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L-groups of crystallographic groups

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Introduction

In their paper [6], Farrell and Hsiang proved the following:

Theorem. *Let N^n be a closed connected flat Riemannian manifold where $n \neq 3, 4$ and let M^n be an aspherical manifold such that $\pi_1(M^n)$ is isomorphic to $\pi_1(N^n)$, then M^n and N^n are homeomorphic.*

This is equivalent to the statement that the structure set $\mathscr{S}_{\mathrm{TOP}}(N)$ consists of a single element. The structure set appears in the surgery exact sequence [16]: $\ldots \to [\Sigma N, G/\mathrm{TOP}] \to L_{n+1}(\pi_1 N) \to \mathscr{S}_{\mathrm{TOP}}(N) \to [N, G/\mathrm{TOP}] \to L_n(\pi_1 N)$, and the result is proved by showing that the homomorphisms $[\Sigma^i N, G/\mathrm{TOP}] \to L_{n+i}(\pi_1 N)$ are bijections.

We would like to prove a similar result for certain stratified spaces. Let Γ be a crystallographic group acting on $\mathbb{R}^n$ and consider the orbit space $\mathbb{R}^n/\Gamma$. If Γ is torsion-free, $\mathbb{R}^n/\Gamma$ is an n-dimensional closed flat Riemannian manifold, and Farrell and Hsiang's result will apply to this. If Γ has torsion, $\mathbb{R}^n/\Gamma$ is a stratified space. The following is our conjecture.

Conjecture. *If a stratified space is homotopy equivalent to $\mathbb{R}^n/\Gamma$ in some nice way, then it is homeomorphic to $\mathbb{R}^n/\Gamma$.*

This paper is the first step toward this conjecture. As with the Farrell-Hsiang theorem this conjecture is approached by showing the functions in appropriate "stratified" exact sequences are bijections. Our main result is a partial computation of one of the terms in these exact sequences, specifically the L-groups of Γ. Actually we need to do the computation for a slightly larger class of groups. Let Λ be a finitely generated group which maps onto a crystallographic group Γ of rank n with a finite kernel; in other words, Λ acts by isometries on $\mathbb{R}^n$ discretely, virtually faithfully, with a compact orbit space. Let W_Λ be a free contractible Λ-space. Then the map $p\colon (\mathbb{R}^n \times W_\Lambda)/\Lambda \to \mathbb{R}^n/\Lambda$ has point inverses $p^{-1}(x) = W_\Lambda/\Lambda_x$, which are classifying spaces for "isotropy" subgroups Λ_x. Quinn has defined Ω-spectra $\mathbb{L}(X)$ whose homotopy groups are the surgery obstruction groups $L_i(\pi_1 X)$ [7]. This functor can be applied fibrewise to obtain a "sheaf" of spectra $\mathbb{L}(p) \to \mathbb{R}^n/\Lambda$, with fibre over x, $\mathbb{L}(P^{-1}(x))$. Next Quinn has defined [9] homology groups with spectral sheaf coefficients $H_*(\mathbb{R}^n/\Lambda;\mathbb{L}(p))$. For technical reasons we use a definition of $\mathbb{L}$ using the Poincaré chain complex of Ranicki [12, 13]. The homotopy groups are the limits $L_i^{-\infty}$ of Ranicki's lower L-theory $L_i^{\langle -m\rangle}$ [11] which may differ from L_i possibly by 2-torsion. The following is our main theorem.

Theorem (4.11). *If Λ is a finitely generated group which acts by isometries on $\mathbb{R}^n$ discretely, virtually faithfully, with compact quotient, then there is a natural isomorphism modulo 2-torsion:*

$$a\colon H_*(\mathbb{R}^n/\Lambda;\mathbb{L}(p))\to L_*^{-\infty}(\Lambda).$$

The map a is essentially Quinn's assembly map. To explain the calculational significance of the theorem, let us recall that there is an Atiyah-Hirzebruch type spectral sequence [9, Theorem 8.7] $H_i(\mathbb{R}^n/\Lambda;L_j^{-\infty}(p)) \Rightarrow H_{i+j}(\mathbb{R}^n/\Lambda;\mathbb{L}(p))$. Here $L_j^{-\infty}(p)$ is obtained by applying π_j fibrewise to $\mathbb{L}(p)$, or equivalently applying $L_j^{-\infty}$ to the sheaf of isotropy subgroups Λ_x. Since the isotropy subgroups are finite, the theorem provides a "calculation" of $L^{-\infty}(\Lambda)$ in terms of $L^{-\infty}$ of finite subgroups.

This is the first systematic calculation of surgery obstruction groups for a class of infinite groups with torsion. Previously purely algebraic techniques have been mostly limited to finite groups, and geometric techniques limited to torsion free groups. Here we refine the geometric techniques to reduce calculations for these infinite groups to those for finite groups, which are accessible to the algebra.

The organization of this paper is as follows. In § 1, we prove the splitting lemma for quadratic Poincaré complexes and pairs. In § 2, we introduce quadratic complexes with geometric control in the sense of [8–10]. Using the results in § 1, we prove the stable splitting lemma for geometric quadratic Poincaré complexes and pairs (2.5) and the stable splitting lemma over a manifold (2.11). In § 3, we construct the spectra $\mathbb{L}(X;p)$ and $\mathbb{H}(X;\mathbb{L}(p))$ for stratified systems of fibrations $p\colon E\to X$ [9], define the assembly maps, and prove the characterization theorem (3.9), which characterizes elements of homology groups as certain objects with geometric control. At the same time the shrinking lemma (3.10) can be proved. The key ingredient of the proof of these is an application of the stable splitting lemma of the previous section. We also show in this section that the homotopy groups of $\mathbb{L}(*;B_\Lambda\to *)$ are the limits of Ranicki's lower L-groups of Λ, where B_Λ is the classifying space of Λ. In § 4, the main theorem (4.11) is proved. The proof is like that in [6]. We use induction on the "size" of the action of Λ (= the size of Γ), Farrell and Hsiang's structure theorem of crystallographic groups (4.2), and Dress's induction theory [2]. When the induction on the size of Γ does not work, we need to use the characterization theorem and the shrinking lemma.

This is a work done basically in the author's Ph.D thesis at Virginia Polytechnic Institute and State University. I would like to thank Professor F.S. Quinn for suggesting this problem to me and for providing many invaluable comments.

1. Splitting lemma for quadratic Poincaré complexes and pairs

In this section, we give a sufficient condition to splitting quadratic Poincaré pairs defined by Ranicki. We use notations and sign conventions used in [13]. R denotes a ring with 1 and an involution. Let $c=(f\colon C\to D, (\delta\psi,\psi))$ be an $(n+1)$-dimensional quadratic Poincaré pair over R, and C' (resp. D') be an n-dimensional subcomplex of C (resp. an $(n+1)$-dimensional subcomplex of D). We assume that

(1.1) C/C' is n-dimensional, D/D' is $(n+1)$-dimensional, and

(1.2) $f(C')\subset D'$.

Let i_C denote the inclusion map of C' into C, and p_C denote the projection map of C onto C/C'. We fix splittings $j_C\colon C/C'\to C$ and $q_C\colon C\to C'$ of p_C and i_C. These give an identification of C with $C'\oplus(C/C')$, and if we define a chain map $\rho_C\colon (C/C')_r\to(SC')_r=C'_{r-1}$ by $(-)^{r-1}q_C dj_C$, then the boundary map of C is given by a matrix

$$\begin{pmatrix} d & (-)^{r-1}\rho_C \\ 0 & d \end{pmatrix}\colon C'_r\oplus(C/C')_r\to C'_{r-1}\oplus(C/C')_{r-1}$$

under this identification. Here S denotes the suspension of a chain complex. Maps i_D, p_D, j_D, q_D, ρ_D are defined in the same way for D.

Let C'' denote the chain complex $(C/C')^{n-*}$. By the assumption 1.2, f induces a chain map $f'=q_D f i_C\colon C'\to D'$ such that $i_D f'=f i_C$. The algebraic mapping cone $C(f')$ of f' is a subcomplex of the algebraic mapping cone $C(f)$ of f. Define D'' by $(C(f)/C(f'))^{n+1-*}$, or equivalently $C(p_D f j_C)^{n+1-*}$. There is a chain map (inclusion map)

$$f''=\begin{pmatrix}0\\1\end{pmatrix}\colon C''_r=(C/C')^{n-r}\to D''_r=(D/D')^{n+1-r}\oplus(C/C')^{n-r}$$

from C'' to D''. Recall that $(1+T)\psi_0\colon C^{n-*}\to C$, and $((1+T)\delta\psi_0, (-)^r(1+T)\psi_0 f^*)\colon D^{n+1-r}\to C(f)_r=D_r\oplus C_{r-1}$ are chain equivalences. Here $*$ denotes the dual.

(1.3) **Lemma** *(Splitting lemma for quadratic Poincaré pairs). Let c, C', D', C'', D'' be as above. We further assume that*

(1.4) $$H_i(p_C(1+T)\psi_0 p_C^*)=0 \quad \textit{for } i<0,$$

and

(1.5) $$H_i(p_D((1+T)\delta\psi_0 p_D^*, \quad f(1+T)\psi_0 p_C^*))=0 \quad \textit{for } i<0.$$

Then there are two adjoining $(n+1)$-dimensional quadratic Poincaré triads over R:

$$\left(\mathcal{T}_1:\ \begin{array}{ccc} B & \longrightarrow & C^! \\ \downarrow & \searrow & \downarrow \\ C' & \longrightarrow & D' \end{array},\ \Psi_1\right) \quad \text{and} \quad \left(\mathcal{T}_2:\ \begin{array}{ccc} B & \longrightarrow & C'' \\ \downarrow & \searrow & \downarrow \\ C^! & \longrightarrow & D'' \end{array},\ \Psi_2\right)$$

whose union is homotopy equivalent to c.

Proof. We give explicit matrices describing the quadratic structures and the equivalence. This will be necessary later when we will have to check that these satisfy certain size estimate. B and $C^!$ in the lemma are defined to be $S^{-1}C((i_C, (1+T)\psi_0 p_C^*)\colon C'\oplus C''\to C)$ and $S^{-1}C((i_D, (1+T)\delta\psi_0 p_D^*, f(1+T)\psi_0 p_C^*)\colon D'\oplus D''\to D)$ respectively. Here S^{-1} denotes the desuspension. We have a chain map $g^! = f'\oplus f\oplus f''\colon B_r = C'_r\oplus C_{r+1}\oplus C''_r \to C^!_r = D'_r\oplus D_{r+1}\oplus D''_r$. Notice that, since B and $C^!$ are chain equivalent to $S^{-1}C(p_C(1+T)\psi_0 p_C^*)$ and $S^{-1}C(p_D((1+T)\delta\psi_0 p_D^*, f(1+T)\psi_0 p_C^*))$, the assumptions 1.4 and 1.5 imply that B and $C^!$ are chain equivalent to non-negative chain complexes. We define an n-dimensional quadratic structure $(\delta\bar{\psi}, -\bar{\psi})$ on the pair $g^!\colon B\to C^!$ as follows:

$$\delta\bar{\psi}_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-)^{n-r-s}T\delta\psi_{s-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (s>1)$$

$$: C^{!n-r-s} = D'^{n-r-s}\oplus D^{n+1-r-s}\oplus C''^{n-r-s} \to C^!_r = D'_r\oplus D_{r+1}\oplus D''_r$$

$$\delta\bar{\psi}_0 = \begin{pmatrix} 0 & q_D(1+T)\delta\psi_0 & (-)^{nr+r}\rho' \\ 0 & 0 & (-)^{nr+r+1}j' \\ 0 & 0 & 0 \end{pmatrix}$$

$$: C^{!n-r} = D'^{n-r}\oplus D^{n+1-r}\oplus D''^{n-r} \to C^!_r = D'_r\oplus D_{r+1}\oplus D''_r$$

$$\bar{\psi}_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-)^{n-r-s-1}T\psi_{s-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (s>1)$$

$$: B^{n-1-r-s} = C'^{n-1-r-s}\oplus C^{n-r-s}\oplus C''^{n-1-r-s} \to B_r = C'_r\oplus C_{r+1}\oplus C''_r$$

$$\bar{\psi}_0 = \begin{pmatrix} 0 & q_C(1+T)\psi_0 & (-)^{nr}\rho_C \\ 0 & 0 & (-)^{nr+1}j_C \\ 0 & 0 & 0 \end{pmatrix}$$

$$: B^{n-1-r} = C'^{n-1-r}\oplus C^{n-r}\oplus C''^{n-1-r} \to B_r = C'_r\oplus C_{r+1}\oplus C''_r,$$

where $\rho' = (\rho_D, q_D f j_C)\colon D''^{n-r} = (D/D')_{r+1}\oplus (C/C')_r \to D'_r$ and $j' = (j_D, 0)\colon D''^{n-r}\to D_{r+1}$. A direct calculation shows that this is Poincaré. And the duality implies that B is $(n-1)$-dimensional and $C^!$ is n-dimensional. Thus $(g^!\colon B\to C^!)$ is an n-dimensional quadratic Poincaré pair. Now the triads in the lemma are given by

$$\left(\mathcal{T}_1:\ \begin{array}{ccc} B & \xrightarrow{g^!} & C^! \\ g'_C\downarrow & \searrow^{0} & \downarrow g'_D \\ C' & \xrightarrow[f']{} & D' \end{array},\ \Psi_1=(\bar\psi,0,-\delta\bar\psi,0)\right)$$

$$\left(\mathcal{T}_2:\ \begin{array}{ccc} B & \xrightarrow{g''_C} & C'' \\ g^!\downarrow & \searrow^{0} & \downarrow f'' \\ C^! & \xrightarrow[g''_D]{} & D'' \end{array},\ \Psi_2=(\bar\psi,\delta\bar\psi,0,0)\right)$$

where $g'_C=(-1,0,0)$, $g'_D=(-1,0,0)$, $g''_C=(0,0,1)$, $g''_D=(0,0,1)$. These are $(n+1)$-dimensional quadratic Poincaré triads. There union is

$$\begin{aligned}(F=f'\oplus g^!\oplus(-f'')\colon (C'\cup_B C'')_r = C'_r\oplus B_{r-1}\oplus C''_r \to (D'\cup_{C^!}D'')_r \\ = D'_r\oplus C^!_{r-1}\oplus D''_r,\quad (0\cup_{\delta\bar\psi}0,0\cup_{\bar\psi}0))\end{aligned}$$

where $C'\cup_B C''$ and $D'\cup_{C^!}D''$ are push-outs of

$$C' \xleftarrow{g'_C} B \xrightarrow{g''_C} C''$$

$$D' \xleftarrow{g'_D} C^! \xrightarrow{g''_D} D'',$$

i.e., $C({}^t(g'_C,g''_C))$ and $C({}^t(g'_D,g''_D))$ respectively, and

$$\begin{aligned}(0\cup_{\delta\bar\psi}0)_s &= 0\oplus 0\oplus \delta\psi_s\oplus 0\oplus 0\colon\ D'^{n+1-r}\oplus D'^{n-r}\oplus D^{n+1-r}\oplus D''^{n-r}\oplus D''^{n+1-r} \\ &\to D'_r\oplus D'_{r-1}\oplus D_r\oplus D''_{r-1}\oplus D''_r \\ (0\cup_{\bar\psi}0)_s &= 0\oplus 0\oplus \psi_s\oplus 0\oplus 0\colon\ C'^{n-r}\oplus C'^{n-1-r}\oplus C^{n-r}\oplus C''^{n-1-r}\oplus C''^{n-r} \\ &\to C'_r\oplus C'_{r-1}\oplus C_r\oplus C''_{r-1}\oplus C''_r.\end{aligned}$$

Chain equivalences

$$\begin{aligned}{}^t(0,0,1,0,0)&\colon\ C\to C'\cup_B C'',\ \text{and} \\ {}^t(0,0,1,0,0)&\colon\ D\to D'\cup_{C^!}D''\end{aligned}$$

make the following diagram commute:

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ \downarrow & & \downarrow \\ C'\cup_B C'' & \xrightarrow[F]{} & D'\cup_{C^!}D'', \end{array}$$

and

$$ {}^{t}(0,0,1,0,0)\,\psi_s(0,0,1,0,0)=(0\cup_{\bar{\psi}}0)_s $$
$$ {}^{t}(0,0,1,0,0)\,\delta\psi_s(0,0,1,0,0)=(0\cup_{\delta\bar{\psi}}0)_s, $$

for each $s\geqq 0$. Therefore the union is homotopy equivalent to the original quadratic Poincaré pair, and the lemma is proved. □

(1.6) *Remarks.* (1) If c is strictly n-dimensional, then the conditions 1.4 and 1.5 are satisfied when the images of C'^n and D'^{n+1} by $(1+T)\psi_0 q_C^*$ and $(1+T)\delta\psi_0 q_D^*$ lie in C'_0 and D'_0, respectively.

(2) If C, C', D', C/C', D/D' are free, then the above argument can be carried through in the category of free chain complexes.

(3) If the splitting of the boundary is already given, we can construct a splitting with the given splitting of the boundary.

2. Geometric chain complexes and the stable splitting lemma

Geometric modules, geometric morphisms, and geometric chain complexes are defined in [10, § 1]. Since these are essential to this paper, we review them at first. Let p: $E\to X$ be a map, where X is a metric space. This is called the control map.

(2.1) **Definitions** (Quinn). A *geometric* $\mathbb{Z}$*-module on* E *(generated by a set* S*)* is a free module $\mathbb{Z}[S]$ together with a map of the basis f: $S\to E$. We usually require that geometric $\mathbb{Z}$-modules be *locally finite* in the sense that every point in E has a neighborhood whose preimage by f in S is finite.

A *geometric morphism* h: $\mathbb{Z}[S]\to\mathbb{Z}[T]$ of geometric $\mathbb{Z}$-modules with f: $S\to E$ and g: $T\to E$ is a sum $\Sigma m_j\rho_j^{(x,y)}$, where $m_j\in\mathbb{Z}$, $x\in S$, $y\in T$, and $\rho_j^{(x,y)}$ is a path: $[0,t_j^{(x,y)}]\to E$ which starts at $f(x)$ and ends at $g(y)$. Here $t_j^{(x,y)}$ is a non-negative real number. We require that the sum is locally finite in the sense that for each $x\in S$ there are only finitely many paths $\rho_j^{(x,y)}$ with non-zero coefficient, and for each $y\in T$ there are only finitely many paths $\rho_j^{(x,y)}$ with non-zero coefficient. In a morphism we allow a deletion or an insertion of a path with coefficient 0. Morphisms are composed by composing the component paths and multiplying coefficients: let $h=\Sigma m_j\rho_j^{(x,y)}$: $\mathbb{Z}[S]\to\mathbb{Z}[T]$, $k=\Sigma n_i\mu_i^{(y,8)}$: $\mathbb{Z}[T]\to\mathbb{Z}[U]$, then $kh=\Sigma n_i m_j\mu_i^{(y,8)}\rho_j^{(x,y)}$. Here the Moore composition of paths is used, and we write compositions of paths from right to left.

A *homotopy* ($\sim$) of a morphism is obtained by changing all the paths in the morphism by homotopy keeping the endpoints fixed.

A *morphism has radius* ε *(in* X*)* if each path with non-zero coefficient in the morphism has image in X (via p) inside the closed ball of radius ε about its starting point. Similarly, a *homotopy of morphism has radius* ε if the homotopy of each path has image in X inside the closed ball of radius ε about the starting point of the path. The symbol $\underset{\varepsilon}{\sim}$ denotes a homotopy of radius ε.

Suppose W is a subset of X. The *restriction* $\mathbb{Z}[S]|W$ *of a geometric module* $\mathbb{Z}[S]$, f: $S\to E$, *to* W is $\mathbb{Z}[(pf)^{-1}(W)]$. The *restriction* $h|W$ *of a morphism* h is obtained by throwing away all the paths with initial point outside of $p^{-1}(W)$.

A geometric morphism $h: \mathbb{Z}[S] \to \mathbb{Z}[T]$ of radius ε is an *ε-isomorphism (with support W)* if there is a geometric morphism $k: \mathbb{Z}[T] \to \mathbb{Z}[S]$ of radius ε such that there are ε homotopies (=homotopies of radius ε) $kh \underset{\varepsilon}{\sim} 1$, $hk \underset{\varepsilon}{\sim} 1 (kh|W \underset{\varepsilon}{\sim} 1, hk|W \underset{\varepsilon}{\sim} 1)$. Here 1's denote the "identity" morphisms made up of appropriate constant paths with coefficient 1. According to this definition, an ε-isomorphism with support W may *not* be an ε-isomorphism. It is an ε-isomorphism if $W = X$.

(2.2) *Remarks.* (1) We sometimes do not mention the map of the basis f: $S \to E$ of a geometric $\mathbb{Z}$-module $\mathbb{Z}[S]$ and pretend as if the basis points were in E.

(2) Suppose E has a universal cover $\tilde{E}$. Then by taking the pull-back $\tilde{S} \to E$ of geometric $\mathbb{Z}$-module $\mathbb{Z}[S]$ on E, we get a geometric $\mathbb{Z}$-module on $\tilde{E}$, which is naturally a free $\mathbb{Z}[\pi_1 E]$-module generated by S. A geometric morphism between geometric $\mathbb{Z}$-modules on E induces a $\mathbb{Z}[\pi_1 E]$-module homomorphism.

(2.3) **More Definitions.** A *geometric $\mathbb{Z}$-module chain complex C on E (with support $K \subset X$)* is a sequence of morphisms of geometric $\mathbb{Z}$-modules on E:

$$C: \ldots \to C_i \xrightarrow{d_i} C_{i-1} \xrightarrow{d_{i-1}} \ldots$$

such that $d_{i-1} d_i \sim 0 (d_{i-1} d_i | K \sim 0)$. Here 0 denotes the empty morphism. Such *C has radius* ε if all d_i's have radius ε and there are homotopies $d^2 \sim 0 (d^2 | K \sim 0)$ of radius ε. The word "support" may not give a correct impression. It does *not* mean that C lies over K, but it means that, roughly speaking, C is a chain complex when restricted to K and may *not* be so outside of K. Thus C is a geometric chain complex if it has support X. Similar remarks will apply to the followings.

A *chain map f* (with support K) between geometric $\mathbb{Z}$-module chain complexes C, D (with support K) consists of geometric morphisms f_i: $C_i \to D_i$ such that $df \sim fd (df|K \sim fd|K)$. It has *radius* ε if all f_i's have radius ε and there are homotopies $df \sim fd (df|K \sim fd|K)$ of radius ε.

A *chain homotopy (with support $K \subset X$)* between two chain maps f, g: $C \to D$ (with support K) is a collection $\{H_i\}$ of geometric morphisms H_i: $C_i \to D_{i+1}$ such that $dH_i + H_{i-1} d \sim f_i - g_i ((dH_i + H_{i-1} d)|K \sim (f_i - g_i)|K)$ for all i. It has radius ε if all the H_i's have radius ε and there are ε homotopies $dH_i + H_{i-1} d \underset{\varepsilon}{\sim} f_i - g_i ((dH_i + H_{i-1} d)|K \underset{\varepsilon}{\sim} (f_i - g_i)|K)$.

A chain map f: $C \to D$ (with support K) is a *chain equivalence (with support K)* if there are a chain map g: $D \to C$ (with support K) and chain homotopies $gf \sim 1$ and $fg \sim 1$ (with support K). It is an *ε-chain equivalence (with support K)* if f, g and the two chain homotopies have radius ε.

A *chain contraction of C with support $K \subset X$ (of radius ε)* is a chain homotopy $\{s_i: C_i \to C_{i+1}\}$ with support K between 1 and 0: $C \to C$ (of radius ε).

The *dual* $\mathbb{Z}[S]^*$ of $\mathbb{Z}[S]$ is $\mathbb{Z}[S]$ itself. When we have a geometric morphism $h = \Sigma m_j \rho_j^{(x,y)}$: $\mathbb{Z}[S] \to \mathbb{Z}[T]$, its *dual* h^*: $\mathbb{Z}[T]^* \to \mathbb{Z}[S]^*$ is defined to be $h^* = \Sigma m_j \mu_j^{(y,x)}$, where $\mu_j^{(y,x)}(t) = \rho_j^{(x,y)}(t_j^{(x,y)} - t)$.

The *tensor product over* $\mathbb{Z}$ of a geometric $\mathbb{Z}$-module $\mathbb{Z}[S]$, $f: S \to E$, on E and a geometric $\mathbb{Z}$-module $\mathbb{Z}[T]$, $g: T \to F$, on F, denoted $\mathbb{Z}[S] \otimes \mathbb{Z}[T]$, is a geometric $\mathbb{Z}$-module $\mathbb{Z}[S \times T]$, $f \times g: S \times T \to E \times F$, on $E \times F$. The *tensor product* $h \otimes k$ of geometric morphisms $h = \Sigma m_j \rho_j^{(x,x')}: \mathbb{Z}[S] \to \mathbb{Z}[S']$ (on E) and $k = \Sigma n_i \mu_i^{(y,y')}: \mathbb{Z}[T] \to \mathbb{Z}[T']$ (on F) is defined to be $\Sigma m_j n_i \rho_j^{(x,x')} \otimes \mu_i^{(y,y')}$, where $\rho_j^{(x,x')} \otimes \mu_i^{(y,y')}$ is a path from (x,y) to (x',y') in $E \times F: [0, t_j + u_i] \to E \times F$ sending t to $(\rho_j^{(x,x')}(t), y)$ if $0 \leqq t \leqq t_j$ and to $(x', \mu_i^{(y,y')}(t - t_j))$ if $t_j \leqq t \leqq t_j + u_i$ $(\rho_j^{(x,x')}: [0, t_j] \to E$ and $\mu_i^{(y,y')}: [0, u_i] \to F)$. (See Remark 2.2(1).) The morphism $(hh') \otimes (kk')$ is homotopic to $(h \otimes k)(h' \otimes k')$.

In § 1, we studied quadratic Poincaré complexes. Now we define geometric quadratic Poincaré complexes. Again $p: E \to X$ is the control map.

(2.4) **Definitions.** An *n-dimensional geometric $\mathbb{Z}$-module quadratic complex* $c = (C, \psi)$ *on* E with support K consists of the underlying strictly n-dimensional geometric $\mathbb{Z}$-module chain complex C on E with support K and a quadratic structure $\psi = \{\psi_s | s \geqq 0\}$. Here ψ_s is a set of geometric morphisms: $C^{n-r-s} = (C_{n-r-s})^* \to C_r$ $(r \in \mathbb{Z})$ such that

$$(*) \qquad [d\psi_s + (-)^r \psi_s d^* + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1} T\psi_{s+1})] | K \sim 0$$
$$: C^{n-r-s-1} | K \to C_r \qquad (s \geqq 0).$$

T above sends $f: C^p \to C_q$ to $(-)^{pq} f^*: C^q \to C_p$. Such a complex c is *Poincaré* if

$$(1+T)\psi_0: C^{n-*} \to C$$

is a chain equivalence with support K. When the support is X, we do not mention the support. It *has radius* ε if C, all the morphisms ψ_s, and all the homotopies $(*)$ above have radius ε (and $(1+T)\psi_0$ is an ε-chain equivalence with support K when it is Poincaré).

A *map* (resp. *homotopy equivalence*) of n-dimensional geometric $\mathbb{Z}$-module quadratic complexes on E (with support K)

$$f: (C, \psi) \to (C', \psi')$$

is a chain map (resp. chain equivalence) $f: C \to C'$ (with support K) such that

$$(**) \qquad f\psi_s f^* \sim \psi_s' (f\psi_s f^* | K \sim \psi_s' | K).$$

A homotopy equivalence f is an *ε-homotopy equivalence (with support K)* if $f: C \to C'$ is an ε-chain equivalence (with support K) and the homotopies $(**)$ have radius ε.

Similarly, *geometric symmetric Poincaré complexes (of radius ε)* and *geometric quadratic* or *symmetric Poincaré pairs* and *triads (of radius ε)* can be defined by replacing everything in the standard definition by geometric objects (of radius ε) and replacing the necessary identities by homotopies (of radius ε). If M is a PL manifold of dimension m, then a PL triangulation of M produces an m-dimensional geometric $\mathbb{Z}$-module symmetric Poincaré complex on M. See Ranicki [13] and Quinn [10]. This will be denoted $\sigma^*(M)$. *Unions* and *tensor products* of geometric complexes and pairs are defined by the same formulae as the standard cases. Then the following is the main technical result of this section.

(2.5) **Lemma.** (Stable splitting lemma for geometric quadratic Poincaré pairs). *Fix n and $p: E \to X$. Given any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $(n+1)$-dimensional geometric $\mathbb{Z}$-module quadratic Poincaré pair c on E of radius δ in X and a subset Y of X, the $(n+2)$-dimensional geometric $\mathbb{Z}$-module quadratic Poincaré pair $\sigma^*(S^1) \otimes c$ on $S^1 \times E$ is ε-homotopy equivalent to a union of two $(n+2)$-dimensional geometric $\mathbb{Z}$-module quadratic Poincaré triads of radius ε lying over Y^ε and $X - Y^{-\varepsilon}$ respectively. Here we use the control map $p' = (\text{constant map}) \times p: S^1 \times E \to \{pt.\} \times X = X$.*

To prove this, we need the following local "folding" argument (cf. [1] § 14).

(2.6) **Lemma.** *If a geometric $\mathbb{Z}$-module chain complex (C, d) on E of radius δ in X has a chain contraction with support $W (\subset X)$ of radius δ and satisfies the following:*

$$C_l | W = 0 \quad \text{for } l < k,$$

then C is 2δ-chain equivalent to a geometric $\mathbb{Z}$-module chain complex (C', d') on E of radius δ in X such that

(1) *$C'_l | W^{-\delta} = 0$ for $l \leqq k$,*
(2) *$C' | X - W = C | X - W$, and*
(3) *there exists a chain contraction of C' with support $W^{-2\delta}$ of radius 3δ.*

Proof. Let $\{s_i\}$ denote the chain contraction of C with support W, and let i, j, r, q denote the following canonical inclusion morphisms and projection morphisms:

$$C_k | X - W^{-\delta} \underset{r}{\overset{i}{\rightleftarrows}} C_k \underset{j}{\overset{q}{\rightleftarrows}} C_k | W^{-\delta}.$$

We have homotopies $d_{k+1} s_k j \sim j$ and $d_k j \sim 0$. Define (C', d') as follows: $C'_l = C_l$ $(l \neq k,\ k+2)$, $C'_k = C_k | X - W^{-\delta}$, $C'_{k+2} = C_{k+2} \oplus C_k | W^{-\delta}$; $d'_k = d_k i$, $d'_{k+1} = p\, d_{k+1}$, $d'_{k+2} = (d_{k+2}, s_k j)$, $d'_{k+3} = {}^t(d_{k+3}, 0)$, $d'_l = d_l$ otherwise. Let T be a chain complex with $T_{k+2} = T_{k+1} = C_k | W^{-\delta}$, $T_l = 0$ otherwise, and $d_T = 1$: $T_{k+2} \to T_{k+1}$; and let T' be the desuspension of T. There are obvious chain equivalences f: $C \to C \oplus T$ and h: $C' \oplus T' \to C'$. Define a chain map g: $C \oplus T \to C' \oplus T'$ by

$$g_l = 1 \quad \text{if } l \neq k+1$$

$$g_{k+1} = \begin{pmatrix} 1 & s_k j \\ q\, d_{k+1} & 0 \end{pmatrix}: C_{k+1} \oplus C_k | W^{-\delta} \to C_{k+1} \oplus C_k | W^{-\delta}.$$

Since g_{k+1} can be decomposed as

$$g_{k+1} \sim \begin{pmatrix} 1 & 0 \\ q\, d_{k+1} & -1 \end{pmatrix} \begin{pmatrix} 1 & s_k j \\ 0 & 1 \end{pmatrix}$$

g is a δ-isomorphism. Composing f, g, and h, we obtain a 2δ-chain equivalence between C and C'. A desired chain contraction $\{s'_l\}$ of C' is defined by $s'_l = 0$ for $l \leqq k$, $s'_{k+1} = {}^t(s_{k+1},\ q\, d_{k+1})$, $s'_{k+2} = (s_{k+2},\ -s_{k+2} s_{k+1} s_k j)$, and $s'_l = s_l$ for $l \geqq k+3$. □

Using this lemma repeatedly, we get the following:

(2.7) **Corollary.** *Fix n and p. Given any $\varepsilon>0$, there exists $\delta>0$ such that for any n-dimensional geometric $\mathbb{Z}$-module chain complex C on E of radius δ, C is ε chain equivalent to an n-dimensional geometric $\mathbb{Z}$-module chain complex C' on E of radius ε satisfying:*

(1) $C'_0|W^{-\varepsilon}=\ldots=C'_{n-2}|W^{-\varepsilon}=0$

(2) $C|X-W=C'|X-W$

(3) *there exists a geometric morphism s: $C'_{n-1}\to C'_n$ of radius ε such that*

$$d_n s|W^{-\varepsilon} \underset{\varepsilon}{\sim} 1 \quad \text{and} \quad s d_n|W^{-\varepsilon} \underset{\varepsilon}{\sim} 1.$$

It is in general impossible to finish this elimination in the last two layers, but if we "stabilize" everything, we can avoid the difficulty as follows. Let (A, d_A) be the underlying geometric $\mathbb{Z}$-module chain complex on S^1 of $\sigma^*(S^1)$: $0\to A_1\to A_0\to 0$. Then $A\otimes C'$ is a strictly $(n+1)$-dimensional $\mathbb{Z}$-module chain complex on $S^1\times E$ of radius ε in X:

$$0\to A_1\otimes C'_n \xrightarrow{\begin{pmatrix}(-)^n d^n_A\otimes 1\\ 1\otimes d_n\end{pmatrix}} (A_0\otimes C'_n)\oplus(A_1\otimes C'_{n-1}) \xrightarrow{\begin{pmatrix}1\otimes d_n & (-)^n d_A\otimes 1\\ 0 & 1\otimes d_{n-1}\end{pmatrix}} (A_0\otimes C'_{n-1})\oplus(A_1\otimes C'_{n-2})\to$$

and $(A\otimes C')_i|W^{-\varepsilon}=0$ if $i\leqq n-2$. The morphisms

$$(1\otimes s'_i)\oplus(1\otimes s'_{i-1})\colon (A_0\otimes C'_i)\oplus(A_1\otimes C'_{i-1})\to(A_0\otimes C'_{i+1})\oplus(A_1\otimes C'_i),$$

where $s'_{n-1}=s$, and $s'_i=0$ if $i\neq n-1$, define a chain contraction of $A\otimes C'$ with support $W^{-\varepsilon}$ and radius ε. Use 2.6 again to get a strictly $(n+1)$-dimensional geometric $\mathbb{Z}$-module chain complex $\bar{C}$ on $S^1\times E$ of radius ε in X such that $\bar{C}_i|W^{-2\varepsilon}=0$ if $i\leqq n-1$; the boundary map $d\colon \bar{C}_{n+1}\to\bar{C}_n$ is given by the following matrix:

$$d=\begin{pmatrix}(-)^n d_A\otimes 1 & 1\otimes sj\\ 1\otimes d_n & 0\end{pmatrix}$$
$$:(A_1\otimes C'_n)\oplus(A_0\otimes C'_{n-1}|W^{-2\varepsilon})\to(A_0\otimes C'_n)\oplus(A_1\otimes C'_{n-1}),$$

where j is the inclusion morphism of $C'_{n-1}|W^{-2\varepsilon}$ into C'_{n-1}. Now we consider the following ε-isomorphism of $\bar{C}_{n+1}$ to itself:

$$h=\begin{pmatrix}1 & (-)^n d_A\otimes sj\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & -1\otimes sj\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1\otimes q d_n & 1\end{pmatrix}\begin{pmatrix}1 & 1\otimes sj\\ 0 & 1\end{pmatrix},$$

where q is the projection morphism of C'_{n-1} onto $C'_{n-1}|W^{-2\varepsilon}$. Then $h|W^{-3\varepsilon}\sim d|W^{-3\varepsilon}$. If we replace the boundary map $d\colon \bar{C}_{n+1}\to\bar{C}_n$ by dh^{-1}, we get a new geometric $\mathbb{Z}$-module chain complex $\bar{C}'$, which is ε-isomorphic to $\bar{C}$ and the boundary map dh^{-1} is homotopic to the identity when restricted to $W^{-3\varepsilon}$. Now we can delete $\bar{C}'_{n+1}|W^{-4\varepsilon}$ and $\bar{C}'_n|W^{-4\varepsilon}$ from $\bar{C}'$. Thus we obtain the following:

(2.8) **Lemma.** *Fix n and p. Given any $\varepsilon>0$, there exists $\delta>0$ such that for any strictly n-dimensional geometric $\mathbb{Z}$-module chain complex C on E of radius δ in X which has a chain contraction with support W and radius δ, there exists a strictly $(n+1)$-dimensional geometric $\mathbb{Z}$-module chain complex $\tilde{C}$ on $S^1\times E$ of radius ε in X satisfying*

(1) *$\tilde{C}$ lies over $X-W^{-\varepsilon}$,*
(2) *$\tilde{C}$ is ε-chain equivalent to $A\otimes C$, and*
(3) *$\tilde{C}|X-W=(A\otimes C)|X-W$.*

Proof of 2.5. Suppose an $(n+1)$-dimensional pair $(f\colon C\to D, (\delta\psi,\psi))$ of radius δ is given. Define subcomplexes C', D' of C, D as follows:

$$C'_i=C_i|Y^{-(i+1)\delta}$$
$$D'_i=D_i|Y^{-i\delta}.$$

Then 1.1 and 1.2 are automatic, and 1.4 and 1.5 are also satisfied. (See Remark 1.6(1).) So formulae in the previous section will give a splitting into two $(n+1)$-dimensional triads. The only defect is that the common boundary pieces B and $C^!$ of Lemma 1.3 may lie all over X. It is easy to cut off the portion of B lying over $Y^{-n\delta}$; B is chain equivalent to $S^{-1}C(p_C(1+T)\psi_0 p_C^*)$. Similarly for $C^!$. Next notice that $p_C(1+T)\psi_0 p_C^*$ is a chain equivalence with support $X-Y^{\delta}$. The formula on p. 167 of [15] produces a chain contraction of $S^{-1}C(p_C(1+T)\psi_0 p_C^*)$ with support $X-Y^{3\delta}$ and radius 3δ. Now, using Lemma 2.8, we can eliminate the portion of $S^{-1}C(p_C(1+T(\psi_0 p_C^*)$ lying over $X-Y^{\delta'}$ for some $\delta'>0$ which depends only on δ and n. Apply the same argument to $C^!$. This ends the proof of Lemma 2.5. □

The notion of pairs$(=2$-ads) and triads$(=3$-ads) naturally extends to "$(k+2)$-ads" (Ranicki [13]) or "higher algebraic bordisms" (Weiss [17]). A $(k+2)$-ad x of chain complexes is a collection of

(1) the *underlying chain complex* $|x|$,
(2) $k+1$ $(k+1)$-ads $\partial_0 x,\dots,\partial_k x$ satisfying

$$\partial_j\partial_i x=\partial_i\partial_{j+1}x \quad \text{if } 0\leqq i\leqq j<k,$$

(3) a pair $\bigcup\partial_i x\to|x|$.

Let x be a $(k+2)$-ad. When $i_1<\dots<i_l$, we define $\partial_{\{i_1,\dots,i_l\}}x$ by $\partial_{i_1}\partial_{i_2}\dots\partial_{i_l}x$.

An *n-dimensional quadratic Poincaré $(k+2)$-ad* x is a $(k+2)$-ad x of chain complexes together with structure maps

$$(\psi_\alpha)_s\colon |\partial_\alpha x|^{n-|\alpha|-r-s}\to|\partial_\alpha x|, \quad \alpha\subset\{0,1,\dots,k\}$$

such that

(1) $\partial_i x$ is an $(n-1)$-dimensional quadratic Poincaré $(k+1)$-ad for each i, and
(2) $(\bigcup\partial_i x\to|x|, (\psi_\phi, \bigcup\psi_i))$ is an n-dimensional quadratic Poincaré pair.

Here $|\alpha|$ denotes the size of the subset α.

Let X be a metric space and $p\colon E\to X$ be a map. If we use geometric $\mathbb{Z}$-module chain complexes on E, we can define *geometric $\mathbb{Z}$-module quadratic Poincaré n-ads on E.* Such a thing has *radius* ε (resp. *support* $K\subset X$) if all the involved chain complexes, chain maps, chain homotopies, and structure maps are replaced by those of radius ε in X (resp. support K). An n-ad x is called *special* if $\partial_{\{0,1,\ldots,n-2\}}x=0$. Recall that an "$n$-ad" in the usual sense is a topological space together with $n-1$ subsets. We call this a *topological n-ad* to distinguish it from n-ads of (quadratic) chain complexes. Let $(E,\partial_* E)$ be a topological n-ad. A *geometric* $\mathbb{Z}$*-module quadratic Poincaré n-ad x on* $(E,\partial_* E)$ is a geometric $\mathbb{Z}$-module quadratic Poincaré n-ad on E such that $\partial_i x$ lies over $\partial_i E$ for each i.

Let us consider the following problem. Let M be an n-dimensional compact manifold with a PL-triangulation L, and fix a metric on M. Also let p: $E\to M$ be a map. Suppose each n-simplex Δ of L is given a geometric $\mathbb{Z}$-module quadratic Poincaré $(n+2)$-ad x_Δ of dimension m on a topological $(n+2)$-ad $(p^{-1}(\Delta), p^{-1}(\partial_*\Delta))$ such that

(2.9) (compatibility) if two distinct n-simplices Δ and Δ' have a common $(n-1)$-face, $\partial_i\Delta=\partial_j\Delta'$, then $\partial_i x_\Delta=-\partial_j x_{\Delta'}$.

We would like to glue all these $(n+2)$-ads to get an m-dimensional geometric $\mathbb{Z}$-module quadratic Poincaré pair on E. We can also consider a problem of the inverse direction. Notice that there is a small difficulty. Ranicki's formula allows us to glue things only along a codimension 1 boundary piece, so we have to be careful about the order of glueing. There are several ways to avoid this difficulty; we do it in the following way. First we glue locally so that the local blocks behave nicely, and then we glue the blocks (which can be glued in any order). When we split something, we first split it into several blocks so that each block is over a union of simplices in a controlled way, and we split each block into the desired pieces. More precisely, assume that the triangulation L of M is the first barycentric subdivision of another triangulation K. For each vertex v of K, consider its star $S(v)$ in L, or the dual cone. Two such dual cones are either disjoint or meet along codimension 1 cell(s). The glueing and splitting problem over $S(v)$ can be solved by looking at the link $L(v)$ of v in L. Note that $L(v)$ is an $(n-1)$-dimensional sphere and the triangulation is the first barycentric subdivision of another. Thus we can keep on reducing the dimension until the link becomes a circle, and in this case there is an obvious order of 2-simplices and glueing and splitting can be done. Thus we have:

(2.10) **Theorem.** (Glueing over a manifold). *Let L be the barycentric subdivision of a PL-triangulation of a compact n-dimensional manifold M and p: $E\to M$ be a map. And suppose each n-simplex Δ is given an m-dimensional geometric $\mathbb{Z}$-module quadratic Poincaré special $(n+2)$-ad on $(p^{-1}(\Delta)$, $p^{-1}(\partial_*\Delta))$ which are compatible on common faces (in the sense of 2.9). Then one can glue them together to get an m-dimensional geometric $\mathbb{Z}$-module quadratic Poincaré pair on $(E,p^{-1}(\partial M))$.*

(2.11) **Theorem.** (Stable splitting lemma over a manifold). *Let us fix m and p: $E\to M$ and let L be the first barycentric subdivision of a PL-triangulation of the compact manifold M. Fix a metric of M. Let ε be any positive number. Then there exists $\delta>0$ such that any m-dimensional geometric $\mathbb{Z}$-module quadratic Poincaré pair on $(E,p^{-1}(\partial M))$ of radius δ can be split into pieces each of which* *has radius ε and lies over an ε-neighborhood of the corresponding simplex of L, after tensored with a sufficiently large number of $\sigma^*(S^1)$'s.*

(2.12) *Remark.* If a splitting of the boundary is already given, then the result can be arranged to have the given splitting of the boundary.

3. Surgery spaces and assembly maps

Let $p: E \to X$ be a map, where X is a metric space. In this section we first construct an Ω-spectrum $\mathbb{L}(X;p)$ which might be called a "controlled $L^{-\infty}$-theory spectrum". If X is a single point $*$, then $\mathbb{L}(*;\ E \to *)$ is homotopy equivalent to the $L^{-\infty}$-theory spectrum $\mathbb{L}(E)$, whose homotopy groups are the limits $L_*^{-\infty}(\pi_1 E)$ of Ranicki's lower L-groups $L_*^{\langle -j \rangle}(\pi_1 E)$ [11]. Next we describe homology $H_*(X;\mathbb{L}(p))$. It is a sort of generalized homology with local coefficients, defined by Quinn [9, §8]. Given a space X and a spectrum $\mathscr{S}$, the usual (constant coefficient) generalized homology groups $H_*(X;\mathscr{S})$ is the homotopy groups of an Ω-spectrum $\lim_{n\to\infty} \Omega^n(\mathscr{S}_n \times X/\{\text{base point}\} \times X) = \lim_{n\to\infty} \Omega^n((\bigcup_{x\in X} \mathscr{S}_n \times \{x\})/(\bigcup_{x\in X}\{\text{base point}\} \times \{x\}))$. In our case, we apply the $\mathbb{L}^{-\infty}$-theory spectrum functor $\mathbb{L}(-)$ to each fiber of $p: E \to X$, and define $H_*(X;\mathbb{L}(p))$ to be the homotopy groups of an Ω-spectrum $\mathbb{H}(X;\mathbb{L}(p)) = \lim_{n\to\infty} \Omega^n((\bigcup_{x\in X} \mathbb{L}_{-n}(p^{-1}(x)) \times \{x\})/(\bigcup_{x\in X}\{\text{base point}\} \times \{x\}))$. Actually, we assume that X is a polyhedron and apply $\mathbb{L}(-)$ blockwise to p and use $\mathbb{L}(p^{-1}(\Delta)) \times \Delta$ as a building block of $\mathbb{H}(X;\mathbb{L}(p))$, where Δ is a simplex of X. When p is sufficiently close to being a fibration, these two approaches are equivalent [ibid.]. Lastly we show that the homology spectrum $\mathbb{H}(X;\mathbb{L}(p))$ is homotopy equivalent to the controlled $L^{-\infty}$-theory spectrum $\mathbb{L}(X;p)$, for certain maps p (Theorem 3.9). This will be used to prove the main theorem (4.11) in section 4.

Now let us begin defining $\mathbb{L}(X;p)$. A point in $\mathbb{L}_n(X;p)$ is, roughly speaking, an $(n+l)$-dimensional geometric $\mathbb{Z}$-module quadratic Poincaré complexes on $\mathbb{R}^l \times E$ whose sufficiently high suspension (defined below) is cobordant to one with arbitrarily small radius measured in X. For example, an n-dimensional surgery problem between PL-manifolds $M \to N$ together with a map $N \to E$ inducing an isomorphism on π_1 produces such a quadratic Poincaré complex on E; one can make the radius as small as one likes by choosing sufficiently fine PL triangulations. Actually, by successively taking barycentric subdivisions of M and N, one gets a sequence of quadratic Poincaré complexes with radius converging to 0. This defines a "path" in the "space of quadratic Poincaré complexes" $\mathbb{P}_n(X;p)$.

There are cases in which we need a restriction map $\mathbb{L}(X,p) \to \mathbb{L}(W;p|W)$ for an open subset W of X. Note that, when we restrict a geometric quadratic Poincaré complex with support the whole space X to W, the result is generally *not* a quadratic Poincaré complex on $p^{-1}(W)$ with support W because it is damaged near the frontier of W in X. So, when we define $\mathbb{L}(X;p)$ and $\mathbb{P}(X;p)$, we need to consider complexes with support not necessarily the whole space.

This surely complicates the construction. Fortunately, we can simplify the construction, as long as we deal with compact X's. See Remark (3.3) below. One place where we might use restriction maps is the proof of theorem 3.9. There are at least two ways of establishing a homotopy equivalence between $\mathbb{H}(X;\mathbb{L}(p))$ and $\mathbb{L}(X;p)$, one employed in this article and one employed by Quinn in [9]. In Quinn's way, one needs to check that $\mathbb{L}$ satisfy a certain restriction axiom. In this article, theorem 3.9 will be proved without using restriction maps, but the full definition will be given for completeness.

Fix $p\colon E \to X$ and an integer n. A *primitive k-simplex x of degree n with radius ε and support $K \subset X$* is an $(n+k+l)$-dimensional geometric $\mathbb{Z}$-module quadratic Poincaré special $(k+2)$-ad on $\mathbb{R}^l \times E$ with radius ε in X and support $K \subset X$, for some non-negative integer l. The composition $\mathbb{R}^l \times E \to E \to X$ is the control map. We require that x be locally finite and have bounded radius in $\mathbb{R}^l \times X$ via $1 \times p$. If x has radius ε and support K, then its faces $\partial_0 x, \ldots, \partial_k x$ are primitive $(k-1)$-simplices of degree n with radius ε and support K.

For a primitive k-simplex of degree n with radius ε and support K, we have the following operations.

(1) *Reduction.* Suppose $\varepsilon' \geqq \varepsilon$ and $K' \subset K \subset X$. Then x can be regarded as a primitive k-simplex of degree n with radius ε' and support K'. This is called a reduction of x.

(2) *External suspension.* $\sigma^*(S^1) \otimes x$ gives an $(n+k+l+1)$-dimensional geometric $\mathbb{Z}$-module quadratic Poincaré special $(k+2)$-ad on $S^1 \times \mathbb{R}^l \times E$. Lifting everything into the infinite cyclic cover $\mathbb{R} \times \mathbb{R}^l \times E = \mathbb{R}^{l+1} \times E$, we obtain a new primitive k-simplex of degree n. The result is denoted by Σx, and called the external suspension. $\Sigma^m x = \Sigma \Sigma \ldots \Sigma x$ has the same radius and support as x.

Now we define the space of quadratic Poincaré ads.

(3.1) **Definition.** $\mathbb{P}_n(X;p)$ is the Δ-set with simplices (which will be called *elaborate simplicies*) defined inductively: an elaborate 0-simplex is a primitive 0-simplex of degree n, i.e., a strictly $(n+l)$-dimensional geometric $\mathbb{Z}$-module quadratic Poincaré special 2-ad (=complex) on $\mathbb{R}^l \times E$ for some l (with unrestricted compact support and radius). An elaborate k-simplex σ consists of an underlying primitive k-simplex $|\sigma|$ of degree n, together with $k+1$ elaborate $(k-1)$-simplices $\partial_0 \sigma, \ldots, \partial_k \sigma$. We require these to satisfy the usual $\partial_i \partial_j$ identities, and in addition require that the external suspension of a reduction of the underlying primitive $(k-1)$-simplex $|\partial_i \sigma|$ of $\partial_i \sigma$ be equal to the i-th face $\partial_i |\sigma|$ of the underlying primitive k-simplex $|\sigma|$. The *support* and *radius* of an elaborate simplex are those of its underlying primitive simplex.

We are not interested in $\mathbb{P}_n(X;p)$ itself; it is contractible, because there is no restriction on radius and support in the definition. If we required that the support be equal to X and the radius be arbitrary, then we would have the usual $L^{-\infty}$-group spectrum. In the following definition, we are going to put a restriction on both the support and the radius. Suppose K is a compact subset of X, ε is a positive number, and N is a positive integer, then $\mathbb{P}_n(X, K, p, \varepsilon)$ $(\mathbb{P}_n(X, K, p, \varepsilon)^{(N)})$ denotes the subset of $\mathbb{P}_n(X;p)$ made up of all the simplices with support containing K, radius not exceeding ε (and the dimension of the underlying primitive simplex less than or equal to N).

(3.2) **Definition.** We define a Δ-set $\mathbb{L}_n(X;p)$ as follows. Let Δ^k be a k-simplex with the obvious Δ-set structure and let $[0,\infty)$ have the triangulation with vertices at integer points. The obvious ordering of vertices makes $[0,\infty)$ into a Δ-set. A k-simplex of $\mathbb{L}_n(X;p)$ is defined to be a Δ-map $\Delta^k \otimes [0,\infty) \to \mathbb{P}_n(X;p)$ which satisfies the following condition: there are an increasing sequence of compact sets $K_i \subset X$ with $\bigcup$ (interior of K_i) $= X$, a sequence ε_i of numbers monotone decreasing to 0, and a positive integer N, such that the image of $\Delta^k \otimes [i,\infty)$ lies in $\mathbb{P}_n(X, K_i, p, \varepsilon_i)^{(N)}$. Here $\otimes$ denotes the geometric product of Δ-sets [14].

(3.3) *Remarks.* (1) When X is compact, we may replace $\mathbb{P}_n(X, K_i, p, \varepsilon)^{(N)}$ by $\mathbb{P}_n(X, X, p, \varepsilon)^{(N)}$ in the above. In this case, we write $\mathbb{P}'_n(X, p, \varepsilon)$ (resp. $\mathbb{P}'_n(X, p, \varepsilon)^{(N)}$) instead of $\mathbb{P}_n(X, X, p, \varepsilon)$ (resp. $\mathbb{P}_n(X, X, p, \varepsilon)^{(N)}$). The union $\bigcup\{\mathbb{P}'_n(X, p, \varepsilon) \mid \varepsilon > 0\}$ (resp. $\bigcup\{\mathbb{P}'_n(X, p, \varepsilon)^{(N)}\}$) is denoted $\mathbb{P}'_n(X;p)$ (resp. $\mathbb{P}'_n(X;p)^{(N)}$).

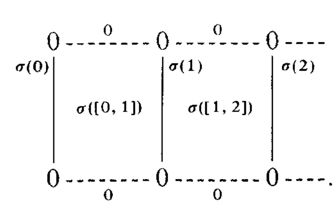
(2) $\mathbb{P}_n(X;p)$ satisfies the Kan condition. As was mentioned above , this fact itslef has no importance at all. What is important is the construction used to prove it. Note that the same proof works for $\mathbb{P}'_n(X;p)$. For example, suppose we have a "horn" of k-simplices of $\mathbb{P}_n(X;p)$, $y_0, \dots, y_k$, satisfying $\partial_j y_i = -\partial_i y_{i+1}$ $(0 \leqq i \leqq j < k)$. After some necessary suspensions, we can fit $y_0, \dots, y_k$ together to produce a new k-simplex y_{k+1}. There is an obvious cobordism (product cobordism) between y_{k+1} and itself, and this gives a $(k+1)$-simplex x such that $\partial_i x = y_i$ $(i = 0, 1, \dots, k+1)$. The radius and support of x depend on those of y_i's $(i = 0, 1, \dots, k)$. Therefore, $\mathbb{L}_n(X;p)$ also satisfies the Kan condition.

The next result describes the spectrum structure. Unfortunately, our subscripts for $\mathbb{L}$ do not coincide with the indexing for spectra.

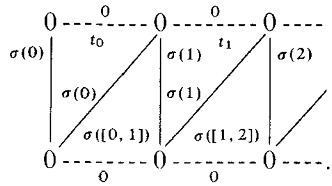
(3.4) **Theorem.** *There is a natural homotopy equivalence* T: $\Omega\mathbb{L}_n(X;p) \to \mathbb{L}_{n+1}(X;p)$.

Proof. A k-simplex of $\Omega\mathbb{L}_n(X;p)$ is a Δ-map σ: $\Delta^k \otimes [0,\infty) \otimes I \to \mathbb{P}_n(X;p)$. We define $T\sigma$: $\Delta^k \otimes [0,\infty) \to \mathbb{P}_{n+1}(X;p)$ as follows. Let τ be an m-simplex of $\Delta^k \otimes [0,\infty)$. $\sigma(\tau \otimes I)$ consists of several simplices of $\mathbb{P}_n(X;p)$. We can fit these together after some necessary suspensions and the result is an m-simplex of $\mathbb{P}_{n+1}(X;p)$, since $\sigma(\tau \otimes 0) = \sigma(\tau \otimes 1) = 0$. This defines a k-simplex $T\sigma$ of $\mathbb{L}_{n+1}(X;p)$.

We will show that T is a homotopy equivalence. First of all, note that each 0-simplex of $\mathbb{P}_{n+1}(X;p)$ can be naturally regarded as a 1-simplex of $\mathbb{P}_n(X;p)$ with two 0 faces, and a 0-simplex σ: $[0,\infty) \to \mathbb{P}_{n+1}(X;p)$ of $\mathbb{L}_{n+1}(X;p)$ can be expressed as in the following picture.



This itself is not a 0-simplex of $\Omega\mathbb{L}_n(X;p)$, since this picture is not triangulated. By inserting trivial cobordisms as in 3.3(2), we can triangulate this picture:



Here t_i denotes the trivial (=product) cobordism between $\sigma(i)$ and itself. Or one can supply a more formal construction: suppose $\sigma([i,i+1])\subset \mathbb{P}_{n+1}(X, K_i, p, \varepsilon_i)$, then one can easily construct a "simplicial" map $\Delta^1\otimes[i,i+1]\to\mathbb{P}_n(X,K_i,p,\varepsilon_i)$ by sending one of the two 2-simplices of $\Delta^1\otimes[i,i+1]$ to the 2-simplex $\sigma([i,i+1])$ whose edges are $\sigma(i)$, $\sigma(i+1)$ and 0, and the other to the 1-simplex $\sigma(i)$. Then use the relative Δ-map approximation theorem of Rourke and Sanderson [14, theorem 5.3] to get a Δ-map. Here we use the fact that $\mathbb{P}_n(X,K_i,p,\varepsilon_i)$ is Kan (3.3). Observe that this gives the same map as above (up to homotopy). Similar argument will be used often later, and will be called the "triangulation argument". Now this defines a 0-simplex $[0,\infty)\otimes I\to\mathbb{P}_n(X;p)$ of $\Omega\mathbb{L}_n(X,p)$. If we apply T to this, then the result is different from the original only by trivial cobordisms; therefore these two can be connected by a 1-simplex of $\mathbb{L}_{n+1}(X;p)$; i.e., T maps into every component. Next consider an element of the relative homotopy group $\pi_j(T)$. By the Kan condition it is represented by a map ρ: $\Delta^j\otimes[0,\infty)\to\mathbb{P}_{n+1}(X;p)$ such that $\rho|\partial_i\Delta^j=0$ for $i<j$ and $\rho|\partial_j\Delta^j=T\sigma$ for some σ: $\Delta^{j-1}\otimes[0,\infty)\otimes I\to\mathbb{P}_n(X;p)$. We need a deformation of ρ rel $\partial_j\rho$ to a map in the image of T. An extension σ': $\Delta^j\otimes[0,\infty)\otimes I\to\mathbb{P}_n(X;p)$ of σ can be constructed by first letting $\sigma'(\tau)=0$ for τ in $(\bigcup_{i<j}\partial_i\Delta^j)\otimes[0,\infty)\otimes I\cup\Delta^j\otimes[0,\infty)\otimes\{0,1\}$ and then using the triangulation argument. There is an obvious cobordism which gives a simplex connecting ρ and $T\sigma'$. □

Now let us consider a special case when X is a single point $*$. In this case, $\mathbb{L}_j(*;E\to *)$ is homotopy equivalent to $\mathbb{P}'_j(*;E\to *)$. We denote $\mathbb{P}'_j(*;E\to *)$ by $\mathbb{L}_j(E)$.

(3.5) **Proposition.** *There is a natural isomorphism*

$$\theta\colon\ L_n^{-\infty}(G)\to\pi_n\mathbb{L}(E)$$

where G is the fundamental group of E.

Here $L_n^{-\infty}$ is the direct limit $\lim_{j\to\infty} L_n^{\langle -j\rangle}$ of Ranicki's lower L-groups [11]. $L_n^{\langle -j\rangle}(G)$ is defined to be the kernel of the product of projection maps

$$L^{(1)}_{n+j+1}(\mathbb{Z}^{j+1}\times G)\to\prod^{j+1} L^{(1)}_{n+j+1}(\mathbb{Z}^{j}\times G).$$

The map $L_n^{\langle -j\rangle}\to L_n^{\langle -j-1\rangle}$ is induced by the map

$$\sigma^*(S^1)\otimes\colon\ L^{(1)}_{n+j+1}(\mathbb{Z}^{j+1}\times G)\to L^{(1)}_{n+j+2}(\mathbb{Z}^{j+2}\times G).$$

Proof of 3.5. First let us define θ. An element of $L_n^{\langle -j\rangle}(G)$ can be represented by a free $(n+j+1)$-dimensional quadratic Poincaré complex over $\mathbb{Z}[\mathbb{Z}^{j+1}\times G]$.

Represent this by a geometric $\mathbb{Z}$-module quadratic Poincaré complex on $T^{j+1} \times E$, where T^{j+1} is a $(j+1)$-torus. The pull-back of this on $\mathbb{R}^{j+1} \times E$ has a bounded radius in the $\mathbb{R}^{j+1}$ coordinates. So we can regard this as an n-simplex of $\mathbb{L}_0(E)$ with zero boundary, and hence as an element of $\pi_n \mathbb{L}_0(E)$. $(\sigma^*(S^1)\otimes)$ on the left side corresponds to Σ on the right, so this defines a homomorphism θ of the direct limit.

Next we show that θ is onto. Take an element of $\pi_n \mathbb{L}(E)$ and represent it by a locally finite geometric $\mathbb{Z}$-module quadratic Poincaré complex c on $\mathbb{R}^j \times E$ with bounded radius on $\mathbb{R}^j$ for some j. Split c along $\{0\} \times \mathbb{R}^{j-1} \times E$ without taking the external suspension; we obtain a splitting $c \sim c_+ \cup_b c_-$, where b may lie all over $\mathbb{R}^j \times E$. We proved that the geometric $\mathbb{Z}$-module quadratic Poincaré complex $\sigma^*(S^1)\otimes b$ on $S^1 \times \mathbb{R}^j \times E$ is homotopy equivalent to another complex d which lies on $S^1 \times \{0\} \times \mathbb{R}^{j-1} \times E$. So Σc on $\mathbb{R} \times \mathbb{R}^j \times E$ has a splitting $c'_+ \cup_{d'} c'_-$, where d' is the pull-back of d in $\mathbb{R} \times \{0\} \times \mathbb{R}^{j-1} \times E$. Note that $\Sigma d'$, the pull-back of $\sigma^*(S^1)\otimes d'$ (on $\mathbb{R} \times S^1 \times \mathbb{R}^{j-1} \times E$) on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^{j-1} \times E$ also has a splitting $d'_+ \cup_{d'} d'_-$. Here $+$ and $-$ are with respect to the second $\mathbb{R}$. We claim that Σc and $\Sigma d'$ are cobordant. $\Sigma c \oplus (-\Sigma d')$ is cobordant to $(c'_+ \cup_{d'} (-d'_+)) \oplus (c'_- \cup_{-d'} (-d'_-))$. Since $c'_+ \cup_{d'} (-d'_+)$ (resp. $c'_- \cup_{-d'} (-d'_-)$) lies over $\mathbb{R} \times [0, \infty) \times \mathbb{R}^{j-1} \times E$ (resp. $\mathbb{R} \times (-\infty, 0] \times \mathbb{R}^{j-1} \times E$), the next lemma implies that these are cobordant to 0, and hence Σc is cobordant to $\Sigma d'$. Repeat this process until one gets $\Sigma^j e'$, where e is a geometric $\mathbb{Z}$-module quadratic Poincaré complex on $T^j \times E$, and e' is the pull-back of e on $\mathbb{R}^j \times E$. This e represents an element of $L_n^{-\infty}(G)$. Obviously θ sends this element to $[c] \in \pi_n \mathbb{L}(E)$.

(3.6) **Lemma.** *Any locally finite $\mathbb{Z}$-module quadratic Poincaré complex on $\mathbb{R}^j \times E$ which lies over $[0, \infty) \times \mathbb{R}^{j-1} \times E$ is locally finitely cobordant to zero.*

Proof. Let c be such a complex, and t denote the parrallel translation of $\mathbb{R} \times \mathbb{R}^{j-1} \times E$ defined by $t(x, y, z) = (x+1, y, z)$. Then c is cobordant to a locally finite complex

$$c \oplus [(-t\, c) \oplus (t^2 c)] \oplus [(-t^3 c) \oplus (t^4 c)] \oplus \dots$$
$$= [c \oplus (-t\, c)] \oplus [(t^2 c) \oplus (-t^3 c)] \oplus \dots$$

which is cobordant to zero. □

Let us go back to the proof of 3.5. We will prove the injectivity of θ. Pick an element x in the kernel of θ. Its image is cobordant to 0. Apply the same argument to this null cobordism as in the onto part. This will show that $x = 0$. □

This justifies the following notation:

3.7 Notation. $L_n^{-\infty}(E) := \pi_n \mathbb{L}(E)$.

Next we describe homology $H_*(X; \mathbb{L}(p))$ defined by Quinn [9, §8], in terms of Δ-sets. We assume X is a finite polyhedron and p is a simplicial stratified system of fibrations. We fix a Δ-set K with realization $|K|$ equal to X; for example, the first derived of a triangulation of X gives such a Δ-set. $H_*(X; \mathbb{L}(p))$ does not depend on the choice of K [ibid.]. Recall that $\mathbb{L}(-)$ is a covariant functor from spaces to spectra of Δ-sets (3.4). We apply $\mathbb{L}(-)$ blockwise to p. Define a Δ-set $\mathbb{L}_n(p)$ by:

$$\{\bigcup_{\Delta\in K}\mathbb{L}_n(p^{-1}(\Delta))\otimes\Delta\}/\sim,$$

where a simplex $\Delta\in K$ is given the obvious Δ-set structure, and the equivalence relation $\sim$ is generated by : a simplex in $\mathbb{L}_n(p^{-1}(\partial_j\Delta))\otimes\partial_j\Delta$ is identified with its image in $\mathbb{L}_n(p^{-1}(\Delta))\otimes\Delta$. The realization $|\mathbb{L}_n(p)|$ is homeomorphic to:

$$\{\bigcup_{\Delta\in K}|\mathbb{L}_n(p^{-1}(\Delta))|\times|\Delta|\}/\sim,$$

since $\mathbb{L}_n(p^{-1}(\Delta))$ is Kan. Here $\sim$ is induced by the identification of simplices given above. Denote by p_* the natural projection of $|\mathbb{L}_n(p)|$ to X. Also the section i: $X\to|\mathbb{L}_n(p)|$ is defined by fitting together the base points of the pieces. Further the structure maps $|\mathbb{L}_{-n}(p^{-1}(\Delta))|\to|\Omega\mathbb{L}_{-n-1}(p^{-1}(\Delta))|\simeq\Omega|\mathbb{L}_{-n-1}(p^{-1}(\Delta))|$ fit together to make $|\mathbb{L}_{-n}(p)|$ into an ex-spectrum ([9, p. 423]):

$$|\mathbb{L}_{-n}(p)|\to\Omega_X(|\mathbb{L}_{-n-1}(p)|).$$

For a subpolyhedron Y of X, the composition

$$|\mathbb{L}_{-n}(p)|\to\Omega_X(|\mathbb{L}_{-n-1}(p)|)\xrightarrow{\text{inclusion}}\Omega(|\mathbb{L}_{-n-1}(p)|/i(X))$$

induces a map:

$$|\mathbb{L}_{-n}(p)|/i(X)\cup p_*^{-1}(Y)\to\Omega(|\mathbb{L}_{-n-1}(p)|/i(X)\cup p_*^{-1}(Y)).$$

Taking Ω^{n-j} of this we have a map:

$$\Omega^{n-j}(|\mathbb{L}_{-n}(p)|/i(X)\cup p_*^{-1}(Y))\to\Omega^{n-j+1}(|\mathbb{L}_{-n-1}(p)|/i(X)\cup p_*^{-1}(Y)).$$

We replace this by a Δ-map between Δ-sets, applying singular complex functor S and then applying forgetful functor F [14].

(3.8) **Definition.** The homology spectrum $\mathrm{H}(X, Y;\mathbb{L}(p))$ is an Ω-spectrum of Δ-sets defined by

$$\mathbb{H}_j(X, Y;\mathbb{L}(p))=\lim_{n\to\infty}FS\Omega^{n-j}(|\mathbb{L}_{-n}(p)|/i(X)\cup p_*^{-1}(Y)).$$

The homology groups are the homotopy groups of this spectrum.

The functor $\mathbb{L}(-)$ which was used to construct $\mathbb{H}(X, Y;\mathbb{L}(p))$ is homotopy invariant; in fact, the homotopy type of $\mathbb{L}(E)$ depends only on the fundamental group $\pi_1 E$ (3.5). Therefore, according to Quinn [9, p. 421], $\mathbb{H}(-;\mathbb{L}(-))$ is a homology theory on the category of polyhedra with stratified systems of fibrations. From now on we always assume p as such. The following is the main theorem of this section.

(3.9) **Theorem.** (Characterization theorem). *Let p: $E\to X$ be a polyhedral stratified system of fibrations on a finite polyhedron X. Then there is a homotopy equivalence A_j: $\mathbb{H}_j(X;\mathbb{L}(p))\to\mathbb{L}_{-j}(X;p)$.*

Proof. One way to prove this is to use Characterization theorem (8.5) of Quinn [9]. But, since we will also need something more explicit (3.10 below), we give a definition of the map A, called the assembly map, and prove that it is a homotopy equivalence.

A k-simplex of $FS\Omega^{n-j}(|\mathbb{L}_{-n}(p)|/i(X))$ is a map $\rho\colon S^{n-j}\times\Delta^k\to|\mathbb{L}_{-n}(p)|/i(X)$. By modifying ρ a little, if necessary, we may assume that there exist a compact codimension 0 submanifold V of $S^{n-j}\times\Delta^k$ and a cellular map $\rho'\colon V\to|\mathbb{L}_{-n}(p)|$ with respect to a triangulation of V such that ρ sends the complement of $\mathrm{int}(V)$ to the base point $[i(X)]$ and $\rho|V$ factors through ρ'. We may assume the triangulation of V is a derived of another, and regard V as a Δ-set. By a simplex-wise inductive application of the relative Δ-map approximation theorem of Rourke and Sanderson [14, theorem 5.3], we may assume ρ' is a (realization of) a Δ-map, because $\mathbb{L}_{-n}(p^{-1}(\Delta))\otimes\Delta$ is Kan. A further application of the approximation theorem produces a homotopic Δ-map, also denoted by ρ', into a subcomplex $\bigcup\mathbb{L}_{-n}(p^{-1}(\Delta))\times FG(\Delta)$ of $\mathbb{L}_{-n}(p)$, because $\mathbb{L}_{-n}(p^{-1}(\Delta))$ is Kan. For each $(n-j+k)$-dimensional simplex Δ of V, $\rho'(\Delta)$ is given a structure of $(-j+k+l)$-dimensional geometric $\mathbb{Z}$-module quadratic Poincaré special $(n-j+k+2)$-ad on $\mathbb{R}^l\times E$ with radius measured in $X=$ the diameter of $p_*\rho'(\Delta)$. Glueing all these, after taking external suspensions if necessary, we obtain a $(-j+k+l')$-dimensional geometric $\mathbb{Z}$-module quadratic Poincaré special $(k+2)$-ad on $\mathbb{R}^{l'}\times E$ with radius max $\{$diameter of $p_*\rho'(\Delta)|\Delta\in V\}$, which is a simplex of $\mathbb{P}_{-j}(X;p)$. Note that, since p is a stratified system of fibrations, each piece can be shrunk to produce a copy of a smaller radius lying over a simplex Δ' of a barycentric subdivision of $p_*\rho'(\Delta)$ (use a lift to E of a shrinking map $p_*\rho'(\Delta)\to\Delta'$), and we fill in the resultant gaps with fine product cobordisms and glue these together to obtain a simplex with a smaller radius. More precisely, let V' denote the barycentric subdivision of V. There is an obvious simplicial map $f\colon V'\to V$ homotopic to the identity which maps, for each $(n-j+k)$-simplex Δ of V, one of the $(n-j+k)$-simplex Δ' of V' contained in Δ to Δ and all the others to faces of Δ. Apply the argument above to approximate (relatively) the composition $\rho' f\colon V'\to|\mathbb{L}_{-n}(p)|$ by a Δ-map. Then use a lift of the homotopy $1_V\simeq f$ to shrink things. This is called the "barycentric subdivision shrinking argument". There is a cobordism between the original and the new simplex, and repeated application of this barycentric subdivision shrinking argument, together with the triangulation argument, generates a map $\Delta^k\otimes[0,\infty)\to\mathbb{P}_{-j}(X;p)$. Since the radius goes to 0 as $t\to\infty$, this defines a simplex of $\mathbb{L}_{-j}(X;p)$. This defines the desired map A_j.

In the following proof, we will not mention taking barycentric subdivisions or external suspensions when we glue or split things for simplicity. First we will show that A_j maps into every component. Let σ be a 0-simplex of $\mathbb{L}_{-j}(X;p)$; since X is compact, we may assume that σ is a map from $[0,\infty)$ to $\mathbb{P}'_{-j}(X;p)^{(N)}$ with a sequence ε_i monotone decreasing to 0 such that $\sigma([i,\infty))\subset\mathbb{P}_{-j}(X,p,\varepsilon_i)^{(N)}$ for each i, where N is some positive integer. Embed X in S^{n-j} for a sufficiently large n, and let W and r denote a regular neighborhood of X in S^{n-j} and the retraction: $W\to X$. Let $p'\colon E'\to W$ be the stratified system of fibrations obtained as the pull-back of p by r; E' retracts to E, and p' is an extension of p and has the advantage that the base space is a manifold. $\sigma(0)$ can now be regarded as an $(l-j)$-dimensional geometric $\mathbb{Z}$-module quadratic Poincaré complex on $\mathbb{R}^l \times E'$ with radius ε_0 (with respect to p'). Let us split (stably) $\sigma(0)$ into pieces lying over the simplices of W. Let ε be any positive number; the stable splitting lemma gives $\delta > 0$ such that if $\sigma(0)$ has radius δ, then each split piece lies over an ε-neighborhood of the corresponding simplex of W. We may assume that this is the case. If $\varepsilon_0 > \delta$, then we can replace σ by another simplex $\sigma'(t) = \sigma(t+i)$ in the same component as σ, where i is chosen to be sufficiently large so that $\varepsilon_i < \delta$. If we had chosen ε sufficiently small at the beginning, we can construct a map of p' to itself of radius ε which sends $(p')^{-1}$ (ε-neighborhood of Δ) to $(p')^{-1}(\Delta)$ for each simplex $\Delta \in W$. This map induces ε-isomorphisms which make each split piece lie exactly over the corresponding simplex of W. The retraction $E' \to E$, then, induces isomorphisms that make each piece lying on $\mathbb{R}^l \times E$ for some l. Thus we have a triangulation of W such that each simplex of W is given a geometric $\mathbb{Z}$-module quadratic Poincaré ad on $\mathbb{R}^l \times E$. Since there are only trivial ads over the simplices in ∂W, we can associate trivial ads to the simplices in $S^{n-j} - \mathrm{int}(W)$ and define a 0-simplex of $FS\Omega^{n-j}(|\mathbb{L}_{-n}(p)|/i(X))$. We will show that A_j sends ρ into the same component as σ. First of all, by construction, $(A_j\rho)(0)$ and $\sigma(0)$ can be joined by a 1-simplex in $\mathbb{P}'_{-j}(X;p)$. This is the first step of the inductive construction of a 1-simplex τ connecting σ and $A_j\rho$. Assume we have constructed a map from $\partial_0\Delta^1 \otimes [0,\infty) \cup \Delta^1 \otimes [0,m] \cup \partial_1\Delta^1 \otimes [0,\infty)$ for some integer m, giving σ and $A_j\rho$ on each end. We can extend this over $\Delta^1 \otimes [m,m+1]$ by applying the "barycentric subdivision" shrinking argument to the union $\sigma([m,m+1]) \cup \tau(\Delta^1 \otimes \{m\}) \cup A_j\rho([m,m+1])$ to get $\tau(\Delta^1 \otimes \{m+1\})$ and then applying the triangulation argument to the resulted cobordism to fill in the square $\Delta^1 \otimes [m,m+1]$. This inductively constructs the desired homotopy. Thus A_j maps into every component.

Next we will show that the relative homotopy groups $\pi_k(A_j)$ vanish for all k. Its element is represented by a map σ: $\Delta^k \otimes [0,\infty) \to \mathbb{P}'_{-j}(X;p)$ such that $\sigma|\partial_i\Delta^k = 0$ for $i<k$ and $\sigma|\partial_k\Delta^k = A_j\rho$ for some ρ: $S^{n-j} \times \partial_k\Delta^k \to |\mathbb{L}_{-n}(p)|/i(X)$ such that $\rho(S^{n-j} \times \partial\partial_k\Delta^k) = [i(X)]$. We may assume that ρ maps the complement of the interior of some codimension 0 submanifold V of $S^{n-j} \times \partial_k\Delta^k$ to $i(X)$, and hence restricts to a map ρ': $V \to |\mathbb{L}_{-n}(p)|$. Roughly speaking, $\sigma(0)$ is a Poincaré pair whose boundary is split into pieces lying over simplices of V. To find a k-simplex ρ_σ of $FS\Omega^{n-j}(|\mathbb{L}_{-n}(p)|i(X))$ whose image $A_j\rho_\sigma$ by A_j is homotopic to σ fixing the boundary, we want to apply the stable splitting lemma to $\sigma(0)$. As before we assume that X is a subcomplex of S^{n-j}. This time let W denote a regular neighborhood of X in $S^{n-j} \times \Delta^k$, where S^{n-j} is identified with $S^{n-j} \times$ {the center of the $(k-1)$-simplex $\partial_k\Delta^k$}. We extend p to the pull-back p': $E' \to W$ using the retraction $W \to X$ and split things here as before. Unfortunately, the splitting of the boundary of $\sigma(0)$ is not over $W_0 = W \cap (S^{n-j} \times \partial_k\Delta^k)$ but over V. We remedy this as follows, changing ρ by homotopy. First apply the "barycentric subdivision" shrinking argument to each split piece to get a homotopic map, denoted by ρ again, for which the radius of each split piece is very small. Consider the composition $p_*\rho'$: $V \to X$. Choose sufficiently large m, so that the mapping cylinder M of $p_*\rho'$ can be embedded in $\Sigma^m S^{n-j} \times \partial_k\Delta^k \times [0,1]$ so that $V \subset \Sigma^m S^{n-j} \times \partial_k\Delta^k \times \{0\}$ and $X \subset \Sigma^m S^{n-j} \times \partial_k\Delta^k \times \{1\}$.

Take a regular neighborhood N of M in $\Sigma^m S^{n-j} \times \partial_k \Delta^k \times [0,1]$, then N retracts to M and then to X. Use this to get a pull-back $E^* \to N$ of $E \to X$. At 0, $\Sigma^m \rho'$ gives a geometric $\mathbb{Z}$-module quadratic Poincaré complex on E^* split over simplices of $\Sigma^m S^{n-j} \times \partial_k \Delta^k \times \{0\}$, and at 1, consider Σ^m of the boundary of $\sigma(0)$ regarded to be on E^*. Since these are essentially the same thing, these can be connected by a cobordism. Actually, if one places sufficiently many layers of product cobordisms along the mapping cylinder, we may assume that the radius of the cobordism is very small. If everything is sufficiently small, then we can apply the stable splitting lemma over N, to get a homotopy of ρ to a new map for which the splitting is over a regular neighborhood of X in $\Sigma^m S^{n-j} \times \partial_k \Delta^k$. Thus, from the beginning, we may assume that the splitting of the boundary of $\sigma(0)$ is over W_0. Now we can apply the stable splitting lemma to obtain a map $\rho_\sigma: S^{n-j} \times \Delta^k \to |\mathbb{L}_{-n}(p)|/i(X)$. ρ_σ is a k-simplex of $FS\Omega^{n-j}(|\mathbb{L}_{-n}(p)|/i(X))$ with all faces at the base point except ρ. The same argument as in the first part gives a homotopy between σ and $A_j \rho_\sigma$ fixing the boundary. This completes the proof. □

The proof actually gives the following:

(3.10) **Corollary.** (Shrinking lemma). *Let p be as in 3.9. Then, given any positive integer n, there exists a positive number ε such that for any $0<\delta<\varepsilon$, there is a function ("shrinking function") S: $\mathbb{P}_j'(X,p,\delta)^{(n)} \to \mathbb{L}_j(X;p)$ such that the map $\mathbb{P}_j'(X,p,\delta)^{(n)} \to \mathbb{P}_j'(X;p)$ induced by the following composition is homotopic to the inclusion map:*

$$\mathbb{P}_j'(X,p,\delta)^{(n)} \xrightarrow{S} \mathbb{L}_j(X;p) \xrightarrow{\text{restriction to } 0} \mathbb{P}_j(X;p).$$

4. Crystallographic groups

We begin this section by reviewing some work on crystallographic groups by Farrell and Hsiang in [6]. Γ is *crystallographic* if it is a discrete co-compact subgroup of $E(n)$, the group of rigid motions of Euclidean n-space. Identify $\mathbb{R}^n$ with the group of translations of $\mathbb{R}^n$, then $E(n) = \mathbb{R}^n \rtimes O(n)$. The intersection of Γ and $\mathbb{R}^n$ is the maximal abelian subgroup of Γ with finite index, which is denoted by A and is called the *translation subgroup* of Γ. The finite factor group Γ/A is called the *holonomy group* of Γ. The *rank* of Γ is the rank of A. For any positive integer $s, \Gamma_s = \Gamma/sA$ and $A_s = A/sA$. T and T_n denote the infinite cyclic group and the finite cyclic group of order n respectively.

4.1 Examples. (1) D_∞ will denote the infinite dihedral group; i.e., $D_\infty \subset E(1)$ is the subgroup generated by $x \mapsto x+1$ and $x \mapsto -x$ (where $x \in \mathbb{R}$). It is a semi-direct product $T \rtimes T_2$, where T_2 acts on T via multiplication by -1.

(2) See [6, p. 658] for the definition of 2-dimensional crystallographic groups of type 1, 2 and 3. The holonomy group of a crystallographic group of type 2 or 3 is $T_2 \oplus T_2$.

(3) A crystallographic group Γ of rank $n \geqq 2$ with holonomy group G is called *special* if there exist a crystallographic group $\tilde{\Gamma}$ of rank $m \geqq 1$ and an infinite sequence of positive integers s such that, if H is a maximal hyper-elementary subgroup of Γ_s, either

(4.1.1) H projects to a proper subgroup of G via the canonical map $\Gamma_s \to G$; or

(4.1.2) H is conjugate to a hyperelementary subgroup H' and there is a group surjection η: $q^{-1}(H') \to \tilde{\Gamma}$ together with a η-equivariant affine surjection h: $\mathbb{R}^n \to \mathbb{R}^m$ such that

$$|dh(X)| \leqq (2/\sqrt{s})\,|X|$$

for each vector X tangent to $\mathbb{R}^n$, where q: $\Gamma \to \Gamma_s$ is the natural projection and $|\ |$'s are the Euclidean metrics on $\mathbb{R}^m$ and $\mathbb{R}^n$. Note that $\tilde{\Gamma}$ is required to be independent of the choice of s and H.

Farrell and Hsiang showed that 2-dimensional crystallographic groups of type 1, 2 or 3 are special [6, theorem 4.2].

On the other hand, T and D_∞ are not special, because they have rank 1. But a similar statement as above holds true for D_∞ when we replace "hyper-elementary" by "elementary". In fact, let $\tilde{\Gamma} = D_\infty$ and $\{s\}$ be the set of all odd primes; then the only maximal elementary subgroups of $(D_\infty)_s = T_s \rtimes T_2$ (dihedral group of order $2s$) are $H_1 = T_s$ and $H_2 = T_2$ up to conjugacy. The image of H_1 in T_2 is $\{1\}$, so H_1 satisfies (4.1.1). For H_2, $q^{-1}(H_2) = (sT) \rtimes T_2$, so we can let η be the obvious isomorphism: $(st) \rtimes T_2 \to T \rtimes T_2 = \tilde{\Gamma}$, and h be the linear map $h(x) = (1/s)\,x$ $(x \in \mathbb{R})$. Thus H_2 satisfies (4.1.2).

The following results of Farrell and Hsiang will play a key role in the proof of our main theorem.

(4.2) **Theorem.** ([6, theorem 1.1], [5, theorem 3.1]). *Let Γ be a crystallographic group of rank n and holonomy group G, then either*

(i) *$\Gamma = \Gamma' \rtimes T$ where Γ' is a crystallographic subgroup of rank $n-1$; or*

(ii) *there is an infinite sequence of positive integers $s \equiv 1 \bmod |G|$ such that any hyperelementary subgroup of Γ_s which projects onto G (via the canonical map) projects isomorphically onto G; or*

(iii) *G is an elementary abelian 2-group and*

(a) *if $|G| = 2$, then $\Gamma = A \rtimes T_2$ and $T_2 = G$ acts on A via multiplication by -1, and*

(b) *if $|G| > 2$, then Γ maps epimorphically onto a crystallographic group of type 2 or 3.*

(4.3) **Lemma.** ([6, lemma 1.2]). *Let ϕ: $\Gamma \to \Gamma'$ be an epimorphism between crystallographic groups Γ of rank n and Γ' of rank m. Then there exists a ϕ-equivariant affine surjection F: $\mathbb{R}^n \to \mathbb{R}^m$.*

Theorem 4.2 above is stated in a slightly different form from [6, theorem 1.1]; this stronger version is implicitly used in the proof of [6, theorem 5.1]. On p. 665 of [6], a crystallographic group $\hat{\Gamma}$ is replaced by $\bar{\Gamma}$, via an epimorphism ψ: $\hat{\Gamma} \to \bar{\Gamma}$. This epimorphism should not increase the order of the holonomy group. In the case of possibility (ii) of 4.2 (with $\hat{\Gamma}$ used as Γ), ψ is the identity map. In the case of possibility (iii)-(a), one uses the obvious epimorphism from $\hat{\Gamma}$ to $(T \oplus T) \rtimes T_2$; so both have the same holonomy group. In the case of possibility (iii)-(b), $\hat{\Gamma}$ has holonomy group of order $\geqq 4$; so the epimorphism does not increase the order of holonomy group.

Suppose Γ satisfies 4.2(ii). Theorem 4.1 in [6] and an argument in the proof of theorem 4.4 [ibid.] shows that Γ is special; Γ itself is used as $\tilde{\Gamma}$. If Γ satisfies the possibility (iii) of 4.2, then Γ maps epimorphically onto a crystallographic group of type 1, 2 or 3 with the order of the holonomy group $\leqq |G|$, or Γ is $T \rtimes T_2$. Therefore we can rephrase 4.2 as follows:

(4.4) **Corollary.** *Let Γ be a crystallographic group of rank $n \geqq 1$, then either*

(1) *$\Gamma = \Gamma' \rtimes T$ where Γ' is a crystallographic subgroup of rank $n-1$; or*

(2) *there is an epimorphism from Γ onto a special crystallographic group which does not increase the order of the holonomy group; or*

(3) *Γ is isomorphic to D_∞.*

Next let us consider a slightly larger class of groups. Suppose Λ is a finitely generated group which maps onto a crystallographic group Γ of rank n with a finite kernel. For example, if Λ is a virtually abelian group of rank n, then, by lemma 1.2 in [5], there is a surjection $\Lambda \to \Gamma$ onto a crystallographic group of rank n with a finite kernel. Via this surjection, Λ acts by isometries on $\mathbb{R}^n$ discretely, virtually faithfully, with compact orbit space. The orbit space $\mathbb{R}^n/\Lambda$ is equal to $\mathbb{R}^n/\Gamma$. The action of Γ may not be free, since Γ may have torsion, but its translation subgroup A acts on $\mathbb{R}^n$ freely and the orbit space is a flat torus T^n. The holonomy group G of Γ acts on T^n as a group of isometries so that $\mathbb{R}^n/\Gamma = T^n/G$. Therefore the orbit space can be viewed as the orbit space of a finite group action on a compact smooth manifold. If (H) is the conjugacy class of a subgroup H of G, then $Y_{(H)}$ denotes the subset of T^n/G consisting of the points x such that the isotropy subgroup at a point of T^n lying in the orbit x is in (H). It is observed in [6] that $\{Y_{(H)}\}$ gives a stratification of T^n/G.

Let W_Λ denote a free contractible Λ-complex. Λ acts freely on $\mathbb{R}^n \times W_\Lambda$ diagonally.

(4.5) **Proposition.** *The projection $p\colon (\mathbb{R}^n \times W_\Lambda)/\Lambda \to \mathbb{R}^n/\Lambda$ is a stratified system of fibrations.*

Proof. Let $\beta\colon \Lambda \to \Gamma$ denote the surjection and $r\colon \mathbb{R}^n \to \mathbb{R}^n/\Lambda$ the quotient map. For each orbit $x \in \mathbb{R}^n/\Lambda = \mathbb{R}^n/\Gamma$, define Λ_x and Γ_x to be the isotropy subgroups of Λ and Γ at a point $\bar{x} \in r^{-1}(x)$. Λ_x and Γ_x are well-defined up to conjugacy. The map p has point inverses $p^{-1}(x) = W_\Lambda/\Lambda_x$, which are classifying spaces of Λ_x. Since $\Lambda_x = \beta^{-1}(\Gamma_x)$, this proposition will be proved if one can show that (Γ_x) is constant on each component of each stratum $Y_{(H)}$. Suppose x is in $Y_{(H)}$, and let $[\bar{x}] \in T^n$ denote the orbit of $\bar{x}$ by A. The isotropy subgroup of G at $[\bar{x}]$ is H. Let π denote the quotient map: $\Gamma \to G$. Then $\pi|\Gamma_x\colon \Gamma_x \to H$ is an isomorphism. This implies that Γ_x is locally constant on $Y_{(H)}$, and it in turn implies the proposition. □

Again let β be the surjection $\Lambda \to \Gamma$. Let us study the geometric implication of corollary 4.4(1), (2), and (3). First suppose (1) $\Gamma = \Gamma' \rtimes T$. Then by 4.3, the epimorphism $\phi\colon \Gamma \to T$ induces a ϕ-equivariant affine surjection $F\colon \mathbb{R}^n \to \mathbb{R}$. F is also $(\phi\beta)$-equivariant and defines a map $\bar{F}\colon \mathbb{R}^n/\Lambda \to \mathbb{R}/T = S^1$; $\bar{F}$ is a fibre bundle with fibre $\mathbb{R}^{n-1}/\Lambda'$, where $\mathbb{R}^{n-1}$ is a fibre of F and $\Lambda' = \ker(\phi\beta) = \beta^{-1}(\Gamma')$. $(\mathbb{R}^n \times W_\Lambda)/\Lambda$ also fibres over S^1 with fibre $(\mathbb{R}^{n-1} \times W_\Lambda)/\Lambda'$. Then p

restricts to a stratified system of fibrations between the fibres p': $(\mathbb{R}^{n-1} \times W_A)/A' \to \mathbb{R}^{n-1}/A'$.

Next suppose Γ satisfies 4.4(2) (resp. (3)), and let ϕ be the surjection $\Gamma \to \Gamma'$ (resp. the identity map: $D_\infty \to D_\infty$). Let $\tilde{\Gamma}$, s, η, h be as in 4.1(3) with Γ replaced by Γ'. Suppose H is a maximal hyperelementary (resp. elementary) subgroup of Γ'_s. Define a subgroup A_H of A (Γ_H of Γ) by $A_H = \beta^{-1}\phi^{-1}q^{-1}(H)$ ($\Gamma_H = \phi^{-1}q^{-1}(H)$). Γ_H is a subgroup of Γ of finite index, and hence a crystallographic group. Thus A_H maps onto a crystallographic group, and we can consider p_H: $(\mathbb{R}^n \times W_A)/A_H \to \mathbb{R}^n/A_H$. Here the complex W_A is used as W_{A_H}.

If H satisfies (4.1.1), then the holonomy group of $q^{-1}(H)$ has a strictly smaller order than Γ'.

Next suppose that H satisfies (4.1.2). Replace H by H'. The surjection ϕ induces a ϕ-equivariant affine surjection f: $\mathbb{R}^n \to \mathbb{R}^l$. The maps f and h induce a surjection

$$\alpha: \mathbb{R}^n/A_H = \mathbb{R}^n/\Gamma_H \to \mathbb{R}^l/q^{-1}(H) \to \mathbb{R}^m/\tilde{\Gamma}.$$

The composition hf: $\mathbb{R}^n \to \mathbb{R}^m$ will be denoted by $\tilde{\alpha}$. We consider the composition αp_H; the map p_H can be understood by studying αp_H and the restriction of p_H to the preimages of points of $\mathbb{R}^m/\tilde{\Gamma}$. For an orbit $x \in \mathbb{R}^m/\tilde{\Gamma}$, $\tilde{\Gamma}_x$ denotes the isotropy subgroup of $\tilde{\Gamma}$ at a point $\tilde{x} \in \mathbb{R}^m$ in the orbit x, as usual. Define a subgroup $A_H(x)$ of $A_H(\Gamma_H(x)$ of $\Gamma_H)$ by $A_H(x) = (\eta\phi\beta|A_H)^{-1}(\tilde{\Gamma}_x)$ $(\Gamma_H(x) = (\eta\phi|\Gamma_H)^{-1}(\tilde{\Gamma}_x))$; i.e., $A_H(x)(\Gamma_H(x))$ is the set of elements of $A_H(\Gamma_H)$ which leaves the affine subspace $\mathbb{R}^{n-m} = \tilde{\alpha}^{-1}(\tilde{x})$ of $\mathbb{R}^n$ invariant. $A_H(x)$ and $\Gamma_H(x)$ are well-defined up to conjugacy. The point inverses $(\alpha p_H)^{-1}(x)$ of αp_H are $(\mathbb{R}^{n-m} \times W_A)/A_H(x)$. Note that αp_H is a stratified system of fibrations, and that we can use the same filtration of $\mathbb{R}^m/\tilde{\Gamma}$ and the same neighborhoods of strata as those for the projection $\tilde{p}$: $(\mathbb{R}^m \times W_{\tilde{\Gamma}})/\tilde{\Gamma} \to \mathbb{R}^m/\tilde{\Gamma}$, since A_H is determined by $\tilde{\Gamma}_x$. The actions of $A_H(x)$ and $\Gamma_H(x)$ on $\mathbb{R}^{n-m}$ give a homomorphism

$$A_H(x) \xrightarrow{\beta'} \Gamma_H(x) \xrightarrow{\gamma} E(n-m),$$

where β' is the restriction of β. Let $\Gamma_H^*(x)$ denote the image of this in $E(n-m)$.

(4.6) **Proposition.** *$\Gamma_H^*(x)$ is a crystallographic subgroup of $E(n-m)$ and the kernel of $A_H(x) \to \Gamma_H^*(x)$ is finite.*

Proof. Let K be the kernel of $\eta\phi|\Gamma_H$: $\Gamma_H \to \tilde{\Gamma}$. Since K is a normal subgroup of a crystallographic group Γ_H, K is also a crystallographic group. In fact there is a K-invariant $(n-m)$-dimensional affine subspace V of $\mathbb{R}^n$ on which K acts discretely and faithfully with compact quotient. (See Farkas [3], theorem 17.) Since all the parallels of $\mathbb{R}^{n-m}$ in $\mathbb{R}^n$ are of the form $\tilde{\alpha}^{-1}(pt)$, they are all invariant under the action of K, and hence V is contained in one of these, say $\tilde{\alpha}^{-1}(x_0)$ for some $x_0 \in \mathbb{R}^m$. Both V and $\tilde{\alpha}^{-1}(x_0)$ have the same dimension, so they are actually equal. Therefore $\tilde{\alpha}^{-1}(x_0)/K$ is compact. On the other hand, the actions of K on $\tilde{\alpha}^{-1}(\tilde{x})$ and on $\tilde{\alpha}^{-1}(x_0)$ are affinely equivalent. So $\mathbb{R}^{n-m}/\gamma(K)$ is also compact. Since $\Gamma_H^*(x)$ contains $\gamma(K)$, $\mathbb{R}^{n-m}/\Gamma_H^*(x)$ is compact. $\Gamma_H^*(x)$ is obviously discrete in $E(n-m)$; therefore, $\Gamma_H^*(x)$ is a crystallographic subgroup of $E(n-m)$. Now $|\mathrm{Ker}(\gamma)| = [K \cdot \mathrm{Ker}(\gamma)\colon K] \leq [\Gamma_H(x)\colon K] = |\tilde{\Gamma}_x| < \infty$, so

$\mathrm{Ker}(\gamma)$ is finite. Since the kernel of β is finite by assumption, so is the kernel of $\gamma\beta'$. □

Thus p_H restricts to p_x: $(\mathbb{R}^{n-m} \times W_A)/A_H(x) \to \mathbb{R}^{n-m}/A_H(x)$, where A_H maps onto a crystallographic group $\Gamma_H^*(x)$ with finite kernel and acts on $\mathbb{R}^{n-m}$ via the action of $\Gamma_H^*(x)$. Note that $n-m$ is strictly smaller than n.

Next we observe that $H_*(\mathbb{R}^n/A; \mathbb{L}(p))$ and $L_*^{-\infty}((\mathbb{R}^n \times W_A)/A)$ satisfy elementary and hyperelementary induction. Let β: $A \to \Gamma$ be as before, and suppose we have an epimorphism ψ: $\Gamma \to G$ onto a finite group G (not necessarily the holonomy group of Γ). Of course, what we have in mind is the composition $q\phi$: $\Gamma \to \Gamma' \to \Gamma_s'$ above, where Γ' is a special crystallographic group. For a subgroup H of G, let $A_H = \beta^{-1}\psi^{-1}(H)$ and $\Gamma_H = \psi^{-1}(H)$. Let $\mathscr{S}$ denote the category of the subgroups of G and conjugations and Ab denote the category of abelian groups.

Let us define a Mackey functor M: $\mathscr{S} \to Ab$. For a subgroup H of G, $M(H)$ is defined to be $H_j(\mathbb{R}^n/A_H; \mathbb{L}(p_H))$, where p_H is the projection $(\mathbb{R}^n \times W_A)/A_H \to \mathbb{R}^n/A_H$. Suppose $f = (H, g, K)$ is a morphism from H to K, i.e., g is an element of G such that $g^{-1}Hg \subset K$. Pick an element $\lambda \in A$ such that $\psi\beta(\lambda) = g$. Then the actions of λ on $\mathbb{R}^n \times W_A$ and $\mathbb{R}^n$ induce a map $f_\#$: $p_H \to p_K$ between stratified systems of fibrations; i.e., we have a commutative diagram:

$$\begin{array}{ccc} (\mathbb{R}^n \times W_A)/A_H & \xrightarrow{f_\#} & (\mathbb{R}^n \times W_A)/A_K \\ \Big\downarrow{\scriptstyle p_H} & & \Big\downarrow{\scriptstyle p_K} \\ \mathbb{R}^n/A_H & \xrightarrow[f_\#]{} & \mathbb{R}^n/A_K. \end{array}$$

We have the following two operations corresponding to $f_\#$:

(1) (functorial image f_*: $\mathbb{P}(\mathbb{R}^n/A_H; p_H) \to \mathbb{P}(\mathbb{R}^n/A_K; p_K)$). If $\mathbb{Z}[S]$, h: $S \to (\mathbb{R}^n \times W_A)/A_H$, is a geometric $\mathbb{Z}$-module on $(\mathbb{R}^n \times W_A)/A_H$, then a geometric $\mathbb{Z}$-module $f_*(\mathbb{Z}[S], h)$ on $(\mathbb{R}^n \times W_A)/A_K$ is defined to be $\mathbb{Z}[S]$ with $f_\# h$: $S \to (\mathbb{R}^n \times W_A)/A_K$. If $k = \Sigma m_\rho \rho$ is a morphism between geometric $\mathbb{Z}$-modules on $(\mathbb{R}^n \times W_A)/A_H$, then $f_* k = \Sigma m_\rho f_\# \rho$ is the functorial image of k. These induce a map f_*: $\mathbb{P}(\mathbb{R}^n/A_H; p_H) \to \mathbb{P}(\mathbb{R}^n/A_K; p_K)$.

(2) (pullback f^*: $\mathbb{P}(\mathbb{R}^n/A_K; p_K) \to \mathbb{P}(\mathbb{R}^n/A_H; p_H)$). If $\mathbb{Z}[S]$, h: $S \to (\mathbb{R}^n \times W_A)/A_K$, is a geometric $\mathbb{Z}$-module on $(\mathbb{R}^n \times W_A)/A_K$, then $f^*(\mathbb{Z}[S], h)$ is $(\mathbb{Z}[S^*], h^*)$, where h^*: $S^* \to (\mathbb{R}^n \times W_A)/A_H$ is the pullback of h. Pullbacks of morphisms are also defined by pullbacks of paths in the obvious way. These define a map f^*: $\mathbb{P}(\mathbb{R}^n/A_K; p_K) \to \mathbb{P}(\mathbb{R}^n/A_H; p_H)$.

Obviously, f_* does not increase the radius. On the other hand, f^* may increase the radius in general; but f^* does not increase the radius measured in $\mathbb{R}^n$, so it does not increase the radii of things which have sufficiently small radius. Therefore f_* and f^* induce maps $\mathbb{L}(\mathbb{R}^n/A_H; p_H) \to \mathbb{L}(\mathbb{R}^n/A_K; p_K)$ and $\mathbb{L}(\mathbb{R}^n/A_K; p_K) \to \mathbb{L}(\mathbb{R}^n/A_H; p_H)$. By the characterization theorem, these induce the desired maps f_*: $M(H) \to M(K)$ and f^*: $M(K) \to M(H)$. M is a bifunctor.

(4.7) **Proposition.** *M satisfies the double coset formula, and hence is a Mackey functor.*

Proof. Let L and L' be subgroups of a subgroup H of G, and suppose H has a double coset decomposition $H=\bigcup_{i=1}^{k} Lg_iL'$, $g_i\in H$. Let $\Lambda_H=\bigcup_{i=1}^{k}\Lambda_L\bar{g}_i\Lambda_{L'}$ be a corresponding double coset decomposition of Λ_H, where $\bar{g}_i$ is an element of Λ_H such that $\psi\beta(\bar{g}_i)=g_i$. Let P be the pullback:

$$\begin{array}{ccc} P & \longrightarrow & (\mathbb{R}^n\times W_A)/\Lambda_{L'} \\ \downarrow & & \downarrow{\scriptstyle (L',e,H)_*} \\ (\mathbb{R}^n\times W_A)/\Lambda_L & \longrightarrow & (\mathbb{R}^n\times W_A)/\Lambda_H , \end{array}$$

i.e.,

$$P=\{([x],[y])\in(\mathbb{R}^n\times W_A)/\Lambda_L\times(\mathbb{R}^n\times W_A)/\Lambda_{L'}\ [x]_{\Lambda_H}=[y]_{\Lambda_H}\in(\mathbb{R}^n\times W_A)/\Lambda_H\},$$

where $x\in\mathbb{R}^n\times W_A$ and [ ] denotes the corresponding orbit. Then it is easily verified that the map

$$\bigcup_{i=1}^{k}(\mathbb{R}^n\times W_A)/\Lambda_{L\cap g_iL'g_i^{-1}}\to P;$$
$$[x]_{\Lambda_{L\cap g_iL'g_i^{-1}}}\mapsto([x],[\bar{g}_i^{-1}x])$$

is a Λ_H-isomorphism. Therefore we obtain a pullback diagram:

$$\begin{array}{ccc} \bigcup_{i=1}^{k}(\mathbb{R}^n\times W_A)/\Lambda_{L\cap g_iL'g_i^{-1}} & \xrightarrow{(L\cap g_iL'g_i^{-1},\,g_i^{-1},\,L')_*} & (\mathbb{R}^n\times W_A)/\Lambda_{L'} \\ \downarrow{\scriptstyle (L\cap g_iL'g_i^{-1},\,e,\,L)_*} & & \downarrow{\scriptstyle (L',e,H)_*} \\ (\mathbb{R}^n\times W_A)/\Lambda_L & \xrightarrow[(L,e,H)_*]{} & (\mathbb{R}^n\times W_A)/\Lambda_H \end{array}$$

and the double coset formula is easily derived from this. □

Dress's equivariant Witt ring $GW(H,\mathbb{Z})$ [2] acts on $M(H)$ by tensor product. Recall that $GW(H,\mathbb{Z})$ is constructed using H-spaces. An H-space is a $\mathbb{Z}$-free (left) $\mathbb{Z}H$-module N together with a symmetric H-invariant non-singular form $f\colon N\times N\to\mathbb{Z}$. Let $N^*=\mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$, then N^* is also a (left) $\mathbb{Z}H$-module. An element $h\in H$ acts on N^* by $h\cdot\alpha(y)=\alpha(h^{-1}\cdot y)$ for $\alpha\in N^*$, $y\in N$. By letting $(C_N)_0=N^*$ and $(C_N)_i=0$ for $i\neq 0$, we have a $\mathbb{Z}$-module chain complex C_N. We define a 0-dimensional symmetric Poincaré structure $\phi_f\colon N\to N^*$ by $\phi_f(x)=f(x,-)$. By assumption ϕ_f is an isomorphism. If $\{e_1,\dots,e_m\}$ is a (free $\mathbb{Z}$-module) basis of N^* and $\mathbb{Z}[S]$, $h\colon S\to(\mathbb{R}^n\times W_A)/\Lambda_H$, is a geometric $\mathbb{Z}$-module, then $N^*\otimes_{\mathbb{Z}}\mathbb{Z}[S]$ is a geometric $\mathbb{Z}$-module $\mathbb{Z}[\{e_1,\dots,e_m\}\times S]$ on $(\mathbb{R}^n\times W_A)/\Lambda_H$, where a basis element $e_i\otimes x$ is sent to $h(x)$ for $x\in S$. Since Λ_H maps onto H, Λ_H acts on N^*. Λ_H is supposed to act diagonally on the free $\mathbb{Z}\Lambda_H$-module corresponding to this geometric $\mathbb{Z}$-module on $(\mathbb{R}^n\times W_A)/\Lambda_H$. This can be done through the following definition of morphisms. Suppose $a\colon M\to N$ is a $\mathbb{Z}H$-homomorphism of $\mathbb{Z}$-free $\mathbb{Z}H$-modules with $\mathbb{Z}$-bases $\{d_i\}$, $\{e_k\}$ and b:

$\mathbb{Z}[T]\to\mathbb{Z}[S]$ is a morphism of geometric $\mathbb{Z}$-modules on $(\mathbb{R}^n\times W_\Lambda)/\Lambda_H$. Let $\rho_j^{(x,y)}$ be a path in the morphism b. Pick a point $\tilde{x}$ (resp. $\tilde{y}$)$\in\mathbb{R}^n\times W_\Lambda$ in the orbit x (resp. y)$\in(\mathbb{R}^n\times W_\Lambda)/\Lambda_H$. Lift this path to a path in $\mathbb{R}^n\times W_\Lambda$ from $\tilde{x}$ to $\tilde{y}'$. Let $g\in\Lambda_H$ be the unique covering transformation which sends $\tilde{y}$ to $\tilde{y}'$. Write $g^{-1}\cdot a(d_i)=\Sigma\, a_{ijk}e_k$. Then consider a sum of paths $\Sigma\, a_{ijk}\,\rho_{ijk}^{(d_i\otimes x,\, e_k\otimes y)}$, where $\rho_{ijk}^{(d_i\otimes x,\, e_k\otimes y)}=\rho_j^{(x,y)}$. The tensor product $a\otimes b$ is a morphism from $M\otimes_{\mathbb{Z}}\mathbb{Z}[T]$ to $N\otimes_{\mathbb{Z}}\mathbb{Z}[S]$ defined as the sum of these sums for all $\rho_j^{(x,y)}$'s in b. The tensor product formula in [13, § 1.9], viewed as above, allows us to take tensor products with (C_N,ϕ_f), and this induces the desired action of $GW(H,\mathbb{Z})$ on $M(H)$. The following theorem can be proved in the same way as the proof of [4, theorem 2.3].

(4.8) **Theorem.** *M is a $GW(-,\mathbb{Z})$-module.*

Another Mackey functor M': $\mathcal{S}\to Ab$ can be defined by setting $M'(H) = L_j^{-\infty}((\mathbb{R}^n\times W_\Lambda)/\Lambda_H)$. f_* and f^* are defined in the same way as for M. We can prove:

(4.9) **Theorem.** *M' is a $GW(-,\mathbb{Z})$-module.*

As an immediate consequence of 4.8 and 4.9, we have the following theorem. See [2].

(4.10) **Theorem.** *If F is the family of the conjugacy classes of maximal hyperelementary subgroups of G. Then the following sequences are exact.*

$$0\to H_j(\mathbb{R}^n/\Lambda;\mathbb{L}(p))\xrightarrow{(\mathrm{res}_H)_H}\bigoplus_{H\in F}H_j(\mathbb{R}^n/\Lambda_H;\mathbb{L}(p_H))\longrightarrow\bigoplus_{H,K,g}H_j(\mathbb{R}^n/\Lambda_{H\cap gKg^{-1}};\mathbb{L}(p_{H\cap gKg^{-1}}))$$

$$0\to L_j^{-\infty}((\mathbb{R}^n\times W_\Lambda)/\Lambda)\xrightarrow{(\mathrm{res}_H)_H}\bigoplus_{H\in F}L_j^{-\infty}((\mathbb{R}^n\times W_\Lambda)/\Lambda_H)\longrightarrow\bigoplus_{H,K,g}L_j^{-\infty}((\mathbb{R}^n\times W_\Lambda)/\Lambda_{H\cap gKg^{-1}}).$$

If F is the family of the conjugacy classes of maximal elementary subgroups of G, then these are exact when each term is tensored with $\mathbb{Z}[1/2]$.

The following is the main result of this paper.

(4.11) **Theorem.** *Suppose Λ is a finitely generated group which acts by isometries on $\mathbb{R}^n$ discretely, virtually faithfully, with compact quotient. Let p be the projection $(\mathbb{R}^n\times W_\Lambda)/\Lambda\to\mathbb{R}^n/\Lambda$, where W_Λ is a contractible free Λ-complex. Then there is a natural isomorphism*

$$1\otimes a\colon\ \mathbb{Z}[1/2]\otimes H_j(\mathbb{R}^n/\Lambda;\mathbb{L}(p))\to\mathbb{Z}[1/2]\otimes L_j^{-\infty}(\Lambda).$$

Proof. The composition

$$\mathbb{H}_{-j}(\mathbb{R}^n/\Lambda;\mathbb{L}(p))\xrightarrow{A_{-j}}\mathbb{L}_j(\mathbb{R}^n/\Lambda;p)\xrightarrow{F}\mathbb{P}_j(\mathbb{R}^n/\Lambda;p)$$

has image in $\mathbb{P}_j'(\mathbb{R}^n/\Lambda;p)=\mathbb{P}_j(*;(\mathbb{R}^n\times W_\Lambda)/\Lambda\to *)=\mathbb{L}_j((\mathbb{R}^n\times W_\Lambda)/\Lambda)$, where A_{-j} is the assembly map (3.9) and F is the restriction to 0; so it defines a map

$\mathbb{H}_{-j}(\mathbb{R}^n/\Lambda;\mathbb{L}(p))\to\mathbb{L}_j((\mathbb{R}^n\times W_\Lambda)/\Lambda)$. This induces the desired map a: $H_j(\mathbb{R}^n/\Lambda;\mathbb{L}(p))\to L_j^{-\infty}(\Lambda)$.

Note that the theorem is a statement on the action of Λ on $\mathbb{R}^n$ and not on the group itself. We will prove the theorem inductively on the "size" $(n,h(\Gamma))$ of the action of Λ, where Γ is the image of Λ in $E(n)$ and $h(\Gamma)$ is the holonomy number of Γ; i.e., $h(\Gamma)$ is the order of the holonomy group of Γ if $\operatorname{rank}(\Gamma)=1$, and $h(\Gamma)$ is the minimum order of the holonomy group of a crystallographic group of rank $\geqq 2$ onto which Γ can map epimorphically if $\operatorname{rank}(\Gamma)\geqq 2$. We use the lexicographic order for the pairs $(n,h(\Gamma))$. If $n=0$, then $\mathbb{R}^n/\Lambda$ is a single point. Since $\mathbb{H}_{-j}(*;\mathbb{L}(p))\simeq\mathbb{L}_j(*;p)$ $(=\mathbb{L}_j(W_\Lambda/\Lambda))$ by 3.9, the theorem is obvious in this case. So assume that $n\geqq 1$.

First suppose that Γ satisfies (1) of 4.4. We use the notation in the remarks after 4.4. Recall that $\mathbb{R}^n/\Lambda$ fibres over S^1 with fibre diffeomorphic to $\mathbb{R}^{n-1}/\Lambda'$. The homology group $H_j(\mathbb{R}^n/\Lambda,\ \mathbb{R}^{n-1}/\Lambda';\mathbb{L}(p))$ is isomorphic to $H_{j-1}(\mathbb{R}^{n-1}/\Lambda';\mathbb{L}(p'))$. Thus we have a commutative ladder:

$$\begin{array}{ccccc}
\ldots\to & H_j(\mathbb{R}^{n-1}/\Lambda';\mathbb{L}(p')) & \longrightarrow & H_j(\mathbb{R}^n/\Lambda;\mathbb{L}(p)) & \\
 & \big\downarrow & & \big\downarrow & \\
\ldots\to & L_j^{-\infty}(\Lambda') & \longrightarrow & L_j^{-\infty}(\Lambda) &
\end{array}$$

$$\begin{array}{ccccc}
\to & H_{j-1}(\mathbb{R}^{n-1}/\Lambda';\mathbb{L}(p')) & \longrightarrow & H_{j-1}(\mathbb{R}^{n-1}/\Lambda';\mathbb{L}(p')) & \to\ldots \\
 & \big\downarrow & & \big\downarrow & \\
\to & L_{j-1}^{-\infty}(\Lambda') & \longrightarrow & L_{j-1}^{-\infty}(\Lambda') & \to\ldots
\end{array}$$

The first row is the exact sequence for the pair $(\mathbb{R}^n/\Lambda,\ \mathbb{R}^{n-1}/\Lambda')$; and the second row is induced from the well-known exact sequence due to Wall, Shaneson, Farrell and Hsiang. By induction hypothesis and 5-lemma, $1\otimes a$ is proved to be an isomorphism.

Next suppose Γ satisfies 4.4(2). Choose an epimorphism ϕ: $\Gamma\to\Gamma'$ onto a crystallographic group Γ' of rank $l\geqq 2$ whose holonomy group G' has order equal to $h(\Gamma)$. Apply corollary 4.4 to Γ'. As $\operatorname{rank}(\Gamma')\geqq 2$, the possibility (3) does not occur. In the case of possibility (1), Γ' maps epimorphically onto T, and hence so does Γ. We have already observed that the theorem holds true in this case. In the case of possibility (2), Γ' maps epimorphically onto a special crystallographic group Γ'' with holonomy group G'', and $|G''|\leqq|G|$. Γ'' is special, so it has rank $\geqq 2$. As $|G'|$ is the minimum, $|G''|=|G'|=h(\Gamma)$. So we may assume from the beginning that Γ' is special. So there exist a crystallographic group $\tilde{\Gamma}$ of rank $m\geqq 1$ and an infinite sequence of positive integers s such that any maximal hyperelementary subgroup of Γ_s' satisfies (4.1.1) or (4.1.2) with Γ and G replaced by Γ' and G'.

We first show that the map being considered is injective. Suppose y is an element of the kernel. We will show that $y=0$. We can regard $2^r y$ to be an element of $H_j(\mathbb{R}^n/\Lambda;\mathbb{L}(p))$, for some r. Represent $2^r y$ by a 0-cell ρ of $\mathbb{H}_{-j}(\mathbb{R}^n/\Lambda;\mathbb{L}(p))$. $A_{-j}\rho(0)$ represents the image $a(2^r y)$ by a, which represents 0 in $\mathbb{Z}[1/2]\otimes L_j^{-\infty}((\mathbb{R}^n\times W_\Lambda)/\Lambda)$. By choosing a sufficiently large r, we may assume that $A_{-j}\rho(0)$ represents 0 in $L_j^{-\infty}((\mathbb{R}^n\times W_\Lambda)/\Lambda)$. The Kan condition implies that there exists a 1-simplex σ of $\mathbb{L}_j((\mathbb{R}^n\times W_\Lambda)/\Lambda)$ which connects $A_{-j}\rho(0)$ and 0. Pull σ back to $\mathbb{R}^n\times W_\Lambda$ and δ be the radius measured in $\mathbb{R}^n$. It is a finite number. Choose a positive number s in (2) sufficiently large so that

$$2\delta K/\sqrt{s}<\varepsilon$$

where K is the Lipschitz constant of the affine surjection $f\colon \mathbb{R}^n\to\mathbb{R}^l$ induced by the epimorphism $\phi\colon \Gamma\to\Gamma'$ and ε is the positive number posited in 3.10, where we consider $\mathbb{P}_j'(\mathbb{R}^m/\tilde{\Gamma},\tilde{p},-)^{(\dim\sigma)}$. Note that ε depends not only on $\tilde{\Gamma}$ but also on the dimension of σ, so we cannot choose s which works for all y's. Now we have a commutative diagram:

$$\begin{array}{ccc} 0\to H_j(\mathbb{R}^n/\Lambda;\mathbb{L}(p)) & \xrightarrow{(\mathrm{res}_H)_H} & \bigoplus_H H_j(\mathbb{R}^n/\Lambda_H;\mathbb{L}(p_H))\to\dots \\ \Big\downarrow a & & \Big\downarrow \oplus a_H \\ 0\to L_j^{-\infty}((\mathbb{R}^n\times W_\Lambda)/\Lambda) & \xrightarrow[(\mathrm{res}_H)_H]{} & \bigoplus_H L_j^{-\infty}((\mathbb{R}^n\times W_\Lambda)/\Lambda_H)\to\dots \end{array},$$

where each row comes from the restriction maps corresponding to the conjugacy classes of the maximal hyperelementary subgroups H of Γ_s' (4.10). For H which projects to a proper subgroup of G', the size of the action of Λ_H is strictly smaller than the size of the action of Λ, because Λ_H maps onto $q^{-1}(H)$ and the order of the holonomy group of $q^{-1}(H)$ is strictly smaller than $|G'|$ $=h(\Gamma)$. So, by induction hypothesis, $1\otimes a_H$ is an isomorphism, and hence $\mathrm{res}_H(2^r y)=0$. For H which projects onto G', we have a shrinking map $\alpha\colon \mathbb{R}^n/\Lambda_H\to\mathbb{R}^m/\tilde{\Gamma}$, after replacing H by H'. Notice that the map a_H factors through $H_j(\mathbb{R}^m/\tilde{\Gamma};\mathbb{L}(\alpha p_H))$, by regarding things controlled in $\mathbb{R}^n/\Lambda_H$ to be controlled in $\mathbb{R}^m/\tilde{\Gamma}$ via α, and that the image of $\mathrm{res}_H(2^r y)$ in $H_j(\mathbb{R}^m/\tilde{\Gamma};\mathbb{L}(\alpha p_H))$ which is represented by $\mathrm{res}_H(A_{-j}\rho(0))$ is 0, due to smallness of σ in $\mathbb{R}^m/\tilde{\Gamma}$ and the shrinking lemma. Now we claim that the map

$$\mathbb{Z}[1/2]\otimes H_j(\mathbb{R}^n/\Lambda_H;\mathbb{L}(p_H))\to\mathbb{Z}[1/2]\otimes H_j(\mathbb{R}^m/\tilde{\Gamma};\mathbb{L}(\alpha p_H))$$

is an isomorphism; this will imply that $\mathrm{res}_H(2^r y)=0$.

Recall that

$$H_*(X;\mathbb{L}(p))=[S^*,\lim FS\Omega^j(|\mathbb{L}_{-j}(p)|/i(X))].$$

To prove the claim, we show that the map

$$\lim FS\Omega^j(|\mathbb{L}_{-j}(p_H)|/(\mathbb{R}^n/\Lambda_H))\to\lim FS\Omega^j(|\mathbb{L}_{-i}(\alpha p_H)|/(\mathbb{R}^m/\tilde{\Gamma}))$$

is a homotopy equivalence up to 2-torsion. For the convenience of the proof, we replace $\mathbb{L}(p)$ by

$$\mathbb{L}'(p)=\{\bigcup_\Delta \mathbb{L}(p^{-1}(v_\Delta))\otimes\Delta\}/\sim,$$

where v_Δ is a carefully chosen vertex of a simplex Δ of the control space. See [9, proof of 8.6]. Recall that $(\alpha p_H)^{-1}(v_\Delta) = (\mathbb{R}^{n-m} \times W_\Delta)/\Lambda_H(v_\Delta)$, and that the size of the action of $\Lambda_H(v_\Delta)$ is strictly smaller than that of Λ. Therefore, by induction, $\mathbb{L}((\alpha p_H)^{-1}(v_\Delta)) \simeq \mathbb{H}(\mathbb{R}^{n-m}/\Lambda_H(v_\Delta); \mathbb{L}(p_\Delta))$ modulo 2-torsion. Here p_Δ is the stratified system of fibrations $(\mathbb{R}^{n-m} \times W_\Delta)/\Lambda_H(v_\Delta) \to \mathbb{R}^{n-m}/\Lambda_H(v_\Delta)$. So, modulo 2-torsion,

$$\begin{aligned}
&\lim_j FS\Omega^j(|\mathbb{L}'_{-j}(p_H)|/(\mathbb{R}^n/\Lambda_H)) \\
&\quad = \lim_{i+k} FS\Omega^{i+k}\{(\bigcup_{\Delta' \subset \mathbb{R}^n/\Lambda_H} |\mathbb{L}_{-i-k}((p_H)^{-1}(v_{\Delta'}))| \times \Delta')/\sim\}/(\mathbb{R}^n/\Lambda_H) \\
&\quad \simeq \lim_i FS\Omega^i\{\bigcup_{\Delta \subset \mathbb{R}^m/\tilde{\Gamma}} \lim_k \Omega^k(|\mathbb{L}'_{-k-i}(p_\Delta)|/(\mathbb{R}^{n-m}/\Lambda_H(v_\Delta))) \times \Delta/\sim)/(\mathbb{R}^m/\tilde{\Gamma}) \\
&\quad \simeq \lim_i FS\Omega^i((\bigcup_\Delta |\mathbb{H}_i(\mathbb{R}^{n-m}/\Lambda_H(v_\Delta); \mathbb{L}(p_\Delta))| \times \Delta)/\sim)/(\mathbb{R}^m/\tilde{\Gamma}) \\
&\quad \simeq \lim_i FS\Omega^i((\bigcup |\mathbb{L}_{-i}((\alpha p_H)^{-1}(v_\Delta))| \times \Delta)/\sim)/(\mathbb{R}^m/\tilde{\Gamma}) \\
&\quad = \lim_i FS\Omega^i(|\mathbb{L}'_{-i}(\alpha p_H)|/(\mathbb{R}^m/\tilde{\Gamma})).
\end{aligned}$$

Thus the claim is proved. Now we have proved that $\mathrm{res}_H(2^r y) = 0$ for all H. Since the product of restriction maps $(\mathrm{res}_H)_H$ is injective, this implies that $2^r y$, and hence y, is 0; i.e., $1 \otimes a$ is injective.

The onto part is similar. Pick an element y of $\mathbb{Z}[1/2] \otimes L_j^{-\infty}((\mathbb{R}^n \times W_\Delta)/\Lambda)$. Without loss of generality we may assume that y belongs to $L_j^{-\infty}((\mathbb{R}^n \times W_\Delta)/\Lambda)$. Represent y by a geometric $\mathbb{Z}$-module quadratic Poincaré complex y^* on $(\mathbb{R}^n \times W_\Delta)/\Lambda$. We want to show that the restriction image $\mathrm{res}_H(y)$ of y in each $\mathbb{Z}[1/2] \otimes L_j^{-\infty}((\mathbb{R}^n \times W_\Delta)/\Lambda_H)$ lies in the image of $1 \otimes a_H$. If the size of the action of Λ_H is strictly smaller than that of Λ, then by induction hypothesis this is the case. If not, then there is a shrinking map $\alpha\colon \mathbb{R}^n/\Lambda_H \to \mathbb{R}^m/\tilde{\Gamma}$ for some crystallographic group $\tilde{\Gamma}$ of rank $m \geqq 1$, and we can make the radius of the restriction image $\mathrm{res}_H(y^*)$ of y^* arbitrarily small on $\mathbb{R}^m/\tilde{\Gamma}$, by choosing a very large integer s. (Thus s is chosen after y^* is picked up and $\tilde{\Gamma}$ is independent of s and H, as in the injectivity part.) Now the characterization theorem implies that $\mathrm{res}_H(y)$ comes from an element of $\mathbb{Z}[1/2] \otimes H_j(\mathbb{R}^m/\tilde{\Gamma}; \mathbb{L}(\alpha p_H))$, and hence from an element y_H of $\mathbb{Z}[1/2] \otimes H_j(\mathbb{R}^n/\Lambda_H; \mathbb{L}(p_H))$. Finally $(y_H)_H$ is the image of an element x of $\mathbb{Z}[1/2] \otimes H_j(\mathbb{R}^n/\Lambda; \mathbb{L}(p))$ by the restriction map. Here we need to use the next column of the diagram, which is already known to be isomorphic or at least injective. (This is why we proved injectivity part first.) Since restriction maps are injective, $1 \otimes a$ sends x to y, and this completes the onto part in case 4.4(2).

Lastly, if Γ satisfies (3), we use elementary induction instead of hyper-elementary induction. This is the only place where we need to use $\mathbb{Z}[1/2]\otimes$. Now the induction step is completed, and the theorem is proved. □

Note that, if Γ contains no element of order 2, then D_∞ never shows up in the induction steps. Therefore, for Λ acting on $\mathbb{R}^n$ with no elements of period 2, the theorem holds true without taking tensor products $\mathbb{Z}[1/2]\otimes$.

As a final remark, let us relate this result to the usual assembly map. Consider the following diagram:

$$\begin{array}{ccccc} H_*(B_\Lambda;\mathbb{L}(1)) & \xrightarrow{f} & H_*(\mathbb{R}^n/\Lambda;\mathbb{L}(p)) & \xrightarrow{a} & L_*^{-\infty}(\Lambda) \\ & \searrow^{p_*} & \downarrow g & & \\ & & H_*(\mathbb{R}^n/\Lambda;\mathbb{L}(1)) & & \end{array}$$

where $B_\Lambda = (\mathbb{R}^n \times W_\Lambda)/\Lambda$ is a classifying space of Λ. The composition af is the usual assembly map. This diagram is induced by a diagram of morphisms of stratified system of fibrations:

$$\begin{array}{ccccc} (1: B_\Lambda \to B_\Lambda) & \to & (p: B_\Lambda \to \mathbb{R}^n/\Lambda) & \to & (B_\Lambda \to *) \\ & \searrow & \downarrow & & \\ & & (1: \mathbb{R}^n/\Lambda \to \mathbb{R}^n/\Lambda). & & \end{array}$$

Note that the point inverses of p are the classifying spaces of the isotropy subgroups of Λ, which are finite, and hence are $\mathbb{Z}[1/k]$-acyclic, where k is any integer such that k and the order of these finite subgroups are coprime. This implies that p is an ordinary-$\mathbb{Z}[1/k]$-homology isomorphism, and hence p_* is a $(\mathbb{Z}[1/k]\otimes)$-isomorphism. And this, in return, implies that f is a split $(\mathbb{Z}[1/k]\otimes)$-injection. The main theorem states that a is a $(\mathbb{Z}[1/2]\otimes)$-isomorphism. Therefore we get:

(4.12) **Corollary.** *Suppose Λ is a group which acts by isometries on $\mathbb{R}^n$ discretely, virtually faithfully, with compact quotient, and k is an integer which is coprime with the order of each isotropy subgroup of Λ. Then the assembly map*

$$\mathbb{Z}[1/2, 1/k] \otimes H_*(B_\Lambda; \mathbb{L}(1)) \to \mathbb{Z}[1/2, 1/k] \otimes L_*^{-\infty}(\Lambda)$$

is a split injection.

This is a stronger form of the Novikov conjecture, which states that the assembly map is injective when tensored with $\mathbb{Q}$.

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