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Inventiones mathematicae

The minimal number of Seifert circles equals the braid index of a link

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Introduction

Alexander proved that any oriented link diagram can be transformed into a closed braid by an ambient isotopy ([1, 2]). But we note that his transformation does not keep invariant the number of Seifert circles and the writhe between the original diagram and the obtained closed braid.

In this paper we will give an alternative method which keeps invariant the number of Seifert circles and the writhe (Theorem 1).

The existence of such a transformation gives that $s(L) \geqq b(L)$ for any oriented link L (Theorem 2), where $s(L)$ denotes the minimal number of Seifert circles of all diagrams for L and $b(L)$ denotes the braid index of L.

It is obvious that $s(L) \leqq b(L)$, therefore we have the following theorem.

Theorem 3. *For any oriented link* L, $s(L)=b(L)$.

Two estimates for the degree of a variable of the two variable Jones polynomial are given in [3], [5] and [6]. We will show that they are essentially equivalent to each other.

1. Notations and definitions

Let D be an oriented link diagram. We define the writhe $wr(D)$ of D by $wr(D) = \sum_c \operatorname{sign}(c)$ where c ranges all of the crossings of D and $\operatorname{sign}(c)$ is defined as in Fig. 1. Let $s(D)$ be the number of Seifert circles of D, where Seifert circles of D are the circles obtained by smoothing all the crossings of D as in Fig. 2.

For a braid b, let $\hat{b}$ be the closed braid diagram got by closing b, i.e. tying the top ends to the bottom ends of b as in Fig. 3, $e(b)$ be the exponent sum of b, and $n(b)$ be the number of strings of b.

Let C and C' be oriented circles on the 2-sphere S^2, we say that C and C' are coherent (anti-coherent resp.) iff $[C]=[C']$ $(-[C']$ resp.$) \in H_1(A)$, where A is the annulus bounded by C and C' on S^2. Let $C_1, \ldots, C_n$ be mutually disjoint

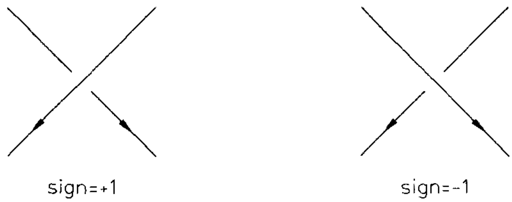


Fig. 1

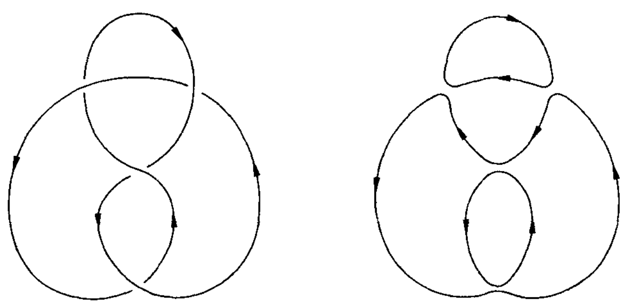


Fig. 2

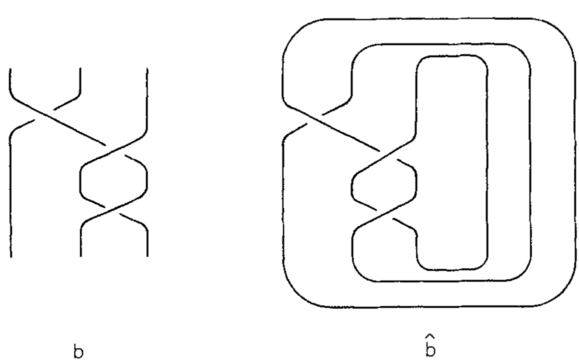


Fig. 3

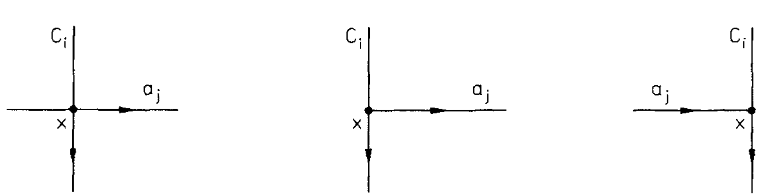
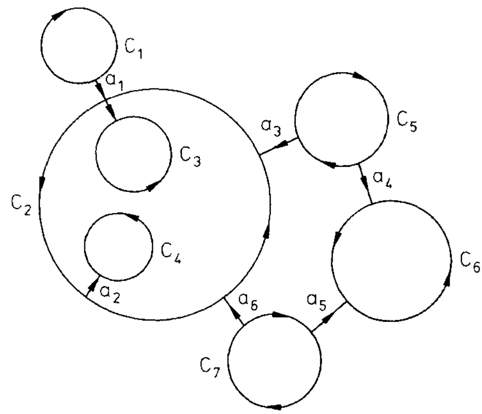


Fig. 4



$w(C_1)=1,\ w(C_2)=2,\ w(C_3)=2,\ w(C_4)=1,\ w(C_5)=1,\ w(C_6)=3,\ w(C_7)=1$

Fig. 5

oriented circles on S^2, and suppose that each circle has a positive integer weight and let $w(C_i)$ denote the weight of C_i. Let $a_1, \ldots, a_r$ be mutually disjoint oriented simple arcs on S^2 such that for all a_j, if $x \in (C_i \cap a_j) \cup \partial a_j$ then a neighbourhood of x is diffeomorphic to one of the situations in Fig. 4.

Then we say that $\{C_1, \ldots, C_n;\ a_1, \ldots, a_r\}$ is a system of weighted Seifert circles. See an example of a system in Fig. 5.

Let $\mathscr{S}$ denote the set of all systems of weighted Seifert circles.

Let $S=\{C_1, \ldots, C_n;\ a_1, \ldots, a_r\}$ be an element of $\mathscr{S}$. Then we will give a method of producing a number of link diagrams from S.

First we replace every C_i with an arbitrary $w(C_i)$-string closed braid. Secondly we replace every a_j with an arbitrary $[w(C_{i_1})+\ldots+w(C_{i_m})]$-string braid so that, it wedges into each $w(C_{i_k})$-string braid C_{i_k}, where $C_{i_1}, \ldots, C_{i_m}$ are the circles of S which intersect a_j. So, we obtain a number of link diagrams from S.

If a link diagram D is obtained from S by using the method as the above, then we say that D is derived from S. There is an example of a diagram in Fig. 6 which is derived from the system in Fig. 5.

Let $\mathscr{D}(S)$ denote the set of link diagrams which are derived from S. For each integer t, let $\mathscr{D}(S,t)=\{D \in \mathscr{D}(S) \mid wr(D)=t\}$. Let $\mathscr{L}(S,t)$ ($\mathscr{L}(S)$ resp.) denote the set of all oriented link types which are presented by a diagram in $\mathscr{D}(S,t)$ ($\mathscr{D}(S)$ resp.).

It is shown easily that for any oriented link diagram D, there exists a system $S \in \mathscr{S}$ such that $D \in \mathscr{D}(S)$. We can choose $S=\{C_1, \ldots, C_n;\ a_1, \ldots, a_r\}$ for such a system, where $C_1, \ldots, C_n$ are the Seifert circles of D and $a_1, \ldots, a_r$ are placed at each crossing of the diagram and $w(C_i)=1$ for all C_i.

Let $S=\{C_1, \ldots, C_n;\ a_1, \ldots, a_r\} \in \mathscr{S}$. We define the total weight $w(S)$ of S by $w(S)=w(C_1)+\ldots+w(C_n)$.

For each positive integer p, let $\mathscr{S}_p=\{S \in \mathscr{S} \mid w(S)=p\}$, and let $T_p=\{C;\ \} \in \mathscr{S}_p$ where $w(C)=p$. Clearly $\mathscr{D}(T_p)$ is the set of all p-string closed braid diagrams. We call T_p the trivial system of total weight p.

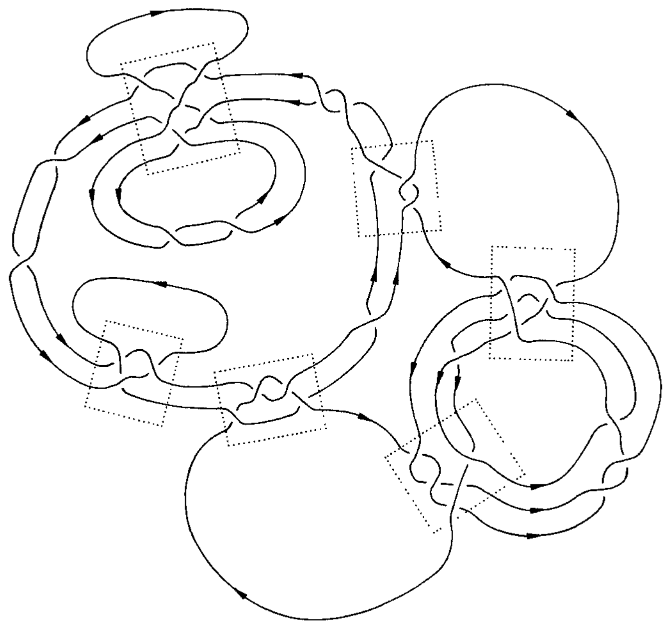


Fig. 6

2. Bunching operations

We define two operations, the bunching operations of type I and type II, by the following.

Let $S=\{C_1,\ldots,C_n;\ a_1,\ldots,a_r\}$ be an element of $\mathcal{S}$.

If there are two coherent circles C_i and C_j such that $\mathrm{Int}(A)\cap\{C_1\cup\ldots\cup C_n\}=\emptyset$, where A is the annulus bounded by C_i and C_j on S^2, then we define a new system S' as follows.

Let C be an abstract circle and $f\colon A\to C$ be a continuous map such that $f|_{C_i}$ and $f|_{C_j}$ are homeomorphisms, if $A\cap a_k\neq\emptyset$ then $f(A\cap a_k)=\{\text{one point}\}$ and if $C_i\cap a_k\neq\emptyset$, $C_j\cap a_l\neq\emptyset$ and $a_k\neq a_l$ then $f(C_i\cap a_k)\neq f(C_j\cap a_l)$. Then the quotient space $(S^2\cup C)/f(x)\sim x$ is a 2-sphere, let $S'=(S\cup C)/f(x)\sim x$ with $w(C)=w(C_i)+w(C_j)$.

We say that S' is derived from S by applying the bunching operation of type I to C_i and C_j. See Fig. 7.

If there are two anti-coherent circles C_i and C_j of S and a band b on S^2 such that $b\cap S=\partial b\cap(C_i\cup C_j)=d_i\cup d_j$ and $b\cap\{a_1\cup\ldots\cup a_r\}=\emptyset$, where d_i and d_j are subarcs of C_i and C_j respectively, then let $S'=\{C_1,\ldots,\check{C}_i,\ldots,\check{C}_j,\ldots,C_n,C;\ a_1,\ldots,a_r\}$. Here $\check{C}_i$, $\check{C}_j$ means the deleting of these circles, $C=(C_i\cup C_j\cup\partial b)$-$\mathrm{Int}(d_i\cup d_j)$, and the orientation of C is determined from those of C_i and C_j naturally with $w(C)=w(C_i)+\omega(C_j)$.

We say that S' is derived from S by applying the bunching operation of type II to C_i and C_j. See Fig. 8.

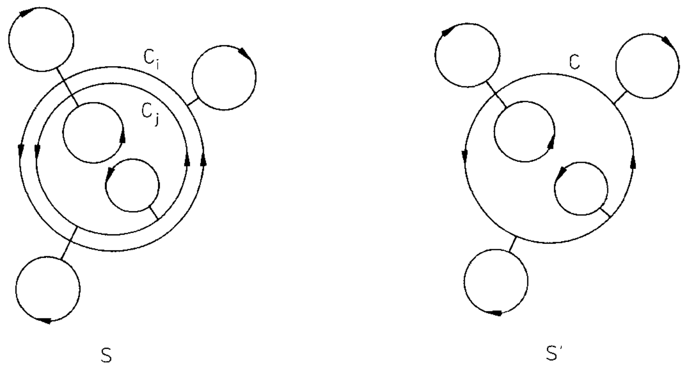


Fig. 7

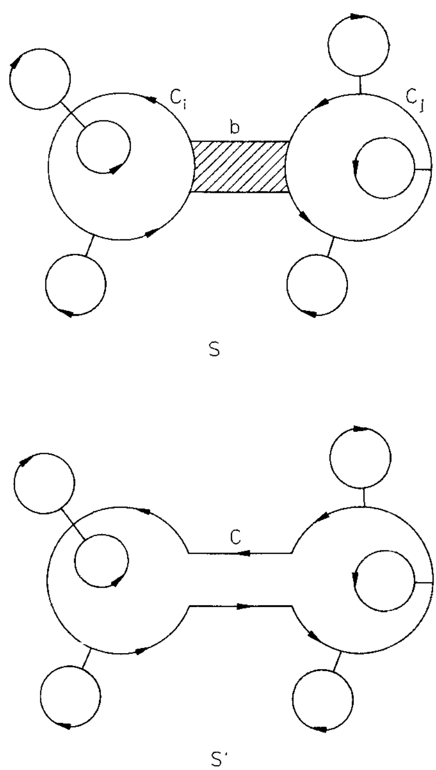


Fig. 8

3. Lemma and Theorem 1

Lemma. *Let S and S' be elements of $\mathscr{S}$. If S' is derived from S by the bunching operation of type I or type II then $w(S)=w(S')$ and $\mathscr{L}(S,t)\subset\mathscr{L}(S',t)$ for any integer t.*

Proof. It is obvious that $w(S)=w(S')$ by the definition of the bunching operations. We will show the latter claim.

In the case of type I, we can show easily that $\mathscr{D}(S,t)\subset\mathscr{D}(S',t)$. Therefore $\mathscr{L}(S,t)\subset\mathscr{L}(S',t)$.

In the case of type II, let C_i and C_j be the circles of S to which the bunching operation of type II is applied. Let b, d_i, d_j be as in the above description. Let N_k be a regular neighbourhood of C_k such that $N_k \cap \{C_1 \cup \ldots \cup C_n\} = C_k$ for $k=i,j$. Let C_i^+ be the component of ∂N_i which intersects b. Let C_j^- be the component of ∂N_j which does not intersect b. Let b_1 be a thin band which joins C_i and C_j^-. Let b_2 be a thick band which joins C_i^+ and C_j, as in Fig. 9. Let

$$C_i' = \{C_i \cup C_j^- \cup \partial b_1 - \mathrm{Int}\{(C_i \cup C_j^-) \cap \partial b_1\},$$
$$C_j' = \{C_j \cup C_i^+ \cup \partial b_2\} - \mathrm{Int}\{(C_j \cup C_i^+) \cap \partial b_2\},$$

$d_i' = C_i \cap b_1$, $d_j' = C_j \cap b_2$, $e_i = C_i' - C_i$ and $e_j = C_j' - C_j$.

Let $S'' = \{C_1, \ldots, \check{C}_i, \ldots, \check{C}_j, \ldots, C_n, C_i', C_j';\ a_1, \ldots, a_r\}$, where the orientations of C_i' and C_j' are defined from the those of C_i and C_j naturally with $w(C_i') = w(C_i)$ and $w(C_j') = w(C_j)$. See Fig. 10.

For any element D of $\mathscr{D}(S,t)$, we can deform D to a element D' of $\mathscr{D}(S'',t)$ under a regular isotopy as in Fig. 11. We call this deformation the bunching deformation.

Figure 11 is understood as the following.

We can assume that d_i' and d_j' are replaced by trivial braids when we produce D from S. Let τ_i and τ_j denote those trivial braids. First we stretch out τ_j to τ_j' and set it on e_j. Secondly we stretch out τ_i to τ_i' and set it on e_i. Let e be e_i (e_j resp.) and τ be τ_i (τ_j resp.). If an arc a of S intersects e then the braid β placed at a is changed by the following map φ,

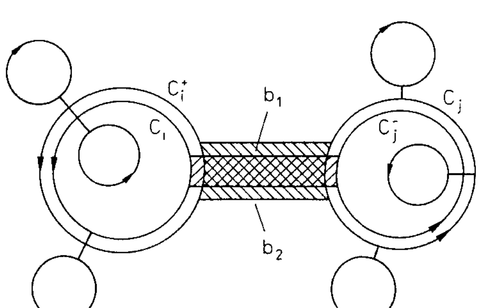


Fig. 9

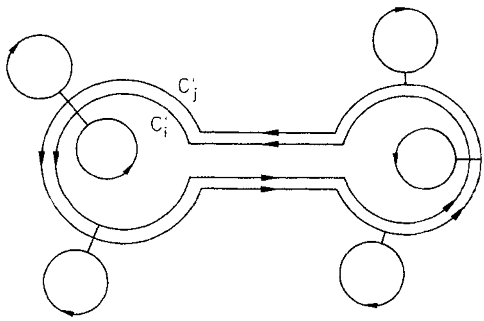


Fig. 10

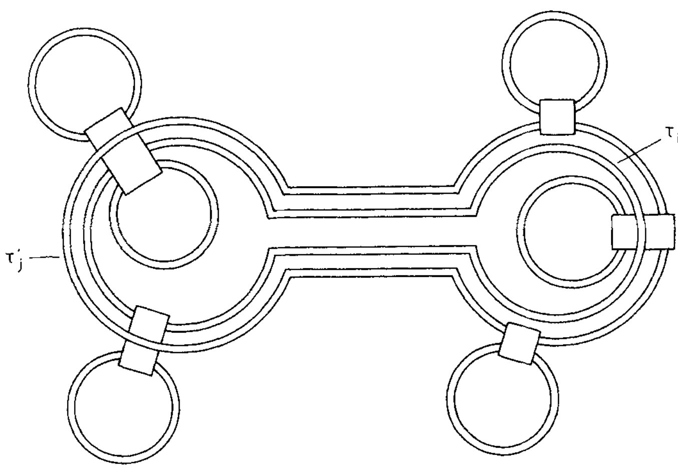


Fig. 11

$$\varphi\colon B_{p+q} \to B_{p+q+r},$$
$$\sigma_k \mapsto \sigma_k \qquad \text{if } 1 \leqq k \leqq p-1,$$
$$\sigma_p \mapsto \rho^{-1}\sigma_p\rho \qquad \text{where } \rho = \prod_{l=p+1}^{p+r} \sigma_l$$
$$\sigma_k \mapsto \sigma_{k+r} \qquad \text{if } p+1 \leqq k \leqq p+q-1, \quad \text{(Fig. 13)},$$

where σ_k is the generator of the braid group which exchanges the k-th string and $(k+1)$-th string by a right hand twist (Fig. 12),

$$p = \sum_{\substack{C \cap a' \neq \emptyset \\ C \in S}} w(C), \qquad q = \sum_{\substack{C \cap a'' \neq \emptyset \\ C \in S}} w(C),$$

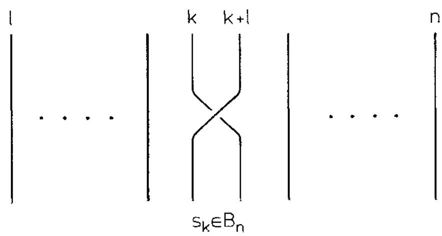


Fig. 12

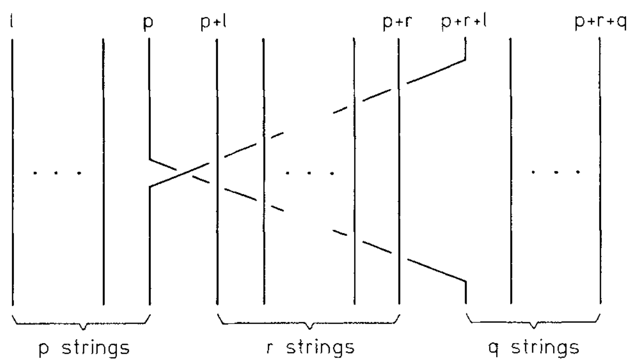


Fig. 13

where a' (a'' resp.) is the component of $a-e$ which contains the starting point (terminal point resp.) of the oriented arc a; and $r=n(\tau)$.

S' is derived from S'' by the bunching operation of type I, therefore $D' \in \mathscr{D}(S', t)$.

We shall note that φ keeps invariant the exponent sum of the braids. Therefore $wr(D)=wr(D')$.

This completes the proof of the lemma. □

We note that the bunching operations decrease the number of circles of a system.

The next Theorem asserts that any oriented link diagram can be deformed to a closed braid by the bunching deformations.

Theorem 1. *If S is a non trivial system then there is a pair of circles of S to which the bunching operation of type I or type II can be applied.*

Proof. Let $S=\{C_1, \dots, C_n;\ a_1, \dots, a_r\}$. We cut the S^2 along $C_1, \dots, C_n$, then there is a piece P which is not a disk because S is not trivial.

In the case that P is an annulus, let $C_i \cup C_j = \partial P$. If C_i and C_j are coherent then the bunching operation of type I can be applied to them. If C_i and C_j are anti-coherent then the bunching operation of type II can be applied to them.

In the case thet P is not an annulus, we cut P along $a_1, \ldots, a_r$. We can assume that every piece is a disk, because if there is a piece which is not a disk then we can add oriented arcs to S to cut the piece into disks. Then there is a piece Q such that $\#(\partial Q \cap (C_1 \cup \ldots \cup C_n)) \geqq 4$ because P is neither an annulus nor a disk. We note that $\#(\partial Q \cap (C_1 \cup \ldots \cup C_n))$ will always be even because of the orientation of arcs.

Let $\partial Q \cap (C_1 \cup \ldots \cup C_n) = E_1 \cup \ldots \cup E_m$, where $E_1, \ldots, E_m$ are on ∂Q in this order, $m \geqq 4$, and m is even. Let $E_j \subset C_{i_j}$. Then C_{i_j} and $C_{i_{j+1}}$ are coherent $(1 \leqq j \leqq m-1)$ therefore C_{i_j} and $C_{i_{j+2}}$ are anti-coherent $(1 \leqq j \leqq m-2)$. We claim that $C_{i_1} \neq C_{i_3}$ or $C_{i_2} \neq C_{i_4}$. Assume that $C_{i_1} = C_{i_3}$ and $C_{i_2} = C_{i_4}$. Let D_j be the disk bounded by C_{i_j} which does not contain Q $(j=1,2)$. Then $Q \cup D_1 \cup D_2$ contains a surface of genus 1, a contradiction.

Hence the bunching operation of type II can be applied to one of the pairs (C_{i_1}, C_{i_3}) and (C_{i_2}, C_{i4}).

This completes the proof of Theorem 1. □

4. Applications

By the lemma and Theorem 1, we get the following Theorem.

Theorem 2. *For any* $S \in \mathscr{S}_p$ *and integer* t, $\mathscr{L}(S,t) \subset \mathscr{L}(T_p, t)$.

This Theorem means that $b(L) \leqq s(L)$ for any oriented link L. Then we get Theorem 3 in the introduction.

Let $\underline{d}(L)$ ($\bar{d}(L)$ resp.) be the lowest (highest resp.) degree of $P_L(l,m)$ about the variable l, where $P_L(l,m)$ is the two variable Jones polynomial of L ([4]). It is shown in [3, 5] and [6] that for any oriented link L, the following inequalities hold.

$$m(L) \leqq \underline{d}(L) \leqq \bar{d}(L) \leqq M(L), \tag{1}$$

$$m'(L) \leqq \underline{d}(L) \leqq \bar{d}(L) \leqq M'(L), \tag{2}$$

where m, M, m' and M' are defined by the following.

$$m(L) = \max\{wr(D) - (s(D)-1) \mid D \text{ is a diagram for } L\},$$
$$M(L) = \min\{wr(D) + (s(D)-1) \mid D \text{ is a diagram for } L\},$$
$$m'(L) = \max\{e(b) - (n(b)-1) \mid \hat{b} \text{ is a closed braid for } L\},$$
$$M'(L) = \min\{e(b) + (n(b)-1) \mid \hat{b} \text{ is a closed braid for } L\}.$$

The next Corollary means that the two inequalities (1) and (2) are essentially equivalent to each other.

Corollary. *For any oriented link* L, $m(L) = m'(L)$ *and* $M(L) = M'(L)$.

Proof. It is obvious that $m'(L) \leqq m(L)$ and $M(L) \leqq M'(L)$ because $wr(\hat{b}) = e(b)$ and $s(\hat{b}) = n(b)$ for any braid b.

To show the reverse inequalities of the above, we assume that D_1 and D_2 are diagrams for L such that $m(L) = wr(D_1) - (s(D_1)-1)$ and $M(L) = wr(D_2) + (s(D_2)-1)$. By Theorem 2 we can transform D_1 and D_2 to a closed braid $\hat{b}_1$

and $\hat{b}_2$ respectively such that $wr(D_1)=wr(\hat{b}_1)$, $s(D_1)=s(\hat{b}_1)$, $wr(D_2)=wr(\hat{b}_2)$ and $s(D_2)=s(\hat{b}_2)$. Then we get that $m'(L)\geqq m(L)$ and $M(L)\geqq M'(L)$. □

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