. Consequently, $\operatorname{Arg}_*\mathbf{X}$ is a multiple-valued continuous function of $\mathbf{X}$. If $(\operatorname{Arg}_*\mathbf{X})_0$ is one of the values of $\operatorname{Arg}_*\mathbf{X}$, the other values are clearly

$$(\operatorname{Arg}_*\mathbf{X})_m = (\operatorname{Arg}_*\mathbf{X})_0 + 2\pi m \qquad (m = \pm 1,\ \pm 2,\ \ldots). \tag{5.6}$$

Let $\mathbf{X}(t)$ $(0 \leqslant t \leqslant T)$ be some curve in the group $\mathfrak{G}$, i. e., a continuous function of t, $0 \leqslant t \leqslant T$, such that $\mathbf{X}(t) \in \mathfrak{G}$ for these t-values. Let $\rho_j(t)$ be the eigenvalues of $\mathbf{X}(t)$ of the first kind, defined by continuity. The increment $\Delta \operatorname{Arg}\rho_j(t) = \operatorname{Arg}\rho_j(T) - \operatorname{Arg}\rho_j(0)$ is clearly single-valued, and so the increment of the matrix argument

$$\Delta \operatorname{Arg}_*\mathbf{X}(t) = \sum_{j=1}^{k} [\operatorname{Arg}\rho_j(T) - \operatorname{Arg}\rho_j(0)] \tag{5.7}$$

is also single-valued. If $\mathbf{X}(t)$ is a closed curve ($\mathbf{X}(0) = \mathbf{X}(T)$) it follows from (5.6) that $\Delta \operatorname{Arg}_*\mathbf{X}(t) = 2p\pi$, where p is an integer, called the index of the closed curve.

If the curve $\mathbf{X}(t)$ is subjected to a continuous deformation, the right-hand side of (5.7), i. e., $\Delta \operatorname{Arg}_*\mathbf{X}(t)$, varies continuously. Thus the index of a closed curve $\mathbf{X}(t) \in \mathfrak{G}$ remains invariant under any continuous deformation. The index of a "constant curve" $\mathbf{X}(t) \equiv \mathbf{X}_0$ is clearly zero. Consequently, a closed curve may be contracted to a point only if its index is zero. The next theorem, whose proof is omitted, implies, in particular, that the converse is also true:

Theorem I. a) The group $\mathfrak{G}$ is arcwise-connected, i. e., for any matrices $\mathbf{X}_1 \in \mathfrak{G}$ and $\mathbf{X}_2 \in \mathfrak{G}$ there exists a curve $\mathbf{X}(t) \in \mathfrak{G}$ $(0 \leqslant t \leqslant T)$ joining the matrices in $\mathfrak{G}$: $\mathbf{X}(0) = \mathbf{X}_1$, $\mathbf{X}(T) = \mathbf{X}_2$. b) A closed curve $\mathbf{X}(t) \in \mathfrak{G}$ may be contracted to a point if and only if its index is zero. Two curves $\mathbf{X}_1(t)$ and $\mathbf{X}_2(t)$ with common endpoints may be deformed continuously into one another, without displacing their endpoints, if and only if $\Delta \operatorname{Arg}_*\mathbf{X}_1(t) = \Delta \operatorname{Arg}_*\mathbf{X}_2(t)$.

It is assumed, of course, that the deformations in question preserve membership in the group $\mathfrak{G}$.

It can be shown that many functions other than (5.5) possess all the properties listed above. Any such function will be called the argument of a symplectic matrix. More precisely:

Definition I. Let $\operatorname{Arg}\mathbf{X}$ be any real countable-valued function, defined on matrices $\mathbf{X} \in \mathfrak{G}$, such that

1) $\operatorname{Arg}\mathbf{X}$ is defined for any $\mathbf{X} \in \mathfrak{G}$;

2) if $(\operatorname{Arg}\mathbf{X})_0$ is any value of $\operatorname{Arg}\mathbf{X}$, the other values are

$$(\operatorname{Arg}\mathbf{X})_m = (\operatorname{Arg}\mathbf{X})_0 + 2m\pi \qquad (m = \pm 1,\ \pm 2,\ \ldots);$$

3) each of the values $(\operatorname{Arg}\mathbf{X})_m$ is a continuous function of $\mathbf{X} \in \mathfrak{G}$;

4) there exists a closed curve $\mathbf{U}(t) \in \mathfrak{G}$ of index 1 such that $\Delta \operatorname{Arg}\mathbf{U}(t) = 2\pi$.

Any such function will be called an a r g u m e n t on the symplectic group.

Property 3 is understood here in the following sense: for any matrix $\mathbf{X}_0 \in \mathfrak{G}$ and any fixed value of $\operatorname{Arg}\mathbf{X}_0$, say $(\operatorname{Arg}\mathbf{X}_0)_m$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\mathbf{X}-\mathbf{X}_0| < \delta$ there is a value of $\operatorname{Arg}\mathbf{X}$, say $(\operatorname{Arg}\mathbf{X})_m$, for which $|(\operatorname{Arg}\mathbf{X})_m - (\operatorname{Arg}\mathbf{X}_0)_m| < \varepsilon$.

It follows easily from the definition that for any continuous matrix-function $\mathbf{X}(t) \in \mathfrak{G}$ the values

$$(\operatorname{Arg}\mathbf{X}(t))_m = (\operatorname{Arg}\mathbf{X}(t))_0 + 2m\pi \qquad (m = 0, \pm 1, \pm 2, \ldots)$$

generate a countable set of continuous functions. Thus the increment

$$\Delta \operatorname{Arg}\mathbf{X}(t)\Big|_0^{t_0} = [\operatorname{Arg}\mathbf{X}(t_0)]_m - [\operatorname{Arg}\mathbf{X}(0)]_m$$

is well-defined, justifying the above formulation of property 4.

In the sequel, the word "curve" will always stand for "continuous curve," i. e., a continuous mapping of the interval $[0, T]$ into the group $\mathfrak{G}$.

The definition may be rephrased as follows. Suppose we have a continuous mapping of $\mathfrak{G}$ onto a circle, with the following property: there exists a closed curve $\mathbf{U}(t) \in \mathfrak{G}$ $(0 \leqslant t \leqslant T)$ of index 1 such that as t varies from $t=0$ to $t=T$ the circle is described once in the positive sense. By definition, $\operatorname{Arg}\mathbf{X}$ is the argument (in the conventional sense) of the point on the circle which is the image of the matrix $\mathbf{X}$.

The following theorem shows that the various argument concepts of Definition I are equivalent in a certain "topological" sense.

T h e o r e m II. If $\mathbf{X}(t)$ is a closed curve of index p, then $\Delta \operatorname{Arg}\mathbf{X}(t) = 2p\pi$. Two curves $\mathbf{X}_1(t)$ and $\mathbf{X}_2(t)$ with common endpoints $(\mathbf{X}_1(0) = \mathbf{X}_2(0))$ may be deformed continuously into one another in $\mathfrak{G}$, without displacing their endpoints, if and only if

$$\Delta \operatorname{Arg}\mathbf{X}_1(t) = \Delta \operatorname{Arg}\mathbf{X}_2(t). \tag{5.8}$$

In particular, a closed curve $\mathbf{X}(t)$ may be contracted to a point only if $\Delta \operatorname{Arg}\mathbf{X}(t) = 0$.

Before proceeding to the proof, we adopt some notational conventions. Two curves $\mathbf{X}'(t)$ and $\mathbf{X}''(t)$ are said to be h o m o t o p i c $(\mathbf{X}'(t) \sim \mathbf{X}''(t))$ if they have common endpoints and each may be deformed continuously into the other in $\mathfrak{G}$. We shall write $\mathbf{X}(t) \sim \mathbf{I}$ if $\mathbf{X}(t)$ is a closed curve that can be contracted to a point. The curve $\mathbf{X}(t)$ described in the opposite direction will be denoted by $\mathbf{X}(t)^{[-1]}$, so that $\mathbf{X}(t)^{[-1]} = \mathbf{X}(T-t)$ $(0 \leqslant t \leqslant T)$. The p r o d u c t of two curves $\mathbf{X}_1(t)$ and $\mathbf{X}_2(t)$, such that the endpoint of the first is the initial point of the second $(\mathbf{X}_1(T) = \mathbf{X}_2(0))$, is defined as the curve formed by tracing $\mathbf{X}_1(t)$ and $\mathbf{X}_2(t)$ in succession:

$$\mathbf{X}(t) = \mathbf{X}_1(t) \times \mathbf{X}_2(t) = \begin{cases} \mathbf{X}_1(2t) & \text{for } 0 \leqslant t \leqslant \frac{T}{2}, \\ \mathbf{X}_2(2t-T) & \text{for } \frac{T}{2} \leqslant t \leqslant T. \end{cases}$$

Let $\mathbf{X}(t)$ be a closed curve. We denote

$$\mathbf{X}(t)^{[m]} = \mathbf{X}(t) \times \ldots \times \mathbf{X}(t) \ (m \text{ factors}), \quad \mathbf{X}(t)^{[-m]} = (\mathbf{X}(t)^{[m]})^{[-1]}.$$

We shall use various elementary properties of these operations without further ado; for example, $\mathbf{X}(t)^{[m]} \times \mathbf{X}(t)^{[l]} = \mathbf{X}(t)^{[m+l]}$, where m and l are integers.

Note that

$$\mathbf{X}(t) \sim \mathbf{X}(t) \times \mathbf{X}(T).$$

To prove this, we define a deformation by varying the parametrization of the curve: the variable point must traverse the entire curve $\mathbf{X}(t)$ in a time $0 \leqslant t \leqslant T - \Delta$, and "halt" for a time Δ at the point $\mathbf{X}(T)$. If $\Delta = \frac{1}{2}T$, the result is the curve $\mathbf{X}(t) \times \mathbf{X}(T)$. Thus, as Δ varies from $\Delta = 0$ to $\Delta = \frac{1}{2}T$ we obtain the required deformation. Similarly, $\mathbf{X}(t) \sim \mathbf{X}(0) \times \mathbf{X}(t)$.

In addition,

$$\Delta \operatorname{Arg} [\mathbf{X}_1(t) \times \mathbf{X}_2(t)] = \Delta \operatorname{Arg} \mathbf{X}_1(t) + \Delta \operatorname{Arg} \mathbf{X}_2(t).$$

Proof of Theorem II. Let $\mathbf{X}(t)$ be a closed curve of index p, i. e., $\Delta \operatorname{Arg}_* \mathbf{X}(t) = 2p\pi$. Let $\mathbf{U}(t)$ be the curve figuring in the definition. Join the points $\mathbf{X}(0)$ and $\mathbf{U}(0)$ of $\mathfrak{G}$ by a curve $\mathbf{Z}(t)$, $\mathbf{Z}(0) = \mathbf{X}(0)$, $\mathbf{Z}(T) = \mathbf{U}(0)$ (that this is possible follows from the fact that $\mathfrak{G}$ is arcwise-connected; see Theorem I, a). Set (Figure 14)

$$\mathbf{V}(t) = \mathbf{X}(t) \times \mathbf{Z}(t) \times \mathbf{U}(t)^{[-p]} \times \mathbf{Z}(t)^{[-1]}.$$

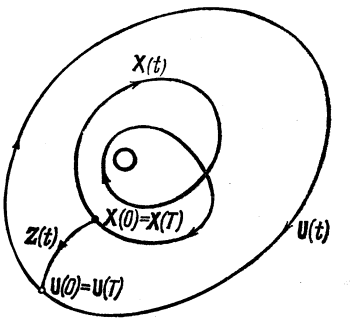


FIGURE 14.

It follows from the definition that $\mathbf{V}(t)$ is a closed curve. We have

$$\Delta \operatorname{Arg}_* \mathbf{V}(t) = \Delta \operatorname{Arg}_* \mathbf{X}(t) + \Delta \operatorname{Arg}_* \mathbf{Z}(t) + \\ + \Delta \operatorname{Arg}_* \mathbf{U}(t)^{[-p]} - \Delta \operatorname{Arg}_* \mathbf{Z}(t),$$

and since

$$\Delta \operatorname{Arg}_* \mathbf{U}(t)^{[-p]} = -p \Delta \operatorname{Arg}_* \mathbf{U}(t) = -2p\pi,$$

it follows that $\Delta \operatorname{Arg}_* \mathbf{V}(t) = 0$, so that the index of $\mathbf{V}(t)$ is zero. By Theorem I, b, $\mathbf{V}(t)$ may be contracted to a point. A suitable continuous deformation doing this does not change $\Delta \operatorname{Arg} \mathbf{V}(t)$, and so $\Delta \operatorname{Arg} \mathbf{V}(t) = 0$. Thus

$$\Delta \operatorname{Arg} \mathbf{V}(t) = \Delta \operatorname{Arg} \mathbf{X}(t) + \Delta \operatorname{Arg} \mathbf{Z}(t) + \Delta \operatorname{Arg} \mathbf{U}(t)^{[-p]} - \Delta \operatorname{Arg} \mathbf{Z}(t) = 0$$

and, using property 4 of Definition I, we obtain

$$\Delta \operatorname{Arg} \mathbf{U}(t)^{[-p]} = -p \Delta \operatorname{Arg} \mathbf{U}(t) = -2p\pi.$$

Hence $\Delta \operatorname{Arg} \mathbf{X}(t) = 2p\pi$ and the first part of Theorem II is proved.

Let $\mathbf{X}_1(t) \sim \mathbf{X}_2(t)$. A continuous deformation of the curve not displacing the endpoints does not change the arguments, and so condition (5.8) is satisfied. Conversely, suppose that $\mathbf{X}_1(t)$, $\mathbf{X}_2(t)$ are curves with common endpoints satisfying (5.8). We must show that $\mathbf{X}_1(t) \sim \mathbf{X}_2(t)$. Let $\mathbf{X}(t)$ denote the closed curve

$$\mathbf{X}(t) = \mathbf{X}_1(t) \times \mathbf{X}_2(t)^{[-1]}.$$

By (5.8), $\Delta \operatorname{Arg} \mathbf{X}(t) = 0$. If $\mathbf{X}(t)$ has nonzero index, then by what we have already proved $\Delta \operatorname{Arg} \mathbf{X}(t) \neq 0$. Thus the index of $\mathbf{X}(t)$ vanishes, so that $\Delta \operatorname{Arg}_* \mathbf{X}(t) = 0$, and $\Delta \operatorname{Arg}_* \mathbf{X}_1(t) = \Delta \operatorname{Arg}_* \mathbf{X}_2(t)$. It follows from the definition of $\operatorname{Arg}_* \mathbf{X}$ and Theorem I, b that $\mathbf{X}_1(t) \sim \mathbf{X}_2(t)$, proving Theorem II.

We shall need the following simple proposition.

Theorem III. a) *Let* $\mathbf{Y}(t)$, $\mathbf{Z}(t)$ $(0 \leqslant t \leqslant T)$ *be arbitrary closed curves in the group* $\mathfrak{G}$ *and* $\operatorname{Arg} \mathbf{X}$ *an arbitrary argument on* $\mathfrak{G}$. *Then*

$$\Delta \operatorname{Arg} [\mathbf{Y}(t) \mathbf{Z}(t)] = \Delta \operatorname{Arg} \mathbf{Y}(t) + \Delta \operatorname{Arg} \mathbf{Z}(t). \tag{5.9}$$

b) *Formula (5.9) is also true when* $\mathbf{Y}(t)$ *is a closed curve,* $\mathbf{Y}(0) = \mathbf{Y}(T) = \mathbf{I}_{2k}$, *and* $\mathbf{Z}(t)$ *is an arbitrary curve.*

Proof. We have

$$\mathbf{Y}(t) \sim \mathbf{Y}(t) \times \mathbf{Y}(T), \qquad \mathbf{Z}(t) \sim \mathbf{Z}(0) \times \mathbf{Z}(t).$$

Consequently,

$$\mathbf{Y}(t) \mathbf{Z}(t) \sim [\mathbf{Y}(t) \times \mathbf{Y}(T)] [\mathbf{Z}(0) \times \mathbf{Z}(t)] = \mathbf{Y}(t) \mathbf{Z}(0) \times \mathbf{Y}(T) \mathbf{Z}(t).$$

By Theorem II, the increments of arguments of homotopic curves are equal, and so

$$\Delta \operatorname{Arg} \mathbf{Y}(t)\mathbf{Z}(t) = \Delta \operatorname{Arg}[\mathbf{Y}(t)\mathbf{Z}(0)] + \Delta \operatorname{Arg}[\mathbf{Y}(T)\mathbf{Z}(t)]. \tag{5.10}$$

Since $\mathfrak{G}$ is arcwise-connected (Theorem I, a) $\mathbf{Z}(0)$ can be joined continuously to $\mathbf{I}_{2k}$. Hence the curve $\mathbf{Y}(t)\mathbf{Z}(0)$ may be deformed continuously into $\mathbf{Y}(t)$. Consequently $\Delta \operatorname{Arg}[\mathbf{Y}(t)\mathbf{Z}(0)] = \Delta \operatorname{Arg} \mathbf{Y}(t)$. If $\mathbf{Z}(t)$ is a closed curve (case (a)), we show in just the same way that $\Delta \operatorname{Arg} \mathbf{Y}(T)\mathbf{Z}(t) = \Delta \operatorname{Arg} \mathbf{Z}(t)$. This equality is also true when $\mathbf{Z}(t)$ is an arbitrary curve but $\mathbf{Y}(T) = \mathbf{I}_{2k}$ (case (b)). In either case, (5.9) follows from (5.10). Q. E. D.

We now define various argument functions in the sense of Definition I which are easier to compute in terms of the matrix $\mathbf{X}$ than (5.5).

Partition the matrix $\mathbf{X}$ into four $k \times k$ matrices

$$\mathbf{X} = \begin{pmatrix} \mathbf{U}_1 & \mathbf{U}_2 \\ \mathbf{V}_1 & \mathbf{V}_2 \end{pmatrix}$$

and denote

$$\left.\begin{aligned} &\operatorname{Arg}_1 \mathbf{X} = \operatorname{Arg} \det(\mathbf{U}_1 - i\mathbf{V}_1), \quad \operatorname{Arg}_2 \mathbf{X} = \operatorname{Arg} \det(\mathbf{U}_2 - i\mathbf{V}_2), \\ &\operatorname{Arg}_3 \mathbf{X} = \operatorname{Arg} \det(\mathbf{U}_1 + i\mathbf{U}_2), \quad \operatorname{Arg}_4 \mathbf{X} = \operatorname{Arg} \det(\mathbf{V}_1 + i\mathbf{V}_2). \end{aligned}\right\} \tag{5.11}$$

Let $\mathbf{S}$ and $\mathbf{T}$ be arbitrary fixed symplectic matrices, and set

$$\operatorname{Arg}'_j \mathbf{X} = \operatorname{Arg}_j \mathbf{SXT}, \quad \operatorname{Arg}''_j \mathbf{X} = -\operatorname{Arg}_j \mathbf{SX}^*\mathbf{T} \qquad (j = 1, 2, 3, 4). \tag{5.12}$$

Theorem IV /224m, u/. Any of the functions (5.11), (5.12) satisfies conditions 1 through 4 in Definition I, and is thus an argument on the symplectic group. All these arguments, including* $\operatorname{Arg}_* \mathbf{X}$, *are equivalent in the following sense: For any two argument functions* $\operatorname{Arg}' \mathbf{X}$ *and* $\operatorname{Arg}'' \mathbf{X}$, *there is a constant* $c > 0$ *such that for any curve* $\mathbf{X}(t) \in \mathfrak{G}$

$$|\Delta \operatorname{Arg}' \mathbf{X}(t) - \Delta \operatorname{Arg}'' \mathbf{X}(t)| \leqslant c. \tag{5.13}$$

Examples can be given of arguments on $\mathfrak{G}$ which are not equivalent (in the sense of (5.13)) to $\operatorname{Arg}_* \mathbf{X}$ (and therefore to any of the arguments (5.11), (5.12)).

We now show that any argument on $\mathfrak{G}$ may serve to define the index of a strongly stable Hamiltonian.

Let $\mathbf{X}(t)$ be the matrizant of some strongly stable equation (5.1) $(0 \leqslant t \leqslant T)$. By assumption, all the eigenvalues of the matrix $\mathbf{X}_0 = \mathbf{X}(T)$ lie on the unit circle and there are no coinciding eigenvalues of different kinds.

Any matrix $\mathbf{X} \in \mathfrak{G}$ with this property will be called strongly stable. Let $\mathbf{X}(t)$ $(T \leqslant t \leqslant 2T)$ be a curve whose initial point is the monodromy matrix $\mathbf{X}_0$, such that $\mathbf{X}(t)$ is strongly stable for $T \leqslant t < 2T$, and $\mathbf{X}(2T) = \mathbf{I}_{2k}$ is the identity matrix. We shall call the resulting matrix-function $\mathbf{X}(t)$, which agrees with the matrizant for $0 \leqslant t \leqslant T$ and is defined subject to the above restrictions for $T < t \leqslant 2T$, a complemented matrizant. By construction, the complemented matrizant is a closed curve: $\mathbf{X}(2T) = \mathbf{X}(0) = \mathbf{I}_{2k}$. On the other hand, by (5.5) the expression on the left of (5.3) is $\Delta \operatorname{Arg}_* \mathbf{X}(t)\Big|_0^{2T}$.

* Thanks to condition 1, this implies in particular that each of the determinants in (5.11) is nonzero for any $\mathbf{X} \in \mathfrak{G}$.

Thus, the index n of a strongly stable Hamiltonian coincides with the index of a suitably defined complemented matrizant:

$$\Delta \operatorname{Arg}_* \mathbf{X}(t)\Big|_{t=0}^{2T} = 2n\pi. \tag{5.14}$$

Using Theorem IV, we obtain the following:

Theorem V. Let $\operatorname{Arg}\mathbf{X}$ *be any argument on the symplectic group. Then the index* n *of a strongly stable Hamiltonian is given by*

$$\Delta \operatorname{Arg}\mathbf{X}(t)\Big|_{t=0}^{2T} = 2n\pi, \tag{5.15}$$

where $\mathbf{X}(t)$ *is the corresponding complemented matrizant.*

In many cases, formula (5.15) with one of the arguments (5.11), (5.12) is more convenient for calculating the index of a strongly stable Hamiltonian than formula (5.4), which requires evaluation of the eigenvalues $\rho_j(t)$. In addition, since the argument $\operatorname{Arg}\mathbf{X}$ in (5.15) is arbitrary (in particular, it may be any of the arguments (5.11), (5.12)), we see that (5.15) defines different (but equivalent) characteristics of Hamiltonians in one stability domain.

To end this subsection, we give the definition of the argument in Gel'fand and Lidskii /132/. Any real $2k\times 2k$ matrix $\mathbf{X}$ may be expressed (uniquely) in polar form

$$\mathbf{X} = \mathbf{R}\mathbf{Y}, \tag{5.16}$$

where $\mathbf{R}$ is a symmetric positive definite matrix and $\mathbf{Y}$ an orthogonal matrix. If $\mathbf{X}$ is symplectic, it can be shown that $\mathbf{R}$ and $\mathbf{Y}$ are also symplectic, and $\mathbf{Y}$ has the special form

$$\mathbf{Y} = \begin{pmatrix} \mathbf{Y}_1 & \mathbf{Y}_2 \\ -\mathbf{Y}_2 & \mathbf{Y}_1 \end{pmatrix},$$

where $\mathbf{U} = \mathbf{Y}_1 + i\mathbf{Y}_2$ is a unitary matrix. Consequently, $\det \mathbf{Y} = e^{i\varphi}$, where φ is a real number. The number φ is the Gel'fand-Lidskii argument of the symplectic matrix $\mathbf{X}$:

$$\operatorname{Arg}_0 \mathbf{X} = \varphi. \tag{5.17}$$

It follows from (5.11) that

$$\operatorname{Arg}_0 \mathbf{X} = \operatorname{Arg}_3 \mathbf{Y}(\mathbf{X}),$$

where $\mathbf{Y}(\mathbf{X}) = \mathbf{Y}$ is the matrix in (5.16). It can be shown that $\operatorname{Arg}_0 \mathbf{X}$ is an argument in the sense of Definition I. Indeed, Gel'fand and Lidskii define the index of a strongly stable Hamiltonian through formula (5.14), with $\operatorname{Arg}_* \mathbf{X}$ replaced by $\operatorname{Arg}_0 \mathbf{X}$. This definition furnishes a simple proof of Theorem V, with $\operatorname{Arg}_* \mathbf{X}$ replaced by $\operatorname{Arg}_0 \mathbf{X}$ and the index of a closed curve defined in terms of $\operatorname{Arg}_0 \mathbf{X}$.

5.5. Structure of the set of strongly stable Hamiltonians and some related sets

Let $\rho_1, \ldots, \rho_k$ be the eigenvalues of the first kind of a matrix $\mathbf{X} \in \mathfrak{G}$. Then its eigenvalues of the second kind are $\rho_1^{-1}, \ldots, \rho_k^{-1}$ (§2.6). The set of all eigenvalues of $\mathbf{X}$, with their kinds specified, will be called the labeled spectrum of $\mathbf{X}$ denoted by $\zeta = \zeta(\mathbf{X})$. The set of all $\zeta(\mathbf{X})$ $(\mathbf{X} \in \mathfrak{G})$ will be denoted by Σ. In other words, an element of Σ is a set of $2k$ complex numbers $\rho_1, \ldots, \rho_k$ (called multipliers of the first kind) and $\rho_1^{-1}, \ldots, \rho_k^{-1}$ (multipliers of the second kind), satisfying the conditions:

1) $0 < |\rho_j| \leqslant 1$ (for multipliers of the first kind);

2) if ρ_j is a multiplier of the first kind in ζ, and $|\rho_j| < 1$, then $\bar{\rho}_j \in \zeta$.

Thus, the multipliers of the first kind may be distributed arbitrarily over the unit circle and on the intervals $(-1, 0)$ and $(0, 1)$, whereas in the region $|\rho| < 1, \operatorname{Im} \rho \neq 0$, they must be symmetric with respect to the real axis.

It is easy to see that any set satisfying these conditions is the labeled spectrum of some matrix $\mathbf{X} \in \mathfrak{G}$. We can write any labeled spectrum as

$$\zeta = \{\rho_1, \ldots, \rho_k; \rho_1^{-1}, \ldots, \rho_k^{-1}\}$$

with the multipliers of the first kind written first, followed by those of the second kind. The order of the multipliers of the first kind is inessential, and we shall identify the elements

$$\zeta_1 = \{\rho_1', \ldots, \rho_k'; (\rho_1')^{-1}, \ldots, (\rho_k')^{-1}\} \quad \text{and} \quad \zeta_2 = \{\rho_1'', \ldots, \rho_k''; (\rho_1'')^{-1}, \ldots, (\rho_k'')^{-1}\}$$

if there exists a permutation

$$\begin{pmatrix} 1, & 2, & \ldots, & k \\ s_1, & s_2, & \ldots, & s_k \end{pmatrix}$$

such that $\rho_j' = \rho_{s_j}''$ $(j = 1, \ldots, k)$.

We now introduce the natural topology in Σ: the ε-neighborhood of a point $\zeta_0 = \{\rho_1^{(0)}, \ldots, \rho_k^{(0)}; \ldots\}$ is defined as the set of all $\zeta = \{\rho_1, \ldots, \rho_k; \ldots\} \in \Sigma$ such that $|\rho_j - \rho_{s_j}^{(0)}| < \varepsilon$ $(j = 1, \ldots, k)$, where $s_1, \ldots, s_k$ is any permutation of the numbers $1, \ldots, k$.

A neighborhood of a point ζ_0 is defined to be any subset of Σ containing some ε-neighborhood of ζ_0. It is easily checked that Σ thereby becomes a topological space /76a/.

The above "identification" of multipliers of the first (hence also the second) kind has the following consequence. If $\zeta(t) = \{\rho_1(t), \ldots, \rho_k(t); \ldots\}$ is a closed curve in the space Σ, the functions $\rho_j(t)$ are not necessarily closed curves on the complex plane. All that we can say is that $\rho_j(T) = \rho_{s_j}(0)$, where $s_1, \ldots, s_k$ is some permutation of the numbers $1, \ldots, k$. We are obliged to adopt this definition of ζ, "identifying" multipliers of one kind, since if a matrix $\mathbf{X}$ describes a closed curve in $\mathfrak{G}$ then its eigenvalues do not necessarily describe closed curves on the complex plane.

We now agree on the following notation. Let $\mathfrak{M}$ be some subset of $\mathfrak{L}$, say $\mathfrak{M} = \mathcal{G}$, $\mathfrak{M} = \mathcal{H}$ or $\mathfrak{M} = \Gamma_0$, where $\mathcal{G}$ and $\mathcal{H}$ are the sets of strongly stable

and strongly unstable Hamiltonians and Γ_0 is their common boundary. The corresponding set of monodromy matrices will be denoted by $\tilde{\mathfrak{M}}$, $\tilde{\mathfrak{M}} \in \mathfrak{G}$. The corresponding set in Σ will be denoted by $\hat{\mathfrak{M}}$, $\hat{\mathfrak{M}} = \zeta(\tilde{\mathfrak{M}})$. Apart from the sets $\tilde{\mathcal{O}}$, $\tilde{\mathcal{H}}$, $\tilde{\Gamma}_0$; $\hat{\mathcal{O}}$, $\hat{\mathcal{H}}$, $\hat{\Gamma}_0$, we shall also consider the sets m_α, $\tilde{m}_\alpha$, $\hat{m}_\alpha (\alpha > 0)$, corresponding to equations (5.1) all of whose solutions admit the estimate

$$|\mathbf{x}(t)| < ce^{\alpha t} \qquad (0 \leqslant t < \infty),$$

i. e., whose multipliers lie within the disk of radius e^α, and the sets M_α, $\tilde{M}_\alpha$, $\hat{M}_\alpha$, corresponding to equations (5.1) for which at least one multiplier is outside the disk with radius e^α.

This notation is clarified in the following table.

TABLE

$\mathfrak{L} = \{\mathbf{H}(t)\}$	$\mathfrak{G}(t) = \{\mathbf{X}(t)\}$	$\mathfrak{G} = \{\mathbf{X}\}$	$\Sigma = \{\zeta\}$	
$\mathcal{O}$	$\mathcal{O}$	$\tilde{\mathcal{O}}$	$\hat{\mathcal{O}}$	All multipliers on the unit circle; none of them multiple multipliers of different kinds (strong stability)
$\mathcal{H}$	$\mathcal{H}$	$\tilde{\mathcal{H}}$	$\hat{\mathcal{H}}$	Some multipliers not on the unit circle (strong instability)
Γ_0	Γ_0	$\tilde{\Gamma}_0$	$\hat{\Gamma}_0$	All multipliers on the unit circle, some of them multiple multipliers of different kinds
Γ	Γ	$\tilde{\Gamma}$	$\hat{\Gamma}$	There exist multiple multipliers of different kinds
m_α	m_α	$\tilde{m}_\alpha$	$\hat{m}_\alpha$	All multipliers inside the disk of radius $e^\alpha > 1$
M_α	M_α	$\tilde{M}_\alpha$	$\hat{M}_\alpha$	Some multipliers outside the disk of radius $e^\alpha > 1$

This table also lists the set $\Gamma(\tilde{\Gamma}, \hat{\Gamma})$ which we shall encounter later.

The mappings

$$\mathfrak{L} \to \mathfrak{G} \to \Sigma \tag{5.18}$$

introduced above are continuous, by the theorems in § II. 1.7 and § III. 2.10. Consequently, the pre-images in $\mathfrak{L}$ and $\mathfrak{G}$ of a domain (connected open set) in Σ are open sets.

Now let $\hat{\mathfrak{M}}$ be a domain in Σ, $\tilde{\mathfrak{M}}$ its pre-image in $\mathfrak{G}$ and $\mathfrak{M}$ its pre-image in $\mathfrak{L}$ under the mappings (5.18). We wish to determine the structure of the sets $\tilde{\mathfrak{M}}$ and $\mathfrak{M}$, in other words, to find the number of connected components of the open sets $\tilde{\mathfrak{M}}$ and $\mathfrak{M}$. In analytical terms, this is simply the problem of studying the structure of a set of Hamiltonians of equations (5.1) and the set of their monodromy matrices, assuming their multipliers to have certain prescribed properties.

It turns out that $\hat{\mathfrak{M}}$ is always a domain, but $\mathfrak{M}$ is the union of finitely or countably many disjoint domains, depending on certain properties of the set $\hat{\mathfrak{M}}$.

To state these properties, it will be convenient to define an argument on Σ by

$$\operatorname{Arg}\zeta = \sum_{j=1}^{k} \operatorname{Arg}\rho_j.$$

For any curve $\zeta(t)$ $(0 \leqslant t \leqslant T)$, the increment

$$\Delta \operatorname{Arg}\zeta(t) = \sum_{j=1}^{k} \Delta \operatorname{Arg}\rho_j(t) \Big|_0^T$$

is uniquely defined. If $\zeta(t)$ is a closed curve, then clearly

$$\Delta \operatorname{Arg}\zeta(t) = 2m\pi,$$

where m is an integer. This integer is clearly invariant under any continuous deformation of the curve $\zeta(t)$. Hence, if a closed curve can be continuously contracted to a point in Σ, then $m = 0$.

Let $\hat{\mathfrak{M}}$ be a domain in Σ. There are two possibilities:

(a) for any curve $\zeta(t) \in \hat{\mathfrak{M}}$ $(0 \leqslant t \leqslant T)$,

$$\Delta \operatorname{Arg}\zeta(t) = 0;$$

(b) there is a curve $\zeta(t) \in \hat{\mathfrak{M}}\,(0 \leqslant t \leqslant T)$ such that

$$\Delta \operatorname{Arg}\zeta(t) = 2m\pi \neq 0. \tag{5.19}$$

In the second case, we may assume without loss of generality that $m > 0$ (otherwise we need only replace $\zeta(t)$ by $\zeta(t)^{[-1]}$). The smallest possible integer $m > 0$ in (5.19), when $\zeta(t)$ is allowed to range over all curves in $\hat{\mathfrak{M}}$, will be called the winding number of the domain $\hat{\mathfrak{M}}$ in Σ.

Theorem I. 1) *Assume that case (a) holds. Then the pre-image* $\hat{\mathfrak{M}} \subset \mathfrak{G}$ *of the domain* $\hat{\mathfrak{M}}$ *is a simply-connected domain in* $\mathfrak{G}$.*

2) *Assume that case (b) holds. Then the pre-image* $\hat{\mathfrak{M}} \subset \mathfrak{G}$ *is a multiply-connected domain in* $\mathfrak{G}$.

3) *For any curve* $\zeta(t)$ $(0 \leqslant t \leqslant T)$ *in* Σ, *there exists a curve* $\mathbf{X}(t)$ *in* $\mathfrak{G}$ *such that* $\zeta[\mathbf{X}(t)] = \zeta(t)$ *for any* t $(0 \leqslant t \leqslant T)$.

The reader may find a proof of this theorem in /224m/.

Consider case (a). Thanks to part 1 of Theorem I, we can proceed as in the stable case, associating with any Hamiltonian $\mathbf{H}(t) \in \mathfrak{M}$ an integer — its index. Indeed, let $\mathbf{Z}_0$ be any matrix (fixed from now on) in the domain $\hat{\mathfrak{M}}$ or on its boundary. Recall that $\hat{\mathfrak{M}}$ is the pre-image in $\mathfrak{G}$ of the set $\hat{\mathfrak{M}}$ under the mapping (5.18). Let $\mathbf{Z}(t)$ $(0 \leqslant t \leqslant T)$ be a curve joining $\mathbf{Z}_0$ to $\mathbf{I}_{2k}$: $\mathbf{Z}(0) = \mathbf{Z}_0$, $\mathbf{Z}(T) = \mathbf{I}_{2k}$. Let

* A domain $\hat{\mathfrak{M}}$ is simply-connected in $\mathfrak{G}$ if any curve $\mathbf{X}(t) \in \hat{\mathfrak{M}}\,(0 \leqslant t \leqslant T)$ may be contracted to a point by a continuous deformation which need not remain in $\hat{\mathfrak{M}}$ (but must remain in $\mathfrak{G}$). Otherwise, the domain is said to be multiply-connected in $\mathfrak{G}$.

$$\mathbf{X}=\Phi[\mathbf{H}(t)]$$

be the monodromy matrix of the equation (5.1) with Hamiltonian $\mathbf{H}(t)$. In $\tilde{\mathfrak{M}}$ we can join the matrix $\mathbf{X}=\Phi[\mathbf{H}(t)]$ to $\mathbf{X}_0$ by a curve $\mathbf{Y}(t)$ $(0\leqslant t\leqslant T)$:

$$\mathbf{Y}(0)=\mathbf{X},\ \mathbf{Y}(T)=\mathbf{Z}_0,\ \mathbf{Y}(t)\in\tilde{\mathfrak{M}}\quad\text{for}\quad 0\leqslant t<T. \tag{5.20}$$

Let $\mathbf{X}(t)$ $(0\leqslant t\leqslant T)$ be the matrizant of the equation with Hamiltonian $\mathbf{H}(t)$. The matrix-function

$$\mathbf{X}'(t)=\mathbf{X}(t)\times\mathbf{Y}(t)\times\mathbf{Z}(t)$$

formed by connecting the curves $\mathbf{X}(t)$, $\mathbf{Y}(t)$, $\mathbf{Z}(t)$ in succession will be called a complemented matrizant. If $\tilde{\mathfrak{M}}=\hat{\mathfrak{G}}$, the definition coincides with that given earlier — let $\mathbf{Z}_0$ be the matrix $\mathbf{Z}_0=\mathbf{I}_{2k}$ and set $\mathbf{Z}(t)\equiv\mathbf{I}_{2k}$.

The curve $\mathbf{X}'(t)$ is closed — it begins and ends at the point $\mathbf{I}_{2k}$. Therefore,

$$\Delta\operatorname{Arg}\mathbf{X}'(t)=2n\pi, \tag{5.21}$$

where n is an integer. This definition of n involves the arbitrary choice of the curve $\mathbf{Y}(t)$. In fact, however, the number n is independent of this choice. Let $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ be two different curves satisfying (5.20). Then $\mathbf{Y}(t)=\mathbf{Y}_1(t)\,\mathbf{Y}_2(t)^{[-1]}$ is a closed curve, one point $\mathbf{Y}(T)=\mathbf{Z}_0$ of which may lie on the boundary of the domain $\tilde{\mathfrak{M}}$ while all other points are in the interior. Since the group $\mathfrak{G}$ is locally homeomorphic to a euclidean space, we may slightly deform the curve $\mathbf{Y}(t)$ so that it lies entirely in the interior of $\tilde{\mathfrak{M}}$. Then, by part 1 of Theorem I, it may be contracted to a point. Consequently, $\Delta\operatorname{Arg}\mathbf{Y}(t)=0$, and so

$$\Delta\operatorname{Arg}\mathbf{Y}_1(t)=\Delta\operatorname{Arg}\mathbf{Y}_2(t).$$

Since by (5.21)

$$2n\pi=\Delta\operatorname{Arg}\mathbf{X}(t)+\Delta\operatorname{Arg}\mathbf{Y}(t)+\Delta\operatorname{Arg}\mathbf{Z}(t),$$

it follows that the number n in (5.21) is independent of the choice of the curve satisfying (5.20).

The number n is defined as the index of the Hamiltonian $\mathbf{H}(t)\in\mathfrak{M}$.

Theorem II. 1) *Assume that case (a) holds. Then the pre-image* $\mathfrak{M}\subset\mathfrak{L}$ *of the domain* $\tilde{\mathfrak{M}}$ *is the union of countably many disjoint domains* $\mathfrak{M}_n$ $(n=0,\pm1,\pm2,\ldots)$, *each of which is the set of all Hamiltonians with index n and labeled spectrum* ζ *belonging to the set* $\tilde{\mathfrak{M}}$. *The image of each of the domains* $\mathfrak{M}_n$ *under the mapping (5.18) is* $\mathfrak{M}$.

2) *Assume that case (b) holds. Then the pre-image* $\mathfrak{M}\subset\mathfrak{L}$ *of the domain* $\tilde{\mathfrak{M}}$ *is the union of m domains* $\mathfrak{M}_1,\ \ldots,\ \mathfrak{M}_m$.

The proof may be found in /224m/.

Theorem II enables one to determine the number of connected components of a set of Hamiltonians defined by certain properties of their multipliers. For example, consider the set $\hat{\mathfrak{G}}^{(+)}$ of all strongly stable

labeled spectra of multiplier type $\mu=(+, \ldots, +)$ (in other words, in all labeled spectra (5.17) in $\hat{\mathcal{O}}^{(+)}$ the multipliers of the first kind lie along the upper unit semicircle). It is clear that $\hat{\mathcal{O}}^{(+)}$ is an open set for which case (a) holds. By part 1 of Theorem I, the corresponding set $\mathcal{O}^{(+)}$ of Hamiltonians is the union of countably many domains $\mathcal{O}_n^{(+)}$, each of which is the set of all Hamiltonians with index n and multiplier type $\mu=(+, \ldots, +)$. This is true for any multiplier type μ of strongly stable Hamiltonians. Indeed, let $\zeta(t)$ $(0\leqslant t\leqslant T)$ be a closed curve in $\hat{\mathcal{O}}^{(\mu)}$. For each t $(0\leqslant t\leqslant T)$, number the multipliers of the first kind on the upper unit semicircle in order of increasing argument. Since by the definition of $\hat{\mathcal{O}}^{(\mu)}$ multipliers of different kinds do not "meet" as t varies $(0\leqslant t\leqslant T)$, each point $\rho_j(t)$ $(0\leqslant t\leqslant T)$ moves along the upper unit semicircle and $\rho_j(0)=\rho_j(T)$ $(j=1, \ldots, k)$. Consequently,

$$\Delta \operatorname{Arg} \mathbf{X}(t)=\sum_{j=1}^{k} \Delta \operatorname{Arg} \rho_j(t)=0.$$

By Theorem I, $\mathcal{O}^{(\mu)}$ is the union of infinitely many domains $\mathcal{O}_n^{(\mu)}$, each of which is the set of all Hamiltonians with index n and multiplier type μ. We have thus proved the Gel'fand-Lidskii theorem (§5.3).

We now consider the set $\mathcal{H}$.

If $k=1$, there are two types of labeled spectra, say ν_1 and ν_2, as shown in Figure 15. Let $\hat{\mathcal{H}}^{(\nu_1)}$ and $\hat{\mathcal{H}}^{(\nu_2)}$ denote the corresponding sets of labeled spectra. Thus, all $\zeta\in\hat{\mathcal{H}}^{(\nu_1)}$ have the form $\zeta=\{\beta, \beta^{-1}\}$ $(0<\beta<1)$, and all $\zeta\in\hat{\mathcal{H}}^{(\nu_2)}$ the form $\zeta=\{-\beta, -\beta^{-1}\}$ $(0<\beta<1)$. Each of these sets clearly corresponds to case (a). Part 1 of Theorem I implies

Theorem III. If $k=1$, the set of strongly unstable Hamiltonians $\mathcal{H}$ is the union of countably many domains $\mathcal{H}_n^{(\nu)}$ $(n=0, \pm1, \pm2, \ldots, \nu=\nu_1, \nu_2)$ where $\mathcal{H}_n^{(\nu)}$ is the set of all Hamiltonians $\mathbf{H}(t)$ of index n and type of labeled spectrum ν_j $(j=1, 2)$. For any n and any labeled spectrum $\zeta_0\in\hat{\mathcal{H}}^{(\nu_j)}$ $(j=1, 2)$, there is a Hamiltonian $\mathbf{H}(t)\in\mathcal{H}_n^{(\nu_j)}$ with labeled spectrum ζ_0. The set $\tilde{\mathcal{H}}$ of corresponding monodromy matrices is the union of two domains $\tilde{\mathcal{H}}^{(\nu_1)}$ and $\tilde{\mathcal{H}}^{(\nu_2)}$ which are simple-connected in $\mathfrak{G}$.

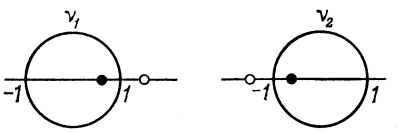


FIGURE 15.

Now consider the case $k>1$.

Theorem IV. If $k>1$, the set of strongly unstable Hamiltonians $\mathcal{H}$ is a domain in $\mathfrak{L}$, and the corresponding set $\tilde{\mathcal{H}}$ of monodromy matrices is a domain in $\mathfrak{G}$.

Proof. We shall show that when $k>1$ the set $\tilde{\mathcal{H}}$ is a domain and its winding number is 1. The assertion of our theorem will then follow from part 2 of Theorem I. Obviously, $\tilde{\mathcal{H}}$ is an open set; we must show that it is connected. Let

$$\zeta_0 = \left\{\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}; 2, 2, \ldots, 2\right\}.$$

We shall join an arbitrary point $\zeta \in \hat{\mathcal{H}}$ to ζ_0 by a continuous curve in $\hat{\mathcal{H}}$. We first assume that ζ contains a multiplier of the first kind on the positive real axis. This multiplier may be transferred to the point $\rho_0 = \frac{1}{2}$ without ζ leaving $\hat{\mathcal{H}}$. Then all other multipliers of the first kind may be transferred to this point. Throughout we obviously remain in $\hat{\mathcal{H}}$, since during the deformation process at least one of the multipliers, namely $\rho_0^{-1} = 2$, is always outside the unit circle.

If ζ contains no multipliers on the positive real axis, but there is at least one quadruple of multipliers lying neither on the unit circle nor on the negative real axis, any such quadruple may be brought continuously to the positive real axis, remaining throughout in $\hat{\mathcal{H}}$, and the resulting point joined as shown previously to ζ_0.

It remains to discuss the case in which all multipliers of ζ not on the unit circle lie on the negative real axis. If there are at least four such multipliers, some quadruple may be displaced into the domain $|\rho| \neq 1$, $\operatorname{Im}\rho \neq 0$, and we return to the preceding case. If there are only two, the other multipliers must lie on the unit circle ($k > 1$!). A pair of the latter multipliers (one of the first kind, the other of the second) may be transferred to the negative real axis and the resulting quadruple displaced into the region $|\rho| \neq 1$, $\operatorname{Im}\rho \neq 0$. We again have the preceding case. Thus $\hat{\mathcal{H}}$ is a connected set, hence a domain.

We now show that the winding number of $\hat{\mathcal{H}}$ is 1. To this end, it will suffice to construct a curve $\zeta(t) \in \hat{\mathcal{H}}$ for which $\Delta \operatorname{Arg} \zeta(t) = 2\pi$. For example, we can take

$$\zeta(t) = \left\{\rho_1, \frac{1}{2}, \ldots, \frac{1}{2}; \rho_1^{-1}, 2, \ldots, 2\right\},$$

where $\rho_1 = \exp(2\pi t i/T)$. This completes the proof of Theorem IV.

We now turn to the question of the number of connected components of the set $\mathcal{H} \setminus \Gamma$.

Consider ζ_1 and $\zeta_2 \in \hat{\mathcal{H}} \setminus \hat{\Gamma}$. We shall say that ζ_1 and ζ_2 have the same multiplier type ν if:

1) they have the same multiplier type on the unit circle in the previously defined sense;

2) the number of multiplier pairs on the positive real axis has the same parity for ζ_1 as for ζ_2.

We shall say that $\mathbf{X}_1$ and $\mathbf{X}_2 \in \tilde{\mathcal{H}} \setminus \tilde{\Gamma}$ have the same eigenvalue type if $\zeta(\mathbf{X}_1)$ and $\zeta(\mathbf{X}_2)$ have the same multiplier type.

Theorem V. a) *The set $\hat{\mathcal{H}} \setminus \hat{\Gamma}$ is the union of $N = 2(2^k - 1)$ disjoint domains $\hat{\mathcal{H}}^{(\nu)}$ ($\nu = \nu_1, \ldots, \nu_N$) which are simply-connected in $\sum$.*

b) *The set $\tilde{\mathcal{H}} \setminus \tilde{\Gamma}$ is the union of domains $\tilde{\mathcal{H}}^{(\nu)}$ ($\nu = \nu_1, \ldots, \nu_N$), simply-connected in $\mathfrak{G}$, each of which is the set of all matrices $\mathbf{X} \in \tilde{\mathcal{H}} \setminus \tilde{\Gamma}$ of a single eigenvalue type; moreover, $\tilde{\mathcal{H}}^{(\nu)}$ is the pre-image in $\mathfrak{G}$ of the domain $\hat{\mathcal{H}}^{(\nu)}$.*

c) *The set* $\mathscr{H}\setminus\Gamma$ *is a union of countably many domains* $\mathscr{H}_n^{(\nu)}$ $(n=0, \pm 1, \pm 2, \ldots; \nu=\nu_1, \ldots, \nu_N)$; *for every labeled spectrum* $\zeta_0 \in \hat{\mathscr{H}}^{(\nu)}$ *and any* n $(n=0, \pm 1, \pm 2, \ldots)$ *there is a Hamiltonian* $\mathbf{H}(t) \in \mathscr{H}_n^{(\nu)}$ *whose labeled spectrum is* ζ_0.

Proof. Let us determine the number of distinct types of labeled spectra for $\zeta \in \hat{\mathscr{H}}$. All the types ν may be divided into two classes:

1) types ν whose labeled spectra have an even number of multipliers in the interval $(-1, 0)$;

2) types ν for which this number is odd.

Deform ζ so that quadruples of multipliers are displaced, without multipliers of different kinds ever meeting, from the region $|\rho| \neq 1$ onto the positive real axis. After this deformation, elements ζ of the first class will have $2j$ multipliers on the unit circle, all others on the positive real axis $(0 \leqslant j < k)$. Elements ζ of the second class will have 2 multipliers on the negative real axis, $2j$ on the unit circle, and all others on the positive real axis $(0 \leqslant j < k)$. Since the $2j$ multipliers of the first and second kinds may be distributed on the unit circle in 2^j ways (see end of §5.2), the total number of "unstable" types ν is

$$N = 2\sum_{j=0}^{k-1} 2^j = 2(2^k - 1).$$

Figure 16 illustrates the six types ν for $k=2$; the first three are in class 1, the others in class 2.

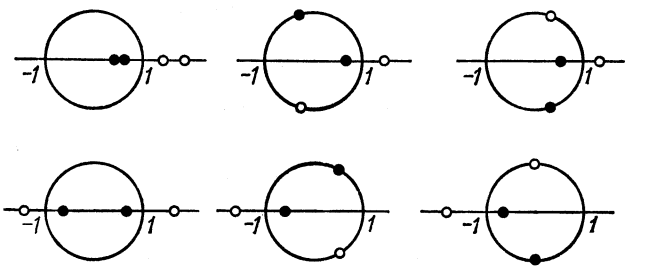


FIGURE 16.

Let $\hat{\mathscr{H}}^{(\nu)}$ denote the set of all labeled spectra of some fixed type ν; then

$$\hat{\mathscr{H}}\setminus\hat{\Gamma} = \bigcup_{\nu} \hat{\mathscr{H}}^{(\nu)}.$$

If a labeled spectrum ζ is subjected to a continuous deformation under which multipliers of different kinds cannot coincide, multipliers may move off the real axis (or onto it) only in fours; thus the type of ζ remains invariant under any such deformation.

On the other hand, it is clear that any two labeled spectra of the same type may be transformed into one another by a deformation of this type,

without leaving the set $\hat{\mathcal{H}} \setminus \hat{\Gamma}$. Thus the sets $\hat{\mathcal{H}}^{(\nu)}$ $(\nu=\nu_1, \ldots, \nu_N)$ are disjoint domains. This proves part (a).

To prove parts (b) and (c), it will suffice (see part 1 of Theorem I) to show that for any domain $\hat{\mathcal{H}}^{(\nu)}$ we have case (a) of Theorem I. Let $\zeta(t)$ $(0 \leqslant t \leqslant T)$ be a closed curve in $\hat{\mathcal{H}}^{(\nu)}$. Along any (not necessarily closed) continuous curve we have

$$\sum \Delta \operatorname{Arg} \rho_j(t) \Big|_0^T = 0, \tag{5.22}$$

where the sum extends over all multipliers of the first kind outside the unit circle, since these multipliers are symmetric with respect to the real axis. Along a closed curve the relation (5.22) holds with the sum over all multipliers of the first kind on the unit circle; this is proved in the same way as for the domains $\tilde{\mathcal{O}}^{(\mu)}$. Thus,

$$\Delta \operatorname{Arg} \zeta(t) = 0.$$

Consequently, case (a) holds for any domain $\hat{\mathcal{H}}^{(\nu)}$, proving parts (b) and (c) of the theorem.

Theorem VI. If $k>1$, the sets $\tilde{M}_\alpha$, M_α, $\tilde{m}_\alpha$ and m_α are domains. If $k=1$, $\tilde{m}_\alpha$ and m_α are domains, $\tilde{M}_\alpha$ is the union of two disjoint domains:

$$\tilde{M}_\alpha = \tilde{M}_\alpha^{(+)} \cup \tilde{M}_\alpha^{(-)},$$

and M_α is the union of countably many disjoint domains $M_{\alpha n}^{(+)}$, $M_{\alpha n}^{(-)}$ $(n=0, \pm 1, \pm 2, \ldots)$.

The proof is analogous to the preceding proofs. For $k=1$, $\hat{M}_\alpha^{(+)}$ is the set of labeled spectra $\zeta=\{\mu; \mu^{-1}\}$ $(0<\mu<e^{-\alpha})$, and $\hat{M}_\alpha^{(-)}$ the set $\zeta=\{-\mu; -\mu^{-1}\}$.

§ 6. STABILITY TESTS FOR CANONICAL EQUATIONS, BASED ON THE THEOREM ON DIRECTIONAL WIDENESS OF STABILITY DOMAINS

6.1. Method for derivation of stability tests

In this section we consider canonical equations

$$\mathbf{J}\frac{d\mathbf{x}}{dt} = \mathbf{H}(t)\,\mathbf{x}, \tag{6.1}$$

where, as before, $\mathbf{H}(t)$ is a real matrix-function and

$$\mathbf{J} = \begin{pmatrix} 0 & -\mathbf{I}_k \\ \mathbf{I}_k & 0 \end{pmatrix}, \quad \mathbf{H}(t)^{\tau} = \mathbf{H}(t) = \mathbf{H}(t+T).$$

Any canonical equation

$$\tilde{\mathbf{J}}\frac{d\mathbf{z}}{dt} = \tilde{\mathbf{H}}(t)\,\mathbf{z}, \tag{6.2}$$

where $\tilde{\mathbf{J}}$ is an arbitrary real nonsingular skew-symmetric matrix and $\tilde{\mathbf{H}}(t)^{\tau}=\tilde{\mathbf{H}}(t)=\tilde{\mathbf{H}}(t+T)$, may be reduced by a nonsingular linear transformation to the form (1.1). Indeed, as shown in §2.3, there is a nonsingular real matrix $\mathbf{S}$ such that

$$\mathbf{S}^{\tau}\mathbf{J}\mathbf{S}=\tilde{\mathbf{J}}.$$

The substitution $\mathbf{x}=\mathbf{S}\mathbf{z}$ brings equation (6.2) to the form (6.1), with

$$\mathbf{H}(t)=(\mathbf{S}^{-1})^{\tau}\tilde{\mathbf{H}}(t)\mathbf{S}^{-1}.$$

Recall (§5.1) that $\mathfrak{L}$ denotes the space of Hamiltonians $\mathbf{H}(t)$ — the set of real symmetric T-periodic $2k\times 2k$ matrix-functions $\mathbf{H}(t)$ with integrable piecewise-continuous elements. The metric in $\mathfrak{L}$ is defined by

$$\rho(\mathbf{H}_1, \mathbf{H}_2)=\int_0^T |\mathbf{H}_1(t)-\mathbf{H}_2(t)|\,dt.$$

Let $\mathbf{H}_1(t)\in\mathfrak{L}$, $\mathbf{H}_2(t)\in\mathfrak{L}$. We shall write

$$\mathbf{H}_1(t)\leqslant\mathbf{H}_2(t),$$

if

$$(\mathbf{H}_1(t)\mathbf{c}, \mathbf{c})\leqslant(\mathbf{H}_2(t)\mathbf{c}, \mathbf{c})$$

for any $2k$-vector $\mathbf{c}$ and any t for which $\mathbf{H}_1(t)$, $\mathbf{H}_2(t)$ are defined. If the strict inequality

$$(\mathbf{H}_1(t)\mathbf{c}, \mathbf{c})<(\mathbf{H}_2(t)\mathbf{c}, \mathbf{c}),$$

holds for all these t and all $\mathbf{c}\neq\mathbf{0}$, we shall write $\mathbf{H}_1(t)<\mathbf{H}_2(t)$. Thus, the notation $\mathbf{H}(t)>\mathbf{0}$ means that $\mathbf{H}(t)$ is a positive definite matrix for all t for which it is defined.

The next definition is important for the sequel.

Definition. A subset $\mathfrak{M}\subset\mathfrak{L}$ is said to be directionally wide if, for any $\mathbf{H}_1(t)\in\mathfrak{M}$, $\mathbf{H}_2(t)\in\mathfrak{M}$ such that $\mathbf{H}_1(t)\leqslant\mathbf{H}_2(t)$ and

$$\mathbf{H}(t, s)\equiv\mathbf{H}_1(t)+s[\mathbf{H}_2(t)-\mathbf{H}_1(t)]\in\mathfrak{M},$$

where s is any number in the interval $0\leqslant s\leqslant 1$, we have $\mathbf{H}(t)\in\mathfrak{M}$ for any Hamiltonian $\mathbf{H}(t)$ satisfying the inequalities

$$\mathbf{H}_1(t)\leqslant\mathbf{H}(t)\leqslant\mathbf{H}_2(t). \tag{6.3}$$

Thus, $\mathfrak{M}$ is a directionally wide set if the fact that an entire "segment" $\mathbf{H}(t, s)$ $(0\leqslant s\leqslant 1)$ with endpoints $\mathbf{H}_1(t)$ and $\mathbf{H}_2(t)$ belongs to the set implies that the whole set of Hamiltonians $\mathbf{H}(t)$ satisfying (6.3) is also in the set. (The set of Hamiltonians in question clearly contains the segment $\mathbf{H}(t, s)$, $0\leqslant s\leqslant 1$.)

To explain this definition, consider instead of $\mathfrak{L}$ a set of points $\mathbf{H}$ on the plane, identifying $\mathbf{H}$ with the appropriate complex number. Let Q^+ be some closed cone with apex at the origin (Figure 17). Introduce a partial order on the plane, writing $\mathbf{H}_1 \leqslant \mathbf{H}_2$ for two points $\mathbf{H}_1$ and $\mathbf{H}_2$ if $\mathbf{H}_2 - \mathbf{H}_1 \in Q^+$. Let Q^- be the set of points symmetric to the points $\mathbf{H} \in Q^+$ with respect to the origin: $\mathbf{H} \in Q^-$ if $(-\mathbf{H}) \in Q^+$ (Figure 17). Thus the cone Q^+ is the set of points $\mathbf{H} \geqslant \mathbf{0}$, and Q^- the set of points $\mathbf{H} \leqslant \mathbf{0}$. The points outside the cones Q^+ and Q^- are incomparable with $\mathbf{H} = \mathbf{0}$. Apply parallel translations to the cones Q^+ and Q^- so that their common apex coincides with some point $\mathbf{H}_0$; denote the new cones by $Q^+(\mathbf{H}_0)$ and $Q^-(\mathbf{H}_0)$ (Figure 17). $Q^+(\mathbf{H}_0)$ is the set of points $\mathbf{H} \geqslant \mathbf{H}_0$, and $Q^-(\mathbf{H}_0)$ the set of points $\mathbf{H} \leqslant \mathbf{H}_0$. Thus the set of points $\mathbf{H}$ such that $\mathbf{H}_1 \leqslant \mathbf{H} \leqslant \mathbf{H}_2$ is a parallelogram with opposite vertices at $\mathbf{H}_1$ and $\mathbf{H}_2$ — the intersection of the cones $Q^+(\mathbf{H}_1)$ and $Q^-(\mathbf{H}_2)$. Using the partial order relation defined on the plane, we can define directionally wide sets in exactly the same way as in the space $\mathfrak{L}$: we say that a set $\mathfrak{M}$ is directionally wide if, whenever the whole segment with endpoints $\mathbf{H}_1$ and $\mathbf{H}_2$, $\mathbf{H}_2 \geqslant \mathbf{H}_1$, lies in $\mathfrak{M}$, the rectangle of all points $\mathbf{H}$ such that $\mathbf{H}_1 \leqslant \mathbf{H} \leqslant \mathbf{H}_2$ lies in $\mathfrak{M}$.

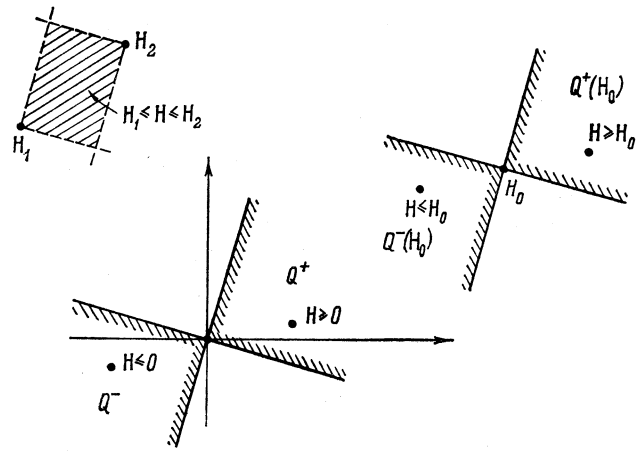


FIGURE 17.

If the segment $\mathbf{H}_1\mathbf{H}_2$ is parallel to a side of the cone Q^+, the rectangle $\mathbf{H}_1 \leqslant \mathbf{H} \leqslant \mathbf{H}_2$ coincides with the segment $\mathbf{H}_1\mathbf{H}_2$. The closer the direction of $\mathbf{H}_1\mathbf{H}_2$ to that of the bisector of Q^+, the "wider" is the rectangle $\mathbf{H}_1 \leqslant \mathbf{H} \leqslant \mathbf{H}_2$. In other words, the "width" of a directionally wide set $\mathfrak{M}$ is associated with its "direction" relative to the cone $\mathbf{H} \geqslant \mathbf{0}$. This is illustrated in Figure 18, which shows three directionally wide sets $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$ and two sets $\mathfrak{M}_4$, $\mathfrak{M}_5$ which are not directionally wide. Note that the set $\mathfrak{M} = \mathfrak{M}_1 \cup \mathfrak{M}_2 \cup \mathfrak{M}_3$ in Figure 18 is also directionally wide. (Any segment $\mathbf{H}_1\mathbf{H}_2$ in this set is necessarily in one of the sets $\mathfrak{M}_1$, $\mathfrak{M}_2$ or $\mathfrak{M}_3$, so that, since these sets are directionally wide, the rectangle $\mathbf{H}_1 \leqslant \mathbf{H} \leqslant \mathbf{H}_2$ is also in the set.) Thus a directionally wide set need not be connected.

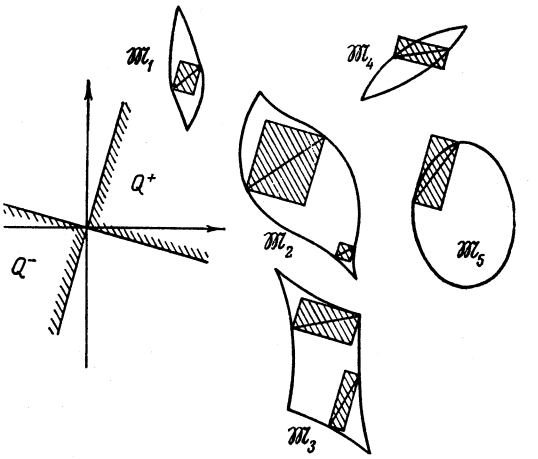


FIGURE 18.

We shall show below that the stability domains $G_n^{(\mu)}$ in the Hamiltonian space (see §3.5) are directionally wide sets. This is of considerable importance for the sequel, for it will provide us with a very simple means of establishing strong stability tests for equation (6.1). Indeed, let $\mathbf{H}_1$ and $\mathbf{H}_2$ be suitably chosen Hamiltonians, which are in some sense simple, such that

1) the equation with Hamiltonian $\mathbf{H}(t, s) = s\mathbf{H}_2(t) + (1-s)\mathbf{H}_1(t)$ is integrable in closed form, say $\mathbf{H}_1(t) = \text{const}$, $\mathbf{H}_2(t) = \text{const}$,

2) the equation is strongly stable for $0 \leqslant s \leqslant 1$.

Then the fact that the stability domains are directionally wide implies that the equation is strongly stable with any Hamiltonian $\mathbf{H}(t)$ such that $\mathbf{H}_1(t) \leqslant \mathbf{H}(t) \leqslant \mathbf{H}_2(t)$.

To illustrate the method, consider the scalar equation (Hill's equation)

$$\ddot{\xi} + p(t)\xi = 0, \tag{6.4}$$

where $p(t+T) \equiv p(t)$. We write equation (6.4) in the form of (6.1) with $k = 1$, where

$$\mathbf{x} = \begin{pmatrix} \xi \\ \dot{\xi} \end{pmatrix}, \quad \mathbf{H}(t) = \begin{Vmatrix} p(t) & 0 \\ 0 & 1 \end{Vmatrix}, \quad \mathbf{J} = \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix}.$$

Let $p_1(t)$, $p_2(t)$ be two functions and $\mathbf{H}_1(t)$, $\mathbf{H}_2(t)$ two Hamiltonians of the above form for $p = p_1$, $p = p_2$. It is clear that the matrix inequality $\mathbf{H}_1(t) \leqslant \mathbf{H}_2(t)$ is equivalent to the scalar inequality $p_1(t) \leqslant p_2(t)$.

The Hamiltonian $\mathbf{H}_1(t) + s[\mathbf{H}_2(t) - \mathbf{H}_1(t)]$ determines an equation (6.4) with the function $p(t, s) = p_1(t) + s[p_2(t) - p_1(t)]$. In terms of equation (6.4), the directional wideness property (which we have yet to prove!) may be rephrased as follows. If equation (6.4) with the function $p(t, s) = sp_2(t) + (1-s)p_1(t)$ is strongly stable for any $0 \leqslant s \leqslant 1$, then it is strongly stable with any function $p(t)$ such that $p_1(t) \leqslant p(t) \leqslant p_2(t)$. For example, let $p_1(t)$

and $p_2(t)$ be constants: $p_1(t)=c_1^2$, $p_2(t)=c_2^2$. Equation (6.4) with constant coefficient $p(t)=c^2\geqslant 0$ has the solutions $\xi_1=\exp(ict)$, $\xi_2=\exp(-ict)$. Thus its multipliers are $\rho_1=\exp(icT)$, $\rho_2=\exp(-icT)$. By Lemma II in §2.6, one of them is a multiplier of the first kind, the other of the second kind. These coincide only when $cT=n\pi$ $(n=0, \pm 1, \pm 2, \ldots)$. If $cT\neq n\pi$, i. e., $n\frac{n^2\pi^2}{T^2}<c^2<\frac{(n+1)^2\pi^2}{T^2}$ for some $\sim$, then equation (6.4) with $p(t)\equiv c^2$ is strongly stable (§1.4). Consequently, equation (6.4) with $p(t, s)=c_1^2+s(c_2^2-c_1^2)$ is strongly stable for all $0\leqslant s\leqslant 1$ if and only if

$$\frac{n^2\pi^2}{T^2}<c_1^2\leqslant c_2^2<\frac{(n+1)^2\pi^2}{T^2}$$

for some $n=0, 1, 2, \ldots$ Since stability domains are directionally wide, it follows that equation (6.4) is strongly stable if

$$\frac{n^2\pi^2}{T^2}<c_1^2\leqslant p(t)\leqslant c_2^2<\frac{(n+1)^2\pi^2}{T^2} \tag{6.5}$$

for some $n=0, 1, 2, \ldots$ The numbers c_1^2, c_2^2 may be as near as desired to $n^2\pi^2/T^2$ and $(n+1)^2\pi^2/T^2$. As a matter of fact, the equation is strongly stable if

$$\frac{n^2\pi^2}{T^2}\leqq p(t)\leqq\frac{(n+1)^2\pi^2}{T^2}. \tag{6.6}$$

This stability test, due to Zhukovskii /147/, will be verified in § VIII. 3.3. We have shown how the directional wideness property yields a test almost as good as Zhukovskii's test. It will be used below in similar fashion to establish various stability tests for the general equation (6.1).

6.2. Directional wideness of stability domains

Theorem. The stability domains are directionally wide. In other words, if the equation

$$\mathbf{J}\frac{d\mathbf{x}}{dt}=\mathbf{H}(t, s)\mathbf{x}, \tag{6.7}$$

where $\mathbf{H}(t, s)=\mathbf{H}_1(t)+s[\mathbf{H}_2(t)-\mathbf{H}_1(t)]$, *is strongly stable for any* $0\leqslant s\leqslant 1$, *then equation (6.1) is stable for any Hamiltonian* $\mathbf{H}(t)$ *satisfying inequalities (6.3).*

Proof. Consider the triangle $\mathbf{H}_1\mathbf{H}\mathbf{H}_2$ in the Hamiltonian space $\mathfrak{L}$, that is to say, the set of all Hamiltonians $\mathbf{H}'=\tau_1\mathbf{H}_1+\tau_2\mathbf{H}_2+\tau\mathbf{H}$, where $\tau_1+\tau_2+\tau=1$, $\tau_1\geqslant 0$, $\tau_2\geqslant 0$, $\tau\geqslant 0$. (Henceforth we shall sometimes omit the argument t of the Hamiltonian, for the sake of brevity.) We shall prove that all Hamiltonians in this triangle, including $\mathbf{H}$, are strongly stable.

We shall continuously deform the triangle $\mathbf{H_1HH_2}$, contracting it to its base $\mathbf{H_1H_2}$. Consider the set of triangles $\Delta(\mu)$ with base $\mathbf{H_1H_2}$ and apex at the point

$$\mathbf{H}(\mu)=\tfrac{1}{2}(\mathbf{H}_1+\mathbf{H}_2)+\mu\left[\mathbf{H}-\tfrac{1}{2}(\mathbf{H}_1+\mathbf{H}_2)\right] \qquad (0\leqslant\mu\leqslant 1) \tag{6.7a}$$

(see Figure 19). We have $\mathbf{H}(1)=\mathbf{H}$, $\mathbf{H}(0)=\frac{1}{2}(\mathbf{H}_1+\mathbf{H}_2)$. Therefore the triangle $\Delta(1)$ is $\mathbf{H_1HH_2}$, and $\Delta(0)$ degenerates into the base $\mathbf{H_1H_2}$. By assumption, all points of the bases $\mathbf{H_1H_2}$ are strongly stable Hamiltonians. (Recall that $\mathbf{H_1H_2}$ is the set of all Hamiltonians $\mathbf{H}(t, s)=s\mathbf{H}_2(t)+(1-s)\mathbf{H}_1(t)$, $0\leqslant s\leqslant 1$.) Since the set of strongly stable Hamiltonians is open (see the definition of strong stability in §5.1), all triangles sufficiently near the base (i.e., those with small $\mu>0$) are also strongly stable. *

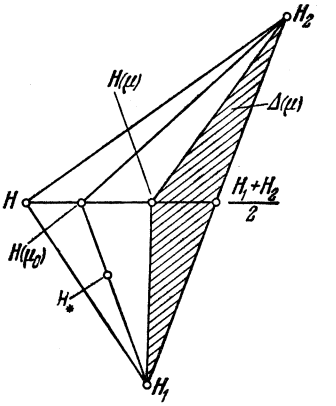


FIGURE 19.

There are two possibilities: (a) all triangles $\Delta(\mu)$ $(0\leqslant\mu\leqslant 1)$ are strongly stable (so that the crucial triangle $\Delta(1)$ is also strongly stable); (b) there exists μ_0 $(0<\mu_0<1)$ such that the triangle $\Delta(\mu_0)$ has at least one point $\mathbf{H}_*$ on one of its lateral sides $\mathbf{H_1H}(\mu_0)$, $\mathbf{H}(\mu_0)\mathbf{H_2}$ which is not strongly stable, but all triangles $\Delta(\mu)$, $0\leqslant\mu<\mu_0$, are strongly stable. In the first case, we are done. We therefore consider the second case.

Consider any triangle $\Delta(\mu)$ with sides $\mathbf{H_1H}(\mu)$ and $\mathbf{H}(\mu)\mathbf{H_2}$. Let $\mathbf{H}(\lambda, \mu)$ be a Hamiltonian which ranges over the side $\mathbf{H_1H}(\mu)$ when λ varies from $\lambda=0$ to $\lambda={}^1/_2$, and over the side $\mathbf{H}(\mu)\mathbf{H_2}$ when λ varies from $\lambda={}^1/_2$ to $\lambda=1$:

$$\mathbf{H}(\lambda,\ \mu)=\begin{cases}\mathbf{H}_1+2\lambda[\mathbf{H}(\mu)-\mathbf{H}_1] & (0\leqslant\lambda\leqslant{}^1/_2),\\ \mathbf{H}(\mu)+2\left(\lambda-\frac{1}{2}\right)[\mathbf{H}_2-\mathbf{H}(\mu)] & ({}^1/_2\leqslant\lambda\leqslant 1).\end{cases}$$

* That is to say, all its points are strongly stable Hamiltonians.

We claim that for each of these expressions $\partial \mathbf{H}/\partial\lambda \geqslant 0$, i. e.,

$$\mathbf{H}_1 \leqslant \mathbf{H}(\mu) \leqslant \mathbf{H}_2 \qquad (0 \leqslant \mu \leqslant 1). \tag{6.8}$$

Replacing $\mathbf{H}$ in (6.7a) by $\mathbf{H}_1$ and $\mathbf{H}_2$ and using (6.3), we obtain

$$\tfrac{1}{2}(\mathbf{H}_1+\mathbf{H}_2)-\tfrac{1}{2}\mu(\mathbf{H}_2-\mathbf{H}_1) \leqslant \mathbf{H}(\mu) \leqslant \tfrac{1}{2}(\mathbf{H}_1+\mathbf{H}_2)+\tfrac{1}{2}\mu(\mathbf{H}_2-\mathbf{H}_1).$$

Since the coefficient of μ in both these inequalities is nonnegative, they remain valid if we replace μ on both sides by 1. This gives (6.8).

Fix an arbitrary μ in the interval $0 \leqslant \mu < \mu_0$. By the definition of μ_0, the Hamiltonian $\mathbf{H}(\lambda, \mu) = \mathbf{H}(t; \lambda, \mu)$ is strongly stable for any $0 \leqslant \lambda \leqslant 1$, so that all the corresponding multipliers lie on the unit circle and there are no coincident multipliers of different kinds. In particular (see Lemma III of §2.6), none of them are equal to 1 or -1.

We shall show that there are disjoint arcs on the upper unit semicircle, each of which contains only multipliers of one kind for any Hamiltonian $\mathbf{H} \in \Delta(\mu)$, $0 \leqslant \mu < \mu_0$, and conversely, every multiplier lies on one of these distinguished arcs.

Consider an equation with Hamiltonian $\mathbf{H}(\lambda, \mu) \in \Delta(\mu)$ $(0 \leqslant \mu < \mu_0)$. Number its multipliers in order of appearance on the upper unit semicircle, from $\rho = 1$ to $\rho = -1$. Assign a plus sign to each multiplier of the first kind and a minus sign to each multiplier of the second kind. Thus, the multipliers of the equation with Hamiltonian $\mathbf{H}(\lambda, \mu) \in \Delta(\mu)$ $(0 \leqslant \mu < \mu_0,\ 0 \leqslant \lambda \leqslant 1)$ will be denoted by $\rho_j^{\pm}(\lambda, \mu)$. We have $|\rho_j^{\pm}(\lambda, \mu)| = 1$, $\rho_j^{\pm}(\lambda, \mu) \neq \pm 1$, $\rho_j^{+}(\lambda, \mu) \neq \rho_k^{-}(\lambda, \mu)$ for $0 \leqslant \mu < \mu_0$, $0 \leqslant \lambda \leqslant 1$.

Consider the equation with Hamiltonian $\mathbf{H}_1 = \mathbf{H}(0, 0)$ and multipliers $\rho_j^{\pm} = \rho_j^{\pm}(0, 0)$. Divide the multipliers into groups so that consecutive multipliers of the same kind are in the same group. For example, let $k = 9$, and suppose the multipliers are arranged as shown in Figure 20. We then have four groups:

$$\{\rho_1^+, \rho_2^+, \rho_3^+\},\quad \{\rho_4^-, \rho_5^-, \rho_6^-,\},\quad \{\rho_7^+\},\quad \{\rho_8^-, \rho_9^-\}.$$

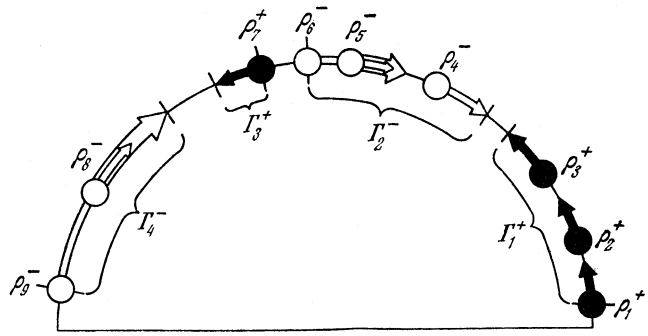


FIGURE 20.

We now begin to deform the Hamiltonians, "moving" along the side $\mathbf{H}_1\mathbf{H}_2$. That is, we consider the Hamiltonian $\mathbf{H}(\lambda, 0) = \mathbf{H}_1 + \lambda(\mathbf{H}_2 - \mathbf{H}_1)$ with multipliers $\rho_j^{\pm}(\lambda, 0)$. Since $\partial\mathbf{H}(\lambda, 0)/\partial\lambda = \mathbf{H}_2 - \mathbf{H}_1 \geqslant 0$, it follows from the results of §1.7 that as λ varies from $\lambda = 0$ to $\lambda = 1$ the multipliers $\rho_j^+(\lambda, 0)$ of the first kind move along the unit circle in the sense of increasing argument, and the multipliers $\rho_j^-(\lambda, 0)$ of the second kind do so in the sense of decreasing argument. Since the segment $\mathbf{H}_1\mathbf{H}_2$ is strongly stable, at no time during this "motion" do multipliers of different kinds meet. Thus, we have an analogous grouping of the multipliers $\rho_j^{\pm}(\lambda, 0)$ for any $0 \leqslant \lambda \leqslant 1$:

$$\{\rho_1^+(\lambda, 0), \rho_2^+(\lambda, 0), \rho_3^+(\lambda, 0)\},$$
$$\{\rho_4^-(\lambda, 0), \rho_5^-(\lambda, 0), \rho_6^-(\lambda, 0)\}, \quad \{\rho_7^+(\lambda, 0)\},$$
$$\{\rho_8^-(\lambda, 0), \rho_9^-(\lambda, 0)\}.$$

Consequently, throughout the interval $0 \leqslant \lambda \leqslant 1$ the multipliers of the first group lie on the closed arc Γ_1^+ with endpoints $\rho_1^+(0, 0)$ and $\rho_3^+(1, 0)$, those of the second group on the arc Γ_2^- with endpoints $\rho_4^-(1, 0)$ and $\rho_6^-(0, 0)$, those of the next two groups on the arcs

$$\Gamma_3^+ = [\rho_7^+(0, 0), \rho_7^+(1, 0)], \quad \Gamma_4^- = [\rho_8^-(1, 0), \rho_9^-(0, 0)].$$

The behavior of the multipliers is analogous when the Hamiltonian is deformed along the sides $\mathbf{H}_1\mathbf{H}(\mu)\mathbf{H}_2$ for $0 \leqslant \mu < \mu_0$ (as long as the equation remains strongly stable), i. e., for Hamiltonians $\mathbf{H}(\lambda, \mu)$ with fixed μ $(0 \leqslant \mu < \mu_0)$ and λ varying from 0 to 1. Indeed, we showed above that $\partial\mathbf{H}(\lambda, \mu)/\partial\lambda \geqslant 0$, for both $0 \leqslant \lambda < 1/2$ and $1/2 < \lambda \leqslant 1$, and hence, by virtue of strong stability, it follows that as λ varies from 0 to 1 the multipliers $\rho_j^{\pm}(\lambda, \mu)$ move along the unit circle in the sense of increasing argument $(\rho_j^+(\lambda, \mu))$ and decreasing argument $(\rho_j^-(\lambda, \mu))$, and moreover multipliers of different kinds never meet. Throughout the process, the arcs $\Gamma_j^{\pm}$ remain the same, for they are determined by the "endpoints" $\mathbf{H}(0, \mu) = \mathbf{H}_1$, $\mathbf{H}(1, \mu) = \mathbf{H}_2$, which are independent of μ.

We have thus shown that for any equation with $\mathbf{H} \in \Delta(\mu)$ $(0 \leqslant \mu < \mu_0)$ the multipliers of the first kind lie on the closed arcs Γ_j^+, those of the second kind on closed arcs Γ_j^-, and the arcs Γ_j^+ and Γ_j^- are disjoint. This proves the assertion made above.

Since $\Gamma_j^{\pm}$ are disjoint closed arcs, it follows that for any Hamiltonian $\mathbf{H} \in \Delta(\mu)$ $(0 \leqslant \mu < \mu_0)$ the distance between multipliers of the first and second kinds is bounded below by some positive number.

On the other hand, by assumption there is a Hamiltonian $\mathbf{H}_*$ on the boundary of $\Delta(\mu_0)$ which is not strongly stable. Hence this Hamiltonian has coincident multipliers of different kinds. But multipliers of the first and second kinds depend continuously on the Hamiltonian (§2.10). Letting $\mathbf{H} \to \mathbf{H}_*$ within the triangle $\Delta(\mu_0)$, we see that multipliers of different kinds must meet in the limit. But this is impossible since the distance between different kinds is bounded away from zero. Thus the second possibility cannot occur, and the proof is complete.

Remark 1. For fixed $\mathbf{H}_1(t)$ and $\mathbf{H}_2(t)$, the Hamiltonians $\mathbf{H}(t)$ satisfying the conditions of the theorem lie in one stability domain (namely, the stability domain containing $\mathbf{H}_1(t)$ and $\mathbf{H}_2(t)$). Indeed, any Hamiltonian $\mathbf{H}(t)$ such that

$\mathbf{H}_1(t) \leqslant \mathbf{H}(t) \leqslant \mathbf{H}_2(t)$, and which is therefore strongly stable, may be joined to $\mathbf{H}_1(t)$, say, moving along the side $\mathbf{HH}_1$. (A suitable deformation is $\mathbf{H}(t, s) = s\mathbf{H}_1(t) + (1-s)\mathbf{H}(t)$.) This means that the Hamiltonians $\mathbf{H}(t)$ and $\mathbf{H}_1(t)$ are in the same stability domain.

Remark 2. The theorem remains valid for equation (6.2), when $\mathbf{J}'$ is an arbitrary nonsingular real skew-symmetric matrix. Indeed, the proof makes no use of the explicit form of the matrix $\mathbf{J}$. The same conclusion may be reached via the substitution at the beginning of §6.1, reducing equation (6.2) to equation (6.1).

Remark 3. The theorem remains valid for complex Hamiltonian equations

$$i^{-1}\mathbf{G}\frac{dz}{dt} = \mathbf{H}(t)z,$$

where $\mathbf{G}^* = \mathbf{G}$, $\det \mathbf{G} \neq 0$, $\mathbf{H}(t)^* = \mathbf{H}(t)$ are complex matrices. The proof is exactly the same except that instead of the upper unit semicircle one considers the whole unit circle.

6.3. Directional convexity of certain subsets of stability domains

In this subsection we shall discuss certain more discriminating properties of stability domains.

Definition 1. A subset $\mathfrak{M} \subset \mathfrak{L}$ is said to be directionally convex* if the conditions

$$\mathbf{H}_1(t) \in \mathfrak{M}, \quad \mathbf{H}_2(t) \in \mathfrak{M}, \quad \mathbf{H}_1(t) \leqslant \mathbf{H}(t) \leqslant \mathbf{H}_2(t),$$

imply that $\mathbf{H}(t) \in \mathfrak{M}$.

This definition recalls the definitions of both convexity and directional wideness. Unlike a convex set, in a directionally convex set the segment $\mathbf{H}_1\mathbf{H}_2$ joining two points $\mathbf{H}_1$ and $\mathbf{H}_2$ of the set does not always lie entirely in the set, unless $\mathbf{H}_1 \geqslant \mathbf{H}_2$ (or $\mathbf{H}_1 \leqslant \mathbf{H}_2$) — an additional condition characterizing the direction of the segment. The distinction from a directionally wide set is that in the latter the set $\mathbf{H}_1 \leqslant \mathbf{H} \leqslant \mathbf{H}_2$ lies in $\mathfrak{M}$ if the segment $\mathbf{H}_1\mathbf{H}_2$ is in $\mathfrak{M}$, whereas in a directionally convex set it is sufficient that the endpoints $\mathbf{H}_1$ and $\mathbf{H}_2$ be in $\mathfrak{M}$. Any directionally convex set is clearly also directionally wide, but the converse is generally untrue: the set $\mathfrak{M}_1 \cup \mathfrak{M}_2$ in Figure 18 (§6.1) is directionally wide, since it is the union of disjoint directionally wide sets, but it is not directionally convex.

*Theorem I. Any stability domain $\mathfrak{G}_n^{(\mu)}$ with the property that no multipliers of different kinds alternate on the upper unit semicircle is directionally convex.***

* [Or (t)-convex.]

** "No alternating multipliers of different kinds" means that either all multipliers on the upper unit semicircle are of the same kind or as we traverse the semicircle from $\rho = 1$ to $\rho = -1$ we first encounter all the multipliers of one kind and then all those of the other.

Corollary. If $k=1, 2$, all stability domains $\mathcal{G}_n^{(\mu)}$ are directionally convex.

The next theorem, which is much more complicated, implies Theorem I as a special case.

Theorem II. Let Γ_j be r disjoint open arcs on the upper unit semicircle. Let $\mathcal{G}_0$ denote the set of all Hamiltonians $\mathbf{H}(t)$ in a fixed stability domain $\mathcal{G}_n^{(\mu)}$ which have the following property. The corresponding equation (6.1) has on the arc Γ_j exactly $m_j^+ \geqslant 0$ multipliers of the first kind and exactly $m_j^- \geqslant 0$ of the second, arranged so that there are no alternations of multipliers of different kinds on the arc and no coincidences of multipliers of different kinds. Moreover, there are no multipliers outside these distinguished arcs, so that $m_1^+ + m_1^- + \ldots + m_r^+ + m_r^- = k$. The numbers m_j^+, m_j^- and the sequence of the different kinds of multipliers on each arc Γ_j are the same for all $\mathbf{H}(t) \in \mathcal{G}_0$.

Then $\mathcal{G}_0$ is directionally convex.

Remark. Theorem I is the special case of Theorem II with $r=1$ and Γ_1 the open upper unit semicircle.

The proof of Theorem II will employ the results of §5.5. *

As before, let $\mathfrak{L}=\{\mathbf{H}(t)\}$ be the set of all real Hamiltonians with the metric (5.2), $\mathbf{X}=\Phi[\mathbf{H}(t)]$ the corresponding monodromy matrix, $\zeta=\zeta[\mathbf{X}]$ its labeled spectrum, i. e., the set of multipliers of the corresponding equation, labeled by their kinds. The mappings $\mathbf{X}=\Phi[\mathbf{H}(t)]$, $\zeta=\zeta[\mathbf{X}]$ are continuous. We let $\zeta[\mathbf{H}(t)]=\zeta[\Phi\{\mathbf{H}(t)\}]$ denote the labeled spectrum of the equation with Hamiltonian $\mathbf{H}(t)$ (the image of $\mathbf{H}(t)$ in Σ under the above mapping).

Let $\zeta(s)$ be a continuous curve in Σ. We shall say that $\zeta(s)$ is increasing for $s_1<s<s_2$ if the arguments of all its multipliers of the first kind on the unit circle which do not coincide with multipliers of the second kind are nondecreasing functions of s $(s_1<s<s_2)$ (i. e., the multipliers of the first kind move along the circle in the positive sense or remain in position; the multipliers of the second kind will then move in the opposite sense). A more rigorous definition is as follows.

Definition 2. Let $\zeta(s) \in \Sigma$ be a continuous curve. Let $\rho_0 = \exp i\varphi_0$ be a multiplier of the first kind of multiplicity $p \geqslant 1$ in the spectrum $\zeta(s_0)$, and suppose that there are no multipliers of the second kind in $\zeta(s_0)$ equal to ρ_0. Suppose that for any $\varepsilon>0$ there exists $\delta>0$ such that if $s_0 \leqslant s \leqslant s_0+\delta$ the spectrum $\zeta(s)$ has exactly p multipliers (automatically of the first kind) on the closed subarc $[\exp i\varphi_0, \exp i(\varphi_0+\varepsilon)]$ of the unit circle. If this is true for any $s_0 \in (s_1, s_2)$ and any ρ_0 as described, then $\zeta(s)$ is an increasing curve for $s_1<s<s_2$.

Lemma. Let $\mathfrak{M} \subset \mathfrak{L}$ be a domain and $\hat{\mathfrak{M}}$ its image in Σ under the mapping (5.18). Let $\zeta_0(s)$ be any curve in Σ having the following properties:

1. *$\zeta_0(s)$ is an increasing curve for $0 \leqslant s \leqslant 1/2$.*
2. *$\zeta_0(s) \in \hat{\mathfrak{M}}$ for $1/2 \leqslant s \leqslant 1$.*
3. *$\zeta_0(s)$ leaves $\hat{\mathfrak{M}}$ in the interval $0<s<1/2$, i. e., $\zeta_0(s_0) \notin \hat{\mathfrak{M}}$ for some s_0 $(0<s_0<1/2)$.*

Suppose that for any curve with these properties

* The proof of Theorem II is difficult and the reader may omit it on a first reading, proceeding directly to §6.4.

$$\Delta \operatorname{Arg} \zeta_0(s) = 2m\pi \neq 0.$$

Then $\mathfrak{M}$ is directionally convex.

Proof. Let $\mathbf{H}_1(t) \in \mathfrak{M}$, $\mathbf{H}_2(t) \in \mathfrak{M}$, $\mathbf{H}_1(t) \leqslant \mathbf{H}(t) \leqslant \mathbf{H}_2(t)$. Suppose that $\mathbf{H}(t) \notin \mathfrak{M}$. Joining $\mathbf{H}_1$ linearly to $\mathbf{H}$ (for $0 \leqslant s \leqslant 1/4$) and $\mathbf{H}$ linearly to $\mathbf{H}_2$ (for $1/4 \leqslant s \leqslant 1/2$), we have

$$\mathbf{H}(t, s) = \begin{cases} (1-4s)\,\mathbf{H}_1(t) + 4s\,\mathbf{H}(t) & \text{for } 0 \leqslant s \leqslant 1/4, \\ (2-4s)\,\mathbf{H}(t) + (4s-1)\,\mathbf{H}_2(t) & \text{for } 1/4 \leqslant s \leqslant 1/2. \end{cases}$$

We have $\partial \mathbf{H}/\partial s \geqslant 0$ for $0 < s < 1/4$ and $1/4 < s < 1/2$. Using Krein's theorem (§ 1.7), we deduce that $\zeta_0(s) = \zeta[\mathbf{H}(t, s)]$ is an increasing curve for $0 < s < 1/2$. Since $\mathfrak{M}$ is a domain, there exists a curve $\mathbf{H}(t, s)$ $(1/2 \leqslant s \leqslant 1)$ joining $\mathbf{H}_2$ in $\mathfrak{M}$ to $\mathbf{H}_1$. The corresponding curve $\zeta_0(s) = \zeta[\mathbf{H}(t, s)]$ $(1/2 \leqslant s \leqslant 1)$ joins the spectrum $\zeta_2 = \zeta[\mathbf{H}_2]$ to $\zeta_1 = \zeta[\mathbf{H}_1]$ in $\hat{\mathfrak{M}}$. Since $\mathbf{H}(t) \notin \mathfrak{M}$, it follows that $\zeta_0\left(\frac{1}{2}\right) \notin \hat{\mathfrak{M}}$. Thus $\zeta_0(s)$ is a closed curve satisfying conditions 1, 2, 3 of the lemma. Therefore $\Delta \operatorname{Arg} \zeta_0(s) = 2\pi m \neq 0$. Consequently, the curve $\zeta_0(s)$ cannot be contracted continuously to a point. (Proof: otherwise, $\Delta \operatorname{Arg} \zeta(s)$ would remain invariant under the relevant deformation; if $\zeta(s) \equiv \text{const}$, it is clear that $\Delta \operatorname{Arg} \zeta(s) \equiv 0$. Thus, if $\zeta(s)$ could be contracted to a point it would follow that $\Delta \operatorname{Arg} \zeta(s) = 0$.)

On the other hand, $\mathfrak{L}$, as a linear space, is simply-connected — any closed curve in $\mathfrak{L}$ may be contracted to a point. Contracting the closed curve $\mathbf{H}(t, s)$ $(0 \leqslant s \leqslant 1)$ to a point, we obtain a deformation contracting the curve $\zeta_0(s) = \zeta[\mathbf{H}(t, s)]$ to a point. This contradiction proves the lemma.

Proof of Theorem II. We divide the proof into several steps.

Step 1. We claim that $\hat{\mathcal{O}}_0$ is a domain. Let $\hat{\mathcal{O}}_0$ denote the set of labeled spectra defined in the theorem — the set of labeled spectra having exactly m_j^+ multipliers of the first kind and m_j^- of the second kind on Γ_j, such that there are no coincident multipliers of different kinds and multipliers of different kinds do not alternate on Γ_j: either Γ_j contains multipliers of one kind only ($m_j^+ = 0$ or $m_j^- = 0$) or when the arc is described in the sense of increasing argument we first encounter all multipliers of one kind and then all those of the other. In addition, $m_1^+ + m_1^- + \ldots + m_r^+ + m_r^- = k$, i. e., all multipliers of $\zeta \in \hat{\mathcal{O}}_0$ lie on the unit circle. It is easy to see that case (a) of §5. 5 holds: for any closed curve $\zeta(t) \in \hat{\mathcal{O}}_0$ $(0 \leqslant t \leqslant 1)$ we have $\Delta \operatorname{Arg} \zeta(t) = 0$. Indeed, as t varies from 0 to 1, each group of multipliers of one kind on each arc Γ_j, say the group $\rho_j(t)$ $(j = 1, \ldots, m_1^+)$ of multipliers of the first kind on Γ_1, moving along some arc $\Gamma_j^+ \subset \Gamma_j$, is mapped onto itself: $\rho_j(1) = \rho_{h_j}(0)$, where $(h_1, \ldots, h_{m_1})$ is some permutation of $(1, \ldots, m^+)$. Thus, for each of these groups,

$$\sum \Delta \operatorname{Arg} \rho_j(t) = \sum_j \arg \rho_j(1) - \sum \arg \rho_{h_j}(0) = 0.$$

Consequently, $\Delta \operatorname{Arg} \zeta(t) = 0$.

By Theorem 2 of §5.5, the pre-image of $\hat{\mathcal{O}}_0$ in $\mathfrak{L}$ is a countable union of domains $\mathcal{O}_0^{(n)}$, each the set of all Hamiltonians of index n with labeled

spectrum $\zeta \in \hat{\mathcal{G}}_0$. One of these sets is a subset of the stability domain under consideration. Since $\mathcal{G}_0$ is the set of all Hamiltonians in $\mathcal{G}_n^{(\mu)}$ with $\zeta \in \hat{\mathcal{G}}_0$, the indices of the Hamiltonians in $\mathcal{G}_0$ are all equal to the relevant number n. Thus $\mathcal{G}_0 = \mathcal{G}_0^{(n)}$ and so $\mathcal{G}_0$ is a domain.

We now apply the lemma, with $\mathfrak{M} = \mathcal{G}_0$, $\hat{\mathfrak{M}} = \hat{\mathcal{G}}_0$.

Step 2. Let $\zeta(s)$ be a closed curve as stipulated in the lemma. It may be deformed in such a way that all its multipliers move only along the unit circle. Indeed, suppose that for $s_1 < s < s_2$ the curve $\zeta(s)$ has a multiplier of the first kind $|\rho^+(s)| < 1$, $|\rho^+(s_1)| = 1$, $|\rho^+(s_2)| = 1$ (clearly, $0 \leqslant s_1 \leqslant s_2 \leqslant 1/2$). Deform the curve $\rho^+(s)$ $(s_1 < s < s_2)$ continuously into a curve lying along the unit circle. Repeating this procedure, we obtain a curve $\zeta(s)$ as required. This curve will have multiple multipliers of different kinds, $\rho^+(s) = \rho^-(s)$ $(s_1 \leqslant s \leqslant s_2)$, and will also be increasing for $0 \leqslant s \leqslant 1/2$.

Let us say that $\zeta(s)$ has two paired multipliers at ρ_0 for $s = s_0$ if for all s sufficiently close to s_0 it has multiple multipliers of different kinds: $\rho^+(s) = \rho^-(s) \equiv \rho(s)$, $|\rho(s)| \equiv 1$ and $\rho(s_0) = \rho_0$.

Thus, if two multipliers of different kinds coincide at a point ρ_0 $(|\rho_0| = 1)$ at some "time" $s = s_0$, but move apart at the "preceding" or "next" instant of time, they are not paired.

Suppose that p multipliers of the first kind and q of the second coincide at $s = s_0$ at a point ρ_0 $(|\rho_0| = 1)$, and that, of these, m multipliers of the first kind and m of the second $(0 \leqslant m \leqslant \min\{p, q\})$ remain multiple for all s sufficiently near s_0. We shall then say that at the point ρ_0 there are m paired multipliers, $p - m$ unpaired multipliers of the first kind and $q - m$ of the second.

Step 3. Let $\varphi_j(s)$ and $\psi_j(s)$ be the arguments of the multipliers of the first and second kinds, respectively, for the curve $\zeta(s)$, defined by continuity. We can represent $\zeta(s)$ graphically by constructing k graphs $\varphi = \varphi_j(s)$ and k graphs $\varphi = \psi_j(s)$ $(0 \leqslant s \leqslant 1)$ (Figure 21). If necessary, we extend the open arcs Γ_j in such a way that, though remaining disjoint, the union of their closure is the entire upper unit semicircle. Let $[\Psi_1, \Psi_2]$ be any of the new arcs thus obtained. On the $s\varphi$-plane, this arc defines a rectangle $0 \leqslant s \leqslant 1$, $\Psi_1 \leqslant \varphi \leqslant \Psi_2$. We shall say that there is a multiplier of the first (second) kind at a point $M(s, \varphi)$ if the graph of one of the functions $\varphi = \varphi_j(s)$ $(\varphi = \psi_j(s))$ passes through the point. Thus we may represent the behavior of the multipliers, moving in the $s\varphi$-plane, by the corresponding graph.

We shall assume that the graphs of the functions $\varphi = \varphi_j(s)$, $\varphi = \psi_j(s)$ have only finitely many points in common with the lines $\varphi = \Psi_1$ and $\varphi = \Psi_2$. This may always be guaranteed by a suitable small deformation of the graphs (i. e., of the curve $\zeta(s)$). Moreover, this deformation may clearly be chosen so that the curve $\zeta(s)$ is again increasing for $0 \leqslant s \leqslant 1/2$.

Thus we may legitimately speak of the number of multipliers on the lines $\varphi = \Psi_1$ and $\varphi = \Psi_2$.

Note that, by assumption, the functions $\varphi = \varphi_j(s)$ are nondecreasing on the subintervals of $0 \leqslant s \leqslant 1/2$ where $\varphi_j(s) \neq \psi_h(s)$ (solid curves in Figure 21) and the functions $\psi_j(s)$ are nonincreasing whenever $\psi_j(s) \neq \varphi_h(s)$ (dashed curves in Figure 21). By condition 2 of the lemma, when $1/2 < s < 1$ and graphs of $\psi_j(s)$ and $\varphi_h(s)$ do not intersect: $\psi_j(s) \neq \varphi_h(s)$.

Step 4. We claim that at least one of our r rectangles has an unpaired multiplier on its side $\varphi = \Psi_1$ or $\varphi = \Psi_2$.

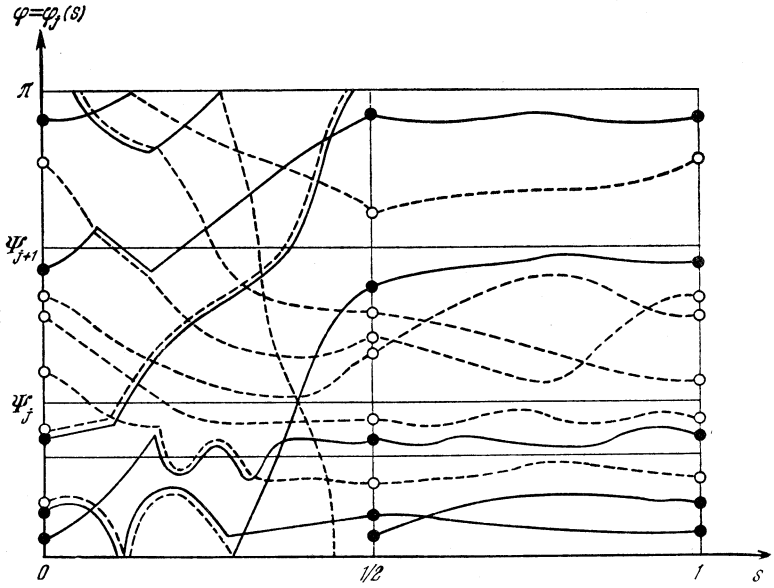


FIGURE 21.

Suppose the contrary: on both sides $\varphi=\Psi_1$ and $\varphi=\Psi_2$ of each rectangle there are either no multipliers at all or only paired multipliers. There is a rectangle in whose interior multipliers of different kinds come together (otherwise, $\zeta(s)\in\mathfrak{M}$ for $0\leqslant s\leqslant 1/2$, contrary to condition 3 of the lemma). Subsequently, these multipliers either move apart or form paired multipliers. The paired multipliers will either leave the rectangle across one of the lines $\varphi=\Psi_1$, Ψ_2 or ultimately move apart. After moving apart, they cannot reach the line $s=1/2$, for then the order of the first and second kinds there would be different from that on $s=0$. Thus the paired multipliers necessarily leave the rectangle across the sides $\varphi=\Psi_1$, Ψ_2. Since the total number of multipliers on the sides $s=0$, $s=1/2$ is the same, the same number of pairs must enter the rectangle across these lines. Connecting each arriving pair within the rectangle to a leaving pair, we obtain a figure in which multipliers cannot leave the rectangle, and therefore must move apart at some time so as to form the multiplier system on $s=1/2$. But we have already seen that paired multipliers cannot reach the line $s=1/2$ after moving apart. This contradiction proves our assertion.

Step 5. Let m_1 denote the number of unpaired multipliers on $\varphi=\Psi_1$, and m_2 the number on $\varphi=\Psi_2$. It follows from Step 4 that at least one of these numbers is positive. We claim that $m_1=m_2$.

To fix ideas, suppose there is a multiplier of the first kind on $\varphi=\Psi_2$. We shall associate it with a certain unpaired multiplier on $\varphi=\Psi_1$. A multiplier of the first kind on $\varphi=\Psi_2$ will invariably leave the arc in question.

(The function $\varphi=\varphi_j(s)$ is nondecreasing if no multipliers of different kinds coincide.) Since we must have the same number of multipliers of the first kind at $s=0$ and at $s=1/2$, it follows that there is an "incoming" multiplier of the first kind on the lines $\varphi=\Psi_1$ or $\varphi=\Psi_2$. This multiplier is either an unpaired multiplier on $\varphi=\Psi_1$ (unpaired multipliers cannot enter the rectangle across the line $\varphi=\Psi_2$) or one of a pair with a multiplier of the second kind. In the first case we are done. In the second, there is an "outgoing" multiplier of the second kind. The assertion is proved if this multiplier is on the line $\varphi=\Psi_1$ and is unpaired (unpaired multipliers of the second kind cannot enter the rectangle across $\varphi=\Psi_2$). Otherwise, it is paired with a multiplier of the first kind. Since the number of multipliers on the line $\varphi=\Psi_1$ and $\varphi=\Psi_2$ is finite, it follows that there is an unpaired multiplier on $\varphi=\Psi_1$. Deleting these "passed" multipliers and continuing the reasoning, we obtain $m_1=m_2$.

Step 6. The number of unpaired multipliers on all the lines Ψ_1 and Ψ_2 is the same, and even: $m_1=m_2=2m>0$. We saw in Step 5 that the number $m_1=m_2$ is the same for all arcs. The multipliers of both kinds on $\varphi=0$ (or $\varphi=\pi$) must be multiple (since the numbers of multipliers of either kind at the points $\rho=\pm 1$ are the same). Consequently, $m_1=m_2=2m$. As shown previously, $2m>0$.

Step 7. We now show that as the multipliers move along the arc (Ψ_1, Ψ_2) on the unit circle the increment of $\operatorname{Arg}\zeta(s)$ is $2m(\Psi_1-\Psi_2)$.

We first note that

$$\Delta \operatorname{Arg}\zeta(s)=\sum_j \Delta \operatorname{Arg}\rho_j^+(s)-\sum_j \Delta \operatorname{Arg}\rho_j^-(s),$$

where ρ_j^+ are the multipliers of the first kind and ρ_j^- those of the second kind on the upper unit semicircle.

Let

$$\rho_j'^+=\exp(i\varphi_j'), \quad \rho_j'^-=\exp(i\psi_j')$$

be the multipliers of the first and second kinds on the arc (Ψ_1, Ψ_2) at $s=0$ (hence also at $s=1$), and

$$\rho_j''^+=\exp(i\varphi_j''), \quad \rho_j''^-=\exp(i\psi_j'')$$

the same for $s=1/2$.

If a multiplier $\rho_j'^+$ of the first kind from the line $s=0$ becomes a multiplier $\rho_{j_1}''^+$ on $s=1/2$ and subsequently a multiplier $\rho_j'^+$ on $s=1$, the corresponding increment is

$$(\varphi_{j_1}''-\varphi_j')+(\varphi_{j_1}'-\varphi_{j_1}'')=\varphi_{j_1}'-\varphi_j'.$$

If a multiplier $\rho_j'^+$ of the first kind from $s=0$ reaches the line $\varphi=\Psi_2$, the increment is

$$\Psi_2-\varphi_j'.$$

Proceeding in this way, we see that in order to calculate the increment of the argument we must assign a term for each multiplier on the perimeter of the rectangle

$$0 \leqslant s \leqslant 1, \quad \Psi_1 \leqslant \varphi \leqslant \Psi_2$$

according to the following rules.

On $s=0$: for a multiplier $\rho_j'^{+}$ we have a term $(-\varphi_j)$, and for ρ_j' a term ψ_j.

On $s=1$: for a multiplier $\rho_j'^{+}$ a term φ_j, for a multiplier ρ_j' a term $(-\psi_j)$. All these terms cancel out.

On the line $\varphi = \Psi_1$:

	1st kind	2nd kind
outgoing multiplier	Ψ_1	$-\Psi_1$
incoming multiplier	$-\Psi_1$	Ψ_1

On $\varphi = \Psi_2$:

	1st kind	2nd kind
outgoing multiplier	Ψ_2	$-\Psi_2$
incoming multiplier	$-\Psi_2$	Ψ_2

Thus, for each paired multiplier on $\varphi = \Psi_1$ or on $\varphi = \Psi_2$ (two outgoing or two incoming multipliers of first and second kind) we have a vanishing term. For each unpaired multiplier on $\varphi = \Psi_2$ (outgoing multiplier of the first kind or incoming multiplier of the second) we have the term Ψ_2; for each unpaired multiplier on $\varphi = \Psi_1$ (incoming multiplier of the first kind or outgoing multiplier of the second) the term $(-\Psi_1)$.

Hence, moving along the arc (Ψ_1, Ψ_2), we obtain the increment

$$\Delta \operatorname{Arg} \zeta(s) = 2m(\Psi_2 - \Psi_1).$$

Summing over all the arcs, we obtain

$$\Delta \operatorname{Arg} \zeta(s) = 2\pi m \neq 0.$$

Applying the lemma, we complete the proof of Theorem II.

Remark. It follows from the proof that the set $\mathcal{O}_0$ in Theorem II may also be defined as the set of all Hamiltonians with index n and the multiplier type described in Theorem II (defined by the number r, the arcs $\Gamma_1, \ldots, \Gamma_r$, and the numbers $m_1^{\pm}, \ldots, m_r^{\pm}$, each pair of which is arranged in the order dictated by the sequence of multipliers of the first and second kinds on the corresponding arc).

Theorem II yields an easy proof of the theorem of §6.2.

Theorems I and II may be applied in a more complicated situation than the theorem of §6.2, when solutions are known for the equations with Hamiltonians $\mathbf{H}_1(t)$ and $\mathbf{H}_2(t)$ but not for the Hamiltonians $\mathbf{H}(t, s) = s\mathbf{H}_2(t) + (1-s)\mathbf{H}_1(t)$ $(0 \leqslant s \leqslant 1)$. Below, however, we shall usually use only the theorem of §6.2.

6.4. Strong stability tests for almost separable canonical systems

Consider equation (6.1) with a scalar Hamiltonian $\mathbf{H}(t)=h(t)\mathbf{I}_{2k}$:

$$\mathbf{J}\frac{d\mathbf{x}}{dt}=h(t)\mathbf{x}. \tag{6.9}$$

Here $h(t)$ is a real scalar integrable piecewise-continuous scalar function of period T. Our goal is to determine the properties of stability domains for Hamiltonians of this type with various functions $h(t)$.

Equation (6.9) is directly integrable. Its matrizant and monodromy matrix are

$$\mathbf{X}(t)=\exp\left[-\mathbf{J}\int_0^t h(t_1)\,dt_1\right],\quad \mathbf{X}(T)=e^{-\tilde{h}\mathbf{J}},\quad \tilde{h}=\int_0^T h(t)\,dt. \tag{6.10}$$

Denote

$$\mathbf{F}=\begin{pmatrix}\mathbf{I}_k & \mathbf{I}_k\\ i\mathbf{I}_k & -i\mathbf{I}_k\end{pmatrix},\quad \mathbf{F}_+=\begin{pmatrix}\mathbf{I}_k\\ i\mathbf{I}_k\end{pmatrix},\quad \mathbf{F}_-=\begin{pmatrix}\mathbf{I}_k\\ -i\mathbf{I}_k\end{pmatrix}. \tag{6.10a}$$

It follows from the equality

$$\mathbf{JF}=\mathbf{F}\begin{pmatrix}-i\mathbf{I}_k & 0\\ 0 & i\mathbf{I}_k\end{pmatrix}$$

that the eigenvalues of $\mathbf{J}$ are $\pm i$, each of multiplicity k. The columns of $\mathbf{F}_-$ are eigenvectors belonging to eigenvalue i, those of $\mathbf{F}_+$ eigenvectors belonging to eigenvalue $(-i)$.

Let $\mathbf{f}_1, \ldots, \mathbf{f}_{2k}$ denote the columns of $\mathbf{F}$: $\mathbf{F}=\|\mathbf{f}_1, \ldots, \mathbf{f}_{2k}\|$. By (1.1), $\langle \mathbf{f}_h, \mathbf{f}_j\rangle = i(\mathbf{J}\mathbf{f}_h, \mathbf{f}_j)$. Consequently,

$$\|\langle \mathbf{f}_h, \mathbf{f}_j\rangle\|_1^{2k}=\|i(\mathbf{J}\mathbf{f}_h, \mathbf{f}_j)\|_1^{2k}=i\mathbf{F}^*\mathbf{J}\mathbf{F}=2\begin{pmatrix}\mathbf{I}_k & 0\\ 0 & -\mathbf{I}_k\end{pmatrix}. \tag{6.11}$$

These equalities show that the vectors $\mathbf{f}_1, \ldots, \mathbf{f}_{2k}$ form a J-orthogonal basis, the subspace spanned by $\mathbf{f}_1, \ldots, \mathbf{f}_k$ is positive definite, that spanned by $\mathbf{f}_{k+1}, \ldots, \mathbf{f}_{2k}$ negative definite. Thus the multipliers of system (6.9), or the eigenvalues of the matrix $\mathbf{X}(t)=\exp(-\tilde{h}\mathbf{J})$, are

$$\rho^{(+)}=e^{i\tilde{h}},\quad \rho^{(-)}=e^{-i\tilde{h}}.$$

$\rho^{(+)}$ is a k-fold multiplier of the first kind (corresponding to the eigensubspace spanned by $\mathbf{f}_1, \ldots, \mathbf{f}_k$) and $\rho^{(-)}$ a k-fold multiplier of the second kind (eigensubspace spanned by $\mathbf{f}_{k+1}, \ldots, \mathbf{f}_{2k}$). Equation (6.9) will be strongly stable if $\rho^{(+)}\neq\rho^{(-)}$, i. e., $\tilde{h}\neq m\pi$, where m is an integer. It follows from the form of the multipliers that the Hamiltonians of these equations may lie only in stability domains $\mathcal{G}_n^{(\mu)}$ (§5.3) for which the multiplier type (§5.2) is

or

$$\mu=(+,\ +,\ \ldots,\ +)$$

$$\mu=(-,\ -,\ \ldots,\ -)$$

(i. e., on the upper half of the unit circle we have multipliers either of the first kind only or of the second kind only). The index n of the stability domains will be determined below.

We now use the directional wideness theorem (§6.2); let the Hamiltonians $\mathbf{H}_1(t)$ and $\mathbf{H}_2(t)$ in the statement of the theorem be

$$\mathbf{H}_1(t)=h_1(t)\,\mathbf{I}_{2k},\quad \mathbf{H}_2(t)=h_2(t)\,\mathbf{I}_{2k}, \tag{6.12}$$

where $h_1(t)$, $h_2(t)$ are T-periodic functions. The condition $\mathbf{H}_1(t)\leqslant\mathbf{H}_2(t)$ holds if and only if $h_1(t)\leqslant h_2(t)$, and we shall assume that this is the case. The Hamiltonian $\mathbf{H}(t, s)=\mathbf{H}_1(t)+s[\mathbf{H}_2(t)-\mathbf{H}_1(t)]$ is

$$\mathbf{H}(t, s)=h(t, s)\,\mathbf{I}_{2k},\ \text{where}\ h(t, s)=h_1(t)+s[h_2(t)-h_1(t)]. \tag{6.13}$$

According to the foregoing discussion, this Hamiltonian is strongly stable for $0\leqslant s\leqslant 1$ if

$$(1-s)\tilde{h}_1+s\tilde{h}_2\neq m\pi\quad\text{for}\quad 0\leqslant s\leqslant 1, \tag{6.14}$$

where

$$\tilde{h}_j=\int_0^T h_j(t)\,dt\qquad (j=1,\ 2). \tag{6.15}$$

Inequality (6.14) is true if and only if, for some integer m,

$$m\pi<\tilde{h}_1\leqslant\tilde{h}_2<(m+1)\,\pi. \tag{6.16}$$

In other words, condition (6.16) implies that the Hamiltonian $\mathbf{H}(t, s)=$ $=\mathbf{H}_1(t)+s[\mathbf{H}_2(t)-\mathbf{H}_1(t)]$ is strongly stable for $0\leqslant s\leqslant 1$. Applying the directional wideness theorem, we obtain the following test.

Test I. Let the Hamiltonian $\mathbf{H}(t)$ *satisfy the conditions*

$$h_1(t)\,\mathbf{I}_{2k}\leqslant\mathbf{H}(t)\leqslant h_2(t)\,\mathbf{I}_{2k}, \tag{6.17}$$

where $h_1(t)$, $h_2(t)$ *are scalar (T-periodic integrable) functions. Let condition (6.16) hold, where the numbers* $\tilde{h}_j$ *are defined by (6.15). Then the canonical equation with Hamiltonian* $\mathbf{H}(t)$ *is strongly stable.*

The roles of $h_1(t)$ and $h_2(t)$ may be fulfilled by the minimum and maximum eigenvalues $h_{\min}(t)$ and $h_{\max}(t)$ of the matrix $\mathbf{H}(t)$. Indeed, by (I. 1.58),

$$h_{\min}(t)\,\mathbf{I}_{2k}\leqslant\mathbf{H}(t)\leqslant h_{\max}(t)\,\mathbf{I}_{2k}.$$

A serious shortcoming of this test is that (for $k > 1$) it is by no means applicable to Hamiltonians from every stability domain. Indeed, by Remark 1 to the theorem of §6.2, it holds only for Hamiltonians of the same stability domain as the Hamiltonians $\mathbf{H}_j(t) = h_j(t)\,\mathbf{I}_{2k}$ $(j = 1, 2)$. As shown above, the multiplier type μ of this stability domain may be one of only two of the possible 2^k types. Let us determine the possible values of the index n (§5.3). It will suffice to consider Hamiltonian $\mathbf{H}(t) = h(t)\,\mathbf{I}_{2k}$, where

$$m\pi < \tilde{h} < (m+1)\pi \qquad \left(\tilde{h} = \int_0^T h(t)\,dt,\ \ m = 0, \pm 1, \pm 2, \ldots\right). \tag{6.18}$$

Indeed, this Hamiltonian may clearly be joined continuously to either $\mathbf{H}_1(t)$ or $\mathbf{H}_2(t)$ while maintaining condition (6.18). Since the deformation also preserves strong stability, these Hamiltonians lie in one stability domain. To determine the index n, we use formula (5.4). The principal value ϑ_j of the argument of $\rho_j^{(+)}(T) = e^{i\tilde{h}}$ is $\tilde{h} - m\pi$ if m is even and $\tilde{h} - (m-1)\pi$ if m is odd. In other words,

$$\vartheta_j = \tilde{h} - \left[\frac{m}{2} - \frac{1-(-1)^m}{4}\right] 2\pi.$$

On the other hand, the multipliers of the first kind of the matrizant of equation (6.9) are

$$\rho_j^{(+)} = \exp i \int_0^T h(t_1)\,dt_1,$$

and so

$$\Delta \operatorname{Arg} \rho_j(t)\Big|_{t=0}^{T} = \int_0^T h(t_1)\,dt_1 = \tilde{h}.$$

By (5.4),

$$2n\pi = \sum_{j=1}^{k} \left[\Delta \operatorname{Arg} \rho_j(t)\Big|_{t=0}^{T} - \vartheta_j\right] = k\left[\frac{m}{2} - \frac{1-(-1)^m}{4}\right] 2\pi.$$

Thus the index is

$$n = k\left[\frac{m}{2} - \frac{1-(-1)^m}{4}\right].$$

It is easy to see that for $k = 1$ and $m = 0, \pm 1, \pm 2, \ldots$ the index n assumes all values: $n = 0, \pm 1, \pm 2, \ldots$ Thus, for $k = 1$ our test yields a Hamiltonian from each stability domain. If $k > 1$ the test may be valid only for Hamiltonians in domains $\mathcal{G}_n^{(\mu)}$ of one of two multiplier types, as shown previously, and the index n is a multiple of k.

We shall now derive a test suitable for any stability domain $\mathcal{G}_n^{(\mu)}$. Consider a canonical equation which separates into k canonical second-order equations with scalar Hamiltonians

$$\mathbf{J}_2\frac{d\mathbf{x}_\varkappa}{dt}=h_\varkappa(t)\,\mathbf{x}_\varkappa \qquad (\varkappa=1,\ \ldots,\ k). \tag{6.19}$$

Here

$$\mathbf{J}_2=\begin{Vmatrix}0 & -1\\ 1 & 0\end{Vmatrix},\quad \mathbf{x}_\varkappa=\begin{pmatrix}\xi_1^{(\varkappa)}\\ \xi_2^{(\varkappa)}\end{pmatrix}.$$

System (6.19) may be written as a single vector equation

$$\mathbf{J}'\frac{d\mathbf{x}}{dt}=\mathbf{H}'(t)\,\mathbf{x} \tag{6.20}$$

where the matrices $\mathbf{J}'$, $\mathbf{H}'(t)$ are direct sums of 2×2 diagonal blocks:

$$\mathbf{J}'=\mathbf{J}_2\dot{+}\mathbf{J}_2\dot{+}\ldots\dot{+}\mathbf{J}_2,\quad \mathbf{H}'(t)=h_1(t)\,\mathbf{I}_2\dot{+}\ldots\dot{+}h_k(t)\,\mathbf{I}_2. \tag{6.21}$$

The vector $\mathbf{x}$ is also a direct sum: $\mathbf{x}=\mathbf{x}_1\dot{+}\ldots\dot{+}\mathbf{x}_k$. If necessary, the substitution $\mathbf{y}=\mathbf{S}\mathbf{x}$, where $\mathbf{S}$ is some nonsingular real $2k\times2k$ matrix such that $\mathbf{S}^\tau\mathbf{J}_{2k}\mathbf{S}=\mathbf{J}'$, reduces equation (6.20) to the standard form (6.1) (see § II. 3.1).

As shown previously in this subsection, equation (6.19) (fixed $\varkappa$) has a multiplier $\exp(i\tilde{h}_\varkappa)$ of the first kind and a multiplier $\exp(-i\tilde{h}_\varkappa)$ of the second kind, where

$$\tilde{h}_\varkappa=\int_0^T h_\varkappa(t)\,dt \qquad (\varkappa=1,\ \ldots,\ k). \tag{6.22}$$

Consequently, equation (6.20) has the same multipliers, of the same kinds. Thus the conditions for system (6.20) to be strongly stable are

$$e^{i\tilde{h}_p}\neq e^{-i\tilde{h}_q},\ \text{i. e.,}\ \ \tilde{h}_p+\tilde{h}_q\neq 2m\pi \qquad (p,\ q=1,\ \ldots,\ k;\ m=0,\ \pm1,\ \pm2,\ \ldots) \tag{6.23}$$

for all $p,\ q=1,\ \ldots,\ k$. Assume that these conditions are satisfied, and define integers $m_{pq}=0,\ \pm1,\ \pm2,\ \ldots$ by

$$2m_{pq}\pi<\tilde{h}_p+\tilde{h}_q<2(m_{pq}+1)\,\pi \qquad (p,\ q=1,\ \ldots,\ k). \tag{6.24}$$

As before, we evaluate the index n. Setting $q=p$ in (6.24), we get $m_{pp}\pi<\tilde{h}_p<(m_{pp}+1)\,\pi$; therefore,

$$\vartheta_p=\arg\rho_p^{(+)}(T)=\arg e^{i\tilde{h}_p}=\tilde{h}_p-\left[\frac{m_{pp}}{2}-\frac{1-(-1)^{m_{pp}}}{4}\right]2\pi,$$

$$2n\pi=\Delta\operatorname{Arg}X(t)\Big|_0^T=-\sum_{p=1}^k\vartheta_p=\sum_{p=1}^k\tilde{h}_p-\sum_{p=1}^k\left\{\tilde{h}_p-\left[\frac{m_{pp}}{2}-\frac{1-(-1)^{m_{pp}}}{4}\right]2\pi\right\}.$$

Hence

$$n=\sum_{p=1}^k\left[\frac{m_{pp}}{2}-\frac{1-(-1)^{m_{pp}}}{4}\right].$$

The bracketed term may be any number, and so the index n itself is also arbitrary. The multiplier type μ is determined by the relative positions of the numbers $\exp(i\tilde{h}_1), \ldots, \exp(ih_k), \exp(-i\tilde{h}_1), \ldots, \exp(-ih_k)$ on the unit circle and so may also be any one of the 2^k possible types.

Thus there are systems (6.20), separable into k second-order systems (6.19), in any stability domain $\delta_n^{(\mu)}$.

Test II. Consider the equation

$$\mathbf{J}'\frac{d\mathbf{x}}{dt} = \mathbf{H}(t)\,\mathbf{x}, \tag{6.25}$$

where $\mathbf{H}(t) = \mathbf{H}'(t) + \mathbf{H}_0(t)$ *and the matrices* $\mathbf{J}'$, $\mathbf{H}'(t)$ *have the form (6.21). Let* $h_1^{(0)}(t)$ *and* $h_2^{(0)}(t)$ *be the minimum and maximum eigenvalues, respectively, of the matrix* $\mathbf{H}_0(t)$, *or any functions such that* $h_1^{(0)}(t)\,\mathbf{I}_{2k} \leqslant \mathbf{H}_0(t) \leqslant h_2^{(0)}(t)\,\mathbf{I}_{2k}$. *Let*

$$\tilde{h}_j^{(0)} = \int_0^T h_j^{(0)}(t)\,dt \qquad (j = 1, 2). \tag{6.26}$$

Suppose that for certain integers $m_{pq} = 0, \pm 1, \pm 2, \ldots$

$$2m_{pq}\pi < \tilde{h}_p + \tilde{h}_q + 2\tilde{h}_1^{(0)} \leqslant \tilde{h}_p + \tilde{h}_q + 2\tilde{h}_2^{(0)} < 2(m_{pq}+1)\pi \tag{6.27}$$

for all $p, q = 1, \ldots, k$, *where* $\tilde{h}_p$, $\tilde{h}_q$ *are defined by (6.22). Then equation (6.25) is strongly stable.*

Proof. Using Remark 2 of §6.2, we apply the directional wideness theorem, with

$$\mathbf{H}_1(t) = \mathbf{H}'(t) + h_1^{(0)}(t)\,\mathbf{I}_{2k}, \quad \mathbf{H}_2(t) = \mathbf{H}'(t) + h_2^{(0)}(t)\,\mathbf{I}_{2k}.$$

Then

$$\mathbf{H}(t, s) = \mathbf{H}_1(t) + s[\mathbf{H}_2(t) - \mathbf{H}_1(t)] = \mathbf{H}'(t) + h(t, s)\,\mathbf{I}_{2k},$$

where

$$h(t, s) = h_1^{(0)}(t) + s[h_2^{(0)}(t) - h_1^{(0)}(t)] \qquad (0 \leqslant s \leqslant 1).$$

The Hamiltonian $\mathbf{H}(t, s)$ has the form (6.21)

$$\mathbf{H}(t, s) = h_1(t, s)\,\mathbf{I}_2 \dotplus \ldots \dotplus h_k(t, s)\,\mathbf{I}_2,$$

where

$$h_\varkappa(t, s) = h_\varkappa(t) + h(t, s) \qquad (\varkappa = 1, \ldots, k).$$

Let

$$\tilde{h}(s) = \int_0^T h(t, s)\,dt = \tilde{h}_1^{(0)} + s[\tilde{h}_2^{(0)} - \tilde{h}_1^{(0)}]. \tag{6.28}$$

The stability conditions (6.24) for $\mathbf{H}(t, s)$ are in this case

$$2m_{pq}\pi < \tilde{h}_p + \tilde{h}_q + 2\tilde{h}(s) < 2(m_{pq}+1)\pi.$$

By (6.27), (6.28) these inequalities are valid for $0 \leqslant s \leqslant 1$. Thus the equation $\mathbf{J}\dot{\mathbf{x}} = \mathbf{H}(t, s)\mathbf{x}$ is strongly stable for $0 \leqslant s \leqslant 1$. Since

$$\mathbf{H}_1(t) \leqslant \mathbf{H}(t) \leqslant \mathbf{H}_2(t),$$

it follows from the directional wideness theorem that equation (6.25) is also strongly stable, Q. E. D.

Remark 1. Suppose that the numbers m_{pp} in inequalities (6.24) and (6.27) are the same. Test I then shows how close the coefficients of equation (6.25) must be to those of a separable strongly stable equation (6.20) in order that equation (6.25) remain strongly stable. It is obviously impossible to improve condition (6.27) by changing the constants on the left- and right-hand sides of the inequalities, since equation (6.20) with the matrix (6.21) is stable but not strongly stable if $\tilde{h}_p + \tilde{h}_q = 2\pi m$ for some p, q. Thus, for any $\varepsilon > 0$ and any $p, q = 1, \ldots, k$ there exist unstable equations (6.25) with matrices $\mathbf{H}(t) = \mathbf{H}'(t) + \mathbf{H}_0(t)$, for which

$$2m_{pq}\pi - \varepsilon < \tilde{h}_p + \tilde{h}_q + 2\tilde{h}_1^{(0)} \leqslant \tilde{h}_p + \tilde{h}_q + 2h_2^{(0)} < 2(m_{pq}+1)\pi,$$

and also unstable equations for which

$$2m_{pq}\pi < \tilde{h}_p + \tilde{h}_q + 2\tilde{h}_1^{(0)} \leqslant \tilde{h}_p + \tilde{h}_q + 2h_2^{(0)} < 2(m_{pq}+1)\pi + \varepsilon.$$

Remark 2. Consider equation (6.2)

$$\tilde{\mathbf{J}}\frac{d\mathbf{z}}{dt} = \tilde{\mathbf{H}}(t)\mathbf{z},$$

where $\tilde{\mathbf{J}}$ is an arbitrary nonsingular skew-symmetric matrix. Suppose that $\tilde{\mathbf{H}}(t)$ may be expressed as $\tilde{\mathbf{H}}(t) = \tilde{\mathbf{H}}'(t) + \tilde{\mathbf{H}}_0(t)$, where $\tilde{\mathbf{H}}_0(t)$ is a "small correction," so that after a suitable linear transformation the equation

$$\tilde{\mathbf{J}}\frac{d\mathbf{z}}{dt} = \tilde{\mathbf{H}}'(t)\mathbf{z} \tag{6.29}$$

separates into k second-order equations (6.19). This means that there exists a substitution $\mathbf{x} = \mathbf{S}\mathbf{z}$, where $\mathbf{S}$ is a nonsingular real matrix, reducing equation (6.29) to (6.20):

$$\mathbf{J}'\frac{d\mathbf{x}}{dt} = \mathbf{H}'(t)\mathbf{x}.$$

Here $\mathbf{J}' = (\mathbf{S}^{-1})^{\tau}\tilde{\mathbf{J}}\mathbf{S}^{-1}$ and $\mathbf{H}'(t) = (\mathbf{S}^{-1})^{\tau}\tilde{\mathbf{H}}'(t)\mathbf{S}^{-1}$ have the form (6.21). Applying the transformation $\mathbf{x} = \mathbf{S}\mathbf{z}$ to the original equation, we get

$$\mathbf{J}'\frac{d\mathbf{x}}{dt} = [\mathbf{H}'(t) + (\mathbf{S}^{-1})^{\tau}\tilde{\mathbf{H}}_0(t)\mathbf{S}^{-1}]\mathbf{x},$$

and this equation may be handled by Test II. Inequalities (6.27) are conditions for strong stability of this equation, where $h_1^{(0)}(t)$ and $h_2^{(0)}(t)$ are any functions such that

$$h_1^{(0)}(t)\,\mathbf{I}_{2k} \leqslant (\mathbf{S}^{-1})^{\tau}\,\tilde{\mathbf{H}}_0(t)\,\mathbf{S}^{-1} \leqslant h_2^{(0)}(t)\,\mathbf{I}_{2k}. \tag{6.30}$$

In particular, $h_1^{(0)}(t)$ and $h_2^{(0)}(t)$ may be the minimum and maximum eigenvalues of the matrix $(\mathbf{S}^{-1})^{\tau}\tilde{\mathbf{H}}_0(t)\,\mathbf{S}^{-1}$.

If $\mathbf{S}$ is an orthogonal matrix, inequalities (6.30) are equivalent to $h_1^{(0)}\mathbf{I}_{2k} \leqslant \tilde{\mathbf{H}}_0 \leqslant h_2^{(0)}\mathbf{I}_{2k}$ (this is readily seen by going over to inequalities for the corresponding quadratic forms). If $\tilde{\mathbf{J}} = \mathbf{J}_{2k}$, there exists an orthogonal matrix $\mathbf{S}$ such that $(\mathbf{S}^{-1})^{\tau}\mathbf{J}_{2k}\mathbf{S}^{-1} = \mathbf{J}'$. We have thus proved that if the equation

$$\mathbf{J}_{2k}\frac{dz}{dt} = \tilde{\mathbf{H}}'(t)\mathbf{z} \tag{6.31}$$

separates, after the linear transformation $\mathbf{x} = \mathbf{S}\mathbf{z}$ with an orthogonal matrix $\mathbf{S}$, into k equations (6.19), then Test II is valid for the equation

$$\mathbf{J}_{2k}\frac{dz}{dt} = [\tilde{\mathbf{H}}'(t) + \tilde{\mathbf{H}}_0(t)]\,\mathbf{z} \tag{6.32}$$

where $h_j^{(0)}(t)$ in (6.26) may be any functions such that

$$h_1^{(0)}(t)\,\mathbf{I}_{2k} \leqslant \tilde{\mathbf{H}}_0(t) \leqslant h_2^{(0)}(t)\,\mathbf{I}_{2k}$$

and $\tilde{h}_j^{(0)}$ are defined by (6.26).

It is frequently convenient to apply Test II in conjunction with the following theorem:

Theorem. Suppose that the canonical equation

$$\mathbf{J}\frac{d\mathbf{x}}{dt} = \mathbf{H}(t)\mathbf{x}$$

is subjected to a linear substitution $\mathbf{x}_1 = \mathbf{W}(t)\,\mathbf{x}$, *where* $\mathbf{W}(t)$ *is an integrable piecewise-continuous symplectic* T*-periodic matrix-function and*

$$\Delta \operatorname{Arg} \mathbf{W}(t)\Big|_0^T = 2n_0\pi.$$

Then:

1) *The transformed equation is also canonical:*

$$\mathbf{J}\dot{\mathbf{x}}_1 = \mathbf{H}_1(t)\,\mathbf{x}_1 \qquad (\mathbf{H}_1(t+T) = \mathbf{H}_1(t) = \mathbf{H}_1(t)^{\tau}).$$

2) *The equations with* $\mathbf{H}(t)$ *and* $\mathbf{H}_1(t)$ *have similar monodromy matrices; in particular, each is strongly stable if and only if the other is.*

3) *If* $\mathbf{H}(t) \in \mathcal{O}_n^{(\mu)}$, *then* $\mathbf{H}_1(t) \in \mathcal{O}_{n+n_0}^{(\mu)}$.

Proof. 1) Let $\mathbf{X}(t)$ be the matrizant of the original equation and $\mathbf{X}_1(t)$ that of the transformed equation. Since $\mathbf{X}_1(t) = \mathbf{W}(t)\,\mathbf{X}(t)\,\mathbf{W}(0)^{-1}$, it follows that $\mathbf{X}_1(t)$ is a symplectic matrix: $\mathbf{X}_1(t) \in \mathfrak{S}$ $(0 \leqslant t \leqslant T)$. Thus the matrizant $\mathbf{X}_1(t)$ is that of a canonical equation (§ II. 3.4). A simple computation shows that

$$\mathbf{H}_1(t) = \mathbf{J}\frac{d\mathbf{W}}{dt}\mathbf{W}^{-1} - \mathbf{JWJHW}^{-1}.$$

Consequently, $\mathbf{H}_1(t+T) = \mathbf{H}_1(t)$.

2) The monodromy matrix $\mathbf{X}_1(T)$ has the form

$$\mathbf{X}_1(T) = \mathbf{W}(0)\,\mathbf{X}(T)\,\mathbf{W}(0)^{-1},$$

so that the monodromy matrices of our two systems are indeed similar. We claim that their spectra (including the kinds of the eigenvalues) are the same. Let $\mathbf{X}(T)\mathbf{a}_j = \rho_j^{(+)}\mathbf{a}_j + \mathbf{a}_{j-1}$ $(j = 1, \ldots, k)$, where $\rho_j^{(+)}$ are the multipliers of the first kind and $\mathbf{a}_0 = \mathbf{0}$. Denoting $\mathbf{b}_j = \mathbf{W}(0)\mathbf{a}_j$, we see that $\mathbf{X}_1(T)\mathbf{b}_j = \rho_j^{(+)}\mathbf{b}_j + \mathbf{b}_{j-1}$. In addition, $\langle \mathbf{b}_h, \mathbf{b}_j\rangle = \langle \mathbf{a}_h, \mathbf{a}_j\rangle$, since $\mathbf{W}(0)$ is a symplectic matrix. Thus the Gram matrices of the root subspaces of identical eigenvalues of $\mathbf{X}_1(T)$ and $\mathbf{X}(T)$ coincide, and so their spectra coincide as to the kinds of the eigenvalues.

3) It follows from what we have proved that the stability domains of $\mathbf{H}(t)$ and $\mathbf{H}_1(t)$ have the same multiplier type.

Let n and n_1 denote the indices of these stability domains. It remains to show that $n_1 = n + n_0$.

Construct a complemented matrizant $\mathbf{X}(t)$ $(0 \leqslant t \leqslant 2T)$ as in §5.4. Let $\mathbf{Y}(t) = \mathbf{X}(t)$, $T \leqslant t \leqslant 2T$, be the corresponding strongly stable curve. As we have just shown, the matrix $\mathbf{W}(0)\,\mathbf{Y}(t)\,\mathbf{W}(0)^{-1}$ is also strongly stable. Thus we can define a complemented matrizant $\mathbf{X}_1(t)$ for $T \leqslant t \leqslant 2T$ by setting $\mathbf{X}_1(t) = \mathbf{W}(0)\,\mathbf{Y}(t)\,\mathbf{W}(0)^{-1}$ (as required, $\mathbf{X}_1(t)$ will then be a continuous function at $t = T$, hence also for $0 \leqslant t \leqslant 2T$, and $\mathbf{X}_1(2T) = \mathbf{I}_{2k}$). By the definition of the index,

$$\Delta \operatorname{Arg} \mathbf{X}_1(t)\Big|_0^{2T} = 2n_1\pi, \quad \Delta \operatorname{Arg} \mathbf{X}(t)\Big|_0^{2T} = 2n\pi.$$

The closed curve $\mathbf{X}_1(t)$ is made up of the curves $\mathbf{W}(t)\,\mathbf{X}(t)\,\mathbf{W}(0)^{-1}$ and $\mathbf{W}(0)\,\mathbf{Y}(t)\,\mathbf{W}(0)^{-1}$. Joining the matrix $\mathbf{W}(0)$ continuously to $\mathbf{I}_{2k}$, we deform the curve $\mathbf{X}_1(t)$ $(0 \leqslant t \leqslant 2T)$ into the curve made up of $\mathbf{W}(t)\,\mathbf{X}(t)$ and $\mathbf{Y}(t)$. Thus,

$$\Delta \operatorname{Arg} \mathbf{X}_1(t)\Big|_0^{2T} = \Delta \operatorname{Arg}\left[\mathbf{W}(t)\,\mathbf{X}(t)\right]\Big|_0^{T} + \Delta \operatorname{Arg} \mathbf{Y}(t)\Big|_T^{2T}.$$

By part (a) of Theorem III in §5.4,

$$\Delta \operatorname{Arg}\mathbf{X}_1(t)\Big|_0^{2T} = \Delta \operatorname{Arg} \mathbf{W}(t)\Big|_0^{T} + \Delta \operatorname{Arg} \mathbf{X}(t)\Big|_0^{T} + \Delta \operatorname{Arg} \mathbf{Y}(t)\Big|_T^{2T}.$$

The sum of the last two terms is $\Delta \operatorname{Arg} \mathbf{X}(t)\Big|_0^{2T}$. Therefore,

$$\Delta \operatorname{Arg} \mathbf{X}_1(t)\Big|_0^{2T} = \Delta \operatorname{Arg} \mathbf{W}(t)\Big|_0^{T} + \Delta \operatorname{Arg} \mathbf{X}(t)\Big|_0^{2T},$$

so that $n_1 = n + n_0$. This completes the proof.

The role of the matrix $\mathbf{W}(t)$ may be filled by

$$\mathbf{W}(t) = \exp\left(-\frac{2\pi t}{T}\mathbf{JC}_0\right),$$

where $\mathbf{C}_0=\mathbf{C}_0^*$ is a real matrix such that $\mathbf{JC}_0$ is reducible to diagonal form with eigenvalues im_j, where m_j are integers. Then $\mathbf{H}_1(t)=\frac{2\pi}{T}\mathbf{C}_0+$ $+\exp\left(-\frac{2\pi t}{T}\mathbf{JC}_0\right)\mathbf{H}(t)\exp\left(\frac{2\pi t}{T}\mathbf{JC}_0\right)$. Any sufficient condition for strong stability applied to $\mathbf{H}_1(t)$ yields some new sufficient condition for strong stability applying to $\mathbf{H}(t)$. Note that part (1) and the second half of part (2) of the theorem remain valid if the condition $\mathbf{W}(t+T)=\mathbf{W}(t)$ is replaced by $\mathbf{W}(t+T)=-\mathbf{W}(t)$.

6.5. Canonical equations with constant coefficients

Later we shall establish some stability tests for canonical equations with "almost-constant" Hamiltonians. Prior to this, we must first devote more attention to canonical equations with constant Hamiltonians. This is the content of this subsection. We shall first show that a suitable nonsingular linear transformation with real coefficients separates a stable equation of order $2k$ with constant coefficients into k strongly stable second-order equations. In detail:

Lemma I. A canonical equation with constant Hamiltonian $\mathbf{H}(t)\equiv\mathbf{C}$ *is strongly stable if and only if there is a real nonsingular matrix* $\mathbf{S}$ *such that*

$$\mathbf{S}^{-1}\mathbf{JCS}=\alpha_1\mathbf{J}_2\dotplus\ldots\dotplus\alpha_k\mathbf{J}_2,\quad \mathbf{S}^*\mathbf{JS}=\mathbf{J}_2\dotplus\ldots\dotplus\mathbf{J}_2, \tag{6.33}$$

where $\alpha_1, \ldots, \alpha_k$ *are any numbers such that* $(\alpha_j+\alpha_h)T\not\equiv 2\pi$ *for all* j, h. *The numbers* $i\alpha_1, \ldots, i\alpha_k$ *are then the eigenvalues of the first kind of the matrix* $\mathbf{J}^{-1}\mathbf{C}$ *(so that* $-i\alpha_1, \ldots, -i\alpha_k$ *are its eigenvalues of the second kind), and the multipliers of the canonical equation of the first kind* $\rho_j^{(+)}$ *and of the second kind* $\rho_j^{(-)}$ *are*

$$\rho_j^{(+)}=\exp(i\alpha_jT),\quad \rho_j^{(-)}=\exp(-i\alpha_jT)\qquad (j=1, \ldots, k).$$

Note that since $(\alpha_j+\alpha_h)T\not\equiv 2\pi$ it follows that there are $\frac{1}{2}k(k+1)$ integers m_{jh} for which

$$2m_{jh}\pi<(\alpha_j+\alpha_h)T<2(m_{jh}+1)\pi. \tag{6.34}$$

Proof. Since the canonical equation

$$\mathbf{J}\frac{dx}{dt}=\mathbf{C}x \tag{6.35}$$

is stable, the eigenvalues of the matrix $\mathbf{J}^{-1}\mathbf{C}$ are pure imaginary and have simple elementary divisors.

By Lemma III of §2.5, there is a nonsingular matrix $\mathbf{S}_0$ such that

$$\left.\begin{aligned}\mathbf{S}_0^{-1}\mathbf{J}^{-1}\mathbf{CS}_0&=i\alpha_1\dotplus(-i\alpha_1)\dotplus\ldots\dotplus i\alpha_k\dotplus(-i\alpha_k),\\ \mathbf{S}_0^*i\mathbf{JS}_0&=(+1)\dotplus(-1)\dotplus\ldots\dotplus(+1)\dotplus(-1).\end{aligned}\right\} \tag{6.36}$$

The first equality shows that the columns of $\mathbf{S}_0$ are eigenvectors of $\mathbf{J}^{-1}\mathbf{C}$. It is readily seen from the details of the proof in §2.5 that for a real matrix $\mathbf{A}=\mathbf{J}^{-1}\mathbf{C}$ we may assume that the eigenvectors belonging to complex conjugate eigenvalues are complex conjugate vectors. Then the matrix $\mathbf{S}_0$ has the form $\mathbf{S}_0=\|\mathbf{f}_1, \bar{\mathbf{f}}_1, \ldots, \mathbf{f}_k, \bar{\mathbf{f}}_k\|$. We can rewrite (6.36) as

$$\mathbf{J}^{-1}\mathbf{C}\mathbf{f}_j=i\alpha_j\mathbf{f}_j, \qquad \langle \mathbf{f}_j, \mathbf{f}_j\rangle=1 \qquad (j=1, \ldots, k), \tag{6.37}$$

$$\langle \mathbf{f}_j, \mathbf{f}_h\rangle=0 \quad (j\neq h), \quad \langle \mathbf{f}_j, \bar{\mathbf{f}}_h\rangle=0 \qquad (j, h=1, \ldots, k). \tag{6.38}$$

Let $\mathbf{f}_j=\mathbf{u}_j+i\mathbf{v}_j$ $(j=1, \ldots, k)$, where $\mathbf{u}_j$, $\mathbf{v}_j$ are real vectors. Since the vectors $\mathbf{f}_1, \bar{\mathbf{f}}_1, \ldots, \mathbf{f}_k, \bar{\mathbf{f}}_k$ are linearly independent, the same is true of the vectors $\mathbf{u}_1, \mathbf{v}_1, \ldots, \mathbf{u}_k, \mathbf{v}_k$. Hence the real matrix

$$\mathbf{S}=\sqrt{2}\,\|\mathbf{u}_1, \mathbf{v}_1, \ldots, \mathbf{u}_k, \mathbf{v}_k\|$$

is nonsingular. Separating the real and imaginary parts of (6.37), we obtain

$$\mathbf{J}^{-1}\mathbf{C}\mathbf{u}_j=-\alpha_j\mathbf{v}_j, \quad \mathbf{J}^{-1}\mathbf{C}\mathbf{v}_j=\alpha_j\mathbf{u}_j \qquad (j=1, \ldots, k),$$

so that

$$\mathbf{J}^{-1}\mathbf{C}\,\|\mathbf{u}_j, \mathbf{v}_j\|=\|\mathbf{u}_j, \mathbf{v}_j\|\,\mathbf{J}_2 \qquad (j=1, \ldots, k). \tag{6.39}$$

These conditions are equivalent to the first part of (6.33). It follows from (6.37), (6.28) that

$$\mathbf{u}_h^{\tau}\mathbf{J}\mathbf{u}_h=0, \quad \mathbf{v}_h^{\tau}\mathbf{J}\mathbf{v}_h=0,$$

$$\mathbf{v}_h^{\tau}\mathbf{J}\mathbf{u}_h=i^{-1}\left\langle \frac{\mathbf{f}_h+\bar{\mathbf{f}}_h}{2}, \frac{\mathbf{f}_h-\bar{\mathbf{f}}_h}{2i}\right\rangle=\frac{1}{2}.$$

On the other hand, by (6.38) the two-dimensional subspaces spanned by $\{\mathbf{u}_j, \mathbf{v}_j\}$ and $\{\mathbf{u}_h, \mathbf{v}_h\}$ are J-orthogonal for $j\neq h$. Hence follows the second part of (6.33).

Conditions (6.33) imply that the substitution $\mathbf{y}=\mathbf{S}\mathbf{x}$ reduces equation (6.35) to equation (6.20) with the Hamiltonian

$$\mathbf{H}'=\alpha_1\mathbf{J}_2\dotplus\ldots\dotplus\alpha_k\mathbf{J}_2.$$

The strong stability conditions (6.24) for this equation have the form (6.34).

Conversely, it follows from (6.33) that $\pm i\alpha_1, \ldots, \pm i\alpha_k$ are the eigenvalues of the matrix $\mathbf{J}^{-1}\mathbf{C}$ and that there are no multiple multipliers of mixed kind among $\rho_j=\exp(\pm i\alpha_j T)$, so that equation (6.35) is strongly stable. This completes the proof of Lemma I.

Henceforth we shall assume that the numbers α_j are indexed so that

$$\alpha_1\leqslant\alpha_2\leqslant\ldots\leqslant\alpha_k.$$

The integers m_{jh} in (6.34) are then uniquely determined by the matrix $\mathbf{C}$.

Remark. Consider the diagonal matrix

$$\mathbf{C}' = \alpha_1 \mathbf{I}_2 \dotplus \ldots \dotplus \alpha_k \mathbf{I}_2. \tag{6.40}$$

We may rewrite (6.33) as

$$\mathbf{S}^T\mathbf{C}\mathbf{S} = \mathbf{C}', \quad \mathbf{S}^T\mathbf{J}\mathbf{S} = \mathbf{J}_2 \dotplus \ldots \dotplus \mathbf{J}_2. \tag{6.41}$$

Indeed, conditions (6.33) may be written as $\mathbf{S}^{-1}\mathbf{J}\mathbf{C}\mathbf{S} = \mathbf{J}'\mathbf{C}'$, $\mathbf{S}^T\mathbf{J}\mathbf{C} = \mathbf{J}'$, where $\mathbf{J}'$ is the matrix of (6.21). Since $\mathbf{J}^{-1} = -\mathbf{J}$, $(\mathbf{J}')^{-1} = -\mathbf{J}'$, we have $\mathbf{S}^{-1}\mathbf{J}^{-1}\mathbf{C}\mathbf{S} = (\mathbf{J}')^{-1}\mathbf{C}'$, $(\mathbf{J}')^{-1} = \mathbf{S}^{-1}\mathbf{J}^{-1}(\mathbf{S}^T)^{-1}$. Substituting $(\mathbf{J}')^{-1}$ into this relation and canceling out $\mathbf{S}^{-1}\mathbf{J}^{-1}$, we obtain the first equality in (6.41). Reversing the arguments, we can derive (6.33) from (6.41).

Lemma II. The canonical equation (6.35) is strongly stable if and only if there exists a symplectic matrix $\mathbf{U}$ *such that*

$$\mathbf{U}^T\mathbf{C}\mathbf{U} = \mathbf{C}'', \tag{6.42}$$

where $\mathbf{C}''$ *is the diagonal matrix with diagonal elements* $\alpha_1, \ldots, \alpha_k; \alpha_1, \ldots, \alpha_k$, *satisfying (6.34) for some integers* m_{jh}. *In this case,* $i\alpha_1, \ldots, i\alpha_k$ *are the eigenvalues of the first kind of the matrix* $\mathbf{J}^{-1}\mathbf{C}$, *and* $-i\alpha_1, \ldots, -i\alpha_k$ *are its eigenvalues of the second kind.*

Proof. Let $\mathbf{D}$ denote a $k \times k$ diagonal matrix with diagonal elements $\delta_1, \ldots, \delta_k$ and $\mathbf{D}'$ the $2k \times 2k$ diagonal matrix with elements $\delta_1, \ldots, \delta_k; \delta_1, \ldots, \delta_k$. Then

$$\mathbf{J}^{-1}\mathbf{D}' = \begin{pmatrix} 0 & \mathbf{D} \\ -\mathbf{D} & 0 \end{pmatrix}.$$

Let $\mathbf{F}$ be the nonsingular matrix defined by (6.10a). Then

$$\begin{pmatrix} 0 & \mathbf{D} \\ -\mathbf{D} & 0 \end{pmatrix} \mathbf{F} = \mathbf{F} \begin{pmatrix} -i\mathbf{D} & 0 \\ 0 & i\mathbf{D} \end{pmatrix}.$$

Consequently, the numbers $i\delta_1, \ldots, i\delta_k; -i\delta_1, \ldots, -i\delta_k$ are the eigenvalues of $\mathbf{J}^{-1}\mathbf{D}'$, the corresponding eigenvectors being the columns of the matrices $\mathbf{F}_+$, $\mathbf{F}_-$ (see (6.10a)). Formula (6.11) shows that the subspace spanned by the columns of $\mathbf{F}_+$ is positive definite while that spanned by the columns of $\mathbf{F}_-$ is negative definite. The first subspace is the sum of eigensubspaces of the eigenvalues $i\delta_1, \ldots, i\delta_k$. Consequently, these are all eigenvalues of the first kind for $\mathbf{J}^{-1}\mathbf{D}'$. Similarly, $-i\delta_1, \ldots, -i\delta_k$ are its eigenvalues of the second kind.

Let $\mathbf{C}$ be a constant Hamiltonian such that equation (6.35) is strongly stable. Let $i\alpha_1, \ldots, i\alpha_k$ be the eigenvalues of the first kind of $\mathbf{J}^{-1}\mathbf{C}$. By Lemma I, we have (6.33), (6.34).

Let $\mathbf{D}$ be the diagonal matrix with diagonal elements $\delta_1 = \alpha_1, \ldots, \delta_k = \alpha_k$. Then $\mathbf{D}' = \mathbf{C}''$.

Consider the canonical equation (6.1) with Hamiltonian $\mathbf{H}(t) \equiv \mathbf{C}'' = \mathbf{D}'$. By the above reasoning, the multipliers of this equation are the numbers $\exp(i\alpha_1 T), \ldots, \exp(i\alpha_k T)$ (first kind) and $\exp(-i\alpha_1 T), \ldots, \exp(-i\alpha_k T)$ (second kind). Since condition (6.34) is satisfied, there are no coinciding multipliers of different kinds, so that the Hamiltonian $\mathbf{H}(t) = \mathbf{C}''$ is also strongly stable. By Lemma I, there exists a nonsingular real matrix $\mathbf{S}'$ such that

$$(\mathbf{S}')^{-1}\mathbf{J}\mathbf{C}''\mathbf{S}' = \alpha_1\mathbf{J}_2 \dotplus \ldots \dotplus \alpha_k\mathbf{J}_2, \quad (\mathbf{S}')^{\tau}\mathbf{J}\mathbf{S}' = \mathbf{J}_2 \dotplus \ldots \dotplus \mathbf{J}_2. \tag{6.43}$$

Thus the right-hand sides of (6.33) and (6.43) for the matrices $\mathbf{C}$ and $\mathbf{C}''$ are identical. Therefore,

$$\mathbf{S}^{-1}\mathbf{J}\mathbf{C}\mathbf{S} = (\mathbf{S}')^{-1}\mathbf{J}\mathbf{C}''\mathbf{S}', \quad \mathbf{S}^{\tau}\mathbf{J}\mathbf{S} = (\mathbf{S}')^{\tau}\mathbf{J}\mathbf{S}'.$$

Denoting $\mathbf{S}(\mathbf{S}')^{-1} = \mathbf{U}$, we have

$$\mathbf{U}^{-1}\mathbf{J}\mathbf{C}\mathbf{U} = \mathbf{J}\mathbf{C}'' \quad \text{and} \quad \mathbf{U}^{\tau}\mathbf{J}\mathbf{U} = \mathbf{J}. \tag{6.44}$$

The second equality shows that $\mathbf{U} \in \mathfrak{G}$, while the first (combined with the second) implies (6.42).

Conversely, if $\mathbf{U} \in \mathfrak{G}$ and (6.42) is true, then (6.44) is also true, whence it follows that the matrices $\mathbf{J}^{-1}\mathbf{C}$ and $\mathbf{J}^{-1}\mathbf{C}''$ have the same eigenvalues, and these are of the same kinds, so that $i\alpha_1, \ldots, i\alpha_k$ are the eigenvalues of the first kind of $\mathbf{J}^{-1}\mathbf{C}$. Since (6.34) holds, it follows that the Hamiltonian $\mathbf{H}(t) = \mathbf{C}$ is strongly stable, proving Lemma II.

It can be shown that all constant Hamiltonians $\mathbf{H}(t) = \mathbf{C}$ for which conditions (6.33) and (6.34) hold with the same numbers m_{jh} belong to the same stability domain. Indeed, it is known that the symplectic group $\mathfrak{G}$ is connected. Joining $\mathbf{U} \in \mathfrak{G}$ by a curve $\mathbf{U}(t)$ to $\mathbf{I}_{2k}$, we see via (6.44) that any such matrix $\mathbf{C}$ may be joined continuously to a matrix $\mathbf{C}''$ of the above type. It is moreover obvious that two diagonal matrices $\mathbf{C}_1''$ and $\mathbf{C}_2''$ (satisfying (6.34) with the same m_{jh}'s) may be transformed into one another without violating (6.34). Thus, any two constant Hamiltonians with the same numbers m_{jh} may be transformed into one another without affecting the strong stability conditions, and so they are in the same stability domain.

The converse, however, is false. It can be shown that two strongly stable constant Hamiltonians satisfying conditions (6.34) with different numbers m_{jh} may belong to the same stability domain. This means that they may be joined by a strongly stable curve, but some points of this curve will not be constant Hamiltonians but time-dependent.

Later we shall need the following simple proposition.

Lemma III. Equation (6.35) with a positive definite constant Hamiltonian is stable. Let $\pm i\alpha_1, \ldots, \pm i\alpha_k$ be the eigenvalues of the matrix $\mathbf{J}^{-1}\mathbf{C}$, with the sign so chosen that $\alpha_j > 0$. Then $i\alpha_1, \ldots, i\alpha_k$ are the eigenvalues of the first kind (so that $-i\alpha_1, \ldots, -i\alpha_k$ are the eigenvalues of the second kind).

Proof. The trivial solution of the equation is stable. This follows immediately from Lyapunov's stability theorem, if we define the Lyapunov function to be the quadratic form $V = (\mathbf{C}\mathbf{x}, \mathbf{x})$ which, by assumption, is positive definite. Indeed,

$$\frac{dV}{dt} = (\mathbf{CJ^{-1}Cx},\ \mathbf{x}) + (\mathbf{Cx},\ \mathbf{J^{-1}Cx}) = -(\mathbf{CJCx},\ \mathbf{x}) + (\mathbf{CJCx},\ \mathbf{x}) \equiv 0,$$

so that the assumption of Lyapunov's theorem is satisfied.

Since we are dealing with a Hamiltonian equation, it follows from the stability of the trivial solution that all the eigenvalues of $\mathbf{J^{-1}C}$ are pure imaginary and have simple elementary divisors. Let $\pm i\alpha_1, \ldots, \pm i\alpha_k$, $\alpha_j > 0$, be these eigenvalues. We shall show that $i\alpha_1, \ldots, i\alpha_k$ are the eigenvalues of the first kind. Let $\mathbf{a}_j$ be eigenvectors such that $\mathbf{J^{-1}Ca}_j = i\alpha_j \mathbf{a}_j$, $\alpha_j > 0$ $(j = 1, \ldots, k)$. Let $i\alpha_j$ be a simple eigenvalue. Then

$$\langle \mathbf{a}_j, \mathbf{a}_j \rangle = i(\mathbf{Ja}_j, \mathbf{a}_j) = -\frac{1}{\alpha_j}(\mathbf{Ja}_j, i\alpha_j \mathbf{a}_j) =$$
$$= -\frac{1}{\alpha_j}(\mathbf{Ja}_j, \mathbf{J^{-1}Ca}_j) = -\frac{1}{\alpha_j}(\mathbf{JJa}_j, \mathbf{Ca}_j) = \frac{1}{\alpha_j}(\mathbf{Ca}_j, \mathbf{a}_j) > 0,$$

i. e., α_j is an eigenvalue of the first kind. Let $i\alpha_1 = \ldots = i\alpha_p = i\alpha$ be an eigenvalue of multiplicity $p \leqslant k$. Then the Gram matrix of the corresponding eigensubspace

$$\|\langle \mathbf{a}_h, \mathbf{a}_j \rangle\|_1^p = \frac{1}{\alpha}\|(\mathbf{Ca}_h, \mathbf{a}_j)\|_1^p$$

is positive definite (since $\mathbf{C} > 0$ and $\alpha > 0$). This proves Lemma III.

Corollary. The Hamiltonian $\varepsilon\mathbf{C}$, where $\mathbf{C} > 0$ and ε is a sufficiently small number, is strongly stable and belongs to the stability domain $G_0^{(\mu)}$ of multiplier type $\mu = (+, \ldots, +)$.

Indeed, it follows from the above arguments that the eigenvalues of the first kind for the matrizant $\mathbf{X}(t) = \exp(\varepsilon t \mathbf{J^{-1}C})$ are $\rho_j(t) = \exp(i\varepsilon\alpha_j t)$ $(j = 1, \ldots, k)$. For sufficiently small $\varepsilon > 0$ and $0 \leqslant t \leqslant T$ all these points move from $\rho = 1$ along the upper unit semicircle. According to (5.4), $n = 0$. Since all multipliers $\rho_j(T)$ of the first kind are on the upper unit semicircle (provided $\varepsilon > 0$ is sufficiently small), it follows that $\mu = (+, \ldots, +)$.

6.6. Strong stability test for canonical equations with almost constant Hamiltonians

Consider the "unperturbed" canonical equation

$$\mathbf{J}\frac{d\mathbf{x}}{dt} = \mathbf{Cx}$$

with constant Hamiltonian $\mathbf{C} = \mathbf{C}^*$, which we assume to be strongly stable, and the "perturbed" equation

$$\mathbf{J}\frac{d\mathbf{x}}{dt} = [\mathbf{C} + \mathbf{H}_0(t)]\,\mathbf{x}. \tag{6.45}$$

The results of the last two subsections provide an immediate answer to the question as to how "small" the perturbation $\mathbf{H}_0(t)$ has to be for equation (6.45) to be strongly stable as well.

Let $\mathbf{S}$ and $\mathbf{U}$ be the matrices whose existence is proved in Lemmas I and II of §6.5, satisfying conditions (6.33) and (6.42). As before, we define integers m_{jh} by (6.34).

Test I. Let $\varphi_1(t)$ and $\varphi_2(t)$ be functions such that either

$$\varphi_1(t)\,\mathbf{I}_{2k} \leqslant \mathbf{S}^{\tau}\mathbf{H}_0(t)\,\mathbf{S} \leqslant \varphi_2(t)\,\mathbf{I}_{2k} \tag{6.46}$$

or

$$\varphi_1(t)\,\mathbf{I}_{2k} \leqslant \mathbf{U}^{\tau}\mathbf{H}_0(t)\,\mathbf{U} \leqslant \varphi_2(t)\,\mathbf{I}_{2k}. \tag{6.47}$$

Equation (6.45) is strongly stable and the Hamiltonian $\mathbf{C}+\mathbf{H}_0(t)$ is in the same stability domain as $\mathbf{C}$ if, besides inequalities (6.34),

$$2m_{jh}\pi < (\alpha_j+\alpha_h)\,T + \int_0^T \varphi_1(t)\,dt \leqslant$$

$$\leqslant (\alpha_j+\alpha_h)\,T + \int_0^T \varphi_2(t)\,dt < 2\,(m_{jh}+1)\,\pi. \tag{6.48}$$

Remark. Equation (6.45) is strongly stable if condition (6.48) holds for numbers m_{jh} which are not necessarily those defined by (6.34). In that case, though, the Hamiltonian $\mathbf{C}$ and $\mathbf{C}+\mathbf{H}_0(t)$ may belong to different stability domains.

Proof. Let condition (6.46) hold. Substitute $\mathbf{y}=\mathbf{S}\mathbf{x}$ (§6.4) in equations (6.35), (6.45). The result is

$$\mathbf{J}'\frac{d\mathbf{y}}{dt} = \mathbf{C}'\mathbf{y}, \tag{6.49}$$

$$\mathbf{J}'\frac{d\mathbf{y}}{dt} = [\mathbf{C}'+\mathbf{S}^{\tau}\mathbf{H}_0(t)\,\mathbf{S}]\,\mathbf{y}. \tag{6.50}$$

Here we have used (6.40) and (6.41). System (6.50) has the form of (6.25) with $\mathbf{H}'(t)=\mathbf{C}'$. Inequalities (6.27) become (6.48). Applying Test II of §6.4, we deduce the desired conclusion for this case.

Now suppose that (6.47) holds. Substitute $\mathbf{y}=\mathbf{U}\mathbf{x}$ in equation (6.45). Using the symplectic condition $\mathbf{U}^{\tau}\mathbf{J}\mathbf{U}=\mathbf{J}$ and conditions (6.42), we obtain the equations

$$\mathbf{J}\frac{d\mathbf{y}}{dt} = \mathbf{C}'\mathbf{y}, \tag{6.51}$$

$$\mathbf{J}\frac{d\mathbf{y}}{dt} = [\mathbf{C}'+\mathbf{U}^{\tau}\mathbf{H}_0(t)\,\mathbf{U}]\,\mathbf{y}. \tag{6.52}$$

Writing equation (6.52) in coordinate notation, we obtain a system which separates into k equations of type (6.19). Thus there exists an orthogonal matrix $\mathbf{S}$ such that the substitution $\mathbf{y}=\mathbf{S}\mathbf{x}$ reduces equation (6.51) to the form (6.20). (In this case the substitution is simply a permutation of the coordinates of $\mathbf{x}$.) Using Remark 2 to Test II in §6.4, we see that Test II

is applicable. Equations (6.51), (6.52) will then coincide respectively with equations (6.25), (6.29), and $h_j^{(0)}(t) = \varphi_j(t)$. Inequalities (6.27) assume the form (6.48), completing the proof.

We now consider a canonical system of k second-order equations; in vector notation:

$$\frac{d^2\mathbf{y}}{dt^2} + [\mathbf{P}_0 + \mathbf{R}(t)]\,\mathbf{y} = \mathbf{0}. \tag{6.53}$$

Here $\mathbf{y}$ is a k-vector, $\mathbf{P}_0$ and $\mathbf{R}(t)$ are real symmetric matrices, $\mathbf{R}(t)$ is a small "perturbation," $\mathbf{R}(t+T) = \mathbf{R}(t)$, $\mathbf{P}_0$ is a constant matrix. The "unperturbed" equation $\ddot{\mathbf{y}} + \mathbf{P}_0\mathbf{y} = \mathbf{0}$ is stable if and only if $\mathbf{P}_0 > \mathbf{0}$, and we assume this condition fulfilled. Let $\omega_1^2 \leqslant \ldots \leqslant \omega_k^2$ denote the eigenvalues of $\mathbf{P}_0$. (The numbers $\omega_\varkappa > 0$ are the natural frequencies of the "unperturbed" equation.)

Let $\mathbf{U}_0$ be a real orthogonal $k \times k$ matrix such that $\mathbf{U}_0^{-1}\mathbf{P}_0\mathbf{U}_0$ is a diagonal matrix, with diagonal elements $\omega_1^2, \ldots, \omega_k^2$ (§ I. 1.14):

$$\mathbf{U}_0^{-1}\mathbf{P}_0\mathbf{U}_0 = \operatorname{diag}(\omega_1^2, \ldots, \omega_k^2).$$

Define real symmetric positive definite matrices $\mathbf{P}_0^{1/2}$, $\mathbf{P}_0^{1/4}$ and $\mathbf{P}_0^{-1/4}$ by

$$\mathbf{P}_0^{1/2} = \mathbf{U}_0 \operatorname{diag}(\omega_1, \ldots, \omega_k)\,\mathbf{U}_0^{-1}, \quad \mathbf{P}_0^{1/4} = \mathbf{U}_0 \operatorname{diag}(\omega_1^{1/2}, \ldots, \omega_k^{1/2})\,\mathbf{U}_0^{-1},$$
$$\mathbf{P}_0^{-1/4} = \mathbf{U}_0 \operatorname{diag}(\omega_1^{-1/2}, \ldots, \omega_k^{-1/2})\,\mathbf{U}_0^{-1}.$$

The substitution

$$\mathbf{x} = \begin{pmatrix} \mathbf{P}_0^{1/4}\,\mathbf{y} \\ \mathbf{P}_0^{-1/4}\,\dot{\mathbf{y}} \end{pmatrix}$$

reduces equation (6.53) to the canonical equation

$$\mathbf{J}\frac{d\mathbf{x}}{dt} = [\mathbf{C} + \mathbf{H}_0(t)]\,\mathbf{x},$$

where

$$\mathbf{C} = \begin{pmatrix} \mathbf{P}_0^{1/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_0^{1/2} \end{pmatrix}, \quad \mathbf{H}_0(t) = \begin{pmatrix} \mathbf{P}_0^{-1/4}\,\mathbf{R}(t)\,\mathbf{P}_0^{-1/4} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

We apply Test II of §6.4 to this equation. The matrix

$$\mathbf{U} = \begin{pmatrix} \mathbf{U}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_0 \end{pmatrix}$$

is symplectic and $\mathbf{U}^\tau\mathbf{C}\mathbf{U}$ is a diagonal matrix with diagonal elements $\omega_1, \ldots, \omega_k; \omega_1, \ldots, \omega_k$. Hence, by Lemma II of §6.5, the numbers $i\omega_1, \ldots, i\omega_k$ are the eigenvalues of the first kind of the matrix $\mathbf{J}^{-1}\mathbf{C}$.

Since $\mathbf{U}$ is orthogonal, the matrices $\mathbf{U}^{\mathrm{T}}\mathbf{H}_0(t)\mathbf{U}$ and $\mathbf{H}_0(t)$ have the same eigenvalues. Let $\lambda_1(t)$ and $\lambda_2(t)$ denote respectively the minimum and maximum eigenvalues of the matrix $\mathbf{P}_0^{-1/4}\mathbf{R}(t)\mathbf{P}_0^{-1/4}$, in other words, the smallest and largest roots of the equation

$$\det[\mathbf{R}(t)-\lambda\mathbf{P}_0^{1/2}]=0. \tag{6.54}$$

Then the corresponding eigenvalues of $\mathbf{H}_0(t)$ will be $\lambda_1^{(-)}(t)=\min[\lambda_1(t),\ 0]$, $\lambda_2^{(+)}(t)=\max[\lambda_2(t),\ 0]$, or

$$\lambda_1^{(-)}(t)=\frac{1}{2}[\lambda_1(t)-|\lambda_1(t)|],\quad \lambda_2^{(+)}(t)=\frac{1}{2}[\lambda_2(t)+|\lambda_2(t)|]. \tag{6.55}$$

The conditions for strong stability (6.23) of the "unperturbed" equation are

$$(\omega_j+\omega_h)T\neq 2\pi m \qquad (m=1,\ 2,\ldots) \tag{6.56}$$

for all $j,\ h=1,\ \ldots,\ k$. Test II of §6.4 now implies

Test II. Suppose that the matrix $\mathbf{P}_0$ in system (6.53) has eigenvalues $\omega_1^2,\ \ldots,\ \omega_k^2$ $(\omega_\varkappa>0)$ and that inequalities (6.56) are satisfied. Define non-negative integers m_{jh} by

$$2\pi m_{jh}<(\omega_j+\omega_h)T<2\pi(m_{jh}+1) \qquad (j,\ h=1,\ \ldots,\ k). \tag{6.57}$$

Define functions $\lambda_1^{(-)}(t)$ and $\lambda_2^{(+)}(t)$ by (6.55). Suppose that, apart from inequalities (6.57), the numbers and functions so defined satisfy the inequalities

$$2\pi m_{jh}<(\omega_j+\omega_h)T+\int_0^T\lambda_1^{(-)}(t)\,dt\leqslant(\omega_j+\omega_h)T+\int_0^T\lambda_2^{(+)}(t)\,dt<2\pi(m_{jh}+1) \tag{6.58}$$

for all $j,\ h=1,\ \ldots,\ k$. Then system (6.53) is strongly stable.

We might point out here that a great many dynamic stability problems for elastic systems, of practical importance, may be reduced to systems of type (6.53), where the matrix $\mathbf{R}(t)=\varphi(t)\mathbf{R}_0$ is a constant matrix $\mathbf{R}_0$ up to a scalar factor. It is then quite easy to evaluate $\lambda_1^{(-)}(t)$ and $\lambda_2^{(+)}(t)$.

Theorem I. Suppose that the canonical systems with constant real symmetric matrices $0<\mathbf{C}_1\leqslant\mathbf{C}_2$ are strongly stable. Then a system with any real symmetric T-periodic Hamiltonian $\mathbf{H}(t)$ such that $\mathbf{C}_1\leqslant\mathbf{H}(t)\leqslant\mathbf{C}_2$ is strongly stable if and only if the numbers m_{jh} in (6.34) are the same for both matrices $\mathbf{C}_1$ and $\mathbf{C}_2$.

Proof. Set $\mathbf{C}(s)=\mathbf{C}_1+s(\mathbf{C}_2-\mathbf{C}_1)$ $(0\leqslant s\leqslant 1)$. We have $\mathbf{C}(0)=\mathbf{C}_1$, $\mathbf{C}(1)=\mathbf{C}_2$ and $d\mathbf{C}/ds=\mathbf{C}_2-\mathbf{C}_1\geqslant 0$. By Lemma III of §6.5, the trivial solution of the canonical equation

$$\mathbf{J}\frac{d\mathbf{x}}{dt}=\mathbf{C}(s)\mathbf{x} \tag{6.59}$$

is stable. Let $\pm i\alpha_1(s),\ \ldots,\ \pm i\alpha_k(s)$ be the eigenvalues of $\mathbf{J}^{-1}\mathbf{C}(s)$, and $\alpha_j(s)>0$ $(j=1,\ \ldots,\ k)$. By Lemma III of §6.5, the numbers $i\alpha_1(s),\ \ldots,\ i\alpha_k(s)$

are the eigenvalues of the first kind. Consequently, the eigenvalues of the first kind for equation (6.59) are $\rho_j^{(+)}(s) = \exp[i\alpha_j(s)T]$ $(j=1, \ldots, k)$. We claim that $\alpha_1(s), \ldots, \alpha_k(s)$ are nondecreasing functions of s.

Since $d\mathbf{C}/ds = \mathbf{C}_2 - \mathbf{C}_1 \geqslant 0$, it follows from §1.7 that the multipliers $\rho_j^{(+)}(s)$ are analytic functions of s and

$$\frac{d}{ds}\arg\rho_j^{(+)}(s) \geqslant 0 \qquad (j=1, \ldots, k) \tag{6.60}$$

for all s such that no multipliers of different kinds coincide. This is true if

$$T[\alpha_j(s)+\alpha_h(s)] \neq 2m\pi \qquad (m=0, \pm 1, \pm 2, \ldots). \tag{6.61}$$

For any fixed s, we can choose T in such a way that (6.61) holds. Thus $\rho_j^{(+)}(s)$, and therefore also $\alpha_j(s)$, are analytic functions of s for any real s. It follows from (6.60) that

$$\frac{d\alpha_j(s)}{ds} \geqslant 0 \qquad (j=1, \ldots, k), \tag{6.62}$$

which verifies our assertion. Note that this may also be proved directly, along the same lines as in §6.5.

It follows from (6.62) that

$$\alpha_j(0) \leqslant \alpha_j(s) \leqslant \alpha_j(1) \quad \text{for} \quad 0 \leqslant s \leqslant 1 \qquad (j=1, \ldots, k).$$

By assumption, the matrices $\mathbf{C}_1$ and $\mathbf{C}_2$ satisfy conditions (6.34) with the same numbers m_{jh}, i. e.,

$$2m_{jh}\pi < [\alpha_j(0)+\alpha_h(0)]T \leqslant [\alpha_j(1)+\alpha_h(1)]T < 2(m_{jh}+1)\pi \quad (j, h=1, \ldots, k),$$

and so, for $0 \leqslant s \leqslant 1$ and all $j, h = 1, \ldots, k$,

$$2m_{jh}\pi < [\alpha_j(s)+\alpha_h(s)]T < 2(m_{jh}+1)\pi, \tag{6.63}$$

whence it follows that system (6.59) is strongly stable for any $0 \leqslant s \leqslant 1$. By the directional wideness theorem (§6.2) any Hamiltonian $\mathbf{H}(t)$ such that $\mathbf{C}_1 \leqslant \mathbf{H}(t) \leqslant \mathbf{C}_2$ is strongly stable. This proves sufficiency.

To prove necessity, we shall assume that not all the conditions (6.34) hold for $\mathbf{C}_1$ and $\mathbf{C}_2$ with the same m_{jh}'s; in other words, for some j, h,

$$2m_{jh}^{(0)}\pi < [\alpha_j(0)+\alpha_h(0)]T < 2m\pi < [\alpha_j(1)+\alpha_h(1)]T < 2(m_{jh}^{(1)}+1)\pi,$$

where m is a natural number. By continuity, for some s_0 $(0 < s_0 < 1)$,

$$[\alpha_j(s_0)+\alpha_h(s_0)]T = 2m\pi, \tag{6.64}$$

so that system (6.59) is weakly stable for $s = s_0$. This completes the proof of Theorem I.

Remark 1. Let $\mathbf{C}_1 < \mathbf{C}_2$ and suppose that the matrices $\mathbf{C}_1$ and $\mathbf{C}$ do not have the same sets of numbers m_{jh}. Then there exists a matrix $\mathbf{H}(t)$ such that $\mathbf{C}_1 < \mathbf{H}(t) < \mathbf{C}$ and the system

$$\mathbf{J}\frac{d\mathbf{x}}{dt} = \mathbf{H}(t)\,\mathbf{x}$$

is unstable. The matrix $\mathbf{H}(t)$ may be chosen sufficiently close to a matrix of the form $\mathbf{C}(s_0)$, where s_0 is the number in (6.64). This follows directly from the definition of weak stability.

Remark 2. The condition $0 < \mathbf{C}_1 \leqslant \mathbf{C}_2$ may of course be replaced by $\mathbf{C}_2 \leqslant \mathbf{C}_1 < 0$.

When one of the matrices $\mathbf{C}_1$ and $\mathbf{C}_2$ is not sign definite, the situation becomes substantially more complicated. Before presenting the relevant theorem, we introduce some notation.

Let $\mathbf{C}$ be some constant strongly stable Hamiltonian and $i\alpha_1, \ldots, i\alpha_k$ the eigenvalues of the first kind of $\mathbf{J}^{-1}\mathbf{C}$. Denote

$$\left.\begin{aligned} \beta_j^+ &= \alpha_j - m_{jj}\frac{\pi}{T}, \quad \text{if} \quad m_{jj} \text{ is even,} \\ \beta_j^- &= (m_{jj}+1)\frac{\pi}{T} - \alpha_j, \quad \text{if} \quad m_{jj} \text{ is odd.} \end{aligned}\right\} \tag{6.65}$$

Then, by (6.34),

$$0 < \beta_j^{\pm} < \frac{\pi}{T}.$$

The eigenvalues of the first kind for the matrizant $\mathbf{X}(t)$ of the equation $\mathbf{J}\dot{\mathbf{x}} = \mathbf{C}\mathbf{x}$ are the numbers $\rho_j^{(+)}(t) = \exp\left(\frac{2\pi i}{T}\alpha_j t\right)$; thus it is easy to determine the multiplier type μ and index n of the stability domain $\mathcal{O}_n^{(\mu)}$ containing the constant Hamiltonian $\mathbf{C}$.

Theorem II. Let $\mathbf{C}_1 \leqslant \mathbf{H}(t) \leqslant \mathbf{C}_2$, *where* $\mathbf{C}_1, \mathbf{C}_2 \in \mathcal{O}_n^{(\mu)}$. *Let the interval* $(0, \pi/T)$ *be divided into* r *subintervals* $\Delta_1, \ldots, \Delta_r$ *such that the following conditions are satisfied:*

1) *On each subinterval,* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *have the same number of numbers* β_j^+ *and* β_h^-.

2) *On each of the subintervals, the numbers* β_j^+ *and* β_h^- *for* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *are arranged in the same order: first the numbers* β_h^+ *and then the* β_h^-, *or vice versa, and* $\beta_j^+ \neq \beta_h^-$ $(j, h = 1, \ldots, k)$.

Then the equation with Hamiltonian $\mathbf{H}(t)$ *is strongly stable and* $\mathbf{H}(t) \in \mathcal{O}_n^{(\mu)}$.

In particular, these conditions are satisfied if for both $\mathbf{C}_1$ *and* $\mathbf{C}_2$

$$0 < \beta_1^+ \leqslant \beta_2^+ \leqslant \ldots \leqslant \beta_l^+ < \beta_{l+1}^- \leqslant \ldots \leqslant \beta_k^- < \frac{\pi}{T}$$

or

$$0 < \beta_1^- \leqslant \beta_2^- \leqslant \ldots \leqslant \beta_l^- < \beta_{l+1}^+ \leqslant \ldots \leqslant \beta_k^+ < \frac{\pi}{T},$$

where l *is one of the numbers* $1, \ldots, k$.

Proof. Given a system with Hamiltonian $\mathbf{C} \in \mathcal{G}_n^{(\mu)}$, the numbers $\rho_j^{(+)} = \exp(i\beta_j T)$ are multipliers of the first kind and $\rho_j^{(-)} = \exp(i\beta_j T)$ multipliers of the second kind, on the upper half of the unit circle. Thus Theorem II is a corollary of Theorem II in §6.3.

6.7. Strong stability tests for $2k$-th order systems (corollaries of tests for second-order systems)

It will now be shown that any strong stability test for a canonical system of second order immediately yields a test for canonical systems of arbitrary even order $2k$. The arguments are analogous to those in §6.4.

Consider a canonical equation, separable into k canonical second-order systems

$$\mathbf{J}_2 \frac{d\mathbf{x}_\varkappa}{dt} = \mathbf{H}^{(\varkappa)}(t)\,\mathbf{x}_\varkappa \qquad (\varkappa = 1, \ldots, k). \tag{6.66}$$

Here $\mathbf{H}^{(\varkappa)}(t)$ are second-order Hamiltonians:

$$\mathbf{H}^{(\varkappa)}(t) = \begin{Vmatrix} \alpha_\varkappa(t) & \beta_\varkappa(t) \\ \beta_\varkappa(t) & \gamma_\varkappa(t) \end{Vmatrix}.$$

System (6.66) may be written as a single vector equation

$$\mathbf{J}' \frac{d\mathbf{x}}{dt} = \mathbf{H}'(t)\,\mathbf{x}, \tag{6.67}$$

where $\mathbf{J}'$ has the form of (6.21); $\mathbf{x} = \mathbf{x}_1 \dot{+} \ldots \dot{+} \mathbf{x}_k$ is the direct sum of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$, and

$$\mathbf{H}'(t) = \mathbf{H}^{(1)}(t) \dot{+} \ldots \dot{+} \mathbf{H}^{(k)}(t). \tag{6.68}$$

Any stability domain $\mathcal{G}_n^{(\mu)}$ (in the space of Hamiltonians for the equation $\mathbf{J}'\dot{\mathbf{x}} = \mathbf{H}(t)\,\mathbf{x}$) contains Hamiltonians of type (6.68). Indeed, this is true, as shown in §6.4, even for scalar Hamiltonians

$$\mathbf{H}_j(t) = h_j(t)\,\mathbf{I}_2.$$

Let $\mathbf{H}_j(t)$ be a second-order Hamiltonian in the stability domain $\mathcal{G}_n^{(\mu)}$ (in the Hamiltonian space for the second-order equation $\mathbf{J}_2\dot{\mathbf{x}} = \mathbf{H}_j(t)\,\mathbf{x}$). The multiplier type μ may be one of two: either $\mu_j = (+)$ (multiplier of the first kind in the upper half-plane) or $\mu_j = (-)$ (multiplier of the first kind in the lower half-plane). We claim that the multiplier type μ and index n of the stability domain $\mathcal{G}_n^{(\mu)}$ containing the Hamiltonian (6.68) are defined by

$$\mu = (\mu_1, \ldots, \mu_k), \qquad n = n_1 + \ldots + n_k. \tag{6.69}$$

Since the set of multipliers of equation (6.67) is the union of the sets of multipliers for all equations (6.66), it follows that the multiplier type of $\mathbf{H}'(t)$ is $\mu=(\mu_1, \ldots, \mu_k)$. (This furnishes another proof of the fact that there are 2^k possible multiplier types for equations of type (6.67).)

Let $\rho_j(t)$ be a multiplier of the first kind for the matrizant $\mathbf{X}_j(t)$ of equation (6.66). Since the matrizant of equation (6.67) is

$$\mathbf{X}(t)=\mathbf{X}_1(t)\dot{+}\ldots\dot{+}\mathbf{X}_k(t), \tag{6.70}$$

it follows that the k eigenvalues of the first kind of the matrizant (6.70) are $\rho_1(t), \ldots, \rho_k(t)$. By (5.4),

$$\Delta \operatorname{Arg}\rho_j(t)\Big|_0^T-\vartheta_j=2n_j\pi, \quad \sum_{j=1}^{k}[\Delta \operatorname{Arg}\rho_j(t)\Big|_0^T-\vartheta_j]=2n\pi.$$

Here ϑ_j are the principal arguments of the numbers $\rho_j(T)$ and n is the index of the domain $\mathcal{O}_n^{(\mu)}$ that contains $\mathbf{H}'(t)$. Consequently, $n=n_1+\ldots+n_k$, as claimed. Since the n_j's are arbitrary integers, the number n is also arbitrary. Thus we have another proof of the fact that there are Hamiltonians of the form (6.68) in any stability domain $\mathcal{O}_n^{(\mu)}$.

Consider an arbitrary canonical equation

$$\mathbf{J}'\frac{d\mathbf{x}}{dt}=\mathbf{H}(t)\,\mathbf{x} \tag{6.71}$$

and express its Hamiltonian as

$$\mathbf{H}(t)=\mathbf{H}'(t)+\mathbf{H}_0(t), \tag{6.72}$$

where $\mathbf{H}'(t)$ is a block-diagonal matrix of the form (6.68). (This representation is of course far from unique.) As in §6.4, let $h_1^{(0)}(t)$ and $h_2^{(0)}(t)$ be any functions such that

$$h_1^{(0)}(t)\,\mathbf{I}_{2k}\leqslant\mathbf{H}_0(t)\leqslant h_2^{(0)}(t)\,\mathbf{I}_{2k}. \tag{6.73}$$

(For example, $h_1^{(0)}(t)$ and $h_2^{(0)}(t)$ may be the minimum and maximum eigenvalues of the matrix $\mathbf{H}_0(t)$.) It follows from (6.72) that

$$\mathbf{H}_1(t)\leqslant\mathbf{H}(t)\leqslant\mathbf{H}_2(t), \tag{6.74}$$

where

$$\mathbf{H}_j(t)=\mathbf{H}'(t)+h_j^{(0)}(t)\,\mathbf{I}_{2k} \qquad (j=1, 2). \tag{6.75}$$

The Hamiltonians $\mathbf{H}_1(t)$ and $\mathbf{H}_2(t)$ have the form of (6.68). The appropriate canonical equations separate into k second-order equations, which may be checked for strong stability by any of the tests relating to second-order systems. The directional wideness theorem now enables us to derive stability conditions for the original equation (6.71). Indeed, setting $\mathbf{H}(t, s)=\mathbf{H}_1(t)+s[\mathbf{H}_2(t)-\mathbf{H}_1(t)]$, we see that $\mathbf{H}(t, s)$ has the form

$$\mathbf{H}(t, s) = \mathbf{H}^{(1)}(t, s) \dotplus \ldots \dotplus \mathbf{H}^{(k)}(t, s), \tag{6.76}$$

where

$$\mathbf{H}^{(j)}(t, s) = \mathbf{H}^{(j)}(t) + \{h_1^{(0)}(t) + s[h_2^{(0)}(t) - h_1^{(0)}(t)]\}\mathbf{I}_2 \quad (j = 1, \ldots, k). \tag{6.77}$$

If $\mathbf{H}^{(j)}(t, s) \in \mathcal{C}_{n_j}^{(\mu_j)}$ for $0 \leqslant s \leqslant 1$, it follows from the directional wideness theorem that equation (6.71) is strongly stable. Moreover, $\mathbf{H}(t, s) \in \mathcal{C}_n^{(\mu)}$, where μ and n are determined by (6.69). By Remark 1 in §6.2, we have also $\mathbf{H}(t) \in \mathcal{C}_n^{(\mu)}$.

The result may be stated as follows.

Theorem I. Express the Hamiltonian of equation (6.71) as (6.72), where $\mathbf{H}'(t)$ is of the form (6.68). Suppose that condition (6.73) holds. Suppose that for any $0 \leqslant s \leqslant 1$ all the second-order canonical equations $\mathbf{J}_2\dot{\mathbf{x}}_j = \mathbf{H}^{(j)}(t, s)\,\mathbf{x}_j$ $(j = 1, \ldots, k)$, where $\mathbf{H}^{(j)}(t, s)$ is given by (6.77), are strongly stable. Then equation (6.71) is strongly stable. If $\mathbf{H}^{(j)}(t, s) \in \mathcal{C}_{n_j}^{(\mu_j)}$, then $\mathbf{H}(t) \in \mathcal{C}_n^{(\mu)}$, where μ and n are given by (6.69).

Analogous reasoning proves the following theorem about the second-order vector equation

$$\frac{d^2\mathbf{y}}{dt^2} + \mathbf{P}(t)\,\mathbf{y} = 0. \tag{6.78}$$

(As usual, $\mathbf{P}(t)^{\mathrm{T}} = \mathbf{P}(t) = \mathbf{P}(t+T)$ is a $k \times k$ matrix.)

*Theorem II. Express the matrix $\mathbf{P}(t)$ as**

$$\mathbf{P}(t) = \operatorname{diag} \| p_1(t), \ldots, p_k(t) \| + \mathbf{R}(t), \tag{6.79}$$

where $\mathbf{R}(t+T) = \mathbf{R}(t)$, $p_j(t+T) = p_j(t)$. Let

$$r_1(t)\,\mathbf{I}_k \leqslant \mathbf{R}(t) \leqslant r_2(t)\,\mathbf{I}_k. \tag{6.80}$$

If the scalar equations

$$\ddot{y}_j + p_j(t, s)\,y_j = 0 \qquad (j = 1, \ldots, k), \tag{6.81}$$

where

$$p_j(t, s) = p_j(t) + r_1(t) + s[r_2(t) - r_1(t)], \tag{6.82}$$

are strongly stable for any $0 \leqslant s \leqslant 1$, then the vector equation (6.78) is also strongly stable.

Theorems I and II point to the special significance of strong stability tests for the second-order canonical equation and for Hill's equation $\ddot{y} + p(t)\,y = 0$. Strong stability tests for these equations will be derived by special methods in Chapter VIII.

* Recall that the first term in (6.79) denotes the diagonal matrix with diagonal elements $p_1(t), \ldots, p_k(t)$.

6.8. Estimate of the small parameter for which the method of averaging may be applied to linear Hamiltonian systems with periodic coefficients

Consider the canonical equation

$$\mathbf{J}\frac{d\mathbf{x}}{dt}=\varepsilon\left[\mathbf{H}(t)+\varepsilon\mathbf{H}_1(t,\varepsilon)\right]\mathbf{x}, \tag{6.83}$$

where ε is a small real parameter and the matrix $\mathbf{H}_1(t,\varepsilon)$ satisfies the condition

$$\lim_{\varepsilon\to 0}\varepsilon\int_0^T|\mathbf{H}_1(t,\varepsilon)|\,dt=0. \tag{6.84}$$

Dropping the "small correction" $\varepsilon\mathbf{H}_1(t,\varepsilon)$ and replacing $\mathbf{H}(t)$ by

$$\mathbf{C}=[\mathbf{H}(t)]_{av}=\frac{1}{T}\int_0^T\mathbf{H}(t)\,dt,$$

we obtain the "simplified" system

$$\mathbf{J}\frac{d\mathbf{y}}{dt}=\varepsilon\mathbf{C}y. \tag{6.85}$$

It is well known (this follows, e. g., from the Krylov-Bogolyubov-Mitropol'skii method of averaging /66b/) that under certain fairly broad conditions the stability of equation (6.85) implies that of equations (6.83) for small values of ε. We are going to prove this statement, and as a by-product we shall find estimates for the relevant values of ε.

Theorem. Let equation (6.85) be strongly stable for all sufficiently small $\varepsilon>0$ ($\varepsilon<0$). Then, given the coefficients of (6.83), one can effectively determine a number $\varepsilon_0>0$ such that equation (6.83) is strongly stable for any $0<\varepsilon<\varepsilon_0$ ($-\varepsilon_0<\varepsilon<0$) and moreover belongs to the same stability domain as equation (6.85).

Proof. We perform the substitution

$$\mathbf{x}=\mathbf{W}(t,\varepsilon)\mathbf{z},$$

where*

$$\mathbf{W}(t,\varepsilon)=\exp\left[\varepsilon\mathbf{J}^{-1}\mathbf{P}(t)\right],\quad \mathbf{P}(t)=\int_0^t[\mathbf{H}(t_1)-\mathbf{C}]\,dt_1.$$

Since $\mathbf{W}(t,\varepsilon)$ is a symplectic matrix (see §1.1), it follows by the theorem of §1.4 that the system

$$\frac{d\mathbf{z}}{dt}=\left[-\mathbf{W}^{-1}\frac{d\mathbf{W}}{dt}+\varepsilon\mathbf{W}^{-1}\mathbf{J}^{-1}(\mathbf{H}+\varepsilon\mathbf{H}_1)\mathbf{W}\right]\mathbf{z}$$

* In fact, the matrix $\mathbf{W}(t,\varepsilon)$ may be any integrable piecewise-continuous symplectic T-periodic matrix-function whose expansion in powers of ε is

$$\mathbf{W}(t,\varepsilon)=\mathbf{I}_{2k}-\varepsilon\mathbf{J}\mathbf{P}(t)+O(\varepsilon^2).$$

is canonical. Moreover, if this system is in some stability domain for $0<\varepsilon<\varepsilon_0$ $(-\varepsilon_0<\varepsilon<0)$, then the original system is in the same stability domain for the same parameter values. Indeed, for sufficiently small ε the curve $\mathbf{W}(t, \varepsilon)$ will lie in an arbitrarily small neighborhood of the matrix $\mathbf{I}_{2k}$, and so $\Delta \operatorname{Arg} \mathbf{W}(t)\,|_0^T = 0$. But then, by continuity, this will be true for any real ε. The assertion now follows from the theorem of §1.4.

Denote

$$\left.\begin{aligned} \mathbf{V}(t, \varepsilon) &= \frac{1}{\varepsilon^2}\frac{d\mathbf{W}}{dt} + \frac{1}{\varepsilon}\mathbf{J}[\mathbf{H}(t) - \mathbf{C}], \\ \mathbf{Q}(t, \varepsilon) &= \mathbf{W}^\tau \mathbf{J}^{-1}\mathbf{V} + \mathbf{W}^\tau \mathbf{H}_1 \mathbf{W} + \frac{1}{\varepsilon}(\mathbf{W}^\tau - \mathbf{I}_{2k})\mathbf{C} + \frac{1}{\varepsilon}\mathbf{W}^\tau \mathbf{H}(\mathbf{W} - \mathbf{I}_{2k}). \end{aligned}\right\} \tag{6.86}$$

Using the fact that for a symplectic matrix $\mathbf{W}^{-1} = \mathbf{J}^{-1}\mathbf{W}^\tau \mathbf{J}$, we express the system for $\mathbf{z}$ as

$$\frac{d\mathbf{z}}{dt} = \varepsilon \mathbf{J}^{-1}[\mathbf{C} + \varepsilon \mathbf{Q}(t, \varepsilon)]\,\mathbf{z}. \tag{6.87}$$

Denote

$$p(t) = |\mathbf{P}(t)|.$$

Then it follows from (6.86) that

$$|\mathbf{Q}(t, \varepsilon)| \leqslant q(t, \varepsilon) \equiv \frac{1}{\varepsilon}[e^{\varepsilon p(t)} - 1]\{e^{\varepsilon p}|\mathbf{H} - \mathbf{C}| + |\mathbf{C}| + e^{\varepsilon p}|\mathbf{H}|\} + \varepsilon^2 e^{2\varepsilon p}|\mathbf{H}_1|. \tag{6.88}$$

Let $i\alpha_1, \ldots, i\alpha_k$ be the eigenvalues of the first kind of $\mathbf{J}^{-1}\mathbf{C}$. System (6.85) is strongly stable for values of ε such that

$$2m_{jh}\pi < \varepsilon(\alpha_j + \alpha_h)T < 2(m_{jh} + 1)\pi \tag{6.89}$$

for all $j, h = 1, \ldots, k$. Here the numbers m_{jh} are 0 or -1. Then, by Test I of §6.6, system (6.87), and hence also system (6.83), are strongly stable for all ε that satisfy (6.89) and the inequalities

$$\begin{aligned} 2m_{jh}\pi < \varepsilon(\alpha_j + \alpha_h)T - \varepsilon^2 \varkappa^2 \int_0^T q(t, \varepsilon)\,dt \leqslant \\ \leqslant \varepsilon(\alpha_j + \alpha_h)T + \varepsilon^2 \varkappa^2 \int_0^T q(t, \varepsilon)\,dt < 2(m_{jh} + 1)\pi, \end{aligned} \tag{6.90}$$

again for all $j, h = 1, \ldots, k$. Here $\varkappa = |\mathbf{U}| = |\mathbf{U}^\tau|$ and $\mathbf{U}$ is the symplectic matrix in (6.42). By (6.84) and (6.88),

$$\lim_{\varepsilon \to 0} \varepsilon \int_0^T q(t, \varepsilon)\,dt = 0$$

and so there exists $\varepsilon_0 > 0$ such that (6.90) holds for $0<\varepsilon<\varepsilon_0$. This completes the proof.

We have actually proved more: the stability of the averaged equation (6.85) implies that of equation (6.83) if $0 < \varepsilon < \varepsilon_0$, i. e., if conditions (6.90) hold.

§ 7. STABILITY TESTS FOR CANONICAL EQUATIONS, BASED ON ESTIMATES FOR THE EIGENVALUES OF BOUNDARY-VALUE PROBLEMS

In this section we present a few tests of the type mentioned in the title, for the most part without proofs. The proofs of Krein's tests (§§7.2 and 7.3) may be found in /166e/.

7.1. Lidskii-Neigauz test

Using an elegant argument, Lidskii and Neigauz /177/ have given a quite simple proof of a condition for all solutions of the canonical equation

$$\ddot{\mathbf{y}} + \mathbf{P}(t)\,\mathbf{y} = \mathbf{0},$$

where $\mathbf{P}(t)$ is "almost" a scalar matrix of the form $m^2\pi^2\mathbf{I}_k/T^2$, to be bounded.

Consider the canonical vector equation

$$\frac{d^2\mathbf{y}}{dt^2} + \left[\frac{m^2\pi^2}{T^2}\,\mathbf{I}_k + \mathbf{Q}(t)\right]\mathbf{y} = 0, \tag{7.1}$$

where m is an integer, $\mathbf{y}$ a k-vector-function, $\mathbf{Q}(t)$ a real symmetric T-periodic matrix-function which is either positive definite ($\mathbf{Q}(t) > \mathbf{0}$) or negative definite ($\mathbf{Q}(t) < \mathbf{0}$). That is to say, $(\mathbf{Q}(t)\,\mathbf{c}, \mathbf{c}) > 0$ for any constant k-vector $\mathbf{c} \neq \mathbf{0}$ and any $0 \leqslant t \leqslant T$, or $(\mathbf{Q}(t)\,\mathbf{c}, \mathbf{c}) < 0$ under these conditions.

Lidskii-Neigauz test. Under the above assumptions, equation (7.1) is strongly stable if, for all t,

$$\int_0^T \left|\sin\frac{m\pi t}{T}\right| \cdot |\mathbf{Q}(t-\tau)|\,dt < \frac{2m\pi}{T}. \tag{7.2}$$

Proof. Let m be an odd integer; to fix ideas, let $\mathbf{Q}(t) > \mathbf{0}$. Consider the equation

$$\frac{d^2\mathbf{y}}{dt^2} + \left[\frac{m^2\pi^2}{T^2}\,\mathbf{I}_k + \lambda\mathbf{Q}(t)\right]\mathbf{y} = 0. \tag{7.3}$$

Using the substitution described in § II. 3.2, we can reduce this equation to a canonical equation $\mathbf{J}\dot{\mathbf{x}} = \mathbf{H}(t, \lambda)\,\mathbf{x}$ of positive type (§4.1) with

$$\left.\begin{aligned} \mathbf{J}=\begin{pmatrix} 0 & -\mathbf{I}_k \\ \mathbf{I}_k & 0 \end{pmatrix}, \quad \mathbf{H}(t,\lambda)=\mathbf{H}_1(t)+\lambda\mathbf{H}_2(t), \\ \mathbf{H}_1(t)=\begin{pmatrix} \frac{m^2\pi^2}{T^2}\mathbf{I}_k & 0 \\ 0 & \mathbf{I}_k \end{pmatrix}, \quad \mathbf{H}_2(t)=\begin{pmatrix} \mathbf{Q}(t) & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}\right\} \tag{7.4}$$

The Hamiltonian $\mathbf{H}(t, \lambda)$ is a nondecreasing function of λ: $\partial\mathbf{H}(t, \lambda)/\partial\lambda \geqslant 0$.

At $\lambda=0$, equation (7.3) has a $2k$-fold multiplier $\rho=-1$ of mixed kind. By §4.3, for small $\lambda>0$ equation (7.3) has k multipliers of the first kind on the lower open unit semicircle and k of the second kind on the upper open unit semicircle. As λ increases, the multipliers of the first kind travel along the unit circle in the counterclockwise sense, those of the second kind in the clockwise sense. By the symmetry of the multipliers relative to the real axis, the first "meeting" of multipliers of different kinds may occur only at the point $\rho=+1$ (for some $\lambda=\lambda_0$). This means that at $\lambda=\lambda_0$ equation (7.3) has a T-periodic solution $\mathbf{y}(t)$.

Writing equation (7.3) for $\lambda=\lambda_0$ in the form of (II. 2.40), with

$$\mathbf{P}=\frac{m^2\pi^2}{T^2}\mathbf{I}_k, \quad \mathbf{h}(t)=-\lambda_0\mathbf{Q}(t)\mathbf{y}(t), \tag{7.5}$$

we see from (II. 2.41) that the T-periodic solution $\mathbf{y}(t)$ satisfies the integral equation

$$\mathbf{y}(t)=-\frac{T\lambda_0}{2m\pi}\int_0^T \sin\left(\frac{\tau m\pi}{T}\right)\mathbf{Q}(t-\tau)\mathbf{y}(t-\tau)\,d\tau. \tag{7.6}$$

Let $|\mathbf{y}(t)|$ achieve its maximum at t_0, i. e., $|\mathbf{y}(t)|\leqslant|\mathbf{y}(t_0)|$. For $t=t_0$, by (7.6),

$$|\mathbf{y}(t_0)|\leqslant\frac{T\lambda_0}{2m\pi}\int_0^T\left|\sin\frac{\tau m\pi}{T}\right|\cdot|\mathbf{Q}(t_0-\tau)|\cdot|\mathbf{y}(t_0-\tau)|\,d\tau\leqslant$$

$$\leqslant\frac{T\lambda_0|\mathbf{y}(t_0)|}{2m\pi}\int_0^T\left|\sin\frac{\tau m\pi}{T}\right||\mathbf{Q}(t_0-\tau)|\,d\tau.$$

Dividing by $|\mathbf{y}(t_0)|\neq 0$, we get

$$\lambda_0\geqslant\Lambda, \text{ where } \Lambda=\frac{2m\pi}{T}\left[\int_0^T\left|\sin\frac{\tau m\pi}{T}\right||\mathbf{Q}(t_0-\tau)|\,dt\right]^{-1}. \tag{7.7}$$

The point λ_0 is the first $\lambda>0$ at which multipliers of different kinds can coincide. Thus, for $0<\lambda<\Lambda$ all the multipliers of the first kind of equation (7.3) lie on the lower open unit semicircle, while those of the second kind are on the upper semicircle. Since no multipliers of different kinds coincide, it follows that equation (7.3) is strongly stable for $0<\lambda<\Lambda$. Setting $\lambda=1$, we see that equation (7.1) is strongly stable provided $1<\Lambda$. This is precisely (7.2).

If m is odd but $\mathbf{Q}(t) < 0$, the reasoning is the same. The only difference is that now when $0 < \lambda < \Lambda$ the multipliers of the first (second) kind will lie on the upper (lower) semicircle.

We now consider the case of even m. Let $\mathbf{Q}(t) > 0$. As λ increases from $\lambda = 0$, the first meeting of multipliers of different kinds will occur at $\rho = -1$. If λ_0 is the corresponding value of λ, equation (7.3) will have an antiperiodic solution

$$\mathbf{y}(t+T) \equiv -\mathbf{y}(t) \tag{7.8}$$

at $\lambda = \lambda_0$. The general solution of the equation

$$\frac{d^2\mathbf{y}}{dt^2} + \frac{m^2\pi^2}{T^2}\mathbf{y} = \mathbf{h}(t)$$

may be written

$$\mathbf{y}(t) = \mathbf{y}_0(t) + \frac{T}{m\pi}\int_0^t \sin\frac{m\pi}{T}(t-\tau)\,\mathbf{h}(\tau)\,d\tau,$$

where

$$\mathbf{y}_0(t) = \mathbf{c}_1 \sin\frac{m\pi}{T}t + \mathbf{c}_2 \cos\frac{m\pi}{T}t,$$

and $\mathbf{c}_1$, $\mathbf{c}_2$ are constant vectors. Applying this formula to equation (7.3), where $\mathbf{h}(t)$ has the form (7.5) and $\lambda = \lambda_0$, we obtain

$$\mathbf{y}(t) = \mathbf{y}_0(t) - \frac{\lambda_0 T}{m\pi}\int_{t_0}^{t} \sin\frac{m\pi}{T}(t-\tau)\,\mathbf{Q}(\tau)\,\mathbf{y}(\tau)\,d\tau.$$

Replacing t by $t+T$, setting $\int_{t_0}^{t+T} = \int_{t_0}^{t} + \int_{t}^{t+T}$, and recalling that m is even and $\mathbf{y}_0(t+T) = \mathbf{y}_0(t)$, we get

$$\mathbf{y}(t+T) = \mathbf{y}(t) - \frac{\lambda_0 T}{m\pi}\int_{t}^{t+T} \sin\frac{m\pi}{T}(t-\tau)\,\mathbf{Q}(\tau)\,\mathbf{y}(\tau)\,d\tau. \tag{7.9}$$

From (7.8) and (7.9) we obtain the necessary integral equation

$$\mathbf{y}(t) = \frac{\lambda_0 T}{2m\pi}\int_{t}^{t+T} \sin\frac{m\pi}{T}(t-\tau)\,\mathbf{Q}(\tau)\,\mathbf{y}(\tau)\,d\tau.$$

Hence, reasoning as before, we deduce (7.7). As before, the inequality $1 < \Lambda$ (which is just (7.2)) is a sufficient condition for equation (7.1) to be strongly stable. Thus the Lidskii-Neigauz test is also valid for even m.

We note that the test (7.2) singles out matrix-functions $\mathbf{H}(t, \lambda)$ (7.4) (for $\lambda = 1$) from the stability domains whose interior contains the Hamiltonian

$\mathbf{H}_1(t)$ of (7.4). If the order k of equation (7.1) is greater than 2, there are not Hamiltonians of this form in every stability domain. Indeed, in this case the multiplier type of the relevant stability domain may be either $\mu=(+, +, \ldots, +)$ or $\mu=(-, -, \ldots, -)$ and none other. And even for these types the index n of the stability domains that contain Hamiltonians $\mathbf{H}_1(t)$ as in (7.4) does not assume all possible integer values (see the analogous reasoning in §6.4). For $k>2$, therefore, the Lidskii-Neigauz test does not pick out Hamiltonians from every stability domain. This is not a shortcoming, but follows from the very formulation of the problem: Hamiltonians corresponding to equation (7.1) with small $\mathbf{Q}(t)>\mathbf{0}$ may lie only in these stability domains. The situation is similar for all the stability tests of this section, based as they are on estimates for the eigenvalues of periodic and antiperiodic boundary-value problems for Hamiltonian equations of positive type.

7.2. Strong stability tests for canonical equations

Consider the canonical equation

$$\mathbf{J}\frac{d\mathbf{x}}{dt}=\lambda\mathbf{H}(t)\,\mathbf{x} \tag{7.10}$$

where $\mathbf{H}(t)$ is a real symmetric T-periodic integrable piecewise-continuous $2k\times 2k$ matrix-function. Denote

$$\mathbf{H}_{\mathrm{av}}=\frac{1}{T}\int_0^T \mathbf{H}(t)\,dt=\begin{Vmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{Vmatrix}. \tag{7.11}$$

Recall (§4.1, Example (B)) that this equation is always of positive type if, for any constant $2k$-vector $\mathbf{c}\neq\mathbf{0}$,

$$(\mathbf{H}(t)\mathbf{c},\ \mathbf{c})\geqslant 0 \qquad (0\leqslant t\leqslant T) \quad \text{and} \quad (\mathbf{H}_{\mathrm{av}}\mathbf{c},\ \mathbf{c})>0. \tag{7.12}$$

It was shown in §4.1 that for any equation (7.10) of positive type the antiperiodic boundary-value problem

$$\mathbf{J}\frac{d\mathbf{x}}{dt}=\lambda\mathbf{H}(t)\,\mathbf{x}, \quad \mathbf{x}(T)=-\mathbf{x}(0)$$

has only real eigenvalues, and these have no finite point of accumulation. Denote the eigenvalues by $\lambda_j=\lambda_j(\mathbf{H})$, and assume that they are indexed so that

$$\ldots\leqslant\lambda_{-2}\leqslant\lambda_{-1}<0<\lambda_1\leqslant\lambda_2\leqslant\ldots$$

Krein has shown that neither of the cases $\lambda_1=\infty$ nor $\lambda_{-1}=-\infty$ is possible, so that our boundary-value problem always has at least one positive and at least one negative eigenvalue /166f/. The proof is extremely complicated and will not be presented here.

It was shown in §4.1 that equation (7.10) is strongly stable for

$$\lambda_{-1}<\lambda<0, \quad 0<\lambda<\lambda_1.$$

Thus a sufficient condition for strong stability of the equation

$$\mathbf{J}\frac{d\mathbf{x}}{dt}=\mathbf{H}(t)\,\mathbf{x} \tag{7.13}$$

is $\lambda_1=\lambda_1(\mathbf{H})>1$. For the sake of brevity we shall call $\mathbf{H}(t)$ a Hamiltonian of positive type if equation (7.10) is of positive type in the sense defined in §4.1. Thus a Hamiltonian $\mathbf{H}(t)$ satisfying (7.12) is of positive type. Krein coined the term Z-test for any strong stability test for equation (7.13) based on the inequality $\lambda_1(\mathbf{H})>1$. We claim that all Hamiltonians $\mathbf{H}(t)$ satisfying some Z-test belong to the stability domain $\mathcal{O}_0^{(+)}$ (see §5.3). Indeed, if $\lambda_1=\lambda_1(\mathbf{H})$, then the eigenvalue $\lambda_1(\varepsilon\mathbf{H})$ of the Hamiltonian $\varepsilon\mathbf{H}$ ($\varepsilon>0$) is λ_1/ε. Thus, $\lambda_1(\varepsilon\mathbf{H})=\frac{1}{\varepsilon}\lambda_1(\mathbf{H})$, i. e., if $\mathbf{H}(t)$ satisfies a Z-test, then any Hamiltonian $\varepsilon\mathbf{H}(t)$ ($0<\varepsilon\leqslant 1$) satisfies the same test. On the other hand, for sufficiently small $\varepsilon>0$ (see §4.3) the Hamiltonian $\varepsilon\mathbf{H}(t)$ is strongly stable and belongs to the stability domain $\mathcal{O}_0^{(+)}$, as asserted.

The converse is also true: If $\mathbf{H}(t)$ is a Hamiltonian of positive type, $\mathbf{H}(t)\in\mathcal{O}_0^{(+)}$, then $\lambda_1(\mathbf{H})>1$. Indeed, we saw in §4.3 that any Hamiltonian $\mathbf{H}(t)$ of positive type is such that $\lambda\mathbf{H}(t)\in\mathcal{O}_0^{(+)}$ for $0<\lambda\leqslant\lambda_0$, where λ_0 is sufficiently small. By Theorem I of §6.3, $\mathcal{O}_0^{(+)}$ is a directionally convex set. Since $\lambda_0\mathbf{H}(t)\in\mathcal{O}_0^{(+)}$, $\mathbf{H}(t)\in\mathcal{O}_0^{(+)}$ and $\lambda_0\mathbf{H}(t)\leqslant\lambda\mathbf{H}(t)\leqslant\mathbf{H}(t)$ for $\lambda_0\leqslant\lambda\leqslant 1$, it follows that $\lambda\mathbf{H}(t)\in\mathcal{O}_0^{(+)}$ for $\lambda_0\leqslant\lambda\leqslant 1$. Thus $\lambda\mathbf{H}(t)\in\mathcal{O}_0^{(+)}$ for any $0<\lambda\leqslant 1$, and this means that $\lambda_1(\mathbf{H})>1$.

We have thus shown that the validity of any Z-test for Hamiltonians of positive type is equivalent to the condition $\mathbf{H}(t)\in\mathcal{O}_0^{(+)}$.

The ten tests presented below are due to Krein /166e/. The proofs are based on estimates for the eigenvalues $\lambda_1(\mathbf{H})$ and $\lambda_{-1}(\mathbf{H})$ /166e/.

Test I. Equation (7.13) with a Hamiltonian of positive type is strongly stable, and $\mathbf{H}(t)\in\mathcal{O}_0^{(+)}$, if (in the notation of (7.11))

$$T^2\operatorname{Tr}(\mathbf{A}_{11}\mathbf{A}_{22}-\mathbf{A}_{12}^2)<2, \tag{7.14}$$

and if $\mathbf{H}(t)$ is an even matrix-function, it is sufficient that

$$T^2\operatorname{Tr}(\mathbf{A}_{11}\mathbf{A}_{22}-\mathbf{A}_{12}^2)<4. \tag{7.15}$$

The constant 4 in the last inequality is best possible.

Corollary 1. If the smallest eigenvalue $h_{\min}(t)$ of $\mathbf{H}(t)$ satisfies the inequality

$$h_{\min}(t)\geqslant a>0 \qquad (0\leqslant t\leqslant T),$$

then equation (7.13) is strongly stable if

$$\operatorname{Tr}(\mathbf{A}_{11}\mathbf{A}_{22}-\mathbf{A}_{12}^2)<\frac{2}{T^2}+\left(k-\frac{2}{\pi^2}\right)a^2, \tag{7.16}$$

and if $\mathbf{H}(t)$ *is even, it is sufficient that*

$$\operatorname{Tr}(\mathbf{A}_{11}\mathbf{A}_{22}-\mathbf{A}_{12}^2) < \frac{4}{T^2}+\left(k-\frac{4}{\pi^2}\right)a^2. \tag{7.17}$$

We recall that $2k$ is the order of system (7.13). Note that inequalities (7.16) and (7.17) are false for $a \geqslant \pi/T$ (see § 6.4), and so, a fortiori, inequalities (7.14) and (7.15) are false for $h_{\min}(t) \geqslant \pi/T$.

Using the theorem of §6.4 and Test 1, we readily prove

Corollary 2. As in Corollary 1, let $h_{\min}(t) \geqslant a > 0$ $(0 \leqslant t \leqslant T)$, *but now suppose that the number* a *satisfies the inequalities*

$$\frac{m\pi}{T} \leqslant a < \frac{(m+1)\pi}{T} \tag{7.18}$$

for some natural number m. *Writing* $\mathbf{H}(t)$ *as*

$$\mathbf{H}(t) = \begin{pmatrix} \mathbf{H}_{11}(t) & \mathbf{H}_{12}(t) \\ \mathbf{H}_{12}(t)^{\tau} & \mathbf{H}_{22}(t) \end{pmatrix}, \tag{7.19}$$

where $\mathbf{H}_{jh}(t)$ $(j, h = 1, 2)$ *are* $k \times k$ *matrices, set*

$$\mathbf{G}_{11}(t) = \mathbf{H}_{11}\cos^2\frac{m\pi}{T}t - \frac{1}{2}(\mathbf{H}_{12}+\mathbf{H}_{12}^{\tau})\sin\frac{2m\pi}{T}t + \mathbf{H}_{22}\sin^2\frac{m\pi}{T}t - \frac{m\pi}{T}\mathbf{I}_k,$$

$$\mathbf{G}_{22}(t) = \mathbf{H}_{11}\sin^2\frac{m\pi}{T}t - \frac{1}{2}(\mathbf{H}_{12}+\mathbf{H}_{12}^{\tau})\sin\frac{2m\pi}{T}t + \mathbf{H}_{22}\cos^2\frac{m\pi}{T}t - \frac{m\pi}{T}\mathbf{I}_k,$$

$$\mathbf{G}_{12}(t) = \mathbf{G}_{21}(t)^{\tau} = \frac{1}{2}(\mathbf{H}_{11}-\mathbf{H}_{22})\sin\frac{2m\pi}{T}t + \mathbf{H}_{12}\cos^2\frac{m\pi}{T}t - \mathbf{H}_{12}^{\tau}\sin^2\frac{m\pi}{T}t,$$

$$\mathbf{A}_{jh}^{(m)} = \frac{1}{T}\int_0^T \mathbf{G}_{jh}(t)\,dt \qquad (j, h = 1, 2).$$

Then equation (7.13) is strongly stable if

$$\operatorname{Tr}[\mathbf{A}_{11}^{(m)}\mathbf{A}_{22}^{(m)} - (\mathbf{A}_{12}^{(m)})^2] < \frac{2}{T^2}+\left(k-\frac{2}{\pi^2}\right)\left(a-\frac{m\pi}{T}\right)^2. \tag{7.20}$$

This test may also be generalized, as shown in §3.2 of /166e/.

Test 2. Equation (7.13) with a Hamiltonian of positive type is strongly stable, and $\mathbf{H}(t) \in \mathcal{O}_0^+$, *if the maximum eigenvalue of the matrix*

$$\left\| \int_0^T |h_{j,\,k+l}(t)|\,dt \right\|_{j,l=1}^{2k}$$

is smaller than 2. It is assumed here that

$$h_{j,\,2k+\varkappa}(t) \equiv h_{j\varkappa}(t) \qquad (j = 1, \ldots, 2k;\ \varkappa = 1, \ldots, k).$$

The next test pertains to equation (7.13) of not necessarily positive type.

Test 3. Assume that the matrix-functions $\mathbf{H}_{11}(t)$ *and* $\mathbf{H}_{22}(t)$ *in (7.19) are nonnegative (in the sense of quadratic forms):* $\mathbf{H}_{11}(t) \geqslant 0$, $\mathbf{H}_{22}(t) \geqslant 0$ $(0 \leqslant t \leqslant T)$. *Let* $\alpha(t)$, $\beta(t)$ *and* $\gamma(t)$ *be periodic functions whose value for each* t *is the absolute value of the maximum (in absolute value) eigenvalue of the matrices* $\mathbf{H}_{11}(t)$, $\mathbf{H}_{12}(t)$ *and* $\mathbf{H}_{22}(t)$, *respectively. Then equation (7.13) is strongly stable if*

$$\int_0^T \beta(t)\,dt + \sqrt{\int_0^T \alpha(t)\,dt \int_0^T \gamma(t)\,dt} < 2. \tag{7.21}$$

7.3. Strong stability tests for second-order canonical vector equations

Consider the vector equation

$$\frac{d^2\mathbf{y}}{dt^2} + \mathbf{P}(t)\,\mathbf{y} = 0, \quad \mathbf{P}(t)^{\mathrm{T}} = \mathbf{P}(t) = \mathbf{P}(t+T) = \|p_{jk}(t)\|_1^k. \tag{7.22}$$

Test 4. If the minimum eigenvalue $p_{\min}(t)$ *of the matrix* $\mathbf{P}(t)$ *satisfies the conditions*

$$p_{\min}(t) \geqslant a^2 > 0 \qquad (0 \leqslant t \leqslant T)$$

and

$$0 < a < \frac{\pi}{T},$$

then equation (7.22) is strongly stable if

$$T\int_0^T \mathrm{Tr}\,\mathbf{P}(t)\,dt < 4 + \left(m - \frac{4}{\pi^2}\right)a^2.$$

Remarks. 1) The natural reduction of equation (7.22) to canonical form gives the Hamiltonian

$$\mathbf{H}(t) = \begin{pmatrix} \mathbf{P}(t) & 0 \\ 0 & \mathbf{I}_k \end{pmatrix}.$$

Under the assumptions of Test 4 (and Tests 6 through 10 below), this Hamiltonian belongs to $\mathcal{G}_0^{(+)}$.

2) Test 4 may also be applied when $a = 0$, provided the Hamiltonian is positive "on the average":

$$\int_0^T (\mathbf{P}(t)\,\mathbf{c}, \mathbf{c})\,dt > 0.$$

We now consider the case that $p_{\min}(t) \geqslant a^2$ $(0 \leqslant t \leqslant T)$ and

$$\frac{m\pi}{T} \leqslant a < \frac{m+1}{T}\pi \tag{7.23}$$

for some natural number m. We define matrix-valued Fourier coefficients by

$$\mathbf{A}_0 = \frac{1}{T}\int_0^T \mathbf{P}(t)\,dt, \qquad \mathbf{A}_l = \frac{2}{T}\int_0^T \mathbf{P}(t)\cos\frac{2l\pi}{T}t\,dt, \quad \mathbf{B}_l = \frac{2}{T}\int_0^T \mathbf{P}(t)\sin\frac{2l\pi}{T}t\,dt.$$

Test 5. If conditions (7.23) are satisfied, then equation (7.22) is strongly stable if

$$\operatorname{Tr}(\mathbf{A}_0 - a^2\mathbf{I}_k)^2 < 8\frac{a^2}{T^2} + \frac{1}{4}\operatorname{Tr}(\mathbf{A}_m^2 + \mathbf{B}_m^2) - 4a\left(a - \frac{m\pi}{T}\right)\operatorname{Tr}\mathbf{A}_0 + \\ + 4a^2\left[\frac{mk\pi}{T}\left(a - \frac{m\pi}{T}\right) + \left(k - \frac{2}{\pi^2}\right)\left(a - \frac{m\pi}{T}\right)^2\right] \tag{7.24}$$

for the natural number m of (7.23).

Remark. If $a = m\pi/T$, inequality (7.24) is simplified:

$$\operatorname{Tr}\left(\mathbf{A}_0 - \frac{m^2\pi^2}{T^2}\mathbf{I}_k\right)^2 < 8\frac{m^2\pi^2}{T^2} + \frac{1}{4}\operatorname{Tr}(\mathbf{A}_m^2 + \mathbf{B}_m^2). \tag{7.25}$$

Test 6. If the matrix-function $\mathbf{P}(t)$ is odd:

$$\mathbf{P}(t) \sim \sum_{l=1}^{\infty} \mathbf{B}_l \sin\frac{2l\pi}{T}t,$$

then equation (7.22) is strongly stable if

$$\sum_{l=1}^{\infty} \frac{1}{l^2}\operatorname{Tr}\mathbf{B}_l^2 < 16\frac{\pi^2}{T^2}.$$

Denote

$$\mathbf{P}_{\text{av}} = \frac{1}{T}\int_0^T \mathbf{P}(t)\,dt, \quad \mathbf{Q}(t) = \int_0^t [\mathbf{P}(t_1) - \mathbf{P}_{\text{av}}]\,dt_1.$$

In the next three tests, we shall say that $\mathbf{P}(t)$ is nonnegative in mean if for some constant k-vector $\mathbf{c}$

$$(\mathbf{P}_{\text{av}}\,\mathbf{c}, \mathbf{c}) \geqslant 0 \quad \text{and} \quad \mathbf{P}(t)\mathbf{c} \not\equiv 0 \qquad (0 \leqslant t \leqslant T).$$

Test 7. If $\mathbf{P}(t)$ is nonnegative in mean, then equation (7.22) is strongly stable provided

$$T\int_0^T \operatorname{Tr}\mathbf{Q}^2\,dt + \frac{1}{24}T^2\operatorname{Tr}\left(\int_0^T \mathbf{P}(t)\,dt\right)^2 < 2.$$

Test 8. If $\mathbf{P}(t)$ *is nonnegative in mean, then equation (7.22) is strongly stable if*

$$T\int_0^T \mathrm{Tr}[\mathbf{Q}(t)^2+\mathbf{P}(t)]\,dt<2.$$

Remark. If moreover $\mathbf{P}(t)-\mathbf{P}_{av}$ is an odd function, we may replace the bound 2 in this inequality by 4.

Test 9. If $\mathbf{P}(t)$ *is nonnegative in mean, then equation (7.22) is strongly stable if*

$$T\left[\int_0^T q_{\max}(t)^2\,dt+\omega_{\max}^2\right]<\frac{1}{4}\pi^2,$$

where $q_{\max}(t)^2$ *and* $\omega_{\max}^2$ *are the maximum eigenvalues of the matrices* $\mathbf{Q}(t)^2$ *and* $\mathbf{P}_{av}$, *respectively.*

Test 10. If $\mathbf{P}(t)$ *is nonnegative in mean, then equation (7.22) is strongly stable if*

$$TM(\mathbf{R})<4,$$

where $M(\mathbf{R})$ *is the maximum eigenvalue of the matrix*

$$\mathbf{R}=\left\|\int_0^T |p_{jh}(t)|\,dt\right\|_1^k.$$

An advantage of this test over Test 9 is that it involves evaluation of the eigenvalues of a constant matrix only.

The reader may find many other sufficient conditions for strong stability in /30/.

We have not discussed the strong stability tests for equations (7.13) and (7.22) that follow from the material of Chapter VIII by direct application of the theorems of §6.7.

Chapter IV

METHODS OF PERTURBATION THEORY (SYSTEMS WITH A SMALL PARAMETER)

This chapter is devoted to systems of linear differential equations with periodic coefficients involving a small parameter ε:

$$\frac{d\mathbf{x}}{dt} = [\mathbf{C} + \varepsilon \mathbf{B}_1(t) + \varepsilon^2 \mathbf{B}_2(t) + \dots]\,\mathbf{x}, \tag{0.1}$$

that is to say, systems whose coefficients are "almost constant." Here $\mathbf{x}$ is the unknown vector, $\mathbf{C}$ a constant matrix, and $\mathbf{B}_j(t) \equiv \mathbf{B}_j(t+T)$ $(j = 1, 2, \dots)$ are piecewise-continuous matrix-functions, integrable over the interval $(0, T)$. All the matrices are $n \times n$, and $\mathbf{x}$ is an n-vector. The series on the right of equation (0.1) is assumed to be convergent for $|\varepsilon| < r_0$, where r_0 is independent of t. By §II. 1. 3, for any fixed t the matrizant $\mathbf{X}(t, \varepsilon)$ is an analytic function of ε in the interval $|\varepsilon| < r_0$.

Many problems of modern engineering and mechanics involve the investigation of systems (0.1), often of high orders. One has to determine the order of increase or decrease of the solutions, to specify regions on the parameter plane in which the solutions satisfy some prescribed exponential law of decrease, and so on. Frequently, the "unperturbed" system (system (0.1) with $\varepsilon = 0$) is stable (but not asymptotically stable), though the perturbed system (0.1) may be unstable even for arbitrarily small $\varepsilon > 0$.

In practice, the coefficients of system (0.1) often depend on certain unknown parameters. Stability (or instability) conditions entail certain relationships between these parameters, which must be determined. These problems reduce to evaluating the characteristic exponents of system (0.1) for small $\varepsilon > 0$; this is the central topic of this chapter (§2). In the very important case when the unperturbed system is conservative (§3), i. e., has pure imaginary characteristic exponents, but the perturbed system is not conservative (dynamic stability of elastic systems, many problems of celestial mechanics), the first approximation usually provides information as to the stability of the system. For this reason we shall devote special attention to the derivation of simple first-approximation formulas.

One could try to determine the characteristic exponents directly, as series in fractional powers of ε. To do this, one must first find the coefficient matrices $\mathbf{X}_1(t), \mathbf{X}_2(t), \dots$ in the expansion of the matrizant in powers of ε:

$$\mathbf{X}(t, \varepsilon) = e^{t\mathbf{C}} + \varepsilon \mathbf{X}_1(t) + \varepsilon^2 \mathbf{X}_2(t) + \dots,$$

then find the coefficients $\Delta_j(\rho)$ in the expansion of the characteristic polynomial in powers of ε:

$$\Delta(\rho, \varepsilon) \equiv \det[X(T, \varepsilon) - \rho I_n] = \Delta_0(\rho) + \Delta_1(\rho)\varepsilon + \dots$$

and then determine its roots $\rho(\varepsilon)$ in the neighborhood of a fixed multiplier $\rho(0)$ of the unperturbed equation which is a root (usually multiple) of the equation $\Delta_0(\rho) = 0$. This procedure involves successive determination of the coefficients of the expansion of $\rho(\varepsilon)$ in fractional powers of ε. The required characteristic exponents will be the quantities $T^{-1}\ln\rho(\varepsilon)$.

This method is extremely cumbersome and of little practical use, especially for systems with many degrees of freedom. Many authors, therefore, beginning from Hill /257/ and Whittaker /92/, have sought effective methods which exploit the specific features of the problem. In §§2 and 3 we shall describe a method for calculating the characteristic exponents, which has proved highly successful in problems with many degrees of freedom.

The idea of the method is illustrated in §1, demonstrating the evaluation of the eigenvalues of a matrix which is an analytic function of ε for small ε. The results of §1 are also used in §§2 and 3. Thus the main results of the chapter are concentrated in §§1–3.

The material of §§4, 5 will not be used later, and the reader may omit these sections on a first reading, proceeding directly to §6. In §4 we describe methods for solving system (0.1), due to Artem'ev /120b/, Erugin /40b, c/, Shtokalo /104/, Hukuhara /261/ and others, and some complements to these methods. These methods resemble the asymptotic methods of Krylov, Bogolyubov and Mitropol'skii /10, 11, 55, 66a, b/ for systems of a considerably more general type. The gist of the methods is that the desired form of the approximate solution yields information about the behavior of the exact solution for small ε as $t \to \infty$. A simple example will clarify the situation. Consider the scalar equation

$$\ddot{y} + (a_0 + a_1\varepsilon + a_2\varepsilon^2 + \dots)^2 y = 0, \tag{0.2}$$

which may be written in the form of (0.1) for $n = 2$ (here a_j are constant coefficients).

Suppose we are able to calculate only a few of the coefficients a_j. The approximate solution formula based on expanding the solution $y(t, \varepsilon)$ in powers of ε,

$$y \approx c_1\left[\sin a_0 t + \varepsilon a_1 t\cos a_0 t + \frac{\varepsilon^2 t^2}{2!}\varphi_2(t) + \dots + \frac{\varepsilon^N t^N}{N!}\varphi_N(t)\right] + \\ + c_2\left[\cos a_0 t - \varepsilon a_1 t\sin a_0 t + \frac{\varepsilon^2 t^2}{2!}\psi_2(t) + \dots + \frac{\varepsilon^N t^N}{N!}\psi_N(t)\right] \tag{0.3}$$

(where $\varphi_2(t), \dots, \varphi_N(t)$; $\psi_2(t), \dots, \psi_N(t)$ are bounded functions on $(-\infty, +\infty)$ which can be determined successively), is in fact useless. Indeed, the presence of "secular terms" (terms involving powers of t) in (0.3)

precludes investigation of the solution as $t \to \infty$. A more satisfactory formula would be one of the form

$$y \approx c_1 \sin [(a_0 + a_1\varepsilon + \ldots + a_N\varepsilon^N)t] + c_2 \cos [(a_0 + a_1\varepsilon + \ldots + a_N\varepsilon^N)t], \quad (0.4)$$

which would imply in obvious fashion that the solutions of (0.2) are bounded on $(-\infty, +\infty)$.

In the case of interest, the coefficients $a_1, a_2, \ldots$ in (0.2) are T-periodic functions of time. The quest for an approximate solution formula like (0.4) (but not like (0.3)!) is the main topic of §§2, 3 (in part) and a full account will be given in §4. It will be seen that for small $\varepsilon > 0$ the solutions of equation (0.2) (with T-periodic coefficients $a_j(t)$) need not be bounded.

In the general case of system (0.1), solution formulas analogous to (0.4) may be obtained by the following approach.

According to the Floquet-Lyapunov theorem (§II. 2.2), the matrizant of system (0.1) may be expressed as

$$\mathbf{X}(t, \varepsilon) = \mathbf{F}(t, \varepsilon) e^{t\mathbf{K}(\varepsilon)}, \text{ where } \mathbf{F}(t+T, \varepsilon) \equiv \mathbf{F}(t, \varepsilon). \quad (0.5)$$

Since $\mathbf{X}(t, 0) = \exp(t\mathbf{C})$ (§II. 1.5) it is natural to conjecture that $\mathbf{F}(t, \varepsilon)$ and $\mathbf{K}(\varepsilon)$ can be represented as series

$$\mathbf{F}(t, \varepsilon) = \mathbf{I}_n + \varepsilon\mathbf{F}_1(t) + \varepsilon^2\mathbf{F}_2(t) + \ldots, \quad \mathbf{K}(\varepsilon) = \mathbf{C} + \varepsilon\mathbf{K}_1 + \varepsilon^2\mathbf{K}_2 + \ldots \quad (0.6)$$

The analog of (0.4) will then have the form

$$\mathbf{X}^{(N)}(t, \varepsilon) = \mathbf{F}^{(N)}(t, \varepsilon) e^{t\mathbf{K}^{(N)}(\varepsilon)},$$

where

$$\mathbf{F}^{(N)}(t, \varepsilon) = \mathbf{I}_n + \varepsilon\mathbf{F}_1(t) + \ldots + \varepsilon^N\mathbf{F}_N(t), \quad \mathbf{K}^{(N)}(\varepsilon) = \mathbf{C} + \varepsilon\mathbf{K}_1 + \ldots + \varepsilon^N\mathbf{K}_N.$$

The basic problem is to determine the coefficients $\mathbf{K}_j$ in the expansion of the exponent matrix $\mathbf{K}(\varepsilon)$.

However, it turns out that the series (0.6) which should represent $\mathbf{F}(t, \varepsilon)$ and $\mathbf{K}(\varepsilon)$ are not always convergent for small ε. These series are convergent (for small ε) in the case specified below as "nonsingular," when the matrix $\mathbf{C}$ has no distinct eigenvalues which are congruent modulo $2\pi i/T$. In the singular case, when such eigenvalues exist (and this is precisely the more interesting case in applications), the representation (0.6) is meaningless. Indeed, the matrix-functions $\mathbf{F}(t, \varepsilon)$ and $\mathbf{K}(\varepsilon)$ defined by the Floquet-Lyapunov theorem and the condition $\mathbf{K}(0) = \mathbf{C}$ need not be analytic in ε at $\varepsilon = 0$; in fact, they may even fail to be continuous at $\varepsilon = 0$. Nevertheless, the singular case may be reduced, as we shall show in §4.2, by a suitable substitution

$$\mathbf{x} = \mathbf{F}_0(t)\,\mathbf{y} \qquad (\mathbf{F}_0(t+T) \equiv \mathbf{F}_0(t))$$

to the nonsingular case. We shall see that the matrix-function $\mathbf{F}_0(t)$ may be chosen in various ways. This indeterminacy will indeed be exploited

below so as to ensure that the series (0.6) will have the maximum possible radius of convergence; this will be accomplished by judicious choice of the branch of the logarithm in the formula

$$\mathbf{K}(0)=\frac{1}{T}\ln \mathbf{X}(T, 0).$$

In §4.7 we shall derive approximate formulas for an inhomogeneous system. In §5 we evaluate the first coefficients of (0.6) for Hill's equation, for a certain third-order equation, and also for a sixth-order equation.

In §§3.2 through 3.4 we apply the results of §2 to almost-canonical systems. The new results will be utilized in § V.4. The last section of the chapter (§6) uses the results of §3 to investigate the stability of periodic motions of nonautonomous systems.

§1. PERTURBATION THEORY FOR MATRIX EQUATIONS

We shall first illustrate the general computational procedure for characteristic exponents by a simple example, evaluating the eigenvalues of a matrix $\mathbf{K}+\varepsilon\mathbf{L}(\varepsilon)$ (analytic in ε) for small ε, on the assumption that the eigenvalues of $\mathbf{K}$ are known. A preliminary discussion of this problem is all the more desirable in that both it and a somewhat more general problem (to be considered later in this section) will be needed later on. The idea of the perturbation theory described in this section carries over almost unchanged to the much more general framework of operator equations (see §1.5) and the method is therefore applicable in many other problems of physics and mechanics, which will not be discussed here.

1.1. Statement of the problem. General case

Suppose we are given an $n\times n$ matrix-function

$$\mathbf{A}(\beta, \varepsilon)=\mathbf{K}-\beta\mathbf{I}_n+\varepsilon\mathbf{L}(\beta, \varepsilon), \tag{1.1}$$

which is analytic in ε and β in the neighborhood of a point $\varepsilon=0, \beta=\beta_0$. Assume that β_0 is an r-fold eigenvalue of the matrix $\mathbf{K}$, so that it satisfies the equation $\det\mathbf{A}(\beta, 0)=0$. The problem is, for small ε, to find the roots $\beta=\beta(\varepsilon)$ of the equation

$$\det\mathbf{A}(\beta, \varepsilon)=0 \tag{1.2}$$

in the neighborhood of β_0. The problem mentioned in the introduction to this subsection (eigenvalue problem for $\mathbf{K}+\varepsilon\mathbf{L}(\varepsilon)$) is the special case in which $\mathbf{L}(\beta, \varepsilon)$ is independent of β.

Before going on to solve the problem, we recall some basic properties of the equation

$$(\mathbf{K}-\beta_0\mathbf{I}_n)\,\mathbf{x}=\mathbf{f}, \tag{1.3}$$

since we shall have to solve it below (see § I. 1.6).

Let d be the nullity of the matrix $\mathbf{K}-\beta_0\mathbf{I}_n$, or the dimension of the eigensubspace of $\mathbf{K}$ belonging to the eigenvalue β_0. Let $\mathbf{a}_j$, $\mathbf{b}_j$ be linearly independent eigenvectors of the matrices $\mathbf{K}$ and $\mathbf{K}^*$:

$$\mathbf{K}\mathbf{a}_j=\beta_0\mathbf{a}_j, \quad \mathbf{K}^*\mathbf{b}_j=\bar{\beta}_0\mathbf{b}_j \qquad (j=1,\ldots,d). \tag{1.4}$$

Then:

1) By Lemma II of § I. 1.16, the entire space $\mathfrak{N}_n$ of complex n-vectors is an orthogonal direct sum

$$\mathfrak{N}_n=\mathfrak{D}\oplus\mathfrak{N}_*, \tag{1.5}$$

where $\mathfrak{D}$ is the range of the matrix (operator) $\mathbf{K}-\beta_0\mathbf{I}_n$ and $\mathfrak{N}_*$ the null subspace of the matrix $(\mathbf{K}-\beta_0\mathbf{I}_n)^*=\mathbf{K}^*-\bar{\beta}_0\mathbf{I}_n$, i. e., the subspace spanned by $\mathbf{b}_1,\ldots,\mathbf{b}_d$.

Let $\mathbf{P}_0$ be some projection (not necessarily orthogonal) on $\mathfrak{D}$. The solvability condition for equation (1.3) is $\mathbf{f}\in\mathfrak{D}$, i. e.,

$$\mathbf{P}_0\mathbf{f}=\mathbf{f}. \tag{1.6}$$

By (1.5), $\mathbf{f}\in\mathfrak{D}$ if and only if

$$(\mathbf{f},\ \mathbf{b}_j)=0 \qquad (j=1,\ldots,d). \tag{1.7}$$

2) Let $\mathbf{p}_1,\ldots,\mathbf{p}_d$; $\mathbf{q}_1,\ldots,\mathbf{q}_d$ be arbitrary vectors such that

$$\det\|(\mathbf{p}_h,\ \mathbf{a}_j)\|_1^d\neq 0, \quad \det\|(\mathbf{q}_h,\ \mathbf{b}_j)\|_1^d\neq 0. \tag{1.8}$$

The operator $\mathbf{P}_0$ may be defined explicitly by

$$\mathbf{P}_0\mathbf{u}=\mathbf{u}-\sum_{h=1}^{d}\zeta_h\mathbf{q}_h, \tag{1.9}$$

where $\mathbf{u}$ is any vector in $\mathfrak{N}_n$ and $\zeta_1,\ldots,\zeta_d$ a solution of the system

$$\sum_{h=1}^{d}(\mathbf{q}_h,\ \mathbf{b}_j)\,\zeta_h=(\mathbf{u},\ \mathbf{b}_j) \qquad (j=1,\ldots,d). \tag{1.10}$$

Setting $\mathbf{u}=\mathbf{e}_1,\ldots,\mathbf{u}=\mathbf{e}_n$ in (1.9) and (1.10), where $\mathbf{e}_1,\ldots,\mathbf{e}_n$ is the standard basis, we find the columns $\mathbf{P}_0\mathbf{e}_\nu$ of the matrix $\mathbf{P}_0$ and hence the matrix $\mathbf{P}_0$ itself. *

Recall that in geometric terms the second inequality of (1.8) means that the intersection of the subspace $\{\mathbf{q}_j\}$ spanned by $\mathbf{q}_1,\ldots,\mathbf{q}_d$ and the

* Once the (standard) basis has been fixed, we may identify the operator $\mathbf{P}_0$ with its matrix relative to the basis.

subspace $\mathfrak{D}$ (orthogonal to $\mathfrak{N}_* = \{\mathbf{b}_j\}$) is zero, so that $\mathfrak{N}_n = \mathfrak{D} \dotplus \{\mathbf{q}_j\}$. The geometric interpretation of the first part of (1.8) is analogous. Thus $\mathbf{P}_0$ projects the space onto $\mathfrak{D}$ "in the direction of" (or "along") the subspace $\{\mathbf{q}_j\}$.

3) Let either of conditions (1.6) or (1.7) be satisfied. Then equation (1.3) has a unique solution $\mathbf{x}_0$ satisfying the relations

$$(\mathbf{x}, \mathbf{p}_h) = 0 \qquad (h = 1, \ldots, d). \tag{1.11}$$

We denote this solution (which is defined for all $\mathbf{f} \in \mathfrak{D}$) by

$$\mathbf{x}_0 = \mathbf{S}_0 \mathbf{f}. \tag{1.12}$$

The vector $\mathbf{x}_0$ may also be defined as

$$\mathbf{x}_0 = (\mathbf{K} - \beta_0 \mathbf{I}_n - \mathbf{V}\mathbf{U}^*)^{-1} \mathbf{f}, \tag{1.13}$$

where

$$\mathbf{U} = \| \mathbf{p}_1, \ldots, \mathbf{p}_d \|, \quad \mathbf{V} = \| \mathbf{q}_1, \ldots, \mathbf{q}_d \|, \tag{1.14}$$

and the inverse matrix in (1.13) indeed exists.

Note that the right-hand side of (1.13), in contrast to that of (1.12), is meaningful for any vector $\mathbf{f}$, not merely for $\mathbf{f} \in \mathfrak{D}$. Thus formula (1.13) extends the operator $\mathbf{S}_0$ to the whole space. Thus we risk no contradiction if we define $\mathbf{S}_0$ on the whole space by

$$\mathbf{S}_0 = (\mathbf{K} - \beta_0 \mathbf{I}_n - \mathbf{V}\mathbf{U}^*)^{-1}.$$

4) The general solution of equation (1.3) is

$$\mathbf{x} = \mathbf{x}_0 + \sum_{\delta=1}^{d} \gamma_\delta \mathbf{a}_\delta, \tag{1.15}$$

where γ_δ are arbitrary complex numbers.

The proof of properties 1—4 may be found in § I. 1.6.

We now return to the problem as stated above. Suppose that equation (1.2) holds for some β and ε. In other words, there is a vector $\mathbf{x} \neq \mathbf{0}$ such that

$$\mathbf{A}(\beta, \varepsilon)\, \mathbf{x} = \mathbf{0}. \tag{1.16}$$

Denote

$$\mathbf{Q} = (\beta - \beta_0)\mathbf{I}_n - \varepsilon \mathbf{L}(\beta, \varepsilon). \tag{1.17}$$

Equation (1.16) may now be written

$$(\mathbf{K} - \beta_0 \mathbf{I}_n)\, \mathbf{x} = \mathbf{Q}\mathbf{x}. \tag{1.18}$$

Consequently, conditions (1.6), (1.7) hold for $\mathbf{f}=\mathbf{Qx}$, i. e.,

$$(\mathbf{K}-\beta_0\mathbf{I}_n)\,\mathbf{x}=\mathbf{P}_0\mathbf{Q}\mathbf{x} \tag{1.19}$$

and

$$(\mathbf{Qx},\ \mathbf{b}_h)=0 \qquad (h=1,\ \ldots,\ d). \tag{1.20}$$

Assuming for the moment that the $\mathbf{x}$ on the right of (1.19) is known, we deduce from (1.15) that

$$\mathbf{x}=\mathbf{x}_0+\mathbf{x}_1,\quad \mathbf{x}_1=\sum_{j=1}^{d}\gamma_j\mathbf{a}_j, \tag{1.21}$$

where $\gamma_1,\ \ldots,\ \gamma_d$ are certain (unknown) numbers, and $\mathbf{x}_0$ is defined by (1.12), which in our case is the formula

$$\mathbf{x}_0=\mathbf{S}_0\mathbf{P}_0\mathbf{Q}\mathbf{x}. \tag{1.22}$$

To express $\mathbf{x}$ in terms of $\mathbf{x}_1$, we substitute $\mathbf{x}_0=\mathbf{x}-\mathbf{x}_1$ into (1.22):

$$\mathbf{x}-\mathbf{S}_0\mathbf{P}_0\mathbf{Q}\mathbf{x}=\mathbf{x}_1. \tag{1.23}$$

The matrices (operators) $\mathbf{S}_0$, $\mathbf{P}_0$ depend only on the matrix $\mathbf{K}$ and not on ε and β. By (1.17), $\mathbf{Q}$ is arbitrarily small for sufficiently small ε and $\beta-\beta_0$. Consequently, for sufficiently small ε and $\beta-\beta_0$,

$$\det(\mathbf{I}_n-\mathbf{S}_0\mathbf{P}_0\mathbf{Q})\neq 0,$$

so that equation (1.23) is solvable for $\mathbf{x}$. Therefore

$$\mathbf{x}=(\mathbf{I}_n-\mathbf{S}_0\mathbf{P}_0\mathbf{Q})^{-1}\mathbf{x}_1. \tag{1.24}$$

We first substitute $\mathbf{x}_1$ from (1.21) into (1.24), and then insert (1.24) into (1.20), to obtain equations for $\gamma_1,\ldots,\ \gamma_d$:

$$\sum_{j=1}^{d}\varphi_{jh}(\varepsilon,\ \beta)\,\gamma_j=0 \qquad (h=1,\ \ldots,\ d), \tag{1.25}$$

where

$$\varphi_{jh}(\varepsilon,\ \beta)=(\mathbf{Q}\,[\mathbf{I}_n-\mathbf{S}_0\mathbf{P}_0\mathbf{Q}]^{-1}\mathbf{a}_j,\ \mathbf{b}_h). \tag{1.26}$$

Since $\mathbf{x}\neq\mathbf{0}$, it follows from (1.24) that $\mathbf{x}_1\neq\mathbf{0}$. Consequently, equations (1.25) have a nontrivial solution, and so

$$\det\|\,\varphi_{jh}(\varepsilon,\ \beta)\,\|_1^d=0. \tag{1.27}$$

This is the required equation for the roots $\beta=\beta(\varepsilon)$ of equation (1.2) in the neighborhood of β_0. We shall show below that this equation indeed provides

a simple derivation of the "principal" corrections distinguishing the roots $\beta(\varepsilon)$ from β_0. For the moment, suffice it to say that the order d of the determinant (1.27) is often considerably smaller than the order n of the determinant (1.2).

We have shown that (1.27) follows from (1.2). We now claim that, conversely, any root $\beta=\beta(\varepsilon)$ of equation (1.27) sufficiently near β_0 is also a root of equation (1.2). It will suffice to reverse the above reasoning, proving in reverse order each of the numbered formulas from (1.27) back to (1.16) inclusive. It is clear that equation (1.16) with $\mathbf{x}\neq\mathbf{0}$ implies (1.2).

If (1.27) is true, we can find a nontrivial solution $\gamma_1, \ldots, \gamma_d$ of equations (1.25) and define a vector $\mathbf{x}_1$ by (1.21) and a vector $\mathbf{x}$ by (1.24). (The inverse in (1.24) exists for sufficiently small $\beta-\beta_0, \varepsilon$.) Since $\mathbf{x}_1\neq\mathbf{0}$, we have also $\mathbf{x}\neq\mathbf{0}$. Define $\mathbf{x}_0$ by (1.22). Then (1.24) implies (1.23), and hence also the first part of (1.21). By the definition of $\mathbf{S}_0$, it follows from (1.22) that $(\mathbf{K}-\beta_0\mathbf{I}_n)\mathbf{x}_0=\mathbf{P}_0\mathbf{Q}\mathbf{x}$. The truth of (1.20) is obvious, for these equalities are simply another notation for (1.25). It follows from (1.20) that $\mathbf{P}_0\mathbf{Q}\mathbf{x}=\mathbf{Q}\mathbf{x}$. Thus (1.18) follows from (1.17), and hence also (1.16) is true, Q. E. D.

The final result is as follows.

Lemma. Let the $n\times n$ matrix (1.1) be analytic in the neighborhood of $\varepsilon=0$, $\beta=\beta_0$, where β_0 is an r-fold eigenvalue of the matrix $\mathbf{K}$. Define the number d, vectors $\mathbf{a}_j$, $\mathbf{b}_j$, $\mathbf{p}_j$, $\mathbf{q}_j$ $(j=1, \ldots, d)$, and matrices $\mathbf{P}_0$, $\mathbf{S}_0$ as above. Define functions $\varphi_{jh}(\varepsilon, \beta)$ by (1.26), where $\mathbf{Q}$ has the form (1.17). Then, in a sufficiently small neighborhood of $\varepsilon=0$, $\beta=\beta_0$, equation (1.2) holds for the values of ε and β satisfying equation (1.27), and only for these values.

To illustrate, let us consider the case that β_0 is a simple eigenvalue $(r=d=1)$. For small $\varepsilon, \beta-\beta_0$, we have from (1.26)

$$\varphi_{jh}(\varepsilon, \beta)=(\mathbf{Q}\mathbf{a}_j, \mathbf{b}_h)+\ldots=(\beta-\beta_0)(\mathbf{a}_j, \mathbf{b}_h)-\varepsilon(\mathbf{L}(\beta, \varepsilon)\mathbf{a}_j, \mathbf{b}_h)+\ldots, \qquad (1.28)$$

where the dots stand for quantities of second order relative to ε and $\beta-\beta_0$. When $d=1$, (1.27) is the equation

$$\varphi_{11}(\varepsilon, \beta)=0.$$

Thus the required root is

$$\beta=\beta_0+\varepsilon\frac{(\mathbf{L}(\beta_0, 0)\mathbf{a}_1, \mathbf{b}_1)}{(\mathbf{a}_1, \mathbf{b}_1)}+O(\varepsilon^2). \qquad (1.29)$$

Later (§1.3) we shall evaluate β with accuracy up to $O(\varepsilon^4)$.

1.2. Case of simple elementary divisors

If the elementary divisors of β_0 belonging to $\mathbf{K}$ are simple, i. e., $d=r$, then by § I. 1.6

$$\det\|(\mathbf{a}_j, \mathbf{b}_h)\|_1^d\neq 0. \qquad (1.30)$$

Lemma. If the elementary divisors of the matrix $\mathbf{K}$ *belonging to the eigenvalue* β_0 *are simple, then all* r *roots of equation (1.2) that tend to* β_0 *at* $\varepsilon=0$ *are defined as implicit functions of* ε *by equation (1.27), where*

$$\varphi_{jh}(\varepsilon, \beta)=(\beta-\beta_0)(\mathbf{a}_j, \mathbf{b}_h)-\varepsilon(\mathbf{L}\mathbf{a}_j, \mathbf{b}_h)+\varepsilon^2(\mathbf{L}(\mathbf{I}_n-\mathbf{S}_0\mathbf{P}_0\mathbf{Q})^{-1}\mathbf{S}_0\mathbf{P}_0\mathbf{L}\mathbf{a}_j, \mathbf{b}_h) \quad (1.31)$$

and $\mathbf{L}=\mathbf{L}(\beta, \varepsilon)$. *As before,* $\mathbf{Q}$ *is given by (1.17).*

Formula (1.31) is much more convenient than (1.26), since the terms of first order in ε and $\beta-\beta_0$ appear separately and explicitly, and are moreover quite simple in form. We emphasize, however, that (1.31) is valid only when the elementary divisors are simple ($d=r$).

Remark. If (1.30) is true, we can always choose vectors $\mathbf{a}_j$, $\mathbf{b}_h$ to satisfy the following normalizing conditions:

$$(\mathbf{a}_j, \mathbf{b}_h)=\delta_{jh}=\begin{cases} 0 & \text{for } j\neq h, \\ 1 & \text{for } j=h. \end{cases} \quad (1.32)$$

Proof. Define operators $\mathbf{P}_0$ and $\mathbf{S}_0$ as in §1.1, setting $\mathbf{p}_h=\mathbf{b}_h$, $\mathbf{q}_h=\mathbf{a}_h$. By (1.30), conditions (1.8) are satisfied. The choice of $\mathbf{q}_h=\mathbf{a}_h$ implies that

$$\mathbf{P}_0\mathbf{a}_h=0 \qquad (h=1, \ldots, d). \quad (1.33)$$

(This follows at once from (1.9), (1.10) for $\mathbf{u}=\mathbf{a}_k$, for since the solution $\zeta_1, \ldots, \zeta_d$ is unique we have $\zeta_h=0$ for $h\neq k$, $\zeta_k=1$.) The choice of $\mathbf{p}_h=\mathbf{b}_h$ implies that the solution $\mathbf{x}_0=\mathbf{S}_0\mathbf{f}$ of equation (1.4) satisfying (1.11) is orthogonal to the subspace $\mathfrak{N}_*=\{\mathbf{b}_h\}$. Since this is true for any $\mathbf{f}\in\mathfrak{D}$, it follows that $\mathbf{S}_0$ maps $\mathfrak{D}$ into itself (see (1.5)). This means that for any vector $\mathbf{z}\in\mathfrak{N}_n$

$$(\mathbf{S}_0\mathbf{P}_0\mathbf{z}, \mathbf{b}_h)=0 \qquad (h=1, \ldots, d). \quad (1.34)$$

The relations (1.33), (1.34) enable us to reduce (1.26) to the more convenient form (1.31). Setting $\mathbf{N}=\mathbf{S}_0\mathbf{P}_0\mathbf{Q}$, we have

$$(\mathbf{I}_n-\mathbf{N})^{-1}=\mathbf{I}_n+\mathbf{N}+\mathbf{N}^2+\ldots=\mathbf{I}_n+\mathbf{N}(\mathbf{I}_n-\mathbf{N})^{-1}. \quad (1.35)$$

From (1.34) for $\mathbf{z}=\mathbf{Q}(\mathbf{I}_n-\mathbf{N})^{-1}\mathbf{a}_j$ it follows that

$$(\mathbf{N}(\mathbf{I}_n-\mathbf{N})^{-1}\mathbf{a}_j, \mathbf{b}_h)=0 \qquad (j, h=1, \ldots, d). \quad (1.36)$$

From (1.26), (1.17) we have

$$\varphi_{jh}=(\beta-\beta_0)([\mathbf{I}_n-\mathbf{N}]^{-1}\mathbf{a}_j, \mathbf{b}_h)+\varepsilon(\mathbf{L}[\mathbf{I}_n-\mathbf{N}]^{-1}\mathbf{a}_j, \mathbf{b}_h). \quad (1.37)$$

After we insert $(\mathbf{I}_n-\mathbf{N})^{-1}$ from (1.35) and use (1.36), the first term becomes

$$(\beta-\beta_0)(\mathbf{a}_j, \mathbf{b}_h). \quad (1.38)$$

Substituting $(\mathbf{I}_n-\mathbf{N})^{-1}=\mathbf{I}_n+(\mathbf{I}_n-\mathbf{N})^{-1}\mathbf{N}$, we bring the second term in (1.37) to the form

$$\varepsilon(\mathbf{La}_j, \mathbf{b}_h)+\varepsilon^2(\mathbf{L}[\mathbf{I}_n-\mathbf{N}]^{-1}\mathbf{S}_0\mathbf{P}_0\mathbf{La}_j, \mathbf{b}_h), \tag{1.39}$$

since by (1.33)

$$\mathbf{Na}_j=\mathbf{S}_0\mathbf{P}_0[(\beta-\beta_0)\mathbf{I}_n+\varepsilon\mathbf{L}]\mathbf{a}_j=\varepsilon\mathbf{S}_0\mathbf{P}_0\mathbf{La}_j.$$

Inserting (1.38) and (1.39) into (1.37), we obtain (1.31), proving the lemma.

1.3. Case of simple eigenvalues*

Lemma. Under the assumptions of the lemma of §1.1, let β_0 *be a simple root of the equation* $\det\mathbf{A}(\beta, 0)=\det[\mathbf{K}-\beta\mathbf{I}_n]=0$. *Let* $\mathbf{a}_1, \mathbf{b}_1$ *be corresponding eigenvectors of the matrices* $\mathbf{K}$ *and* $\mathbf{K}^*$ $(\mathbf{Ka}_1=\beta_0\mathbf{a}_1,\ \mathbf{K}^*\mathbf{b}_1=\bar{\beta}_0\mathbf{b}_1)$, *normalized to* $(\mathbf{a}_1, \mathbf{b}_1)=1$. *Define an operator* $\mathbf{P}_0$ *for arbitrary* $\mathbf{x}$ *by*

$$\mathbf{P}_0\mathbf{x}=\mathbf{x}-(\mathbf{x}, \mathbf{b}_1)\mathbf{a}_1$$

(i.e., $\mathbf{P}_0=\mathbf{I}_n-\mathbf{a}_1\mathbf{b}_1^*$*). Define an operator* $\mathbf{S}_0$ *for arbitrary vectors* $\mathbf{f}$ *satisfying the condition* $(\mathbf{f}, \mathbf{b}_1)=0$ *as follows:* $\mathbf{y}=\mathbf{S}_0\mathbf{f}$ *if and only if* $(\mathbf{K}-\beta_0\mathbf{I}_n)\mathbf{y}=\mathbf{f}$, $(\mathbf{y}, \mathbf{b}_1)=0$. *(The vector* $\mathbf{y}=\mathbf{S}_0\mathbf{f}$ *may be defined as* $\mathbf{y}=(\mathbf{K}-\beta_0\mathbf{I}_n-\mathbf{a}_1\mathbf{b}_1^*)^{-1}\mathbf{f}$.*) Then the root* $\beta=\beta(\varepsilon)$ *of the equation* $\det\mathbf{A}(\beta, \varepsilon)=0$ *tending to* β_0 *at* $\varepsilon=0$ *is an analytic function of* ε *at* $\varepsilon=0$, *and*

$$\beta=\beta_0+\varepsilon(\mathbf{La}_1, \mathbf{b}_1)+\varepsilon^2(\mathbf{LS}_0\mathbf{P}_0\mathbf{La}_1, \mathbf{b}_1)+ \\ +\varepsilon^3(\mathbf{LS}_0\mathbf{P}_0[(\mathbf{La}_1, \mathbf{b}_1)\mathbf{I}_n+\mathbf{L}]\mathbf{S}_0\mathbf{P}_0\mathbf{La}_1, \mathbf{b}_1)+O(\varepsilon^4), \tag{1.40}$$

where $\mathbf{L}=\mathbf{L}(\beta, \varepsilon)$.

Proof. Formula (1.40) may be derived from the equation $\varphi_{11}(\varepsilon, \beta)=0$ by substituting

$$(\mathbf{I}_n-\mathbf{S}_0\mathbf{P}_0\mathbf{Q})^{-1}=\mathbf{I}_n+\mathbf{S}_0\mathbf{P}_0\mathbf{Q}+(\mathbf{S}_0\mathbf{P}_0\mathbf{Q})^2+\ldots$$

in the expression (1.31) for $\varphi_{11}(\varepsilon, \beta)$ and solving the resulting equation for β.

Note that if $\mathbf{L}=\mathbf{L}(\varepsilon)$ is independent of β the right-hand side of (1.40) is the expansion of the root $\beta=\beta(\varepsilon)$ in powers of ε. If $\mathbf{L}=\mathbf{L}(\beta, \varepsilon)$ depends on β, it is quite easy to derive the first coefficients of this expansion from equation (1.40).

1.4. Case of multiple elementary divisors

The lemma of §1.1 is true in the general case, but if β_0 has multiple elementary divisors it is not always convenient. For in that case it is

* The reader may omit §§ 1.3 — 1.5 (which will not be needed in this chapter) and proceed directly to § 2.

sometimes quite difficult to isolate the "principal" part of the functions $\varphi_{jh}(\varepsilon, \beta)$ as done in (1.31), and even when this is possible it may involve prohibitive manipulations. It is more convenient to increase the order of the determinant (1.27) to obtain a similar equation in which the "principal" part appears explicitly. This can be done if we consider not only eigenvectors $\mathbf{a}_j$, $\mathbf{b}_h$ of $\mathbf{K}$ and $\mathbf{K}^*$ but also their root vectors.

As before, let β_0 be an r-fold root of the equation (1.2). Then (§ I. 1.10) each of the equations

$$(\mathbf{K}-\beta_0\mathbf{I}_n)^r\mathbf{a}=0, \qquad (\mathbf{K}^*-\bar{\beta}_0\mathbf{I}_n)^r\mathbf{b}=0$$

has exactly r linearly independent solutions (root vectors), which we denote by $\mathbf{a}_1, \ldots, \mathbf{a}_r$; $\mathbf{b}_1, \ldots, \mathbf{b}_r$. Any eigenvector is of course also a root vector. We shall therefore assume, in accordance with the notation of §1.1, that the first d of these vectors are eigenvectors of $\mathbf{K}$ and $\mathbf{K}^*$, so that (1.4) holds. If the root β_0 has only simple elementary divisors, then $r=d$ and we are back to the case considered in §1.2.

Let $\mathfrak{L}$ and $\mathfrak{L}_*$ denote the root subspaces of $\mathbf{K}$ and $\mathbf{K}^*$, respectively, belonging to the eigenvalues β_0 and $\bar{\beta}_0$. Thus $\mathfrak{L}$ and $\mathfrak{L}_*$ are the subspaces spanned by $\mathbf{a}_1, \ldots, \mathbf{a}_r$ and $\mathbf{b}_1, \ldots, \mathbf{b}_r$, respectively.

By Corollary 2 in § I. 1. 13,

$$\det \|(\mathbf{a}_j, \mathbf{b}_h)\|_1^r \neq 0. \tag{1.41}$$

Instead of the previous operator $\mathbf{P}_0$, we define a projection $\mathbf{P}$ by

$$\mathbf{Pu}=\mathbf{u}-\sum_{h=1}^{r}\zeta_h\mathbf{a}_h, \tag{1.42}$$

where $\mathbf{u}$ is an arbitrary vector and the numbers ζ_h form a solution of the system

$$\sum_{h=1}^{r}(\mathbf{a}_h, \mathbf{b}_j)\zeta_h=(\mathbf{u}, \mathbf{b}_j) \qquad (j=1, \ldots, r).$$

It is readily seen that $\mathbf{Pu}=0$ for any $\mathbf{u}\in\mathfrak{L}$ and $\mathbf{Pu}=\mathbf{u}$ for any vector $\mathbf{u}$ orthogonal to $\mathfrak{L}_*$. Thus $\mathbf{P}$ is a projection onto the subspace $\mathfrak{M}$ orthogonal to $\mathfrak{L}_*$ "along" $\mathfrak{L}$. By § I. 1. 13 (Corollary I), $\mathfrak{M}$ is the direct sum of the root subspaces of $\mathbf{K}$ belonging to all eigenvalues other than β_0. According to §§ I. 1. 10 and I. 1.13 (formulas (I. 1.39) and (I. 1.55)), the space $\mathfrak{R}_n$ is the direct sum (not necessarily orthogonal) of $\mathfrak{L}$ and $\mathfrak{M}$, and the orthogonal direct sum of $\mathfrak{L}_*$ and $\mathfrak{M}$:

$$\mathfrak{R}_n=\mathfrak{L}\dotplus\mathfrak{M}, \quad \mathfrak{R}_n=\mathfrak{L}_*\oplus\mathfrak{M}. \tag{1.43}$$

$\mathfrak{M}$ is an invariant subspace of $\mathbf{K}$ and moreover the operator $\mathbf{K}-\beta_0\mathbf{I}_n$ is invertible in $\mathfrak{M}$. Set

$$\mathbf{S}=\begin{cases}(\mathbf{K}-\beta_0\mathbf{I}_n)^{-1} & \text{on } \mathfrak{M},\\ 0 & \text{on } \mathfrak{L}.\end{cases} \tag{1.44}$$

Explanation: Let $\mathbf{f}$ be any vector. The equation

$$(\mathbf{K}_0-\beta_0\mathbf{I}_n)\,\mathbf{x}=\mathbf{Pf} \tag{1.45}$$

has a solution, which is moreover unique, provided $\mathbf{x}\in\mathfrak{M}$, or, in other words, provided

$$(\mathbf{x},\ \mathbf{b}_j)=0 \qquad (j=1,\ \ldots,\ r) \tag{1.46}$$

(we have used the second part of (1.43)). This solution is precisely the defined value of the operator $\mathbf{S}$ for the vector $\mathbf{f}$.

Define r^2 functions by

$$\psi_{jh}(\varepsilon,\ \beta)=([\mathbf{K}+\varepsilon\mathbf{L}(\beta,\ \varepsilon)-\beta\mathbf{I}_n]\,\mathbf{a}_j,\ \mathbf{b}_h)+ \\ +\varepsilon^2(\mathbf{L}(\beta,\ \varepsilon)\,[\mathbf{I}_n-\mathbf{SQ}]^{-1}\,\mathbf{SL}(\beta,\ \varepsilon)\,\mathbf{a}_j,\ \mathbf{b}_h), \tag{1.47}$$

where $\mathbf{Q}$ is defined by (1.17). The functions (1.47) are certainly defined for sufficiently small $|\varepsilon|$, $|\beta-\beta_0|$.

Lemma. Let the $n\times n$ matrix $\mathbf{K}+\varepsilon\mathbf{L}(\beta,\ \varepsilon)$ be analytic in the neighborhood of $\varepsilon=0$, $\beta=\beta_0$, where β_0 is an eigenvalue of $\mathbf{K}$ of multiplicity r. Define operators $\mathbf{P}$ and $\mathbf{S}$ as above, and r^2 functions $\psi_{jh}(\varepsilon,\ \beta)$ by (1.47). Then all r roots of equation (1.2) that tend to β_0 at $\varepsilon=0$ are defined as implicit functions of ε by the equation

$$\det\|\psi_{jh}(\varepsilon,\ \beta)\|_1^r=0. \tag{1.48}$$

Proof. Define an operator $\mathbf{R}$ by

$$\mathbf{I}_n=\mathbf{R}+\mathbf{P}. \tag{1.49}$$

By definition of $\mathbf{P}$, the decomposition (1.43) induces a decomposition (1.49) of the identity operator, so that $\mathbf{R}$ is the projection onto $\mathfrak{L}$ along $\mathfrak{M}$.

Let (1.2) be true for certain ε, β in a sufficiently small neighborhood of $\varepsilon=0$, $\beta=\beta_0$. Then there is a vector $\mathbf{x}\neq\mathbf{0}$ such that (1.16) holds. As before, we rewrite (1.16) as (1.18), where $\mathbf{Q}$ is given by (1.17).

Since the operators $\mathbf{P}$ and $\mathbf{R}$ commute with $\mathbf{K}$, we see from (1.18) that

$$(\mathbf{K}-\beta_0\mathbf{I}_n)\,\mathbf{Px}=\mathbf{PQx}, \tag{1.50}$$
$$(\mathbf{K}-\beta_0\mathbf{I}_n)\,\mathbf{Rx}=\mathbf{RQx}. \tag{1.51}$$

By virtue of (1.49), equations (1.50) and (1.51) are equivalent to (1.18). By the definition of S, it follows from (1.50) that $\mathbf{Px}=\mathbf{SPQx}$, and as $\mathbf{SP}=\mathbf{S}$,

$$\mathbf{Px}=\mathbf{SQx}. \tag{1.52}$$

Replacing $\mathbf{Px}$ by $\mathbf{x}-\mathbf{Rx}$ (see (1.49)), we obtain

$$(\mathbf{I}_n-\mathbf{SQ})\,\mathbf{x}=\mathbf{Rx}. \tag{1.53}$$

The operator $\mathbf{Q}$ is small for small $|\varepsilon|$ and $|\beta-\beta_0|$. Thus, for small $|\varepsilon|$ and $|\beta-\beta_0|$, the operator $\mathbf{I}_n-\mathbf{SQ}$ is invertible, and from (1.53) we have

$$\mathbf{x}=(\mathbf{I}_n-\mathbf{SQ})^{-1}\mathbf{Rx}. \tag{1.54}$$

Inserting (1.54) into the right-hand side of (1.51), we get

$$\mathbf{NRx}=0, \tag{1.55}$$

where

$$\mathbf{N}=\mathbf{R}\,[\mathbf{K}-\beta_0\mathbf{I}_n-\mathbf{Q}\,(\mathbf{I}_n-\mathbf{SQ})^{-1}]\,\mathbf{R} \tag{1.56}$$

(we have used the relations $\mathbf{R}^2=\mathbf{R}$, $\mathbf{KR}=\mathbf{RK}$).

Now, since

$$(\mathbf{I}_n-\mathbf{SQ})^{-1}=\mathbf{I}_n+\mathbf{SQ}\,(\mathbf{I}_n-\mathbf{SQ})^{-1}$$

and $\mathbf{Q}=(\beta-\beta_0)\,\mathbf{I}_n-\varepsilon\mathbf{L}$ (where $\mathbf{L}=\mathbf{L}(\beta,\ \varepsilon)$), substitution in (1.56) gives

$$\mathbf{N}=\mathbf{R}\,[(\mathbf{K}-\beta\mathbf{I}_n+\varepsilon\mathbf{L})-\mathbf{QSQ}\,(\mathbf{I}_n-\mathbf{SQ})^{-1}]\,\mathbf{R}. \tag{1.57}$$

By the definition of $\mathbf{S}$, we have $\mathbf{S}=\mathbf{PS}=\mathbf{SP}$, $\mathbf{RS}=\mathbf{SR}=\mathbf{0}$. Therefore $\mathbf{RQS}=$ $=\mathbf{R}\,(-\varepsilon\mathbf{L})\,\mathbf{S}$, $\mathbf{SQR}=\mathbf{S}\,(-\varepsilon\mathbf{L})\,\mathbf{R}$. Using these relations, we obtain*

$$\mathbf{RQSQ}\,(\mathbf{I}_n-\mathbf{SQ})^{-1}\,\mathbf{R}=\mathbf{R}\,(-\varepsilon\mathbf{L})\,\mathbf{SQ}\,(\mathbf{I}_n-\mathbf{SQ})^{-1}\,\mathbf{R}=$$
$$=\mathbf{R}\,(-\varepsilon\mathbf{L})\,(\mathbf{I}_n-\mathbf{SQ})^{-1}\,\mathbf{SQR}=\varepsilon^2\mathbf{RL}\,(\mathbf{I}_n-\mathbf{SQ})^{-1}\,\mathbf{SLR}.$$

Finally, from (1.57) we have

$$\mathbf{N}=\mathbf{R}\,[\mathbf{K}-\beta\mathbf{I}_n+\varepsilon\mathbf{L}-\varepsilon^2\mathbf{L}\,(\mathbf{I}_n-\mathbf{SQ})^{-1}\,\mathbf{S}]\,\mathbf{R}. \tag{1.58}$$

Clearly, $\mathbf{Rx}\neq\mathbf{0}$. Indeed, were $\mathbf{Rx}=\mathbf{0}$ it would follow from (1.54) that $\mathbf{x}=\mathbf{0}$, contrary to assumption.

We now show that (1.48) follows from (1.55), (1.58).

In general, let $\mathbf{N}$ be some linear operator in $\mathfrak{R}_n$ and $\mathbf{R}$ a projection on the subspace $\mathfrak{L}$. The subspace $\mathfrak{L}$ is invariant under the operator $\mathbf{RNR}$. As an operator in $\mathfrak{L}$, this operator is known as the restriction of $\mathbf{N}$ to $\mathfrak{L}$; it will be denoted by $\mathbf{N}_{\mathfrak{L}}$.

In our case, $\mathbf{RNR}=\mathbf{N}$, and the vectors $\mathbf{a}_1,\ \ldots,\ \mathbf{a}_r$ form a basis for $\mathfrak{L}$. The matrix $\|\nu_{jh}\|_1^r$ of the operator relative to this basis is defined as usual (see §I.1.7):

$$\mathbf{N}_{\mathfrak{L}}\mathbf{a}_j=\mathbf{RNa}_j=\sum_{h=1}^{r}\nu_{hj}\mathbf{a}_h \qquad (j=1,\ \ldots,\ r). \tag{1.59}$$

* These operations remove the "undesirable" terms in the expression for ψ_{jh}, proportional to $(\beta-\beta_0)^2$ and $\varepsilon(\beta-\beta_0)$.

In our case it follows from (1.55), (1.58) that $\mathbf{N}_{\mathfrak{L}}\mathbf{x}_1 = 0$, where $\mathbf{x}_1 = \mathbf{R}\mathbf{x} \neq 0$. Therefore

$$\det \| v_{jh} \|_1^r \neq 0. \tag{1.60}$$

Equation (1.60) is equivalent to (1.48). Indeed, by (1.59),

$$\| (\mathbf{R}\mathbf{N}\mathbf{a}_j,\ \mathbf{b}_h) \|_1^r = \| v_{jh} \|_1^r \| (\mathbf{a}_j,\ \mathbf{b}_h) \|_1^r. \tag{1.61}$$

According to § I. 1. 13, the decomposition $\mathbf{I}_n = \mathbf{R}^* + \mathbf{P}^*$ (which follows from (1.49)) induces a decomposition $\mathfrak{N}_n = \mathfrak{L}_* \dotplus \mathfrak{M}_*$, where $\mathfrak{M}_*$ is the sum of all root subspaces of the operator $\mathbf{K}^*$ for all eigenvalues except $\overline{\beta}_0$. Thus $\mathbf{R}^*$ is a projection on $\mathfrak{L}_*$, i. e., $\mathbf{R}^*\mathbf{b}_h = \mathbf{b}_h$. Therefore, $(\mathbf{R}\mathbf{N}\mathbf{a}_j,\ \mathbf{b}_h) = (\mathbf{N}\mathbf{a}_j,\ \mathbf{R}^*\mathbf{b}_h) = (\mathbf{N}\mathbf{a}_j,\ \mathbf{b}_h)$, and it follows from (1.61) that

$$\| v_{jh} \|_1^r = \| (\mathbf{N}\mathbf{a}_j,\ \mathbf{b}_h) \|_1^r \cdot \| (\mathbf{a}_j,\ \mathbf{b}_h) \|^{-1}. \tag{1.62}$$

Consequently, equation (1.60) is equivalent to the equation

$$\det \| (\mathbf{N}\mathbf{a}_j,\ \mathbf{b}_h) \|_1^r = 0,$$

which is precisely (1.48).

We have shown that condition (1.2) implies (1.48), provided $|\varepsilon|$ and $|\beta - \beta_0|$ are sufficiently small. We now prove the converse. Suppose that (1.48) is true. Because of (1.62), equation (1.60) also holds, where, as before, $\| v_{jh} \|_1^r$ is the matrix of the operator $\mathbf{N}_{\mathfrak{L}}$ relative to the basis $\{\mathbf{a}_j\}$ and $\mathbf{N}$ is the operator (1.56). Since $\det \mathbf{N}_{\mathfrak{L}} = 0$, there is a vector $\mathbf{x}_1 \in \mathfrak{L}$ such that $\mathbf{N}_{\mathfrak{L}}\mathbf{x}_1 = 0$ or, in other words, such that

$$\mathbf{N}\mathbf{x}_1 = 0. \tag{1.63}$$

Set

$$\mathbf{x} = (\mathbf{I}_n - \mathbf{S}\mathbf{Q})^{-1}\mathbf{x}_1. \tag{1.64}$$

Then

$$\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2, \quad \text{where} \quad \mathbf{x}_2 = \mathbf{S}\mathbf{Q}\mathbf{x}. \tag{1.65}$$

Since $\mathbf{P}\mathbf{S} = \mathbf{S}$, we have $\mathbf{P}\mathbf{x}_2 = \mathbf{x}_2$, so that $\mathbf{x}_2 \in \mathfrak{M}$. By the first decomposition of (1.43),

$$\mathbf{x} = \mathbf{R}\mathbf{x}, \quad \mathbf{x}_2 = \mathbf{P}\mathbf{x}. \tag{1.66}$$

Thus (1.64) implies (1.54), and moreover (1.63) is equivalent to (1.55). From (1.54), (1.55) and (1.56) we deduce (1.51). From (1.54) we get (1.53), whence (via (1.49)) we deduce (1.52). Since $\mathbf{S} = \mathbf{S}\mathbf{P}$ we have $\mathbf{P}\mathbf{x} = \mathbf{S}\mathbf{P}\mathbf{Q}\mathbf{x}$; hence, by (1.44), we get (1.50). Combining (1.50) and (1.51), we obtain (1.18) and thus also (1.16). By (1.64), $\mathbf{x} \neq 0$, since $\mathbf{x}_1 \neq 0$. Thus (1.2) follows from (1.16) and the proof is complete.

Remark. It follows from what we have proved that equation (1.48) is equivalent to the equation $\det \mathbf{N}_\beta = 0$, where $\mathbf{N}$ is given by (1.58). The last equation does not involve a fixed basis, and for this reason it is sometimes more convenient to use than (1.48).

The order r of the determinant (1.48) is often much lower than the order n of (1.2), and this is the advantage of equation (1.48) over the original equation (1.2). It is important in this connection that the functions ψ_{jh} have the special structure shown in (1.47) — their principal terms are immediately apparent (first term in (1.47)). The second term is of higher order. This facilitates determination of the principal coefficients in the expansion of β in (generally fractional) powers of ε.

The special case of a single elementary Jordan matrix is given special consideration:

Corollary. Under the assumptions of the lemma, suppose that β_0 corresponds to a single elementary Jordan matrix in the canonical form of $\mathbf{K}$. In other words, the equation $\mathbf{Kx} = \beta_0 \mathbf{x}$ has only one linearly independent solution. Let $\mathbf{a}_1, \ldots, \mathbf{a}_r$ be the corresponding cyclic basis:

$$\mathbf{Ka}_1 = \beta_0 \mathbf{a}_1, \quad \mathbf{Ka}_2 = \beta_0 \mathbf{a}_2 + \mathbf{a}_1, \quad \ldots, \quad \mathbf{Ka}_r = \beta_0 \mathbf{a}_r + \mathbf{a}_{r-1}.$$

Let $\mathbf{b}_1, \ldots, \mathbf{b}_r$ be vectors in the root subspace of $\mathbf{K}^*$ belonging to the root $\bar{\beta}_0$, such that

$$(\mathbf{a}_j, \mathbf{b}_h) = \delta_{jh} \qquad (j, h = 1, \ldots, r).$$

Set

$$\gamma = (\mathbf{L}(\beta_0, 0)\mathbf{a}_1, \mathbf{b}_r).$$

Let $\gamma \neq 0$. Then the r roots $\beta_j(\varepsilon)$ of equation (1.2) are

$$\beta_j(\varepsilon) = \beta_0 - (\varepsilon\gamma)_j^{1/r} + O(|\varepsilon|^{2/r}),$$

where $(\varepsilon\gamma)_j^{1/r}$ are the different values of the root $\sqrt[r]{\varepsilon\gamma}$.

1.5. Generalizations

The lemmas established in §§ 1.1 through 1.4 are special cases of more general theorems of perturbation theory which are worth citing here, even though they are somewhat beyond the scope of our book. To do this we need a few of the simplest concepts of operator theory (see, e. g., /78/). In a Hilbert space $\mathfrak{H}$, consider the operator

$$\mathbf{A}(\beta, \varepsilon) = \mathbf{K} - \beta\mathbf{I} + \varepsilon\mathbf{L}(\beta, \varepsilon),$$

where $\mathbf{I}$ is the identity operator, $\mathbf{L}(\beta, \varepsilon)$ a bounded operator which is analytic in β, ε at the point $(\beta_0, 0)$, $\mathbf{K}$ a closed (not necessarily bounded) operator

whose domain is dense in $\mathfrak{H}$, with an isolated eigenvalue β_0 of finite multiplicity. (In other words, β_0 is the only point of the spectrum of $\mathbf{K}$ in a sufficiently small disk $|\lambda-\beta_0|<\delta$.)

Consider the problem of determining the points (β, ε) in the neighborhood of $(\beta_0, 0)$ for which the equation

$$\mathbf{A}(\beta, \varepsilon)\mathbf{x}=\mathbf{0}$$

has a nontrivial solution $\mathbf{x}=\mathbf{x}(\beta, \varepsilon)$. If the operator $\mathbf{L}=\mathbf{L}(\varepsilon)$ is independent of β, this is simply the problem of finding the eigenvalues $\beta=\beta(\varepsilon)$ of the operator $\mathbf{K}+\varepsilon\mathbf{L}(\varepsilon)$ in the neighborhood of $\varepsilon=0$.

It is easy to see that under these assumptions all the proofs of §§1.1 through 1.4 remain valid after slight terminological adjustments.

Theorem. The assertions of the lemmas in §§1.1 — 1.4 remain valid if we replace the words "matrix $\mathbf{K}$*" by "operator* $\mathbf{K}$*" and so on. The operators* $\mathbf{S}_0$, $\mathbf{S}$ *are assumed to be bounded.*

As a matter of fact, the lemmas remain valid in various other cases, for example, when the operator $\mathbf{L}(\beta, \varepsilon)$ is nonlinear, or linear but unbounded, and subordinate in a certain sense to the operator $\mathbf{K}$.

§2. COMPUTATION OF CHARACTERISTIC EXPONENTS FOR A SYSTEM OF GENERAL FORM

2.1. Characteristic exponents of unperturbed system

Consider the vector equation (0.1), written as

$$\frac{d\mathbf{x}}{dt}=[\mathbf{C}+\varepsilon\mathbf{B}(t, \varepsilon)]\mathbf{x}. \tag{2.1}$$

Let us assume that the "unperturbed" equation (i.e., equation (2.1) with $\varepsilon=0$)

$$\frac{d\mathbf{x}}{dt}=\mathbf{C}\mathbf{x} \tag{2.2}$$

has an r-fold characteristic exponent α_0.

Recall (§II.2.5) that the characteristic exponent may be defined as a complex number α_0 for which there is a solution to the equation

$$\mathbf{x}(t)=e^{\alpha_0 t}\mathbf{u}(t), \tag{2.3}$$

where $\mathbf{u}(t)$ is a T-periodic vector-function. The characteristic exponents are defined up to terms $2\pi i m/T$, where m is an integer. This statement, previously proved by suitable choice of the branch of logarithm for the monodromy matrix (§§II.2.5 and I.2.6), follows very easily from the definition as phrased here. Indeed, let

$$\alpha_1=\alpha_0+\frac{2\pi i}{T}m.$$

Then we can write (2.3) as

$$\mathbf{x}(t)=e^{\alpha_1 t}\mathbf{u}_1(t) \qquad \left(\mathbf{u}_1(t)=\exp\left(-\frac{2\pi i}{T}mt\right)\mathbf{u}(t)\right),$$

where $\mathbf{u}_1(t)$ is clearly also T-periodic. Thus α_1 is also a characteristic exponent.

Thus, for any r-fold characteristic exponent α_0 of equation (2.2) we have a class of eigenvalues λ_j of the matrix $\mathbf{C}$, congruent modulo $2\pi i/T$:

$$\lambda_j=\alpha_0+\frac{2\pi i}{T}m_j \qquad (j=1,\ \ldots,\ r;\ m_j=0,\ \pm1,\ \pm2,\ \ldots). \tag{2.4}$$

It is our aim in this section to derive equations for the characteristic equations of equation (2.1) which tend at $\varepsilon=0$ to α_0.

We shall say that a characteristic exponent α_0 of equation (2.1) has dimension s if α_0 corresponds to s linearly independent solutions (2.3) of system (2.2). In other words, the dimension s is equal to the number of elementary divisors of the matrix $\mathbf{C}$ belonging to eigenvalues of the form (2.4), or the number of linearly independent eigenvectors belonging to these eigenvalues. The dimension s cannot exceed the multiplicity r ($s\leqslant r$), and $s=r$ if and only if all the eigenvalues of $\mathbf{C}$ congruent to α_0 modulo $2\pi i/T$ have simple elementary divisors. It is this case that is most frequently met with in applications, and we shall discuss it below in greater detail. When $s=r$ the characteristic exponent α_0 will be called an r-fold characteristic exponent of simple type.

Let $\mathbf{c}_1,\ \ldots,\ \mathbf{c}_s$ and $\mathbf{d}_1,\ \ldots,\ \mathbf{d}_s$ denote eigenvectors of the matrix $\mathbf{C}$ and $\mathbf{C}^*$ corresponding to the s eigenvalues (2.4) and their conjugates. In our new notation, *

$$\left.\begin{aligned} &\mathbf{C}\mathbf{c}_\sigma=\lambda_{(\sigma)}\mathbf{c}_\sigma, \quad \lambda_{(\sigma)}=\alpha_0+\frac{2\pi i}{T}m_{(\sigma)}, \\ &\mathbf{C}^*\mathbf{d}_\sigma=\bar{\lambda}_{(\sigma)}\mathbf{d}_\sigma, \quad \bar{\lambda}_{(\sigma)}=\bar{\alpha}_0-\frac{2\pi i}{T}m_{(\sigma)} \qquad (\sigma=1,\ \ldots,\ s). \end{aligned}\right\} \tag{2.5}$$

The linearly independent solutions $\mathbf{x}_\sigma(t)$ of equation (2.2) that can be expressed in the form (2.3) are now written

$$\mathbf{x}_\sigma(t)=e^{\lambda_{(\sigma)}t}\mathbf{c}_\sigma \qquad (\sigma=1,\ \ldots,\ s). \tag{2.6}$$

The substitution

$$\mathbf{x}=e^{\alpha_0 t}\mathbf{y}$$

reduces equation (2.2) to

$$\frac{d\mathbf{y}}{dt}-(\mathbf{C}-\alpha_0\mathbf{I}_n)\mathbf{y}=\mathbf{0}. \tag{2.7}$$

* Recall that in keeping with the notation introduced in § I.1.12 $\lambda_{(\sigma)}$ denotes the eigenvalue corresponding to an elementary Jordan matrix in the canonical form of $\mathbf{C}$.

Equation (2.7) has s linearly independent T-periodic solutions $\mathbf{y}_1(t), \ldots, \mathbf{y}_s(t)$; by (2.6), we can write

$$\mathbf{y}_\sigma(t) = \exp\left(\frac{2\pi i}{T} m_{(\sigma)} t\right) \mathbf{c}_\sigma \qquad (\sigma = 1, \ldots, s). \tag{2.8}$$

It is readily seen by a direct check that the adjoint system of (2.7), in vector notation,

$$\frac{d\mathbf{z}}{dt} + (\mathbf{C}^* - \bar{\alpha}_0 \mathbf{I}_n)\mathbf{z} = 0, \tag{2.9}$$

where $\mathbf{C}^*$ is the hermitian conjugate of $\mathbf{C}$, has s linearly independent solutions $\mathbf{z}_1(t), \ldots, \mathbf{z}_s(t)$, which may be expressed as

$$\mathbf{z}_\sigma(t) = \exp\left(\frac{2\pi i}{T} m_{(\sigma)} t\right) \mathbf{d}_\sigma \qquad (\sigma = 1, \ldots, s). \tag{2.10}$$

Let $\mathfrak{L}_2(0, T)$ be the space of vector-functions $\mathbf{u}(t)$, $0 \leqslant t \leqslant T$, for which the norm

$$\|\mathbf{u}\| = \sqrt{\frac{1}{T}\int_0^T |\mathbf{u}(t)|^2\, dt} \tag{2.11}$$

is finite, where $|\mathbf{u}(t)|$ is defined by formula (I. 1. 2).

For any $\mathbf{u}, \mathbf{v} \in \mathfrak{L}_2(0, T)$, we define

$$((\mathbf{u}, \mathbf{v})) = \frac{1}{T}\int_0^T (\mathbf{u}(t), \mathbf{v}(t))\, dt, \tag{2.12}$$

where $(\mathbf{u}, \mathbf{v})$ is the scalar product of the vectors $\mathbf{u} = \mathbf{u}(t)$ and $\mathbf{v} = \mathbf{v}(t)$ (see (I.1.1)). Since

$$|(\mathbf{u}, \mathbf{v})| \leqslant |\mathbf{u}||\mathbf{v}| \leqslant \tfrac{1}{2}(|\mathbf{u}|^2 + |\mathbf{v}|^2),$$

the integral (2.12) exists. We shall call $((\mathbf{u}, \mathbf{v}))$ the averaged scalar product of $\mathbf{u}(t)$ and $\mathbf{v}(t)$.

It is easy to see that the averaged scalar product has the usual properties of a scalar product (see § I. 1. 1). In particular, we have the Cauchy-Bunyakovskii inequality for vector-functions:

$$|((\mathbf{u}, \mathbf{v}))| \leqslant \|\mathbf{u}\|\,\|\mathbf{v}\|.$$

Let $\mathbf{Q}(t)$ be a piecewise-continuous bounded matrix-function on $[0, T]$. Then $\mathbf{Q}(t)\mathbf{u}(t) \in \mathfrak{L}_2(0, T)$, $|\mathbf{Q}(t)\mathbf{u}(t)| \leqslant |\mathbf{Q}(t)||\mathbf{u}(t)|$, and

$$\|\mathbf{Qu}\|^2 \equiv \frac{1}{T}\int_0^T |\mathbf{Q}(t)\mathbf{u}(t)|^2\, dt \leqslant \max_{[0,T]} |\mathbf{Q}(t)|^2 \cdot \frac{1}{T}\int_0^T |\mathbf{u}(t)|^2\, dt,$$

so that

$$\| \mathbf{Qu} \| \leqslant \sqrt{\max_{[0,\,T]} |\mathbf{Q}(t)|^2}\, \| \mathbf{u} \|. \tag{2.13}$$

2.2. The operators S and P. Two lemmas

Together with the linearly independent systems $\mathbf{y}_1(t), \ldots, \mathbf{y}_s(t)$; $\mathbf{z}_1(t), \ldots, \mathbf{z}_s(t)$, we define vector-functions $\mathbf{v}_1(t), \ldots, \mathbf{v}_s(t)$; $\mathbf{w}_1(t), \ldots, \mathbf{w}_s(t)$ as arbitrary T-periodic functions such that

$$\det \| ((\mathbf{y}_j, \mathbf{v}_h)) \|_1^s \neq 0, \quad \det \| ((\mathbf{z}_j, \mathbf{w}_h)) \|_1^s \neq 0. \tag{2.14}$$

For example, we can take $\mathbf{v}_h = \mathbf{y}_h$, $\mathbf{w}_h = \mathbf{z}_h$, but in the sequel (§2.6) we shall adopt a different choice. (Since $\mathbf{y}_1, \ldots, \mathbf{y}_s$; $\mathbf{z}_1, \ldots, \mathbf{z}_s$ are linearly independent systems, the determinants (2.14) for $\mathbf{v}_h = \mathbf{y}_h$, $\mathbf{w}_h = \mathbf{z}_h$ are Gram determinants and so do not vanish; see §I. 1. 8.) It follows from (2.14) that both systems, $\mathbf{v}_1, \ldots, \mathbf{v}_s$ and $\mathbf{w}_1, \ldots, \mathbf{w}_s$, are linearly independent. *

Consider the inhomogeneous vector equation

$$\frac{d\mathbf{y}}{dt} - (\mathbf{C} - \alpha_0 \mathbf{I}_n)\, \mathbf{y} = \mathbf{u}(t), \tag{2.15}$$

where $\mathbf{u}(t)$ is T-periodic. Let us assume first that $\mathbf{u}(t)$ is orthogonal in the sense of the scalar product (2.12) to all T-periodic solutions of equation (2.9), i. e., it satisfies the conditions

$$((\mathbf{u}, \mathbf{z}_\sigma)) = 0 \qquad (\sigma = 1, \ldots, s), \tag{2.16}$$

where $\mathbf{z}_\sigma(t)$ are defined by (2.10). Under the additional assumptions

$$((\mathbf{y}, \mathbf{v}_\sigma)) = 0 \qquad (\sigma = 1, \ldots, s), \tag{2.17}$$

equation (2.15) has a unique T-periodic solution (see Lemma II in §II. 2.10), which we denote by $\mathbf{y} = \mathbf{Su}(t)$. Recall that according to part 2 of that lemma there exists a positive number υ, depending on the matrix $\mathbf{C} - \alpha_0 \mathbf{I}_n$, such that

$$|\mathbf{y}(t)| \leqslant \upsilon \| \mathbf{u} \| \qquad (0 \leqslant t \leqslant T), \text{ where } \mathbf{y}(t) = \mathbf{Su}(t). \tag{2.18}$$

The norm $\| \mathbf{u} \|$ is defined by (2.11).

As a solution of equation (2.15), the function $\mathbf{y}(t) = \mathbf{Su}(t)$ is continuous and has a continuous (or piecewise-continuous) derivative.

By the definition of $\mathbf{S}$,

$$((\mathbf{Su}, \mathbf{v}_\sigma)) = 0 \qquad (\sigma = 1, \ldots, s).$$

Suppose that $\mathbf{u}(t)$ does not satisfy conditions (2.16). Choose constants $\zeta_1, \ldots, \zeta_s$ in such a way that the conditions (2.16) will hold for

* This may be proved in exactly the same way as the analogous statement for vectors in § I.1.4.

$$\mathbf{Pu}(t)=\mathbf{u}(t)-\sum_{j=1}^{s}\zeta_j\mathbf{w}_j(t), \tag{2.19}$$

i. e. ,

$$((\mathbf{Pu},\ \mathbf{z}_\sigma))=0 \qquad (\sigma=1,\ \ldots,\ s). \tag{2.20}$$

By (2.14), the constants $\zeta_1,\ \ldots,\ \zeta_s$ are uniquely determined by the equations

$$((\mathbf{u},\ \mathbf{z}_\sigma))-\sum_{j=1}^{s}\zeta_j((\mathbf{w}_j,\ \mathbf{z}_\sigma))=0 \qquad (\sigma=1,\ \ldots,\ s). \tag{2.21}$$

Formulas (2.19), (2.21) define the operator $\mathbf{P}$ for any vector-function $\mathbf{u}\in\mathfrak{L}_2(0,\ T)$.

By the definition of this operator, conditions (2.16) are equivalent to

$$\mathbf{Pu}=\mathbf{u}. \tag{2.22}$$

Indeed, if conditions (2.16) hold, it follows from (2.14), (2.21) that $\zeta_1=\ldots=\zeta_s=0$, and from (2.19) that $\mathbf{Pu}=\mathbf{u}$. Conversely, (2.22) and (2.19) and the linear independence of the vector-functions $\mathbf{w}_1,\ \ldots,\ \mathbf{w}_s$ imply that $\zeta_1=\ldots=\zeta_s=0$. From (2.21) we now deduce (2.16).

We claim that for any vector-function $\mathbf{u}\in\mathfrak{L}_2(0,\ T)$

$$\mathbf{P}^2\mathbf{u}=\mathbf{Pu}.$$

Indeed, $\mathbf{P}^2\mathbf{u}=\mathbf{P}(\mathbf{Pu})$ is defined by (2.19), (2.21) with $\mathbf{u}$ replaced by $\mathbf{Pu}$. Since $((\mathbf{Pu},\ \mathbf{z}_\sigma))=0$ $(\sigma=1,\ \ldots,\ s)$, it follows from (2.21) that $\zeta_1=\ldots=\zeta_s=0$ Hence it follows from (2.19) that $\mathbf{P}^2\mathbf{u}=\mathbf{Pu}$.

It follows from (2.20) that for any T-periodic vector-function $\mathbf{u}\in\mathfrak{L}_2(0,\ T)$ there exists a T-periodic solution of the equation

$$\frac{d\mathbf{y}}{dt}-(\mathbf{C}-\alpha_0\mathbf{I}_n)\mathbf{y}=\mathbf{Pu}(t), \tag{2.23}$$

and if we require in addition fulfillment of conditions (2.17) then $\mathbf{y}$ is uniquely determined. In accordance with our notation, $\mathbf{y}=\mathbf{SPu}(t)$.

Note that $\mathbf{Su}$ is not defined for arbitrary $\mathbf{u}\in\mathfrak{L}_2(0,\ T)$, but as shown above $\mathbf{SPu}$ is always defined.

Lemma 1. There exists a constant $\delta>0$ such that for any vector-function $\mathbf{u}\in\mathfrak{L}_2(0,\ T)$*

$$\|\mathbf{Pu}\|\leqslant\delta\|\mathbf{u}\|. \tag{2.24}$$

Proof. Let δ_1 denote the norm of the matrix $[\|((\mathbf{w}_j,\ \mathbf{z}_h))\|_1^s]^{-1}$ (which exists by virtue of (2.14)). By (2.21),

$$\sqrt{\sum_{\sigma=1}^{s}\zeta_\sigma^2}\leqslant\delta_1\sqrt{\sum_{\sigma=1}^{s}((\mathbf{u},\ \mathbf{z}_\sigma))^2}.$$

* This lemma is a special case of an elementary theorem in the theory of linear operators.

Since $|((\mathbf{u}, \mathbf{z}_\sigma))|^2 \leqslant \|\mathbf{u}\|^2 \|\mathbf{z}_\sigma\|^2$ (see end of §2.1), it follows that

$$\sqrt{\sum \zeta_\sigma^2} \leqslant \delta_1 \sqrt{\sum \|\mathbf{z}_\sigma\|^2} \|\mathbf{u}\|.$$

From (2.19), we have

$$\|\mathbf{Pu}\| \leqslant \|\mathbf{u}\| + \sum |\zeta_\sigma| \|\mathbf{w}_\sigma\| \leqslant \|\mathbf{u}\| + \sqrt{\sum \zeta_\sigma^2 \cdot \sum \|\mathbf{w}_\sigma\|^2}.$$

Thus the estimate (2.24) holds with the constant

$$\delta = 1 + \delta_1 \sqrt{\sum \|\mathbf{z}_\sigma\|^2 \sum \|\mathbf{w}_\sigma\|^2}.$$

Lemma II. Consider the vector equation

$$\mathbf{y} - \mathbf{SPQ}(t)\,\mathbf{y} = \mathbf{f}(t), \tag{2.25}$$

where $\mathbf{S}$ *and* $\mathbf{P}$ *are the operators defined above,* $\mathbf{Q}(t)$ *is an integrable piecewise-continuous* T*-periodic matrix-function and* $\mathbf{f}(t)$ *a continuous* T*-periodic vector-function. If* $\mathbf{Q}(t)$ *is sufficiently small, to be precise,*

$$\|\mathbf{Q}\| \equiv \sqrt{\frac{1}{T}\int_0^T |\mathbf{Q}(t)|^2\, dt} < \frac{1}{v\delta}, \tag{2.26}$$

where v *is the number from (2.18) and* δ *the constant of (2.24), then equation (2.25) has a unique solution, which is a continuous* T*-periodic vector-function satisfying the estimate*

$$|\mathbf{y}(t)| \leqslant \frac{1}{1 - v\delta \|\mathbf{Q}\|} \|\mathbf{f}\| \qquad (0 \leqslant t \leqslant T). \tag{2.27}$$

Proof. We write equation (2.25) as

$$[\mathbf{I}_n - \mathbf{SPQ}(t)]\,\mathbf{y} = \mathbf{f}(t)$$

and expand the solution in a formal power series

$$\mathbf{y} = [\mathbf{I}_n - \mathbf{SPQ}(t)]^{-1}\mathbf{f}(t) = [\mathbf{I}_n + \mathbf{SPQ} + (\mathbf{SPQ})^2 + \ldots]\,\mathbf{f}(t), \tag{2.28}$$

or

$$\mathbf{y} = \mathbf{f}(t) + \mathbf{f}_1(t) + \mathbf{f}_2(t) + \ldots, \tag{2.29}$$

where

$$\mathbf{f}_1(t) = \mathbf{SPQf}(t), \ \ldots, \ \mathbf{f}_N(t) = (\mathbf{SPQ})^N \mathbf{f}(t) = \mathbf{SP}[\mathbf{Q}(t)\,\mathbf{f}_{N-1}(t)].$$

The functions $\mathbf{f}_1, \mathbf{f}_2, \ldots$ are continuous T-periodic vector-functions by the properties of $\mathbf{S}$. Using Lemma I and formulas (2.18), (2.13), we obtain

$$|\mathbf{f}_1(t)| \leqslant v\|\mathbf{PQf}\| \leqslant v\delta\|\mathbf{Qf}\| \leqslant v\delta\|\mathbf{Q}\|\|\mathbf{f}\|,$$

. . . . . .

$$\mathbf{f}_N(t)| \leqslant v\|\mathbf{PQf}_{N-1}\| \leqslant v\delta\|\mathbf{Qf}_{N-1}\| \leqslant v\delta\|\mathbf{Q}\|\|\mathbf{f}_{N-1}\| \leqslant (v\delta\|\mathbf{Q}\|)^N \cdot \|\mathbf{f}\| \quad (0 \leqslant t \leqslant T).$$

By (2.26), the series (2.29) is uniformly convergent. The vector-function $\mathbf{y}(t)$ is continuous, as the limit of a uniformly convergent series of continuous functions. Moreover,

$$|\mathbf{y}(t)| \leqslant [1 + v\delta\|\mathbf{Q}\| + (v\delta\|\mathbf{Q}\|)^2 + \ldots]\|\mathbf{f}\| = \frac{1}{1 - v\delta\|\mathbf{Q}\|}\|\mathbf{f}\| \qquad (0 \leqslant t \leqslant T).$$

Our vector-function $\mathbf{y}(t)$ is such that

$$\mathbf{y} - \mathbf{SPQy} = \mathbf{f} + \mathbf{SPQf} + (\mathbf{SPQ})^2\mathbf{f} + \ldots - \mathbf{SPQf} - (\mathbf{SPQ})^2\mathbf{f} - \ldots = \mathbf{f},$$

so that it satisfies equation (2.25). This proves the lemma.

Note that equation (2.25) may also have a solution when inequality (2.26) fails to hold. In any case that a solution exists, we shall denote it by

$$\mathbf{y} = [\mathbf{I}_n - \mathbf{SPQ}(t)]^{-1}\mathbf{f}(t).$$

2.3. Theorem on computation of characteristic exponents

The following theorem uses the definitions and formulas of §§2.1 and 2.2.

Theorem. Assume that equation (2.1) with $\varepsilon = 0$ (i.e., equation (2.2)) has an r-fold characteristic exponent α_0 of dimension s. Define an $s \times s$ matrix $\|\varphi_{jh}\|$ whose elements

$$\varphi_{jh} = ((\mathbf{Q}[\mathbf{I}_n - \mathbf{SPQ}]^{-1}\mathbf{y}_j, \mathbf{z}_h)) \qquad (j, h = 1, \ldots, s), \tag{2.30}$$

where $\mathbf{Q} = (\alpha_0 - \alpha)\mathbf{I}_n + \varepsilon\mathbf{B}(t, \varepsilon)$, are functions of the parameters ε and α. Then the characteristic exponents $\alpha = \alpha(\varepsilon)$ of equation (2.1) that tend to α_0 at $\varepsilon = 0$ are defined for sufficiently small $|\varepsilon|$ as implicit functions of ε by the equation

$$\Phi(\alpha, \varepsilon) = 0. \tag{2.31}$$

Recall that the vectors $\mathbf{y}_1(t), \ldots, \mathbf{y}_s(t)$ and $\mathbf{z}_1(t), \ldots, \mathbf{z}_s(t)$ are defined by (2.5), (2.8) and (2.10), and

$$\{\mathbf{I}_n - \mathbf{SP}[(\alpha_0 - \alpha)\mathbf{I}_n + \varepsilon\mathbf{B}(t, \varepsilon)]\}^{-1}\mathbf{y}_j(t)$$

denotes the solution $\mathbf{w}(t)$ of the equation

$$\mathbf{w}(t) - \mathbf{SP}[(\alpha_0 - \alpha)\mathbf{I}_n + \varepsilon\mathbf{B}(t, \varepsilon)]\mathbf{w}(t) = \mathbf{y}_j(t), \tag{2.32}$$

where the operators $\mathbf{S}$ and $\mathbf{P}$ were defined at the beginning of §2.2. As shown in Lemma II of §2.2, for sufficiently small $|\alpha_0-\alpha|$ and $|\varepsilon|$ this solution exists and may be determined by successive approximations as a series

$$\mathbf{w}(t)=\sum_{m=0}^{\infty}\{\mathbf{SP}[(\alpha_0-\alpha)\mathbf{I}_n+\varepsilon\mathbf{B}(t,\ \varepsilon)]\}^m\mathbf{y}_j(t).$$

Proof. Let $\alpha=\alpha(\varepsilon)$ be a characteristic exponent of equation (2.1) such that $\alpha(0)=\alpha_0$. We shall show that (2.31) holds for sufficiently small ε. By assumption, equation (2.1) has a solution

$$\mathbf{x}=e^{\alpha t}\mathbf{y},$$

where $\mathbf{y}=\mathbf{y}(t,\ \varepsilon)$ is a T-periodic function of t. Substituting into (2.1), we get

$$\frac{d\mathbf{y}}{dt}-(\mathbf{C}-\alpha_0\mathbf{I}_n)\mathbf{y}=[(\alpha_0-\alpha)\mathbf{I}_n+\varepsilon\mathbf{B}(t,\ \varepsilon)]\mathbf{y}. \tag{2.33}$$

By Lemma I of §II. 2. 10, this implies*

$$(([(\alpha_0-\alpha)\mathbf{I}_n+\varepsilon\mathbf{B}(t,\ \varepsilon)]\mathbf{y},\ \mathbf{z}_h))=0 \qquad (h=1,\ \ldots,\ s), \tag{2.34}$$

where the $\mathbf{z}_h$ are defined by (2.10). Since (2.16) and (2.22) are equivalent for any $\mathbf{u}(t)$, the right-hand side of (2.33) is invariant under the operator $\mathbf{P}$:

$$\frac{d\mathbf{y}}{dt}-(\mathbf{C}-\alpha_0\mathbf{I}_n)\mathbf{y}=\mathbf{P}[(\alpha_0-\alpha)\mathbf{I}_n+\varepsilon\mathbf{B}(t,\ \varepsilon)]\mathbf{y}. \tag{2.35}$$

If (2.34) holds, then by part 3 of the above-mentioned lemma

$$\mathbf{y}=\mathbf{y}^{(0)}(t,\ \varepsilon)+\mathbf{y}^{(1)}(t,\ \varepsilon), \tag{2.36}$$

where

$$\mathbf{y}^{(1)}(t,\ \varepsilon)=\sum_{j=1}^{s}\zeta_j(\varepsilon)\mathbf{y}_j(t). \tag{2.37}$$

Here the vectors $\mathbf{y}_j(t)$ are defined by (2.8) and $\zeta_j(\varepsilon)$ are coefficients independent of t. We may assume, by Lemma II of § II. 2. 10, that a particular solution $\mathbf{y}^{(0)}(t,\ \varepsilon)$ of equation (2.35) is (uniquely) determined by the conditions

$$((\mathbf{y}^{(0)},\ \mathbf{v}_j))=0 \qquad (j=1,\ \ldots,\ s),$$

where $\mathbf{v}_j$ are the vectors defined at the beginning of §2.2.

Substituting (2.36) into the left-hand side of (2.35), we obtain

$$\frac{d\mathbf{y}^{(0)}}{dt}-(\mathbf{C}-\alpha_0\mathbf{I}_n)\mathbf{y}^{(0)}=\mathbf{P}[(\alpha_0-\alpha)\mathbf{I}_n+\varepsilon\mathbf{B}(t,\ \varepsilon)]\mathbf{y}. \tag{2.38}$$

* To obtain (2.34), multiply (2.33) (scalar product!) by $\mathbf{z}_h$, integrate over the period and use the fact that the $\mathbf{z}_h$ are T-periodic solutions of equation (2.9).

By the definition of **S** (§2.2),

$$\mathbf{y}^{(0)}(t, \varepsilon) = \mathbf{SP}[(\alpha_0 - \alpha)\mathbf{I}_n + \varepsilon \mathbf{B}(t, \varepsilon)]\,\mathbf{y}(t, \varepsilon). \tag{2.39}$$

Substituting this expression into (2.36) and rearranging, we get

$$\{\mathbf{I}_n - \mathbf{SP}[(\alpha_0 - \alpha)\mathbf{I}_n + \varepsilon \mathbf{B}(t, \varepsilon)]\}\,\mathbf{y}(t, \varepsilon) = \mathbf{y}^{(1)}(t, \varepsilon). \tag{2.40}$$

The characteristic exponents of equation (2.1) are continuous functions of ε, and so, for sufficiently small $|\varepsilon|$, the difference $|\alpha_0 - \alpha(\varepsilon)|$ may be made arbitrarily small. Therefore, for at least these values of $|\varepsilon|$, $|\alpha_0 - \alpha(\varepsilon)|$, equation (2.40) has a solution (see Lemma II, §2.2)

$$\mathbf{y}(t, \varepsilon) = \{\mathbf{I}_n - \mathbf{SP}[(\alpha_0 - \alpha)\mathbf{I}_n + \varepsilon \mathbf{B}(t, \varepsilon)]\}^{-1}\mathbf{y}^{(1)}(t, \varepsilon). \tag{2.41}$$

Substituting this formula for $\mathbf{y}(t, \varepsilon)$ into the orthogonality conditions (2.34), we obtain a system of algebraic equations for $\zeta_1(\varepsilon), \ldots, \zeta_s(\varepsilon)$:

$$\sum_{j=1}^{s} \varphi_{jh}\zeta_j(\varepsilon) = 0 \qquad (h = 1, \ldots, s), \tag{2.42}$$

where φ_{jh} have the form (2.30). This system has nontrivial solutions (and only these constitute a solution to the problem, by (2.41)) if and only if equation (2.31) holds, as claimed.

Now let the numbers ε, $\alpha(\varepsilon)$ be such that (2.31) holds and the operator in curly brackets in (2.40) is invertible (e.g., $|\varepsilon|$, $|\alpha_0 - \alpha(\varepsilon)|$ are sufficiently small). Then system (2.41) defines a nontrivial solution $\zeta_1(\varepsilon), \ldots, \zeta_s(\varepsilon)$. With these coefficients, we use formulas (2.37) and (2.41) to define a vector-function $\mathbf{y}(t, \varepsilon)$. By Lemma II of §2.2, it is T-periodic and satisfies equation (2.40). Defining $\mathbf{y}^{(0)}$ by (2.36), we write (2.40) in the form (2.39). By the definition of **S**, equation (2.38) is true. Substituting the value

$$\mathbf{y}^{(0)} = \mathbf{y} - \sum_{j=1}^{s} \zeta_j \mathbf{y}_j(t)$$

found from (2.36), (2.37) into (2.38), we obtain (2.35).

On the other hand, it follows from (2.42) that

$$((\,[(\alpha_0 - \alpha)\mathbf{I}_n + \varepsilon \mathbf{B}(t, \varepsilon)]\,\mathbf{y}(t, \varepsilon),\ \mathbf{z}_\sigma(t))) = 0 \qquad (\sigma = 1, \ldots, s).$$

This means that we can drop the operator **P** on the right of (2.35). Consequently, $\mathbf{y}(t, \varepsilon)$ satisfies equation (2.33).

Then the vector-function

$$\mathbf{x}(t, \varepsilon) = e^{\alpha(\varepsilon)t}\mathbf{y}(t, \varepsilon)$$

is a solution of equation (2.1) and $\alpha = \alpha(\varepsilon)$ is a characteristic exponent of equation (2.1), with $\alpha(0) = \alpha_0$. Q. E. D.

Remarks. I. The function $\Phi(\varepsilon, \alpha)$ is analytic at the point $(0, \alpha_0)$ in all its arguments, and $\Phi(0, \alpha_0) = 0$. It follows from implicit function theory

that equation (2.31) defines one or more functions $\alpha=\alpha(\varepsilon)$ which take the value α_0 at $\varepsilon=0$ and are analytic functions of $\varepsilon^{1/q}$, where q is an integer.

II. The order of the determinant on the left of equation (2.31), which defines the characteristic exponents of the perturbed equation (2.1) in implicit fashion, is equal to the dimension s and does not exceed the multiplicity r of the characteristic exponent of the unperturbed equation (2.2). In applications, s is often much smaller than the order n of system (2.1), which is an important factor in computational work.

III. The proof of the theorem does not use the fact that the unperturbed system has constant coefficients. It is easy to formulate an analogous theorem for the case that the unperturbed equation is a system with periodic coefficients and solutions are known for the unperturbed equation and its adjoint, corresponding to the characteristic exponent in which we are interested.

IV. Equation (2.31) defines functions $\alpha=\alpha(\varepsilon)$ not necessarily for small ε, but for the values of ε such that (2.30) is meaningful. This is the case if the corresponding equation (2.32), which is a special form of integral equation, is solvable. Basing ourselves on this equation, we can work out computational procedures for evaluation of characteristic exponents, suitable for not necessarily small ε.

V. Recall that the operators $\mathbf{S}$ and $\mathbf{P}$ figuring in the theorem depend on the choice of the vector-functions $\mathbf{v}_1, \ldots, \mathbf{v}_s; \mathbf{w}_1, \ldots, \mathbf{w}_s$ introduced in §2.2.

2.4. Computation of characteristic exponents in the general case ($s \leqslant r$)*

Let $s \leqslant r$, so that the characteristic exponent α_0 will generally have multiple elementary divisors.

Using the expansion (2.28) and the notation for $\mathbf{Q}$, we develop the expression (2.30) for φ_{jh} as follows:

$$\varphi_{jh}=((\,[\mathbf{Q}+\mathbf{QSPQ}+\mathbf{QSPQSPQ}+\ldots]\,\mathbf{y}_j,\ \mathbf{z}_h)) \qquad (j,\ h=1,\ \ldots,\ s). \tag{2.43}$$

We wish to derive a detailed formula for the second-order term in $\alpha_0-\alpha$ and ε.

We shall utilize the expansion

$$\mathbf{B}(t,\ \varepsilon)=\mathbf{B}(t,\ 0)+\varepsilon\frac{\partial \mathbf{B}(t,\ 0)}{\partial\varepsilon}+\ldots=\mathbf{B}_1(t)+\varepsilon\mathbf{B}_2(t)+\ldots \tag{2.43a}$$

and matrix-valued Fourier series for the T-periodic matrix-functions $\mathbf{B}_1(t)$ and $\mathbf{B}_2(t)$:

$$\mathbf{B}_1(t)\sim\sum_{m=-\infty}^{\infty}e^{i\omega_m t}\mathbf{B}_1^{(m)},\quad \mathbf{B}_2(t)\sim\sum_{m=-\infty}^{\infty}e^{i\omega_m t}\mathbf{B}_2^{(m)}, \tag{2.43b}$$

where $\omega_m=2\pi m/T$.

* The reader interested only in the case $s=r$ may proceed directly to § 2.6.

Denoting $m' = m_{(j)}$, $m'' = m_{(h)}$, inserting (2.8), (2.10) and using (2.12), we obtain

$$((\mathbf{B}_1(t)\,\mathbf{y}_j,\ \mathbf{z}_h)) = \frac{1}{T}\int_0^T \Big(\sum_m e^{i\omega_m t}\mathbf{B}_1^{(m)} e^{i\omega_{m'} t}\,\mathbf{c}_j,\ e^{i\omega_{m''} t}\,\mathbf{d}_h\Big)\,dt =$$

$$= \frac{1}{T}\int_0^T \sum_m e^{i(\omega_m+\omega_{m'}-\omega_{m''})t}\,(\mathbf{B}_1^{(m)}\mathbf{c}_j,\ \mathbf{d}_h)\,dt = \Big(\mathbf{B}_1^{(m_{(h)}-m_{(j)})}\mathbf{c}_j, \mathbf{d}_h\Big).$$

Similarly, we have

$$((\mathbf{B}_2(t)\,\mathbf{y}_j,\ \mathbf{z}_h)) = \Big(\mathbf{B}_2^{(m_{(h)}-m_{(j)})}\mathbf{c}_j,\ \mathbf{d}_h\Big),$$

$$((\mathbf{y}_j,\ \mathbf{z}_h)) = (\mathbf{c}_j,\ \mathbf{d}_h)\,\delta_{m_{(j)},\,m_{(h)}} = \begin{cases} 0, & \text{if} \quad m_{(h)} \neq m_{(j)}, \\ (\mathbf{c}_j,\ \mathbf{d}_h), & \text{if} \quad m_{(h)} = m_{(j)}. \end{cases}$$

Finally, we find the desired expression for φ_{jh}:

$$\begin{aligned} \varphi_{jh} = (\alpha_0-\alpha)\,\delta_{m_{(j)},\,m_{(h)}}(\mathbf{c}_j,\ \mathbf{d}_h) + \varepsilon\Big(\mathbf{B}_1^{(m_{(h)}-m_{(j)})}\mathbf{c}_j,\ \mathbf{d}_h\Big) + \\ + \varepsilon^2\Big[\Big(\mathbf{B}_2^{(m_{(h)}-m_{(j)})}\mathbf{c}_j,\ \mathbf{d}_h\Big) + ((\mathbf{B}_1(t)\,\mathbf{SPB}_1(t)\,\mathbf{y}_j,\ \mathbf{z}_h))\Big] + (\alpha_0-\alpha)^2((\mathbf{SPy}_j,\ \mathbf{z}_h)) + \\ + \varepsilon(\alpha_0-\alpha)[((\mathbf{SPB}_1(t)\,\mathbf{y}_j,\ \mathbf{z}_h)) + ((\mathbf{B}_1(t)\,\mathbf{SPy}_j,\ \mathbf{z}_h))] + \\ + O(\varepsilon^3) + O[\varepsilon^2(\alpha-\alpha_0)] + \ldots + O[(\alpha-\alpha_0)^s] \qquad (2.44) \\ (j,\ h = 1,\ \ldots,\ s). \end{aligned}$$

Formula (2.43) readily yields expressions for the third-order terms.

If the elementary divisors are simple, formulas (2.44) may be simplified (see below). If the elementary divisors are multiple, evaluation of the roots of equation (2.31) is quite tedious, but nonetheless feasible, as the example presented below will be demonstrated.

In computations we shall need the relations

$$((\mathbf{z}_j,\ \mathbf{z}_h)) = \begin{cases} 0, & \text{if} \quad m_{(j)} \neq m_{(h)}, \\ (\mathbf{d}_j,\ \mathbf{d}_h), & \text{if} \quad m_{(j)} = m_{(h)}, \end{cases} \qquad (2.45)$$

which are proved as before:

$$((\mathbf{z}_j,\ \mathbf{z}_h)) = \frac{1}{T}\int_0^T (\mathbf{z}_j,\ \mathbf{z}_h)\,dt = \frac{1}{T}(\mathbf{d}_j,\ \mathbf{d}_h)\int_0^T \exp[i(\omega_{(j)}-\omega_{(h)})\,t]\,dt,$$

giving (2.45). Vectors $\mathbf{d}_j$ with identical numbers $m_{(j)}$ are eigenvectors belonging to the same eigenvalue of the matrix $\mathbf{C}$. We may therefore assume that they are orthonormal (Lemma II of §I. 1. 5):

$$(\mathbf{d}_{j'},\ \mathbf{d}_{h'}) = \delta_{j'h'} \qquad (\text{for } m_{(j')} = m_{(h')}).$$

Assuming this condition fulfilled, we have

$$((\mathbf{z}_j,\ \mathbf{z}_h)) = \delta_{jh} \qquad (j,\ h = 1,\ \ldots,\ s). \qquad (2.46)$$

2.5. Example

We return once again (§ II. 3.8) to the variational equations for the periodic motion of a flexible shaft with a heavy symmetrically attached flywheel, revolving at half the critical angular velocity. In the new notation, the system of equations (II. 3.53) is

$$\frac{d\mathbf{x}}{dt}=[\mathbf{C}+\varepsilon\mathbf{B}_1(t)+\varepsilon^2\mathbf{B}_2]\,\mathbf{x}, \tag{2.47}$$

where

$$\mathbf{C}=\begin{pmatrix}0 & \mathbf{I}_3\\ -\mathbf{P}_0 & 0\end{pmatrix},\quad \mathbf{B}_1(t)=\begin{pmatrix}0 & 0\\ -\mathbf{P}_1(t) & 0\end{pmatrix},\quad \mathbf{B}_2=\begin{pmatrix}0 & 0\\ -\mathbf{P}_2 & 0\end{pmatrix},$$

$$\mathbf{P}_0=\begin{Vmatrix}1&0&0\\0&1&0\\0&0&0\end{Vmatrix},\quad \mathbf{P}_1(t)=\begin{Vmatrix}0 & 0 & \sin\frac{1}{2}t\\ 0 & 0 & -\cos\frac{1}{2}t\\ \sin\frac{1}{2}t & -\cos\frac{1}{2}t & -2\delta\sin\frac{1}{2}t\end{Vmatrix},\quad \mathbf{P}_2=\begin{Vmatrix}0&0&0\\0&0&0\\0&0&4/3\end{Vmatrix}.$$

The elementary divisors of $\mathbf{C}$ are λ^2, $\lambda-i$, $\lambda-i$, $\lambda+i$, $\lambda+i$ (Example 4 in § I. 1.19). Number the eigenvalues of $\mathbf{C}$ in accordance with the elementary divisors:

$$\lambda_{(1)}=0,\quad \lambda_{(2)}=\lambda_{(3)}=i,\quad \lambda_{(4)}=\lambda_{(5)}=-i.$$

They are all congruent modulo $2\pi i/T=i/2$ and form a single class of eigenvalues of $\mathbf{C}$, corresponding to the zero characteristic exponent $\alpha_0=0$ of the unperturbed system — the equation $\dot{\mathbf{x}}=\mathbf{C}\mathbf{x}$:

$$\lambda_{(\sigma)}=0+m_{(\sigma)}\frac{i}{2}\qquad (\sigma=1,\ \ldots,\ 5;\ m_{(1)}=0,\ m_{(2)}=m_{(3)}=2,\ m_{(4)}=m_{(5)}=-2). \tag{2.48}$$

Thus $r=6$, $s=5$.

The eigenvectors of the matrices $\mathbf{C}$ and $\mathbf{C}^*$ are

$$\mathbf{c}_1=\xi_3\mathbf{e}_3,\quad \mathbf{c}_2=\eta_1(\mathbf{e}_1+i\mathbf{e}_4),\quad \mathbf{c}_3=\eta_2(\mathbf{e}_2+i\mathbf{e}_5),\quad \mathbf{c}_4=\zeta_1(\mathbf{e}_1-i\mathbf{e}_4),\quad \mathbf{c}_5=\zeta_2(\mathbf{e}_2-i\mathbf{e}_5),$$
$$\mathbf{d}_1=\varphi_6\mathbf{e}_6,\quad \mathbf{d}_2=\chi_1(\mathbf{e}_1+i\mathbf{e}_4),\quad \mathbf{d}_3=\chi_2(\mathbf{e}_2+i\mathbf{e}_5),\quad \mathbf{d}_4=\psi_1(\mathbf{e}_1-i\mathbf{e}_4),\quad \mathbf{d}_5=\psi_2(\mathbf{e}_2-i\mathbf{e}_5).$$

Normalize these vectors in such a way that all of them, except those with a multiple elementary divisor, form a biorthonormal system:

$$(\mathbf{c}_\sigma,\ \mathbf{d}_\rho)=\delta_{\sigma\rho}\qquad (\sigma,\ \rho=1,\ \ldots,\ 5;\ \sigma+\rho>2).$$

We have

$$\mathbf{c}_1=\mathbf{e}_3,\quad \mathbf{d}_1=\mathbf{e}_6,\quad \mathbf{c}_2=\mathbf{d}_2=\bar{\mathbf{c}}_4=\bar{\mathbf{d}}_4=\frac{1}{\sqrt{2}}(\mathbf{e}_1+i\mathbf{e}_4),\quad \mathbf{c}_3=\mathbf{d}_3=\bar{\mathbf{c}}_5=\bar{\mathbf{d}}_5=\frac{1}{\sqrt{2}}(\mathbf{e}_2+i\mathbf{e}_5).$$

Moreover, $(\mathbf{c}_1,\ \mathbf{d}_1)=0$.

Referring to formula (2.44), we see that

$$\mathbf{B}_1^{(1)}=\bar{\mathbf{B}}_1^{(-1)}=\begin{pmatrix}0 & 0\\ -\mathbf{P}_1^{(1)} & 0\end{pmatrix},\quad \mathbf{P}_1^{(1)}=\frac{1}{2}\begin{Vmatrix}0&0&-i\\0&0&-1\\-i&-1&2\delta i\end{Vmatrix};\quad \mathbf{B}_2^{(0)}=\mathbf{B}_2,$$

and all other Fourier coefficients of $\mathbf{B}_1(t)$ and $\mathbf{B}_2$ vanish. By (2.48), $m_{(\rho)}-m_{(\sigma)}$ must have one of the values $-4, -2, 0, 2, 4$, and so

$$(\mathbf{B}_1^{(m_{(\rho)}-m_{(\sigma)})}\mathbf{c}_\sigma, \mathbf{d}_\rho)=0 \qquad (\sigma, \rho=1, \ldots, 5).$$

We now use the formulas (2.44) to evaluate φ_{jh} up to second-order terms (with respect to ε and $\alpha-\alpha_0$) inclusive. The two terms in (2.44) have been found. For the third, we note that

$$\mathbf{B}_2^{(m_{(\rho)}-m_{(\sigma)})}=\begin{cases}\mathbf{B}_2, & \text{if} \quad m_{(\rho)}=m_{(\sigma)},\\ 0, & \text{if} \quad m_{(\rho)}\neq m_{(\sigma)}.\end{cases}$$

Of the quantities $\left(\mathbf{B}_2^{(m_{(\rho)}-m_{(\sigma)})}\mathbf{c}_\sigma, \mathbf{d}_\rho\right)$ $(\sigma, \rho=1, \ldots, 5)$, only the first does not vanish:

$$(\mathbf{B}^{(0)}{}_2\mathbf{c}_1, \mathbf{d}_1)=-\frac{4}{3}.$$

With an eye to further computations, we write $\mathbf{y}_\sigma(t)$ and $\mathbf{z}_\sigma(t)$ according to (2.8) and (2.10):

$$\begin{aligned}&\mathbf{y}_1=\mathbf{c}_1, \quad \mathbf{y}_2=e^{it}\mathbf{c}_2, \quad \mathbf{y}_3=e^{it}\mathbf{c}_3, \quad \mathbf{y}_4=e^{-it}\mathbf{c}_4, \quad \mathbf{y}_5=e^{-it}\mathbf{c}_5,\\ &\mathbf{z}_1=\mathbf{d}_1, \quad \mathbf{z}_2=e^{it}\mathbf{d}_2, \quad \mathbf{z}_3=e^{it}\mathbf{d}_3, \quad \mathbf{z}_4=e^{-it}\mathbf{d}_4, \quad \mathbf{y}_5=e^{-it}\mathbf{d}_5.\end{aligned}$$

Proceeding to the fourth term in (2.44), we denote

$$\chi_{\sigma\rho}=((\mathbf{B}_1(t)\,\mathbf{SPB}_1(t)\,\mathbf{y}_\sigma, \mathbf{z}_\rho)) \qquad (\sigma, \rho=1, \ldots, 5).$$

Beginning with $\chi_{1\rho}$, we first compute

$$\mathbf{B}_1\mathbf{y}_1=e^{i\frac{t}{2}}\mathbf{u}+e^{-i\frac{t}{2}}\bar{\mathbf{u}}, \quad \mathbf{u}=\frac{1}{2}\begin{pmatrix}0\\0\\0\\i\\1\\-2\delta i\end{pmatrix}.$$

By (2.19) with $\mathbf{w}_1=\mathbf{z}_1, \ldots, \mathbf{w}_5=\mathbf{z}_5$,

$$\mathbf{PB}_1\mathbf{y}_1=\mathbf{B}_1\mathbf{c}_1-\sum_{\rho=1}^{5}((\mathbf{B}_1\mathbf{c}_1, \mathbf{z}_\rho))\,\mathbf{z}_\rho=\mathbf{B}_1\mathbf{c}_1,$$

since $\mathbf{z}_1, \ldots, \mathbf{z}_5$ contain only e^0, e^{it}, e^{-it}, and $\mathbf{B}_1\mathbf{c}_1$ involves only $e^{i\frac{t}{2}}, e^{-i\frac{t}{2}}$. We can now write down an inhomogeneous equation (2.15) for the vector-function $\mathbf{SPB}_1\mathbf{y}_1=\mathbf{SB}_1\mathbf{c}_1$:

$$\frac{dy}{dt}-\mathbf{C}\mathbf{y}=\mathbf{PB}_1\mathbf{y}_1=e^{i\frac{t}{2}}\mathbf{u}+e^{-i\frac{t}{2}}\bar{\mathbf{u}}.$$

We shall assume that $\mathbf{SPB_1y_1}$ has the form

$$\mathbf{y} = \mathbf{SPB_1y_1} = e^{i\frac{t}{2}}\mathbf{v} + e^{-i\frac{t}{2}}\overline{\mathbf{v}}.$$

Substituting into the inhomogeneous equation, we obtain

$$\mathbf{v} = \left(\frac{i}{2}\mathbf{I_6} - \mathbf{C}\right)^{-1}\mathbf{u}.$$

The computation gives

$$\mathbf{SPB_1y_1} = e^{i\frac{t}{2}}\begin{bmatrix} \frac{2}{3}i \\ \frac{2}{3} \\ 4\delta i \\ -\frac{1}{3} \\ \frac{1}{3}i \\ -2\delta \end{bmatrix} + e^{-i\frac{t}{2}}\begin{bmatrix} -\frac{2}{3}i \\ \frac{2}{3} \\ -4\delta i \\ -\frac{1}{3} \\ -\frac{1}{3}i \\ -2\delta \end{bmatrix}.$$

Conditions (2.17) clearly hold for $\mathbf{v_1} = \mathbf{y_1}, \ldots, \mathbf{v_5} = \mathbf{y_5}$:

$$((\mathbf{SPB_1y_1}, \mathbf{y_\sigma})) = 0 \qquad (\sigma = 1, \ldots, 5).$$

We then compute

$$\mathbf{B_1}(t)\,\mathbf{SPB_1}(t)\,\mathbf{y_1} = e^{-it}\begin{bmatrix} 0 \\ 0 \\ 0 \\ -2\delta \\ -2\delta i \\ 4\delta^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4\delta \\ 0 \\ \frac{4}{3} - 8\delta^2 \end{bmatrix} + e^{it}\begin{bmatrix} 0 \\ 0 \\ 0 \\ -2\delta \\ 2\delta i \\ 4\delta^2 \end{bmatrix}.$$

Finally, multiplying this vector by each of $\mathbf{z_1}, \mathbf{z_2}, \mathbf{z_3}, \mathbf{z_4}, \mathbf{z_5}$ in succession, we shall compute only the free terms, since the last operation in calculating of $\chi_{\sigma\rho}$ is averaging of the scalar product. Thus,

$$\chi_{11} = \frac{4}{3} - 8\delta^2, \quad \chi_{12} = \sqrt{2}\,\delta i, \quad \chi_{13} = \sqrt{2}\,\delta, \quad \chi_{14} = -\sqrt{2}\,\delta i, \quad \chi_{15} = \sqrt{2}\,\delta.$$

Similarly,

$$\chi_{21} = -\sqrt{2}\,\delta, \quad \chi_{22} = \frac{5}{9}i, \quad \chi_{23} = \frac{4}{9}, \quad \chi_{24} = \chi_{25} = 0,$$
$$\chi_{31} = -\sqrt{2}\,\delta i, \quad \chi_{32} = -\frac{4}{9}, \quad \chi_{33} = \frac{5}{9}i, \quad \chi_{34} = \chi_{35} = 0,$$

$$\chi_{41}=-\sqrt{2}\,\delta,\quad \chi_{42}=\chi_{43}=0,\quad \chi_{44}=-\frac{5}{9}i,\quad \chi_{45}=\frac{4}{9},$$
$$\chi_{51}=\sqrt{2}\,\delta i,\quad \chi_{52}=\chi_{53}=0,\quad \chi_{54}=-\frac{4}{9},\quad \chi_{55}=-\frac{5}{9}i.$$

Now for the fifth term in (2.44). We compute first

$$\mathbf{P}y_1=y_1-\sum_{\rho=1}^{5}((y_1, z_\rho))\,z_\rho=c_1$$

and then $\mathbf{SP}y_1$, from the equation

$$\frac{dy}{dt}-Cy=\mathbf{P}y_1=c_1.$$

Since the right-hand side of this inhomogeneous equation is orthogonal "on the average" to $z_1, \ldots, z_5$, i.e., satisfies the necessary and sufficient conditions for solvability, it follows that the solution exists and may be written

$$y=\mathbf{SP}y_1=\xi_3 e_3-e_6.$$

The constant ξ_3 is determined from (2.17):

$$((\mathbf{SP}y_1, y_\sigma))=0 \qquad (\sigma=1, \ldots, 5),$$

whence $\xi_3=0$. Thus $\mathbf{SP}y_1=-e_6$. We note that $\mathbf{P}y_\sigma=0$ for $\sigma=2, 3, 4, 5$, since

$$\mathbf{P}y_\sigma=y_\sigma-\sum_{\rho=1}^{5}((y_\sigma, z_\rho))\,z_\rho=y_\sigma-z_\sigma=0 \qquad (\sigma=2, 3, 4, 5).$$

Hence, only one of the quantities $(\mathbf{SP}y_\sigma, z_\rho))$ is not zero:

$$((\mathbf{SP}y_1, z_1))=-1, \quad \text{but} \quad ((\mathbf{SP}y_\sigma, z_\rho))=0 \qquad (\sigma+\rho>2).$$

The vectors $\mathbf{SPB}_1 y_\sigma$ evaluated previously contain only components with factors $\exp\left(-\frac{1}{2}it\right)$ and $\exp\left(\frac{1}{2}it\right)$, and so

$$((\mathbf{SPB}_1 y_\sigma, z_\rho))=0 \qquad (\sigma, \rho=1, \ldots, 5)$$

so that the fifth term in (2.44) vanishes.

As for the sixth term, we have

$$((\mathbf{B}_1(t)\,\mathbf{SP}y_\sigma, z_\rho))=0 \qquad (\sigma=2, 3, 4, 5;\ \rho=1, \ldots, 5),$$

since $\mathbf{P}y_\sigma=0$ for $\sigma>1$. On the other hand,

$$((\mathbf{B}_1(t)\,\mathbf{SP}y_1, z_\rho))=0 \qquad (\rho=1, \ldots, 5),$$

since the first vector contains only components with factors $\exp\left(-\frac{1}{2}it\right)$ and $\exp\left(\frac{1}{2}it\right)$. Thus the sixth term in (2.44) is also zero for all j and h.

The subsequent terms in (2.44) are evaluated in similar fashion. For further computations, one must compute (using (2.43)) certain terms in (2.44) up to fourth order in $(\alpha-\alpha_0)$ and ε, which are not written out explicitly in (2.44).

Substituting $\alpha=\varepsilon\gamma$ in (2.31), we bring the left-hand side of that equation to the form $\Phi(\alpha, \varepsilon)=\varepsilon^6\Psi(\gamma, \varepsilon)$, where $\Psi(\gamma, \varepsilon)$ is analytic at $\varepsilon=0$ for any γ. It can be seen from the expression for $\Psi(\gamma, \varepsilon)$ (not given here) that $\Psi(\gamma, 0)=\gamma^4(\gamma^2+8\delta^2)$ and $\partial\Psi/\partial\gamma\neq 0$ for $\varepsilon=0$, $\gamma=\gamma_0$, where $\gamma_0=2\sqrt{2\delta}i$ is a root of the equation $\Psi(\gamma, 0)=0$. By the implicit function theorem, there exists a function $\gamma(\varepsilon)$, analytic at $\varepsilon=0$, such that $\Psi[\gamma(\varepsilon), \varepsilon]\equiv 0$ and $\gamma(0)=\gamma_0$. Thus equation (2.3) has a root $\alpha_1(\varepsilon)=2\sqrt{2}\delta i\varepsilon+O(\varepsilon^2)$. Setting $\alpha=\varepsilon\gamma_0+\varepsilon^2\gamma$, we similarly determine the coefficient $\gamma_1=\gamma(0)$ in the expansion $\alpha_1(\varepsilon)=\varepsilon\gamma_0+$ $+\varepsilon^2\gamma_1+O(\varepsilon^3)$, and so on. Similarly, we evaluate the other three roots $\alpha_2(\varepsilon), \alpha_3(\varepsilon), \alpha_4(\varepsilon)$ of equation (2.31) corresponding to the roots $\gamma=-2\sqrt{2\delta}i$ and $\gamma=0$ of the equation $\Psi(\gamma, 0)=0$. Since (2.47) is a variational equation and a canonical equation, it has an at least double zero characteristic exponent. Finally, we obtain the following expression for the characteristic exponents of equation (2.47):

$$\alpha_{1,2}=\pm\sqrt{2\varepsilon}i\left[2\delta+\varepsilon^2\left(\frac{1}{8\delta}-8\delta\right)\right]+O(\varepsilon^4),\quad \alpha_{3,4}=\pm\frac{1}{9}\varepsilon^2 i+O(\varepsilon^4),\qquad \alpha_{5,6}\equiv 0.$$

Note that α_1, α_2, α_3, α_4 are analytic functions of ε at $\varepsilon=0$. It can be shown (see the analogous arguments below, §3.1) that α_1, α_2, α_3, α_4 are pure imaginary for all sufficiently small real ε.

2.6. Computation of characteristic exponents of simple type ($s=r$). Formulas for first approximation

If the dimension of the characteristic exponent of the unperturbed equation is equal to its multiplicity ($s=r$), i. e., α_0 is a characteristic exponent of simple type, then the formulas for the matrix elements in equation (2.31) may be considerably simplified. In this case, which holds if and only if the matrix $\mathbf{C}$ has no multiple elementary divisors for the class of eigenvalues (2.4), the eigenvectors $\mathbf{c}_1, \ldots, \mathbf{c}_r$ are selected by the normalizing conditions (I. 1.16)*

$$(\mathbf{c}_j, \mathbf{d}_h)=\delta_{jh}\qquad (j, h=1, \ldots, r). \tag{2.49}$$

Then, proceeding as for (2.46), we obtain

* We note that if the eigenvalues of $\mathbf{C}$ are distinct, conditions (2.49) for $j\neq h$ are automatically satisfied. Indeed, $\lambda_j(\mathbf{y}_j, \mathbf{z}_h)=(\mathbf{C}\mathbf{y}_j, \mathbf{z}_h)=(\mathbf{y}_j, \mathbf{C}^*\mathbf{z}_h)=(\mathbf{y}_j, \overline{\lambda}_h\mathbf{z}_h)=\lambda_h(\mathbf{y}_j, \mathbf{z}_h)$. Hence it follows that $(\mathbf{y}_j, \mathbf{z}_h)=0$ for $\lambda_j\neq\lambda_h$. If the elementary divisors are not simple (i.e., $s<r$), the matrices $\|(\mathbf{c}_j, \mathbf{d}_h)\|$, $\|((\mathbf{y}_j, \mathbf{z}_h))\|$ are singular, so that the normalizations (2.49), (2.50) are impossible.

$$((\mathbf{y}_j, \mathbf{z}_h)) = \delta_{jh} \qquad (j, h = 1, \ldots, r), \tag{2.50}$$

where $\mathbf{y}_1(t), \ldots, \mathbf{y}_r(t)$; $\mathbf{z}_1(t), \ldots, \mathbf{z}_r(t)$ are defined by formulas (2.8) and (2.10) with $s = r$.

We now choose the vector-functions $\mathbf{v}_h$, $\mathbf{w}_h$ in the definitions of the operators $\mathbf{S}$ and $\mathbf{P}$ as follows:

$$\mathbf{v}_h(t) = \mathbf{z}_h(t), \quad \mathbf{w}_h(t) = \mathbf{y}_h(t) \qquad (h = 1, \ldots, r;\ 0 \leqslant t \leqslant T). \tag{2.51}$$

Conditions (2.14) are satisfied, since by (2.50) the determinants in (2.14) are equal to unity.

Formula (2.19) for the operator $\mathbf{P}$ becomes

$$\mathbf{Pu}(t) = \mathbf{u}(t) - \sum_{j=1}^{r} ((\mathbf{u}, \mathbf{z}_j))\, \mathbf{y}_j(t). \tag{2.52}$$

Indeed, it follows from equations (2.21) by (2.51), (2.50) that

$$\zeta_j = ((\mathbf{u}, \mathbf{z}_j)) \qquad (j = 1, \ldots, r).$$

The operator $\mathbf{P}$ now has the properties

$$\mathbf{P}\mathbf{y}_j = 0 \qquad (j = 1, \ldots, r), \tag{2.53}$$

$$((\mathbf{u}, \mathbf{z}_j)) = 0 \qquad (j = 1, \ldots, r), \quad \text{if} \quad \mathbf{Pu} = \mathbf{u}. \tag{2.54}$$

Indeed, (2.53) follows from (2.52) and (2.50). Property (2.54) follows from (2.52) since the vectors $\mathbf{y}_1(t), \ldots, \mathbf{y}_r(t)$ are linearly independent.

Given that (2.51) holds, we may define the operator $\mathbf{S}$ as follows. Let $\mathbf{y}_0$ be an arbitrary solution of equation (2.23), where $\mathbf{u} \in \mathfrak{L}_2(0, T)$ is any vector-function. Then

$$\mathbf{SPu} = \mathbf{y}_0 - \sum_{j=1}^{r} ((\mathbf{y}_0, \mathbf{z}_j))\, \mathbf{y}_j(t). \tag{2.55}$$

Indeed, denoting the right-hand side of this equality by $\mathbf{y}$, we see that $\mathbf{y}$ satisfies equation (2.23) and moreover conditions (2.17) hold with $\mathbf{v}_1 = \mathbf{z}_1, \ldots, \mathbf{v}_r = \mathbf{z}_r$. By the definition of $\mathbf{S}$, we have $\mathbf{y} = \mathbf{SPu}$.

It follows from (2.55) that for any $\mathbf{u} \in \mathfrak{L}_2(0, T)$,

$$\mathbf{PSPu} = \mathbf{SPu}. \tag{2.56}$$

Indeed, by (2.53), $\mathbf{PSPu} = \mathbf{Py}_0$. By (2.52), the expression for $\mathbf{Py}_0$ is precisely the right-hand side of (2.55), so that (2.56) is true. From (2.56) and (2.54) it follows that

$$((\mathbf{SPu}, \mathbf{z}_j)) = 0 \qquad (j = 1, \ldots, r) \tag{2.57}$$

for any $\mathbf{u} \in \mathfrak{L}_2(0, T)$.

We now set

$$\sigma_{jh} = ((\mathbf{B}(t, 0)\mathbf{y}_j, \mathbf{z}_h)) = (\mathbf{B}_1^{(m_h - m_j)}\mathbf{c}_j, \mathbf{d}_h) \qquad (j, h = 1, \ldots, r), \tag{2.58}$$

where $\mathbf{c}_1, \ldots, \mathbf{c}_r$ and $\mathbf{d}_1, \ldots, \mathbf{d}_r$ are linearly independent eigenvectors of the matrices $\mathbf{C}$ and $\mathbf{C}^*$ belonging to the characteristic exponent α_0 of simple type, normalized so that

$$(\mathbf{d}_j, \mathbf{d}_h) = \delta_{jh}, \qquad (\mathbf{c}_j, \mathbf{d}_h) = \delta_{jh} \qquad (j, h = 1, \ldots, r),$$

the constant matrices $\mathbf{B}_1^{(m)}$ are the Fourier coefficients of the function $\mathbf{B}_1(t) = \mathbf{B}(t, 0)$:

$$\mathbf{B}_1^{(m)} = \frac{1}{T}\int_0^T \exp\left(-\frac{2\pi i}{T} mt\right)\mathbf{B}(t, 0)\, dt \tag{2.58a}$$

and, finally, the numbers $m_1, \ldots, m_r$ are defined by (2.4). Note that when $r = s$, in the notation adopted in §2.1, $\lambda_{(\sigma)} = \lambda_\sigma$, $m_{(\sigma)} = m_\sigma$.

Theorem. Assume that equation (2.2) has an r-fold characteristic exponent α_0 of simple type. Then equation (2.1) has exactly r characteristic exponents $\alpha_1(\varepsilon), \ldots, \alpha_r(\varepsilon)$ tending to α_0 at $\varepsilon = 0$, which have expansions in fractional powers of ε, beginning with first-order terms

$$\alpha_\rho(\varepsilon) = \alpha_0 + \beta_\rho \varepsilon + O\left(\varepsilon^{1+\frac{1}{q_\rho}}\right) \qquad (\rho = 1, \ldots, r). \tag{2.59}$$

The coefficients $\beta_1, \ldots, \beta_r$ (some of which may coincide) are the roots of the equation

$$\det \| \sigma_{jh} - \beta\delta_{jh} \|_1^r = 0, \tag{2.60}$$

and $q_1, \ldots, q_r$ are natural numbers — the multiplicities of the corresponding roots β_ρ. If β_ρ is a simple root, then $q_\rho = 1$ and the corresponding function $\alpha_\rho(\varepsilon)$ is an analytic function of ε at $\varepsilon = 0$.

Proof. We expand the function $\varphi_{jh} = \varphi_{jh}(\alpha, \varepsilon)$ (2.30) in powers of ε, showing only first-order terms explicitly.

We have

$$\begin{aligned}\varphi_{jh}(\alpha, \varepsilon) &= \varphi_{jh}(\alpha, 0) + \varepsilon((\mathbf{B}[\mathbf{I}_n - \mathbf{SP}(\alpha_0 - \alpha)]^{-1}\mathbf{y}_j, \mathbf{z}_h)) + \ldots = \\ &= \varphi_{jh}(\alpha, 0) + \varepsilon((\mathbf{B}(t, 0)\mathbf{y}_j, \mathbf{z}_h)) + \varepsilon\psi_{jh}(\alpha, \varepsilon) \qquad (j, h = 1, \ldots, r),\end{aligned}$$

where $\psi_{jh}(\alpha, \varepsilon)$ is a series in powers of ε, $\alpha - \alpha_0$ without free term, convergent for sufficiently small $|\varepsilon|$, $|\alpha - \alpha_0|$. We have

$$\begin{aligned}&\varphi_{jh}(\alpha, 0) = (\alpha_0 - \alpha)((\mathbf{[I}_n - \mathbf{SP}(\alpha_0 - \alpha)]^{-1}\mathbf{y}_j, \mathbf{z}_h)) = \\ &= (\alpha_0 - \alpha)\left\{((\mathbf{y}_j, \mathbf{z}_h)) + \sum_{m=1}^{\infty}(\alpha_0 - \alpha)^m((\mathbf{[SP]}^m\mathbf{y}_j, \mathbf{z}_h))\right\} \qquad (j, h = 1, \ldots, r).\end{aligned}$$

By (2.57), for $m \geqslant 1$,

$$(([\mathbf{SP}]^m \mathbf{y}_j,\ \mathbf{z}_h)) = ((\mathbf{SP}\,[\mathbf{SP}]^{m-1}\mathbf{y}_j,\ \mathbf{z}_h)) = 0 \qquad (j,\ h=1,\ \ldots,\ r),$$

and so, using (2.50), we obtain

$$\varphi_{jh}(\alpha,\ 0) = (\alpha_0 - \alpha)\,\delta_{jh} \qquad (j,\ h=1,\ \ldots,\ r).$$

Therefore, in the notation of (2.58),

$$\varphi_{jh}(\alpha,\ \varepsilon) = -(\alpha-\alpha_0)\,\delta_{jh} + \varepsilon\sigma_{jh} + \varepsilon\psi_{jh}(\alpha,\ \varepsilon) \qquad (j,\ h=1,\ \ldots,\ r). \tag{2.61}$$

It is apparent from this expression that the function $\Phi(\alpha, \varepsilon) = \det \| \varphi_{jh}(\alpha, \varepsilon) \|_1^r$ in (2.31) satisfies the condition

$$\frac{d^r \Phi(\alpha,\ 0)}{d\alpha^r} = (-1)^r r! \neq 0.$$

Thus, by the implicit function theorem /99/, equation (2.31) ($s=r$) defines r values of $\alpha = \alpha(\varepsilon)$ represented by series in fractional powers of ε.

Define a new variable β by

$$\alpha - \alpha_0 = \beta\varepsilon. \tag{2.62}$$

Then

$$\psi_{jh}(\alpha,\ \varepsilon) = \varepsilon\tilde{\psi}_{jh}(\beta,\ \varepsilon),$$

where $\tilde{\psi}_{jh}$ are series in ε and β which converge for sufficiently small $|\varepsilon|, |\beta|$. Dividing by ε^r, we bring equation (2.31) to the form

$$\Psi(\beta,\ \varepsilon) \equiv \det \| \beta\delta_{jh} - \sigma_{jh} - \varepsilon\tilde{\psi}_{jh}(\beta,\ \varepsilon) \|_1^r = 0.$$

We have $\Psi(\beta_\rho,\ 0) = 0$, where β_ρ are the roots of equation (2.60). In the neighborhood of each root β_ρ the function $\beta = \beta(\varepsilon)$ may be expanded in fractional powers of ε:

$$\beta(\varepsilon) = \beta_\rho + O\left(\varepsilon^{\frac{1}{q_\rho}}\right) \qquad (\rho = 1,\ \ldots,\ r).$$

Substituting into (2.62), we obtain (2.59), proving the theorem.

2.7. Computation of characteristic exponents of simple type ($s=r$). Second and higher approximations

We shall continue to use the operators **S** and **P** as defined in §2.6. Let us start from (2.30).

Define an operator $\mathbf{N}$ by

$$\mathbf{N}=\mathbf{SPQ}. \tag{2.63}$$

Then

$$[\mathbf{I}_n-\mathbf{N}]^{-1}=\mathbf{I}_n+\mathbf{N}+\mathbf{N}^2+\ldots=\mathbf{I}_n+\mathbf{N}[\mathbf{I}_n-\mathbf{N}]^{-1}. \tag{2.64}$$

Substituting this expression into (2.30), we get

$$\varphi_{jh}=((\mathbf{Q}\mathbf{y}_j, \mathbf{z}_h))+((\mathbf{QN}[\mathbf{I}_n-\mathbf{N}]^{-1}\mathbf{y}_j, \mathbf{z}_h)) \qquad (j, h=1, \ldots, r). \tag{2.65}$$

It follows from (2.57) that for any $\mathbf{v}(t)\in\mathfrak{L}_2(0, T)$

$$((\mathbf{N}\mathbf{v}, \mathbf{z}_j))=0 \qquad (j=1, \ldots, r).$$

Using this and inserting the value of $\mathbf{Q}$ from (2.30), we transform the second term in (2.65) thus:

$$((\mathbf{QN}[\mathbf{I}_n-\mathbf{N}]^{-1}\mathbf{y}_j, \mathbf{z}_h))=\varepsilon((\mathbf{BN}[\mathbf{I}_n-\mathbf{N}]^{-1}\mathbf{y}_j, \mathbf{z}_h))=\varepsilon((\mathbf{B}[\mathbf{I}_n-\mathbf{N}]^{-1}\mathbf{N}\mathbf{y}_j, \mathbf{z}_h)).$$

It follows from (2.53) that

$$\mathbf{N}\mathbf{y}_j=\varepsilon\mathbf{SPB}\mathbf{y}_j \qquad (j=1, \ldots, r).$$

Therefore,

$$((\mathbf{QN}[\mathbf{I}_n-\mathbf{N}]^{-1}\mathbf{y}_j, \mathbf{z}_h))=\varepsilon^2((\mathbf{B}[\mathbf{I}_n-\mathbf{N}]^{-1}\mathbf{SPB}\mathbf{y}_j, \mathbf{z}_h)).$$

Substituting into (2.65) and using (2.50), we finally obtain

$$\varphi_{jh}(\alpha, \varepsilon)=(\alpha_0-\alpha)\delta_{jh}+\varepsilon((\mathbf{B}\mathbf{y}_j, \mathbf{z}_h))+ \\ +\varepsilon^2((\mathbf{B}\{\mathbf{I}_n-\mathbf{SP}[(\alpha_0-\alpha)\mathbf{I}_n+\varepsilon\mathbf{B}]\}^{-1}\mathbf{SPB}\mathbf{y}_j, \mathbf{z}_h)) \qquad (j, h=1, \ldots, r). \tag{2.66}$$

The theorem of 2.3 now implies

Theorem I. Assume that equation (2.1) with $\varepsilon=0$ (i.e., equation (2.2)) has an r-fold characteristic exponent α_0 of simple type ($\mathfrak{s}=r$). Let $\|\varphi_{jh}\|$ be the $r\times r$ matrix with elements (2.66), where $\mathbf{B}=\mathbf{B}(t, \varepsilon)$ and the operators $\mathbf{S}$, $\mathbf{P}$ are as defined in §2.6. The characteristic exponents $\alpha=\alpha(\varepsilon)$ of equation (2.1) that tend to α_0 at $\varepsilon=0$ are defined for sufficiently small $|\varepsilon|$, $|\alpha-\alpha_0|$ as implicit functions of ε by the equation $\Phi(\alpha, \varepsilon)\equiv\det\|\varphi_{jh}\|_1^r=0$.

Remarks I — IV following the theorem of §2.3 remain valid here too.

Remark. It follows from (2.65) that if the normalization conditions (2.50) are not assumed, Theorem I remains valid provided the first term in (2.66) is replaced by

$$(\alpha_0-\alpha)((\mathbf{y}_j, \mathbf{z}_h))=(\alpha_0-\alpha)(\mathbf{c}_j, \mathbf{d}_h)\ \delta_{m_j m_h}.$$

The matrices $\|(\mathbf{c}_j, \mathbf{d}_h)\|$ and $\|((\mathbf{y}_j, \mathbf{z}_h))\|$ are then nonsingular.

Formula (2.66) at once implies an expression for φ_{jh}, accurate up to and including third-order terms. Indeed, by the theorem of §2.6, $\alpha-\alpha_0=\beta\varepsilon+O(\varepsilon^{1+\delta})$, where $\delta>0$ and β is a root of equation (2.60). Thus, by (2.66),

$$\varphi_{jh}(\alpha,\varepsilon)=(\alpha_0-\alpha)\,\delta_{jh}+\varepsilon((\mathbf{By}_j,\mathbf{z}_h))+\varepsilon^2((\mathbf{BSPBy}_j,\mathbf{z}_h))+ \\ +\varepsilon^3((\mathbf{BSP}[-\beta\mathbf{I}_n+\mathbf{B}]\,\mathbf{SPBy}_j,\mathbf{z}_h))+O(\varepsilon^{3+\delta}). \quad (2.67)$$

Here $\delta>0$, and if β is a simple root (or a multiple root with simple elementary divisors), then $\delta=1$.

Let us consider in greater detail the case of a simple characteristic exponent ($s=r=1$). By the theorem of §2.3, $\alpha(\varepsilon)$ may be expanded in powers of ε and by the theorem of §2.6 the equation for $\alpha=\alpha(\varepsilon)$ is $\varphi_{11}=0$, where φ_{11} is defined by (2.66). From (2.67) we at once deduce

$$\alpha(\varepsilon)=\alpha_0+\varepsilon((\mathbf{By}_1,\mathbf{z}_1))+\varepsilon^2((\mathbf{BSPBy}_1,\mathbf{z}_1))+ \\ +\varepsilon^3((\mathbf{BSP}[-\beta_1\mathbf{I}_n+\mathbf{B}]\,\mathbf{SPBy}_1,\mathbf{z}_1))+O(\varepsilon^4), \quad (2.68)$$

where $\beta_1=((\mathbf{B}(t,0)\mathbf{y}_1,\mathbf{z}_1))=(\mathbf{B}_1^{(m)}\mathbf{c}_1,\mathbf{d}_1)$.

It is also easy to determine the higher-order terms. The procedure is as follows. The value of $\alpha(\varepsilon)$, accurate up to the required number of terms, is substituted into (2.66) and the expression in curly brackets is then expanded in powers of ε, retaining terms up to the desired order. Since the third term in (2.66) has a factor ε^2, the equation $\varphi_{11}=0$ will yield $\alpha(\varepsilon)$ up to two orders higher. For example, by (2.66),

$$\alpha=\alpha_0+\varepsilon((\mathbf{By}_1,\mathbf{z}_1))+O(\varepsilon^2)=\alpha_0+\varepsilon\beta_1+O(\varepsilon^2).$$

Since the relevant term in (2.66) is $\{\ \}^{-1}=\mathbf{I}_n+O(\varepsilon)$, we substitute this into (2.66), to obtain (2.68). Again setting

$$\mathbf{N}=\mathbf{SP}[(\alpha_0-\alpha)\mathbf{I}_n+\varepsilon\mathbf{B}],$$

we have $\mathbf{N}=O(\varepsilon)$,

$$\{\mathbf{I}_n-\mathbf{N}\}^{-1}=\mathbf{I}_n+\mathbf{N}+\mathbf{N}^2+\mathbf{N}^3+O(\varepsilon^4),$$

$$\mathbf{N}=\mathbf{SP}\{-[\varepsilon((\mathbf{By}_1,\mathbf{z}_1))+\varepsilon^2((\mathbf{BSPBy}_1,\mathbf{z}_1))+\varepsilon^3((\mathbf{BSP}[-\beta_1\mathbf{I}_n+ \\ +\mathbf{B}]\,\mathbf{SPBy}_1,\mathbf{z}_1))]\,\mathbf{I}_n+\varepsilon\mathbf{B}\}+O(\varepsilon^4),$$

$$\mathbf{N}^2=\{\mathbf{SP}[-\varepsilon((\mathbf{By}_1,\mathbf{z}_1))\,\mathbf{I}_n-\varepsilon^2((\mathbf{BSPBy}_1,\mathbf{z}_1))\,\mathbf{I}_n+\varepsilon\mathbf{B}\}^2+O(\varepsilon^4),$$

$$\mathbf{N}^3=\varepsilon^3\{\mathbf{SP}[-\beta_1\mathbf{I}_n+\mathbf{B}(t,0)]\}^3+O(\varepsilon^4).$$

Substituting these expressions into (2.66) and using the equation $\varphi_{11}=0$, we get

$$\alpha(\varepsilon)=\alpha_0+\varepsilon((\mathbf{By}_1,\mathbf{z}_1))+\varepsilon^2((\mathbf{BSPBy}_1,\mathbf{z}_1))+ \\ +\varepsilon^3((\mathbf{BSP}[-((\mathbf{By}_1,\mathbf{z}_1))\,\mathbf{I}_n+\mathbf{B}]\,\mathbf{SPBy}_1,\mathbf{z}_1))+ \\ +\varepsilon^4[((\mathbf{BSPBy}_1,\mathbf{z}_1))\,((\mathbf{B}(\mathbf{SP})^2\mathbf{By}_1,\mathbf{z}_1))+ \\ +((\mathbf{B}\{\mathbf{SP}[-((\mathbf{By}_1,\mathbf{z}_1))\,\mathbf{I}_n+\mathbf{B}]\}^2\,\mathbf{SPBy}_1,\mathbf{z}_1))]+O(\varepsilon^5), \quad (2.69)$$

where, as before,

$$\mathbf{B} = \mathbf{B}(t, \varepsilon) = \mathbf{B}_1(t) + \varepsilon \mathbf{B}_2(t) + \varepsilon^2 \mathbf{B}_3(t) + \varepsilon^3 \mathbf{B}_4(t) + \ldots$$

This procedure may be continued at will, defining $\alpha(\varepsilon)$ to any desired approximation. The result may be summarized as follows.

Theorem II. Let α_0 be a simple characteristic exponent of equation (2.1) with $\varepsilon = 0$ (i.e., $s = r = 1$). Then the characteristic exponent $\alpha(\varepsilon)$ of equation (2.1) that tends to α_0 at $\varepsilon = 0$ is an analytic function of ε at $\varepsilon = 0$ and may be expanded as in (2.69), where **S** *and* **P** *are the operators defined in §2.6.*

In the case $s = r > 1$ the arguments become much more complicated, but even then the characteristic exponents are determined to within any degree of approximation from the equation

$$\Phi(\alpha, \varepsilon) = \det \| \varphi_{jh}(\alpha, \varepsilon) \|_1^r = 0.$$

In this case it is convenient to replace the equation $\Phi(\alpha, \varepsilon) = 0$ by

$$\Psi(\beta, \varepsilon) = 0, \tag{2.70}$$

where

$$\Psi(\beta, \varepsilon) = \det \| \psi_{jh}(\beta, \varepsilon) \|_1^r \tag{2.71}$$

and

$$\psi_{jh}(\beta, \varepsilon) = \frac{\varphi_{jh}(\alpha, \varepsilon)}{\varepsilon}, \quad \beta = \frac{\alpha - \alpha_0}{\varepsilon}. \tag{2.72}$$

We shall show that $\psi_{jh}(\beta, \varepsilon)$ are analytic functions of β and ε at any point $\varepsilon = 0$, $\beta = \beta_0$, where β_0 is arbitrary. The expression (2.66) for $\varphi_{jh}(\alpha, \varepsilon)$ yields

$$\begin{aligned} \psi_{jh}(\beta, \varepsilon) = &-\beta\delta_{jh} + ((\mathbf{B}\mathbf{y}_j, \mathbf{z}_h)) + \\ &+ \varepsilon((\mathbf{B}\{\mathbf{I}_n - \varepsilon\mathbf{S}\mathbf{P}(-\beta\mathbf{I}_n + \varepsilon\mathbf{B})\}^{-1}\mathbf{S}\mathbf{P}\mathbf{B}\mathbf{y}_j, \mathbf{z}_h)) \\ &(j, h = 1, \ldots, r). \end{aligned} \tag{2.73}$$

Here $\mathbf{B} = \mathbf{B}(t, \varepsilon)$ is analytic in ε at $\varepsilon = 0$; the functions $\mathbf{y}_j = \mathbf{y}_j(t)$, $\mathbf{z}_h = \mathbf{z}_h(t)$ are as in (2.8), (2.10); it is evident from (2.73) that $\psi_{jh}(\beta, \varepsilon)$ are indeed analytic functions of β and ε at any point $\beta = \beta_0$, $\varepsilon = 0$.

Let us find the first coefficients in the expansions of $\psi_{jh}(\beta, \varepsilon)$ in powers of β and ε. Since

$$\varepsilon\mathbf{B}(t, \varepsilon) = \varepsilon\mathbf{B}_1(t) + \varepsilon^2\mathbf{B}_2(t) + \ldots, \tag{2.74}$$

we deduce from (2.73) that

$$\psi_{jh}(\beta, \varepsilon) = -\beta\delta_{jh} + \sigma_{jh} + \varepsilon[\tau_{jh} + \ldots], \tag{2.75}$$

where the dots stand for terms of first order in β and ε, the coefficients σ_{jh} are determined by (2.58) and

$$\left.\begin{aligned} &\tau_{jh}=\tau'_{jh}+\tau''_{jh},\\ &\tau'_{jh}=((\mathbf{B}_2\mathbf{y}_j,\ \mathbf{z}_h))=(\mathbf{B}_2^{(m_h-m_j)}\mathbf{c}_j,\ \mathbf{d}_h),\\ &\tau''_{jh}=((\mathbf{B}_1\mathbf{S}\mathbf{P}\mathbf{B}_1\mathbf{y}_j,\ \mathbf{z}_h))\\ &\qquad (j,\ h=1,\ 2,\ \ldots,\ r). \end{aligned}\right\} \tag{2.76}$$

It follows from (2.73) that the expression in square brackets in (2.75) is an analytic function of ε and β at $\varepsilon=0$.

Consider equation (2.70). When $\varepsilon=0$ it is the same as (2.60). Let β_0 be a root of equation (2.60). We have reduced the determination of the characteristic exponents $\alpha(\varepsilon)$ to determination of the roots $\beta(\varepsilon)$ of equation (2.70) for small ε which tend to β_0 at $\varepsilon=0$. This problem was considered in §1. Indeed, it follows from the formulas (2.75) for the functions $\psi_{jh}(\varepsilon)$ that equation (2.70) coincides with (1.2), where the matrix

$$\mathbf{A}(\beta,\ \varepsilon)=\|\psi_{jh}(\beta,\ \varepsilon)\|$$

has the properties described in §1.1. The matrix $\mathbf{K}$ in (1.1) is $\mathbf{A}(0,\ 0)$. Thus we infer from (2.75) that in our case

$$\mathbf{K}=\|\sigma_{jh}\|_1^2. \tag{2.77}$$

Comparing (2.73) and (1.1), we have

$$\mathbf{L}(\beta,\ \varepsilon)=\|\lambda_{jh}(\beta,\ \varepsilon)\|, \tag{2.78}$$

where

$$\lambda_{jh}(\beta,\ \varepsilon)=((\mathbf{B}\{\mathbf{I}_n-\varepsilon\mathbf{S}\mathbf{P}(-\beta\mathbf{I}_n+\varepsilon\mathbf{B})\}^{-1}\mathbf{S}\mathbf{P}\mathbf{B}\mathbf{y}_j,\ \mathbf{z}_h))+\left(\left(\frac{\mathbf{B}-\mathbf{B}_1}{\varepsilon}\mathbf{y}_j,\ \mathbf{z}_h\right)\right).$$

It follows that in our case the matrix $\mathbf{L}(\beta, 0)=\mathbf{L}_0$ is independent of β:

$$\mathbf{L}_0=\|\tau_{jh}\|. \tag{2.79}$$

Applying the lemmas of §§1.2 and 1.3 and the corollary at the end of §1.4 to equation (2.70) and recalling the relationship (2.72) of $\alpha=\alpha(\varepsilon)$ and $\beta=\beta(\varepsilon)$, we obtain:

Theorem III. Let equation (2.2) have an r-fold characteristic exponent α_0 of simple type.

Let $\lambda_1,\ \ldots,\ \lambda_r$ be the class of eigenvalues of the matrix $\mathbf{C}$ which are congruent to α_0 modulo $2\pi i/T$, define integers m_j by (2.4), and let $\mathbf{c}_j,\ \ldots,\ \mathbf{c}_r,\ \mathbf{d}_1,\ \ldots,\ \mathbf{d}_r$ be corresponding eigenvectors of $\mathbf{C}$ and $\mathbf{C}^*$:

$$\mathbf{C}\mathbf{c}_j=\lambda_j\mathbf{c}_j,\quad \mathbf{C}^*\mathbf{d}_j=\bar{\lambda}_j\mathbf{d}_j\quad (j=1,\ \ldots,\ r),$$

normalized by the conditions

$$(\mathbf{c}_j,\ \mathbf{d}_h)=\delta_{jh}\quad (j,\ h=1,\ \ldots,\ r).$$

Define r^2 *numbers* σ_{jh} $(j, h=1, \ldots, r)$ *by formulas (2.58), (2.58a), and* r^2 *numbers* τ_{jh} $(j, h=1, \ldots, r)$ *by formulas (2.76). In these formulas,* $\mathbf{S}$ *and* $\mathbf{P}$ *are the operators defined in §2.6,* $\mathbf{B}_1$, $\mathbf{B}_2$ *are the coefficients in (2.74), and* $\mathbf{B}_2^{(m)}$ *the Fourier coefficients of* $\mathbf{B}_2(t)$:

$$\mathbf{B}_2(t) \sim \sum_m \mathbf{B}_2^{(m)} \exp\left(\frac{2\pi i t m}{T}\right).$$

Let β_0 *be some root of equation (2.60), of multiplicity* r_0, *to which (theorem of §2.6) correspond* r_0 *characteristic exponents* $\alpha_j(\varepsilon)$, *and let* s_0 *be the dimension of the eigensubspace of* $\mathbf{K}$ *belonging to the eigenvalue* β_0.

a) *Let* β_0 *be a simple root* $(r_0 = s_0 = 1)$. *Define corresponding (r-dimensional) eigenvectors* $\mathbf{a}$, $\mathbf{b}$ *of the matrices* $\mathbf{K}$ *and* $\mathbf{K}^*$:

$$\mathbf{K}\mathbf{a} = \beta_0 \mathbf{a}, \quad \mathbf{K}^*\mathbf{b} = \bar{\beta}_0 \mathbf{b},$$

normalized so that

$$(\mathbf{a}, \mathbf{b}) = 1.$$

Set $\gamma_0 = (\mathbf{L}_0\mathbf{a}, \mathbf{b})$. *Then the characteristic exponent* $\alpha(\varepsilon)$ *that corresponds (theorem of §2.6) to* β_0 *admits the expansion**

$$\alpha(\varepsilon) = \alpha_0 + \beta_0\varepsilon + \gamma_0\varepsilon^2 + O(\varepsilon^3).$$

b) *Let* $r_0 > 1$, *but suppose that all elementary divisors of* $\mathbf{K}$ *belonging to the eigenvalue* β_0 *are simple* $(r_0 = s_0 > 1)$. *Define corresponding (r-dimensional) eigenvectors* $\mathbf{a}_j$, $\mathbf{b}_j$ *of the matrices* $\mathbf{K}$ *and* $\mathbf{K}^*$:

$$\mathbf{K}\mathbf{a}_j = \beta_0 \mathbf{a}_j, \quad \mathbf{K}^*\mathbf{b}_j = \bar{\beta}_0 \mathbf{b}_j \quad (j = 1, \ldots, s_0),$$

normalized so that

$$(\mathbf{a}_j, \mathbf{b}_h) = \delta_{jh} \quad (j, h = 1, \ldots, s_0).$$

Let γ_k *be a root of the equation*

$$\det \| (\mathbf{L}_0\mathbf{a}_j, \mathbf{b}_h) - \gamma\delta_{jh} \|_{j,h=1}^{s_0} = 0$$

of multiplicity q_k. *For each pair of roots* (β_0, γ_k) *we have* q_k *characteristic exponents*

$$\alpha_x(\varepsilon) = \alpha_0 + \beta_0\varepsilon + \gamma_k\varepsilon^2 + O(\varepsilon^{2+1/q_k}), \tag{2.80}$$

which may be expanded in (generally fractional) powers of ε. **

c) *Let* $r_0 > 1$ *and suppose that to the eigenvalue* β_0 *there belongs one elementary divisor (elementary Jordan matrix) of the matrix* $\mathbf{K}$ $(r_0 > 1, s_0 = 1)$. *Construct a cyclic basis*

$$\mathbf{K}\mathbf{a}_1 = \beta_0\mathbf{a}_1, \quad \mathbf{K}\mathbf{a}_2 = \beta_0\mathbf{a}_2 + \mathbf{a}_1, \quad \ldots, \quad \mathbf{K}\mathbf{a}_{r_0} = \beta_0\mathbf{a}_{r_0} + \mathbf{a}_{r_0-1}.$$

* Recall that in this case $\alpha(\varepsilon)$ is an analytic function of ε at $\varepsilon = 0$ (theorem of §2.6).

** If γ_k is a simple root $(q_k = 1)$, the (unique) corresponding characteristic exponent (2.80) is analytic in ε at $\varepsilon = 0$ (lemma of § 1.3).

Let $\mathbf{b}_1, \ldots, \mathbf{b}_{r_0}$ *be vectors in the root subspace of* $\mathbf{K}^*$ *belonging to* $\overline{\beta}_0$, *such that*

$$(\mathbf{a}_j, \mathbf{b}_{ih}) = \delta_{jh} \quad (j, h = 1, \ldots, r_0).$$

Set $\gamma_0 = (\mathbf{L}_0\mathbf{a}_1, \mathbf{b}_{r_0})$.

Then the corresponding r_0 *characteristic exponents* $\alpha_j(\varepsilon)$ *admit the expansions*

$$\alpha_j(\varepsilon) = \alpha_0 + \beta_0\varepsilon + (\gamma_0\varepsilon)_j^{1/r_0}\varepsilon + O(\varepsilon^{2+1/r_0}),$$

where $(\gamma_0\varepsilon)_j^{1/r_0}$ *are the different values of the root* $\sqrt[r_0]{\gamma_0\varepsilon}$.

Using the theorem of §1.4, one can establish an analogous result for the case $r_0 > s_0 > 1$, when the eigenvalue β_0 corresponds to several elementary Jordan matrices in the canonical form of $\mathbf{K}$; however, the formulas are much more cumbersome.

We end this subsection with a remark. The theorems of this section reduce computation of the characteristic exponents to the problems considered in §1. This is done in such a way that if r is the multiplicity of the characteristic exponent α_0 of the "unperturbed" system and $r > 1$, one obtains an algebraic equation of degree r for the "first" r corrections. When α_0 is a characteristic exponent of simple type, the relevant equation is (2.60). If some of the roots of this equation are distinct, so that "splitting" occurs, then, performing the appropriate substitution and again applying the lemmas of §1, we can derive a similar equation of degree $r_0 \leqslant r$ for the next correction to each of the "split-off" roots of multiplicity β_0. The same may again be done for each of the new equations, and so on.

In practice, it is sometimes simpler not to use the splitting procedure just described, but to compute φ_{jh} from formulas (2.30) or (2.66), limiting the computation to the desired degree of approximation (e.g., using formula (2.67), in which we have a term $O(\varepsilon^{3+\delta})$), and then determining the characteristic exponents from the equation $\det \| \varphi_{jh} \| = 0$.

2.8. Example

Consider the following equation (see /62b/, p. 201):

$$\frac{d^3\xi}{dt^3} + \frac{d\xi}{dt} = \varepsilon\left[-F(M\cos t + N\sin t)\xi - b\frac{d\xi}{dt} + \frac{d^2\xi}{dt^2}\right].$$

Here $F(\varphi)$ is an even function, $\varepsilon > 0$ a "small" parameter, $b > 0$. We express this equation in the form (0.1):

$$\frac{d\mathbf{x}}{dt} = [\mathbf{C} + \varepsilon\mathbf{B}_1(t)]\,\mathbf{x},$$

where the vector $\mathbf{x}$ and matrices $\mathbf{C}$, $\mathbf{B}_1(t)$ are

$$\mathbf{x}=\begin{pmatrix}\xi_1\\ \dot{\xi}_1\\ \ddot{\xi}_1\end{pmatrix},\quad \mathbf{C}=\begin{Vmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{Vmatrix},\quad \mathbf{B}_1(t)=\begin{Vmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ -F & -b & -1\end{Vmatrix},\ (\mathbf{B}_1(t+2\pi)=\mathbf{B}_1(t)).$$

Since the eigenvalues of $\mathbf{C}$ are distinct ($\lambda_1=0$, $\lambda_2=i$, $\lambda_3=-i$), the three-fold characteristic exponent $\alpha_0=0$ is of simple type. The eigenvectors $\mathbf{c}_j$ and $\mathbf{d}_j$ of $\mathbf{C}$ and $\mathbf{C}^*$ are

$$\mathbf{c}_1=\begin{pmatrix}\eta_1\\ 0\\ 0\end{pmatrix},\quad \mathbf{c}_2=\bar{\mathbf{c}}_3=\begin{pmatrix}-\zeta_3\\ -\zeta_3 i\\ \zeta_3\end{pmatrix},\quad \mathbf{d}_1=\begin{pmatrix}\chi_1\\ 0\\ \chi_1\end{pmatrix},\quad \mathbf{d}_2=\bar{\mathbf{d}}_3=\begin{pmatrix}0\\ -\psi_3 i\\ \psi_3\end{pmatrix}.$$

Normalizing these vectors so that $(\mathbf{d}_j, \mathbf{d}_j)=1$ and $(\mathbf{c}_j, \mathbf{d}_h)=\delta_{jh}$ $(j, h=1, 2, 3)$, we have

$$\mathbf{c}_1=\sqrt{2}\begin{pmatrix}1\\ 0\\ 0\end{pmatrix},\quad \mathbf{c}_2=\bar{\mathbf{c}}_3=\frac{\sqrt{2}}{2}\begin{pmatrix}-1\\ -i\\ 1\end{pmatrix};\quad \mathbf{d}_1=\frac{\sqrt{2}}{2}\begin{pmatrix}1\\ 0\\ 1\end{pmatrix},\quad \mathbf{d}_2=\bar{\mathbf{d}}_3=\frac{\sqrt{2}}{2}\begin{pmatrix}0\\ -i\\ 1\end{pmatrix}.$$

We now determine the vectors $\mathbf{B}_1^{(m)}\mathbf{c}_j$ $(m=m_h-m_j;\ j, h=1, 2, 3)$. Recalling that

$$\lambda_1=0+0\cdot i,\quad \lambda_2=0+1\cdot i,\quad \lambda_3=0-1\cdot i,$$

so that $m_1=0$, $m_2=1$, $m_3=-1$, we have

$$\mathbf{B}_1^{(0)}\mathbf{c}_1=\sqrt{2}\begin{pmatrix}0\\ 0\\ -F^{(0)}\end{pmatrix},\quad \mathbf{B}_1^{(1)}\mathbf{c}_1=\mathbf{B}_1^{(-1)}\mathbf{c}_1=\mathbf{B}_1^{(-1)}\mathbf{c}_2=\mathbf{B}_1^{(1)}\mathbf{c}_3=0,$$

$$\mathbf{B}_1^{(0)}\mathbf{c}_2=\frac{\sqrt{2}}{2}\begin{pmatrix}0\\ 0\\ F^{(0)}+bi-1\end{pmatrix},\quad \mathbf{B}_1^{(-2)}\mathbf{c}_2=\frac{\sqrt{2}}{2}\begin{pmatrix}0\\ 0\\ F^{(-2)}\end{pmatrix},$$

$$\mathbf{B}_1^{(2)}\mathbf{c}_3=\frac{\sqrt{2}}{2}\begin{pmatrix}0\\ 0\\ F^{(2)}\end{pmatrix},\quad \mathbf{B}_1^{(0)}\mathbf{c}_3=\frac{\sqrt{2}}{2}\begin{pmatrix}0\\ 0\\ F^{(0)}-bi-1\end{pmatrix}.$$

Here $F^{(m)}$ are the complex Fourier coefficients of the function $F(\varphi)$ $(\varphi=M\cos t+N\sin t)$:

$$F(\varphi)\sim\sum_{m=-\infty}^{\infty}e^{imt}F^{(m)}.$$

Since $F(\varphi)$ is even, the coefficients $F^{(1)}$ and $F^{(-1)}$ vanish. The matrix (2.58) is

$$\|\sigma_{jh}\|_1^3=\left\|(\mathbf{B}_1^{(m_h-m_j)}\mathbf{c}_j, \mathbf{d}_h)\right\|_1^3=\begin{Vmatrix}(\mathbf{B}_1^{(0)}\mathbf{c}_1, \mathbf{d}_1) & (\mathbf{B}_1^{(1)}\mathbf{c}_1, \mathbf{d}_2) & (\mathbf{B}_1^{(-1)}\mathbf{c}_1, \mathbf{d}_3)\\ (\mathbf{B}_1^{(-1)}\mathbf{c}_2, \mathbf{d}_1) & (\mathbf{B}_1^{(0)}\mathbf{c}_2, \mathbf{d}_2) & (\mathbf{B}_1^{(-2)}\mathbf{c}_2, \mathbf{d}_3)\\ (\mathbf{B}_1^{(1)}\mathbf{c}_3, \mathbf{d}_1) & (\mathbf{B}_1^{(2)}\mathbf{c}_3, \mathbf{d}_2) & (\mathbf{B}_1^{(0)}\mathbf{c}_3, \mathbf{d}_3)\end{Vmatrix}=$$

$$=\begin{Vmatrix}-F^{(0)} & 0 & 0\\ 0 & \frac{1}{2}[F^{(0)}+bi-1] & \frac{1}{2}F^{(-2)}\\ 0 & \frac{1}{2}F^{(2)} & \frac{1}{2}[F^{(0)}-bi-1]\end{Vmatrix}.$$

This matrix is clearly hermitian. Equation (2.60) becomes

$$\det \| \sigma_{jh} - \beta \delta_{jh} \|_1^3 = \\ = -(\beta + F^{(0)}) \left[\beta^2 - (F^{(0)} - 1)\beta + \frac{1}{4}(F^{(0)} - 1)^2 + \frac{1}{4} b^2 - \frac{1}{4} F^{(2)} F^{(-2)} \right] = 0.$$

Let β_1, β_2 be the roots of the bracketed polynomial and $\beta_3 = -F^{(0)}$. If the roots β_1, β_2, β_3 are distinct, it follows from the theorem of §2.6 that the characteristic exponents $\alpha_j(\varepsilon)$ are $\alpha_j(\varepsilon) = \beta_j \varepsilon + O(\varepsilon^2)$, $j = 1, 2, 3$.

We now evaluate the characteristic exponents in the second approximation. Here $\mathbf{B}_2(t) \equiv \mathbf{0}$. Denote

$$\chi_{jh} = ((\mathbf{B}_1(t)\,\mathbf{SPB}_1(t)\,\mathbf{y}_j,\ \mathbf{z}_h)) = \\ = \frac{1}{2\pi} \int_0^{2\pi} e^{-im_h t} (\mathbf{B}_1 \mathbf{SPB}_1 e^{im_j t}\,\mathbf{c}_j,\ \mathbf{d}_h)\, dt \qquad (j,\ h = 1,\ 2,\ 3).$$

Recalling that

$$\mathbf{B}_1^{(k)} = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -F^{(k)} & -b\delta_{k0} & -\delta_{k0} \end{Vmatrix},$$

we calculate

$$\mathbf{B}_1(t)\, e^{im_j t}\,\mathbf{c}_j = \sum_{k=-\infty}^{\infty} \mathbf{B}_1^{(k)} e^{i(k+m_j)t}\,\mathbf{c}_j = \sum_k e^{ikt} \mathbf{B}_1^{(k-m_j)} \mathbf{c}_j.$$

This gives

$$\mathbf{B}_1(t)\, e^{im_1 t}\,\mathbf{c}_1 = -\sqrt{2} \sum_k e^{ikt} F^{(k)} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\mathbf{B}_1(t)\, e^{im_2 t}\,\mathbf{c}_2 = \frac{\sqrt{2}}{2} \sum_k e^{ikt} [F^{(k-1)} + b\delta_{k-1,\,0} i - \delta_{k-1,\,0}] \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\mathbf{B}_1(t)\, e^{im_3 t}\,\mathbf{c}_3 = \frac{\sqrt{2}}{2} \sum_k e^{ikt} [F^{(k+1)} - b\delta_{k+1,\,0} i - \delta_{k+1,\,0}] \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

By (2.52), we have

$$\mathbf{PB}_1(t)\, e^{im_1 t}\,\mathbf{c}_1 = \mathbf{PB}_1(t)\,\mathbf{c}_1 = \\ = \mathbf{B}_1(t)\,\mathbf{c}_1 - ((\mathbf{B}_1(t)\,\mathbf{c}_1,\ \mathbf{z}_1))\,\mathbf{y}_1 - ((\mathbf{B}_1(t)\,\mathbf{c}_1,\ \mathbf{z}_2))\,\mathbf{y}_2 - ((\mathbf{B}_1(t)\,\mathbf{c}_1,\ \mathbf{z}_3))\,\mathbf{y}_3$$

and, in view of (2.8) and (2.10),

$$\mathbf{PB}_1(t)\,e^{im_1 t}\mathbf{c}_1=\mathbf{B}_1(t)\,\mathbf{c}_1-(\mathbf{B}_1^{(0)}\mathbf{c}_1,\ \mathbf{d}_1)-(\mathbf{B}_1^{(1)}\mathbf{c}_1,\ \mathbf{d}_2)\,e^{it}\mathbf{c}_2-(\mathbf{B}_1^{(-1)}\mathbf{c}_1,\ \mathbf{d}_3)\,e^{-it}\mathbf{c}_3.$$

But $\mathbf{B}_1^{(1)}=\mathbf{B}_1^{(-1)}=0$, and so

$$\mathbf{PB}_1(t)\,e^{im_1 t}\mathbf{c}_1=-\sqrt{2}\sum_{k\neq 0,\,\pm 1}{}'\,e^{ikt}F^{(k)}\begin{pmatrix}0\\0\\1\end{pmatrix}+\sqrt{2}\,F^{(0)}\begin{pmatrix}1\\0\\-1\end{pmatrix}.$$

Similarly, we find that

$$\mathbf{PB}_1(t)\,e^{im_2 t}\mathbf{c}_2=\frac{\sqrt{2}}{2}\sum_{k\neq 0,\,\pm 1}{}'\,F^{(k-1)}e^{ikt}\begin{pmatrix}0\\0\\1\end{pmatrix}+$$
$$+\frac{\sqrt{2}}{4}F^{(-2)}e^{-it}\begin{pmatrix}1\\-i\\1\end{pmatrix}+\frac{\sqrt{2}}{4}(F^{(0)}+bi-1)\,e^{it}\begin{pmatrix}1\\i\\1\end{pmatrix},$$
$$\mathbf{PB}_1(t)\,e^{im_3 t}\mathbf{c}_3=\frac{\sqrt{2}}{2}\sum_{k\neq 0,\,\pm 1}F^{(k+1)}\,e^{ikt}\begin{pmatrix}0\\0\\1\end{pmatrix}+$$
$$+\frac{\sqrt{2}}{4}(F^{(0)}-bi-1)\,e^{-it}\begin{pmatrix}1\\-i\\1\end{pmatrix}+\frac{\sqrt{2}}{4}F^{(2)}e^{it}\begin{pmatrix}1\\i\\1\end{pmatrix}.$$

We have set apart the terms with $k=0,\ \pm 1$ in order to facilitate construction of the resolvent operator $\mathbf{S}$. Consider the inhomogeneous equation

$$\frac{d\mathbf{x}}{dt}-\mathbf{Cx}=\mathbf{Pu}(t)\qquad\left(\mathbf{Pu}\sim\sum_{k\neq 0,\,\pm 1}e^{ikt}\mathbf{u}_k\right).$$

We shall look for its solution as

$$\mathbf{x}\sim\sum_{k\neq 0,\,\pm 1}e^{ikt}\mathbf{x}_k.$$

Substituting this series and comparing the coefficients of like harmonics, we obtain

$$(ik\mathbf{I}_3-\mathbf{C})\,\mathbf{x}_k=\mathbf{u}_k,$$

whence

$$\mathbf{x}_k=(ik\mathbf{I}_3-\mathbf{C})^{-1}\mathbf{u}_k=\frac{i}{k(k^2-1)}\begin{Vmatrix}1-k^2 & ik & 1\\ 0 & -k^2 & ik\\ 0 & -ik & -k^2\end{Vmatrix}\qquad(k\neq 0,\ \pm 1).$$

Solution of the inhomogeneous equations $\dot{\mathbf{x}}-\mathbf{Cx}=\mathbf{Pv}(t)$, whose right-hand side contains only free terms and first harmonics, involves arbitrary constants, which are determined by the uniqueness of the required periodic solution:

$$((\mathbf{SPB}_1(t)\,e^{im_jt}\mathbf{c}_j,\ \mathbf{z}_h))=0 \qquad (h=1,\ 2,\ 3).$$

We thus obtain

$$\mathbf{SPB}_1(t)\,e^{im_1t}\mathbf{c}_1=-i\sqrt{2}\sum_{k\neq 0,\,\pm 1}{}'\frac{F^{(k)}e^{ikt}}{k(k^2-1)}\begin{pmatrix}1\\ik\\-k^2\end{pmatrix}-\sqrt{2}F^{(0)}\begin{pmatrix}0\\1\\0\end{pmatrix},$$

$$\mathbf{SPB}_1(t)\,e^{im_2t}\mathbf{c}_2=i\frac{\sqrt{2}}{2}\sum_{k\neq 0,\,\pm 1}{}'\frac{F^{(k-1)}e^{ikt}}{k(k^2-1)}\begin{pmatrix}1\\ik\\-k^2\end{pmatrix}+$$
$$+\frac{\sqrt{2}}{8}F^{(-2)}e^{-it}\begin{pmatrix}3i\\1\\i\end{pmatrix}+\frac{\sqrt{2}}{8}(F^{0}+bi-1)e^{it}\begin{pmatrix}-3i\\1\\-i\end{pmatrix},$$

$$\mathbf{SPB}_1(t)\,e^{im_3t}\mathbf{c}_3=i\frac{\sqrt{2}}{2}\sum_{k\neq 0,\,\pm 1}{}'\frac{F^{(k+1)}e^{ikt}}{k(k^2-1)}\begin{pmatrix}1\\ik\\-k^2\end{pmatrix}+$$
$$+\frac{\sqrt{2}}{8}(F^{(0)}-bi-1)\,e^{-it}\begin{pmatrix}3i\\1\\i\end{pmatrix}+\frac{\sqrt{2}}{8}F^{(2)}e^{it}\begin{pmatrix}-3i\\1\\-i\end{pmatrix}.$$

Turning to the expressions for the quantities χ_{jh}, we see that the only term of interest in the vector $\mathbf{B}_1(t)\,\mathbf{v}(t)$, where

$$\mathbf{v}(t)=\mathbf{SPB}_1(t)\,e^{im_jt}\mathbf{c}_j,$$

is the m_h-th harmonic. Express the vector $\mathbf{B}_1(t)\,\mathbf{v}(t)$ as

$$\mathbf{B}_1(t)\,\mathbf{v}(t)=\sum_{\varkappa}e^{i\varkappa t}\mathbf{B}_\varkappa\sum_{k}e^{ikt}\mathbf{v}_k=\sum_{\varkappa,\,k}e^{i(\varkappa+k)t}\mathbf{B}_\varkappa\mathbf{v}_k.$$

As stated, we are interested only in terms with $\varkappa+k=m_h$, or $\varkappa=m_h-k$. The calculation gives

$$\chi_{11}=i\sum_{k\neq 0,\,\pm 1}{}'\frac{F^{(-k)}F^{(k)}}{k(k^2-1)}+bF^{(0)},\quad \chi_{12}=i\sum{}'\frac{F^{(1-k)}F^{(k)}}{k(k^2-1)},$$
$$\chi_{13}=i\sum{}'\frac{F^{(-1-k)}F^{(k)}}{k(k^2-1)},\qquad \chi_{21}=-\frac{1}{2}i\sum{}'\frac{F^{(-k)}F^{(k-1)}}{k(k^2-1)},$$
$$\chi_{22}=-\frac{1}{2}i\sum{}'\frac{F^{(1-k)}F^{(k-1)}}{k(k^2-1)}+\frac{1}{8}i(F^{(0)}+bi-1)(3F^{(0)}+bi+1),$$
$$\chi_{23}=-\frac{1}{2}i\sum{}'\frac{F^{(-1-k)}F^{(k-1)}}{k(k^2-1)}-\frac{1}{8}iF^{(-2)}(3F^{(0)}-bi+1),$$
$$\chi_{31}=-\frac{1}{2}i\sum{}'\frac{F^{(-k)}F^{(k+1)}}{k(k^2-1)},\quad \chi_{32}=-\frac{1}{2}i\sum{}'\frac{F^{(1-k)}F^{(k+1)}}{k(k^2-1)}+$$
$$+\frac{1}{8}iF^{(2)}(3F^{(0)}+bi+1),$$
$$\chi_{33}=-\frac{1}{2}i\sum{}'\frac{F^{(-1-k)}F^{(k+1)}}{k(k^2-1)}-\frac{1}{8}i(F^{(0)}-bi-1)(3F^{(0)}-bi-1).$$

In the second approximation, equation (2.31) for the characteristic exponents $\alpha(\varepsilon)$ ($\alpha(0)=0$) of our example is

$$\det\|\varepsilon\sigma_{jh}+\varepsilon^2\chi_{jh}-\alpha\delta_{jh}\|_1^3=0.$$

§3. COMPUTATION OF CHARACTERISTIC EXPONENTS OF CANONICAL (HAMILTONIAN) AND ALMOST-CANONICAL SYSTEMS

3.1. Stability conclusions based on approximate characteristic exponents

We return to the vector equation

$$\frac{d\mathbf{x}}{dt} = [\mathbf{C} + \varepsilon \mathbf{B}_1(t) + \varepsilon^2 \mathbf{B}_2(t) + \dots]\mathbf{x}, \tag{3.1}$$

where $\mathbf{x}$ is a $2k$-vector, $\mathbf{C}$ a constant $2k \times 2k$ matrix; $\mathbf{B}_1(t)$, $\mathbf{B}_2(t)$, ... are T-periodic $2k \times 2k$ matrix-functions with the same properties as in (0.1). The bracketed series is convergent for sufficiently small $|\varepsilon|$, in the same sense as before. In contradistinction to (0.1), we now assume that equation (3.1) is Hamiltonian for real ε, i. e., the matrices $\mathbf{C}$ and $\mathbf{B}_1(t)$, $\mathbf{B}_2(t)$, ... are $\tilde{J}$-antihermitian for all $t\,(0 \leqslant t \leqslant T)$ (§ III. 1.1):

$$\mathbf{C} = \tilde{\mathbf{J}}^{-1}\mathbf{H}_0 \quad (\mathbf{H}_0^* = \mathbf{H}_0); \quad \mathbf{B}_j(t) = \tilde{\mathbf{J}}^{-1}\mathbf{H}_j(t)$$
$$(\mathbf{H}_j(t)^* = \mathbf{H}_j(t), \quad j = 1, 2, \dots), \quad \tilde{\mathbf{J}}^* = -\tilde{\mathbf{J}}, \quad \det \tilde{\mathbf{J}} \neq 0.$$

In practice, these matrices will themselves be real, but certain of the subsequent manipulations may make them complex. For the sake of generality, therefore, we shall allow the coefficients of equation (3.1) to be complex from the start, and point out the specific features of the real case.

Suppose that the "unperturbed" Hamiltonian equation

$$\tilde{\mathbf{J}}\frac{d\mathbf{x}}{dt} = \mathbf{H}_0\mathbf{x} \tag{3.2}$$

has a pure imaginary r-fold characteristic exponent $\alpha_0 = i\omega_0$, or an r-fold multiplier $\rho_0 = \exp(i\omega_0)$. Then the matrix $\tilde{\mathbf{J}}^{-1}\mathbf{H}_0$ has exactly r eigenvalues λ_j congruent to $i\omega_0$ modulo $2\pi i/T$. In this case, for sufficiently small ε equation (3.1) has exactly r characteristic exponents in the neighborhood of $i\omega_0$ which tend to $i\omega_0$ at $\varepsilon = 0$. According to the general theory, these characteristic exponents may be expanded in series

$$\alpha_j(\varepsilon) = i\omega_0 + \beta_j\tau_j^{p_j} + \gamma_j\tau_j^{p_j+1} + \dots \qquad (j = 1, \dots, r), \tag{3.3}$$

where $\beta_j \neq 0$, $\tau_j = \varepsilon^{1/q_j}$ is the arithmetical value of the root, p_j is a natural number. Assuming $\varepsilon > 0$, we have written out in (3.3) all the r expansions, distinguishing between the values of the roots by the coefficients.

In applications one is mainly interested in stability. If $\operatorname{Re}\beta_j \neq 0$ for some j, then $\operatorname{Re}\alpha_j(\varepsilon) \neq 0$ for sufficiently small $\varepsilon > 0$. Since the spectrum of characteristic exponents of a Hamiltonian system is symmetric about the imaginary axis (§ II. 3.5), there exists j' such that $\operatorname{Re}\alpha_{j'}(\varepsilon) > 0$, so that the trivial solution of system (3.1) is unstable for sufficiently small $\varepsilon > 0$. This will certainly be true if one of the q_j's is at least 3. If all the β_j's are pure imaginary (and so $q_j \leqslant 2$ for all j) but some of the coefficients in the expansions (3.3) are not, we may again conclude that the trivial solution is unstable.

On the other hand, if successive computation of the expansion coefficients constantly yields pure imaginary values, it would seem at first sight that no conclusions may be drawn as to stability at any fixed step of the process. However, it is easy to see that if at some step we obtain pure imaginary and distinct coefficients, we may legitimately infer that the trivial solution is stable for sufficiently small $\varepsilon > 0$. Let us state this more rigorously.

Let q be the lowest common multiple of $q_1, \ldots, q_r$. Modifying the notation, we can rewrite (3.3) as

$$\alpha_j(\varepsilon) = i\omega_0 + \beta_j \varepsilon^{p/q} + \gamma_j \varepsilon^{(p+1)/q} + \ldots \qquad (j = 1, \ldots, r). \tag{3.4}$$

Here at least one of the coefficients $\beta_1, \ldots, \beta_r$ is not zero. The theorem stated below does not assume that equation (3.1) is Hamiltonian — it may be any equation whose characteristic equation is reversible (for example, t-invariant; see § II. 3. 9).

Theorem. Assume that for real ε equation (3.1) has a reversible characteristic equation. Let $i\omega_0$ be a pure imaginary r-fold characteristic exponent of equation (3.1) for $\varepsilon = 0$ and let (3.4) be the corresponding r characteristic exponents of equation (3.1) for $\varepsilon > 0$. Suppose that when the coefficients $\beta_j, \gamma_j, \ldots$ in (3.4) are evaluated successively, they are all pure imaginary up to some step (which is possible only if $q \leqslant 2$) and at this step all the coefficients of like powers of ε are distinct. Then for sufficiently small $\varepsilon > 0$,

$$\operatorname{Re} \alpha_j(\varepsilon) = 0, \quad \alpha_j(\varepsilon) \neq \alpha_h(\varepsilon) \qquad (j, h = 1, \ldots, r;\ j \neq h).$$

Proof. Let us denote the partial sums of the series (3.3) corresponding to the distinct coefficients by $\alpha_j'(\varepsilon)$ and their remainders by $\alpha_j''(\varepsilon)$ $(j = 1, \ldots, r)$, so that

$$\alpha_j(\varepsilon) = \alpha_j'(\varepsilon) + \alpha_j''(\varepsilon), \quad |\alpha_j''(\varepsilon)| = o(|\alpha_j'(\varepsilon) - \alpha_h'(\varepsilon)|) \ (j, h = 1, \ldots, r;\ j \neq h). \tag{3.5}$$

Consider a disk centered at $i\omega_0$, in whose interior there are no other eigenvalues of $\widetilde{\mathbf{J}}^{-1}\mathbf{H}_0$. For sufficiently small ε, this disk will contain only r eigenvalues of (3.3). Now the matrix $\mathbf{K}(\varepsilon)$ (§ III. 4.6) is J-antihermitian (§ III. 1.1), and its spectrum is symmetric with respect to the imaginary axis. Therefore, if the statement of the theorem is untrue, there are at least two eigenvalues $\alpha_{j_1}, \alpha_{h_1}$ such that, for sufficiently small ε,

$$\operatorname{Im} \alpha_{j_1}(\varepsilon) = \operatorname{Im} \alpha_{h_1}(\varepsilon) \quad (j_1 \neq h_1).$$

Hence

$$\operatorname{Im}[\alpha_{j_1}'(\varepsilon) - \alpha_{h_1}'(\varepsilon)] = -\operatorname{Im}[\alpha_{j_1}''(\varepsilon) - \alpha_{h_1}''(\varepsilon)],$$

and since by assumption $\operatorname{Re} \alpha_{j_1}'(\varepsilon) = \operatorname{Re} \alpha_{h_1}'(\varepsilon)$, it follows that

$$|\alpha_{j_1}'(\varepsilon) - \alpha_{h_1}'(\varepsilon)| = |\operatorname{Im}[\alpha_{j_1}'(\varepsilon) - \alpha_{h_1}'(\varepsilon)]| \leqslant |\alpha_{j_1}''(\varepsilon)| + |\alpha_{h_1}''(\varepsilon)|,$$

contradicting (3.5). Q. E. D.

If the statement of the theorem is true for all characteristic exponents of system (3.1), its trivial solution is stable (§ II. 2.5).

3.2. Formulas for characteristic exponents of Hamiltonian systems

Suppose that for $\varepsilon=0$ equation (3.1) has a pure imaginary r-fold characteristic exponent $\alpha_0=i\omega_0$ of simple type. To compute the corresponding r characteristic exponents for small $\varepsilon\neq 0$ (in a first approximation), we shall use the results of §2.6.

Let $\lambda_1, \ldots, \lambda_r$ be the class of eigenvalues of the matrix $\mathbf{C}$ congruent to $i\omega_0$ modulo $2\pi i/T$, and let $\mathbf{c}_1, \ldots, \mathbf{c}_r, \mathbf{d}_1, \ldots, \mathbf{d}_r$ be corresponding linearly independent eigenvectors of $\mathbf{C}$ and $\mathbf{C}^*$:

$$\mathbf{C}\mathbf{c}_j=\lambda_j\mathbf{c}_j,\quad \mathbf{C}^*\mathbf{d}_j=\bar{\lambda}_j\mathbf{d}_j \qquad (j=1, \ldots, r). \tag{3.6}$$

Denote

$$\gamma_{jh}=\langle \mathbf{c}_j,\ \mathbf{c}_h\rangle = i(\tilde{\mathbf{J}}\mathbf{c}_j,\ \mathbf{c}_h). \tag{3.7}$$

The eigenvectors $\mathbf{c}_1, \ldots, \mathbf{c}_r$ satisfy the conditions (see lemma of § II. 2. 4)

$$\gamma_{jh}=0 \quad \text{for} \quad m_j\neq m_h \qquad (j, h=1, \ldots, r).$$

By Lemma I of § III. 2.7, we may index the numbers λ_j and select the vectors $\mathbf{c}_1, \ldots, \mathbf{c}_r$ in such a way that the matrix $\|\gamma_{jh}\|$ will be

$$\|\gamma_{jh}\|=\begin{pmatrix}\mathbf{I}_p & 0\\ 0 & -\mathbf{I}_q\end{pmatrix} \qquad (p+q=r). \tag{3.8}$$

Assuming that this is so, we see that $\lambda_1, \ldots, \lambda_p$ are the eigenvalues of the first kind and $\lambda_{p+1}, \ldots, \lambda_{p+q}$ those of the second kind of the Hamiltonian matrix $\mathbf{C}$, $p+q=r$. If λ_j is a pure imaginary eigenvalue of $\mathbf{C}$, we may take the eigenvector of $\mathbf{C}^*$ belonging to the eigenvalue $\bar{\lambda}_j=-\lambda_j$ to be $\tilde{\mathbf{J}}\mathbf{c}_j$. Indeed, write the first of equalities (3.6) as $\tilde{\mathbf{J}}^{-1}\mathbf{H}_0\mathbf{c}_j=\lambda_j\mathbf{c}_j$ $(\bar{\lambda}_j=-\lambda_j)$; multiplying on the left by $\tilde{\mathbf{J}}$, we obtain $(\mathbf{H}_0\tilde{\mathbf{J}}^{-1})(\tilde{\mathbf{J}}\mathbf{c}_j)=\lambda_j\tilde{\mathbf{J}}\mathbf{c}_j$, i. e., $\mathbf{C}^*\tilde{\mathbf{J}}\mathbf{c}_j=\bar{\lambda}_j\tilde{\mathbf{J}}\mathbf{c}_j$ $(j=1, \ldots, r)$. Since $\det\tilde{\mathbf{J}}\neq 0$, the vectors $\tilde{\mathbf{J}}\mathbf{c}_1, \ldots, \tilde{\mathbf{J}}\mathbf{c}_r$ are linearly independent and so constitute the required system of eigenvectors of $\mathbf{C}^*$, as asserted.

However, a reference to §2. 6 will show that we cannot put $\mathbf{d}_h=\tilde{\mathbf{J}}\mathbf{c}_h$, since the normalization conditions (2.49) would then fail to hold: $(\mathbf{c}_j, \mathbf{d}_h)=-(\tilde{\mathbf{J}}\mathbf{c}_j,\ \mathbf{c}_h)=i\langle \mathbf{c}_j,\ \mathbf{c}_h\rangle=i\gamma_{jh}\neq\delta_{jh}$. We therefore put $\mathbf{d}_h=i\gamma_{hh}\tilde{\mathbf{J}}\mathbf{c}_h$. Then $(\mathbf{c}_j, \mathbf{d}_h)=i\gamma_{jh}(-i\gamma_{hh})=\delta_{jh}$, so that conditions (2.49) are satisfied.

We want to use the theorem of §2.6. To this end, we evalute the elements σ_{jh} from formula (2.58). Recalling that $\mathbf{B}_1(t)=\tilde{\mathbf{J}}^{-1}\mathbf{H}_1(t)$, we obtain $\sigma_{jh}=(\mathbf{B}_1^{(m_h-m_j)}\mathbf{c}_j,\ \mathbf{d}_h)=-i\gamma_{hh}\sigma_{jh}^{(0)}$, where

$$\sigma_{jh}^{(0)}=-(\mathbf{H}_1^{(m_h-m_j)}\mathbf{c}_j,\ \mathbf{c}_h) \tag{3.9}$$

and $\mathbf{H}_1^{(m)}$ are the Fourier coefficients of the matrix-function $\mathbf{H}_1(t)$. Equation (2.60), that is, the equation $\det \| i\gamma_{hh}\sigma_{jh}^{(\varphi)} - \beta\delta_{jh} \| = 0$, now becomes

$$\det \| \sigma_{jh}^{(\varphi)} + \nu\gamma_{jh} \| = 0, \tag{3.10}$$

and $\beta = i\nu$.

The final result may be stated as follows.

Assume that for $\varepsilon = 0$ equation (3.1) has an r-fold pure imaginary characteristic exponent $i\omega_0$ of simple type $(r = s)$, corresponding to a class $\lambda_1, \ldots, \lambda_r$ of pure imaginary eigenvalues of the matrix $\mathbf{C} = \tilde{\mathbf{J}}^{-1}\mathbf{H}_0$ which are congruent to $i\omega_0$ modulo $2\pi i/T$. Then the r characteristic exponents $\alpha_1(\varepsilon), \ldots, \alpha_r(\varepsilon)$ that tend to $i\omega_0$ at $\varepsilon = 0$ are given by

$$\alpha_j(\varepsilon) = i\omega_0 + i\nu_j\varepsilon + O(\varepsilon^{1+\delta_j}), \tag{3.11}$$

where ν_j are the roots of equation (3.10). The numbers $\sigma_{jh}^{(\varphi)}$, γ_{jh} and integers m_j are defined by formulas (3.9), (3.8), and (3.7), where $\mathbf{c}_j$ are the eigenvalues of the matrix $\mathbf{C} = \tilde{\mathbf{J}}^{-1}\mathbf{H}_0$ (i.e., $\mathbf{C}\mathbf{c}_j = \lambda_j\mathbf{c}_j$, $j = 1, \ldots, r$) satisfying the normalization conditions (3.6), and $\mathbf{H}_1^{(m)}$ are the Fourier coefficients of the matrix-function $\mathbf{H}_1(t)$:

$$\mathbf{H}_1(t) \sim \sum_m e^{\frac{2\pi i m t}{T}} \mathbf{H}_1^{(m)}.$$

If ν_j is a simple root of equation (3.10), then $\alpha_j(\varepsilon)$ is an analytic function of ε and $\delta_j = 1$ in (3.11). If it is an l-fold root, we have l expansions (3.11) with $\delta_j = 1/l$.

In similar fashion, all the other results of §2 carry over to Hamiltonian systems. We shall not state these results since they will not be needed in the sequel. The results of this and the following subsections will be used in Chapter V.

3.3. Vector equation of second order

Consider a system of k scalar second-order equations, in vector notation

$$\mathbf{M}\frac{d^2\mathbf{y}}{dt^2} + [\mathbf{P}_0 + \varepsilon\mathbf{P}_1(t) + \varepsilon^2\mathbf{P}_2(t) + \ldots]\mathbf{y} = \mathbf{0}, \tag{3.12}$$

where $\mathbf{y}$ is a k-vector, $\mathbf{M}$ and $\mathbf{P}_0$ are real symmetric positive definite (§I. 1. 14) $k \times k$ matrices: $\mathbf{M} = \mathbf{M}^T > 0$, $\mathbf{P}_0 = \mathbf{P}_0^T > 0$. Then the eigenvalues $\omega_1^2, \ldots, \omega_k^2$ of the matrix $\mathbf{M}^{-1}\mathbf{P}_0$ are also positive. The $k \times k$ matrix-functions $\mathbf{P}_1(t)$, $\mathbf{P}_2(t)$, ... have real T-periodic integrable piecewise-continuous elements the union of whose sets of discontinuity points is at most a finite subset of the interval $[0, T]$. The series on the right of (3.12) is assumed to be convergent for $|\varepsilon| < r_0$, where r_0 is independent of t; ε is a real parameter.

Equation (3.12) is canonical if

$$\mathbf{P}_l(t)^{\tau}=\mathbf{P}_l(t) \qquad (0\leqslant t\leqslant T,\ l=1,\ 2,\ldots)$$

and t-invariant if, say,

$$\mathbf{P}_l(-t)=\mathbf{P}_l(t) \qquad (0\leqslant t\leqslant T,\ l=1,\ 2,\ldots).$$

We shall not assume these conditions in the sequel: system (3.12) may be only almost-canonical.

Introducing a $2k$-vector $\mathbf{x}=\mathbf{y}+\dot{\mathbf{M}}\mathbf{y}$, we write equation (3.12) as

$$\frac{d\mathbf{x}}{dt}=\left\{\begin{pmatrix}\mathbf{0} & \mathbf{M}^{-1}\\ -\mathbf{P}_0 & \mathbf{0}\end{pmatrix}+\varepsilon\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ -\mathbf{P}_1(t) & \mathbf{0}\end{pmatrix}+\ldots\right\}\mathbf{x}, \tag{3.13}$$

which corresponds to (3.1) with

$$\mathbf{C}=\begin{pmatrix}\mathbf{0} & \mathbf{M}^{-1}\\ -\mathbf{P}_0 & \mathbf{0}\end{pmatrix},\ \mathbf{B}_1(t)=\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ -\mathbf{P}_1(t) & \mathbf{0}\end{pmatrix},\ \mathbf{H}_1(t)=\begin{pmatrix}\mathbf{P}_1(t) & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}. \tag{3.14}$$

For the eigenvectors of $\mathbf{C}$, we have

$$\begin{pmatrix}\mathbf{0} & \mathbf{M}^{-1}\\ -\mathbf{P}_0 & \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}=\lambda\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix},\ \text{i. e.,}\ \mathbf{M}^{-1}\mathbf{b}=\lambda\mathbf{a},\ -\mathbf{P}_0\mathbf{a}=\lambda\mathbf{b},$$

so that $\mathbf{b}=\lambda\mathbf{M}\mathbf{a}$, and the vectors $\mathbf{a}$ are defined by

$$\mathbf{M}^{-1}\mathbf{P}_0\mathbf{a}=\omega^2\mathbf{a} \qquad (\omega^2=-\lambda^2).$$

Thus $\mathbf{a}$ and ω^2 are eigenvectors and eigenvalues, respectively, of the matrix $\mathbf{M}^{-1}\mathbf{P}_0$. Assuming that $\omega_1,\ldots,\omega_k$ are positive, we index the eigenvalues $\lambda=\pm i\omega$ of $\mathbf{C}$ as follows:

$$\lambda_\nu=i\omega_\nu \quad (\nu=\pm1,\ldots,\pm k),$$

where $\omega_{-\nu}=-\omega_\nu$. The eigenvectors $\mathbf{c}_\nu$ of $\mathbf{C}$ are

$$\mathbf{c}_\nu=\begin{pmatrix}\mathbf{a}_\nu\\ i\omega_\nu\mathbf{M}\mathbf{a}_\nu\end{pmatrix} \qquad (\nu=\pm1,\ \ldots,\ \pm k);$$

here $\mathbf{a}_{-\nu}=\mathbf{a}_\nu$ ($\mathbf{a}_{|\nu|}$ are the eigenvectors of the matrix $\mathbf{M}^{-1}\mathbf{P}_0$). We evaluate γ_{jh} from (3.7):

$$\gamma_{jh}=i\left(\begin{Vmatrix}\mathbf{0} & -\mathbf{I}_k\\ \mathbf{I}_k & \mathbf{0}\end{Vmatrix}\begin{pmatrix}\mathbf{a}_{|j|}\\ i\omega_j\mathbf{M}\mathbf{a}_{|j|}\end{pmatrix},\ \begin{pmatrix}\mathbf{a}_{|h|}\\ i\omega_h\mathbf{M}\mathbf{a}_{|h|}\end{pmatrix}\right)=(\omega_j+\omega_h)(\mathbf{M}\mathbf{a}_{|j|},\ \mathbf{a}_{|h|})$$
$$(j,\ h=\pm1,\ \ldots,\ \pm k).$$

Index the eigenvectors $\mathbf{a}_1,\ \ldots,\ \mathbf{a}_k$ of $\mathbf{M}^{-1}\mathbf{P}_0$ so that $2\omega_\varkappa(\mathbf{M}\mathbf{a}_\varkappa,\ \mathbf{a}_\varkappa)=1$ $(\varkappa=1,\ \ldots,\ k)$. By the lemma of § III. 2.4, $\gamma_{jh}=0$ for $j\neq h$; hence

$$\gamma_{jh}=\begin{cases} 0 & \text{for } j\neq h,\\ 1 & \text{for } j=h>0,\\ -1 & \text{for } j=h<0.\end{cases}$$

We determine σ_{jh} from formulas (3.9) and (3.14):

$$\sigma_{jh}=-\left(\begin{pmatrix}\mathbf{P}_1^{(m_h-m_j)} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\mathbf{a}_{|j|}\\ i\omega_j\mathbf{M}\mathbf{a}_{|j|}\end{pmatrix},\begin{pmatrix}\mathbf{a}_{|h|}\\ i\omega_h\mathbf{M}\mathbf{a}_{|h|}\end{pmatrix}\right)=-(\mathbf{P}_1^{(m_h-m_j)}\mathbf{a}_{|j|},\ \mathbf{a}_{|h|})$$
$$(j,\ h=\pm 1,\ \ldots,\ \pm k).$$

The matrix $\|\sigma_{jh}\|_{-k}^{k}$ is hermitian. Indeed, if $\mathbf{P}_1(t)^{\tau}=\mathbf{P}_1(t)$, then $[\mathbf{P}_1^{(m)}]^*=\mathbf{P}_1^{(-m)}$, and so

$$\sigma_{hj}=-(\mathbf{P}_1^{(m_j-m_h)}\mathbf{a}_{|h|},\ \mathbf{a}_{|j|})=-(\mathbf{a}_{|h|},\ \mathbf{P}_1^{(m_h-m_j)}\mathbf{a}_{|j|})=$$
$$=-\overline{(\mathbf{P}_1^{(m_h-m_j)}\mathbf{a}_{|j|},\ \mathbf{a}_{|h|})}=\bar{\sigma}_{jh}\qquad(j,\ h=\pm 1,\ \ldots,\ \pm k),$$

as claimed.

The final result is as follows.

Let (3.12) be a vector equation whose coefficients satisfy all the conditions enumerated at the beginning of §3.3. Define eigenvectors $\mathbf{a}_1,\ \ldots,\ \mathbf{a}_k$ *of the matrix* $\mathbf{M}^{-1}\mathbf{P}_0$ *by*

$$\mathbf{P}_0\mathbf{a}_{\varkappa}=\omega_{\varkappa}^2\mathbf{M}\mathbf{a}_{\varkappa},\quad(\mathbf{M}\mathbf{a}_{\varkappa},\ \mathbf{a}_{\lambda})=\frac{1}{2\omega_{\varkappa}}\delta_{\varkappa\lambda}\qquad(\varkappa,\lambda=1,\ \ldots,\ k),\tag{3.15}$$

where $\delta_{\varkappa\lambda}$ *is the Kronecker symbol* ($\delta_{\varkappa\varkappa}=1$; $\delta_{\varkappa\lambda}=0$ *if* $\lambda\neq\varkappa$). *Divide the numbers* $\omega_{\varkappa}>0$ *and* $\omega_{-\varkappa}=-\omega_{\varkappa}$ ($\varkappa=1,\ \ldots,\ k$) *into congruence classes modulo* $2\pi/T$. *Let* $\omega_{j_1},\ \ldots,\ \omega_{j_r}$ *be one of these classes and* $\omega^{(0)}$ *an arbitrary number congruent to the numbers of the class modulo* $2\pi/T$. *Thus equation (3.12) with* $\varepsilon=0$ *has an r-fold characteristic exponent* $i\omega^{(0)}$. *Define integers* m_j *by*

$$\omega_j=\omega^{(0)}+\frac{2\pi}{T}m_j\qquad(j=j_1,\ \ldots,\ j_r).\tag{3.16}$$

Using the Fourier series for $\mathbf{P}_1(t)\left(\mathbf{P}_1(t)\sim\sum_m\mathbf{P}_1^{(m)}e^{\frac{2\pi imt}{T}}\right)$, *define numbers* σ_{jh} *by*

$$\sigma_{jh}=-(\mathbf{P}_1^{(m_h-m_j)}\mathbf{a}_{|j|},\ \mathbf{a}_{|h|})\qquad(j,\ h=j_1,\ \ldots,\ j_r).\tag{3.17}$$

Define numbers γ_{jh} *by*

$$\gamma_{jh}=\begin{cases} 0 & \text{for } j\neq h,\\ 1 & \text{for } j=h>0,\\ -1 & \text{for } j=h<0.\end{cases}\tag{3.18}$$

Then:

1) *The r characteristic exponents of equation (3.12) that tend to* $\varepsilon=0$ *at* $i\omega^{(0)}$ *are given by*

$$\alpha_j(\varepsilon) = i\omega^{(0)} + i\nu_j\varepsilon + O(\varepsilon^{1+\delta}) \qquad (\delta > 0;\ j = j_1, \ldots, j_r), \tag{3.19}$$

where the numbers ν_j *are the roots of the equation*

$$\det \| \sigma_{jh} + \nu\gamma_{jh} \|_{j,\, h = j_1, \ldots, j_r} = 0. \tag{3.20}$$

2) *If* ν_j *is a simple root,* $\alpha_j(\varepsilon)$ *is an analytic function of* ε *and* $\delta = 1$. *If* $l < r$ *of the numbers* ν_j *coincide, we have* $\delta = l^{-1}$ *in the corresponding expansions (3.19).*

3) *If* $\mathbf{P}_1(t)^\tau = \mathbf{P}_1(t)$, *the matrix* $\| \sigma_{jh} \|$ *is hermitian.*

Remarks and special cases.

a) If the class consists of a single number $\omega^{(0)}$, the corresponding ν in (3.20) is, up to sign,

$$\nu = (\mathbf{P}_1^{(0)}\mathbf{a}^{(0)}, \mathbf{a}^{(0)}) \qquad (\mathbf{P}_1^{(0)} = [\mathbf{P}_1(t)]_{\mathrm{av}}), \tag{3.21}$$

where $\mathbf{a}^{(0)}$ is the eigenvector defined by

$$\mathbf{P}_0\mathbf{a}^{(0)} = (\omega^{(0)})^2\mathbf{M}\mathbf{a}^{(0)}, \qquad 2\omega^{(0)}(\mathbf{M}\mathbf{a}^{(0)}, \mathbf{a}^{(0)}) = 1.$$

It is clear that ν is real.

b) The matrix $\| \gamma_{jh} \|$ is diagonal; its diagonal elements are $+1$ and -1. If $\| \gamma_{jh} \| = \mathbf{I}_r$ or $\| \gamma_{jh} \| = -\mathbf{I}_r$ and $\mathbf{P}_1(t)^\tau = \mathbf{P}_1(t)$, then $\nu_1, \ldots, \nu_r$, are the eigenvalues of the hermitian matrix $\| \sigma_{jh} \|$ and are therefore real. Referring to §III. 2.5, we see that if all the congruence classes of ω_j's modulo $2\pi/T$ consist of numbers of the same kind, i. e., of the numbers $\omega_1, \ldots, \omega_k$ or of $-\omega_1, \ldots, -\omega_k$, then all the corrections to the characteristic exponents are pure imaginary. If system (3.12) is canonical (Hamiltonian) or t-invariant, it follows from the theorem of §3.1 that the trivial solution of system (3.12) will then be stable for sufficiently small $|\varepsilon|$.

If there is at least one class containing ω_j's of different kinds, i. e., with indices of opposite signs, then the real part of at least one of the above-mentioned corrections may be positive. The trivial solution of equation (3.12) will then be unstable for sufficiently small $|\varepsilon|$.

c) If we transform equation (3.12) ($\varepsilon = 0$) to normal coordinates, i. e.,

$$\mathbf{M} = \begin{pmatrix} q_1 & & 0 \\ & \ddots & \\ 0 & & q_k \end{pmatrix}, \qquad \mathbf{P}_0 = \begin{pmatrix} p_1 & & 0 \\ & \ddots & \\ 0 & & p_k \end{pmatrix}$$

are diagonal matrices, then by (3.15)

$$\omega_\varkappa = \sqrt{\frac{p_\varkappa}{q_\varkappa}}, \qquad \mathbf{a}_\varkappa = \frac{1}{\sqrt{2\omega_\varkappa q_\varkappa}}\mathbf{e}_\varkappa \qquad (\varkappa = 1, \ldots, k).$$

Let

$$\mathbf{P}_1^{(m)} = \| \pi_{jh}^{(m)} \|_1^k, \quad \text{i. e.,} \quad \mathbf{P}_1(t) \sim \sum_{m=-\infty}^{\infty} \exp\left(m\frac{2\pi i}{T}t\right) \| \pi_{jh}^{(m)} \|_1^k.$$

Then by (3.17)

$$\sigma_{jh}=-\frac{1}{2\sqrt{\omega_{|j|}\omega_{|h|}q_{|j|}q_{|h|}}}\pi_{|j||h|}^{(m_h-m_j)} \qquad (j,\ h=\pm 1,\ \ldots,\ \pm k). \tag{3.22}$$

3.4. Second-order vector equation with gyroscopic term

We now consider a more general system of k scalar second-order equations (in vector notation):

$$\mathbf{M}\frac{d^2\mathbf{y}}{dt^2}+[\mathbf{Q}_0+\varepsilon\mathbf{Q}_1(t)+\varepsilon^2\mathbf{Q}_2(t)+\ldots]\frac{d\mathbf{y}}{dt}+[\mathbf{P}_0+\varepsilon\mathbf{P}_1(t)+\varepsilon^2\mathbf{P}_2(t)+\ldots]\mathbf{y}=\mathbf{0}, \tag{3.23}$$

where the constant coefficients are real $k\times k$ matrices such that

$$\mathbf{M}=\mathbf{M}^{\tau}>0, \quad \mathbf{Q}_0^{\tau}=-\mathbf{Q}_0, \quad \mathbf{P}_0^{\tau}=\mathbf{P}_0,$$

and the analytical properties of the other coefficients are as in §3.3; the T-periodic matrix-functions $\mathbf{Q}_l(t)$ and $\mathbf{P}_l(t)$ $(l=1, 2, \ldots)$ are again assumed to be real, though this is not essential.

Define a $2k$-vector by $\mathbf{z}=\mathbf{y}+\mathbf{M}\dot{\mathbf{y}}$ and write the vector equation (3.23) as

$$\frac{d\mathbf{z}}{dt}=\begin{pmatrix}\mathbf{0} & \mathbf{M}^{-1}\\ -\mathbf{P}(t,\varepsilon) & -\mathbf{Q}(t,\varepsilon)\mathbf{M}^{-1}\end{pmatrix}\mathbf{z},$$

where

$$\mathbf{P}(t,\varepsilon)=\mathbf{P}_0+\varepsilon\mathbf{P}_1(t)+\ldots, \quad \mathbf{Q}(t,\varepsilon)=\mathbf{Q}_0+\varepsilon\mathbf{Q}_1(t)+\ldots$$

Multiplying by the constant nonsingular skew-symmetric matrix

$$\tilde{\mathbf{J}}=\begin{pmatrix}-\mathbf{Q}_0 & -\mathbf{I}_k\\ \mathbf{I}_k & \mathbf{0}\end{pmatrix},$$

we write this equation as

$$\tilde{\mathbf{J}}\frac{d\mathbf{z}}{dt}=\tilde{\mathbf{H}}(t,\varepsilon)\mathbf{z}\equiv\begin{pmatrix}\mathbf{P}(t,\varepsilon) & -\mathbf{Q}_0\mathbf{M}^{-1}+\mathbf{Q}(t,\varepsilon)\mathbf{M}^{-1}\\ \mathbf{0} & \mathbf{M}^{-1}\end{pmatrix}\mathbf{z}. \tag{3.24}$$

The matrix $\mathbf{H}(t, 0)$ is symmetric, and so equation (3.24) for $\varepsilon=0$ is of canonical type (II.3.4). The substitution $\mathbf{x}=\mathbf{S}\mathbf{z}$, where $\mathbf{S}$ is defined by formula (II.3.5),

$$\mathbf{S}=\begin{pmatrix}\mathbf{I}_k & \mathbf{0}\\ \frac{1}{2}\mathbf{Q}_0 & \mathbf{I}_k\end{pmatrix},$$

reduces equation (3.24) to the form (3.1):

$$\frac{d\mathbf{x}}{dt}=[\mathbf{C}+\varepsilon\mathbf{B}_1(t)+\ldots]\,\mathbf{x}, \tag{3.25}$$

where

$$\mathbf{C}=\begin{pmatrix} -\frac{1}{2}\mathbf{M}^{-1}\mathbf{Q}_0 & \mathbf{M}^{-1} \\ -\mathbf{P}_0+\frac{1}{4}\mathbf{Q}_0\mathbf{M}^{-1}\mathbf{Q}_0 & -\frac{1}{2}\mathbf{Q}_0\mathbf{M}^{-1} \end{pmatrix},$$

$$\mathbf{B}_1(t)=\begin{pmatrix} 0 & 0 \\ -\mathbf{P}_1(t)+\frac{1}{2}\mathbf{Q}_1(t)\,\mathbf{M}^{-1}\mathbf{Q}_0 & -\mathbf{Q}_1(t)\,\mathbf{M}^{-1} \end{pmatrix}.$$

Equation (3.25) may be derived directly from equation (3.23) by the substitution

$$\mathbf{x}=\begin{pmatrix} \mathbf{y} \\ \frac{1}{2}\mathbf{Q}_0\mathbf{y}+\mathbf{M}\dot{\mathbf{y}} \end{pmatrix}.$$

The matrix $\mathbf{C}$ is Hamiltonian, $\mathbf{C}=-\mathbf{J}\mathbf{H}_0$, where $\mathbf{H}_0^{\tau}=\mathbf{H}_0$. Note that the matrices $\mathbf{Q}_l(t)$ are not assumed to be skew-symmetric, or the matrices $\mathbf{P}_l(t)$ symmetric ($l=1, 2, \ldots$). Thus the systems (3.25) and (3.23) are only almost-canonical.

Proceeding as in §3.3, we find the eigenvectors of $\mathbf{C}$:

$$\mathbf{c}_\nu=\begin{pmatrix} \mathbf{f}_\nu \\ \left[\frac{1}{2}\mathbf{Q}_0+\lambda_\nu\mathbf{M}\right]\mathbf{f}_\nu \end{pmatrix},$$

where the vectors $\mathbf{f}_\nu$ and numbers λ_ν are defined by the equation

$$(\lambda_\nu^2\mathbf{M}+\lambda_\nu\mathbf{Q}_0+\mathbf{P}_0)\,\mathbf{f}_\nu=0 \qquad (\nu=\pm 1, \ldots, \pm k).$$

We determine γ_{jh} and σ_{jh} from (3.7) and (3.9):

$$\gamma_{jh}=-i\,([\mathbf{Q}_0+(\lambda_j-\bar{\lambda}_h)\,\mathbf{M}]\,\mathbf{f}_j,\ \mathbf{f}_h),$$
$$\sigma_{jh}=-\left([\mathbf{P}_1^{(m_h-m_j)}+\lambda_j\mathbf{Q}_1^{(m_h-m_j)}]\,\mathbf{f}_j,\ \mathbf{f}_h\right).$$

The final result is as follows.

Let the coefficients of the vector equation (3.23) satisfy the conditions enumerated at the beginning of §3.4. Assume that for $\varepsilon=0$ equation (3.23) has only bounded solutions, in particular, that all roots of the equation

$$\det(\lambda^2\mathbf{M}+\lambda\mathbf{Q}_0+\mathbf{P}_0)=0 \tag{3.26}$$

are pure imaginary. Then, if $\lambda_\varkappa=i\omega_\varkappa$ is a root of equation (3.26), so is $\lambda_{-\varkappa}=-i\omega_\varkappa=i\omega_{-\varkappa}$ ($\varkappa=1, \ldots, k$). The eigenvectors $\mathbf{f}_\varkappa$ satisfying the equations

$$(-\omega_\varkappa^2\mathbf{M}+i\omega_\varkappa\mathbf{Q}_0+\mathbf{P}_0)\,\mathbf{f}_\varkappa=0 \qquad (\varkappa=\pm 1, \ldots, \pm k),$$

may be so chosen and indexed that they satisfy the normalization conditions

$$\gamma_{jh}=([(\omega_j+\omega_h)\mathbf{M}-i\mathbf{Q}_0]\mathbf{f}_j,\ \mathbf{f}_h)=\begin{cases}0, & j\neq h,\\ 1, & j=h>0,\\ -1, & j=h<0.\end{cases} \tag{3.27}$$

Divide the numbers $\omega_\varkappa>0$ *and* $\omega_{-\varkappa}=-\omega_\varkappa$ *into congruence classes modulo* $2\pi/T$. *Let* $\omega_{j_1},\ \ldots,\ \omega_{j_r}$ *be one of these classes and* $\omega^{(0)}$ *any number congruent to them modulo* $2\pi/T$. *Define integers* m_j *by*

$$\omega_j=\omega^{(0)}+\frac{2\pi}{T}m_j \qquad (j=j_1,\ \ldots,\ j_r)$$

and numbers σ_{jh} *by*

$$\sigma_{jh}=-([\mathbf{P}_1^{(m_h-m_j)}+i\omega_j\mathbf{Q}_1^{(m_h-m_j)}]\mathbf{f}_j,\ \mathbf{f}_h) \qquad (j,\ h=j_1,\ \ldots,\ j_r), \tag{3.28}$$

where $\mathbf{P}_1^{(m)}$, $\mathbf{Q}_1^{(m)}$ *are the Fourier coefficients of the matrix-functions* $\mathbf{P}_1(t)$ *and* $\mathbf{Q}_1(t)$:

$$\mathbf{P}_1(t)\sim\sum_{m=-\infty}^{\infty}\exp\left(m\frac{2\pi i}{T}t\right)\mathbf{P}_1^{(m)},\quad \mathbf{Q}_1(t)\sim\sum_{m=-\infty}^{\infty}\exp\left(m\frac{2\pi i}{T}t\right)\mathbf{Q}_1^{(m)}.$$

Then the r *characteristic exponents of equation (3.23) that tend to* $i\omega^{(0)}$ *at* $\varepsilon=0$ *are given by*

$$\alpha_j(\varepsilon)=i\omega^{(0)}+i\nu_j\varepsilon+O(\varepsilon^{1+\delta}) \qquad (\delta>0;\ j=j_1,\ \ldots,\ j_r), \tag{3.29}$$

where the ν_j*'s are the roots of the equation*

$$\det\|\sigma_{jh}+\nu\gamma_{jh}\|_{j,\,h=j_1,\ \ldots,\ j_r}=0. \tag{3.30}$$

If ν_j *is a simple root, then* $\alpha_j(\varepsilon)$ *is an analytic function of* ε $(\delta=1)$. *If* $l<r$ *of the* ν_j*'s are equal, we may put* $\delta=l^{-1}$ *in (3.29) for the corresponding* j*'s.*

Special case. If $\mathbf{Q}_0=\mathbf{0}$ but $\mathbf{Q}_1(t)\not\equiv\mathbf{0}$, the conclusions of §3.3 remain valid, but

$$\sigma_{jh}=-([\mathbf{P}_1^{(m_h-m_j)}+i\omega_j\mathbf{Q}_1^{(m_h-m_j)}]\mathbf{a}_{|j|},\ \mathbf{a}_{|h|}). \tag{3.31}$$

§4. COMPUTATION OF EXPONENT MATRIX $\mathbf{K}(\varepsilon)$ AND MATRIZANT

4.1. Notation. Singular and nonsingular cases

We begin by introducing some notation to be used throughout this section. As usual, we shall refer to system (0.1) with $\varepsilon=0$ as the

unperturbed system. The eigenvalues of the matrix $\mathbf{C}$ will be denoted by $\lambda_1, \ldots, \lambda_n$. The multipliers of the unperturbed system are

$$\rho_\nu = e^{\lambda_\nu T} \qquad (\nu = 1, \ldots, n) \tag{4.1}$$

(some of these may coincide). Divide the eigenvalues λ_ν into congruence classes modulo $2\pi i/T$. Eigenvalues $\lambda_{j_1}, \ldots, \lambda_{j_r}$ of one class correspond to an r-fold multiplier of the unperturbed system

$$\rho^{(0)} = \rho_j = e^{\lambda_j T} \qquad (j = j_1, \ldots, j_r). \tag{4.2}$$

For each class $\lambda_{j_1}, \ldots, \lambda_{j_r}$, let α be any number congruent to its members modulo $2\pi i/T$. Thus, for any fixed class,

$$\lambda_j = \alpha + m_j \frac{2\pi i}{T} \qquad (j = j_1, \ldots, j_r). \tag{4.3}$$

We have thus defined integers m_j $(j = 1, \ldots, n)$ which will figure constantly in what follows. Note that the m_j's are not uniquely defined, since the numbers α may be chosen in many ways. It is easy to see that for any other choice of α all numbers m_j for a fixed class are modified by adding the same integer.

Definition. If at least one of the congruence classes $\lambda_{j_1}, \ldots, \lambda_{j_r}$ consists of distinct numbers $\lambda_j \neq \lambda_h$ $(j, h = j_1, \ldots, j_r)$, we shall say that the singular case holds for system (0.1).

If each class $\lambda_{j_1}, \ldots, \lambda_{j_r}$ consists of equal numbers, we shall say that the nonsingular case holds.

The singular case is clearly exceptional, the nonsingular case being the rule. In fact, the most common case is that in which each of the classes contains a single number. * It is precisely this case that is most interesting for applications. We are now going to show that the singular case may be reduced to the nonsingular case.

4.2. Reduction of singular case to nonsingular

Given the matrix $\mathbf{C}$ of system (0.1), we construct a constant $n \times n$ matrix $\mathbf{C}_0$ satisfying the following conditions:

1) $\mathbf{C}\mathbf{C}_0 = \mathbf{C}_0\mathbf{C}$;

2) all the eigenvalues of $\mathbf{C}_0$ have the form $2\pi m i/T$ and their elementary divisors are simple (m denotes integers);

3) the matrix $\mathbf{K}_0 = \mathbf{C} - \mathbf{C}_0$ is in the nonsingular case.

We can define $\mathbf{C}_0$ as follows. Let

$$\mathbf{S}^{-1}\mathbf{C}\mathbf{S} = \mathbf{Q}_1(\lambda_{(1)}) \dotplus \ldots \dotplus \mathbf{Q}_s(\lambda_{(s)}) \tag{4.4}$$

* In more rigorous terms, this means that in the n^2-dimensional space of matrices the set of matrices $\mathbf{C}$ corresponding to the singular case (and even the set for which at least one of the classes $\{\lambda_j\}$ consists of more than one number) has dimension less than n^2. In practice, this implies that in the usual case, when the matrix $\mathbf{C}$ depends on certain parameters, the singular case defines curves rather than domains on the plane of any two parameters.

be the Jordan normal form of $\mathbf{C}$ (I. 1.46). Here $\mathbf{Q}_\sigma(\lambda_{(\sigma)})$ is a $k_\sigma \times k_\sigma$ elementary Jordan matrix, $\lambda_{(1)}, \ldots, \lambda_{(s)}$ are the eigenvalues of $\mathbf{C}$, each appearing as many times as it has elementary divisors. We now define $\mathbf{C}_0$ by

$$\mathbf{S}^{-1}\mathbf{C}_0\mathbf{S} = \mathbf{R}_1 \dot{+} \ldots \dot{+} \mathbf{R}_s, \tag{4.5}$$

where each $\mathbf{R}_\sigma$ is a $k_\sigma \times k_\sigma$ diagonal matrix, obtained from $\mathbf{Q}_\sigma$ by replacing all ones by zeros and the numbers $\lambda_{(\sigma)}$ by $2m_\sigma\pi i/T$:

$$\mathbf{R}_\sigma = \frac{2\pi m_\sigma i}{T}\mathbf{I}_{k_\sigma} \qquad (\sigma = 1, \ldots, s). \tag{4.6}$$

Example. We shall construct $\mathbf{C}_0$ for the case that the matrix $\mathbf{S}^{-1}\mathbf{C}\mathbf{S}$ (the Jordan form of $\mathbf{C}$) is

$$\mathbf{S}^{-1}\mathbf{C}\mathbf{S} = \begin{pmatrix} \begin{Vmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{Vmatrix} & & \\ & \lambda_1 & \\ & & \begin{Vmatrix} \lambda_4 & 1 & 0 \\ 0 & \lambda_4 & 1 \\ 0 & 0 & \lambda_4 \end{Vmatrix} \end{pmatrix} = \begin{Vmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{Vmatrix} \dot{+} \| \lambda_1 \| \dot{+} \begin{Vmatrix} \lambda_4 & 1 & 0 \\ 0 & \lambda_4 & 1 \\ 0 & 0 & \lambda_4 \end{Vmatrix},$$

where $\lambda_4 \equiv \lambda_1 \pmod{2\pi i/T}$, $\lambda_1 \neq \lambda_4$, and the empty spaces stand for zeros. The matrix

$$\mathbf{S}^{-1}(\mathbf{C} - \mathbf{C}_0)\mathbf{S} = \begin{pmatrix} \begin{Vmatrix} \alpha & 1 \\ 0 & \alpha \end{Vmatrix} & & \\ & \alpha & \\ & & \begin{Vmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{Vmatrix} \end{pmatrix}$$

has a six-fold eigenvalue α with three elementary divisors.

Note that according to the notation of § I. 1.12 the eigenvalues of $\mathbf{C}$ are either

$$\lambda_1 = \lambda_2 = \lambda_3 \neq \lambda_4 = \lambda_5 = \lambda_6$$

(indexed in order, not counting multiplicity) or

$$\lambda^{(1)} = \lambda_1, \quad \lambda^{(2)} = \lambda_4$$

(indexing only distinct eigenvalues) or

$$\lambda_{(1)} = \lambda_1, \ \lambda_{(2)} = \lambda_1, \ \lambda_{(3)} = \lambda_4$$

(indexing in accordance with elementary divisors). In this case, the six eigenvalues of $\mathbf{C}$ form a single class. Letting α be any number congruent to the members of the class modulo $2\pi i/T$, we define the integers m_1, m_4 by

$$\lambda_1 = \alpha + m_1\frac{2\pi i}{T}, \qquad \lambda_4 = \alpha + m_4\frac{2\pi i}{T}.$$

The matrix $\mathbf{C}_0$ is defined by

$$\mathbf{S}^{-1}\mathbf{C}_0\mathbf{S}=\frac{2\pi i}{T}\left[\left\|\begin{matrix} m_1 & 0 \\ 0 & m_1 \end{matrix}\right\|\dot{+}\|\, m_1\,\|\dot{+}\left\|\begin{matrix} m_4 & 0 & 0 \\ 0 & m_4 & 0 \\ 0 & 0 & m_4 \end{matrix}\right\|\right].$$

The reader should note that our definition of $\mathbf{C}_0$ is equivalent to the following. Let $\lambda_{j_1}, \ldots, \lambda_{j_r}$ be a class of eigenvalues and let the integer corresponding to λ_j in (4.3) be m_j. We define the matrix $\mathbf{C}_0$ as an operator on the root subspace $\mathfrak{L}_j$ of λ_j by

$$\mathbf{C}_0\mathbf{f}=\frac{2\pi i}{T}m_j\mathbf{f} \qquad (\mathbf{f}\in\mathfrak{L}_j). \tag{4.7}$$

This is obviously the same as formula (4.6). The matrix $\mathbf{S}^{-1}(\mathbf{C}-\mathbf{C}_0)\mathbf{S}$ will then have the Jordan form obtained from $\mathbf{S}^{-1}\mathbf{C}\mathbf{S}$ by replacing each λ_j by the corresponding α. Consequently, to each class of eigenvalues $\lambda_{j_1}, \ldots, \lambda_{j_r}$ of $\mathbf{C}$ there corresponds an r-fold eigenvalue α of $\mathbf{K}_0=\mathbf{C}-\mathbf{C}_0$. Thus condition 3 holds. That conditions 1 and 2 hold is obvious.

Substituting

$$\mathbf{x}=e^{t\mathbf{C}_0}\mathbf{y} \qquad \left(y=\begin{pmatrix}\eta_1\\ \ldots \\ \eta_n\end{pmatrix}\right) \tag{4.8}$$

in system (0.1), we arrive at the system

$$\frac{d\mathbf{y}}{dt}=[\mathbf{K}_0+\varepsilon\mathbf{D}_1(t)+\varepsilon^2\mathbf{D}_2(t)+\ldots]\,\mathbf{y}, \tag{4.9}$$

where

$$\mathbf{K}_0=\mathbf{C}-\mathbf{C}_0, \quad \mathbf{D}_l(t)=e^{-t\mathbf{C}_0}\mathbf{B}_l(t)\,e^{t\mathbf{C}_0} \qquad (l=1,\ 2,\ \ldots) \tag{4.10}$$

(using condition 1). By condition 2,

$$e^{(t+T)\,\mathbf{C}_0}=e^{t\mathbf{C}_0}, \tag{4.11}$$

and therefore

$$\mathbf{D}_l(t+T)=\mathbf{D}_l(t) \qquad (l=1,\ 2,\ \ldots). \tag{4.12}$$

System (4.9) has the same form as (0.1). By condition 3, however, the nonsingular case holds for system (4.9).

4.3. Reduction of singular case to nonsingular with real coefficients*

Suppose that the coefficients of system (0.1) are real. It is natural to ask whether one can choose the matrix $\mathbf{C}_0$ so that the transformed

* The contents of this subsection will not be used in the sequel and the reader may omit it.

system (4.9) will also have real coefficients. It will suffice, of course, if we can make the matrix $\mathbf{C}_0$ itself real. Now it follows from (4.7) that the matrix $\mathbf{C}_0$ is determined by the choice of the numbers m_j, i. e., by the choice of the numbers α in (4.3). We must therefore choose these numbers in such a way that the set $\{\alpha\}$ is symmetric about the real axis (since the α's are the eigenvalues of $\mathbf{K}_0$, which must be real).

Lemma 1. Suppose that the matrix $\mathbf{C}$ *in (0.1) is real and its eigenvalues* λ_ν *are such that*

$$2\operatorname{Im}\lambda_\nu \neq \pm\frac{2\pi}{T}m \qquad (\nu=1,\ \ldots,\ n;\ m=1,\ 3,\ 5,\ \ldots). \tag{4.13}$$

Then the numbers α *in (4.3) may be chosen in such a way that the set* $\{\alpha\}$ *is symmetric about the real axis. For any such choice of the* α*'s, the matrix* $\mathbf{C}_0$ *is real (and hence so are the coefficients of system (4.9)) and the nonsingular case holds for system (4.9).*

Proof. Let λ be an eigenvalue of $\mathbf{C}$, and suppose that

$$\operatorname{Im}\lambda \equiv 0 \left(\operatorname{mod}\frac{2\pi}{T}\right).$$

For this class one can clearly choose a real α.

Now suppose that

$$\operatorname{Im}\lambda \not\equiv 0 \left(\operatorname{mod}\frac{2\pi}{T}\right). \tag{4.14}$$

Then the eigenvalues λ and $\bar{\lambda}$ are in different classes. Indeed, if

$$\lambda \equiv \bar{\lambda} \left(\operatorname{mod}\frac{2\pi i}{T}\right),$$

then for some natural number m

$$2\operatorname{Im}\lambda = \pm\frac{2\pi}{T}m.$$

By (4.14), m is odd, contradicting condition (4.13) of the lemma. Thus λ and $\bar{\lambda}$ are in different classes. Assigning some number α for the class $\{\lambda\}$, we assign the number $\bar{\alpha}$ for $\{\bar{\lambda}\}$. The set $\{\alpha\}$ will then be symmetric about the real axis.

Now let $\{\alpha\}$ be any set satisfying this property. We claim that $\mathbf{C}_0$ as constructed above will be real. It will suffice to show that for any vector $\mathbf{f}$

$$\overline{\mathbf{C}_0\mathbf{f}} = \mathbf{C}_0\bar{\mathbf{f}}, \tag{4.15}$$

where $\bar{\mathbf{f}}$ is the vector whose components are the complex conjugates of those of $\mathbf{f}$. Indeed, set $\mathbf{f}=\mathbf{e}_\nu$ $(\nu=1,\ldots,n)$ (§ I. 1.4) in (4.15). The vector $\mathbf{C}_0\mathbf{e}_\nu$ is the ν-th column of the matrix $\mathbf{C}_0$, and it follows from (4.15) that this vector is real, so that $\mathbf{C}_0$ is indeed a real matrix.

Now consider the matrix $\mathbf{C}$. Since it is real, any nonreal eigenvalues must appear in pairs of complex conjugates λ, $\bar{\lambda}$. If $\mathbf{f} \in \mathfrak{L}_\lambda$, so that for some natural number q

$$(\mathbf{C}-\lambda \mathbf{I}_n)^q \mathbf{f} = 0,$$

then, going over to complex conjugates, we have

$$(\mathbf{C}-\bar{\lambda}\mathbf{I}_n)^q \bar{\mathbf{f}} = 0,$$

so that $\bar{\mathbf{f}} \in \mathfrak{L}_{\bar{\lambda}}$. Let $\bar{\mathfrak{L}}_\lambda$ denote the subspace of all vectors complex conjugate to the vectors $\mathbf{f}$; then we have just shown that

$$\bar{\mathfrak{L}}_\lambda = \mathfrak{L}_{\bar{\lambda}}. \tag{4.16}$$

Suppose first that the eigenvalues λ and $\bar{\lambda}$ of $\mathbf{C}$ ($\bar{\lambda} \neq \lambda$) belong to one class:

$$\lambda - \bar{\lambda} = \frac{2\pi i}{T} m, \quad \text{i. e.,} \quad 2 \operatorname{Im} \lambda = \pm \frac{2\pi}{T} m.$$

By (4.13), m is an even number. Since

$$\lambda = \operatorname{Re}\lambda + i \operatorname{Im}\lambda = \operatorname{Re}\lambda \pm \frac{2\pi i}{T} \frac{m}{2}, \tag{4.17}$$

the numbers λ and $\operatorname{Re}\lambda$ are congruent modulo $2\pi i/T$. Then the corresponding α is real. Indeed, it follows from (4.17) that

$$\alpha \equiv \operatorname{Re}\lambda \left(\operatorname{mod} \frac{2\pi i}{T}\right),$$

and so

$$\bar{\alpha} \equiv \operatorname{Re}\lambda \left(\operatorname{mod} \frac{2\pi i}{T}\right)$$

and if $\alpha \neq \bar{\alpha}$ it would follow that $\alpha \equiv \bar{\alpha} (\operatorname{mod} 2\pi i/T)$, which is impossible by the definition of the numbers α.

If λ_j is one of the numbers of the class, say

$$\lambda_j = \alpha + \frac{2\pi i}{T} m_j,$$

then it follows from

$$\bar{\lambda}_j = \alpha - \frac{2\pi i}{T} m_j$$

that $\bar{\lambda}_j$ is in the same class and $\bar{m}_j = -m_j$. By the definition of $\mathbf{C}_0$ we have (4.7)

$$\overline{\mathbf{C}_0\mathbf{f}} = -\frac{2\pi i}{T}\overline{m_j}\overline{\mathbf{f}} = \frac{2\pi i}{T}m_j\overline{\mathbf{f}} = \mathbf{C}_0\overline{\mathbf{f}}. \tag{4.18}$$

Thus (4.15) is valid for the vectors in question.

It remains to consider the case that λ and $\bar{\lambda}$ are in different classes. Then the classes in question consist of complex conjugate eigenvalues and the symmetry of the set $\{\alpha\}$ implies that the corresponding numbers are complex conjugates α and $\bar{\alpha}$. The numbers corresponding to complex conjugates λ_j and $\overline{\lambda_j}$ in (4.3) are m_j and $-m_j$. Thus (4.18) holds for vectors $\mathbf{f}\in\mathfrak{L}_{\lambda_j}$, $\overline{\mathbf{f}}\in\mathfrak{L}_{\bar{\lambda}_j}$.

We have thus shown that (4.15) holds for the vectors of any root subspace of $\mathbf{C}_0$. Since $\mathfrak{R}_n$ is the direct sum of all root subspaces (§ I. 1.10), it follows that (4.15) is true for any vector, so that $\mathbf{C}_0$ is real.

We now show that the nonsingular case holds for system (4.9). The eigenvalues of the matrix $\mathbf{K}_0=\mathbf{C}-\mathbf{C}_0$ are the numbers $\{\alpha\}$. It follows from (4.4) and (4.5) that the matrix $\mathbf{S}^{-1}\mathbf{K}_0\mathbf{S}$ has the same decomposition into elementary Jordan matrices as $\mathbf{C}$, except that each λ_j must be replaced by the corresponding α, i. e., the numbers $\{\alpha\}$ are the eigenvalues of $\mathbf{K}_0$. Now suppose that the singular case holds. Then there exist α_1 and α_2 such that

$$e^{\alpha_1 T} = e^{\alpha_2 T} \qquad (\alpha_1 \neq \alpha_2). \tag{4.19}$$

Let

$$\lambda_j \equiv \alpha_1, \quad \lambda_h \equiv \alpha_2 \left(\mathrm{mod}\,\frac{2\pi i}{T}\right).$$

Then by (4.19)

$$\lambda_j \equiv \lambda_h \left(\mathrm{mod}\,\frac{2\pi i}{T}\right),$$

i. e., we have two different numbers α_1 and α_2 for the same class, contradicting the choice of the α's and proving Lemma I.

We now consider the case excluded in Lemma I. Suppose that the coefficients of system (0.1) are real and the singular case holds. We divide the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\mathbf{C}$ into two types: an eigenvalue λ_j is of type (A), if there exist no odd integer N_j and natural number μ_j such that

$$\mathrm{Im}\,\lambda_j = \frac{N_j}{2^{\mu_j}}\frac{2\pi}{T},$$

and of type (B) if such numbers exist. Set

$$\mu = \max \mu_j,$$

where the maximum extends over all λ_j of type (B); if there are none, we set $\mu=0$ (if $\mu=0$, the assumptions of Lemma I are valid). Denote $M=2^{\mu}$ and divide the numbers λ_j into congruence classes modulo $2\pi i/MT$. For each class, choose a number α' congruent to its members modulo $2\pi i/MT$, and define integers m_j' by

$$\lambda_j = \alpha' + \frac{2\pi i}{MT} m_j'. \tag{4.20}$$

Lemma II. Let λ *be an eigenvalue of the matrix* $\mathbf{C}$, $\mathfrak{L}_\lambda$ *the corresponding root subspace and* m' *the corresponding integer in (4.20). Define a matrix* $\mathbf{C}_0$ *by*

$$\mathbf{C}_0 \mathbf{f} = \frac{2\pi i}{MT} m' \mathbf{f} \quad \text{for} \quad \mathbf{f} \in \mathfrak{L}_\lambda$$

(λ *runs through all eigenvalues of* $\mathbf{C}$*). Then* $\exp(t\mathbf{C}_0)$ *is an MT-periodic matrix-function and the substitution (4.8) reduces system (0.1) to a system (4.9) for which the nonsingular case holds, and its coefficients (4.10) are MT-periodic. The numbers* α' *may be chosen in such a way that the set* $\{\alpha'\}$ *is symmetric about the real axis and for any such choice of these numbers the matrix* $\mathbf{C}_0$ *and the coefficients of system (4.9) are real.*

Proof. Let λ be an eigenvalue of $\mathbf{C}$. If

$$\operatorname{Im}\lambda \equiv 0 \left(\operatorname{mod} \frac{2\pi}{MT}\right),$$

we can clearly choose a real α' for the class. If

$$\operatorname{Im}\lambda \not\equiv 0 \left(\operatorname{mod} \frac{2\pi}{MT}\right), \tag{4.21}$$

then the eigenvalues λ and $\bar{\lambda}$ are in different classes. Indeed, otherwise

$$\lambda \equiv \bar{\lambda} \left(\operatorname{mod} \frac{2\pi i}{MT}\right),$$

whence

$$\operatorname{Im}\lambda = \frac{\pi}{MT} N$$

for some integer N. By (4.21), N is odd, contradicting the definition of μ. With different classes $\{\lambda\}$ and $\{\bar{\lambda}\}$ we associate complex conjugates α' and $\bar{\alpha}'$. We have thus shown that the numbers α' may be chosen so that the set $\{\alpha'\}$ is symmetric about the real axis. Assuming now that α' are any numbers satisfying this condition, we continue the proof as in Lemma I.

4.4. Floquet-Lyapunov theorem for systems analytic in the parameter

Let $\mathbf{Y}(t, \varepsilon)$ be the matrizant of equation (4.9) and $\Delta(\varepsilon)$ the discriminant of the characteristic equation

$$\det[\mathbf{Y}(T, \varepsilon) - \rho \mathbf{I}_n] = 0.$$

The roots of this equation (the multipliers of equation (4.9)) will be denoted by $\rho_1(\varepsilon), \ldots, \rho_n(\varepsilon)$. Let $\alpha_1, \ldots, \alpha_n$ denote the eigenvalues of the matrix $\mathbf{K}_0$, counting multiplicities. Since

$$\mathbf{Y}(T, 0) = e^{T\mathbf{K}_0},$$

it follows that

$$\alpha_\nu = \frac{1}{T} \ln \rho_\nu(0) \qquad (\nu = 1, \ldots, n) \tag{4.22}$$

for suitable values of the logarithm. Suppose that system (4.9) is in the nonsingular case. Then for each multiple eigenvalue of $\mathbf{Y}(T, 0)$, say

$$\rho_{j_1}(0) = \ldots = \rho_{j_r}(0),$$

we have an eigenvalue of $\mathbf{K}_0$,

$$\alpha_{j_1} = \ldots = \alpha_{j_r},$$

of the same multiplicity. Distributing the eigenvalues $\rho_\nu(0)$ over the Riemann surface of $\ln\zeta$ according to (4.22), we see that eigenvalues $\rho_\nu(0)$ which are multiple on the complex plane are also multiple on the Riemann surface. As in §I. 3.3, let us classify the roots of the equation

$$\Delta(\varepsilon) = 0 \tag{4.23}$$

as extraneous and nonextraneous. Recall that ε_0 is a nonextraneous root if, whenever one moves along some path from $\varepsilon = 0$ to ε_0 in the domain $|\varepsilon| < |\varepsilon_0|$, some of the multiple roots $\rho_\nu(\varepsilon_0)$ will lie on different sheets of the Riemann surface. Otherwise ε_0 is an extraneous root (for the details, see §I. 3.3).

Theorem. Suppose that the nonsingular case holds for equation (4.9). The matrizant $\mathbf{Y}(t, \varepsilon)$ *of equation (4.9) may be expressed as*

$$\mathbf{Y}(t, \varepsilon) = \mathbf{F}(t, \varepsilon) e^{t\mathbf{K}(\varepsilon)}, \tag{4.24}$$

where $\mathbf{F}(t+T, \varepsilon) \equiv \mathbf{F}(t, \varepsilon)$ *and*

$$\begin{aligned} \mathbf{F}(t, \varepsilon) &= \mathbf{I}_n + \varepsilon\mathbf{F}_1(t) + \varepsilon^2\mathbf{F}_2(t) + \ldots, \\ \mathbf{K}(\varepsilon) &= \mathbf{K}_0 + \varepsilon\mathbf{K}_1 + \varepsilon^2\mathbf{K}_2 + \ldots \end{aligned} \tag{4.25}$$

are analytic functions of ε *at* $\varepsilon = 0$. *The series (4.25) are convergent in the disk* $|\varepsilon| < |\varepsilon_0| \leqslant r_0$, *where* ε_0 *is the smallest (in modulus) nonextraneous root of equation (4.23)* ($|\varepsilon_0| \leqslant r_0$). *The condition* $\mathbf{K}(0) = \mathbf{K}_0$ *and the requirement that* $\mathbf{K}(\varepsilon)$ *be analytic uniquely determine the matrix-functions* $\mathbf{F}(t, \varepsilon)$ *and* $\mathbf{K}(\varepsilon)$ *in (4.24).*

It follows from the theorem that the series (4.25) are convergent in $|\varepsilon| < r_0$, where r_0 is the radius of convergence of the coefficients of equation (4.9), provided this disk contains no nonextraneous roots of equation (4.23).

Proof. The proof duplicates that of the Floquet-Lyapunov theorem in § II. 2.2, with references to lemmas of § I. 3. The matrizant $\mathbf{Y}(t, \varepsilon)$ of equation (4.9) is analytic in ε and $\mathbf{Y}(T, 0) = \exp(T\mathbf{K}_0)$. Since system (4.9) is in the nonsingular case, the eigenvalues $T\alpha^{(\varkappa)}$ of the matrix $\mathbf{A}_0 = T\mathbf{K}_0$ satisfy the inequalities $T\alpha^{(\varkappa)} \not\equiv T\alpha^{(\lambda)} \pmod{2\pi i}$. By the lemma of § I. 3.1, by suitable choice of the numbers $m_\varkappa$ in formula (I. 3.1) for $\mathbf{Y} = \mathbf{Y}(T, \varepsilon)$, we deduce that the matrix

$$\mathbf{K}(\varepsilon) = \frac{1}{T} \ln \mathbf{Y}(T, \varepsilon)$$

is analytic in ε and $\mathbf{K}(0) = \mathbf{K}_0$. By the lemma of § I. 3. 2, the matrix $\mathbf{K}(\varepsilon)$ is analytic in the disk $|\varepsilon| < |\varepsilon_0| \leqslant r_0$.

Then

$$\mathbf{F}(t, \varepsilon) = \mathbf{Y}(t, \varepsilon) e^{-t\mathbf{K}(\varepsilon)} \tag{4.26}$$

is analytic in ε in the disk $|\varepsilon| < |\varepsilon_0| \leqslant r_0$ for fixed $t\,(0 \leqslant t \leqslant T)$. As in § II. 2.2, we check that $\mathbf{F}(t+T, \varepsilon) \equiv \mathbf{F}(t, \varepsilon)$. Finally, the uniqueness of $\ln \mathbf{Y}(\varepsilon)$ subject to conditions 1 and 2 in the lemma of § I. 3.1 implies the uniqueness of $\mathbf{F}(t, \varepsilon)$ and $\mathbf{K}(\varepsilon)$. This completes the proof.

Remark 1. If equation (4.9) was obtained by the substitution (4.8) from an equation (0.1), then $\mathbf{K}_0$ and the series (4.25) are uniquely defined. In fact, they depend only on the choice of the numbers α for each class $\lambda_{j_1}, \ldots, \lambda_{j_r}$ of eigenvalues of $\mathbf{C}$. The different series (4.25) may well have different radii of convergence. By a special choice of the α's (in other words, suitable choice of the matrix $\mathbf{C}_0$), we can increase the radius of convergence of (4.25). Later (§4.6) we shall return to this question.

Remark 2. We have proved the theorem for the matrizant $\mathbf{Y}(t, \varepsilon)$. An arbitrary fundamental matrix $\tilde{\mathbf{Y}}(t, \varepsilon)$ for system (4.9) may be expressed as (§ II. 2.2)

$$\tilde{\mathbf{Y}}(t, \varepsilon) = \tilde{\mathbf{F}}(t, \varepsilon) e^{t\tilde{\mathbf{K}}(\varepsilon)}, \tag{4.27}$$

where

$$\tilde{\mathbf{F}}(t, \varepsilon) = \mathbf{F}(t, \varepsilon)\tilde{\mathbf{Y}}(0, \varepsilon), \quad \tilde{\mathbf{K}}(\varepsilon) = \tilde{\mathbf{Y}}(0, \varepsilon)^{-1}\mathbf{K}(\varepsilon)\tilde{\mathbf{Y}}(0, \varepsilon). \tag{4.27a}$$

The matrices $\tilde{\mathbf{F}}(t, \varepsilon)$, $\tilde{\mathbf{K}}(\varepsilon)$ need not be analytic in the disk $|\varepsilon| < |\varepsilon_0|$. But if $\tilde{\mathbf{Y}}(0, \varepsilon)$ is analytic in some disk $|\varepsilon| < r_1$, where $r_1 \geqslant |\varepsilon_0|$, and $\det \tilde{\mathbf{Y}}(0, \varepsilon) \neq 0$ in this disk, then $\tilde{\mathbf{F}}(t, \varepsilon)$ and $\tilde{\mathbf{K}}(\varepsilon)$ are analytic in $|\varepsilon| < |\varepsilon_0|$. This follows from (4.27a).

Later we shall see that in certain cases it is more convenient to determine not the coefficients $\mathbf{F}_j(t)$, $\mathbf{K}_j$ in (4.25) but rather the coefficients $\tilde{\mathbf{F}}_j(t)$, $\tilde{\mathbf{K}}_j$ in the expansions

$$\left.\begin{aligned}\tilde{\mathbf{F}}(t,\ \varepsilon)&=\mathbf{I}_n+\varepsilon\tilde{\mathbf{F}}_1(t)+\varepsilon^2\tilde{\mathbf{F}}_2(t)+\ldots,\\ \tilde{\mathbf{K}}(\varepsilon)&=\tilde{\mathbf{K}}_0+\varepsilon\tilde{\mathbf{K}}_1+\varepsilon^2\tilde{\mathbf{K}}_2+\ldots\end{aligned}\right\}\qquad(4.28)$$

The reason is that it may be possible to choose the coefficients $\mathbf{V}_j$ in the series

$$\tilde{\mathbf{Y}}(0,\ \varepsilon)=\mathbf{I}_n+\varepsilon\mathbf{V}_1+\varepsilon^2\mathbf{V}_2+\ldots\qquad(4.28a)$$

in such a way as to simplify the computation of the matrices

$$\tilde{\mathbf{F}}_j(t),\quad \tilde{\mathbf{K}}_j\qquad(j=1,\ 2,\ \ldots).$$

Note that for different choices of $\tilde{\mathbf{K}}_0$ the series (4.28) will generally have different radii of convergence. As follows from the theorem proved above, the value of ε_0 for (4.28) depends on the classification of the roots of equation (4.23) as extraneous and nonextraneous, using instead of (4.22)

$$\tilde{\alpha}_\nu=\frac{1}{T}\ln\rho_\nu(0)\qquad(\nu=1,\ \ldots,\ n),$$

where $\tilde{\alpha}_\nu$ are the eigenvalues of the matrix $\tilde{\mathbf{K}}_0$. Unlike (4.22), these relations may involve different branches of the logarithm.

If we somehow know (§ I. 3.3) that a specific choice of the branches of the logarithm yields the largest possible value for ε_0, this choice uniquely defines the matrix $\tilde{\mathbf{K}}_0$ and the expansions (4.28) with the largest possible radius of convergence.

4.5. Determination of coefficients of expansions (4.25). First and second computational procedures

Substituting (4.24) into equation (4.9), we see that $\mathbf{F}(t,\ \varepsilon)$ satisfies the equation

$$\frac{d\mathbf{F}}{dt}=[\mathbf{K}_0+\varepsilon\mathbf{D}_1(t)+\varepsilon^2\mathbf{D}_2(t)+\ldots]\,\mathbf{F}(t,\ \varepsilon)-\mathbf{F}(t,\ \varepsilon)\,\mathbf{K}(\varepsilon).\qquad(4.29)$$

Inserting the series (4.25) in this equation and comparing the coefficients of like powers of ε, we obtain a sequence of matrix equations

$$\left.\begin{aligned}\frac{d\mathbf{F}_1}{dt}&=\mathbf{K}_0\mathbf{F}_1(t)-\mathbf{F}_1(t)\,\mathbf{K}_0+\mathbf{D}_1(t)-\mathbf{K}_1,\\ \frac{d\mathbf{F}_j}{dt}&=\mathbf{K}_0\mathbf{F}_j(t)-\mathbf{F}_j(t)\,\mathbf{K}_0+\{[\mathbf{D}_1(t)\,\mathbf{F}_{j-1}(t)+\ldots+\mathbf{D}_{j-1}(t)\,\mathbf{F}_1(t)]-\\ &\quad-[\mathbf{F}_{j-1}(t)\,\mathbf{K}_1+\ldots+\mathbf{F}_1(t)\,\mathbf{K}_{j-1}]+\mathbf{D}_j(t)\}-\mathbf{K}_j\qquad(j=2,\ 3,\ \ldots).\end{aligned}\right\}\qquad(4.30)$$

These equations allow recursive determination of $\mathbf{K}_1,\ \mathbf{F}_1(t),\ \mathbf{K}_2,\ \mathbf{F}_2(t),\ \ldots$. The matrices $\mathbf{K}_j$ are chosen in such a way that each $\mathbf{F}_j(t)$ is periodic. This choice is simpler if the original system has either $\mathbf{K}_0=\mathbf{0}$ or $\mathbf{K}_0=\alpha\mathbf{I}_n$

(where α is a number); for then, integrating the j-th equation (4.30) over the period, we obtain

$$\mathbf{K}_j = \{[\mathbf{D}_1(t)\mathbf{F}_{j-1}(t) + \ldots + \mathbf{D}_{j-1}(t)\mathbf{F}_1(t)] - \\ -[\mathbf{F}_{j-1}(t)\mathbf{K}_1 + \ldots + \mathbf{F}_1(t)\mathbf{K}_{j-1}] + \mathbf{D}_j(t)\}_{\text{av}},$$

where $\{\ldots\}_{\text{av}}$ denotes the mean value of the matrix-function $\{\ldots\}$. If $\mathbf{K}_0 \neq 0$ and the matrices $\mathbf{F}_1(t), \ldots, \mathbf{F}_{j-1}(t)$; $\mathbf{K}_1, \ldots, \mathbf{K}_{j-1}$ have been determined, we have a matrix equation for $\mathbf{F}_j(t)$, $\mathbf{K}_j$:

$$\frac{d\mathbf{Z}}{dt} = \mathbf{K}_0\mathbf{Z} - \mathbf{Z}\mathbf{K}_0 + \mathbf{\Phi}(t) - \mathbf{L}. \tag{4.31}$$

Here $\mathbf{Z} = \mathbf{F}_j(t)$, $\mathbf{L} = \mathbf{K}_j$ and $\mathbf{\Phi}(t+T) = \mathbf{\Phi}(t)$. The unknowns in this equation are $\mathbf{Z} = \mathbf{Z}(t)$ and $\mathbf{L}$, where $\mathbf{Z}(t)$ must be a T-periodic function. It is sufficient to demand (in view of the uniqueness of the solution to (4.31)) that

$$\mathbf{Z}(T) = \mathbf{Z}(0). \tag{4.32}$$

Since $\mathbf{F}_j(0) = \mathbf{0}$ for $j = 1, 2, \ldots$, we must have in our case

$$\mathbf{Z}(0) = \mathbf{0}. \tag{4.33}$$

This condition uniquely determines a solution of equation (4.31). It follows from the theorem of §4.4 that there is a matrix $\mathbf{L}$ satisfying (4.32). We now see how to determine $\mathbf{L}$ and $\mathbf{Z}(t)$.

First procedure for determination of $\mathbf{Z} = \mathbf{F}_j(t)$ and $\mathbf{L} = \mathbf{K}$ from equations (4.30). So as not to encumber the exposition, we consider equation (4.31) instead of (4.30). Integrating (4.31) over the period, we get

$$\mathbf{L} = \mathbf{\Phi}_{\text{av}} + \mathbf{K}_0\mathbf{Z}_{\text{av}} - \mathbf{Z}_{\text{av}}\mathbf{K}_0, \tag{4.34}$$

where

$$\mathbf{\Phi}_{\text{av}} = \frac{1}{T}\int_0^T \mathbf{\Phi}(t)\,dt, \quad \mathbf{Z}_{\text{av}} = \frac{1}{T}\int_0^T \mathbf{Z}(t)\,dt.$$

Thus, instead of $\mathbf{L}$ we must find $\mathbf{Z}_{\text{av}}$. We shall view the matrix $\mathbf{Z}$ as an n^2-vector, and define a linear operator $\mathbf{R}$ in the n^2-dimensional space of $n \times n$ matrices by

$$\mathbf{R}\mathbf{Z} = \mathbf{K}_0\mathbf{Z} - \mathbf{Z}\mathbf{K}_0.$$

This operator may of course be represented by an $n^2 \times n^2$ matrix; it is singular ($\det \mathbf{R} = 0$) since, for example, $\mathbf{R}\mathbf{K}_0 = \mathbf{0}$. *

Treating the matrices $\mathbf{\Phi}(t)$ and $\mathbf{L}$ as n^2-vectors, we write equation (4.31) as

$$\frac{d\mathbf{Z}}{dt} = \mathbf{R}\mathbf{Z} + \mathbf{\Phi}(t) - \mathbf{L}.$$

* We are assuming, of course, that $\mathbf{K}_0 \neq 0$.

According to § II. 1.4, the solution for initial conditions (4.33) is

$$\mathbf{Z}(t)=e^{t\mathbf{R}}\int_0^t e^{-\tau\mathbf{R}}\,[\mathbf{\Phi}(\tau)-\mathbf{L}]\,d\tau.$$

Inserting $\mathbf{L}=\mathbf{\Phi}_{av}+\mathbf{R}\mathbf{Z}_{av}$ from (4.34) and integrating, we obtain

$$\mathbf{Z}(t)=\int_0^t e^{(t-\tau)\mathbf{R}}\,[\mathbf{\Phi}(\tau)-\mathbf{\Phi}_{av}]\,d\tau+(\mathbf{I}_{n^2}-e^{t\mathbf{R}})\,\mathbf{Z}_{av}. \tag{4.35}$$

(We have used the formula

$$\left(\frac{\mathbf{I}_{n^2}-e^{t\mathbf{R}}}{\mathbf{R}}\right)\mathbf{R}=\mathbf{I}_{n^2}-e^{t\mathbf{R}}$$

for the singular matrix $\mathbf{R}$, understanding the expression in parentheses as a series.) It follows from the condition $\mathbf{Z}(T)=\mathbf{0}$ (see (4.32)) that the matrix $\mathbf{C}=\mathbf{Z}_{av}$ satisfies the equation

$$(e^{T\mathbf{R}}-\mathbf{I}_{n^2})\,\mathbf{C}=\int_0^T e^{(T-\tau)\mathbf{R}}\,[\mathbf{\Phi}(\tau)-\mathbf{\Phi}_{av}]\,d\tau. \tag{4.36}$$

Since $\det(e^{T\mathbf{R}}-\mathbf{I}_{n^2})=0$, the solution of this equation is not unique. Let $\mathbf{C}$ be any solution. Then (§ I. 1.6)

$$\mathbf{Z}_{av}=\mathbf{C}+\mathbf{C}_0,$$

where

$$(e^{T\mathbf{R}}-\mathbf{I}_{n^2})\,\mathbf{C}_0=\mathbf{0}.$$

By the lemma of § I. 1.6,

$$\mathbf{R}\mathbf{C}_0=\mathbf{0}.$$

Therefore, $\mathbf{L}=\mathbf{\Phi}_{av}+\mathbf{R}\mathbf{Z}_{av}=\mathbf{\Phi}_{av}+\mathbf{R}\mathbf{C}$, i. e.,

$$\mathbf{L}=\mathbf{\Phi}_{av}+\mathbf{K}_0\mathbf{C}-\mathbf{C}\mathbf{K}_0. \tag{4.37}$$

For any t,

$$(e^{t\mathbf{R}}-\mathbf{I}_{n^2})\,\mathbf{C}_0=\left(t\mathbf{R}+\frac{1}{2}t^2\mathbf{R}^2+\dots\right)\mathbf{C}_0=\mathbf{0},$$

and so it follows from (4.35) that

$$\mathbf{Z}(t)=\int_0^t e^{(t-\tau)\mathbf{R}}\,[\mathbf{\Phi}(\tau)-\mathbf{\Phi}_{av}]\,d\tau+(\mathbf{I}_{n^2}-e^{t\mathbf{R}})\,\mathbf{C}. \tag{4.38}$$

Thus the solution to our problem is provided (though not yet in final form) by formulas (4.37), (4.38), where $\mathbf{C}$ is an arbitrary solution of equation (4.36).

We now want to find an explicit formula for the operator $e^{t\mathbf{R}}$. In fact, we shall show that for any $n\times n$ matrix $\mathbf{A}$

$$e^{t\mathbf{R}}\mathbf{A}=e^{t\mathbf{K}_0}\mathbf{A}e^{-t\mathbf{K}_0}. \tag{4.39}$$

Setting

$$\mathbf{X}(t)=e^{t\mathbf{K}_0}\mathbf{A}e^{-t\mathbf{K}_0},$$

we see that

$$\frac{d\mathbf{X}}{dt}=\mathbf{K}_0\mathbf{X}-\mathbf{X}\mathbf{K}_0\equiv\mathbf{R}\mathbf{X},\quad \mathbf{X}(0)=\mathbf{A}.$$

Hence (§II. 1.5) we have $\mathbf{X}(t)=e^{t\mathbf{R}}\mathbf{A}$, proving (4.39).

Using formula (4.39), we rewrite equations (4.36), (4.38) as

$$\mathbf{C}-e^{-T\mathbf{K}_0}\mathbf{C}e^{T\mathbf{K}_0}=\int_0^T e^{-\tau\mathbf{K}_0}\left[\mathbf{\Phi}(\tau)-\mathbf{\Phi}_{\text{av}}\right]e^{\tau\mathbf{K}_0}\,d\tau, \tag{4.40}$$

$$\mathbf{Z}(t)=e^{t\mathbf{K}_0}\left\{\int_0^t e^{-\tau\mathbf{K}_0}\left[\mathbf{\Phi}(\tau)-\mathbf{\Phi}_{\text{av}}\right]e^{\tau\mathbf{K}_0}\,d\tau-\left[\mathbf{C}-e^{-t\mathbf{K}_0}\mathbf{C}e^{t\mathbf{K}_0}\right]\right\}e^{-t\mathbf{K}_0}. \tag{4.41}$$

To summarize: the solution $\{\mathbf{Z}(t),\ \mathbf{L}\}$ of equation (4.31) is derived as follows: (I) find some solution $\mathbf{C}$ of equation (4.40); (II) determine $\mathbf{L}$ and $\mathbf{Z}(t)$ from formulas (4.37) and (4.41). This automatically guarantees $\mathbf{Z}(t+\mathbf{T})=\mathbf{Z}(t)$.

If $\mathbf{K}_0$ is a scalar matrix (multiple of the identity), the integral on the right of (4.40) vanishes, so we may take $\mathbf{C}=\mathbf{0}$, $\mathbf{L}=\mathbf{\Phi}_{\text{av}}$. This is the case already treated above. This method of determining $\mathbf{F}_j(t)$, $\mathbf{K}_j$ from equations (4.30) will be referred to as the first procedure.

We now consider a second procedure, which is applicable in case the coefficients $\mathbf{\Phi}_m$ of the Fourier series

$$\mathbf{\Phi}(t)\sim\sum_{m=-\infty}^{\infty}\exp\left(im\frac{2\pi}{T}t\right)\mathbf{\Phi}^{(m)}$$

satisfy the condition*

$$|\mathbf{\Phi}^{(m)}|=O\left(\frac{1}{|m|}\right)\qquad(m=\pm1,\ \pm2,\ \ldots).$$

This condition is certainly valid in the frequently encountered case that $\mathbf{D}(t,\ \varepsilon)$ in equation (4.9) is a trigonometric polynomial. Formally expanding the unknown $\mathbf{Z}(t)$ in Fourier series,

* This condition may be weakened.

$$\mathbf{Z}(t) \sim \sum_{m=-\infty}^{\infty} \exp\left(im\frac{2\pi}{T}t\right)\mathbf{Z}^{(m)} \tag{4.42}$$

and comparing the Fourier coefficients, we obtain a sequence of matrix equations:

$$\left.\begin{aligned} \frac{2\pi mi}{T}\mathbf{Z}^{(m)} &= \mathbf{K}_0\mathbf{Z}^{(m)} - \mathbf{Z}^{(m)}\mathbf{K}_0 + \mathbf{\Phi}^{(m)} \qquad (m \neq 0), \\ 0 &= \mathbf{K}_0\mathbf{Z}^{(0)} - \mathbf{Z}^{(0)}\mathbf{K}_0 + \mathbf{\Phi}^{(0)} - \mathbf{L}. \end{aligned}\right\} \tag{4.43}$$

The operator $\mathbf{RZ} = \mathbf{K}_0\mathbf{Z} - \mathbf{ZK}_0$ in the n^2-dimensional linear space of $n \times n$ matrices has n^2 eigenvalues (/21/, Chapter VIII)

$$\alpha_g - \alpha_h \qquad (g, h = 1, \ldots, n),$$

where $\alpha_1, \ldots, \alpha_n$ are the eigenvalues of $\mathbf{K}_0$. Since we are dealing with the nonsingular case,

$$\alpha_g - \alpha_h \neq m\frac{2\pi i}{T} \qquad (g, h = 1, \ldots, n;\ m = \pm 1, \pm 2, \ldots).$$

Consequently, the equation

$$m\frac{2\pi i}{T}\mathbf{W} = \mathbf{K}_0\mathbf{W} - \mathbf{WK}_0 + \mathbf{G} \tag{4.44}$$

with $m \neq 0$ is solvable for $\mathbf{W}$; denoting a solution of this equation by

$$\mathbf{W} = \mathbf{R}_m(\mathbf{G}), \tag{4.45}$$

we deduce from the first equation of (4.43)

$$\mathbf{Z}^{(m)} = \mathbf{R}_m(\mathbf{\Phi}^{(m)}). \tag{4.46}$$

Thus $\mathbf{Z}^{(m)}$ $(m \neq 0)$ is uniquely determined by (4.46). It follows from (4.33) that

$$\mathbf{Z}^{(0)} = -\sum_{m \neq 0} \mathbf{Z}^{(m)}. \tag{4.47}$$

The series (4.47) is convergent. Indeed, using the above operator $\mathbf{R}$, we have*

$$\mathbf{R}_m = \left(\mathbf{R} - m\frac{2\pi i}{T}\mathbf{I}_{n^2}\right)^{-1},$$

i. e.,

$$|\mathbf{R}_m| = O\left(\frac{1}{|m|}\right).$$

* As before, we are treating $\mathbf{R}_m$ as an $n^2 \times n^2$ matrix.

Therefore

$$|\mathbf{Z}^{(m)}| \leqslant |\mathbf{R}_m|\,|\mathbf{\Phi}^{(m)}| = O\left(\frac{1}{m^2}\right),$$

and this shows that the series (4.47) is indeed convergent. Hence so is the series (4.42).

To summarize, the second procedure for determining $\mathbf{Z}(t)$, $\mathbf{L}$ from equation (4.31) (and hence also $\mathbf{F}_j(t)$, $\mathbf{K}_j$ from equations (4.30)) consists in the following steps: (I) determine $\mathbf{Z}^{(m)}$ $(m \neq 0)$ from equations (4.43); (II) determine $\mathbf{Z}^{(0)} = \mathbf{Z}_{av}$ from (4.47) and $\mathbf{L}$ from (4.34); (III) find $\mathbf{Z}(t)$ from (4.42).

Both first and second procedures allow inductive determination of the coefficients of the series (4.25) in the matrizant formula (4.24).

4.6. Determination of coefficients of expansions (4.28). Third procedure

Instead of the matrizant $\mathbf{Y}(t, \varepsilon)$, we shall now determine some fundamental matrix (4.27), choosing the coefficients $\mathbf{V}_j$ in (4.28a) in such a way as to simplify the computations involved in determining the coefficients of (4.28).

We are going to show that a suitable choice of $\mathbf{V}_1, \mathbf{V}_2, \ldots$ will bring us back to equation (4.31) for $\mathbf{Z}(t) = \tilde{\mathbf{F}}_j(t)$, $\mathbf{L} = \tilde{\mathbf{K}}_j$, in which $\mathbf{Z}(t)$ will now satisfy not condition (4.33) but only condition (4.32) and the condition $\mathbf{Z}_{av} = \mathbf{0}$. It will then follow from formula (4.34) that $\mathbf{L} = \mathbf{\Phi}_{av}$; this simple computation of $\mathbf{L}$ is the main advantage of this approach.

Substituting (4.27) into equation (4.9) and proceeding as at the beginning of §4.5, we obtain equation (4.30) with $\mathbf{F}_j(t)$ replaced by $\tilde{\mathbf{F}}_j(t)$ and $\mathbf{K}_j$ by $\tilde{\mathbf{K}}_j$ $(j = 1, 2, \ldots)$. (As is apparent from the second formula of (4.28), the matrix $\mathbf{K}_0$ is unchanged.) Thus, we again obtain (4.31) and (4.32) for inductive determination of the matrices $\mathbf{Z}(t) = \tilde{\mathbf{F}}_j(t)$, $\mathbf{L} = \tilde{\mathbf{K}}_j$, except that now in general $\mathbf{Z}(0) = \tilde{\mathbf{F}}_l(0) \neq \mathbf{0}$. The value of $\mathbf{Z}(0) = \tilde{\mathbf{F}}_l(0) = \mathbf{V}_l$ is determined from the condition

$$\mathbf{Z}^{(0)} = [\tilde{\mathbf{F}}_l(t)]_{av} = \mathbf{0}. \tag{4.48}$$

Then, in the second equation of (4.43), we have $\mathbf{L} = \mathbf{\Phi}^{(0)}$ at each step, so that

$$\begin{aligned} \tilde{\mathbf{K}}_l &= [\mathbf{D}_l + (\mathbf{D}_1\tilde{\mathbf{F}}_{l-1} + \ldots + \mathbf{D}_{l-1}\tilde{\mathbf{F}}_1) - (\tilde{\mathbf{F}}_{l-1}\tilde{\mathbf{K}}_1 + \ldots + \tilde{\mathbf{F}}_1\tilde{\mathbf{K}}_{l-1})]_{av} = \\ &= [\mathbf{D}_l + (\mathbf{D}_1\tilde{\mathbf{F}}_{l-1} + \ldots + \mathbf{D}_{l-1}\tilde{\mathbf{F}}_1)]_{av} \quad (l = 1, 2, \ldots), \end{aligned} \tag{4.49}$$

since by (4.48) $(\tilde{\mathbf{F}}_1)_{av} = \ldots = (\tilde{\mathbf{F}}_{l-1})_{av} = \mathbf{0}$. Setting $l = 1$, we have

$$\tilde{\mathbf{K}}_1 = (\mathbf{D}_1)_{av}.$$

Condition (4.46) remains valid. As before, we can determine $\tilde{\mathbf{F}}_l(t)$, $\tilde{\mathbf{K}}_l$ successively. Using the formula

$$[\mathbf{A}(t)\,\mathbf{B}(t)]_{av} = \sum_m \mathbf{A}_{-m}\mathbf{B}_m,$$

where

$$\mathbf{A}(t) \sim \sum \exp\left(im\frac{2\pi}{T}t\right)\mathbf{A}_m, \quad \mathbf{B}(t) \sim \sum \exp\left(im\frac{2\pi}{T}t\right)\mathbf{B}_m,$$

we readily derive simple formulas for the matrices $\tilde{\mathbf{F}}_l(t)$, $\tilde{\mathbf{K}}_l$ $(l=1, 2, 3)$.

We present these formulas, indicating the final results and the computational procedure.

Suppose that the nonsingular case holds for the system of differential equations (4.9). Then system (4.9) has a fundamental matrix $\tilde{\mathbf{Y}}(t, \varepsilon)$ expressible in the form (4.27), where the matrices $\tilde{\mathbf{F}}(t+T, \varepsilon) \equiv \tilde{\mathbf{F}}(t, \varepsilon)$ and $\tilde{\mathbf{K}}(\varepsilon)$ are analytic functions of ε at $\varepsilon=0$ and the first coefficients of the expansions (4.28) are given by the following formulas:

$$\left.\begin{aligned}
&\tilde{\mathbf{K}}_1 = [\mathbf{D}_1(t)]_{av} = \mathbf{D}_1^{(0)}, \quad \tilde{\mathbf{F}}_1(t) = \sum_{m\neq 0} \exp\left(im\frac{2\pi}{T}t\right)\mathbf{R}_m(\mathbf{D}_1^{(m)}),\\
&\tilde{\mathbf{K}}_2 = \mathbf{D}_2^{(0)} + \sum_{m\neq 0} \mathbf{D}_1^{(-m)}\mathbf{R}_m(\mathbf{D}_1^{(m)}),\\
&\Phi^{(m)} = \mathbf{D}_2^{(m)} - \mathbf{R}_m(\mathbf{D}_1^{(m)})\,\tilde{\mathbf{K}}_1 + \sum_{j+k=m}[\mathbf{D}_1^{(k)}\mathbf{R}_j(\mathbf{D}_1^{(j)})],\\
&\tilde{\mathbf{F}}_2(t) = \sum_{m\neq 0} \exp\left(im\frac{2\pi}{T}t\right)\mathbf{R}_m(\Phi^{(m)}),\\
&\tilde{\mathbf{K}}_3 = \mathbf{D}_3^{(0)} + \sum_{m\neq 0}[\mathbf{D}_1^{(-m)}\mathbf{R}_m(\Phi^{(m)}) + \mathbf{D}_2^{(-m)}\mathbf{R}_m(\mathbf{D}_1^{(m)})].
\end{aligned}\right\} \tag{4.50}$$

Here $\mathbf{W} = \mathbf{R}_m(\mathbf{G})$ denotes the solution of the equation

$$m\frac{2\pi i}{T}\mathbf{W} = \mathbf{K}_0\mathbf{W} - \mathbf{W}\mathbf{K}_0 + \mathbf{G}.$$

If $\mathbf{D}_l(t)$ $(l=1, 2, \ldots)$ are real matrices, the formulas may be simplified:

$$\left.\begin{aligned}
&\tilde{\mathbf{K}}_2 = \mathbf{D}_2^{(0)} + 2\mathrm{Re}\sum_{m=1}^{\infty}\bar{\mathbf{D}}_1^{(m)}\mathbf{R}_m(\mathbf{D}_1^{(m)}),\\
&\tilde{\mathbf{K}}_3 = \mathbf{D}_3^{(0)} + 2\mathrm{Re}\sum_{m=1}^{\infty}[\bar{\mathbf{D}}_1^{(m)}\mathbf{R}_m(\Phi^{(m)}) + \bar{\mathbf{D}}_2^{(m)}\mathbf{R}_m(\mathbf{D}_1^{(m)})].
\end{aligned}\right\} \tag{4.51}$$

General formulas for $\tilde{\mathbf{K}}_j$ and $\tilde{\mathbf{F}}_j(t)$ may also be derived, but they are too cumbersome to include here.

All we need to prove is that $\tilde{\mathbf{F}}(t, \varepsilon)$ and $\tilde{\mathbf{K}}(\varepsilon)$ are analytic functions, i. e., that the series (4.28) are convergent. * We shall show that they converge for sufficiently small ε, and estimate the radius of convergence.

We first estimate the solution of equation (4.31) in terms of $\Phi(t)$. Let $\mathfrak{M}$ denote the set of matrix-functions $\Phi(t)$ with absolutely convergent Fourier series

$$\Phi(t) = \sum_{m=-\infty}^{\infty} \exp\left(im\frac{2\pi}{T}t\right)\Phi^{(m)}.$$

* Previously, we deduced the convergence of these series from that of the series (4.28a). Since the matrices $\mathbf{V}_j$ are now constructed by a special device at each step, the convergence of the series (4.28a) is now no longer evident.

We denote the norm of $\Phi(t)$ by $|\Phi|$:

$$|\Phi| = \sum_{m=-\infty}^{\infty} |\Phi^{(m)}|. \tag{4.52}$$

Lemma. If $\Phi(t) \in \mathfrak{M}$, there exists a unique solution $\{Z(t), L\}$ of equation (4.31) satisfying the conditions (4.32) and $Z_{av} = 0$. For this solution, $L = \Phi_{av}$, $Z(t) \in \mathfrak{M}$ and

$$|Z(t)| \leqslant \alpha |\Phi(t) - \Phi_{av}|, \tag{4.53}$$

where α is a constant whose precise value will be specified below.

Proof. The matrix equations (4.43) have been shown to have solutions (4.46) for any $\Phi^{(m)}$, if $m \neq 0$. Consequently, there is a positive constant α_m such that

$$|Z^{(m)}| = |R_m(\Phi^{(m)})| \leqslant \alpha_m |\Phi^{(m)}|.$$

Here α_m is some upper bound for the norm of the operator R_m. We claim that a suitable choice is

$$\alpha_m = \frac{1}{\frac{2\pi|m|}{T} - 2|K_0|}, \quad \text{if} \quad \frac{\pi|m|}{T} > |K_0|. \tag{4.54}$$

It follows from equation (4.43) that

$$\frac{2\pi|m|}{T}|Z^{(m)}| \leqslant 2|K_0||Z^{(m)}| + |\Phi^{(m)}|;$$

we have

$$|Z^{(m)}| \leqslant \frac{1}{\frac{2\pi|m|}{T} - 2|K_0|}|\Phi^{(m)}|, \quad \text{if} \quad |m| > \frac{T}{\pi}|K_0|.$$

We may thus define α_m by (4.54). Set

$$\alpha = \max_{|m| \geqslant 1} \alpha_m.$$

In view of (4.48), we have

$$|Z(t)| = \sum_{\substack{m=-\infty \\ (m \neq 0)}}^{\infty} |Z^{(m)}| \leqslant \sum \alpha_m |\Phi^{(m)}| \leqslant \alpha \sum |\Phi^{(m)}| = \alpha |\Phi(t) - \Phi_{av}|.$$

Uniqueness follows in obvious fashion from (4.53).

Remark. In any specific case (given K_0), the values of α_m may be determined effectively. The role of α_m may also be played by any other upper bound for the norm of R_m (§I. 1.2), viewing the latter as an $n^2 \times n^2$ matrix. It is not necessary to find the exact maximum of α_m—all we need is some number α greater than all the α_m's.

Let us return to equation (4.30). Suppose we have an upper bound for the norm of $\mathbf{D}_j(t)$: $|\mathbf{D}_j(t)| \leqslant \delta_j$ (where $\delta_1 > 0$) and also know the radius of convergence r of the series for $\delta(\varepsilon)$:

$$\delta(\varepsilon) = \sum_{j=1}^{\infty} \delta_j \varepsilon^j. \tag{4.55}$$

The norm (4.52) satisfies the inequalities

$$|\Phi_1 \Phi_2| \leqslant |\Phi_1| |\Phi_2|, \quad |\Phi_{av}| \leqslant |\Phi|.$$

Assuming that

$$|\tilde{\mathbf{K}}_j| \leqslant \beta_j, \quad |\tilde{\mathbf{F}}_j(t)| \leqslant \gamma_j \qquad (j = 1, \ldots, l-1),$$

and using the previous remarks, we deduce from (4.49)

$$|\tilde{\mathbf{K}}_l| < \delta_l + (\delta_1 \gamma_{l-1} + \ldots + \delta_{l-1} \gamma_1).$$

We may therefore take

$$\beta_1 = \delta_1, \ \beta_l = \delta_l + (\delta_1 \gamma_{l-1} + \ldots + \delta_{l-1} \gamma_1) \qquad (l = 2, 3, \ldots). \tag{4.56}$$

By (4.53), we obtain from (4.30)

$$\begin{aligned} |\tilde{\mathbf{F}}_l(t)| \leqslant \alpha \big| [\mathbf{D}_1(t) \tilde{\mathbf{F}}_{l-1}(t) + \ldots + \mathbf{D}_{l-1}(t) \tilde{\mathbf{F}}_1(t)] - \\ - [\tilde{\mathbf{F}}_{l-1}(t) \mathbf{K}_1 + \ldots + \tilde{\mathbf{F}}_1(t) \mathbf{K}_{l-1}] + \mathbf{D}_l(t) - \mathbf{K}_l \big| < \\ < \alpha [(\delta_1 \gamma_{l-1} + \ldots + \delta_{l-1} \gamma_1) + (\gamma_{l-1} \beta_1 + \ldots + \gamma_1 \beta_{l-1}) + \delta_l + \beta_l] \qquad (l = 2, 3, \ldots), \\ |\tilde{\mathbf{F}}_1(t)| \leqslant \alpha (\beta_1 + \delta_1) = 2\alpha\delta_1. \end{aligned}$$

We may thus define γ_l by

$$\begin{aligned} &\gamma_1 = 2\alpha\delta_1, \\ &\gamma_l = \alpha [(\delta_1 \gamma_{l-1} + \ldots + \delta_{l-1} \gamma_1) + (\gamma_{l-1} \beta_1 + \ldots + \gamma_1 \beta_{l-1}) + \delta_l + \beta_l] \quad (l = 2, 3, \ldots). \end{aligned} \tag{4.57}$$

Define

$$\beta(\varepsilon) = \sum_{l=1}^{\infty} \beta_l \varepsilon^l, \quad \gamma(\varepsilon) = \sum_{l=1}^{\infty} \gamma_l \varepsilon^l.$$

It follows from (4.56) that

$$\sum_{l=1}^{\infty} \beta_l \varepsilon^l = \sum_{l=1}^{\infty} \delta_l \varepsilon^l + \sum_{l=2}^{\infty} (\delta_1 \gamma_{l-1} + \ldots + \delta_{l-1} \gamma_1) \varepsilon^l,$$

which may be written

$$\beta = \delta(1 + \gamma). \tag{4.58}$$

Similarly, we deduce from (4.57)

$$\gamma=\alpha[\delta\gamma+\beta\gamma+\delta+\beta]=\alpha(\beta+\delta)(1+\gamma).$$

These equations define β and γ. The series (4.28) are certainly convergent in the disk $|\varepsilon|<r_0$ in which these functions are analytic.

From (4.58) and the last equation, we have

$$\alpha\beta^2+(\alpha\delta-1)\beta+\delta=0.$$

Hence

$$\beta=\frac{1}{2\alpha}\left[1-\alpha\delta-\sqrt{(1-\alpha\delta)^2-4\alpha\delta}\right] \tag{4.59}$$

(the + sign of the root may be rejected, since $\beta(0)=0$). The radius of convergence of the series $\beta(\varepsilon)$ is determined by the singular point of the function $\beta(\varepsilon)$ in the disk $|\varepsilon|<r$ nearest the origin, where, as we recall, r is the radius of convergence of the series for $\delta(\varepsilon)$. Noting that the function $\gamma(\varepsilon)=[\beta(\varepsilon)-\delta(\varepsilon)]\,\delta(\varepsilon)^{-1}$ is analytic at zero since $\delta_1=\beta_1>0$, we see that the series (4.28) are certainly convergent in a disk of radius less than r containing no zeros of the function under the radical in (4.59). Summarizing the result, we have the following proposition.

Suppose that the coefficients of equation (4.9) satisfy the inequalities

$$|\mathbf{D}_l|=\sum_{m=-\infty}^{\infty}|\mathbf{D}_l^{(m)}|\leqslant\delta_l \qquad (l=1, 2, \ldots)$$

and let r be the radius of convergence of the series (4.55). Define a positive number α by

$$\alpha=\max_m \alpha_m.$$

Then the series (4.28) are convergent in the disk $|\varepsilon|\leqslant\min[r, |\varepsilon_0|]$, where ε_0 is the root of least absolute value of the equation

$$[1-\alpha\delta(\varepsilon)]^2-4\alpha\delta(\varepsilon)=0. \tag{4.60}$$

If system (4.9) depends linearly on the parameter ε ($\mathbf{D}_2(t)=\mathbf{D}_3(t)=\ldots=0$), equation (4.60) becomes

$$\alpha^2\delta_1^2\varepsilon^2-6\alpha\delta_1\varepsilon+1=0,$$

so that

$$\varepsilon_0=\frac{3-2\sqrt{2}}{\delta_1\alpha}.$$

Analogous arguments will yield estimates for the radii of convergence of the series (4.25).

4.7. Solution of inhomogeneous system of equations with periodic coefficients

We wish to solve the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{B}(t, \varepsilon)\,\mathbf{x} + \mathbf{f}(t), \tag{4.61}$$

on the following assumptions:

a) the matrix $\mathbf{B}(t, \varepsilon)$ is analytic in ε at $\varepsilon = 0$:

$$\mathbf{B}(t, \varepsilon) = \mathbf{C} + \varepsilon\mathbf{B}_1(t) + \varepsilon^2\mathbf{B}_2(t) + \ldots,$$

integrable, piecewise-continuous and 2π-periodic in t:

$$\mathbf{B}(t + 2\pi, \varepsilon) = \mathbf{B}(t, \varepsilon);$$

b) the vector-function $\mathbf{f}(t)$ is real, integrable and piecewise-continuous;

c) the corresponding homogeneous system is asymptotically stable for sufficiently small $|\varepsilon|$;

d) the nonsingular case holds for the homogeneous system of (4.61).

We know (see formula (II. 1.17)) that the general solution of an inhomogeneous solution is the sum of the general solution of the homogeneous equation and a particular solution of the inhomogeneous equation. By condition (c), the first term tends to zero as $t \to +\infty$. We shall start from the matrizant $\mathbf{X}(t, \varepsilon)$ of the homogeneous equation, in its representation according to the Floquet-Lyapunov theorem (4.24), (4.25):

$$\left.\begin{aligned} \mathbf{X}(t, \varepsilon) &= \mathbf{F}(t, \varepsilon)\, e^{t\mathbf{K}(\varepsilon)}, \\ \mathbf{F}(t, \varepsilon) &= \mathbf{I}_n + \varepsilon\mathbf{F}_1(t) + \varepsilon^2\mathbf{F}_2(t) + \ldots, \\ \mathbf{K}(\varepsilon) &= \mathbf{K}_0 + \varepsilon\mathbf{K}_1 + \varepsilon^2\mathbf{K}_2 + \ldots \end{aligned}\right\} \tag{4.62}$$

For sufficiently small $|\varepsilon|$,

$$\mathbf{F}(t, \varepsilon)^{-1} = \mathbf{I}_n - \varepsilon\mathbf{F}_1(t) + \varepsilon^2[\mathbf{F}_1(t)^2 - \mathbf{F}_2(t)] + \ldots, \tag{4.63}$$

as may be verified by a direct check. Let us expand $\mathbf{F}(t, \varepsilon)^{-1}$ in Fourier series:

$$\mathbf{F}(t, \varepsilon)^{-1} \sim \sum_{-\infty}^{\infty} e^{imt}\,\mathbf{S}_m(\varepsilon).$$

Assume that $\mathbf{f}(t) = \exp(i\vartheta t)\,\mathbf{f}_1 + \exp(-i\vartheta t)\,\mathbf{f}_2$ (where ϑ is not an integer). We evaluate the integral on the right of (II. 1.17):

$$\begin{aligned} \int_0^t \mathbf{X}(t_1, \varepsilon)^{-1}\,\mathbf{f}(t_1)\,dt_1 &= \int_0^t e^{-t_1\mathbf{K}(\varepsilon)}\,\mathbf{F}(t_1, \varepsilon)^{-1}\,\mathbf{f}(t_1)\,dt_1 = \\ &= \int_0^t \{\textstyle\sum \exp[-\mathbf{K} + i(m + \vartheta)\,\mathbf{I}_n]\,t_1\mathbf{S}_m\mathbf{f}_1\}\,dt_1 + \\ &+ \int_0^t \{\textstyle\sum \exp[-\mathbf{K} + i(m - \vartheta)\,\mathbf{I}_n]\,t_1\,\mathbf{S}_m\mathbf{f}_2\}\,dt_1 = \end{aligned}$$

$$= \sum_{m=-\infty}^{\infty} \{\exp[-\mathbf{K}(\varepsilon)+i(m+\vartheta)\mathbf{I}_n]t-\mathbf{I}_n\}\{-\mathbf{K}(\varepsilon)+i(m+\vartheta)\mathbf{I}_n]^{-1}\mathbf{S}_m\mathbf{f}_1+$$

$$+\sum_{m=-\infty}^{\infty}\{\exp[-\mathbf{K}(\varepsilon)+i(m-\vartheta)\mathbf{I}_n]t-\mathbf{I}_n\}[-\mathbf{K}(\varepsilon)+i(m-\vartheta)\mathbf{I}_n]^{-1}\mathbf{S}_m\mathbf{f}_2.$$

By condition (c), the matrices $-\mathbf{K}(\varepsilon)+i(m\pm\vartheta)\mathbf{I}_n$ are nonsingular for sufficiently small $|\varepsilon|$. By (II. 1.17), the particular solution of system (4.61) defined by zero initial data is

$$\mathbf{x}_1(t)=\mathbf{F}(t,\varepsilon)\left\{\sum_{m=-\infty}^{\infty}[e^{i(m+\vartheta)t}\mathbf{I}_n-e^{t\mathbf{K}(\varepsilon)}][-\mathbf{K}(\varepsilon)+i(m+\vartheta)\mathbf{I}_n]^{-1}\mathbf{S}_m\mathbf{f}_1+\right.$$

$$\left.+\sum_{m=-\infty}^{\infty}[e^{i(m-\vartheta)t}\mathbf{I}_n-e^{t\mathbf{K}(\varepsilon)}][-\mathbf{K}(\varepsilon)+i(m-\vartheta)\mathbf{I}_n]^{-1}\mathbf{S}_m\mathbf{f}_2\right\}.$$

By condition (c),

$$\lim_{t\to+\infty} e^{t\mathbf{K}(\varepsilon)}=\mathbf{0}.$$

Denote

$$\left.\begin{aligned}\mathbf{x}_m^{(1)}(\vartheta)&=[-\mathbf{K}(\varepsilon)+i(m+\vartheta)\mathbf{I}_n]^{-1}\mathbf{S}_m\mathbf{f}_1,\\ \mathbf{x}_m^{(2)}(\vartheta)&=[-\mathbf{K}(\varepsilon)+i(m-\vartheta)\mathbf{I}_n]^{-1}\mathbf{S}_m\mathbf{f}_2\end{aligned}\right\} \tag{4.64}$$

and write $\mathbf{x}_1(t)$ as

$$\mathbf{x}_1(t)=\mathbf{F}(t,\varepsilon)\sum_{m=-\infty}^{\infty}[e^{i(m+\vartheta)t}\mathbf{x}_m^{(1)}(\vartheta)+e^{-i(m-\vartheta)t}\mathbf{x}_m^{(2)}(\vartheta)]+\mathbf{w}_1(t), \tag{4.65}$$

where

$$\lim_{t\to+\infty}\mathbf{w}_1(t)=0.$$

The steps of the procedure may be summarized as follows.

1) Express the matrizant of the equation

$$\frac{d\mathbf{x}}{dt}=\mathbf{B}(t,\varepsilon)\,\mathbf{x}$$

in the form of (4.62).

2) Evaluate the Fourier coefficients of (4.63).

3) Using the equations

$$-\mathbf{K}(\varepsilon)\mathbf{x}_m^{(1)}+i(m+\vartheta)\mathbf{x}_m^{(1)}=\mathbf{S}_m\mathbf{f}_1,\quad -\mathbf{K}(\varepsilon)\mathbf{x}_m^{(2)}+i(m-\vartheta)\mathbf{x}_m^{(2)}=\mathbf{S}_m\mathbf{f}_2$$

determine $\mathbf{x}_m^{(1)}$ and $\mathbf{x}_m^{(2)}$ for all harmonics of the matrix-function $\mathbf{F}(t,\varepsilon)^{-1}$.

4) If

$$\mathbf{f}(t)=\mathbf{f}^{(1)}(t)+\mathbf{f}^{(2)}(t)+\ldots,$$

where each $\mathbf{f}^{(j)}(t)$ has the form specified above, then the particular solution of system (4.61) defined by zero initial data is

$$\mathbf{x}_1(t)=\mathbf{x}^{(1)}(t)+\mathbf{x}^{(2)}(t)+\ldots,$$

where each $\mathbf{x}^{(j)}(t)$ is defined by (4.65).

§5. COMPUTATION OF EXPONENT MATRIX $\mathbf{K}(\varepsilon)$ AND MATRIZANT FOR SYSTEMS AND EQUATIONS OF SPECIAL TYPES

5.1. Third computational procedure

Let us review the results for the general system

$$\frac{d\mathbf{x}}{dt}=[\mathbf{C}+\varepsilon\mathbf{B}_1(t)+\varepsilon^2\mathbf{B}_2(t)+\ldots]\,\mathbf{x} \tag{0.1}$$

with real coefficients:

1. Check whether system (0.1) is in the singular or the nonsingular case (§4.1).
2. In the singular case, the substitution

$$\mathbf{x}=e^{t\mathbf{C}_0}\mathbf{y},$$

where $\mathbf{C}_0$ is determined as in §§4.2 and 4.3, reduces system (0.1) to

$$\frac{d\mathbf{y}}{dt}=[\mathbf{K}_0+\varepsilon\mathbf{D}_1(t)+\varepsilon^2\mathbf{D}_2(t)+\ldots]\,\mathbf{y}, \tag{4.9}$$

whose coefficients are defined by (4.10). We shall assume below that we are dealing with a system (4.9) in the nonsingular case.

We shall look for a fundamental matrix of (4.9) in the form*

$$\tilde{\mathbf{Y}}(t,\varepsilon)=\tilde{\mathbf{F}}(t,\varepsilon)\,e^{t\tilde{\mathbf{K}}(\varepsilon)},$$

where

$$\tilde{\mathbf{F}}(t,\varepsilon)=\mathbf{I}_n+\varepsilon\tilde{\mathbf{F}}_1(t)+\varepsilon^2\tilde{\mathbf{F}}_2(t)+\ldots,\quad \tilde{\mathbf{K}}(\varepsilon)=\mathbf{K}_0+\varepsilon\tilde{\mathbf{K}}_1+\varepsilon^2\tilde{\mathbf{K}}_2+\ldots$$

The problem is to determine the matrices $\tilde{\mathbf{K}}_1, \tilde{\mathbf{K}}_2, \ldots; \tilde{\mathbf{F}}_1(t), \tilde{\mathbf{F}}_2(t), \ldots$.

3. The matrix $\tilde{\mathbf{K}}_1$ is given by

$$\tilde{\mathbf{K}}_1=[\mathbf{D}_1(t)]_{\mathrm{av}}.$$

4. Expand $\mathbf{D}_1(t)$ in Fourier series:

$$\mathbf{D}_1(t)\sim\sum\exp\left(im\frac{2\pi}{T}t\right)\mathbf{D}_1^{(m)}.$$

* Fundamental matrices $\tilde{\mathbf{X}}(t,\varepsilon)$ of equation (0.1) and $\tilde{\mathbf{Y}}(t,\varepsilon)$ of equation (4.9) satisfy the identity $\tilde{\mathbf{X}}(t,\varepsilon)=\exp(t\mathbf{C}_0)\,\tilde{\mathbf{Y}}(t,\varepsilon)$.

$\mathbf{W}=\mathbf{R}_m(\mathbf{D}_1^{(m)})$ denotes the (unique) solution of the equation

$$m\frac{2\pi i}{T}\mathbf{W}=\mathbf{K}_0\mathbf{W}-\mathbf{W}\mathbf{K}_0+\mathbf{D}_1^{(m)}. \tag{5.1}$$

The matrix $\tilde{\mathbf{K}}_2$ is given by

$$\tilde{\mathbf{K}}_2=[\mathbf{D}_2(t)]_{\mathrm{cp}}+2\,\mathrm{Re}\sum_{m=1}^{\infty}\overline{\mathbf{D}_1^{(m)}}\mathbf{R}_m(\mathbf{D}_1^{(m)}). \tag{5.2}$$

5. One now evaluates $\tilde{\mathbf{F}}_1(t)$, $\tilde{\mathbf{F}}_2(t)$, $\Phi^{(m)}$ from (4.50) and $\tilde{\mathbf{K}}_3$ from (4.51), etc. However, if one is interested only in questions of stability, or, more commonly, the type of problem which can be solved on the basis of $\tilde{\mathbf{K}}(\varepsilon)$ alone (order of increase, approach to zero), one can halt at step 3, taking as a first approximation

$$\tilde{\mathbf{K}}(\varepsilon)\approx\mathbf{K}_0+\varepsilon\tilde{\mathbf{K}}_1,$$

or at step 4, with the second approximation

$$\tilde{\mathbf{K}}(\varepsilon)\approx\mathbf{K}_0+\varepsilon\tilde{\mathbf{K}}_1+\varepsilon^2\tilde{\mathbf{K}}_2.$$

The stability and instability domains in the parameter space are usually determined by applying the Routh-Hurwitz criterion to the characteristic equation of the matrix $\tilde{\mathbf{K}}(\varepsilon)$ (in some approximation):

$$\det[\tilde{\mathbf{K}}(\varepsilon)-\varkappa\mathbf{I}_n]=0.$$

It is worth noting that this method is effective when the coefficients of system (0.1) depend on parameters. On the other hand, if they are specified in numerical form, it is simpler to use standard methods to compute the monodromy matrix. Computational procedures simpler than our first and second methods were described in §4.5; these are sometimes more convenient to use.

5.2. Hill's equation; the case $c^2=0$

Hill's equation

$$\ddot{\xi}+[c^2+\varepsilon f(t)]\xi=0 \qquad (f(t+T)\equiv f(t))$$

is important in applications. We shall first give a detailed account of the case $c^2=0$. In this case we write the equation as

$$\ddot{\xi}-\varepsilon^2\varphi(t)\xi=0, \tag{5.3}$$

where $\varphi(t)$ is an integrable and piecewise-continuous (§ II. 1.1) T-periodic function, not necessarily of fixed sign. Let $\mathbf{x}$ be the vector with components ξ and $\dot{\xi}/\varepsilon$; we may then write equation (5.3) as a system

$$\frac{dx}{dt}=\varepsilon\mathbf{A}(t)\,\mathbf{x} \qquad \left(\mathbf{x}=\begin{pmatrix}\xi\\ \dot{\xi}/\varepsilon\end{pmatrix},\ \mathbf{A}(t)=\begin{Vmatrix}0 & 1\\ \varphi(t) & 0\end{Vmatrix}\right). \tag{5.4}$$

By the Floquet-Lyapunov theorem (§4.4) we have the following expression for the matrizant $\mathbf{X}(t,\varepsilon)$ $(\mathbf{X}(0,\varepsilon)\equiv\mathbf{I}_2)$:

$$\mathbf{X}(t,\varepsilon)=\mathbf{F}(t,\varepsilon)\,e^{t\mathbf{K}(\varepsilon)} \qquad (\mathbf{F}(t+T)\equiv\mathbf{F}(t)) \tag{5.5}$$

and by (5.4) this gives a vector differential equation for $\mathbf{F}(t)$:

$$\frac{d\mathbf{F}}{dt}=\varepsilon\mathbf{A}(t)\,\mathbf{F}-\mathbf{F}\mathbf{K}. \tag{5.6}$$

Assuming now that

$$\mathbf{F}(t,\varepsilon)=\sum_{n=0}^{\infty}\varepsilon^n\mathbf{F}_n(t), \quad \mathbf{K}(\varepsilon)=\sum_{n=0}^{\infty}\varepsilon^n\mathbf{K}_n,$$

we note that $\mathbf{X}(t,0)=\mathbf{I}_2$, i. e., $\mathbf{F}_0(t)=\mathbf{I}_2$, $\mathbf{K}_0=\mathbf{0}$. Inserting this in (5.4), and proceeding as at the beginning of §4.5, we obtain

$$\frac{d\mathbf{F}_1}{dt}=\mathbf{A}-\mathbf{K}_1, \quad \frac{d\mathbf{F}_2}{dt}=\mathbf{A}\mathbf{F}_1-\mathbf{F}_1\mathbf{K}_1-\mathbf{K}_2,$$

$$\frac{d\mathbf{F}_3}{dt}=\mathbf{A}\mathbf{F}_2-\mathbf{F}_2\mathbf{K}_1-\mathbf{F}_1\mathbf{K}_2-\mathbf{K}_3, \quad \frac{d\mathbf{F}_4}{dt}=\mathbf{A}\mathbf{F}_3-\mathbf{F}_3\mathbf{K}_1-\mathbf{F}_2\mathbf{K}_2-\mathbf{F}_1\mathbf{K}_3-\mathbf{K}_4, \ \ldots$$

In accordance with our first computational procedure (§4.5), since the functions $\mathbf{F}_j(t)$ are T-periodic, we have

$$\mathbf{K}_1=\mathbf{A}_{av}\equiv\frac{1}{T}\int_0^T\mathbf{A}(t)\,dt, \quad \mathbf{F}_1(t)=\int_0^t[\mathbf{A}(t_1)-\mathbf{K}_1]\,dt_1,$$

$$\mathbf{K}_2=[\mathbf{A}\mathbf{F}_1-\mathbf{F}_1\mathbf{K}_1]_{av}, \quad \mathbf{F}_2(t)=\int_0^t[\mathbf{A}\mathbf{F}_1-\mathbf{F}_1\mathbf{K}_1)-\mathbf{K}_2]\,dt,$$

$$\mathbf{K}_3=[\mathbf{A}\mathbf{F}_2-\mathbf{F}_2\mathbf{K}_1-\mathbf{F}_1\mathbf{K}_2]_{av}, \quad \mathbf{F}_3(t)=\int_0^t[(\mathbf{A}\mathbf{F}_2-\mathbf{F}_2\mathbf{K}_1-\mathbf{F}_1\mathbf{K}_2)-\mathbf{K}_3]\,dt,$$

$$\mathbf{K}_4=[\mathbf{A}\mathbf{F}_3-\mathbf{F}_3\mathbf{K}_1-\mathbf{F}_2\mathbf{K}_2-\mathbf{F}_1\mathbf{K}_3]_{av},$$

$$\mathbf{F}_4(t)=\int_0^t[(\mathbf{A}\mathbf{F}_3-\mathbf{F}_3\mathbf{K}_1-\mathbf{F}_2\mathbf{K}_2-\mathbf{F}_1\mathbf{K}_3)-\mathbf{K}_4]\,dt.$$

We shall use the notation

$$\left.\begin{aligned}&\alpha_0=\varphi_{av}, && \varphi_1(t)=\int_0^t[\varphi(t_1)-\alpha_0]\,dt_1,\\ &\alpha_1=[\varphi_1]_{av}, && \varphi_2(t)=\int_0^t[\varphi_1(t_1)-\alpha_1]\,dt_1,\end{aligned}\right\} \tag{5.7}$$

$$\left.\begin{aligned}
&\alpha_2=[\varphi_2]_{av}, && \varphi_3(t)=\int_0^t[\varphi_2(t_1)-\alpha_2]\,dt_1, \quad \alpha_3=[\varphi_3]_{av},\\
&\beta_0=[\varphi\varphi_2]_{av}, && \psi_0(t)=\int_0^t[\varphi(t_1)\varphi_2(t_1)-\beta_0]\,dt_1,\\
&\delta_0=[\varphi_1^2(t)]_{av}-\alpha_1^2\geqslant 0, && \gamma_0=\beta_0+\alpha_0\alpha_2-\alpha_1^2=2\alpha_0\alpha_2-\delta_0-\alpha_1^2,\\
&\psi_1=[\psi_0]_{av}+3\alpha_0\alpha_3-2\alpha_1\alpha_2, && \psi(t)=\psi_0(t)+\alpha_0\varphi_3(t)-\alpha_1\varphi_2(t).
\end{aligned}\right\} \tag{5.7}$$

Carrying out the relevant computations, we obtain

$$\mathbf{F}(t,\varepsilon)=\begin{Vmatrix} 1+\varphi_2(t)\varepsilon^2+O(\varepsilon^4) & -2\varphi_3(t)\varepsilon^3+O(\varepsilon^5)\\ \varphi_1(t)\varepsilon+\psi_1(t)\varepsilon^3+O(\varepsilon^5) & 1-\varphi_2(t)\varepsilon^2+O(\varepsilon^4)\end{Vmatrix},$$

$$\mathbf{K}(\varepsilon)=\begin{Vmatrix} \alpha_1\varepsilon^2+\gamma_1\varepsilon^4+O(\varepsilon^6) & \varepsilon-2\alpha_2\varepsilon^3+O(\varepsilon^5)\\ \alpha_0\varepsilon+\gamma_0\varepsilon^3+O(\varepsilon^5) & -\alpha_1\varepsilon^2+\gamma_2\varepsilon^4+O(\varepsilon^6)\end{Vmatrix}.$$

Let $\varkappa(\varepsilon)$ and $-\varkappa(\varepsilon)$ denote the characteristic exponents of equation (5.3), i. e., the eigenvalues of the matrix

$$\det[\mathbf{K}(\varepsilon)-\varkappa\mathbf{I}_2]=0. \tag{5.8}$$

By formula (I. 2.18),

$$e^{t\mathbf{K}(\varepsilon)}=\operatorname{ch}[\varkappa(\varepsilon)t]\,\mathbf{I}_2+\frac{1}{\varkappa(\varepsilon)}\operatorname{sh}[\varkappa(\varepsilon)t]\,\mathbf{K}(\varepsilon).$$

Evaluating the roots of equation (5.8), we obtain*

$$\varkappa=\varkappa(\varepsilon)=\sqrt{\alpha_0}\,\varepsilon-\frac{\delta_0}{2\sqrt{\alpha_0}}\varepsilon^3-\frac{8\alpha_0\alpha_2\gamma_0-\delta_0^2}{8\alpha_0\sqrt{\alpha_0}}\varepsilon^5+O(\varepsilon^7).$$

After some manipulations (omitted here), we obtain an expression for the matrizant (5.5):

$$\mathbf{X}(t,\varepsilon)=\mathbf{X}_0(t,\varepsilon)+\varepsilon\mathbf{X}_1(t,\varepsilon)+\varepsilon^2\mathbf{X}_2(t,\varepsilon)+\varepsilon^3\mathbf{X}_3(t,\varepsilon)+\ldots,$$

where

$$\mathbf{X}_0=\begin{Vmatrix} \operatorname{ch}\varkappa t & \frac{1}{\sqrt{\alpha_0}}\operatorname{sh}\varkappa t\\ \sqrt{\alpha_0}\operatorname{sh}\varkappa t & \operatorname{ch}\varkappa t\end{Vmatrix},\quad \mathbf{X}_1=\begin{Vmatrix} \frac{\alpha_1}{\sqrt{\alpha_0}}\operatorname{sh}\varkappa t & 0\\ \varphi_1(t)\operatorname{ch}\varkappa t & -\frac{\alpha_1}{\sqrt{\alpha_0}}\operatorname{sh}\varkappa t+\frac{\varphi_1(t)}{\sqrt{\alpha_0}}\operatorname{sh}\varkappa t\end{Vmatrix},$$

$$\mathbf{X}_2=\begin{Vmatrix} \varphi_2(t)\operatorname{ch}\varkappa t & \frac{1}{\sqrt{\alpha_0}}\left[\frac{\delta_0}{2\alpha_0}-2\alpha_2+\varphi_2(t)\right]\operatorname{sh}\varkappa t\\ \frac{1}{\sqrt{\alpha_0}}\left[\gamma_0+\frac{1}{2}\delta_0+\alpha_1\varphi_1(t)-\alpha_0\varphi_2(t)\right]\operatorname{sh}\varkappa t & -\varphi_2(t)\operatorname{ch}\varkappa t\end{Vmatrix},$$

$$\mathbf{X}_3=\begin{Vmatrix} \frac{1}{\sqrt{\alpha_0}}\left[\gamma_1+\frac{\delta_0\alpha_1}{2\alpha_0}+\alpha_1\varphi_2(t)-2\alpha_0\varphi_3(t)\right]\operatorname{sh}\varkappa t & -2\varphi_3(t)\operatorname{ch}\varkappa t\\ \psi_1(t)\operatorname{ch}\varkappa t & \frac{1}{\sqrt{\alpha_0}}\left[\gamma_2-\frac{\delta_0\alpha_1}{2\alpha_0}-2\alpha_2\varphi_1+\frac{\delta_0\varphi_1}{2\alpha_2}+\alpha_1\varphi_2-\psi_1\right]\operatorname{sh}\varkappa t\end{Vmatrix}.$$

* The number α_0 may have any sign (if $\varphi(t)$ is a real function) and it may be complex (if $\varphi(t)$ is a complex-valued function).

Note that we have deliberately not expressed $\operatorname{ch} \varkappa(\varepsilon) t$ and $\operatorname{sh} \varkappa(\varepsilon) t$ as power series in ε.

The matrizant $\mathbf{X}(t, \varepsilon)$ may be written

$$\mathbf{X}(t, \varepsilon)=\begin{Vmatrix} x_{11}(t, \varepsilon) & x_{12}(t, \varepsilon) \\ x_{21}(t, \varepsilon) & x_{22}(t, \varepsilon) \end{Vmatrix}.$$

The solution $u(t, \varepsilon)$ of equation (5.3) satisfying the initial conditions $u(0, \varepsilon)=1$, $\dot{u}(0, \varepsilon)=0$ is defined as

$$u(t, \varepsilon)=x_{11}(t, \varepsilon), \quad \dot{u}(t, \varepsilon)=\varepsilon x_{21}(t, \varepsilon),$$

and the solution $v(t, \varepsilon)$ with initial conditions $v(0, \varepsilon)=0$, $\dot{v}(0, \varepsilon)=1$ is

$$v(t, \varepsilon)=x_{12}(t, \varepsilon), \quad \dot{v}(t, \varepsilon)=\varepsilon x_{22}(t, \varepsilon).$$

(The dot signifies differentiation with respect to t.)

These fundamental solutions, in expanded notation, are

$$u(t, \varepsilon)=\operatorname{ch} \varkappa t+\varepsilon \frac{\alpha_1}{\sqrt{\alpha_0}} \operatorname{sh} \varkappa t+\varepsilon^2 \varphi_2(t) \operatorname{ch} \varkappa t+$$
$$+\varepsilon^3 \frac{1}{\sqrt{\alpha_0}}\left[\gamma_1+\frac{\delta_0 \alpha_1}{2 \alpha_0}+\alpha_1 \varphi_2(t)-2 \alpha_0 \varphi_3(t)\right] \operatorname{sh} \varkappa t+O\left(\varepsilon^4\right),$$

$$\dot{u}(t, \varepsilon)=\varepsilon \sqrt{\alpha_0} \operatorname{sh} \varkappa t+\varepsilon^2 \varphi_1(t) \operatorname{ch} \varkappa t+$$
$$+\varepsilon^3 \frac{1}{\sqrt{\alpha_0}}\left[\gamma_0+\frac{1}{2} \delta_0+\alpha_1 \varphi_1(t)-\alpha_0 \varphi_2(t)\right] \operatorname{sh} \varkappa t+\varepsilon^4 \psi_1(t) \operatorname{ch} \varkappa t+O\left(\varepsilon^5\right),$$

$$v(t, \varepsilon)=\frac{1}{\sqrt{\alpha_0}} \operatorname{sh} \varkappa t-\varepsilon^2 \frac{1}{\sqrt{\alpha_0}}\left[\frac{\delta_0}{2 \alpha_0}-2 \alpha_2+\varphi_2(t)\right] \operatorname{sh} \varkappa t-2 \varepsilon^3 \varphi_3(t) \operatorname{ch} \varkappa t+O\left(\varepsilon^4\right),$$

$$\dot{v}(t, \varepsilon)=\varepsilon \operatorname{ch} \varkappa t+\varepsilon^2 \frac{1}{\sqrt{\alpha_0}}\left[-\alpha_1+\varphi_1(t)\right] \operatorname{sh} \varkappa t-\varepsilon^3 \varphi_2(t) \operatorname{ch} \varkappa t+$$
$$+\varepsilon^4 \frac{1}{\sqrt{\alpha_0}}\left[\gamma_2-\frac{\delta_0 \alpha_1}{2 \alpha_0}-2 \alpha_2 \varphi_1(t)+\frac{\delta_0}{2 \alpha_2} \varphi_1(t)+\alpha_1 \varphi_2(t)+\psi_1(t)\right] \operatorname{sh} \varkappa t+O\left(\varepsilon^5\right).$$

A direct check shows that

$$\ddot{u}-\varepsilon^2 \varphi(t) u=O\left(\varepsilon^5\right), \quad \ddot{v}-\varepsilon^2 \varphi(t) v=O\left(\varepsilon^5\right).$$

5.3. Hill's equation ($c^2>0$) and Mathieu's equation. Nonsingular case

Let us return to the general case of Hill's equation with a small real parameter:

$$\ddot{\xi}+\left[c^2+\varepsilon f(t)\right] \xi=0 \qquad (c>0,\ f(t+T) \equiv f(t)), \tag{5.9}$$

where $f(t)$ is an integrable piecewise-continuous T-periodic function. We write the equation as a system

$$\frac{d\mathbf{x}}{dt}=[\mathbf{C}+\varepsilon\mathbf{B}_1(t)]\,\mathbf{x}\quad\left(\mathbf{x}=\begin{pmatrix}\xi\\ \dot\xi\end{pmatrix},\quad \mathbf{C}=\begin{Vmatrix}0 & 1\\ -c^2 & 0\end{Vmatrix},\quad \mathbf{B}_1(t)=\begin{Vmatrix}0 & 0\\ -f(t) & 0\end{Vmatrix}\right).$$

The eigenvalues of the matrix $\mathbf{C}$ are $\lambda_1=-ci$, $\lambda_2=+ci$. They are incongruent modulo $2\pi i/T$ if $c^2\neq\frac{\pi^2p^2}{T^2}$, i. e., $ci-(-ci)\neq\frac{2\pi i}{T}p$ (where p is an integer), which we shall indeed assume throughout this subsection; we are thus in the non-singular case. Our third computational procedure (§4.6) then gives

$$\mathbf{K}_0=\mathbf{C},\quad \tilde{\mathbf{K}}_1=[\mathbf{B}_1(t)]_{\mathrm{av}}=-f_{\mathrm{av}}\begin{Vmatrix}0 & 0\\ 1 & 0\end{Vmatrix}.$$

In order to evaluate $\tilde{\mathbf{K}}_2$, $\tilde{\mathbf{K}}_3$, ... we must prepare a matrix solution of (4.44). In expanded notation, this equation is now

$$\begin{aligned}
m\tfrac{2\pi i}{T}w_{11}\quad -c^2w_{12}\quad\quad -w_{21}\quad\quad &=g_{11},\\
w_{11}+m\tfrac{2\pi i}{T}w_{12}\quad\quad\quad -w_{22}\quad &=g_{12},\\
c^2w_{11}\quad\quad +m\tfrac{2\pi i}{T}w_{21}\quad -c^2w_{22}\quad &=g_{21},\\
c^2w_{12}\quad\quad +w_{21}+m\tfrac{2\pi i}{T}w_{22}\quad &=g_{22},
\end{aligned}$$

where w_{ij} and g_{ij} are the elements of the matrices $\mathbf{W}$ and $\mathbf{G}$. The solution $\mathbf{W}=\mathbf{R}_m(\mathbf{G})$ of equation (4.44) is defined by

$$\begin{aligned}
w_{11}(\mathbf{G})&=\frac{1}{4\left(c^2-m^2\frac{\pi^2}{T^2}\right)}\left[\frac{Ti}{m\pi}\left(2m^2\frac{\pi^2}{T^2}-c^2\right)g_{11}+c^2g_{12}+g_{21}-\frac{Ti}{m\pi}c^2g_{22}\right],\\
w_{12}(\mathbf{G})&=\frac{1}{4\left(c^2-m^2\frac{\pi^2}{T^2}\right)}\left[-g_{11}+\frac{Ti}{m\pi}\left(2m^2\frac{\pi^2}{T^2}-c^2\right)g_{12}+\frac{Ti}{m\pi}g_{21}+g_{22}\right],\\
w_{21}(\mathbf{G})&=\frac{1}{4\left(c^2-m^2\frac{\pi^2}{T^2}\right)}\left[-c^2g_{11}+\frac{Ti}{m\pi}c^4g_{12}+\frac{Ti}{m\pi}\left(2m^2\frac{\pi^2}{T^2}-c^2\right)g_{21}+c^2g_{22}\right],\\
w_{22}(\mathbf{G})&=\frac{1}{4\left(c^2-m^2\frac{\pi^2}{T^2}\right)}\left[-\frac{Ti}{m\pi}c^2g_{11}-c^2g_{12}-g_{21}+\frac{Ti}{m\pi}\left(2m^2\frac{\pi^2}{T^2}-c^2\right)g_{22}\right].
\end{aligned}$$

Before applying formula (4.50), we must compute

$$\overline{\mathbf{D}}_1^{(m)}\mathbf{R}_m(\mathbf{D}_1^{(m)})=\begin{Vmatrix}0 & 0\\ -\bar f^{(m)} & 0\end{Vmatrix}\begin{Vmatrix}w_{11} & w_{12}\\ w_{21} & w_{22}\end{Vmatrix}=\begin{Vmatrix}0 & 0\\ -\bar f^{(m)}w_{11}(\mathbf{D}_1^{(m)}) & -\bar f^{(m)}w_{12}(\mathbf{D}_1^{(m)})\end{Vmatrix}=$$
$$=\frac{1}{4\left(c^2-m^2\frac{\pi^2}{T^2}\right)}\begin{Vmatrix}0 & 0\\ \bar f^{(m)}f^{(m)} & \frac{Ti}{m\pi}\bar f^{(m)}f^{(m)}\end{Vmatrix}$$

and then we find that

$$\tilde{\mathbf{K}}_2=\frac{1}{2}\sum_{m=1}^{\infty}\frac{\bar f^{(m)}f^{(m)}}{c^2-m^2\frac{\pi^2}{T^2}}\begin{Vmatrix}0 & 0\\ 1 & 0\end{Vmatrix},$$

where $f^{(m)}$ denotes the m-th Fourier coefficient of $f(t)$ (in complex form). Finally, we evaluate

$$\Phi^{(m)} = \frac{f_{\mathrm{av}} f^{(m)}}{4\left(c^2 - m^2\frac{\pi^2}{T^2}\right)}\begin{Vmatrix} -\frac{Ti}{m\pi} & 0 \\ 1 & 0 \end{Vmatrix} + \frac{1}{4}\sum_{j+l=m}\frac{f^{(l)}f^{(j)}}{c^2 - j^2\frac{\pi^2}{T^2}}\begin{Vmatrix} 0 & 0 \\ 1 & \frac{Ti}{j\pi} \end{Vmatrix}$$

and then

$$\tilde{\mathbf{K}}_3 = 2\mathrm{Re}\sum_{m=1}^{\infty}\overline{\mathbf{D}}_1^{(m)}\mathbf{R}_m(\Phi^{(m)}) =$$
$$= \frac{1}{8}\sum_{m=1}^{\infty}\left\{\frac{f^{(m)}\overline{f}^{(m)}f_{\mathrm{av}}}{\left(c^2 - m^2\frac{\pi^2}{T^2}\right)^2}\left(3 - \frac{T^2c^2}{m^2\pi^2}\right) + \mathrm{Re}\left[\overline{f}^{(m)}\sum_{j+l=1}^{m}\frac{f^{(l)}f^{(j)}}{c^2 - j^2\frac{\pi^2}{T^2}}\right]\right\}\begin{Vmatrix} 0 & 0 \\ 1 & 0 \end{Vmatrix}.$$

As a special case of Hill's equation, we have Mathieu's equation

$$\ddot{\xi} + [c^2 + \varepsilon\cos 2t]\xi = 0 \qquad (c^2 \neq p^2,\ p = 0, 1, 2, \ldots),$$

for which $T = \pi$, $f(t) = \cos 2t$, $f^{(1)} = f^{(-1)} = 1/2$ and all other complex Fourier coefficients (including f_{av}) vanish. A look at the formulas for $\tilde{\mathbf{K}}_1$ and $\tilde{\mathbf{K}}_3$ is sufficient to verify that they vanish. Moreover, we shall show below that $\tilde{\mathbf{K}}_{2q-1} = 0$. Since $(\tilde{\mathbf{F}}_1)_{\mathrm{av}} = (\tilde{\mathbf{F}}_2)_{\mathrm{av}} = \ldots = 0$ and $\mathbf{D}_2(t) = \mathbf{D}_3(t) = \ldots = \mathbf{0}$, we see from (4.49) that (for Hill's equation in general)

$$\tilde{\mathbf{K}}_l = [\mathbf{D}_1\tilde{\mathbf{F}}_{l-1}]_{\mathrm{av}},$$

so that, in particular, for Mathieu's equation,

$$\tilde{\mathbf{K}}_l = 2\,\mathrm{Re}\,(\mathbf{D}_1^{(1)}\tilde{\mathbf{F}}_{l-1}^{(1)}) \qquad (l = 2, 3, \ldots).$$

Thus it follows from (4.30) and (4.45) that

$$\tilde{\mathbf{F}}_{l-1}^{(1)} = \mathbf{R}_1(-\tilde{\mathbf{F}}_{l-2}^{(1)}\tilde{\mathbf{K}}_1 - \ldots - \tilde{\mathbf{F}}_1^{(1)}\tilde{\mathbf{K}}_{l-2}) \qquad (l = 2, 3, \ldots),$$

since

$$(\mathbf{D}_1\tilde{\mathbf{F}}_{l-2})^{(1)} = 0.$$

Since $\tilde{\mathbf{K}}_1 = 0$, it follows that $\tilde{F}_2^{(1)} = 0$, and we see further that $\tilde{\mathbf{K}}_3 = 0$, $\tilde{\mathbf{F}}_4^{(1)} = 0$; by induction,

$$\tilde{\mathbf{F}}_{2q}^{(1)} = 0, \quad \tilde{\mathbf{K}}_{2q-1} = 0 \qquad (q = 1, 2, \ldots).$$

The above formulas yield (remember that we are considering the nonsingular case, $(c^2 \neq p^2)$

$$\tilde{\mathbf{K}}_2 = \frac{1}{8(c^2-1)}\begin{Vmatrix} 0 & 0 \\ 1 & 0 \end{Vmatrix}, \quad \tilde{\mathbf{K}}_4 = \frac{13c^2 - 25}{512(c^2-1)^3(c^2-4)}\begin{Vmatrix} 0 & 0 \\ 1 & 0 \end{Vmatrix},$$

$$\tilde{\mathbf{K}}_6 = \frac{-9c^8 - 467c^6 + 6910c^4 - 25\,322c^2 + 28\,144}{32\,768(c^2-1)^5(c^2-4)^2(c^2-9)}\begin{Vmatrix} 0 & 0 \\ 1 & 0 \end{Vmatrix}.$$

5.4. Hill's equation ($c^2 > 0$) and Mathieu's equation. Singular case

We first consider Hill's equation (5.9) when c^2 is either equal to $p^2\pi^2/T^2$ or close to this number:

$$c^2 = \frac{p^2\pi^2}{T^2} + \mu \qquad (p = 1, 2, \ldots),$$

where μ is a small real number or zero. Assuming that μ is of the order of ε, we set $\mu = \varepsilon\mu_0$. In this "almost-resonant" case, which includes the singular case ("resonance" — $\mu = 0$) as a particular case, Hill's equation

$$\ddot{\xi} + \left\{\frac{p^2\pi^2}{T^2} + \varepsilon\left[\mu_0 + f(t)\right]\right\}\xi = 0 \qquad (f(t+T) = f(t))$$

may be written in vector notation as

$$\frac{d\mathbf{x}}{dt} = [\mathbf{C} + \varepsilon\mathbf{B}_1(t)]\,\mathbf{x}\left(\mathbf{x} = \begin{pmatrix}\xi\\ \dot{\xi}\end{pmatrix},\quad \mathbf{C} = \begin{Vmatrix} 0 & 1 \\ -\frac{p^2\pi^2}{T^2} & 0\end{Vmatrix},\quad \mathbf{B}_1(t) = \begin{Vmatrix} 0 & 0 \\ -\mu_0 - f(t) & 0\end{Vmatrix}\right).$$

The eigenvalues of $\mathbf{C}$ are

$$\lambda_1 = -\frac{2\pi i}{T}\frac{p}{2}, \quad \lambda_2 = \frac{2\pi i}{T}\frac{p}{2},$$

and they are congruent modulo $2\pi i/T$.

First let p be even: $p = 2q$ $(q = 1, 2, \ldots)$. We may then take $\alpha = 0$, $m_1 = -q$, $m_2 = q$. Proceeding as in §4.1, we reduce $\mathbf{C}$ to Jordan form

$$\mathbf{C} = \mathbf{SQS}^{-1}, \quad \mathbf{S} = \begin{Vmatrix} 1 & 1 \\ -\frac{p\pi}{T}i & \frac{p\pi}{T}i\end{Vmatrix}, \quad \mathbf{Q} = \begin{Vmatrix} -\frac{p\pi}{T}i & 0 \\ 0 & \frac{p\pi}{T}i\end{Vmatrix}$$

$$\mathbf{S}^{-1} = \frac{1}{2}\begin{Vmatrix} 1 & \frac{T}{p\pi}i \\ 1 & -\frac{T}{p\pi}i\end{Vmatrix}.$$

Since $\alpha = 0$, we have in this case $\mathbf{C}_0 = \mathbf{C}$, and so

$$\mathbf{K}_0 = \mathbf{0}.$$

We now calculate

$$e^{t\mathbf{C}_0} = \mathbf{S}\begin{Vmatrix} \exp\left(-\frac{p\pi}{T}it\right) & 0 \\ 0 & \exp\left(\frac{p\pi}{T}it\right)\end{Vmatrix}\mathbf{S}^{-1} = \begin{Vmatrix} \cos\frac{p\pi}{T}t & \frac{T}{p\pi}\sin\frac{p\pi}{T}t \\ -\frac{p\pi}{T}\sin\frac{p\pi}{T}t & \cos\frac{p\pi}{T}t\end{Vmatrix}$$

and find, by (4.10),

$$\mathbf{D}_1(t)=\begin{Vmatrix} \frac{\mu_0 T}{2p\pi}\sin\frac{2p\pi t}{T}+\frac{T}{2p\pi}f(t)\sin\frac{2p\pi t}{T} & \frac{\mu_0 T^2}{p^2\pi^2}\sin^2\frac{p\pi t}{T}+\frac{T^2}{p^2\pi^2}f(t)\sin^2\frac{p\pi t}{T} \\ -\mu_0\cos^2\frac{p\pi t}{T}-f(t)\cos^2\frac{p\pi t}{T} & -\frac{\mu_0 T}{2p\pi}\sin\frac{2p\pi t}{T}-\frac{T}{2p\pi}f(t)\sin\frac{2p\pi t}{T} \end{Vmatrix}.$$

As shown at the beginning of §4.5, when $\mathbf{K}_0=\mathbf{0}$ the computations are simpler, and according to the first procedure we have from (4.30)

$$\mathbf{K}_1=[\mathbf{D}_1(t)]_{av},\quad \mathbf{K}_2=[\mathbf{D}_1(t)\mathbf{F}_1(t)]_{av}-[\mathbf{F}_1(t)]_{av}\mathbf{K}_1,$$

where

$$\mathbf{F}_1(t)=\int_0^t \mathbf{D}_1(t_1)\,dt_1-\mathbf{K}_1 t \qquad (\mathbf{F}_1(0)=\mathbf{0}).$$

Since we shall presently consider the case of an arbitrary (not necessarily even) natural number p, we confine ourselves here to the result for Mathieu's equation: $f(t)=\cos 2t$ $(T=\pi)$. For even p:

$$\mathbf{D}_1(t)=\begin{Vmatrix} d_{11}(t) & d_{12}(t) \\ d_{21}(t) & -d_{11}(t) \end{Vmatrix},\quad \mathbf{F}_1(t)=\begin{Vmatrix} f_{11}(t) & f_{12}(t) \\ f_{21}(t) & -f_{11}(t) \end{Vmatrix},$$

$$d_{11}=\frac{\mu_0}{2p}\sin 2pt+\frac{1}{4p}\sin(2p+2)t+\frac{1}{4p}\sin(2p-2)t,$$

$$d_{12}=\frac{\mu_0}{2p^2}-\frac{\mu_0}{2p^2}\cos 2pt+\frac{1}{2p^2}\cos 2t-\frac{1}{4p^2}\cos(2p-2)t-\frac{1}{4p^2}\cos(2p+2)t,$$

$$d_{21}=-\frac{\mu_0}{2}-\frac{\mu_0}{2}\cos 2pt-\frac{1}{2}\cos 2t-\frac{1}{4}\cos(2p-2)t-\frac{1}{4}\cos(2p+2)t,$$

$$f_{11}(t)=-\frac{\mu_0}{4p^2}\cos 2pt+\frac{\mu_0}{4p^2}-\frac{\cos(2p+2)t}{4p(2p+2)}+\frac{1}{4(p^2-1)}-\frac{\cos(2p-2)t}{4p(2p-2)},$$

$$f_{12}(t)=-\frac{\mu_0}{4p^3}\sin 2pt+\frac{1}{4p^2}\sin 2pt-\frac{1}{8p^2(p+1)}\sin(2p+2)t-\frac{\sin(2p-2)t}{8p^2(p-1)},$$

$$f_{21}(t)=-\frac{\mu_0}{4p}\sin 2pt-\frac{1}{4}\sin 2t-\frac{1}{8(p+1)}\sin(2p+2)t-\frac{1}{8(p-1)}\sin(2p-2)t.$$

By the previous formulas, in the case of Mathieu's equation with even p,

$$\mathbf{K}_1=\begin{Vmatrix} 0 & \frac{\mu_0}{2p^2} \\ -\frac{\mu_0}{2} & 0 \end{Vmatrix},$$

$$\mathbf{K}_2=\begin{Vmatrix} 0 & -\frac{3\mu_0^2}{8p^4}-\frac{\mu_0}{4p^2(p^2-1)}-\frac{1}{16p^2(p^2-1)} \\ -\frac{\mu_0^2}{8p^2}-\frac{\mu_0}{4(p^2-1)}+\frac{1}{16(p^2-1)} & 0 \end{Vmatrix}.$$

Now let us consider the "almost-resonant" case for any natural number p (not necessarily even). We begin with Hill's equation. We can no longer set $\alpha=0$, but take instead $\alpha=-p\pi i/T$, $m_1=0$, $m_2=p$. Referring to §4.1, we obtain, with the same matrix $\mathbf{S}$ as before,

$$\mathbf{C}_0 = \mathbf{S}\begin{Vmatrix} 0 & 0 \\ 0 & \frac{2p\pi}{T}i \end{Vmatrix}\mathbf{S}^{-1}, \quad \mathbf{K}_0 = \mathbf{C} - \mathbf{C}_0 = -\frac{pT}{\pi}i\mathbf{I}_2,$$

$$e^{t\mathbf{C}_0} = \mathbf{S}\begin{Vmatrix} 1 & 0 \\ 0 & \exp\frac{2p\pi}{T}it \end{Vmatrix}\mathbf{S}^{-1} = \frac{1}{2}\begin{Vmatrix} 1+\exp\frac{2p\pi it}{T} & \frac{T}{p\pi}i\left(1-\exp\frac{2p\pi it}{T}\right) \\ \frac{p\pi}{T}i\left(\exp\frac{2p\pi it}{T}-1\right) & 1+\exp\frac{2p\pi it}{T} \end{Vmatrix},$$

$$\mathbf{D}_1(t) = \begin{Vmatrix} d_{11}(t) & d_{12}(t) \\ d_{21}(t) & -d_{11}(t) \end{Vmatrix}, \quad d_{11}(t) = \frac{T}{2p\pi}[\mu_0 + f(t)]\sin\frac{2p\pi t}{T},$$

$$d_{12}(t) = \frac{T^2}{2p^2\pi^2}[\mu_0 + f(t)]\left(1-\cos\frac{2p\pi t}{T}\right), \quad d_{21}(t) = -\frac{1}{2}[\mu_0 + f(t)]\left(1+\cos\frac{2p\pi t}{T}\right).$$

We now use the second computational procedure (§4.5). The distinctive feature here is provided by the conditions $\mathbf{F}_l(0) = \mathbf{0}$ together with the conditions $\mathbf{F}_l^{(0)} = \mathbf{0}$ $(l = 1, 2, \ldots)$. For Hill's equation

$$\mathbf{K}_1 = [\mathbf{D}_1(t)]_{\text{av}} = \frac{T}{4p\pi}\begin{Vmatrix} b_p & \frac{T}{p\pi}(2f_{\text{av}} + 2\mu_0 - a_p) \\ -\frac{p\pi}{T}(2f_{\text{av}} + 2\mu_0 + a_p) & -b_p \end{Vmatrix},$$

where a_p and b_p denote the Fourier coefficients of the function $f(t)$:

$$a_p = \frac{2}{T}\int_0^T f(t)\cos\frac{2p\pi t}{T}dt, \quad b_p = \frac{2}{T}\int_0^T f(t)\sin\frac{2p\pi t}{T}dt \qquad (p = 1, 2, \ldots).$$

Again for Hill's equation,

$$\mathbf{F}_1(t) = \frac{T^2}{8p\pi^2}\sum_{k\neq 0}\frac{1}{k}\left(\exp\frac{2k\pi it}{T}-1\right)\times$$

$$\times\begin{Vmatrix} \beta^{(k+p)}-\beta^{(k-p)} & -\frac{T}{p\pi i}(\beta^{(k+p)}+\beta^{(k-p)}-2\beta^{(k)}) \\ \frac{p\pi i}{T}(\beta^{(k+p)}+\beta^{(k-p)}+2\beta^{(k)}) & \beta^{(k-p)}-\beta^{(k+p)} \end{Vmatrix},$$

where $\beta^{(l)} = f^{(l)}$ for $l \neq 0$ and $\beta^{(0)} = \mu_0 + f^{(0)}$, $f^{(0)} \equiv f_{\text{av}}$, $f^{(l)}$ are the complex Fourier coefficients of $f(t)$. As before, we denote $a_p = 2\operatorname{Re}\beta^{(p)}$, $b_p = -2\operatorname{Im}\beta^{(p)}$, $a_0 = 2\beta^{(0)}$, and obtain

$$\mathbf{K}_2 = \frac{T^3}{32p^2\pi^3}\sum_{m=1}^{\infty}\frac{1}{m}\begin{Vmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{Vmatrix},$$

$$k_{11} = 2[(b_{m+p}+b_{m-p})(a_m - a_0) - b_m(a_{m+p}+a_{m-p}-2a_p)],$$

$$k_{12} = \frac{T}{p\pi}\{(a_{m+p}-a_{m-p})[a_{m+p}+a_{m-p}-2(a_m+a_p-a_0)] + (b_{m+p}+b_{m-p}-2b_m)(b_{m+p}-b_{m-p}-2b_p)\},$$

$$k_{21} = -\frac{p\pi}{T}\{(a_{m+p}-a_{m-p})[a_{m+p}+a_{m-p}+2(a_m-a_p-a_0)] + (b_{m+p}+b_{m-p}+2b_m)(b_{m+p}-b_{m-p}-2b_p)\},$$

$$k_{22} = 2[(b_{m+p}+b_{m-p})(a_0 - a_m) + b_m(a_{m+p}+a_{m-p}-2a_p)].$$

For Mathieu's equation ($f(t)=\cos 2t$; $T=\pi$), we have

$$\mathbf{D}_1(t)=\begin{Vmatrix}\delta_{11}(t) & \delta_{12}(t)\\ \delta_{21}(t) & -\delta_{11}(t)\end{Vmatrix},$$

$$\mathbf{F}(t)=\begin{Vmatrix}\varphi'_{11}(t) & \varphi'_{12}(t)\\ \varphi'_{21}(t) & -\varphi'_{11}(t)\end{Vmatrix}\ (p=1),\quad \mathbf{F}(t)=\begin{Vmatrix}\varphi_{11}(t) & \varphi_{12}(t)\\ \varphi_{21}(t) & -\varphi_{11}(t)\end{Vmatrix}\ (p>1);$$

$$\delta_{11}(t)=\frac{\mu_0}{2p}\sin 2pt+\frac{1}{4p}\sin 2(p+1)t+\frac{1}{4p}\sin 2(p-1)t,$$

$$\delta_{12}(t)=\frac{\mu_0}{2p^2}(1-\cos 2pt)+\frac{1}{2p^2}\cos 2t-\frac{1}{4p^2}\cos 2(p+1)t-\frac{1}{4p^2}\cos 2(p-1)t,$$

$$\delta_{21}(t)=-\frac{\mu_0}{2}(1+\cos 2pt)-\frac{1}{2}\cos 2t-\frac{1}{4}\cos 2(p+1)t-\frac{1}{4}\cos 2(p-1)t,$$

$$\varphi'_{11}(t)=\frac{\mu_0}{4}(1-\cos 2t)+\frac{1}{16}(1-\cos 4t),$$

$$\varphi'_{12}(t)=\frac{\mu_0}{4}(1-\sin 2t)-\frac{1}{16}\sin 4t,$$

$$\varphi'_{21}(t)=-\frac{\mu_0}{4}(1+\sin 2t)-\frac{1}{16}\sin 4t,$$

$$\varphi_{11}(t)=\frac{\mu_0}{4p^2}(1-\cos 2pt)-\frac{1}{8p(p+1)}\cos 2(p+1)t-\frac{1}{8p(p-1)}\cos 2(p-1)t+\frac{1}{4(p^2-1)},$$

$$\varphi_{12}(t)=-\frac{\mu_0}{4p^3}\sin 2pt+\frac{1}{4p^2}\sin 2pt-\frac{1}{8p^2(p+1)}\sin 2(p+1)t-\frac{1}{8p^2(p-1)}\sin 2(p-1)t,$$

$$\varphi_{21}(t)=-\frac{\mu_0}{4p}\sin 2pt-\frac{1}{4}\sin 2t-\frac{1}{8(p+1)}\sin 2(p+1)t-\frac{1}{8(p-1)}\sin 2(p-1)t.$$

For $p=1$, we get

$$\mathbf{K}_1=\frac{1}{2}\begin{Vmatrix}0 & \mu_0-\frac{1}{2}\\ -\mu_0-\frac{1}{2} & 0\end{Vmatrix},$$

$$\mathbf{K}_2=\frac{1}{8}\begin{Vmatrix}0 & -3\mu_0^2+\frac{3}{2}\mu_0+\frac{1}{8}\\ -\mu_0^2-\frac{1}{2}\mu_0-\frac{1}{8} & 0\end{Vmatrix},$$

and for $p=2, 3, \ldots$

$$\mathbf{K}_1=\frac{1}{2}\begin{Vmatrix}0 & \frac{1}{p^2}\mu_0\\ -\mu_0 & 0\end{Vmatrix},$$

$$\mathbf{K}_2=\frac{1}{8}\begin{Vmatrix}0 & -\frac{3}{p^4}\mu_0^2-\frac{2}{p^2(p^2-1)}\mu_0-\frac{1}{2p^2(p^2-1)}\\ -\frac{1}{p^2}\mu_0^2-\frac{2}{p^2-1}\mu_0+\frac{1}{2(p^2-1)} & 0\end{Vmatrix},$$

agreeing with our formulas for even p and $\alpha=0$.

Finally, we have formulas for the characteristic exponents $\varkappa_{1,2}$ of Mathieu's equation in the "almost-resonant" case:

a) for $c^2=1+\mu_0\varepsilon$,

$$\varkappa_{1,2}=-i\pm\frac{1}{2}i\left[\left(\varepsilon\mu_0+\frac{\varepsilon}{2}-\frac{\varepsilon^2}{4}\mu_0^2+\frac{3}{8}\mu_0\varepsilon^2+\frac{1}{32}\varepsilon^2\right)\times\right.$$
$$\left.\times\left(\mu_0\varepsilon-\frac{\varepsilon}{2}-\frac{3}{4}\mu_0^2\varepsilon^2+\frac{\mu_0}{8}\varepsilon^2+\frac{\varepsilon^2}{32}\right)\right]^{\frac{1}{2}}+O(\varepsilon^3),$$

b) for $c^2 = p^2 + \mu_0\varepsilon$ $(p = 2, 3, \ldots)$,

$$\varkappa_{1,2} = -pi \pm \frac{i}{2p}\left[\left(\mu_0\varepsilon + \frac{\mu_0^2}{4p^2}\varepsilon^2 + \frac{\mu_0}{2(p^2-1)}\varepsilon^2 + \frac{\varepsilon^2}{8(p^2-1)}\right) \times \right.$$
$$\left. \times \left(\mu_0\varepsilon - \frac{3\mu_0^2}{4p^2}\varepsilon^2 - \frac{\mu_0\varepsilon^2}{2(p^2-1)} - \frac{\varepsilon^2}{8(p^2-1)}\right)\right]^{\frac{1}{2}} + O(\varepsilon^3).$$

5.5. Example of third-order equation

We return once again to Malkin's example (/62b/, p. 201; see §2.8)

$$\frac{d^3\xi_1}{dt^3} + \frac{d\xi_1}{dt} = \varepsilon\left[-F(M\cos t + N\sin t)\xi_1 - b\frac{d\xi_1}{dt} - \frac{d^2\xi_1}{dt^2}\right], \tag{5.10}$$

where $F(\varphi)$ is an even function, $b > 0$, and ε a positive parameter. Express equation (5.10) in the form of (0.1):

$$\frac{d\mathbf{x}}{dt} = [\mathbf{C} + \varepsilon\mathbf{B}_1(t)]\mathbf{x}, \tag{5.11}$$

where

$$\mathbf{x} = \begin{pmatrix} \xi_1 \\ \dot{\xi}_1 \\ \ddot{\xi}_1 \end{pmatrix}, \quad \mathbf{C} = \begin{Vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{Vmatrix}, \quad \mathbf{B}_1(t) = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -F & -b & -1 \end{Vmatrix} = \mathbf{B}_1(t + 2\pi).$$

1) The eigenvalues of $\mathbf{C}$ are $\lambda_1 = 0$, $\lambda_2 = i$, $\lambda_3 = -i$; they are congruent modulo $i(T = 2\pi)$, belong to the same class, with $\alpha = 0$, and the integers m_j are $m_1 = 0$, $m_2 = 0$, $m_3 = -1$. Consequently, system (5.11) is in the singular case.

2) It follows from the remark after formula (4.7) that in our case the matrix $\mathbf{S}^{-1}(\mathbf{C} - \mathbf{C}_0)\mathbf{S}$ is the zero matrix, so that $\mathbf{C}_0 = \mathbf{C}$. Thus $\mathbf{K}_0 = \mathbf{C} - \mathbf{C}_0$ is also the zero matrix, and so to determine the matrix $\mathbf{D}_1(t)$ from (4.10) we have to determine $\exp(t\mathbf{C}_0) = \exp(t\mathbf{C})$. To this end, we can use formulas (I. 2.7) and (I. 2.9) or formula (I. 2.19). However, it is sometimes more convenient to utilize the fact that the matrix $\exp(t\mathbf{C})$ is the matrizant of the system $d\mathbf{x}/dt = \mathbf{C}\mathbf{x}$ with constant coefficients. We now write system (5.11) with $\varepsilon = 0$ in expanded form:

$$\xi_1 = \xi_2, \quad \xi_2 = \xi_3, \quad \xi_3 = -\xi_2$$

and readily find three particular solutions, defined respectively by the initial data

$$\begin{array}{lll} \xi_1(0) = 1, & \xi_1(0) = 0, & \xi_1(0) = 0, \\ \xi_2(0) = 0, & \xi_2(0) = 1, & \xi_2(0) = 0, \\ \xi_3(0) = 0; & \xi_3(0) = 0; & \xi_3(0) = 1. \end{array}$$

These solutions are

$$\begin{array}{lll} \xi_1 = 1, & \xi_1 = \sin t, & \xi_1 = 1 - \cos t, \\ \xi_2 = 0, & \xi_2 = \cos t, & \xi_2 = \sin t, \\ \xi_3 = 0, & \xi_3 = -\sin t, & \xi_3 = \cos t \end{array}$$

and these are the columns of the matrizant of system (5.11) with $\varepsilon = 0$. Thus,

$$e^{t\mathbf{C}} = \begin{Vmatrix} 1 & \sin t & 1-\cos t \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{Vmatrix},$$

and after computing the inverse matrix $\exp(-t\mathbf{C})$ we have, by (4.10),

$$\mathbf{D}_1(t) = \begin{Vmatrix} 1 & -\sin t & 1-\cos t \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{Vmatrix} \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -F & -b & -1 \end{Vmatrix} \begin{Vmatrix} 1 & \sin t & 1-\cos t \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{Vmatrix}.$$

Performing the matrix multiplication, we get

$$\mathbf{D}_1(t) = \begin{Vmatrix} -(1-\cos t)F & -\sin t\,(1-\cos t)F - b\cos t\,(1-\cos t) + \sin t\,(1-\cos t) & d_{13}(t) \\ F\sin t & F\sin^2 t + b\sin t\cos t - \sin^2 t & d_{23}(t) \\ -F\cos t & -F\sin t\cos t - b\cos^2 t + \sin t\cos t & d_{33}(t) \end{Vmatrix},$$

where

$$\begin{aligned} d_{13}(t) &= -(1-\cos t)^2 F - b\sin t\,(1-\cos t) - \cos t\,(1-\cos t), \\ d_{23}(t) &= \sin t\,(1-\cos t)F + b\sin^2 t + \sin t\cos t, \\ d_{33}(t) &= -\cos t\,(1-\cos t)F - b\sin t\cos t - \cos^2 t. \end{aligned}$$

System (4.9) for our example is

$$\frac{d\mathbf{y}}{dt} = \varepsilon \mathbf{D}_1(t)\,\mathbf{y}.$$

3) We now compute $\tilde{\mathbf{K}}_1$. We shall show that since $F(\varphi)$ is even

$$\int_0^{2\pi} F(M\cos t + N\sin t)\cos t\,dt = \int_0^{2\pi} F(M\cos t + N\sin t)\sin t\,dt = 0.$$

Expressing $M\cos t + N\sin t$ as $L\sin(t-t_0)$, we write these equalities as

$$\begin{aligned} \int_0^{2\pi} F(L\sin(t-t_0))\cos t\,dt &= \int_0^{2\pi} F(L\sin t)\cos(t+t_0)\,dt = 0, \\ \int_0^{2\pi} F(L\sin(t-t_0))\sin t\,dt &= \int_0^{2\pi} F(L\sin t)\sin(t+t_0)\,dt = 0. \end{aligned}$$

These equalities are valid because, first

$$\int_0^{2\pi} F(L\sin t)\cos t\,dt = \left[\frac{1}{L}G(L\sin t)\right]_0^{2\pi} = 0,$$

where $G(\varphi)$ is a periodic primitive of $F(\varphi)$, and, second,

$$\int_0^{2\pi} F(L\sin t)\sin t\,dt = 0,$$

since the integrand is odd. Simple manipulations yield

$$\tilde{\mathbf{K}}_1 = [\mathbf{D}_1(t)]_{av} = \begin{Vmatrix} -F_{av} & * & * \\ 0 & \frac{1}{2\pi}\int_0^{2\pi} F(\varphi)\sin^2 t\,dt - \frac{1}{2} & -\frac{1}{4\pi}\int_0^{2\pi} F(\varphi)\sin 2t\,dt + \frac{1}{2}b \\ 0 & -\frac{1}{4\pi}\int_0^{2\pi} F(\varphi)\sin 2t\,dt - \frac{1}{2}b & \frac{1}{2\pi}\int_0^{2\pi} F(\varphi)\cos^2 t\,dt - \frac{1}{2} \end{Vmatrix},$$

where the stars stand for elements whose precise values are immaterial. Setting

$$\tilde{\mathbf{K}}(\varepsilon) \approx \varepsilon\tilde{\mathbf{K}}_1,$$

in the first approximation, we see that in this approximation the characteristic equation of $\tilde{\mathbf{K}}(\varepsilon)$ is

$$\det[\varepsilon\tilde{\mathbf{K}}_1 - \varkappa\mathbf{I}_3] = (\varkappa + F_{av})\left[\varkappa^2 + (1 - F_{av})\varkappa + \right.$$
$$+ \frac{1}{4\pi^2}\left(\int_0^{2\pi} F(\varphi)\sin^2 t\,dt - \pi\right)\left(\int_0^{2\pi} F(\varphi)\cos^2 t\,dt - \pi\right) +$$
$$\left. + \frac{b^2}{4} - \frac{1}{16\pi^2}\left(\int_0^{2\pi} F(\varphi)\sin 2t\,dt\right)^2\right] = 0.$$

The Routh-Hurwitz criterion for this equation is

$$\left.\begin{aligned} &0 < F_{av} \equiv \frac{1}{2\pi}\int_0^{2\pi} F(\varphi)\,dt < 1 \qquad (\varphi = M\cos t + N\sin t), \\ &\left(\int_0^{2\pi} F(\varphi)\sin^2 t\,dt - \pi\right)\left(\int_0^{2\pi} F(\varphi)\cos^2 t\,dt - \pi\right) - \frac{1}{4}\left(\int_0^{2\pi} F(\varphi)\sin 2t\,dt\right)^2 + \\ &\qquad + \pi^2 b^2 > 0. \end{aligned}\right\} \tag{5.12}$$

If these inequalities hold, then for sufficiently small ε the unperturbed motion of this servosystem is asymptotically stable.

4) For a more precise idea of the stability of the system, we must compute $\tilde{\mathbf{K}}_2$. Since $\mathbf{K}_0 = 0$, it follows from equation (4.44) that

$$\mathbf{W} = \mathbf{R}_m(\mathbf{D}_1^{(m)}) = -\frac{i}{m}\mathbf{D}_1^{(m)}.$$

Since $\mathbf{D}_2(t) \equiv \mathbf{0}$, formula (4.51) gives

$$\tilde{\mathbf{K}}_2 = -2\,\mathrm{Re}\, i \sum_{m=1}^{\infty} \frac{1}{m} \overline{\mathbf{D}}_1^{(m)} \mathbf{D}_1^{(m)}.$$

Now write the Fourier series for $\mathbf{D}_1(t)$ as

$$\mathbf{D}_1(t) \sim \mathbf{D}_1^{(0)} + \sum_{m=1}^{\infty} (\mathbf{U}_m \cos mt + \mathbf{V}_m \sin mt)$$

and the formula for $\tilde{\mathbf{K}}_2$ is found to be

$$\tilde{\mathbf{K}}_2 = \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} (\mathbf{V}_m \mathbf{U}_m - \mathbf{U}_m \mathbf{V}_m).$$

In the second approximation, the characteristic equation of $\tilde{\mathbf{K}}(\varepsilon)$ is

$$\det \left[\varepsilon \mathbf{D}_1^{(0)} + \frac{1}{2} \varepsilon^2 \sum_{m=1}^{\infty} \frac{1}{m} (\mathbf{V}_m \mathbf{U}_m - \mathbf{U}_m \mathbf{V}_m) - \varkappa \mathbf{I}_3 \right] = 0.$$

Once again, to check for stability there is no need to evaluate the characteristic exponents $\varkappa_{1,2,3}$ themselves, but only to apply the Routh-Hurwitz conditions to the resulting cubic equation.

5.6. Computation of the exponent matrix of the variational equation for periodic motion of a flexible shaft with heavy symmetrically placed flywheel

Consider the variational equation (2.47). We begin by determining which case (singular or nonsingular) holds. The eigenvalues of $\mathbf{C}$,

$$\lambda_{1,2} = 0, \quad \lambda_{3,4} = +i, \quad \lambda_{5,6} = -i$$

are congruent modulo $2\pi i/T = i/2$ and may be expressed as (4.3)

$$\lambda_j = 0 + m_j \frac{i}{2} \qquad (j = 1, \ldots, 6;\ m_1 = m_2 = 0,\ m_3 = m_4 = 2,\ m_5 = m_6 = -2).$$

Thus the singular case holds.

We now effect the reduction from the singular to the nonsingular case. The matrix $\mathbf{C}$ has one quadratic elementary divisor λ^2 and four simple elementary divisors $\lambda - i$, $\lambda - i$, $\lambda + i$, $\lambda + i$ (see Example 4 in §I.1.19); it may be expressed as

$$\mathbf{C} = \mathbf{S} \begin{Vmatrix} 0 & 1 & & & & \\ 0 & 0 & & & & \\ & & i & & 0 & \\ & & & i & & \\ & & 0 & & -i & \\ & & & & & -i \end{Vmatrix} \mathbf{S}^{-1},$$

where

$$\mathbf{S}=\begin{Vmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & -i & 0 \\ 0 & 0 & 0 & i & 0 & -i \\ 0 & 1 & 0 & 0 & 0 & 0 \end{Vmatrix}, \quad \mathbf{S}^{-1}=\begin{Vmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2}i & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2}i & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2}i & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2}i & 0 \end{Vmatrix}.$$

The matrix $\mathbf{C}_0$ is

$$\mathbf{C}_0=\frac{2\pi}{T}i\mathbf{S}\,\mathrm{diag}\,(m_1,\ \ldots,\ m_6)\,\mathbf{S}^{-1}=\mathbf{S}\begin{Vmatrix} 0 & & & & & \\ & 0 & & & & \\ & & i & 0 & & \\ & & & i & & \\ 0 & & & & -i & \\ & & & & & -i \end{Vmatrix}\mathbf{S}^{-1}=\begin{Vmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{Vmatrix},$$

and we have

$$\mathbf{K}_0=\mathbf{C}-\mathbf{C}_0=\mathbf{S}\begin{Vmatrix} 0 & 1 & & \\ 0 & 0 & 0 & \\ & & & 0 \end{Vmatrix}\mathbf{S}^{-1}=\begin{Vmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{Vmatrix}.$$

The matrices $\exp(t\mathbf{C}_0)$ and $\exp(-t\mathbf{C}_0)$ were determined in Example 1 of § II. 2.1. Using them, we find from (4.10)

$$\mathbf{D}_1(t)=\begin{pmatrix} \mathbf{D}_{11}(t) & \mathbf{0} \\ \mathbf{D}_{21}(t) & -\mathbf{D}_{11}(t)^{\tau} \end{pmatrix},$$

$$\mathbf{D}_{11}(t)=\begin{Vmatrix} 0 & 0 & \sin\frac{t}{2}\sin t \\ 0 & 0 & -\cos\frac{t}{2}\sin t \\ 0 & 0 & 0 \end{Vmatrix}, \quad \mathbf{D}_{21}(t)=\begin{Vmatrix} 0 & 0 & -\sin\frac{t}{2}\cos t \\ 0 & 0 & \cos\frac{t}{2}\cos t \\ -\sin\frac{t}{2}\cos t & \cos\frac{t}{2}\cos t & 2\delta\sin\frac{t}{2} \end{Vmatrix},$$

$$\mathbf{D}_2=\begin{Vmatrix} & & & 0 & & \\ 0 & 0 & -\frac{4}{3} & 0 & 0 & 0 \end{Vmatrix}.$$

The complex Fourier coefficients of $\mathbf{D}_1(t)$ are

$$\mathbf{D}_1^{(1)}=\begin{Vmatrix} 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4}i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{4}i & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ -\frac{1}{4}i & \frac{1}{4} & -\delta i & -\frac{1}{4} & -\frac{1}{4}i & 0 \end{Vmatrix}, \quad \mathbf{D}_1^{(3)}=\begin{Vmatrix} 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4}i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4}i & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{4}i & \frac{1}{4} & 0 & \frac{1}{4} & -\frac{1}{4}i & 0 \end{Vmatrix},$$

$\mathbf{D}_1^{(-1)}=\overline{\mathbf{D}}_1^{(1)}$, $\mathbf{D}_1^{(-3)}=\overline{\mathbf{D}}_1^{(3)}$ and all other coefficients vanish. The only nonzero coefficient for $\mathbf{D}_2$ is obviously $\mathbf{D}_2^{(0)}=\mathbf{D}_2$.

Equation (4.44) is

$$\mathbf{W}+\frac{2}{m}i\mathbf{K}_0\mathbf{W}-\frac{2}{m}i\mathbf{W}\mathbf{K}_0=-\frac{2}{m}i\mathbf{G}$$

or, in expanded form,

$$\|w_{ij}\|_1^6+\frac{2}{m}i\begin{Vmatrix} 0 & 0 & 0 & 0 & 0 & -w_{13} \\ 0 & 0 & 0 & 0 & 0 & -w_{23} \\ w_{61} & w_{62} & w_{63} & w_{64} & w_{65} & w_{66}-w_{33} \\ 0 & 0 & 0 & 0 & 0 & -w_{43} \\ 0 & 0 & 0 & 0 & 0 & -w_{53} \\ 0 & 0 & 0 & 0 & 0 & -w_{63} \end{Vmatrix}=-\frac{2}{m}i\|g_{ij}\|_1^6.$$

Solving this equation, we obtain

$$\mathbf{W}=\mathbf{R}_m(\mathbf{G})=-\frac{2}{m}i\|g_{ij}\|_1^6-\frac{4}{m^2}\begin{Vmatrix} 0 & 0 & 0 & 0 & 0 & -g_{13} \\ 0 & 0 & 0 & 0 & 0 & -g_{23} \\ g_{61} & g_{62} & g_{63} & g_{64} & g_{65} & -g_{33}+g_{66}+\frac{4}{m}g_{63} \\ 0 & 0 & 0 & 0 & 0 & -g_{43} \\ 0 & 0 & 0 & 0 & 0 & -g_{53} \\ 0 & 0 & 0 & 0 & 0 & -g_{63} \end{Vmatrix}.$$

Formulas (4.50) and (4.51) for this example give

$$\tilde{\mathbf{K}}_1=\mathbf{D}_1^{(0)}=0,$$
$$\tilde{\mathbf{K}}_2=\mathbf{D}_2^{(0)}+2\mathrm{Re}\,[\overline{\mathbf{D}}_1^{(1)}\mathbf{R}_1(\mathbf{D}_1^{(1)})+\overline{\mathbf{D}}_1^{(3)}\mathbf{R}_3(\mathbf{D}_1^{(3)})].$$

Multiplying, we obtain

$$\tilde{\mathbf{K}}_2=\begin{Vmatrix} 0 & -\frac{4}{9} & 0 & \frac{5}{9} & 0 & -8\delta \\ \frac{4}{9} & 0 & 2\delta & 0 & \frac{5}{9} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{5}{9} & 0 & -2\delta & 0 & -\frac{4}{9} & 0 \\ 0 & -\frac{5}{9} & 0 & \frac{4}{9} & 0 & -8\delta \\ -2\delta & 0 & -8\delta^2 & 0 & -2\delta & 0 \end{Vmatrix}.$$

Finally, the exponent matrix is, up to terms of order ε^2,

$$\mathbf{K}_0+\varepsilon\tilde{\mathbf{K}}_1+\varepsilon^2\tilde{\mathbf{K}}_2=\left\|\begin{array}{cccccc} 0 & -\frac{4}{9}\varepsilon^2 & 0 & \frac{5}{9}\varepsilon^2 & 0 & -8\delta\varepsilon^2 \\ \frac{4}{9}\varepsilon^2 & 0 & 2\delta\varepsilon^2 & 0 & \frac{5}{9}\varepsilon^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{5}{9}\varepsilon^2 & 0 & -2\delta\varepsilon^2 & 0 & -\frac{4}{9}\varepsilon^2 & 0 \\ 0 & -\frac{5}{9}\varepsilon^2 & 0 & \frac{4}{9}\varepsilon^2 & 0 & -8\delta\varepsilon^2 \\ -2\delta\varepsilon^2 & 0 & -8\delta^2\varepsilon^2 & 0 & -2\delta\varepsilon^2 & 0 \end{array}\right\|.$$

Note that the characteristic equation of the exponent matrix $\mathbf{K}^{(2)}=\mathbf{K}_0+\varepsilon\tilde{\mathbf{K}}_1+\varepsilon^2\tilde{\mathbf{K}}_2$ is

$$\det(\mathbf{K}^{(2)}-\varkappa\mathbf{I}_6)\equiv$$
$$\equiv\varkappa^2\left[\varkappa^4+\varepsilon^2\left(\frac{82}{81}\varepsilon^2-32\delta^2\varepsilon^2+8\delta^2\right)\varkappa^2+\varepsilon^4\left(\frac{1}{81}\varepsilon^4-\frac{32}{81}\delta^2\varepsilon^4+\frac{8}{81}\delta^2\varepsilon^2\right)\right]=0.$$

The roots of this equation furnish approximations to the characteristic exponents.

§6. STABILITY OF PERIODIC MOTIONS OF NON-AUTONOMOUS SYSTEMS

The method of small parameters is widely used in the theory of nonlinear oscillation theory. Originating from the theory of small perturbations in celestial mechanics, the method was first developed in the classical treatise of Poincaré /111b/. Further developments were connected for the most part with Russian schools of research. Foremost among the Russian authors are Lyapunov /60/, Andronov, Vitt and Khaikin /2, 3/, Bulgakov /15/ and Malkin /62a, b/.

In this section we shall use the results of §§2 and 3 to derive fairly simple stability conditions for periodic motions of nonatonomous systems. Since the periodic solutions (motions) themselves are found by the method of small parameters, we shall begin with a brief account of the essentials of this method.

6.1. Differential equations of the generating solution and first corrections. Nonresonant case

Consider a nonlinear system of k second-order differential equations, expressed as a vector equation

$$\mathbf{M}\frac{d^2\mathbf{v}}{dt^2}+\mathbf{Q}_0\frac{d\mathbf{v}}{dt}+\mathbf{P}_0\mathbf{v}=\mathbf{f}(t)+\mu\mathbf{g}(t,\ \mathbf{v},\ \dot{\mathbf{v}},\ \mu), \tag{6.1}$$

where $\mathbf{M}$, $\mathbf{Q}_0$, $\mathbf{P}_0$ are constant real $k\times k$ matrices ($\mathbf{M}$ and $\mathbf{P}_0$ are symmetric, $\mathbf{Q}_0$ is skew-symmetric), $\mathbf{f}(t)$ is an integrable piecewise-continuous vector-function of period $T=2\pi/\omega$, $\mathbf{g}(t, \mathbf{v}, \dot{\mathbf{v}}, \mu)$ is an analytic vector-function of μ at $\mu=0$, T-periodic in t and having sufficiently many derivatives with respect to $\mathbf{v}$ and $\dot{\mathbf{v}}$ in the relevant domain of variables.

We wish to find T-periodic solutions of equation (6.1) which are analytic in μ at $\mu=0$:

$$\mathbf{v}(t, \mu)=\mathbf{v}^{(0)}(t)+\mu\mathbf{v}^{(1)}(t)+\mu^2\mathbf{v}^{(2)}(t)+\dots \tag{6.2}$$

Substituting this into (6.1), we see that the generating solution $\mathbf{v}^{(0)}(t)$ must satisfy the following system of linear differential equations with constant coefficients:

$$\mathbf{M}\frac{d^2\mathbf{v}^{(0)}}{dt^2}+\mathbf{Q}_0\frac{d\mathbf{v}^{(0)}}{dt}+\mathbf{P}_0\mathbf{v}^{(0)}=\mathbf{f}(t), \tag{6.3}$$

and the first and second corrections satisfy the systems

$$\left.\begin{aligned}\mathbf{M}\frac{d^2\mathbf{v}^{(1)}}{dt^2}+\mathbf{Q}_0\frac{d\mathbf{v}^{(1)}}{dt}+\mathbf{P}_0\mathbf{v}^{(1)}&=\mathbf{g}(t, \mathbf{v}^{(0)}, \dot{\mathbf{v}}^{(0)}, 0),\\ \mathbf{M}\frac{d^2\mathbf{v}^{(2)}}{dt^2}+\mathbf{Q}_0\frac{d\mathbf{v}^{(2)}}{dt}+\mathbf{P}_0\mathbf{v}^{(2)}&=\left(\frac{\partial\mathbf{g}}{\partial\mathbf{v}}\right)_0\mathbf{v}^{(1)}+\left(\frac{\partial\mathbf{g}}{\partial\dot{\mathbf{v}}}\right)_0\dot{\mathbf{v}}^{(1)}+\left(\frac{\partial\mathbf{g}}{\partial\mu}\right)_0,\end{aligned}\right\} \tag{6.4}$$

where the zero subscript indicates that the partial derivatives in question are to be evaluated for $\mu=0$, $\mathbf{v}=\mathbf{v}^{(0)}(t)$, $\dot{\mathbf{v}}=\dot{\mathbf{v}}^{(0)}(t)$. The symbols $\partial\mathbf{g}/\partial\mathbf{v}$ and $\partial\mathbf{g}/\partial\dot{\mathbf{v}}$ will denote the matrices of the corresponding partial derivatives:

$$\frac{\partial\mathbf{g}}{\partial\mathbf{v}}=\left\|\frac{\partial g_i}{\partial v_x}\right\|_1^k, \quad \frac{\partial\mathbf{g}}{\partial\dot{\mathbf{v}}}=\left\|\frac{\partial g_i}{\partial\dot{v}_x}\right\|_1^k \qquad \left(\mathbf{v}=\begin{pmatrix}v_1\\ \dots\\ v_k\end{pmatrix}, \ \mathbf{g}=\begin{pmatrix}g_1\\ \dots\\ g_k\end{pmatrix}\right).$$

The homogeneous system corresponding to (6.3) is

$$\mathbf{M}\frac{d^2\mathbf{v}^{(0)}}{dt^2}+\mathbf{Q}_0\frac{d\mathbf{v}^{(0)}}{dt}+\mathbf{P}_0\mathbf{v}^{(0)}=\mathbf{0}. \tag{6.5}$$

We let $\lambda_{\pm\nu}$ ($\lambda_{-\nu}=\bar{\lambda}_\nu$) denote the roots of the characteristic equation

$$\boxed{\det(\lambda^2\mathbf{M}+\lambda\mathbf{Q}_0+\mathbf{P}_0)=0} \tag{6.6}$$

and $\mathbf{a}_\nu$ ($\mathbf{a}_{-\nu}=\bar{\mathbf{a}}_\nu$) will be corresponding eigenvectors:

$$(\lambda_\nu^2\mathbf{M}+\lambda_\nu\mathbf{Q}_0+\mathbf{P}_0)\mathbf{a}_\nu=\mathbf{0} \qquad (\nu=\pm1, \dots, \pm k).$$

We shall consider two cases.

Nonresonant case. System (6.5) has no T-periodic solutions — a necessary and sufficient condition for this is that

$$\lambda_\nu\neq ip\omega \qquad (\nu=\pm1, \dots, \pm k; \ p=\pm1, \pm2, \dots).$$

Indeed, by § II. 2.7 the multipliers (6.5) corresponding to period $T=2\pi/\omega$ must be different from unity, $\exp(2\pi\lambda_s/\omega)\neq 1$, and this implies the above condition. In that case system (6.3) has a unique T-periodic solution. According to /111b/, there exists a unique T-periodic solution (6.2) which is analytic in μ at $\mu=0$.

6.2. Resonant case

Suppose that the homogeneous system (6.5) has $2d$ $(1\leqslant d\leqslant k)$ linearly independent T-periodic solutions. Then we have $2d$ equalities

$$\lambda_j=ip_j\omega \qquad (j=\pm 1,\ \ldots,\ \pm d;\ p_{-j}=-p_j;\ \omega=2\pi/T),$$

and for the other roots of the characteristic equation (6.6)

$$\lambda_h\neq ip\omega \qquad (h=\pm(d+1),\ \ldots,\ \pm k),$$

where p_j and p are integers. We write the $2d$ T-periodic solutions of system (6.5) in complex form:

$$\mathbf{v}_{0j}^{(\pm)}=e^{\pm ip_j\omega t}\mathbf{a}_j \qquad (j=1,\ \ldots,\ d). \tag{6.7}$$

Let us establish existence conditions for T-periodic solutions of system (6.3) in the resonant case. Let $\mathbf{v}(t)$ and $\mathbf{w}(t)$ be vector-functions with integrable piecewise-continuous second derivatives on $[0,\ T]$, satisfying the end conditions

$$\mathbf{v}(T)=\mathbf{v}(0), \qquad \dot{\mathbf{v}}(T)=\dot{\mathbf{v}}(0); \qquad \mathbf{w}(T)=\mathbf{w}(0), \qquad \dot{\mathbf{w}}(T)=\dot{\mathbf{w}}(0).$$

Then

$$\int_0^T(\mathbf{M}\ddot{\mathbf{v}}+\mathbf{Q}_0\dot{\mathbf{v}}+\mathbf{P}_0\mathbf{v},\ \mathbf{w})\,dt=\int_0^T(\mathbf{v},\ \mathbf{M}\ddot{\mathbf{w}}+\mathbf{Q}_0\dot{\mathbf{w}}+\mathbf{P}_0\mathbf{w})\,dt. \tag{6.8}$$

Indeed, by assumption $\mathbf{M}^{\tau}=\mathbf{M}$, $\mathbf{Q}_0^{\tau}=-\mathbf{Q}_0$, $\mathbf{P}_0^{\tau}=\mathbf{P}_0$, and so

$$\int_0^T(\mathbf{M}\ddot{\mathbf{v}}+\mathbf{Q}_0\dot{\mathbf{v}}+\mathbf{P}_0\mathbf{v},\ \mathbf{w})\,dt=\int_0^T(\ddot{\mathbf{v}},\ \mathbf{M}\mathbf{w})\,dt-\int_0^T(\dot{\mathbf{v}},\ \mathbf{Q}_0\mathbf{w})\,dt+\int_0^T(\mathbf{v},\ \mathbf{P}_0\mathbf{w})\,dt.$$

Integrating by parts, once in the second integral and twice in the first, we obtain (6.8). Suppose that system (6.1) has a periodic solution $\mathbf{v}(t,\ \mu)$. Substituting this solution and any one of the $2d$ functions $\mathbf{v}_{0j}^{(\pm)}$ in formula (6.8), we obtain the following equality, which holds identically in μ:

$$\int_0^T(\mathbf{f}(t)+\mu\mathbf{g}(t,\ \mathbf{v},\ \dot{\mathbf{v}},\ \mu),\ \mathbf{v}_{0j}^{(\pm)})\,dt=0 \qquad (j=1,\ \ldots,\ d), \tag{6.9}$$

since

$$\mathbf{M}\ddot{\mathbf{v}}_{0j}^{(\pm)}+\mathbf{Q}_0\dot{\mathbf{v}}_{0j}^{(\pm)}+\mathbf{P}_0\mathbf{v}_{0j}^{(\pm)}=\mathbf{0} \qquad (j=1, \ldots, d).$$

Setting $\mu=0$ in (6.9), we obtain necessary conditions for the existence of a T-periodic solution $\mathbf{v}(t, \mu)$:

$$\int_0^T (\mathbf{f}(t), \mathbf{v}_{0j}^{(+)})\,dt=0 \qquad (j=1, \ldots, d).$$

Assuming that these equalities hold, let us subtract them from the identities (6.9). Dividing by μ, we obtain identities

$$\int_0^T (\mathbf{g}(t, \mathbf{v}, \dot{\mathbf{v}}, \mu), \mathbf{v}_{0j}^{(\pm)})\,dt=0 \qquad (j=1, \ldots, d). \tag{6.10}$$

Setting $\mu=0$ and substituting $\mathbf{v}_{0j}^{(\pm)}$ from (6.7), we see that in the resonant case the generating solution $\mathbf{v}^{(0)}(t)$ must satisfy the following $2d$ conditions:

$$\psi_j\equiv\int_0^T e^{-tp_j\omega t}(\mathbf{g}(t, \mathbf{v}^{(0)}, \dot{\mathbf{v}}^{(0)}, 0), \mathbf{a}_j)\,dt=0 \qquad (j=\pm 1, \ldots, \pm d). \tag{6.11}$$

We shall look for a generating solution $\mathbf{v}^{(0)}(t)$ of the form

$$\mathbf{v}^{(0)}=\mathbf{w}_0(t)+\sum_{j=-d}^{d}\zeta_j e^{ip_j\omega t}\mathbf{a}_j \qquad (\zeta_{-j}=\bar{\zeta}_j,\ p_{-j}=-p_j),$$

where $\mathbf{w}_0(t)$ is some particular periodic solution of system (6.3). Conditions (6.11) will now constitute a system of $2d$ equations for the complex numbers $\zeta_1, \ldots, \zeta_d$ — equations for the "generating amplitudes." Let $\zeta_1^{(0)}, \ldots, \zeta_d^{(0)}$ denote a solution of system (6.11). It can be shown /111b, 62a,b/ that if the Jacobian of system (6.11) does not vanish,

$$\frac{D(\psi_{-d}, \ldots, \psi_d)}{D(\zeta_{-d}, \ldots, \zeta_d)}\Bigg|_{\zeta_j=\zeta_j^{(0)}}\neq 0,$$

then the solution (6.2) exists and is analytic in μ at $\mu=0$.

Equating terms in μ, μ^2, ... in the identities (6.10), we get

$$\chi_j\equiv\int_0^T e^{-tp_j\omega t}\left(\left(\frac{\partial \mathbf{g}}{\partial \mathbf{v}}\right)_0\mathbf{v}^{(1)}+\left(\frac{\partial \mathbf{g}}{\partial \dot{\mathbf{v}}}\right)_0\dot{\mathbf{v}}^{(1)}+\left(\frac{\partial \mathbf{g}}{\partial \mu}\right)_0, \mathbf{a}_j\right)dt=0 \tag{6.12}$$

$$(j=\pm 1, \ldots, \pm d), \ldots,$$

where the zero subscript means that the partial derivatives are evaluated at $\mu=0$, $\mathbf{v}=\mathbf{v}^{(0)}(t)$, $\dot{\mathbf{v}}=\dot{\mathbf{v}}^{(0)}(t)$.

6.3. Variational equation for periodic unperturbed motion

Let (6.2) be the unperturbed motion and

$$\tilde{\mathbf{v}} = \mathbf{v}(t, \mu) + \mathbf{y}.$$

Substituting this into (6.1), noting that $\mathbf{v}(t, \mu)$ satisfies (6.1), and dropping second-order terms in $\mathbf{y}$, we obtain the variational equation

$$\mathbf{M}\frac{d^2\mathbf{y}}{dt^2} + \mathbf{Q}_0\frac{d\mathbf{y}}{dt} + \mathbf{P}_0\mathbf{y} = \mu\left(\frac{\partial \mathbf{g}}{\partial \mathbf{v}}\right)\mathbf{y} + \mu\left(\frac{\partial \mathbf{g}}{\partial \dot{\mathbf{v}}}\right)\frac{d\mathbf{y}}{dt}.$$

In this case, the unperturbed solution (6.2): $\mathbf{v} = \mathbf{v}(t, \mu)$, $\dot{\mathbf{v}} = \dot{\mathbf{v}}(t, \mu)$ is substituted into the partial derivatives after differentiation. Denote

$$\mu\left(\frac{\partial \mathbf{g}}{\partial \mathbf{v}}\right) = -\mu\mathbf{P}(t, \mu), \quad \mu\left(\frac{\partial \mathbf{g}}{\partial \dot{\mathbf{v}}}\right) = -\mu\mathbf{Q}(t, \mu).$$

Expand these as power series in μ:

$$\mu\mathbf{P}(t, \mu) = \mu\mathbf{P}_1(t) + \mu^2\mathbf{P}_2(t) + \ldots, \quad \mu\mathbf{Q}(t, \mu) = \mu\mathbf{Q}_1(t) + \mu^2\mathbf{Q}_2(t) + \ldots,$$

where

$$\mathbf{P}_1(t) = -\left(\frac{\partial \mathbf{g}}{\partial \mathbf{v}}\right)_0, \quad \mathbf{Q}_1(t) = -\left(\frac{\partial \mathbf{g}}{\partial \dot{\mathbf{v}}}\right)_0,$$

the subscript 0 meaning that the derivatives are evaluated at $\mu = 0$, $\mathbf{v} = \mathbf{v}^{(0)}(t)$, $\dot{\mathbf{v}} = \dot{\mathbf{v}}^{(0)}(t)$. We write the vector equation corresponding to the variational equations as

$$\mathbf{M}\frac{d^2\mathbf{y}}{dt^2} + [\mathbf{Q}_0 + \mu\mathbf{Q}_1(t) + \mu^2\mathbf{Q}_2(t) + \ldots]\frac{d\mathbf{y}}{dt} + \\ + [\mathbf{P}_0 + \mu\mathbf{P}_1(t) + \mu^2\mathbf{P}_2(t) + \ldots]\mathbf{y} = 0. \quad (6.13)$$

We thus have a system of linear differential equations with T-periodic coefficients, of the form (3.23).

6.4. Case of distinct multipliers of the unperturbed variational equation

Consider the variational equation (6.13) at $\mu = 0$:

$$\mathbf{M}\frac{d^2\mathbf{y}}{dt^2} + \mathbf{Q}_0\frac{d\mathbf{y}}{dt} + \mathbf{P}_0\mathbf{y} = 0. \quad (6.14)$$

Under the assumptions $\mathbf{M}^\tau = \mathbf{M} > 0$, $\mathbf{P}_0^\tau = \mathbf{P}_0 > 0$, $\mathbf{Q}_0^\tau = -\mathbf{Q}_0$, all roots of its characteristic equation (6.6) are pure imaginary (§3.4):

$$\lambda_\nu = i\omega_\nu \qquad (\nu = \pm 1, \ \ldots, \ \pm k; \ \omega_{-\nu} = -\omega_\nu).$$

Let us assume that all the multipliers of equation (6.14) corresponding to period $T = 2\pi/\omega$,

$$\rho_\nu = e^{i\omega_\nu T} \qquad (\nu = \pm 1, \ \ldots, \ \pm k)$$

are distinct, $\rho_i \neq \rho_\nu$, that is,

$$\omega_i \not\equiv \omega_\nu \pmod{\omega} \qquad (i, \ \nu = \pm 1, \ \ldots, \ \pm k).$$

In other words, each congruence class of the numbers ω_ν modulo $2\pi/T$ consists of a single number. The characteristic exponents of the variational equations are (§3.4)

$$\alpha_j(\mu) = i\omega_j - i\sigma_{jj}(\operatorname{sign} j)\,\mu + O(\mu^2),$$

where

$$\left.\begin{aligned} \sigma_{jj} &= -([\mathbf{P}_1^{(0)} + i\omega_j \mathbf{Q}_1^{(0)}]\,\mathbf{a}_j, \ \mathbf{a}_j), \\ \mathbf{P}_1^{(0)} &= \frac{1}{T}\int_0^T \mathbf{P}_1(t)\,dt, \ \ \mathbf{Q}_1^{(0)} = \frac{1}{T}\int_0^T \mathbf{Q}_1(t)\,dt \end{aligned}\right\} \tag{6.15}$$

and the $\mathbf{a}_j$ are normalized eigenvectors:

$$\left.\begin{aligned} (-\omega_j^2 \mathbf{M} + i\omega_j \mathbf{Q}_0 + \mathbf{P}_0)\,\mathbf{a}_j &= 0, \\ ([2\omega_j \mathbf{M} - i\mathbf{Q}_0]\,\mathbf{a}_j, \ \mathbf{a}_j) &= \operatorname{sign} j \qquad (j = \pm 1, \ \ldots, \ \pm k). \end{aligned}\right\} \tag{6.16}$$

Since the numbers

$$(\mathbf{P}_1^{(0)}\mathbf{a}_j, \ \mathbf{a}_j)$$

are real (and positive), while $\omega_j \operatorname{sign} j > 0$, it follows that

$$\operatorname{Re}\alpha_j(\mu) = -\operatorname{Re}(\mathbf{Q}_1^{(0)}\mathbf{a}_j, \ \mathbf{a}_j)\,\mu + O(\mu^2) \qquad (j = \pm 1, \ \ldots, \ \pm k).$$

Now $\mathbf{a}_{-j} = \bar{\mathbf{a}}_j$, and so

$$(\mathbf{Q}_1^{(0)}\mathbf{a}_{-j}, \ \mathbf{a}_{-j}) = \overline{(\mathbf{Q}_1^{(0)}\mathbf{a}_j, \ \mathbf{a}_j)} \qquad (j = \pm 1, \ \ldots, \ \pm k).$$

Consequently, if

$$\operatorname{Re}(\mathbf{Q}_1^{(0)}\mathbf{a}_\varkappa, \ \mathbf{a}_\varkappa) > 0 \qquad (\varkappa = 1, \ \ldots, \ k), \tag{6.17}$$

then the unperturbed motion (6.2) is asymptotically stable for sufficiently small positive μ, but if at least one of the inequalities

$$\operatorname{Re}(\mathbf{Q}_1^{(0)}\mathbf{a}_{\varkappa'}, \ \mathbf{a}_{\varkappa'}) < 0, \tag{6.18}$$

is true, the unperturbed motion is unstable for sufficiently small positive μ.

Remark. If $\mathbf{Q}_0 = \mathbf{0}$, the $\mathbf{a}_\nu$'s are real and $\mathbf{a}_{-\nu} = \mathbf{a}_\nu$ ($\nu = \pm 1, \ldots, \pm k$). Hence the numbers

$$(\mathbf{Q}_1^{(0)}\mathbf{a}_\varkappa, \mathbf{a}_\varkappa) \qquad (\varkappa = 1, \ldots, k)$$

are real and the symbol Re in (6.17) and (6.18) may be omitted.

The case of multiple multipliers is treated along the lines of §3.4. We shall only cite the final results in the next subsection.

6.5. Summary of final results

Assume that equation (6.1) has a T-periodic solution (6.2).

1 *(Case of simple multipliers). Let the roots* $\pm\omega_1, \ldots, \pm\omega_k$ *of equation (6.15) be simple and not congruent to one another (or, consequently, to* 0*) modulo* $\omega = 2\pi/T$, *so that system (6.14) has only simple multipliers.*

a) *If all the numbers*

$$\vartheta_\varkappa \equiv \operatorname{Re}\left(\left[\frac{1}{T}\int_0^T \left(\frac{\partial \mathbf{g}}{\partial \dot{\mathbf{v}}}\right)_0 dt\right]\mathbf{a}_\varkappa, \mathbf{a}_\varkappa\right) \qquad (\varkappa = 1, \ldots, k) \tag{6.19}$$

are negative, then the unperturbed motion (6.2) is asymptotically stable for sufficiently small μ.

Here $\mathbf{a}_{\pm 1}, \ldots, \mathbf{a}_{\pm k}$ is a system of eigenvectors (6.16), and the zero subscript means that all derivatives in the matrix

$$\frac{\partial \mathbf{g}}{\partial \dot{\mathbf{v}}} = \left\| \frac{\partial g_i}{\partial \dot{v}_\varkappa} \right\|_1^k$$

are evaluated at $\mu = 0$, $\mathbf{v} = \mathbf{v}^{(0)}(t)$, $\dot{\mathbf{v}} = \dot{\mathbf{v}}^{(0)}(t)$. The outer parentheses denote the scalar product of the vectors

$$\left[\frac{1}{T}\int_0^T \left(\frac{\partial \mathbf{g}}{\partial \dot{\mathbf{v}}}\right)_0 dt\right]\mathbf{a}_\varkappa \quad \text{and} \quad \mathbf{a}_\varkappa.$$

b) *If at least one of the numbers* $\vartheta_{\varkappa'}$ *is positive, the unperturbed motion (6.2) is unstable for sufficiently small positive* μ.

Remark. If $\mathbf{Q}_0 = \mathbf{0}$, the symbol Re in (6.19) may be omitted (see end of §3.4).

2 *(Case of multiple multipliers). Assume that system (6.14) has an* r*-fold multiplier corresponding to period* T:

$$\rho^{(0)} = e^{i\omega^{(0)}T}.$$

Let $\omega_{j_1}, \ldots, \omega_{j_r}$ *be the class of roots of equation (6.15) corresponding to this multiplier, i.e.,*

$$\omega_j \equiv \omega^{(0)} \quad \left(\operatorname{mod} \frac{2\pi}{T}\right) \qquad (j = j_1, \ldots, j_r).$$

Define integers m_j by

$$\omega_j=\omega^{(0)}+m_j\frac{2\pi i}{T} \qquad (j=j_1,\ \ldots,\ j_r).$$

Expand the matrix-functions

$$\mathbf{P}_1(t)=-\left(\frac{\partial g}{\partial v}\right)_0 \quad \text{and} \quad \mathbf{Q}_1(t)=-\left(\frac{\partial g}{\partial \dot v}\right)_0$$

in complex Fourier series:

$$\mathbf{P}_1(t)\sim\sum_{m=-\infty}^{\infty}\exp\left(m\frac{2\pi}{T}i\right)\mathbf{P}_1^{(m)},\quad \mathbf{Q}_1(t)\sim\sum_{m=-\infty}^{\infty}\exp\left(m\frac{2\pi}{T}i\right)\mathbf{Q}_1^{(m)}.$$

Define σ_{jh}, γ_{jh} by

$$\sigma_{jh}=-\left(\left[\mathbf{P}_1^{(m_h-m_j)}+i\omega_j\mathbf{Q}_1^{(m_h-m_j)}\right]\mathbf{a}_j,\ \mathbf{a}_h\right) \quad (j,\ h=j_1,\ \ldots,\ j_r), \tag{6.20}$$

$$\mathbf{P}_1^{(m_h-m_j)}=\frac{1}{T}\int_0^T e^{-i(\omega_h-\omega_j)t}\mathbf{P}_1(t)\,dt,\quad \mathbf{Q}_1^{(m_h-m_j)}=\frac{1}{T}\int_0^T e^{-i(\omega_h-\omega_j)t}\mathbf{Q}(t)\,dt,$$

$$\gamma_{jh}=\begin{cases}0 & j\neq h,\\ 1 & j=h>0,\\ -1 & j=h<0\end{cases}$$

and consider the equation

$$\det\|\sigma_{jh}-i\chi\gamma_{jh}\|_{j,\,h=j_1,\ \ldots,\ j_r}=0. \tag{6.21}$$

The r characteristic exponents of the variational equations (6.13) that tend to $i\omega^{(0)}$ at $\mu=0$ satisfy the estimates

$$\alpha_j(\mu)=i\omega^{(0)}+\chi_j\mu+O(\mu^{1+\delta}) \qquad (\delta>0;\ j=j_1,\ \ldots,\ j_r),$$

where the numbers χ_j are the roots of equation (6.21). If the real parts of $\chi_{j_1},\ \ldots,\ \chi_{j_r}$ are negative, and this is true for all multipliers $\rho^{(0)}$, then the unperturbed motion (6.2) is asymptotically stable for sufficiently small μ. If the real part of at least one of the χ_j's for at least one of the classes $\{\omega_{j_1},\ \ldots,\ \omega_{j_r}\}$ is positive, then the unperturbed motion (6.2) is unstable.

6.6. Example

Consider the scalar Van der Pol equation

$$\ddot\xi+\sigma^2\xi=l\sin t+\mu(1-\xi^2)\dot\xi \qquad (\sigma>0,\ l>0),$$

which was investigated in depth by Andronov and Vitt (/2/, pp. 70–84). The roots of the characteristic equation (6.6) are $\lambda = \pm i\sigma$. We shall study the two possible cases.

a) Nonresonant case (σ is not a natural number). Equation (6.3) is

$$\ddot{\xi}^{(0)} + \sigma^2 \xi^{(0)} = l \sin t$$

and it has a unique 2π-periodic ($T = 2\pi$) generating solution

$$\xi^{(0)} = \frac{l}{\sigma^2 - 1} \sin t.$$

Using the first equation of (6.4), which in this case is

$$\ddot{\xi}^{(1)} + \sigma^2 \xi^{(1)} = \left[1 - \frac{l^2}{(\sigma^2-1)^2} \sin^2 t\right] \frac{l}{\sigma^2-1} \cos t,$$

we determine the first correction

$$\xi^{(1)} = \frac{l}{(\sigma^2-1)^2} \left[1 - \frac{l^2}{4(\sigma^2-1)^2}\right] \cos t + \frac{l^3}{4(\sigma^2-1)^3(\sigma^2-9)} \cos 3t.$$

All subsequent terms in the series (6.2) are also uniquely determined from the second equation of (6.4) and from subsequent equations. The required solution is

$$\xi(t, \mu) = \frac{l}{\sigma^2-1} \sin t + \mu \frac{l}{(\sigma^2-1)^2} \left\{\left[1 - \frac{l^2}{4(\sigma^2-1)^2}\right] \cos t + \frac{l^2}{4(\sigma^2-1)(\sigma^2-9)} \cos 3t\right\} + O(\mu^2).$$

To check for stability, we evaluate ϑ by formula (6.19):

$$\vartheta = \frac{1}{2\pi} \int_0^{2\pi} \left\{\frac{\partial}{\partial \dot{\xi}} [(1-\xi^2)\dot{\xi}]\right\}_0 dt = \frac{1}{2\pi} \int_0^{2\pi} \left[1 - \frac{l^2}{(\sigma^2-1)^2} \sin^2 t\right] dt = 1 - \frac{l^2}{2(\sigma^2-1)^2}.$$

By part 1 of §6.5, we conclude that for sufficiently small μ and

$$l^2 > 2(\sigma^2-1)^2 \tag{6.22}$$

the above 2π-periodic solution $\xi(t, \mu)$ is asymptotically stable; it is unstable if the sign of the inequality is reversed.

b) Resonant case ($\sigma = p$, where p is a natural number). Equation (6.3) is

$$\ddot{\xi}^{(0)} + p^2 \xi^{(0)} = l \sin t$$

and it has a 2π-periodic solution for any $p \geqslant 2$; we shall assume henceforth that $p \geqslant 2$. By §6.2,

$$\xi^{(0)} = \frac{l}{p^2-1} \sin t + \zeta_1 \cos pt + \zeta_2 \sin pt.$$

After obvious manipulations, the left-hand sides of equations (6.11) are determined and the equations themselves are written as

$$\left.\begin{aligned}\psi_1 &= -\frac{\pi}{4}p\zeta_2\left[\zeta_1^2+\zeta_2^2+\frac{2l^2}{(p^2-1)^2}-4\right]=0,\\ \psi_2 &= \frac{\pi}{4}p\zeta_1\left[\zeta_1^2+\zeta_2^2+\frac{2l^2}{(p^2-1)^2}-4\right]=0.\end{aligned}\right\} \tag{6.23}$$

For any p and l, the equations have the trivial solution

$$\zeta_1=\zeta_2=0,$$

and if $l<\sqrt{2}(p^2-1)$ they also have the solution

$$\zeta_1=\zeta_1^{(0)},\ \zeta_2=\zeta_2^{(0)}=\mp\sqrt{4-\frac{2l^2}{(p^2-1)^2}-\zeta_1^{(0)2}},$$

where $\zeta_1^{(0)}$ is any real number whose absolute value is at most $[4-2l^2/(p^2-1)^2]^{1/2}$. For the nontrivial solution, the Jacobian of §6.2 vanishes:

$$\frac{D(\psi_1,\psi_2)}{D(\zeta_1,\zeta_2)}\Big|_{\zeta_1=\zeta_1^{(0)},\ \zeta_2=\zeta_2^{(0)}}=0$$

and so §6.2 does not tell us whether there exists a 2π-periodic solution, analytic in μ at $\mu=0$, whose generating solution is

$$\xi^{(0)}=\frac{l}{p^2-1}\sin t+\zeta_1^{(0)}\cos pt+\zeta_2^{(0)}\sin pt.$$

For further investigation, we refer the reader to /2, 62a/.

Consider the first solution of equations (6.23), i. e., $\zeta_1=\zeta_2=0$. The Jacobian does not vanish:

$$\frac{D(\psi_1,\psi_2)}{D(\zeta_1,\zeta_2)}\Big|_{\zeta_1=\zeta_2=0}=-\pi^2p^2\neq 0,$$

and, by §6.2, there exists a solution analytic in μ at $\mu=0$. The first equation of (6.4) yields the first correction for even p:

$$\xi^{(1)}=\frac{l}{(p^2-1)^2}\left[1-\frac{l^2}{4(p^2-1)^2}\right]\cos t+\frac{l^3}{4(p^2-1)(p^2-9)}\cos 3t+\eta_1\cos pt+\eta_2\sin pt,$$

where η_1 and η_2 are defined by (6.12). The stability condition is again (6.22).

6.7. Example

Consider the system of two differential equations (/98/, p. 88)

$$\left.\begin{aligned}\ddot{\eta}_1+\eta_1 &= \mu(1-\eta_1^2-\eta_2^2)\dot{\eta}_1+\mu q\cos t,\\ \ddot{\eta}_2+\omega_2^2\eta_2 &= \mu(1-\eta_1^2-\eta_2^2)\dot{\eta}_2,\end{aligned}\right\} \tag{6.24}$$

where $\mu > 0$, $q < 0$, $\omega_2 > 0$ and $\omega_2 \neq \frac{1}{2}m$ (m is a natural number). The roots of the characteristic equation (6.6) are

$$\lambda_{\pm 1} = \pm i, \quad \lambda_{\pm 2} = \pm \omega_2 i,$$

and since $\omega = 2\pi/T = 1$, we have the resonant case: $p_{-1} = -1$, $p_1 = 1$; $\lambda_{\pm 2} \neq pi$ (see beginning of §6.2). The generating 2π-periodic solution of system (6.24) will be sought in the form

$$\eta_1^{(0)} = \zeta_1 e^{it} + \bar{\zeta}_1 e^{-it} = 2\,\mathrm{Re}\,[\zeta_1 e^{it}] = 2\alpha_1 \cos t - 2\beta_1 \sin t; \quad \eta_2^{(0)} \equiv 0,$$

where $\zeta_1 = \alpha_1 + i\beta_1$. The left-hand sides of system (6.11) are

$$\psi_{-1} = 2\pi\left[\frac{1}{2}q - i\bar{\zeta}_1(1 - \zeta_1\bar{\zeta}_1)\right], \quad \psi_1 = 2\pi\left[\frac{1}{2}q + i\zeta_1(1 - \zeta_1\bar{\zeta}_1)\right].$$

Adding equations (6.11) and subtracting, we obtain

$$q - 2(1 - \alpha_1^2 - \beta_1^2)\beta_1 = 0, \quad 2i\alpha_1(1 - \alpha_1^2 - \beta_1^2) = 0.$$

Hence we see that $\alpha_1 = 0$, while β_1 must be a root of the cubic equation

$$f(\beta_1) = \beta_1^3 - \beta_1 + \frac{1}{2}q = 0. \tag{6.25}$$

If $q < -4\sqrt{3}/9$, this equation has only one real root $\beta_1^0 > 2\sqrt{3}/3 > 1$; if $-4\sqrt{3}/9 < q < 0$, it has one positive root β_1^0 ($1 < \beta_1^0 < 2\sqrt{3}/3$) and two negative roots β_1^* and β_1^{**} ($\beta_1^* < \beta_1^{**}$). Thus the generating solution of system (6.24) is

$$\eta_1^{(0)} = \zeta_1^{(0)} e^{it} + \bar{\zeta}_1^{(0)} e^{-it} = -2\beta_1 \sin t \qquad (\zeta_1^{(0)} = i\beta_1), \quad \eta_2^{(0)} \equiv 0, \tag{6.26}$$

where β_1 satisfies equation (6.25). Compute the Jacobian of system (6.11):

$$D = \frac{D(\psi_{-1}, \psi_1)}{D(\bar{\zeta}_1^{(0)}, \zeta_1^{(0)})} = 4\pi^2\left[(1 - 2\zeta_1^{(0)}\bar{\zeta}_1^{(0)})^2 - \zeta_1^{(0)2}\overline{\zeta_1^{(0)2}}\right] = 4\pi^2(3\beta_1^4 - 4\beta_1^2 + 1).$$

The Jacobian D vanishes only for $\beta_1 = \pm\sqrt{3}/3$ and $\beta_1 = \pm 1$. Since $\beta_1^0 > 1$, it follows that the Jacobian does not vanish at $\beta = \beta_1^0$. Further, the case $\beta_1 = \pm 1$ is impossible, since $f(\mp 1) < 0$. Thus D vanishes if and only if $\beta_1 = -\sqrt{3}/3$, i. e., $q = -4\sqrt{3}/9$. This case requires further investigation and will not be considered here. In all other cases the solution (6.2) exists and is analytic in μ at $\mu = 0$. The system of equations (6.4) for the first correction is

$$\ddot{\eta}_1^{(1)} + \eta_1^{(1)} = -2\beta_1^3 \cos t, \quad \ddot{\eta}_2^{(1)} + \omega_2^2 \eta_2^{(1)} = 0,$$

whence

$$\eta_1^{(1)} = \varkappa_1 e^{it} + \bar{\varkappa}_1 e^{-it} + \frac{1}{4}\beta_1^3 \cos 3t, \quad \eta_2^{(1)} \equiv 0.$$

One now determines the generating amplitude $\varkappa_1$ of the first correction; we shall not do this here, but go on to consider the stability of the solution (6.2).

The variational equations (6.13) for system (6.24) and the generating solution (6.26) are

$$\frac{d^2\mathbf{y}}{dt^2}+[\mu\mathbf{Q}_1(t)+\dots]\frac{d\mathbf{y}}{dt}+[\mathbf{P}_0+\mu\mathbf{P}_1(t)+\dots]\mathbf{y}=\mathbf{0},$$

where

$$\mathbf{P}_0=\begin{Vmatrix}1&0\\0&\omega_1^2\end{Vmatrix},\quad \mathbf{P}_1(t)=-\left(\frac{\partial \mathbf{g}}{\partial \mathbf{v}}\right)_0=\begin{Vmatrix}4\beta_1^2\sin 2t&0\\0&0\end{Vmatrix},$$

$$\mathbf{Q}_1(t)=-\left(\frac{\partial \mathbf{g}}{\partial \dot{\mathbf{v}}}\right)_0=\begin{Vmatrix}-1+2\beta_1^2-2\beta_1^2\cos 2t&0\\0&-1+2\beta_1^2-2\beta_1^2\cos 2t\end{Vmatrix}.$$

Thus,

$$\mathbf{P}_1^{(2)}=\overline{\mathbf{P}}_1^{(-2)}=\begin{Vmatrix}-2\beta_1^2 i&0\\0&0\end{Vmatrix},\quad \mathbf{Q}_1^{(0)}=\begin{Vmatrix}-1+2\beta_1^2&0\\0&-1+2\beta_1^2\end{Vmatrix},$$

$$\mathbf{Q}_1^{(2)}=\mathbf{Q}_1^{(-2)}=\begin{Vmatrix}-\beta_1^2&0\\0&-\beta_1^2\end{Vmatrix},$$

and all the other Fourier coefficients of $\mathbf{P}_1(t)$ and $\mathbf{Q}_1(t)$ vanish.

The roots of equation (6.15) are $\omega_1=-1$, $\omega_1=1$, $\omega_{-2}=-\omega_2$ and ω_2. The eigenvectors $\mathbf{a}_1$ and $\mathbf{a}_2$ of the matrix $\mathbf{P}_0$, normalized by conditions (6.16), are

$$\mathbf{a}_1=\frac{1}{\sqrt{2}}\mathbf{e}_1,\quad \mathbf{a}_2=\frac{1}{\sqrt{2\omega_2}}\mathbf{e}_2.$$

There is a class of two roots $\{\omega_{-1}, \omega_1\}$ which are congruent modulo $2\pi/T=1$, and all the other roots are by assumption $\left(\omega_2\neq\frac{1}{2}m\right)$ congruent neither to these two nor to one another, forming two classes each containing one root: $\{\omega_{-2}\}$ and $\{\omega_2\}$. We begin with the class of two roots, defining the integers m_{-1} and m_1 by

$$\omega_{-1}=0+(-1)\cdot 1,\quad \omega_1=0+1\cdot 1,$$

so that $m_{-1}=1$, $m_1=1$. We evaluate σ_{jk} from (6.20):

$$\sigma_{-1,-1}=\bar{\sigma}_{11}=\frac{i}{2}(2\beta_1^2-1),\quad \sigma_{-11}=\bar{\sigma}_{1,-1}=\frac{1}{2}\beta_1^2 i$$

and set up equation (6.21):

$$\begin{vmatrix}\sigma_{-1,-1}+i\chi&\sigma_{-11}\\\sigma_{1,-1}&\sigma_{11}-i\chi\end{vmatrix}=\left[\chi+\frac{1}{2}(2\beta_1^2-1)\right]^2-\frac{1}{4}\beta_1^4=0.$$

Its roots are

$$\chi_1=\frac{1}{2}(1-\beta_1^2),\quad \chi_2=\frac{1}{2}(1-3\beta_1^2).$$

If $\beta_1=\beta_1^0>1$, χ_1 and χ_2 are negative. If $\beta_1=\beta_1^*<0$ or $\beta_1=\beta_1^{**}<0$ (this is possible only if $q>-4\sqrt{3}/9$), χ_1 is positive, since by (6.25)

$$1-\beta_1^{*2}=\frac{1}{2}\frac{q}{\beta_1^*}>0,\quad 1-\beta_1^{**2}=\frac{1}{2}\frac{q}{\beta_1^{**}}>0.$$

By the conclusion of §6.5, the periodic solution (6.2) with generating amplitudes $-2\beta_1^*$ and $-2\beta_1^{**}$ (6.26) is unstable. With regard to the stability of the periodic solution (6.2) with generating amplitude $2\beta_1^0$, we must refer to the other two classes $\{\omega_{-2}\}$ and $\{\omega_2\}$. Evaluate ϑ_2 from (6.19):

$$\vartheta_2=\left(\left[\frac{1}{T}\int_0^T\left(\frac{\partial g}{\partial \dot v}\right)_0 dt\right]\mathbf{a}_2,\ \mathbf{a}_2\right)=-(\mathbf{Q}_1^{(0)}\mathbf{a}_2,\ \mathbf{a}_2)=\frac{1-2\beta_1^2}{2\omega_2},\quad \vartheta_{-2}=\vartheta_2.$$

If $\beta_1^0=\beta_1^0>1$, ϑ_2 is negative. Thus, the periodic solution (6.2) with generating amplitude $2\beta_1^0$ is asymptotically stable.